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Electromagnetic Theory

⇒ **Unit - I** (Chap. - 1, 2)

(1) Vector Algebra

(2) Electric Field & Potential energy & Properties of moving charge.

⇒ **Unit - II** (Chap. 3)

(3) Electric Field in Matter.

⇒ **Unit - III** (Chap. - 4, 5)

(4) Properties of Magnet.

(5) Magnetic Field in Matter.

⇒ **Unit - IV** (Chap. - 6)

6). (a) Maxwell equations.

(b) E.M. waves

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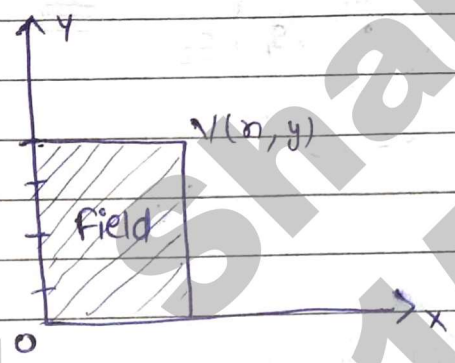
Chapter - 1
Vector Algebra

Field :- The space defined in the frame of reference where physical quantity depends upon coordinates, is said to be field.

Suppose any physical quantity A which depends on the co-ordinates x, y, z. i.e. $A = A(x, y, z)$



We know that physical quantities are of two types- i.e. scalar and vector. On this basis, field is also classified in two types -

(i) Scalar Field (ii) Vector Field.

1) **Scalar Field** :- Those field in which scalar physical quantity depends on coordinates, is known as scalar field. We know that, in scalar quantity only magnitude is known and there is no knowledge about the direction. Therefore, representation of scalar field is not possible by direction i.e. scalar field is represented in the form of magnitude.

Ex. Let, any charged particle is present in F.O.R. electric potential is generated around this charged particle which is represented by surfaces. Suppose, value of electric potential at surface S_1 is ϕ_1 and

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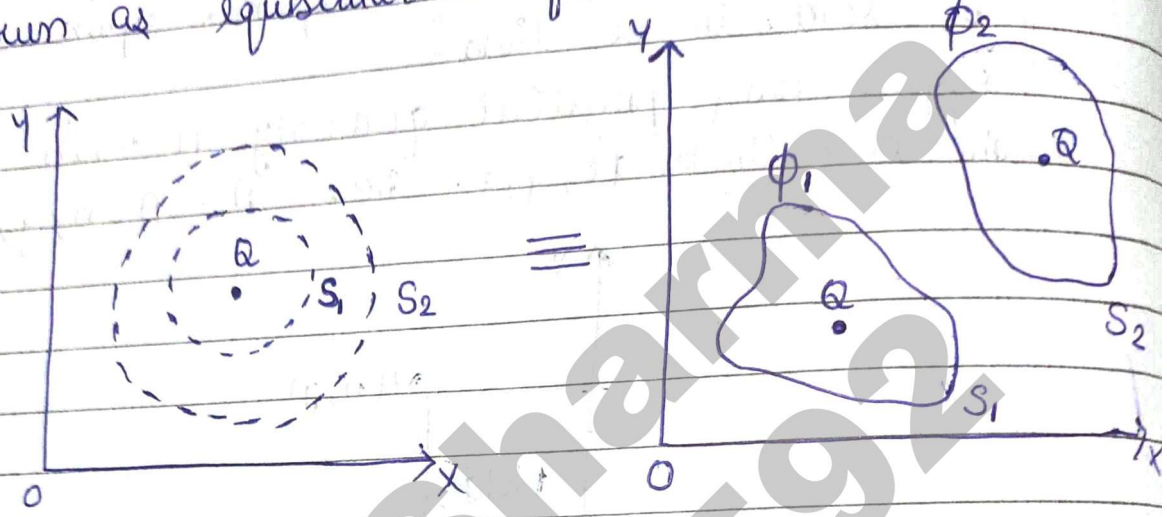
at S_2 surface the value of electric potential is ϕ_2. " If the value of scalar quantity is defined same on any surface then such surface is known as equiscalar surface."



* **Equipotential Surface:-**
Such surface in which the value of electric potential is same at every point, is said to be equipotential surface.

$$V_B - V_A = -\int \vec{E}\cdot d\vec{r}$$

if $V_A = V_B = V$

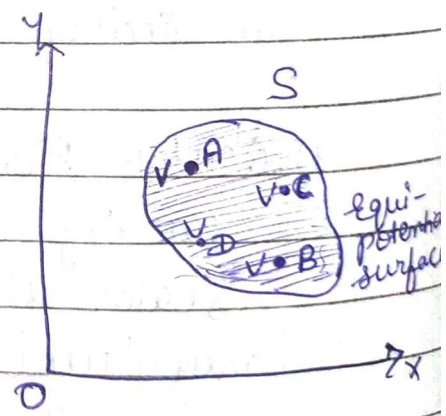
$$\Rightarrow 0 = -\int \vec{E}\cdot d\vec{r}$$

$$\vec{E}\cdot d\vec{r} = 0$$

$$|\vec{E}|\,|d\vec{r}|\cos\theta = 0$$

$$\Rightarrow \theta = \pi/2$$

$$\Rightarrow \vec{E} \perp d\vec{r}$$



This is the special feature of equipotential surface.

2. **Vector Field :-**
The space/place in F.O.R where vector physical quantity depends on co-ordinates is said to be vector field.

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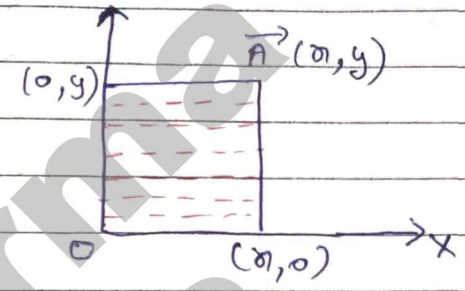
Let any vector physical quantity is $\vec{A}$ which depends on coordinates x, y, z. So, representation of vector field –

$$\boxed{\vec{A} = f(x, y, z)}$$



Ex. – Intensity of electric field,
Intensity of magnetic field,
Intensity of Gravitational field etc.

We know that in vector physical quantity along with magnitude dir^n is also known. Therefore, vector field is represented by the vector lines like electric field is represented by electric flux lines and magnetic field is represented by magnetic field lines. We know that vector lines never intersect each other because at the point of intersection there will be two directions of vector field which is impossible.

→ At any point of vector lines, the tangent line drawn, denotes the direction of vector field.

→ Vector field are of two types –
(i) Uniform vector field
(ii) Non-uniform vector field.

(i) Uniform Vector Field :–
Those vector field in which vector lines are parallel and present at equal distance, present in same direction and magnitude of vector lines is equal, is said to be uniform vector field.

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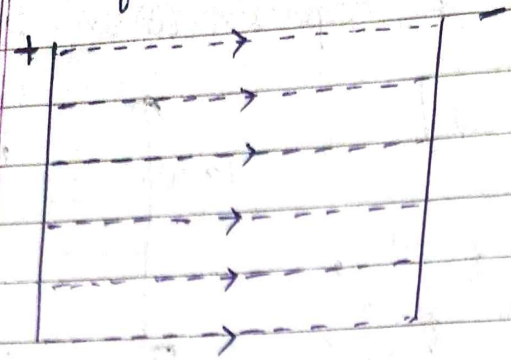
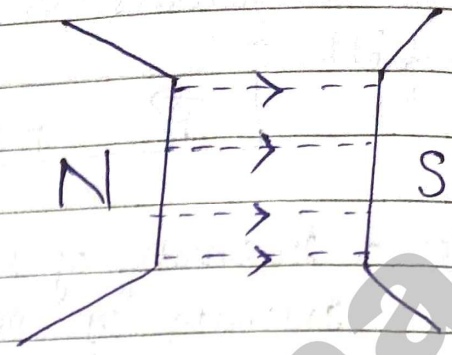
Ex. Uniform electric field.

(ii) ~~Non~~ uniform magnetic field

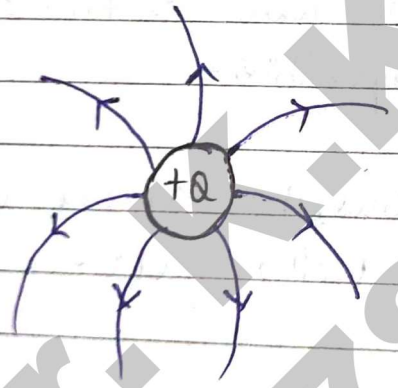




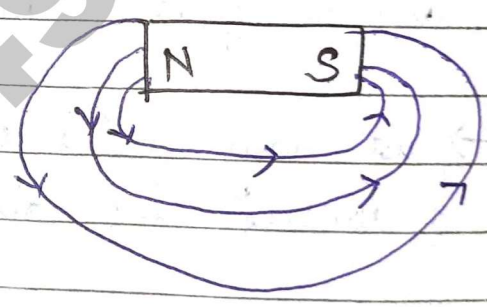
(2) Non- Uniform vector field :-

These vector field in which vector fields are not parallel, in opposite dir^n and present at different distance and different in magnitude, is said to be non- uniform vector field.

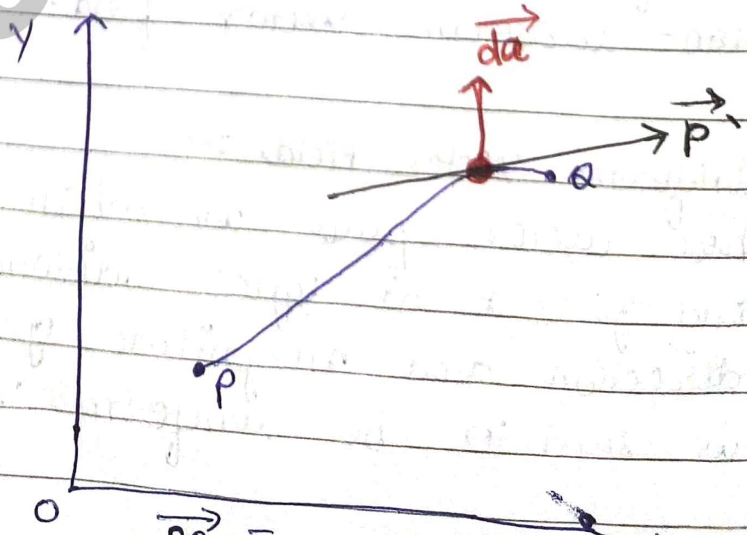
Ex. Non uniform Electric Field.



Non-uniform magnetic field.



Vector flux :-



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To explain the concept of flux, we image a frame of reference. Let, two point P & Q present in this frame of reference. These two points are connected by a path due to which formation of $\vec{PQ}$, a vector line takes place. We know that, from property of vector field, "The tangent drawn at any point of vector field denotes the direction of that vector field." Now, we image a small area element $\vec{da}$. We know that the dir^n of area is perpendicular to the plane.

'No. of vector field lines output from small area, is equal to the magnitude of vector field.

i.e. $$|\vec{A}| = \frac{d\phi}{da} \quad \text{——— (1)}$$

In vector form –

$$\vec{A} = \frac{d\phi}{da}\hat{n} \quad \text{——— (2)}$$

$\because \hat{n}$ = perpendicular unit vector

Dot product of $\hat{n}$ in eq^n (2). (scalar product)

$$\Rightarrow \vec{A}\cdot\hat{n} = \frac{d\phi}{da}\hat{n}\cdot\hat{n} \qquad \because \hat{n}\cdot\hat{n} = 1.$$

$$\Rightarrow \vec{A}\cdot\hat{n}\,da = d\phi$$

$$\because d\hat{a} = \hat{n} = \frac{\vec{da}}{|\vec{da}|} \quad \text{(from property of unit vector)}$$

$$\therefore \hat{n}\,da = \vec{da}$$

$$\Rightarrow \boxed{\vec{A}\cdot\vec{da} = d\phi}$$

on integrating,

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$$\boxed{\phi = \oint_S \vec{A}\cdot\vec{da}} \quad \text{——— (3)}$$

$\oint_S$ = closed surface integration.

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Eqⁿ ③ is said to be mathematical expression of flux.

Condition-01:- If vector field is a electric field.

$$\vec{A} = \vec{E}$$

from eqⁿ ③.

$$\boxed{\phi_E = \oint_S \vec{E} \cdot d\vec{a}} \quad \text{—— ④}$$

Eqⁿ ④ is said to be mathematical expression of electric flux.

Condition-02:- If vector field is a magnetic field, then-

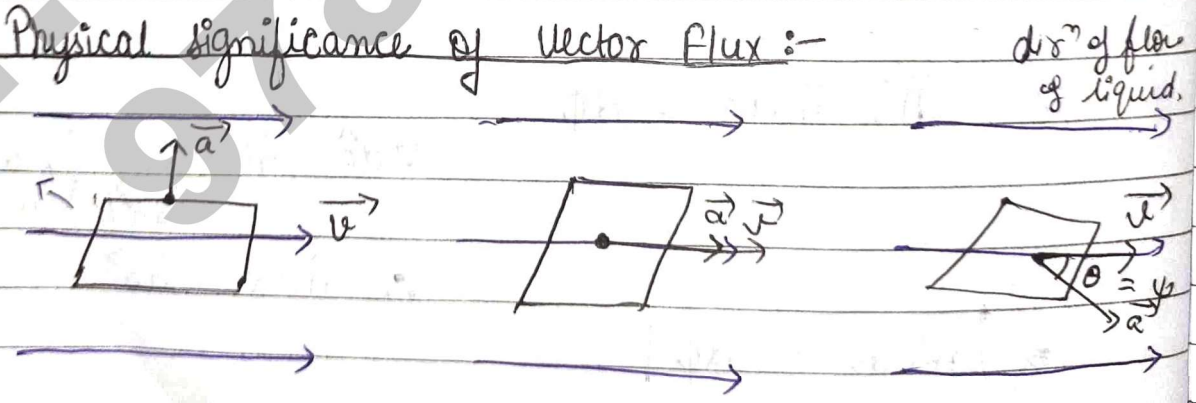
$$\vec{A} = \vec{B}$$

from eqⁿ ③.

$$\boxed{\phi_B = \oint_S \vec{B} \cdot d\vec{a}} \quad \text{—— ⑤}$$

Eqⁿ ⑤ is said to be mathematical expression of magnetic flux.

Physical significance of vector flux:-



(i) $\phi = \vec{v} \cdot \vec{a}$

$\phi = |\vec{v}|\,|\vec{a}| \cos\theta$

$\because \theta = 90^\circ$

$\phi = va \cos 90^\circ$

$\boxed{\phi = 0}$

(ii) $\phi = \vec{v} \cdot \vec{a}$

$\phi = |\vec{v}|\,|\vec{a}| \cos\theta$

$\theta = 0^\circ$

$\phi = va \cos 0^\circ$

$\boxed{\phi = va}$

(iii) $\phi = \vec{v} \cdot \vec{a}$

$\phi = |\vec{v}|\,|\vec{a}| \cos\theta$

$\phi = \psi$

$\boxed{\phi = va \cos\psi}$

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To explain physical significance of flux, we image a liquid which flows by $\vec{v}$ velocity. Let area of this hole is $\vec{A}$. We know that dir^n of area is perpendicular to the plane. Now, we placed the wooden frame in the path of flowing liquid and we see that, the output liquid quantity from hole depends on θ. i.e. for different conditions, the value of magnitude of output fluid is also different. This is the physical significance of flux.

Differential Operator :-

Those operator which shows the derivative term with respect to position, is said to be differential operator. It is denoted by $\vec{\nabla}$.

(a) Cartesian coordinate System :-

That frame of reference in which the coordinate param of particle are present in (x, y, z) form, is said to be cartesian coordinate system.

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

(i) Scalar Factor :-

$$\boxed{h_x^{(h_1)} = 1 \; ; \; h_y^{(h_2)} = 1 \; ; \; h_z^{(h_3)} = 1}$$

(ii) Length Element :-

$$dL_1 = dL_x = h_1 dU_1$$

$$\boxed{dL_1 = dx \; ; \; dL_2 = dy \; ; \; dL_3 = dL_z}$$

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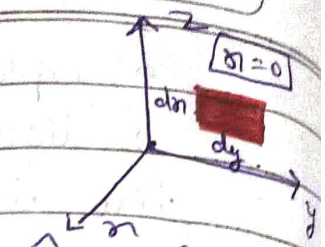


(iii) Surface element :- [2-D]

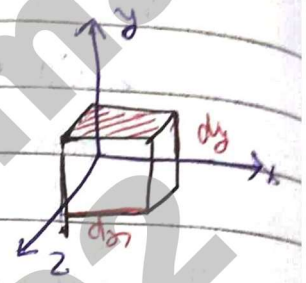
$$\overrightarrow{dS_1} = \overrightarrow{dL_2} \times \overrightarrow{dL_3}$$

$$\overrightarrow{dS_1} = dy\,\hat{j} \times dz\,\hat{k}$$

$$\boxed{dS_1 = dy\,dz\,\hat{i}}$$

$\because \hat{j} \times \hat{k} = \hat{i}$

(iv) Volume element :- [3-D]



$$d\tau = \overrightarrow{dL_1} \cdot (\overrightarrow{dL_2} \times \overrightarrow{dL_3})$$

$$d\tau = dx\,\hat{i} \cdot (dy\,\hat{j} \times dz\,\hat{k})$$

$$d\tau = dx\,dy\,dz\,[\hat{i} \cdot (\hat{j} \times \hat{k})]$$

$$d\tau = dx\,dy\,dz\,(\hat{i} \cdot \hat{i})$$

$$\boxed{d\tau = dx\,dy\,dz}$$

b) Spherical co-ordinate system :-

That frame of reference in which the coordinate parameters of particles are present in (r, θ, ϕ) form, is said to be spherical coordinate system.

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}}$$

(r, θ, ϕ) ; r : 0 to ∞

θ : 0 to π

ϕ : 0 to 2π

) Scalar Factor :-

$$\boxed{h_1 = h_r = 1 \; ; \; h_2 = h_\theta = r \; ; \; h_3 = h_\phi = r\sin\theta}$$

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(ii) Length Element :-

$dL_1 = h_1 dU_1$ | $dL_2 = h_2 dU_2$ | $dL_3 = h_3 dU_3$

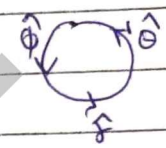
$dL_1 = 1 . dr$ | $dL_2 = r . d\theta$ | $dL_3 = r\sin\theta . d\phi$

$\boxed{dL_1 = dr}$ | $\boxed{\overrightarrow{dL_2} = r d\theta . \hat{\theta}}$ | $\boxed{\overrightarrow{dL_3} = r\sin\theta . d\phi \hat{\phi}}$

length element in r-dir^n. → $\overrightarrow{dL_1} = dr . \hat{r}$

(iii) Surface Element :-

$\overrightarrow{dS_1} = \overrightarrow{dL_2} \times \overrightarrow{dL_3}$; r = constant.



$\overrightarrow{dS_1} = r d\theta . \hat{\theta} \times r\sin\theta \, d\phi \, \hat{\phi}$

$\overrightarrow{dS_1} = r^2 \sin\theta \, d\theta \, d\phi \, (\hat{\theta} . \hat{\phi})$

$\boxed{\overrightarrow{dS_1} = r^2 \sin\theta \, d\theta \, d\phi . \hat{r}}$

$\overrightarrow{dS_3} = \overrightarrow{dL_1} \times \overrightarrow{dL_2}$; ϕ = constant.

$\overrightarrow{dS_3} = dr . \hat{r} \times r \, d\theta . \hat{\theta}$

$\overrightarrow{dS_3} = r . dr . d\theta \, (\hat{r} \times \hat{\theta})$

$\boxed{\overrightarrow{dS_3} = r \, dr \, d\theta \, \hat{\phi}}$

(iv) Volume Element :-

$d\tau = \overrightarrow{dL_1} . (\overrightarrow{dL_2} \times \overrightarrow{dL_3})$

$d\tau = dr . \hat{r} . [r \, d\theta \, \hat{\theta} \times r\sin\theta \, d\phi \, \hat{\phi}]$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi \, [\hat{r} . (\hat{\theta} \times \hat{\phi})]$

$\because \hat{\theta} \times \hat{\phi} = \hat{r}$; $\hat{r} . \hat{r} = 1$

M.T.R.

$\boxed{d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi}$

(c) Cylindrical coordinate system :-

That frame of reference in which the coordinate parameters of particles are present in (ρ, ϕ, z) form, is said to be cylindrical coordinate system.

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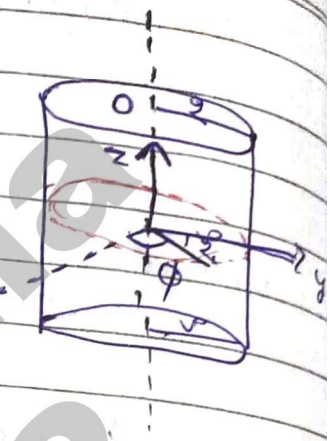
$u_1 = \rho$; $u_2 = \phi$; $u_3 = z$

ρ : 0 to ∞
ϕ : 0 to 2π
z : $-\infty$ to $+\infty$ (height)



(i) Differential operator :-

$$\boxed{\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}}$$

(ii) Scalar factor :-

$$\boxed{h_1 = 1 \; ; \; h_2 = \rho \; ; \; h_3 = 1}$$

(iii) Length Element :-

$\because \hat{z} = \hat{k}$

$\vec{dL_3} = h_3 dU_3 \hat{z}$
$\vec{dL_3} = 1 \cdot dz \cdot \hat{z}$
$\boxed{\vec{dL_3} = dz \, \hat{z}}$

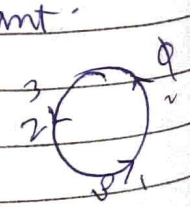
$\vec{dL_1} = h_1 dU_1 \hat{\rho}$
$\vec{dL_1} = 1 \cdot d\rho \, \hat{\rho}$
$\boxed{\vec{dL_1} = d\rho \, \hat{\rho}}$

$\vec{dL_2} = h_2 dU_2 \cdot \hat{\phi}$
$\vec{dL_2} = \rho \, d\phi \cdot \hat{\phi}$
$\boxed{\vec{dL_2} = \rho \, d\phi \cdot \hat{\phi}}$

(iv) Surface Element :-

$\vec{dS_1} = \vec{dL_2} \times \vec{dL_3}$; ρ = constant
$\vec{dS_1} = \rho \, d\phi \cdot \hat{\phi} \times dz \cdot \hat{z}$
$\vec{dS_1} = \rho \, d\phi \, dz \, (\hat{\phi} \times \hat{z})$
$\boxed{\vec{dS_1} = \rho \, d\phi \, dz \cdot \hat{\rho}}$ $\because \hat{\phi} \times \hat{z} = \hat{\rho}$

$\vec{dS_2} = \vec{dL_3} \times \vec{dL_1}$; ϕ = constant
$\vec{dS_2} = dz \cdot \hat{z} \times d\rho \cdot \hat{\rho}$
$\vec{dS_2} = dz \cdot d\rho \, (\hat{z} \times \hat{\rho})$
$\boxed{\vec{dS_2} = d\rho \cdot dz \, \hat{\phi}}$



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$$\vec{dS_3} = \vec{dL_1} \times \vec{dL_2} \quad ; z = \text{constant}$$
$$\vec{dS_3} = d\rho . \hat{\rho} \times \rho . d\phi . \hat{\phi}$$
$$\vec{dS_3} = \rho . d\rho . d\phi \, (\hat{\rho} \times \hat{\phi})$$
$$\boxed{\vec{dS_3} = \rho . d\rho . d\phi . \hat{z}}$$

(v) <u>Volume Element :-</u>

$$d\tau = dV = \vec{dL_1} . (\vec{dL_2} \times \vec{dL_3})$$
$$d\tau = d\rho . \hat{\rho} [\rho d\phi . \hat{\phi} \times dz \, \hat{z}]$$
$$d\tau = \rho d\rho d\phi dz [\hat{\rho} . (\hat{\phi} \times \hat{z})] \quad \because \hat{\phi} \times \hat{z} = \hat{\rho}, \; \hat{\rho} \times \hat{\rho} = 1$$
$$\boxed{d\tau = \rho d\rho d\phi dz}$$

Que.1 In spherical coordinate system, find the value of volume of sphere where a is the radius of sphere.

Que.2 Calculate the volume of cylinder, where the radius of cylinder is 'a' and 'h' is the height of cylinder in cylindrical coordinate system.

NOTE → (1) Divergance and curl are always calculated for vector ~~alg~~ quantity.

(2) The operation sign in divergance is • (dot) and the operation sign in curl is × (cross).

(3) $$\vec{A} . (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

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(4) Laplacian operator :-

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

* Laplacian Eqn :-

$$\boxed{\nabla^2 V = 0}$$

* Poission eqn :-

$$\boxed{\nabla^2 V = \frac{-\rho}{\epsilon_0}}$$; ρ = charge density

(5) The value of curl is zero for conservative field.
Ex. Electrostatic field is a conservative field.
Therefore, the curl of electrostatic field is zero.

$$\boxed{\text{curl } \vec{E} = \vec{\nabla} \times \vec{E} = 0}$$

(6) The value of divergance is always zero for solenoid field.
Ex. Magnetic field is a solenoid field, therefore, the divergance of magnetic field is zero.

$$\boxed{\text{div } \vec{B} = \vec{\nabla} \cdot \vec{B} = 0}$$

(7) The value of curl is zero for irrotational function.

(8) Theorem
- (a) $\boxed{\text{div curl } \vec{A} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0}$
- (b) $\boxed{\text{curl grad } \phi = \vec{\nabla} \times (\vec{\nabla} \phi) = 0}$

Que.2 If $\phi(x, y, z) = x^2 + y^2 + z^2$ then calculate the value of grad ϕ at point (1, 2, 3).

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Sol^n: We know that,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

On operating ϕ

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \quad \text{—— (1)}$$

$\because \phi = x^2 + y^2 + z^2$

d.w.r. to x.

$$\left.\begin{aligned} &\text{(i)} \quad \frac{\partial\phi}{\partial x} = 2x \\ &\text{(ii)} \quad \frac{\partial\phi}{\partial y} = 2y \\ &\text{(iii)} \quad \partial\phi/\partial z = 2z \end{aligned}\right\} \text{—— (2)}$$

from eq^n (1) & (2)

$$\vec{\nabla}\phi = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

at point $(1, 2, 3)$

$$\vec{\nabla}\phi\Big|_{(1,2,3)} = 2(1\hat{i} + 2\hat{j} + 3\hat{k})$$

$\because \vec{\nabla}\phi = \overrightarrow{grad}\,\phi$

$$\boxed{\overrightarrow{grad}\,\phi = 2\hat{i} + 4\hat{j} + 6\hat{k}}$$

$$\because |\overrightarrow{grad}\,\phi| = \sqrt{4+16+36}$$

$$\boxed{|\overrightarrow{grad}\,\phi) = \sqrt{56}}$$

★ NUMERICALS :→

Que.1 Prove that - $\overrightarrow{grad}\left(\frac{1}{r}\right) = \vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

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Solⁿ. We know that,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

on operating $1/r$.

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad (1)$$

$$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

d.w.r. to x.

$$2r\cdot\frac{\partial r}{\partial x} = 2x\cdot 1 + 0 + 0$$

$$\boxed{\frac{\partial r}{\partial x} = x/r} \; ; \; \boxed{\frac{\partial r}{\partial y} = y/r} \; ; \; \boxed{\frac{\partial r}{\partial z} = z/r}$$

Now.

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\cdot\frac{\partial r}{\partial x}$$

$$\rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-x}{r^3}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{r^3} \quad \Big| \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{r^3}$$

from eqⁿ (1).

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-1}{r^3}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\because \vec{\nabla}\left(\frac{1}{r}\right) = \overrightarrow{grad}\left(\frac{1}{r}\right)$$

$$\boxed{\overrightarrow{grad}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}}$$

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Que. 2 Prove that -

$$\boxed{\text{divgrad}\left(\frac{1}{r}\right) = 0}$$

Sol^n. We know that,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (1)}$$

L.H.S. $\text{divgrad}\left(\frac{1}{r}\right) = \vec{\nabla}\cdot\left[\vec{\nabla}\left(\frac{1}{r}\right)\right]$ —— (2).

Now.

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\left(\frac{-1}{r^2}\cdot\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\cdot\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\cdot\frac{\partial r}{\partial z}\right)$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

d.w.r. to x.

$$2r\cdot\frac{\partial r}{\partial x} = 2x\cdot 1 + 0 + 0$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \; ; \; \frac{\partial r}{\partial y} = \frac{y}{r} \; ; \; \frac{\partial r}{\partial z} = z/r.$$

from eq^n (3)

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

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from eq^n ②

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = \vec{\nabla}\cdot\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right]$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}+\frac{1}{r^3}-\frac{3y}{r^4}\frac{\partial r}{\partial y}+\frac{1}{r^3}-\frac{3z}{r^4}\frac{\partial r}{\partial z}\right]$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\frac{1}{r^3}-\frac{3x}{r^4}\cdot\frac{x}{r}+\frac{1}{r^3}-\frac{3y}{r^4}\cdot\frac{y}{r}+\frac{1}{r^3}-\frac{3z}{r^4}\cdot\frac{z}{r}\right]$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\frac{3}{r^3}-\frac{3}{r^5}\left(x^2+y^2+z^2\right)\right]$$

$$\because r^2 = x^2+y^2+z^2$$

$$\Rightarrow \text{div grad}\left(\frac{1}{r}\right) = -\left[\frac{3}{r^3}-\frac{3r^2}{r^5}\right]$$

$$\Rightarrow \boxed{\text{div grad}\left(\frac{1}{r}\right) = 0}$$

NOTE:- $\boxed{\frac{d}{dx}\log_e x = \frac{1}{x}\cdot 1 = \frac{1}{x}}$

Que 3 Prove that - $\boxed{\vec{\nabla}(\log_e r) = \overrightarrow{\text{grad}}(\log_e r) = \frac{\vec{r}}{r^2}}$

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* Theorem - 01 :-

Prove that - $\boxed{\text{curl grad } \phi = 0}$

PROOF :- we know that.

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (1)}$$

on operating scalar function ϕ.

$$\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \quad \text{—— (2)}$$

Now: eq^n (1) × eq^n (2)

$$\vec{\nabla} \times \vec{\nabla}\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right]$$

$$\Rightarrow \text{curl grad } \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$\Rightarrow \text{curl grad } \phi = \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$\boxed{\text{curl grad } \phi = 0}$$

* Theorem - 02 :-

Prove that - $\boxed{\text{div curl } \vec{A} = 0}$

Proof :- we know that.

(i) $$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (1)}$$

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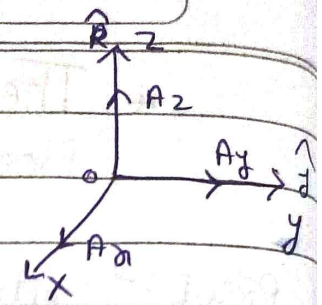
(ii) $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ ——— (2)



eq^n (1) × eq^n (2)

$$\Rightarrow \vec{\nabla}\times\vec{A} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \text{——— (3)}$$

eq^n (1) · eq^n (3)

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\Rightarrow \text{div curl } \vec{A} = \cancel{\frac{\partial^2 A_z}{\partial x\partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x\partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y\partial x}} + \cancel{\frac{\partial^2 A_x}{\partial y\partial z}} + \cancel{\frac{\partial^2 A_y}{\partial z\partial x}} - \cancel{\frac{\partial^2 A_x}{\partial z\partial y}}$$

$$\because \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$$

$$\Rightarrow \boxed{\text{div curl } \vec{A} = 0}$$

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Exam.

Que. Prove that the velocity of particle is irrotational function.

$$\vec{v} = 2x\hat{i} + 2y\hat{j} + \cancel{2z\hat{k}}$$

Sol^n. We know that, from property of vector algebra for irrotational function.

$$\boxed{\text{curl } \vec{v} = 0}$$

Now

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v}$$

$$\because \vec{\nabla} = \hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z$$

$$\because \vec{v} = 2x\hat{i} + 2y\hat{j} + \cancel{2z\hat{k}}$$

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 2y & 0 \end{vmatrix}$$

$$\text{curl } \vec{v} = \hat{i}\left[0 - \frac{\partial}{\partial z}(2y)\right] - \hat{j}\left[0 - \frac{\partial}{\partial z}(2x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(2x)\right]$$

(x, y, z) independent variable $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = \frac{\partial x}{\partial z} = \frac{\partial z}{\partial x} = 0$

$$\boxed{\text{curl } \vec{v} = 0}$$

Exam.

Que. Prove that for any constant vector $\vec{A}$.

$$\boxed{\vec{\nabla}(\vec{A} \cdot \vec{r}) = \vec{A}}$$

Proof:- We know that,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = \text{constant vector}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

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Now $\vec{A}\cdot\vec{r} = xA_x + yA_y + zA_z$

L.H.S.

$$\vec{\nabla}(\vec{A}\cdot\vec{r}) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](\vec{A}\cdot\vec{r})$$

$$\vec{\nabla}(\vec{A}\cdot\vec{r}) = \hat{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{r}) + \hat{j}\frac{\partial}{\partial y}(\vec{A}\cdot\vec{r}) + \hat{k}\frac{\partial}{\partial z}(\vec{A}\cdot\vec{r}) \quad \text{--- (3)}$$

Now

$$\frac{\partial}{\partial x}(\vec{A}\cdot\vec{r}) = \frac{\partial}{\partial x}(xA_x + yA_y + zA_z)$$

$$\frac{\partial}{\partial x}(\vec{A}\cdot\vec{r}) = \frac{\partial}{\partial x}(xA_x + 0 + 0)$$

(i) $\frac{\partial}{\partial x}(\vec{A}\cdot\vec{r}) = A_x \cdot 1 = A_x$

similarly

(ii) $\frac{\partial}{\partial y}(\vec{A}\cdot\vec{r}) = A_y$ | (iii) $\frac{\partial}{\partial z}(\vec{A}\cdot\vec{r}) = A_z$

from eqn (3)

$$\vec{\nabla}(\vec{A}\cdot\vec{r}) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

from eqn (2)

$$\boxed{\vec{\nabla}(\vec{A}\cdot\vec{r}) = \vec{A}}$$

NOTE:- (a) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

(b) $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$

(c) $\vec{\nabla}\times(\phi\vec{A}) = \vec{\nabla}\phi\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

IIT-JAM Que.

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Sol^n. Acq to vector algebra, the value of divergance is zero for solenoidal field.

$$\boxed{\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = 0} \quad \text{—— (1)}$$

Acq, $\vec{A}$ and $\vec{B}$ are two irrotational function; so,

$$\left.\begin{aligned}\vec{\nabla}\times\vec{A} &= 0\\ \vec{\nabla}\times\vec{B} &= 0\end{aligned}\right\} \text{—— (2)}$$

from, identity of vector algebra mathematics.

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

(from eq^n (2))

$$\boxed{\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = 0}$$

Hence, it is clear that $\vec{A}\times\vec{B}$ is a solenoid field.

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Que. Calculate the value of divergance & curl for the following vector field :-

(i) $\vec{F} = \hat{i}(x+y) + \hat{j}(x+y) - \hat{k}\cdot 2z$

(ii) $\vec{F} = \hat{i}y + \hat{j}(x^2+y^2) + \hat{k}\cdot z(x+y)$

Sol^n. (1) We know that -

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (1)}$$

$$\vec{F} = \hat{i}(x+y) + \hat{j}(x+y) - \hat{k} - 2z \quad \text{—— (2)}$$

(a) Divergance -

eq^n (1) & eq^n (2)

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$\text{div}\cdot\vec{F} = 1\cdot\frac{\partial}{\partial x}(x+y) + 1\cdot\frac{\partial}{\partial y}(x+y) - 1\cdot\frac{\partial}{\partial z}(2z)$$

$$\Rightarrow \text{div}\cdot\vec{F} = 1\cdot(1+0) + 1\cdot(0+1) - 1\cdot(2\cdot 1)$$

$$\text{div}\,\vec{F} = 1+1-2 = 2-2$$

$$\boxed{\text{div}\,\vec{F} = 0}$$

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(b) **Curl :-** eq^n ① × eq^n ②

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y) & (x+y) & -2z \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{F} = \hat{i}\left[\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(x+y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-2z) - \frac{\partial}{\partial z}(x+y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x+y)\right]$$

$$\Rightarrow \text{curl } \vec{F} = \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[(1+0)-(0+1)]$$

$$\boxed{\text{curl } \vec{F} = 0}$$

Que. Prove that – $\boxed{\text{divgrad}(\log_e r) = \frac{1}{r^2}}$

Sol^n Proof :- we know that,

$$\cancel{\vec{r} = x\hat{i} + y\hat{j} +}$$

(i) $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ —— ①

(ii) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2 \quad \text{—— ②}$$

d.w.r. to x.

$$2r.\frac{\partial r}{\partial x} = 2x.1 + 0 + 0$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \; ; \; \boxed{\frac{\partial r}{\partial y} = y/r} \; ; \; \boxed{\frac{\partial r}{\partial z} = z/r}$$

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Now. $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$\Rightarrow \vec{\nabla}(\log_e r) = \hat{i}\frac{\partial}{\partial x}(\log_e r) + \hat{j}\frac{\partial}{\partial y}(\log_e r) + \hat{k}\frac{\partial}{\partial z}(\log_e r)$

$\Rightarrow \vec{\nabla}(\log_e r) = \hat{i}\cdot\frac{1}{r}\cdot\frac{\partial r}{\partial x} + \hat{j}\cdot\frac{1}{r}\cdot\frac{\partial r}{\partial y} + \hat{k}\cdot\frac{1}{r}\cdot\frac{\partial r}{\partial z}$

$\Rightarrow \vec{\nabla}(\log_e r) = \hat{i}\cdot\frac{1}{r}\cdot\frac{x}{r} + \hat{j}\cdot\frac{1}{r}\cdot\frac{y}{r} + \hat{k}\cdot\frac{1}{r}\cdot\frac{z}{r}$

$\Rightarrow \vec{\nabla}\cdot\vec{\nabla}(\log_e r) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}\cdot\frac{x}{r^2} + \hat{j}\cdot\frac{y}{r^2} + \hat{k}\cdot\frac{z}{r^2}\right]$

$\Rightarrow \text{div.}\overrightarrow{\text{grad}}\log_e(r) = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$

$\Rightarrow \text{div}\overrightarrow{\text{grad}}(\log_e r) = \frac{1}{r^2} + x\left(\frac{-2}{r^3}\right)\cdot\frac{\partial r}{\partial x} + \frac{1}{r^2} + y\left(\frac{-2}{r^3}\right)\cdot\frac{\partial r}{\partial y}$

$+ \frac{1}{r^2} + z\cdot\left(\frac{-2}{r^3}\right)\cdot\frac{\partial r}{\partial z}$

$\Rightarrow \text{div}\overrightarrow{\text{grad}}(\log_e r) = \frac{3}{r^2} - \frac{2}{r^3}x\cdot\frac{x}{r} - \frac{2}{r^3}y\cdot\frac{y}{r} - \frac{2}{r^3}z\cdot\frac{z}{r}$

$\Rightarrow \text{div.}\overrightarrow{\text{grad}}(\log_e r) = \frac{3}{r^2} - \frac{2}{r^4}(x^2+y^2+z^2)$

$\Rightarrow \text{div.}\overrightarrow{\text{grad}}(\log_e r) = \frac{3}{r^2} - \frac{2}{r^4}\cdot r^2$

$$\boxed{\text{div}\overrightarrow{\text{grad}}(\log_e r) = \frac{1}{r^2}}$$

Que. Prove that :-

(a) $\text{div.}(\vec{A}\times\vec{r}) = 0$; when $\vec{A}$ is a constant vector

Proof :- We know that -

(i) $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ —— (1)

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(ii) $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ — a

(iii) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now.

$$\vec{A}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix}$$

$$\vec{A}\times\vec{r} = \hat{i}(zA_y - yA_z) - \hat{j}(zA_x - xA_z) + \hat{k}(yA_x - xA_y) \quad \text{— (2)}$$

LHS. $\text{div.}(\vec{A}\times\vec{r}) = \vec{\nabla}\cdot(\vec{A}\times\vec{r})$

$$\Rightarrow \text{div.}(\vec{A}\times\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot$$

$$\left[\hat{i}(zA_y - yA_z) - \hat{j}(zA_x - xA_z) + \hat{k}(yA_x - xA_y)\right]$$

$$\Rightarrow \text{div.}(\vec{A}\times\vec{r}) = \frac{\partial}{\partial x}(zA_y - yA_z) + \frac{\partial}{\partial y}(zA_x - xA_z) + \frac{\partial}{\partial z}(yA_x - xA_y)$$

$$\boxed{\text{div.}(\vec{A}\times\vec{r}) = 0}$$

(b) $\text{curl}(\vec{A}\times\vec{r}) = 2\vec{A}$

Sol^n We know that -

(i) $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ — (1)

(ii) $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

(iii) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now

$$\vec{A}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix}$$

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$$\vec{A} \times \vec{r} = \hat{i}(zA_y - yA_z) - \hat{j}(zA_x - xA_z) + \hat{k}(yA_x - xA_y)$$

Now.

$$\text{curl}(\vec{A} \times \vec{r}) = \vec{\nabla} \times (\vec{A} \times \vec{r})$$

$$\text{curl}(\vec{A} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zA_y - yA_z & xA_z - zA_x & yA_x - xA_y \end{vmatrix}$$

$$\Rightarrow \text{curl}(\vec{A} \times \vec{r}) = \hat{i}\left[\frac{\partial}{\partial y}(yA_x - xA_y) - \frac{\partial}{\partial z}(xA_z - zA_x)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(zA_y - yA_z) - \frac{\partial}{\partial x}(yA_x - xA_y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xA_z - zA_x) - \frac{\partial}{\partial y}(zA_y - yA_z)\right]$$

$$\Rightarrow \text{curl}(\vec{A} \times \vec{r}) = \hat{i}\left[(1.A_x - 0) - (0 - 1.A_x)\right] + \hat{j}\left[(1.A_y - 0) - (0 - 1.A_y)\right] + \hat{k}\left[(1.A_z - 0) - (0 - 1.A_z)\right]$$

$$\Rightarrow \text{curl}(\vec{A} \times \vec{r}) = 2A_x\hat{i} + 2A_y\hat{j} + 2A_z\hat{k}$$
$$= 2(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$\boxed{\text{curl}(\vec{A} \times \vec{r}) = 2\vec{A}}$$

V.V. most

Que Prove that -

$$\text{curl curl } \vec{A} = \overset{\text{grad div}}{\cancel{\text{divgrad}}}\ \vec{A} - \nabla^2 \vec{A} \quad \text{OR}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Proof :- We know that ;

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (1)}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{—— (2)}$$

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eq^n ① × eq^n ②

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \text{③}$$

eq^n ① × eq^n ③

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

Now:

$$(\vec{\nabla} \times \vec{\nabla} \times \vec{A})_x = \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$\Rightarrow (\vec{\nabla} \times \vec{\nabla} \times \vec{A})_x = \frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} + \frac{\partial^2 A_x}{\partial x^2}$$

$$\Rightarrow (\vec{\nabla} \times \vec{\nabla} \times \vec{A})_x = \left[-\frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}\right] + \left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial z \partial x}\right]$$

$$\Rightarrow (\vec{\nabla} \times \vec{\nabla} \times \vec{A})_x = -\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] A_x + \frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

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$$\because \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\because \vec{\nabla}\cdot\vec{A} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\Rightarrow (\vec{\nabla}\times\vec{\nabla}\times\vec{A})_x = -\nabla^2 A_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{A})$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = (\vec{\nabla}\times\vec{\nabla}\times\vec{A})_x\hat{i} + (\vec{\nabla}\times\vec{\nabla}\times\vec{A})_y\hat{j} + (\vec{\nabla}\times\vec{\nabla}\times\vec{A})_z\hat{k}$$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \hat{i}\left[-\nabla^2 A_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{A})\right] + \hat{j}\left[-\nabla^2 A_y + \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{A})\right] + \hat{k}\left[-\nabla^2 A_z + \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{A})\right]$$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\nabla^2(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\vec{\nabla}\cdot\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

$$\boxed{\text{curl curl}\,\vec{A} = \text{grad div}\,\vec{A} - \nabla^2\vec{A}}$$

Que. Prove that -

(a) $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$

(b) $\vec{\nabla}\times(\phi\vec{A}) = \vec{\nabla}\phi\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

(c) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

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NOTE :- (i) Laplacian eq^n -

$$\boxed{\nabla^2 V = 0}$$ (without charge field)

(ii) Passion eq^n :-

$$\boxed{\nabla^2 V = -\rho/\epsilon_0}$$ (with charge field)

(iii) Differential form of Gauss law -

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0}$$

Ans sol^n (a) - We know that -

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{--- (1)}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{--- (2)}$$

(i) $$\vec{\nabla} \cdot \vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{--- (3)}$$

(ii) $$\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Now -

$$\vec{\nabla}\phi \cdot \vec{A} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$\vec{\nabla}\phi \cdot \vec{A} = A_x\frac{\partial \phi}{\partial x} + A_y\frac{\partial \phi}{\partial y} + A_z\frac{\partial \phi}{\partial z} \quad \text{--- (4)}$$

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L.H.S.

$$\vec{\nabla}\cdot(\phi\vec{A}) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\phi(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\right]$$

$$\vec{\nabla}\cdot(\phi\vec{A}) = \frac{\partial}{\partial x}(\phi A_x) + \frac{\partial}{\partial y}(\phi A_y) + \frac{\partial}{\partial z}(\phi A_z)$$

$$\Rightarrow \vec{\nabla}\cdot(\phi\vec{A}) = \left(A_x\frac{\partial\phi}{\partial x} + \phi\frac{\partial A_x}{\partial x}\right) + \left(A_y\frac{\partial\phi}{\partial y} + \phi\frac{\partial A_y}{\partial y}\right) + \left(A_z\frac{\partial\phi}{\partial z} + \phi\frac{\partial A_z}{\partial z}\right)$$

$$\Rightarrow \vec{\nabla}\cdot(\phi\vec{A}) = \left[A_x\frac{\partial\phi}{\partial x} + A_y\frac{\partial\phi}{\partial y} + A_z\frac{\partial\phi}{\partial z}\right] + \phi\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] \quad \text{—— (5)}$$

from eq^n (3), (4) & (5)

$$\boxed{\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})}$$

Que. Find the value of intensity of E.F. and charge density for following electric potential.

(a) $V = 3x^2 - y + z$ volt.

(b) $V = 3x^2 + y^2 + 2z^2$ volt.

Sol^n. We know that -

$$\vec{E} = -\vec{\nabla}V = -\overrightarrow{grad}\,V$$

$$\vec{E} = -\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]V$$

$$\vec{E} = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right] \quad \text{—— (1)}$$

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(i) $v = 3x^2 - y + z$ volt ← ①

D.w.r. to x.

$\frac{\partial v}{\partial x} = 6x + 0 + 0$; $\frac{\partial v}{\partial y} = 0 - 1 + 0$; $\frac{\partial v}{\partial z} = 0 - 0 + 1$

$\frac{\partial v}{\partial x} = 6x$; $\frac{\partial v}{\partial y} = -1$; $\frac{\partial v}{\partial z} = +1$

from eq^n ①.

$$\vec{E} = -\left[\hat{i} \cdot 6x + \hat{j}(-1) + \hat{k}(+1)\right]$$

$$\boxed{\vec{E} = -6x\hat{i} + \hat{j} - \hat{k}} \quad \text{—— ②}$$

$$|\vec{E}| = \sqrt{36x^2 + 1 + 1} = \sqrt{36x^2 + 2}$$

→ Charge Density –

we know that -

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\Rightarrow \left[\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[-6x\hat{i} + \hat{j} - \hat{k}\right] = \rho / \epsilon_0$$

$$\Rightarrow \frac{\partial}{\partial x}(-6x) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(1) = \rho / \epsilon_0$$

$$\Rightarrow -6 = \rho / \epsilon_0$$

$$\Rightarrow \boxed{\rho = -6\epsilon_0}$$

2nd method :-

we know that -

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{——— (1)}$$

$\because V = 3x^2 - y + z$ volt.

$$\frac{\partial^2 V}{\partial x^2} = 6 \; ; \; \frac{\partial^2 V}{\partial y^2} = 0 \; ; \; \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = 6 + 0 + 0$$

$$\nabla^2 V = 6 \quad \text{——— (2)}$$

we know that from Passion eq^n -

$$\boxed{\nabla^2 V = -\rho/\epsilon_0} \quad \text{——— (3)}$$

compare eq^n (2) with eq^n (3)

$$-\rho/\epsilon_0 = 6$$

$$\boxed{\rho = -6\epsilon_0}$$

Laplacian and Poisson eq^n :-

Acq to concept of vector algebra mathematics, Calculation of laplacian & Poisson eq^n is done by Maxwell's 1st equation.

from 1st eq^n of maxwell -

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad \text{——— (1)}$$

Relation b/w electric potential and intensity of electric field -

$$\vec{E} = -\vec{\nabla} V \quad \text{——— (2)}$$

from eq^n (1) & (2)

$$\Rightarrow \vec{\nabla} \cdot (-\vec{\nabla} V) = \rho/\epsilon_0$$

$$\vec{\nabla} \cdot \vec{\nabla} V = -\rho/\epsilon_0 \quad \text{——— (3)}$$

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Now. $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\vec{\nabla} V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

from eq^n (3)

$$\Rightarrow \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right] = -\rho/\epsilon_0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = -\rho/\epsilon_0$$

$$\Rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\rho/\epsilon_0}$$

Laplacian operator :- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

M.T.R

$$\boxed{\nabla^2 V = -\rho/\epsilon_0} \quad \text{—— (4)}$$

$\because \rho = \frac{dq}{dv}$ = charge density

This is said to be Poisson's eq^n. This is valid for with charge region.

→ If charge is not present in bounded region then $\boxed{\rho = 0}$.

$$\boxed{\nabla^2 V = 0} \quad \text{—— (5)} \quad (\because \text{from eq}^n \text{ (4)})$$

eq^n (5) is known as Laplacian eq^n.

R.U.

Que. Show that the function $f = x^2 - y^2$ satisfies Laplacian eq^n.

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Solⁿ. We know that - from mathematical expression of laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{—— (1)}$$

on operation function f.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{—— (2)}$$

Now.

$f = x^2 - y^2$

$\frac{\partial \phi}{\partial x} = 2x + 0$

$\frac{\partial^2 \phi}{\partial x^2} = +2$

$\because f = x^2 - y^2$

$\frac{d\phi}{\partial y} = 0 - 2y$

$\frac{\partial^2 \phi}{\partial y^2} = -2$

$\frac{\partial^2 f}{\partial z^2} = 0.$

from eqⁿ (2)

$$\boxed{\nabla^2 f = 0} \quad \text{— (3)}$$

eqⁿ (3) is laplacian eqⁿ therefore function f. satisfies laplacian eqⁿ.

RU-2015

Que. Check function $\phi = b(a - r^2)$ satisfies laplacian eqⁿ - (a, b - constant)

Solⁿ We know that -

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{—— (1)}$$

on operating function ϕ.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad \text{—— (2)}$$

Position vector of particle -

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\boxed{r^2 = x^2 + y^2 + z^2}$$

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(i) $\phi = b(a - r^2)$

$\phi = ab - b(x^2 + y^2 + z^2)$

d.w.r. to x ; a & b = constant

$\frac{\partial \phi}{\partial x} = 0 - b(2x + 0 + 0)$

d.w.r. to x

$\frac{\partial^2 \phi}{\partial x^2} = -b \cdot 2 \cdot 1$

$\boxed{\frac{\partial^2 \phi}{\partial x^2} = -2b}$; $\boxed{\frac{\partial^2 \phi}{\partial y^2} = -2b}$; $\boxed{\frac{\partial^2 \phi}{\partial z^2} = -2b}$

from eqⁿ (2).

$\nabla^2 \phi = -2b + (-2b) + (-2b)$

$\boxed{\nabla^2 \phi = -6b}$ —— (3)

from Poisson's eqⁿ.

$\boxed{\nabla^2 \phi = -\rho/\epsilon_0}$ —— (4)

from eqⁿ (3) & (4)

$-6b = -\rho/\epsilon_0$

$\boxed{\rho = 6\epsilon_0 b}$

It is clear from eqⁿ (3) & (4) —

Electric Potential b/w two parallel plate :-

To calculate electric potential b/w two parallel plate is done by Laplacian and Poisson's eqⁿ. If the value of charge density is zero b/w plates that means without charge region then Laplacian equation is used. and if charge density b/w plates ($\rho \neq 0$). then Poisson's eqⁿ is used.

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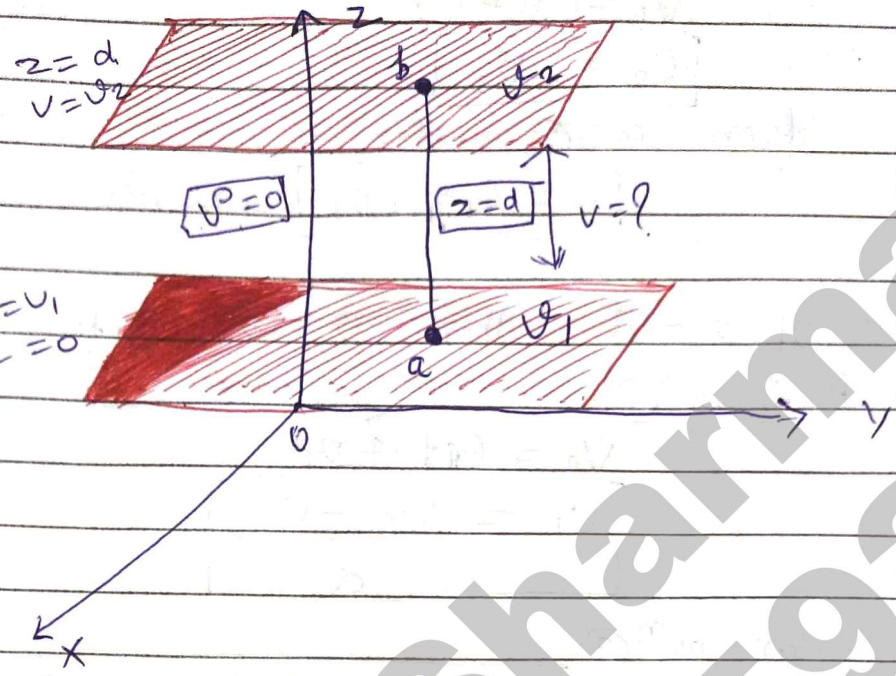
(1) By using Laplacian Equation :—



We know that from Laplacian equation :—

$$\boxed{\nabla^2 V = 0}$$

$$\because \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

In z-dir^n — {plates are present in x-y plane}.

$$\boxed{\frac{\partial^2 V}{\partial z^2} = 0} \quad \text{— (1)}$$

Eq^n (1) is a differential eq^n of second order.

$$\frac{\partial V}{\partial z} = C_1 \quad ; \; C_1 = \text{arbitrary constant.}$$

$$\boxed{V = C_1 z + C_2} \quad \text{— (2)} \quad ; \; C_2 = \text{Integral constant.}$$

Condition — 01 :— If $z = 0$ then $V = V_1$. [for 1st plate]

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from eq^n ②

$$V_1 = C_1 0 + C_2$$

$$\boxed{C_2 = V_1}$$

from eq^n ②

$$\boxed{V = C_1 z + V_1} \quad \text{—— ③}$$

Condition-2:- If $z = d$ then $V = V_2$ [for 2nd plate]

from eq^n ③.

$$V_2 = C_1 d + V_1$$

$$\boxed{C_1 = \frac{V_2 - V_1}{d}}$$

from eq^n ③

M.I.R.

$$\boxed{V = \left(\frac{V_2 - V_1}{d}\right) \cdot z + V_1} \quad \text{—— ④}$$

* Electric Field :-

we know that,

$$\vec{E} = -\vec{\nabla} V$$

$$\because \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{E} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) V$$

$$\vec{E} = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right]$$

In z-dir - (Plate in x-y)

$$\vec{E} = -\hat{k}\frac{\partial V}{\partial z}$$

from eq^n ④

$$\vec{E} = -\hat{k}\frac{\partial}{\partial z}\left[\left(\frac{V_2 - V_1}{d}\right) \cdot z + V_1\right]$$

$\because V_1 = 0$

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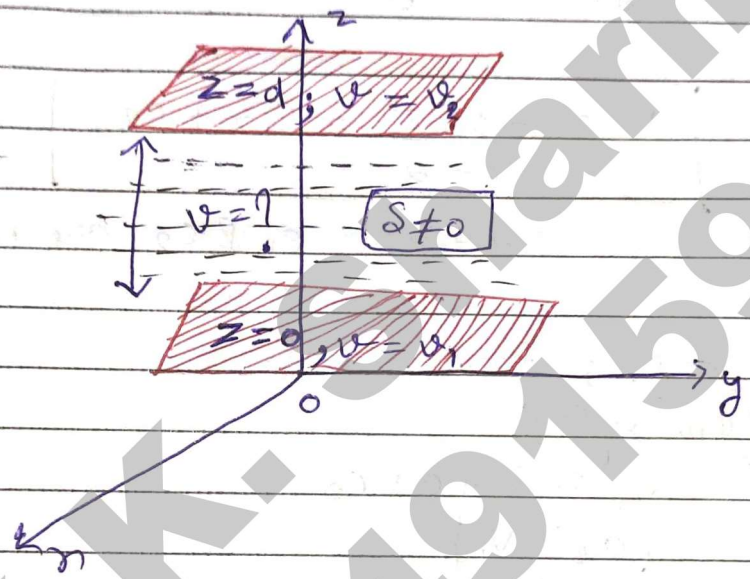
$$\vec{E} = -\hat{k}\left[\left(\frac{V_2 - V_1}{d}\right) \cdot 1 + 0\right]$$

M.T.R.

$$\boxed{\vec{E} = \hat{k}\left(\frac{V_1 - V_2}{d}\right)}$$

(2) Poisson's Equation :-



We know that from Poisson's eqn.

$$\boxed{\nabla^2 V = -\rho/\epsilon_0}$$

$\because$ ρ = charge density

$\because$ $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] V = -\rho/\epsilon_0$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\rho/\epsilon_0 \quad \text{--- (2)}$$

Plate present in x-y plane so $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ --- (3)

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from eq^n (2) & (3)

$$\frac{\partial^2 V}{\partial z^2} = -\rho/\epsilon_0$$

This is an differential eq^n of 2nd order whose sol^n is given by integral method.

$$\frac{\partial V}{\partial z} = \frac{-\rho}{\epsilon_0} z + C_1$$

$$\boxed{V = -\frac{\rho}{\epsilon_0}\frac{z^2}{2} + C_1 z + C_2} \quad \text{—— (4)}$$

C_1 & C_2 are integral constant.

Condition-1: If $z=0$; then $V = V_1$

from eq^n (4)

$$V_1 = 0 + 0 + C_2$$

$$\boxed{C_2 = V_1}$$

from eq^n (4)

$$\boxed{V = \frac{-\rho z^2}{2\epsilon_0} + C_1 z + V_1} \quad \text{—— (5)}$$

Condition 02:- If $z = d$ then $V = V_2$

from eq^n (5)

$$V_2 = \frac{-\rho d^2}{2\epsilon_0} + C_1 d + V_1$$

$$\Rightarrow (V_2 - V_1) + \frac{\rho d^2}{2\epsilon_0} = C_1 . d$$

$$C_1 = \left(\frac{V_2 - V_1}{d}\right) + \frac{\rho d}{2\epsilon_0}$$

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from eq^n (5)

$$V = \frac{-\rho z^2}{2\epsilon_0} + \left[\left(\frac{V_2 - V_1}{d}\right) + \frac{\rho d}{2\epsilon_0}\right] \cdot z + V_1 \quad \text{——— (6)}$$

$$\boxed{V = \frac{\rho z}{2\epsilon_0}(d - z) + \left(\frac{V_2 - V_1}{d}\right) + V_1}$$

★ Electric Field :-

we know that $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right]$$

In z-dir^n. [due to plate]

$$\vec{E} = -\hat{k}\frac{\partial V}{\partial z}$$

from eq^n (6)

$$\vec{E} = -\hat{k}\frac{\partial}{\partial z}\left[\frac{-\rho z^2}{2\epsilon_0} + \left\{\left(\frac{V_2 - V_1}{d}\right) + \frac{\rho d}{2\epsilon_0}\right\} \cdot z + V_1\right]$$

V_1 & V_2 are constant potential.

$$\vec{E} = -\hat{k}\left[\frac{-\rho}{2\epsilon_0} \cdot 2z \cdot 1 + \left\{\frac{V_2 - V_1}{d} + \frac{\rho d}{2\epsilon_0}\right\} \cdot 1 + 0\right]$$

MTR

$$\boxed{\vec{E} = -\hat{k}\left[\frac{\rho}{2\epsilon_0}(d - 2z) + \frac{V_2 - V_1}{d}\right]}$$

R.U.-2013

Que. If potential function $V = 4x^2 + 3y^2 - 9z^2$ volt then calculate the value of electric field at point (1, 2, 3)

Ans. $\sqrt{31} + V/m = E$

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~~Imp~~ Que.1 Define scalar gradient and prove mathematical expression of scalar gradient.

$$\boxed{\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}}$$

Que.2 Write geometrical significance of scalar gradient and prove mathematical expression of it in cartesian coordinate system. 2+7=9

$$\boxed{\text{div}\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

~~Que.3~~

Que.3 Define divergance of vector field and prove the following property by using vector algebra.

$$\boxed{\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A}}$$

Que.4 Prove that mathematical formula of curl in cartesian coordinate system and prove the following property.

$$\boxed{\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A}}$$

Que.5 Write the statement of Gauss divergance theorem and prove it.

Que.6 What is Stoke's curl theorem? Explain it with an example.

Que.7 Prove the Gauss law by using inverse square law.

$$\phi = \frac{\Sigma q}{\epsilon_0}$$

Que.8 Write the concept of solid angle and prove that

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the value of solid angle is 4π for closed spherical surface.

Que.9. Explain the physical significance of line integration of vector field.

Scalar Gradient :-

Acq to concept of vector algebra mathematics, max. change in scalar function with respect to distance, is said to be scalar gradient.
Scalar gradient is a vector quantity which is calculated for scalar function.
Let, a scalar function ϕ. Mathematical representation of its scalar gradient is done as following -

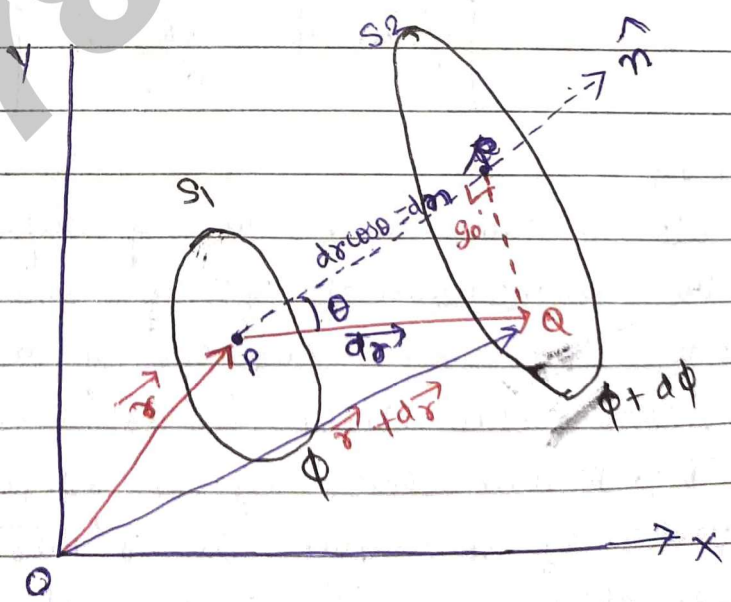
$$\boxed{\text{grad}\,\phi = \vec{\nabla}\phi}$$

So, algebraic expression of scalar gradient -

M.I.R.

$$\boxed{\overrightarrow{\text{grad}}\,\phi = \frac{d\phi}{dn}\hat{n} = \left(\frac{d\phi}{dr}\right)_{max.}\hat{n}}$$

Proof :-



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$$\boxed{dn = dr\cos\theta} \quad \text{——— (1)}$$

change in scalar function ϕ in $\overrightarrow{PQ}$ dir^n.

$$= \frac{d\phi}{dr}$$

from eq^n (1)

$$dn = dr\cos\theta$$

$$dr = dn/\cos\theta$$

change in scalar function ϕ in $\overrightarrow{PQ}$ - dir^n.

$$= \frac{d\phi}{dn/\cos\theta}$$

for maximum change -

~~change in scalar function ϕ in PQ - dir^n.~~

$$= \frac{d\phi}{dn}\cos\theta.$$

$$[\cos\theta]_{max.} = 1 \; ; \; \theta = 0^\circ$$

maximum change in scalar function $\phi = d\phi/dn$.

$$\text{grad}\,\phi = d\phi/dn$$

In vector form -

$$\boxed{\overrightarrow{\text{grad}}\,\phi = \frac{d\phi}{dn}\hat{n}}$$

Theory:-

We imagine a frame of reference to calculate the mathematical eq^n of scalar gradient. Let there are two surfaces S_1 and S_2 are present. Distance b/w them is 'dr' and angle b/w them is 'θ'. Position vector of S_1 surface is $\vec{r}$ and position vector of S_2 surface is $\vec{r} + d\vec{r}$. Scalar function on S_1 surface is ϕ and scalar function on S_2 surface

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is $\phi + d\phi$.

Scalar gradient in cartesian co-ordinate system :-

Those frame of reference in which coordinate parameters are present in (x, y, z) form, is said to be cartesian co-ordinate system.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

in differential form -

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz \quad \text{—— (2)}$$

Now.

$$\overrightarrow{grad}\,\phi = \frac{d\phi}{dn}\hat{n}$$

on multiplying $\vec{dr}$ on both sides,

$$\overrightarrow{grad}\,\phi \cdot \vec{dr} = \frac{d\phi}{dn}\hat{n} \cdot \vec{dr}$$

$$\because \hat{n} \cdot \vec{dr} = |\hat{n}||\vec{dr}|\cos\theta.$$

$$\hat{n} \cdot \vec{dr} = 1 \cdot dr \cdot \cos\theta.$$

$$\because dn = dr\cos\theta \quad (\text{from eq}^n\ (2))$$

$$\hat{n} \cdot \vec{dr} = dn.$$

$$\overrightarrow{grad}\,\phi \cdot \vec{dr} = \frac{d\phi}{dn}\,dn$$

$$\boxed{d\phi = \overrightarrow{grad}\,\phi \cdot \vec{dr}} \quad \text{—— (3)}$$

So Scalar function $\phi = \phi(x; y, z)$

on taking total derivative.

$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \quad \text{—— (4)}$$

We know that scalar gradient is a vector quantity, so, $\vec{g} = \overrightarrow{grad}\,\phi$.

$$\vec{g} = g_x\hat{i} + g_y\hat{j} + g_z\hat{k}.$$

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from eq^n (3)

$$\Rightarrow \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = (g_x \hat{i} + g_y \hat{j} + g_z \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$\Rightarrow \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = g_x dx + g_y dy + g_z dz$$

compare coefficient of dx, dy & dz.

(i) $g_x = \frac{\partial \phi}{\partial x}$ (ii) $g_y = \frac{\partial \phi}{\partial y}$ (iii) $g_z = \frac{\partial \phi}{\partial z}$

put the value in eq^n (5)

$$\vec{g} = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\because \vec{g} = \overrightarrow{grad}\, \phi$$

$$\boxed{\overrightarrow{grad}\, \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}} \quad \text{--- (6)}$$

$$\overrightarrow{grad}\, \phi = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right] \phi$$

$$\because \vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \text{differential operator}$$

M.T.R.

$$\boxed{\overrightarrow{grad}\, \phi = \vec{\nabla} \phi}$$

This is ~~the~~ said to be mathematical expression of scalar gradient.

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* Important Facts :-

(1) Scalar gradient represents the maximum change in scalar function with respect to distance.

(2) Scalar gradient is a general derivative.

(3) Scalar gradient $= \overrightarrow{grad}\,\phi = \vec{N}$ (normal vector)

Ex. :- If $\phi = x^2 + y^2$ then calculate normal vector.

Solⁿ.

$$\vec{N} = \overrightarrow{grad}\,\phi$$
$$\vec{N} = \vec{\nabla}\phi$$
$$\vec{N} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\because \phi = x^2 + y^2$$

$$\boxed{\vec{N} = 2x\hat{i} + 2y\hat{j}}$$

(4) Normal vector $\hat{N} = \dfrac{\vec{N}}{|\vec{N}|}$

$$\because \vec{N} = \overrightarrow{grad}\,\phi = \vec{\nabla}\phi$$

$$\boxed{\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}}$$

Ex. :- $\vec{N} = 2x\hat{i} + 2y\hat{j}$.

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}}$$

$$\hat{N} = \frac{1}{r}(x\hat{i} + y\hat{j}) \quad ; \quad r = \sqrt{x^2 + y^2}.$$

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(5) <u>Directional Derivative</u> :- $\vec{D} = \overrightarrow{grad}\,\phi \cdot \hat{A}$

$\because \overrightarrow{grad}\,\phi = \vec{\nabla}\phi$

$$\boxed{D = \vec{\nabla}\phi \cdot \hat{A}}$$

Ex:- If $\phi = x^2 + y^2$ and $\vec{A} = x\hat{i} + y\hat{j}$ then calculate directional derivative of scalar function ϕ along $\vec{A}$.

Solⁿ:- (i) $\vec{\nabla}\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$\because \phi = x^2 + y^2$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j}$$

(ii) $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$

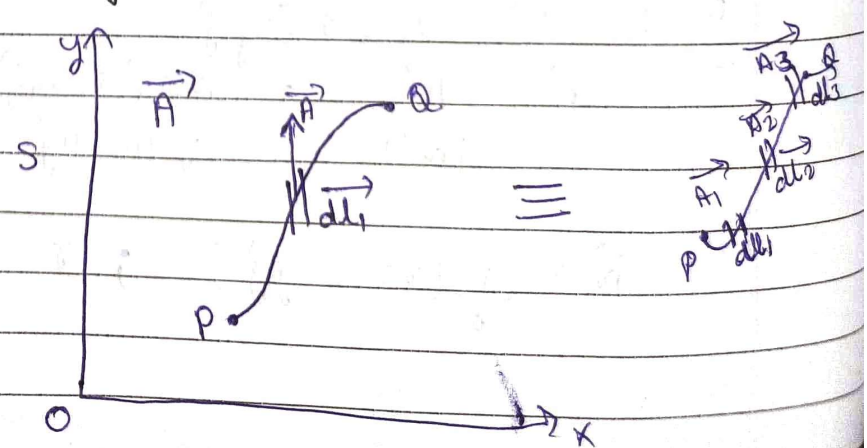
(iii) $D = \vec{\nabla}\phi \cdot \hat{A}$

$D = (2x\hat{i} + 2y\hat{j}) \cdot \dfrac{(x\hat{i} + y\hat{j})}{\sqrt{x^2+y^2}}$

$$\boxed{D = \frac{2x^2 + 2y^2}{\sqrt{x^2+y^2}}}$$

$\because r^2 = x^2 + y^2$

<u>Line Integral of a Vector field</u> :-



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To explain the concept of line integration of vector field, we imagine a frame of reference. In this F.O.R, a vector field $\vec{A}$ is defined. Let two points P and Q are present in F.O.R. These two points are joined by a path which leads to the formation of line PQ. Now, we divide line PQ in length elements. Let

The corresponding vector field to $\vec{dl_n}$ is $\vec{A_n}$.

Now, sum of dot product –

$$r = \vec{A_1}\cdot\vec{dl_1} + \vec{A_2}\cdot\vec{dl_2} + \vec{A_3}\cdot\vec{dl_3} + \ldots\ldots + \vec{A_n}\cdot\vec{dl_n}$$

$$r = \sum_{i=1}^{n} \vec{A_i}\cdot dl_i$$

We know that the sum of discrete function is done by summation and the sum of continuous function is done by integration. If distribution of length element on PQ line is continuous i.e. in micro region then in the place of summation, integration is used.

$$\boxed{r = \int_P^Q \vec{A}\cdot\vec{dl}} \quad \text{— (1)}$$

Eqn (1) is known as mathematical expression of line integration of vector field.

Physical Significance :–

(1) In line integration of vector field, $\vec{A}$ is a general vector field. If this vector field is a force field then the line integration of vector field defines work.

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$$\gamma = \int_P^Q \vec{A} \cdot \vec{dl}$$

if $\vec{A} = \vec{F}$ then $\gamma = w$

$$\boxed{w = \int_P^Q \vec{F} \cdot \vec{dr}}$$

This is an algebraic function of work done. ~~If~~

(2) ~~If~~ in the line integral of vector field, vector field $\vec{A}$ is defined in the form of electric field then the line integral of vector field ~~is~~ represents mathematical formula (Electric Potential).

$$\gamma = \int_P^Q \vec{A} \cdot \vec{dl}$$

if $\vec{A} = \vec{E}$ then $\gamma = v$

$$\boxed{v = \int_P^Q \vec{E} \cdot \vec{dr}}$$

This is mathematical formula of electric potential.

Important facts :-

(1) Line integration of vector field generally depends on path.

(2) Line integration of any closed path is known as circulation. It is denoted by $\oint_C$ or $\oint_l$.

$$\boxed{\gamma = \oint_P^Q \vec{A} \cdot \vec{dl} \rightarrow \text{circulation}}$$

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(3) If vector field present in circulation is a conservative field then the line integration of vector field will be zero. i.e. the value of circulation will be zero.

$$\boxed{\gamma = \oint_P^Q \vec{A} \cdot \vec{dl} = 0 \quad ; \vec{A} = \text{conservative} \quad \text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = 0}$$

In this condition, line integration of vector field doesn't depend on path.

If $\vec{A}$ is non-conservative field in circulation then the line integration depends on path.

Que. $\vec{A} = \hat{i}(4xy - 3x^2z^2) + \hat{j}\,2x^2 - \hat{k}\cdot 2x^3 z$

Prove that the following vector field is a conservative field and find the value of electric potential for following vector field -

Solⁿ. We know that from property of vector algebra mathematics, the value of curl is zero for conservative field. Therefore -

$$\boxed{\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = 0} \quad \text{—} \; (1)$$

Proof:

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Now.

$$\vec{\nabla} \times \vec{A} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

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$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$A_x = 4xy - 3x^2z^2$; $A_y = 2x^2$; $A_z = -2x^3z$

$$\vec{\nabla} \times \vec{A} = \hat{i}(0-0) - \hat{j}(-6x^2z - 6x^2z) + \hat{k}(4x - 4x)$$

$$\boxed{\vec{\nabla} \times \vec{A} = 0}$$

So, the vector field is a conservative field.

~~So,~~

(ii) We know that,

$$\boxed{V = \int \vec{A} \cdot \vec{dl}}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$V = V = \int (A_x dx + A_y dy + A_z dz) + c$$

$$V = \int (4xy - 3x^2z^2)\,dx + \int 2x^2\,dy - \int 2x^3 z\,dz + c$$

$$V = \frac{4x^2y}{2} - \frac{3x^2z^2}{3} + 2x^2y - 2x^3\frac{z^2}{2} + c$$

$$V = 2x^2y - x^3z^2 + 2x^2y - x^3z^2 + c.$$

$$\boxed{V = 2\left[2x^2y - x^3z^2\right] + c}$$ Ans.

Que. 2.

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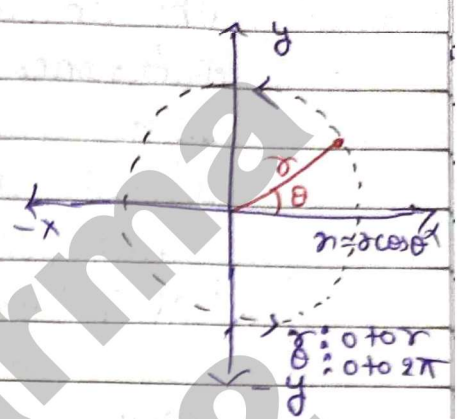
Sol^n. Given, In x-y plane (2-D)

$x^2+y^2 = r^2$; $z=0$; $\gamma = \int \vec{A}\cdot\vec{dl} = ?$

$\vec{A} = -\hat{i}y + \hat{j}x$.

$x = r\cos\theta$; $y = r\sin\theta$

$\vec{dl} = dx\,\hat{i} + dy\,\hat{j}$



Now.

$\vec{A}\cdot\vec{dl} = (-\hat{i}y + \hat{j}x)\cdot(dx\,\hat{i} + dy\,\hat{j})$

$\boxed{\vec{A}\cdot\vec{dl} = -y\,dx + x\,dy}$

$x = r\cos\theta$; $y = r\sin\theta$

d.w.r. to θ ; d.w.r. to θ

$\frac{dx}{d\theta} = r(-\sin\theta).1$; $\frac{dy}{d\theta} = r\cos\theta.1$

$dx = -r\sin\theta\,d\theta$; $dy = r\cos\theta\,d\theta$

$\vec{A}\cdot\vec{dl} = -r\sin\theta.(-r\sin\theta\,d\theta) + r\cos\theta.r\cos\theta\,d\theta$

$\vec{A}\cdot\vec{dl} = +r^2\sin^2\theta\,d\theta + r^2\cos^2\theta\,d\theta$

$\vec{A}\cdot\vec{dl} = r^2(\sin^2\theta + \cos^2\theta)\,d\theta$

$\vec{A}\cdot\vec{dl} = r^2\,d\theta$

Now

$\gamma = \oint_c \vec{A}\cdot\vec{dl}$

$\gamma = \oint_0^{2\pi} r^2\,d\theta$

$\gamma = r^2.[\theta]_0^{2\pi} = r^2(2\pi - 0)$

$\boxed{\gamma = 2\pi r^2}$

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Divergance of Vector Field :-

Acq to concept of vector algebra mathematics, divergance is always calculated of vector function. After calculating divergance, scalar function is defined because dot operation is present in divergance. Divergance is denoted by "div $\vec{A}$ or $\vec{\nabla}\cdot\vec{A}$."

(a) Mathematical Definition :-

For any vector field, in the situation of small volume, the value of flux output from unit volume is said to be divergance of vector field. The meaning of word divergance ~~mean~~ is spread.

we know that from definition of flux.

$$\phi = \oint_S \vec{A}\cdot\vec{ds}$$

so, divergance of vector field $\vec{A}$.

$$\text{div}\cdot\vec{A} = \lim_{dv\to 0} \frac{\phi}{dv}$$

M.T.R

$$\boxed{\text{div}\,\vec{A} = \lim_{dv\to 0} \frac{\oint_S \vec{A}\cdot\vec{ds}}{dv}} \longrightarrow (1)$$

eqn (1) is said to be algebraic expression of divergance of vector field.

(b) Physical significance of divergance :-

To understand the physical significance of divergance

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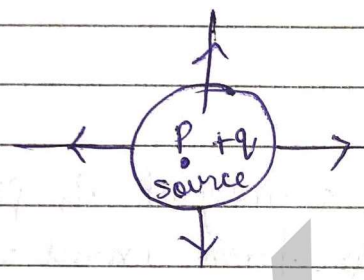
we imagine a liquid flowing with $\vec{v}$ velocity. Now, we consider small volume dv.

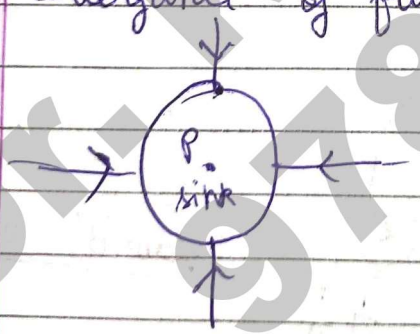
(i) If liquid quantity output from small volume then the small volume acts as a source. In this condition, divergance of flowing liquid found to be positive.

Ex. Acc to concept of electrostatic physics, electric flux lines emitted from positive charge. If +q charge present in small volume dv then this small volume behaves like a source. In this condition, divergance of electric field found positive.



$$\boxed{\text{div } \vec{E} = +ve}$$

(ii) If liquid quantity input in small volume then the small volume acts as sink. In this condition, divergance of flow of liquid found to be negative



$$\boxed{\text{div } \vec{E} = -ve}$$

Ex. If ~~as~~ -ve q charge present in small volume dv then electric flux lines input in small volume that means ~~-ve~~ -q charge behaves like a sink. In this condition, divergance of vector field found to be negative.

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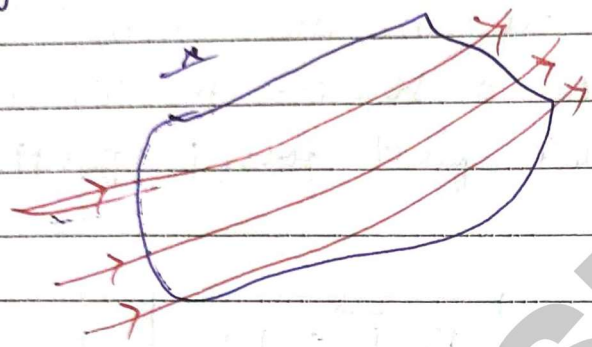
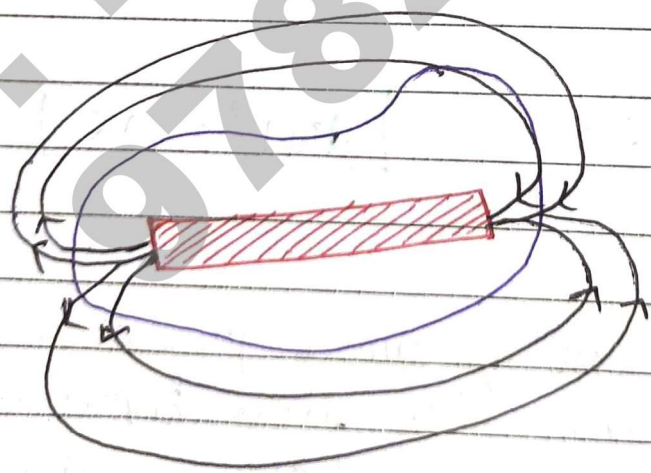
(iii) If in small volume, the value of input liquid is equal to output liquid then in this condition there is no change in small volume i.e. this is known as absence condition of source and sink and in this condition, the divergance of flowing liquid founds zero.

So, we can say that the divergance of vector field can be +ve, -ve or zero.



$$\boxed{\text{div } \vec{V} = 0}$$

Ex. If in small volume, a magnet is present then output magnetic flux lines from small volume and input magnetic flux lines are equal. i.e. The resultant change in small volume is zero. In this condition, the divergance of magnetic field found to be zero.



$$\boxed{\text{div } \vec{B} = 0}$$

This is known as 2nd eqn of maxwell which gives the mathematical verification of the fact that the single pole of magnet ~~isn't possible~~ doesn't exist in universe.

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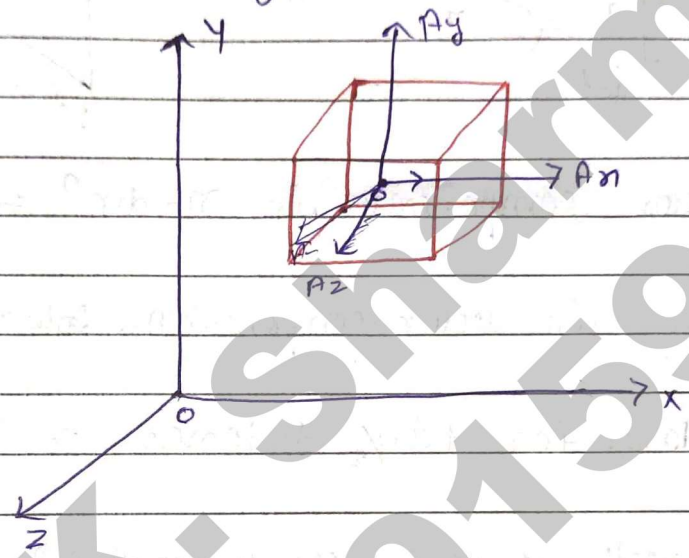
Divergance in Cartesian coordinate System :-

That system of frame of reference in which coordinates are present in (x, y, z) form, is said to be cartesian coordinate system.

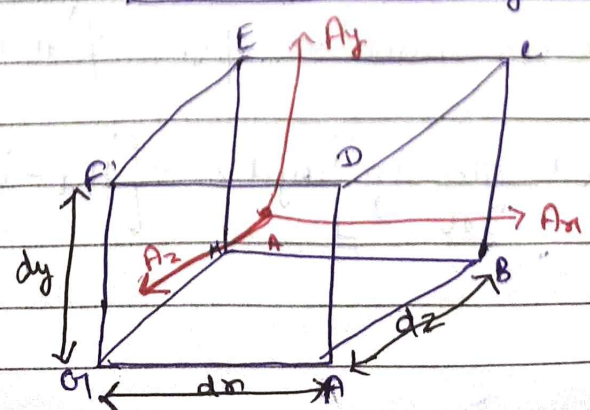


We imagine a vector field $\vec{A}$ in cartesian c.
Let, in x-dirⁿ vector component is A_x, in y-dirⁿ (A_y) and vector component in z-dirⁿ is A_z.

$$\boxed{\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}} \quad \text{—— (1)}$$

Let, length of small volume in x-dirⁿ is dx and width is dy and height is dz.
So, small volume =

$$\boxed{dv = d\tau = dx\,dy\,dz} \quad \text{—— (2)}$$



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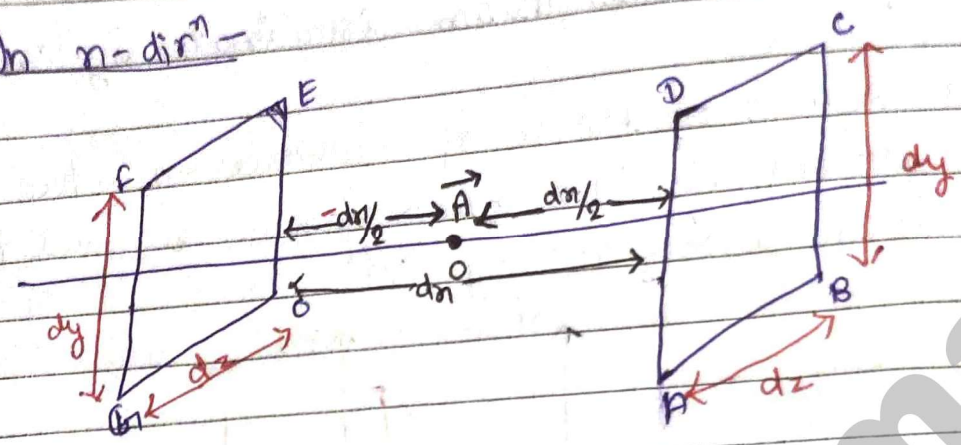
(i) In x-dir^n –



Vector component in x-dir^n = A_x

change in vector component A_x w.r.to x = $\frac{\partial A_x}{\partial x}$

change for $+dx/2$ distance = $\frac{\partial A_x}{\partial x}\left(+\frac{dx}{2}\right)$

Total vector component on surface ABCD = $A_x + \frac{\partial A_x}{\partial x}\left(+\frac{dx}{2}\right)$

$= A_x + \frac{\partial A_x}{\partial x}\frac{dx}{2}$

Output flux from ABCD surface = vector component on surface × area of surface

$$\phi_x' = +\left[A_x + \frac{\partial A_x}{\partial x}\frac{dx}{2}\right]dy\,dz$$

similarly,

input flux on OEFG surface $\phi_x'' = -\left[A_x - \frac{\partial A_x}{\partial x}\frac{dx}{2}\right]dy\,dz$

resultant flux in x-dir^n ⇒ $\phi_x = \phi'_x + \phi''_x$

$$\phi_x = \left[A_x + \frac{\partial A_x}{\partial x}\frac{dx}{2}\right]dy\,dz - \left[A_x - \frac{\partial A_x}{\partial x}\frac{dx}{2}\right]dy\,dz$$

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$$\phi_x = \left[\cancel{A_x} + \frac{\partial A_x}{\partial x}\frac{dx}{2} - \cancel{A_x} + \frac{\partial A_x}{\partial x}\frac{dx}{2}\right] dy\,dz$$

$$\phi_x = \frac{\partial A_x}{\partial x} dx\,dy\,dz$$

$$\because d\tau = dV = dx\,dy\,dz$$

$$\boxed{\phi_x = \frac{\partial A_x}{\partial x} dV}$$

similarly,

(ii) resultant flux in y-dir^n – $\boxed{\phi_y = \frac{\partial A_y}{\partial y} dV}$

iii resultant flux in z-dir^n – $\boxed{\phi_z = \frac{\partial A_z}{\partial z} dV}$

Total output flux from small volume (cuboid).

$$\phi = \phi_x + \phi_y + \phi_z.$$

$$\phi = \frac{\partial A_x}{\partial x} dV + \frac{\partial A_y}{\partial y} dV + \frac{\partial A_z}{\partial z} dV$$

$$\phi = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dV$$

$$\because \phi = \oint_S \vec{A}\cdot\vec{ds}$$

$$\Rightarrow \oint_S \vec{A}\cdot\vec{ds} = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dV.$$

$$\Rightarrow \lim_{dV\to 0} \frac{\oint \vec{A}\cdot\vec{ds}}{dV} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

from definition of divergance,

$$\text{div}\,\vec{A} = \lim_{dV\to 0} \frac{\oint_S \vec{A}\cdot\vec{ds}}{dV}$$

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M.T.R.

$$\boxed{\text{div}\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \quad \text{—— (I)}$$

We know that,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{—— (II)}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{—— (III)}$$

eq^n (II) . eq^n (III)

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{—— (IV)}$$

from eq^n (I) & (IV)

$$\boxed{\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A}}$$

H.P.

Gauss Divergance Theorem :-

This is a fundamental theorem of vector algebra mathematics. By this theorem, surface integration get converted into volume integration.
So, mathematical statement of theorem -

$$\boxed{\oint_S \vec{A}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{A})\,dv}$$

~~Such~~ That fundament theorem in which surface integral of vector field is equal to volume integral of divergance of vector field, is known as Gauss divergance theorem.

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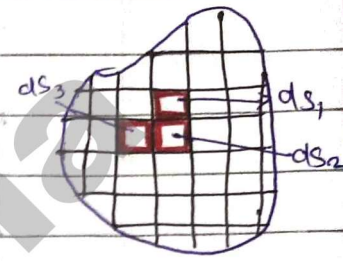
Proof :- We know that, the output flux quantity from any (S) surface is equal to the sum of output flux from small surface element present in that surface.



We know that,

$$\oint_S \vec{A}\cdot\vec{ds} = \oint_{S_1} \vec{A_1}\cdot\vec{ds_1} + \oint_{S_2} \vec{A_2}\cdot\vec{ds_2} + \ldots\ldots$$

$$\oint_S \vec{A}\cdot\vec{ds} = \sum_i \oint \vec{A_i}\cdot\vec{ds_i} \quad \text{——— (1)}$$

from definition of divergance.

$$\text{div}\vec{A} = \lim_{dv\to 0} \frac{\oint \vec{A}\cdot\vec{ds}}{dv}$$

$$\therefore \text{div}\vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$(\vec{\nabla}\cdot\vec{A})dv = \oint_S \vec{A}\cdot\vec{ds}$$

for i^{th} surface element.

$$(\vec{\nabla}\cdot\vec{A_i})dv = \oint_S (\vec{A_i}\cdot\vec{ds_i})$$

for whole surface.

$$\sum_i (\vec{\nabla}\cdot\vec{A_i})dv = \sum_i \oint_S \vec{A_i}\cdot\vec{ds_i} \quad \text{——— (2)}$$

from eq^n (1) & (2)

$$\Rightarrow \oint \vec{A}\cdot\vec{ds} = \sum_i (\vec{\nabla}\cdot\vec{A_i})\, dv$$

If small volume $dv \to 0$ then summation is replaced by integration because we know that

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sum of discrete function is done by summation while sum of continuous function is done by integration.

if $dv \to 0$

$$\boxed{\oint_S \vec{A} \cdot \vec{ds} = \int (\vec{\nabla} \cdot \vec{A})\, dv} \quad \text{—— (3)}$$

eq^n (3) is the mathematical form of Gauss divergance theorem.

Applications of Divergance Theorem:—

(1) Calculation of 1st eq^n of maxwell is done by divergance theorem. 1st eq^n of maxwell is the divergance of electric field.

We know that from divergance theorem.

$$\oint_S \vec{A} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{A})\, dv \quad \text{—— (1)}$$

If $\vec{A} = \vec{E}$ = electric field then from eq^n (1)

$$\boxed{\oint \vec{E} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{E})\, dv} \quad \text{—— (2)}$$

from Gauss statement.

$$\phi = \frac{\Sigma q}{\epsilon} \quad \text{—— (i)}$$

$\because \Sigma q$ = total charge

$$\phi = \oint_S \vec{E} \cdot \vec{ds} \quad \text{—— (ii)}$$

ϕ = flux.

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from eq^n (i) & (ii)

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{\Sigma q}{\epsilon}$$

$\because \Sigma q = q$ (let)

$$\boxed{\oint_S \vec{E} \cdot \vec{ds} = \frac{q}{\epsilon}} \quad \text{—— (3)}$$

from volume charge density.

$$\rho = \frac{dq}{dV} \quad \text{- volume charge density}$$

$$dq = \rho dv$$

on integrating.

$$q = \int_V \rho dv \quad \text{—— (4)}$$

from eq^n (3) & (4)

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon} \int_V \rho dv \quad \text{—— (5)}$$

from eq^n (2) & (5)

$$\int_V (\vec{\nabla} \cdot \vec{E}) dv = \frac{1}{\epsilon} \int_V \rho dv$$

$$\boxed{\int_V (\vec{\nabla} \cdot \vec{E}) dv = \int_V \frac{\rho}{\epsilon} dv}$$ integral form.

comparing coefficients of dv.

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 \\ \text{div. } \vec{E} &= \rho/\epsilon_0 \end{aligned}}$$ differential form

Now,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$$

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$$\vec{\nabla} \cdot \epsilon \vec{E} = \rho$$

displacement vector $\vec{D} = \epsilon \vec{E}$

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

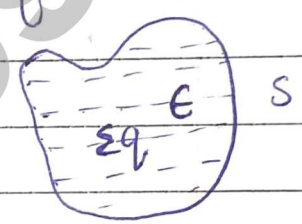
This is the first equation of maxwell.

Derivation of Gauss law from inverse square law:-

① Gauss postulate :-

Acq to electrostatic physics, the ~~value of~~ output flux quantity from a closed surface is equal to the ~~to~~ ratio of total charge present in that surface to permittivity of surface. i.e.

$$\boxed{\phi = \frac{\Sigma q}{\epsilon}}$$



This is the statement of Gauss.

$$\phi = \frac{\Sigma q}{\epsilon_0}$$

$$\phi = \frac{4\pi}{4\pi} \frac{\Sigma q}{\epsilon_0}$$

$$\because K = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \, N\text{-}m^2/C^2$$

$$\boxed{\phi = 4\pi K \Sigma q}$$

Proof :- To verify mathematically Gauss statement, we imagine a closed surface. Let this imaginary surface is S and total charge present in this surface is q due to which production of electric field takes place. Now we imagine a surface element $\vec{ds}$. ~~Electric f~~ Angle b/w E.F. $\vec{E}$ and surface element $\vec{ds}$ is (θ).

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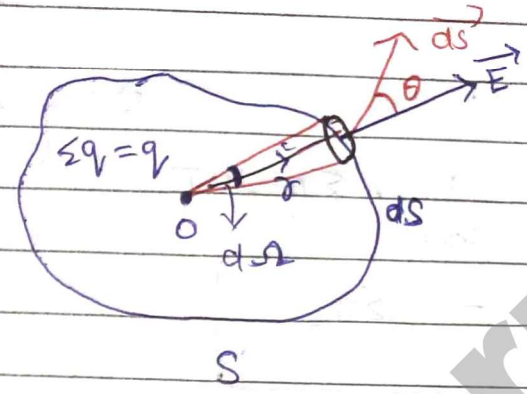
The solid angle designed at point O by surface element $\vec{ds}$ is $d\Omega$.



We know that

$$\phi = \oint_S \vec{E} \cdot \vec{ds} \quad \text{——— (1)}$$

The E.F. intensity due to point charge is following –

$$\vec{E} = \frac{Kq}{r^2}\hat{r} \quad \text{——— (2)}$$

It is clear from eqⁿ (2) that mathematical expression of E.F. intensity follows inverse square law.

from eqⁿ (1) & (2)

$$\phi = \oint_S \frac{Kq}{r^2}\hat{r} \cdot \vec{ds}$$

$$\phi = \oint_S \frac{Kq}{r^2} |\hat{r}| \, |\vec{ds}| \cos\theta$$

$$\phi = \oint_S Kq \cdot 1 \cdot \frac{ds\cos\theta}{r^2}$$

$$\because d\Omega = \frac{ds\cos\theta}{r^2} = \text{solid angle}$$

$$\phi = Kq \oint_S d\Omega$$

We know that the value of solid angle is 4π for closed spherical surface.

$$\Omega = \oint_S d\Omega = 4\pi$$

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$$\phi = kq \cdot 4\pi \qquad \because k = \frac{1}{4\pi\epsilon_0}$$

$$\because q = \Sigma q$$

$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \Sigma q \cdot 4\pi$$

$$\boxed{\phi = \frac{\Sigma q}{\epsilon_0}}$$

Ques. Prove that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$ where c is a constant.

Solⁿ. Given, $\phi(x, y, z) = \text{constant } (c)$

$$\boxed{d\phi = 0} \quad \text{—— (1)}$$

$$\phi = \phi(x, y, z)$$

to take total derivative

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \quad \text{—— (2)}$$

from eqⁿ (1) & (2)

$$\Rightarrow \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0.$$

$$\because \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

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$$\Rightarrow \left[\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right]\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) = 0$$

$$\vec{N} = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}$$ = perpendicular vector

$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\boxed{\vec{N}\cdot\vec{dr} = 0}$$

Que. Find a unit normal vector to the surface $x^2y + xz = 4$ at the point $(2, -1, 2)$.

Solⁿ. we know that from property of gradient.

$$\vec{\nabla}\phi\cdot\vec{dr} = 0$$

$\because \vec{N} = \vec{\nabla}\phi$ = perp. unit vector

Given,

$$\phi = x^2y + xz = 4 \quad \text{surface eqⁿ.}$$

$$x^2y + xz - 4 = 0$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Now

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\vec{N} = \vec{\nabla}\phi = \hat{i}\frac{\partial}{\partial x}(x^2y + xz - 4) + \hat{j}\frac{\partial}{\partial y}(x^2y + xz - 4) + \hat{k}\frac{\partial}{\partial z}(x^2y + xz - 4)$$

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$$\vec{N} = (2xy + z)\hat{i} + (x^2)\hat{j} + x\hat{k}$$

$$\cancel{\vec{N}|_{(2,-1,2)}} \quad \vec{N}\Big|_{(2,-1,2)} = [2 \times 2 \times (-1) + 2]\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\vec{N} = -2\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{-2\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{4+16+4}}$$

$$\hat{N} = \frac{-2(\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{6 \times 4}} = \frac{-\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

.. Most Imp.

Prove that -

(a) curl grad $\phi = \vec{\nabla} \times \vec{\nabla}\phi = 0$

(b) div curl $\vec{A} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$

Curl of Vector field :-

Acq to property of vector algebra mathematics, curl is always calculated of vector functions and we know that after calculation of curl we get a vector quantity because cross (X) operation is used.

To define the curl of vector field, line integration is used. Line integration of closed path is said to be circulation.

$$\boxed{\Gamma = \oint_C \vec{A} \cdot \vec{dl}} \quad \longleftarrow (1)$$

Let a closed path KLMNK whose ~~sides~~ sides are dx and dy. and the area of this small closed

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path is $\vec{dr}$. We know that the dir^n of area is perpendicular to the surface. So, curl of vector field $\vec{A}$.

$$|\text{curl}\,\vec{A}| = \lim_{da \to 0} \left[\frac{\gamma}{da} \right]_{max}$$

from eq^n ①.

$$|\text{curl}\,\vec{A}| = \lim_{da \to 0} \left[\frac{\oint \vec{A} \cdot \vec{dl}}{da} \right]_{max} \quad \text{—— ②}$$

$$\because |\text{curl}\,\vec{A}| = (\text{curl}\,\vec{A}) \cdot \hat{n}$$
$$= |\text{curl}\,\vec{A}|\,|\hat{n}| \cos 0^\circ$$
$$= |\text{curl}\,\vec{A}| (1)$$

from eq^n ②

$$(\text{curl}\,\vec{A}) \cdot \hat{n} = \lim_{da \to 0} \left[\frac{\oint_c \vec{A} \cdot \vec{dl}}{da} \right]_{max} \quad \text{—— ③}$$

"~~Line int Ratio~~

"The max. value of ratio of line integration of vector field and area in small area condition is known as magnitude of curl of vector field.

* Physical significance of curl of vector field :—

To understand the physical significance of curl of vector field we imagine a container and fill liquid in it. Now we do a hole in the lower surface of this container due to which the ~~value~~ quantity of liquid is out from lower surface by doing rotational motion. We know that curl always find for rotational function

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For irrotational function, the value of curl is always zero. This is the physical signiface of curl of vector field.

* Curl in Cartesian Coordinate System :-

Such frame of reference in which coordinates of particle are present in (x, y, z) form is called cartesian coordinate system.

We know that -

(i) from definition of differential operator -

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{--- (1)}$$

(ii) vector field in component form -

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{--- (2)}$$

vector product of eqn (1) & (2)

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \text{--- (3)}$$

To calculate the ~~vecto~~ mathematically curl of vector field, we imagine a closed ~~loop~~ path. Let this closed path is whose small sides are dx and dy. The area of this small closed path is whose dirn is $\hat{k}$.

Now, we image a F.O.R. on

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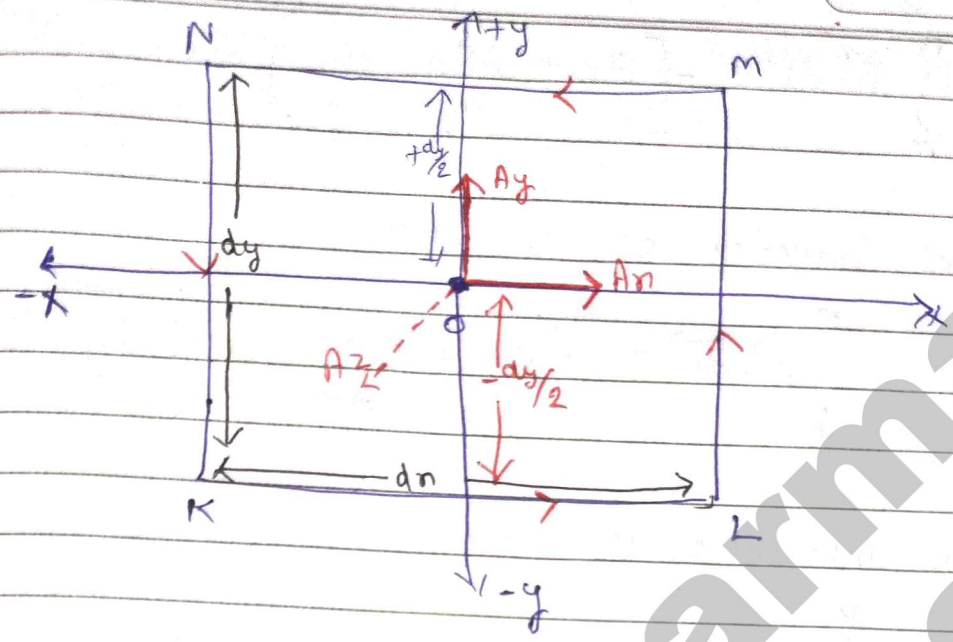

$$\oint \vec{A}\cdot\vec{dl} = \oint_{KLMNK} \vec{A}\cdot\vec{dl} = \int_K^L \vec{A}\cdot\vec{dl} + \int_L^M \vec{A}\cdot\vec{dl} + \int_M^N \vec{A}\cdot\vec{dl} + \int_N^K \vec{A}\cdot\vec{dl} \quad \text{——— (4)}$$

(i) Line integration of K-L line :-

Vector component in x dir^n $= A_x$

change in vector comp. A_x w.r. to y $= \dfrac{\partial A_x}{\partial y}$

change in vector comp. A_x for K-L line $= -\dfrac{\partial A_x}{\partial y}\dfrac{dy}{2}$

Total vector comp. present in K-L line $= A_x + \left(-\dfrac{\partial A_x}{\partial y}\dfrac{dy}{2}\right)$

$$= A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2}$$

Line Integral of line K-L = vector comp. × length of line

$$\int_K^L \vec{A}\cdot\vec{dl} = \left[A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2}\right]dx$$

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for line MN -

(ii) $$\int_M^N \vec{A}\cdot\vec{dl} = -\left[A_x + \frac{\partial A_x}{\partial y}\frac{dy}{2}\right]dx$$

(iii) line integral of line LM -

$$\int_L^M \vec{A}\cdot\vec{dl} = \left[A_y + \frac{\partial A_y}{\partial x}\frac{dx}{2}\right]dy$$

(iv) line integral of line NK

$$\int_N^K \vec{A}\cdot\vec{dl} = -\left[A_y - \frac{\partial A_y}{\partial x}\frac{dx}{2}\right]dy$$

on putting values in eqn (4).

$$\oint_C \vec{A}\cdot\vec{dl} = \left[A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2}\right]dx + \left[A_y + \frac{\partial A_y}{\partial x}\frac{dx}{2}\right]dy$$

$$-\left[A_x + \frac{\partial A_x}{\partial y}\frac{dy}{2}\right]dx - \left[A_y - \frac{\partial A_y}{\partial x}\frac{dx}{2}\right]dy$$

$$\oint_C \vec{A}\cdot\vec{dl} = \left[A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2} - A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2}\right]dx$$

$$+\left[A_y + \frac{\partial A_y}{\partial x}\frac{dx}{2} - A_y + \frac{\partial A_y}{\partial x}\frac{dx}{2}\right]dy$$

$$\oint_C \vec{A}\cdot\vec{dl} = \left[-1\cdot\frac{\partial A_x}{\partial y}dy\right]dx + \left[1\cdot\frac{\partial A_y}{\partial x}dx\right]dy$$

$$\oint_C \vec{A}\cdot\vec{dl} = \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]dx\,dy$$

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$\because da = dx\,dy$

$$\oint \vec{A}\cdot\vec{dl} = \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] da$$

$$\lim_{da\to 0} \frac{\oint \vec{A}\cdot\vec{dl}}{da} = \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

from definition of curl -

$$\left[|\text{curl}\,\vec{A}|\right]_z = \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \quad \text{--- (5)}$$

Vector component in x- dir^n -

$$\left[|\text{curl}\,\vec{A}|\right]_x = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]$$

Vector component in y- dir^n -

$$\left[|\text{curl}\,\vec{A}|\right]_y = \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]$$

So, curl of vector -

$$\text{curl}\,\vec{A} = \left[|\text{curl}\,\vec{A}|\right]_x \hat{i} + \left[|\text{curl}\,\vec{A}|\right]_y \hat{j} + \left[|\text{curl}\,\vec{A}|\right]_z \hat{k}$$

put values -

$$\text{curl}\,\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{j} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{k} \quad \text{--- (6)}$$

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from eq^n ③ & (6)

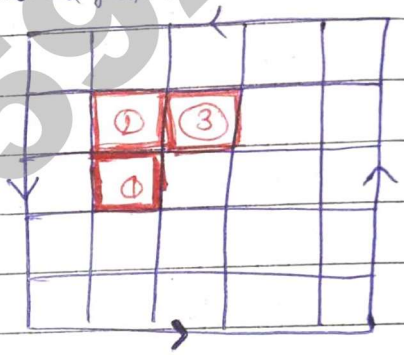
$$\boxed{\text{curl } \vec{A} = \vec{\nabla} \times \vec{A}} \quad \text{—— (7)}$$

Stoke's Curl Theorem :—

The theorem defined in vector algebra mathematics, which changes/convert surface integration to line integration and line integration to surface integration, is said to be Stoke's curl theorem.

$$\boxed{\oint_l \vec{A} \cdot \vec{dl} = \oint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{da}}$$

To mathematical verification of stoke's curl theorem, we imagine a closed path.



Proof: $\oint_C \vec{A} \cdot \vec{dl} = \oint_C \vec{A_1} \cdot \vec{dl_1} + \oint_C \vec{A_2} \cdot \vec{dl_2}$

$$+ \oint_C \vec{A_3} \cdot \vec{dl_3} + \ldots$$

$$+ \oint_C \vec{A_n} \cdot \vec{dl_n}$$

$$\boxed{\oint_C \vec{A} \cdot \vec{dl} = \sum_{i=1}^{n} \oint_C \vec{A_i} \cdot \vec{dl_i}} \quad \text{—— (1)}$$

we know that

$$|\text{curl } \vec{A}| = \lim_{da \to 0} \frac{\oint \vec{A} \cdot \vec{dl}}{da}$$

$$(\text{curl } \vec{A}) \cdot \hat{n} = \lim_{da \to 0} \frac{\oint_C \vec{A} \cdot \vec{dl}}{da}$$

of small closed path

$$\therefore \text{curl } \vec{A} = \vec{\nabla} \times \vec{A}$$

To this closed path, $\vec{A}$ vector field is present. This closed path is made up of small closed path. Therefore, line integration of closed path is equal to sum of line integration

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$$\Rightarrow (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da = \oint_c \vec{A} \cdot \vec{dl}$$

$$\because \vec{da} = \hat{n}\, da \quad \left[\hat{da} = \hat{n} = \frac{\vec{da}}{|\vec{da}|} \right]$$

$$\Rightarrow (\vec{\nabla} \times \vec{A}) \cdot \vec{da} = \oint_c \vec{A} \cdot \vec{dl}$$

for i^{th} particle -

$$\Rightarrow (\vec{\nabla} \times \vec{A}) \cdot \vec{da_i} = \oint_c \vec{A_i} \cdot \vec{dl_i}$$

for whole cross path -

$$\Rightarrow \sum_{i=1}^{n} (\vec{\nabla} \times \vec{A_i}) \cdot \vec{da_i} = \sum_{i=1}^{n} \oint_c \vec{A_i} \cdot \vec{dl_i} \quad \text{--- (2)}$$

from eq^n (1) & (2)

$$\oint_c \vec{A} \cdot \vec{dl} = \sum_{i=1}^{n} (\vec{\nabla} \times \vec{A_i}) \cdot \vec{da_i}$$

if $\vec{da_i} \to 0$ then we use $\int$ in place Σ

$$\boxed{\oint_c \vec{A} \cdot \vec{dl} = \oint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{da}} \quad \text{--- (3)}$$

Question :- By using Stoke's curl theorem, prove 3rd and 4th eq^n of Maxwell.

Sol^n (a) 3rd eq^n of Maxwell :-

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{E} &= \frac{-d\vec{B}}{dt} = -\dot{\vec{B}} \\ &\text{OR} \\ \text{curl } \vec{E} &= -\dot{\vec{B}} \end{aligned}}$$

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Proof :- maxwell's 3rd equation is defined by Faraday's law. So, Faraday's law.

$$e = \frac{-d\phi}{dt} \quad \text{—} (1)$$

$\because e = e.m.f.$

$\because \phi = \text{flux}.$

We know that following relation is b/w induced e.m.f. and intensity of electric field.

$$e = \oint_l \vec{E} \cdot \vec{dl} \quad \text{—} (2)$$

from eq^n (1) & (2)

$$\oint_l \vec{E} \cdot \vec{dl} = \frac{-d\phi}{dt}$$

$$\phi = \oint_S \vec{B} \cdot \vec{da}$$

$$\oint_l \vec{E} \cdot \vec{dl} = \frac{-d}{dt}\left[\oint_S \vec{B} \cdot \vec{da}\right]$$

$$\oint_l \vec{E} \cdot \vec{dl} = -\oint_S \frac{\vec{dB}}{dt} \cdot \vec{da} \qquad \because \dot{\vec{B}} = \frac{\vec{dB}}{dt}$$

$$\oint_l \vec{E} \cdot \vec{dl} = -\oint_S \dot{\vec{B}} \cdot \vec{da} \quad \text{—} (3)$$

from stoke's curl theorem –

$$\oint_l \vec{A} \cdot \vec{dl} = \oint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{da}$$

~~Put~~ if $\vec{A} = \vec{E}$ then

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$$\oint_l \vec{E}\cdot\vec{dl} = \oint_s (\vec{\nabla}\times\vec{E})\cdot\vec{da} \quad \text{—— (4)}$$

from eqn (3) & (4)

$$\oint_s (-\vec{\nabla}\times\vec{E})\cdot\vec{da} = -\oint_s \dot{\vec{B}}\cdot\vec{da}$$

on comparing -

$$\oint_s (\vec{\nabla}\times\vec{E})\cdot\vec{da} = \oint_s \dot{\vec{B}}\cdot\vec{da}$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}}$$

This is 3rd equation of Maxwell.

(b) <u>Fourth eqn of Maxwell :-</u>

$$\boxed{\begin{array}{c}\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \\ \text{or} \\ \text{curl } \vec{B} = \mu_0\vec{J}\end{array}}$$

<u>Proof:-</u> To prove ~~mo~~ 4th eqn of maxwell, ampere circuital law is used.

$$\oint_l \vec{B}\cdot\vec{dl} = \mu_0 \Sigma I$$

$\because \Sigma I = I$ (let)

$$\oint_l \vec{B}\cdot\vec{dl} = \mu_0 I \quad \text{—— (1)}$$

from definition of electric current density

$$\vec{J} = \frac{dI}{\vec{da}}$$

$$dI = \vec{J}\cdot\vec{da} \longrightarrow \text{~~from~~ integration for total current}$$

$$I = \oint_s \vec{J}\cdot\vec{da} \quad \text{—— (2)}$$

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from eqⁿ ① & ②

$$\oint_l \vec{B}\cdot\vec{dl} = \mu_0 \oint_S \vec{J}\cdot\vec{da} \quad \text{—— ③}$$

from stoke's curl theorem,

$$\oint_l \vec{B}\cdot\vec{dl} = \oint_S (\vec{\nabla}\times\vec{B})\cdot\vec{da} \quad \text{—— ④}$$

from eqⁿ ③ & ④

$$\oint_S (\vec{\nabla}\times\vec{B})\cdot\vec{da} = \mu_0 \oint_S \vec{J}\cdot\vec{da}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

V.V most

Que If S is a ~~surface~~ such that a closed path which enclosed volume V where vector field is $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$ then prove that $\oint_S \vec{F}\cdot\vec{ds} = 6V$

Solⁿ We know that from divergance theorem -

$$\oint_S \vec{A}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{A})\,dv$$

If $\vec{A} = \vec{F}$ then

$$\Rightarrow \oint_S \vec{F}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{F})\,dv \quad \text{—— ①}$$

now.

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + 2y\hat{j} + 3z\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{F} = 1\cdot\frac{\partial}{\partial x}(x) + 1\cdot\frac{\partial}{\partial y}(2y) + 1\cdot\frac{\partial}{\partial z}(3z)$$

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$$\vec{\nabla}\cdot\vec{F} = 1 + 2.1 + 3.1$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 6} \quad \text{——— (2)}$$

from eq^n (1) & (2).

$$\oint_S \vec{F}\cdot\vec{ds} = \int_V 6\,dv$$

$$\oint_S \vec{F}\cdot\vec{ds} = 6\int_V dv$$

$$\boxed{\oint_S \vec{F}\cdot\vec{ds} = 6V}$$

Que. Prove that vector field $\vec{F} = [(4xy - 3x^2z)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}]$ is a conservative field. Find the value of electric potential in the electric field. Ans $[\vec{\nabla}\times\vec{F} = 0;\ V = (x^2y - x^2z^2) + c]$.

Soln. We know that for conservative field.

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = 0$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{F} = \hat{i}[4xy - 3x^2z] + 2x^2\hat{j} - 2x^3z\hat{k}.$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - 3x^2z & 2x^2 & -2x^2z \end{vmatrix}$$

$$\vec{\nabla}\times\vec{F} = \hat{i}\left[\cancel{\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)}\right] - \hat{j}$$

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$\frac{\partial}{\partial x}(4xy - 3x^2z)]$

$$\vec{\nabla} \times \vec{F} = \hat{i}\left[\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-2x^3z - \right.$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z)\right]$$

$$\boxed{\vec{\nabla} \times \vec{F} = 0}$$

We know that

$$\text{curl grad } V = \vec{\nabla} \times \vec{\nabla} V = 0$$

$\because V =$ scalar function (electropotential)

$\because \text{grad } V = \vec{\nabla} V =$ let] $\times$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = 0 \quad \text{(ii)}$$

compare eqn (i) with eqn (ii) $\Rightarrow \vec{F} = \vec{\nabla} V$

$$\Rightarrow F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}.$$

(a) $\frac{\partial V}{\partial x} = F_x$

$\partial V = F_x \partial x$

$V_x = V = \int F_x \partial x$

(b) $V_y = \int F_y dy$.

(c) $V_z = \int F_z dz$.

$$\boxed{V = V_x + V_y + V_z + c}$$

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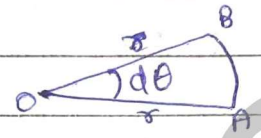
Que. Prove that -

(i) $$\int \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] dv = \oint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{ds}$$

(ii) $$\nabla^2 r^n = n(n+1) r^{n-2}$$

Concept of Solid Angle :-

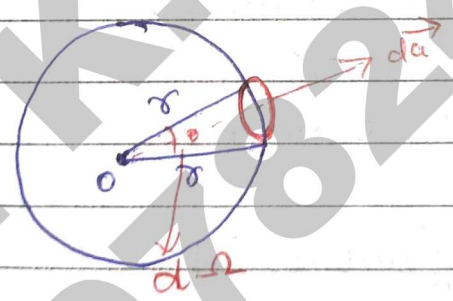
$$\text{angle} = \frac{\text{arc}}{\text{radius}} \longrightarrow \text{Radian (dimensionless)}$$



$$d\theta = \frac{AB}{r}$$

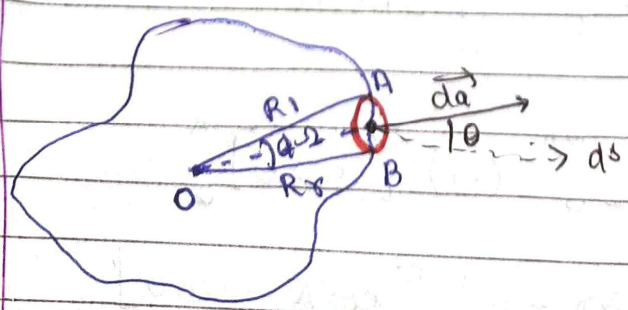
* $$\text{Solid angle} = \frac{\text{area}}{(\text{radius})^2} \longrightarrow \text{Steradian (dimensionless)}$$

(1) Symmetric Surface :-



$$\boxed{d\Omega = \frac{da}{r^2}} \quad \text{—— (1)}$$

(2) Asymmetric Surface :-



$$da = ds \cos\theta$$

from eqn (1)

$$\boxed{d\Omega = \frac{ds \cos\theta}{r^2}}$$

$$[\cos\theta]_{max} = 1$$

$$\boxed{[d\Omega]_{max} = \frac{ds}{r^2}}$$

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The angle designed in 2-D is known as plane angle and the angle designed in 3-D is known as solid angle. Unit of solid angle is steradian. It is a dimensionless quantity. We know that the value of surface element is following.

$$[da]_{u_1 = \text{constant}} = h_2 h_3 du_2 du_3$$

$u_1 = r$; $u_2 = \theta$; $u_3 = \phi$
$h_1 = 1$; $h_2 = r$; $h_3 = r\sin\theta$

$$da = r \cdot r\sin\theta \cdot d\theta \cdot d\phi$$
$$da = r^2 \sin\theta \, d\theta \, d\phi$$

from eq^n ①

$$d\Omega = \frac{da}{r^2}$$

$$d\Omega = \frac{\cancel{r^2} \sin\theta \, d\theta \, d\phi}{\cancel{r^2}}$$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

on integrating,

$$\Omega = \int \sin\theta \, d\theta \, d\phi$$

$\theta : 0$ to π ; $\phi : 0$ to 2π

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$$

$$\Omega = [\phi]_0^{2\pi} (-1) [\cos\theta]_0^{\pi}$$

$$\Omega = -[2\pi - 0][\cos\pi - \cos 0]$$
$$\Omega = -2\pi[-1-1]$$

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$$\Omega = -2\pi(-2)$$
$$\boxed{\Omega = 4\pi}$$

Imp. Exam

Que. P.T. $\rightarrow \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d = \oint_S [\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi] \cdot \vec{ds}$

Solⁿ.

Sol: we know that,
from Gauss divergance theorem -

$$\boxed{\int_V (\vec{\nabla} \cdot \vec{A})\, d\tau = \oint_S \vec{A} \cdot \vec{ds}} \quad \text{—— (1)}$$

For mathematical verification of this theorem, we imagine two scalar function ϕ & ψ. we know that, scalar gradient is a vector quantity.

$$\overrightarrow{grad}\, \psi = \vec{\nabla} \psi = \text{vector quantity}.$$

Let $\phi\, \overrightarrow{grad}\, \psi = \phi \vec{\nabla} \psi = \vec{A}$
from eqⁿ (1).

$$\Rightarrow \int_V [\vec{\nabla} \cdot \phi \vec{\nabla} \psi]\, d\tau = \oint_S \phi \vec{\nabla} \psi \cdot \vec{ds} \quad \text{—— (2)}$$

Now

$$\vec{\nabla} \cdot \phi \vec{\nabla} \psi = \vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi (\vec{\nabla} \cdot \vec{\nabla} \psi)$$

$$\vec{\nabla} \cdot \phi \vec{\nabla} \psi = \vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi$$

from eqⁿ (2)

$$\int_V [\vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi]\, d\tau = \oint_S \phi \vec{\nabla} \psi \cdot \vec{ds} \quad \text{—— (3)}$$

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replace ϕ in place of ψ and ψ in place of ϕ

$$\int_V \left[\vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \nabla^2 \psi\right] d\tau = \oint_S \phi \vec{\nabla}\psi \cdot \vec{ds} \quad \text{—— (3)}$$

Put

$$\int_V \left[\vec{\nabla}\psi \cdot \vec{\nabla}\phi + \psi \nabla^2 \phi\right] d\tau = \oint_S \psi \vec{\nabla}\phi \cdot \vec{ds} \quad \text{—— (4)}$$

Now eq^n (3) − eq^n (4)

$$\Rightarrow \int_V \left[\vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \nabla^2 \psi\right] d\tau - \int_V \left[\vec{\nabla}\psi \cdot \vec{\nabla}\phi + \psi \nabla^2 \phi\right] d\tau$$

$$= \oint_S \phi \vec{\nabla}\psi \cdot \vec{ds} - \oint_S \psi \vec{\nabla}\phi \cdot \vec{ds}$$

$$\Rightarrow \int_V \left[\cancel{\vec{\nabla}\phi \cdot \vec{\nabla}\psi} + \phi \nabla^2 \psi - \cancel{\vec{\nabla}\psi \cdot \vec{\nabla}\phi} - \psi \nabla^2 \phi\right] d\tau$$

$$= \oint_S \left[\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi\right] \cdot \vec{ds}$$

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$\Rightarrow \boxed{\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d\tau = \oint_S \left(\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi\right) \cdot \vec{ds}}$$

V.V.Imp

Que. Prove that electric potential $\phi = K/r$ in E.F. satisfies Laplacian eq^n. where K is a constant and r is a position vector of particle.

Proof

we know that,

from Laplacian eq^n −

$$\boxed{\nabla^2 V = 0}$$

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Electric potential $V = \phi$

$$\boxed{\nabla^2 \phi = 0} \quad \text{—— (1)}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

d.w.r. to x

$$2r \cdot \frac{\partial r}{\partial x} = 2x \cdot 1$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \; ; \; \frac{\partial r}{\partial y} = \frac{y}{r} \; ; \; \frac{\partial r}{\partial z} = \frac{z}{r}$$

L.H.S.

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad \text{—— (2)}$$

Now.

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)$$

$$\because \phi = K/r$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(K/r\right)\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = K\frac{\partial}{\partial x}\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right]$$

$$\because \frac{\partial r}{\partial x} = x/r$$

$$\frac{\partial^2 \phi}{\partial x^2} = -K\frac{\partial}{\partial x}\left(\frac{+x}{r^3}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -K\left[\frac{1}{r^3} + \left(\frac{-3}{r^4}\cdot\frac{x}{r}\right)x\right]$$

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$$\frac{\partial^2 \phi}{\partial x^2} = -K\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

Similarly,

$$\frac{\partial^2 \phi}{\partial y^2} = -K\left[\frac{1}{r^3} - \frac{3y^2}{r^5}\right]$$

$$\frac{\partial^2 \phi}{\partial z^2} = -K\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right]$$

from eq^n (2)

$$\nabla^2 \phi = -K\left[\frac{3}{r^3} - \frac{3}{r^5}(x^2+y^2+z^2)\right]$$

$$\nabla^2 \phi = -K\left[\frac{3}{r^3} - \frac{3}{r^5} r^2\right]$$

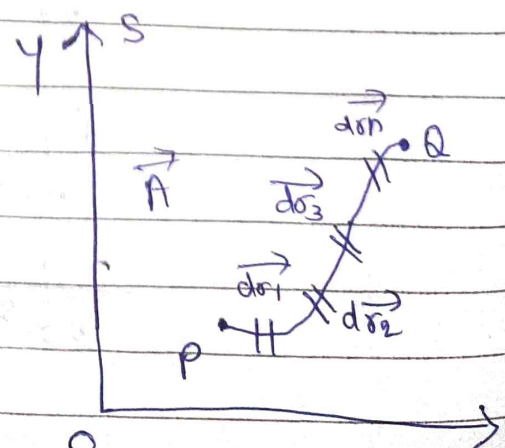
$$\boxed{\nabla^2 \phi = 0}$$

UNIT-I

Potential Function :-

Electric potential is explained by line integral of vector field. To define line integration of vector field, we imagine a F.O.R.-(S). Let two points P and Q are present in this F.O.R. These points are connected by a path due to a which a PQ line is formed. So, line integration of PQ line is -

$$W = V = \int_P^Q \vec{A} \cdot \vec{dr} \quad \text{——— (1)}$$



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