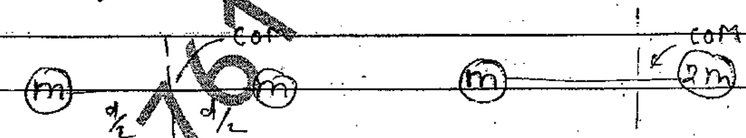
Unit - II

Centre of Mass

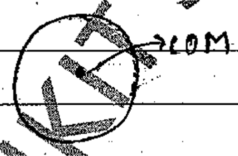
Centre of mass :-

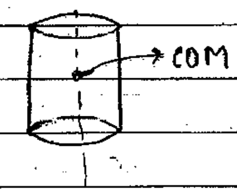
Centre of mass of a body or system is described as a point at which whole of mass of body or system is concentrated.

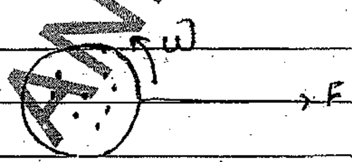
→ Centre of mass of body is displaced where maxm mass is present



→ Centre of mass for symmetric body is present in centre of body.







$$\vec{F}_{net} = m\vec{a}_{com}$$
$$\vec{P} = m\vec{v}_{com}$$

→ Sum of product of mass of the particle and position vector of that particle is zero relative to centre of mass.

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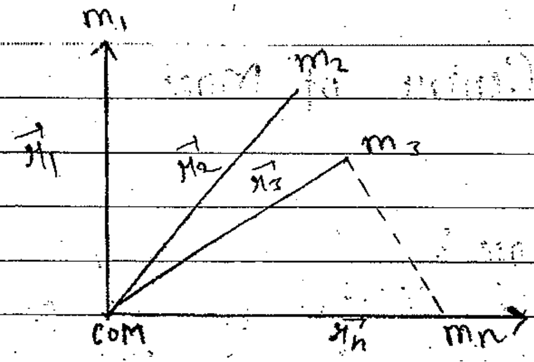

$$m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \cdots + m_n\vec{r_n} = 0$$

$$\sum_{i=1}^{n} m_i\vec{r_i} = 0 \quad \text{---(i)}$$

Differentiate eq. (1) w.r.t. 't'

$$\sum_{i=1}^{n} m_i \frac{d\vec{r_i}}{dt} = 0$$

$$\sum_{i=1}^{n} m_i\vec{v_i} = 0$$

$$\boxed{\sum_{i=1}^{n} \vec{p_i} = 0}$$

Linear momentum of a particle in a system will be zero.

Position vector of Centre of mass :- ($\vec{R}_{com}$)

Let a system having ~~m~~ 'n' no. of particles. Position vector w.r.t. centre of mass is $\vec{R}_{cm}$

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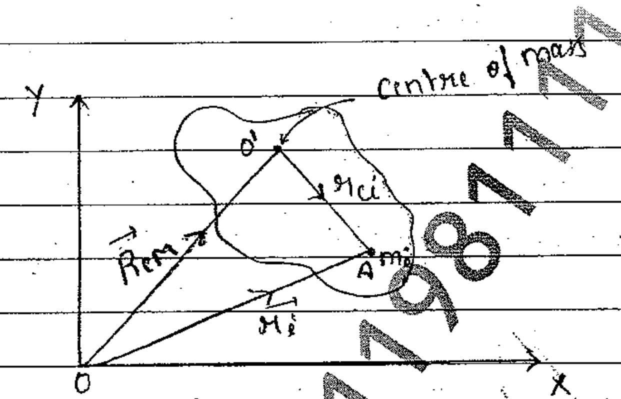
In this system i^{th} particle of mass m_i is at point A and position vector is $\vec{r_i}$ w.r.t. origin and position vector w.r.t. centre of mass $\vec{r_{ci}}$



By vector triangle law $\Delta OO'A$

$$\vec{OA} = \vec{OO'} + \vec{O'A}$$

$$\vec{r_i} = \vec{R}_{com} + \vec{r_{ci}} \quad —(1)$$

multiply eq. (1) by m_i

$$m_i \vec{r_i} = m_i \vec{R}_{cm} + m_i \vec{r_{ci}}$$

In whole body.

$$\sum_{i=1}^{n} m_i \vec{r_i} = \sum_{i=1}^{n} m_i \vec{R}_{cm} + \sum_{i=1}^{n} m_i \vec{r_{ci}}$$

By definition of centre of mass

$$\sum_{i=1}^{n} m_i \vec{r_{ci}} = 0$$

$$\sum_{i=1}^{n} m_i \vec{r_i} = \vec{R}_{cm} \sum_{i=1}^{n} m_i$$

$$\vec{R}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{\sum_{i=1}^{n} m_i}$$

let whole mass of body $= M$

$$\sum_{i=1}^{n} m_i = M$$

$$\Rightarrow \boxed{\vec{R}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{M}} \quad \text{--- (2)}$$

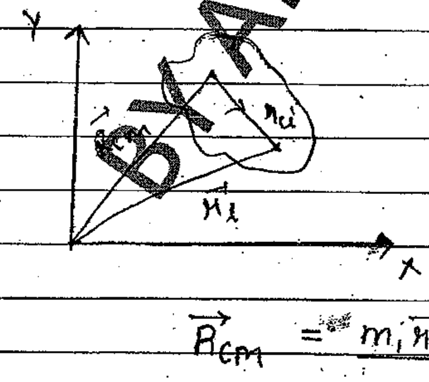
eq. (2) represent position vector of centre of mass

Position vector of centre of mass of two particle system

let a system consist of two particles

Position vector of centre of mass in Co-ordinate form :-

let a system having 'n' no. of particles mass distribution in this system is discontinuous.



$$\vec{R}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \cdots + m_n\vec{r_n}}{m_1 + m_2 + \cdots + m_n} \quad \text{--- (1)}$$

Position vector R_{cm} of COM :- $x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$ —(2)

Similarly

$$\left.\begin{aligned} r_1 &= x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \\ r_2 &= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \\ &\vdots \\ r_n &= x_n\hat{i} + y_n\hat{j} + z_n\hat{k} \end{aligned}\right\} (3)$$

let total mass of body $= M$

$$m_1 + m_2 + m_3 + m_4 + \dots + m_n = M$$

put value of 2 and 3 in eq. (1)

$$x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k} = \frac{m_1(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + m_2(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) + \dots + m_n(x_n\hat{i} + y_n\hat{j} + z_n\hat{k})}{m_1 + m_2 + m_3 + \dots + m_n}$$

On comparing coefficient of $\hat{i}, \hat{j}, \hat{k}$

position vector of R_{com} in x-dirn

$$x_{com} = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n = M} \quad \text{—(4)}$$

in y-dirn

$$y_{com} = \frac{m_1y_1 + m_2y_2 + \dots + m_ny_n}{M} \quad \text{—(5)}$$

in z-dirn

$$z_{com} = \frac{m_1z_1 + m_2z_2 + \dots + m_nz_n}{M} \quad \text{—(6)}$$

Note :-

$\prod_{i=1}$ (गुणा)

$\sum_{i=1}^{n}$ (summation) (discrete form)

$\int$ (continuous) (add)

eq. (4), (5) and (6) can be also represented like this

$$x_{com} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

$$y_{com} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

$$z_{com} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

If mass distribution is continuous, then summation (Σ) replaced by integration ($\int$) and individual mass replaced by element mass (dm)

let in a body element mass is dm and position vector is $\vec{r}$ then position vector of centre of mass is ($\vec{R}_{cm}$) is represent as given below

$$\vec{R}_{cm} = \frac{1}{M} \int \vec{r}\, dm$$

Component of centre of mass for continuous distribution

$$x_{com} = \frac{1}{M} \int x\, dm$$

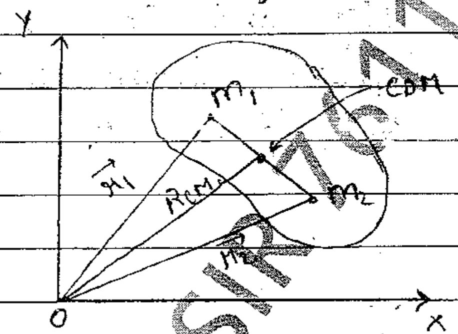
$$y_{com} = \frac{1}{M} \int y\, dm$$

$$z_{com} = \frac{1}{M} \int z\, dm$$

Note : 1. On changing origin of position vector of centre of mass a thin particle will also change but real meaning of centre of mass will not change.

Position Vector of COM for two particle system :-

Let a body / System consist 2 particles of mass m_1 & m_2, position vector w.r.t. origin are $\vec{r_1}$ and $\vec{r_2}$ respectively. position vector of COM w.r.t. origin is $\vec{R}_{cm}$.



We know that,

$$\vec{R}_{COM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{m_1 + m_2 + \dots + m_n}$$

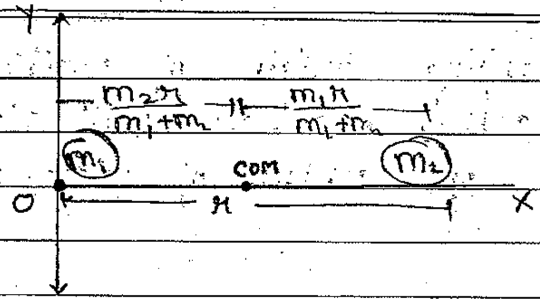
$$\vec{R}_{CM} = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

for two particles $i = 1$ to $n = 2$

$$\vec{R}_{COM} = \frac{\sum_{i=1}^{2} m_i r_i}{\sum_{i=1}^{2} m_i}$$

$$\boxed{\vec{R}_{COM} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}}$$

Note:

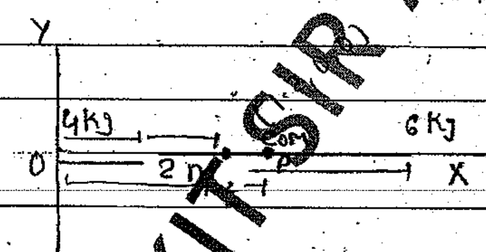


$$X_{COM} = \frac{m_1 \times 0 + m_2 r}{m_1 + m_2} = \frac{m_2 r}{m_1 + m_2}$$

Numerical

Q.1. let two masses 4kg and 6kg are seprated by 2m then find the distance from origin to centre of mass and co-ordinate of COM.

Sol^n



$$X_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$X_{COM} = \frac{4 \times 0 + 6 \times 2}{4 + 6} = \frac{12}{10}$$

$$\boxed{X_{COM} = 1.2\text{ m}}$$

Co-ordinates, $R_{COM} = (X_{COM}, Y_{COM}, Z_{COM})$
$= (1.2, 0, 0)$

Velocity of Centre of mass :- ($\vec{V}_{COM}$)

Let distribution of particle in a system is discontinuous and position vector of COM is $\vec{R}_{CM}$.

The Rate of change in position of COM is known as Velocity of Centre of mass.

We know that

$$\vec{R}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{\sum_{i=1}^{n} m_i} \quad \text{--- (1)}$$

Differentiate eq. (1) w.r.t. 't'

$$\frac{d\vec{R}_{CM}}{dt} = \frac{\sum_{i=1}^{n} m_i \frac{d\vec{r_i}}{dt}}{\sum_{i=1}^{n} m_i}$$

$$\vec{V}_{COM} = \frac{\sum_{i=1}^{n} m_i \vec{v_i}}{\sum_{i=1}^{n} m_i}$$

$$\vec{V}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{v_i}}{M}$$

$$M\vec{V}_{CM} = \sum_{i=1}^{n} m_i \vec{v_i}$$

$$\vec{P}_{CM} = \sum_{i=1}^{n} P_i$$

$$\boxed{\vec{P}_{CM} = P_1 + P_2 + P_3 + \cdots + P_n} \quad \text{---(2)}$$

On basis of eq. (2) we can say that momentum of COM is equal to individual addition of momentum of individual particle in system

Acceleration of Centre of mass :-

Rate of change of Velocity of COM is called Acceleration of centre of mass.

We know that

$$\vec{V}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{V}_i}{\sum_{i=1}^{n} m_i} \quad \text{---(1)}$$

D. (1) w.r.t 't'

$$\frac{d\vec{V}_{CM}}{dt} = \frac{\sum_{i=1}^{n} m_i \frac{d\vec{V}_i}{dt}}{M}$$

$$\boxed{\vec{a}_{CM} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{a}_i} \quad \longleftarrow (2)$$

eq^n (2) is known as Acceleration of COM

$$M\vec{a}_{CM} = \sum_{i=1}^{n} m_i \vec{a}_i$$

$\because F = ma$

$$\vec{F}_{COM} = \sum_{i=1}^{n} \vec{F}_i$$

If a ~~large~~ small no. of particles are present in a body then then it is valid

→ If large no. of particles are in body then they will apply action-reaction law on each other

$$\vec{F}_{CM} = \sum_{i=1}^{n} \left[\vec{F}_i + \sum_{i \neq k} \vec{F}_{ik} \right]$$

Action- Rxn force is always zero

so

$$\sum_{i \neq k} \vec{F}_{ik} = 0$$

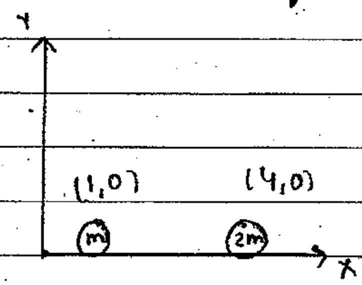
$$\boxed{\vec{F}_{CM} = \sum_{i=1}^{n} \vec{F}_i}$$ (Force of COM)

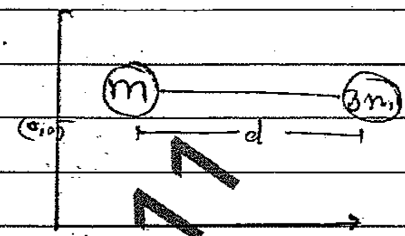
$$\boxed{\vec{a}_{CM} = \frac{1}{M} \sum_{i=1}^{n} \vec{F}_i}$$

Acceleration of COM is $\frac{1}{M}$ times of sum of external force on individual particle in system

Numerical

Q.2. Find COM of given e.g.





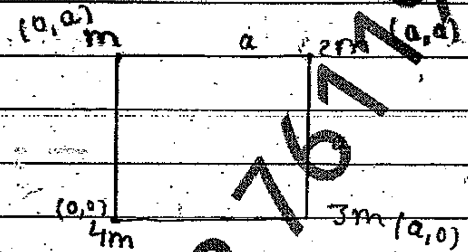
$$X_{com} = \frac{m\times 1 + 2m\times 4}{3m}$$

$$= \frac{9m}{3m}$$

$$\boxed{X_{com} = 3}$$

$$X_{com} = \frac{m\times 0 + 3md}{4m}$$

$$X_{com} = \frac{3md}{4m} = \frac{3d}{4}$$



$$X_{com} = \frac{ma + 2ma + 3ma}{m + 2m + 3m + 4m}$$

$$X_{cM} = \frac{5ma}{10m} = \frac{a}{2}$$

$$Y_{com} = \frac{ma + 2ma + 0 + 0}{10m}$$

$$Y_{com} = \frac{3a}{10}$$

$$\vec{R}_{cM} = \left(\frac{a}{2}, \frac{3}{10}a, 0\right)$$

Q.3. The position vector of two particle of mass 6g and 2g are $6\hat{i}-7\hat{j}$ and $2\hat{i}+10\hat{j}-8\hat{k}$. Determine position of COM.

Sol^n.

Given, $m_1 = 6g$; $\vec{r_1} = 6\hat{i}-7\hat{j}$
$m_2 = 2g$; $\vec{r_2} = 2\hat{i}+10\hat{j}-8\hat{k}$

Find, $\vec{R}_{cm} = ?$

Sol^n

We know that

$$\vec{R}_{com} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{\sum m_i}$$

For 2-particle system

$$\vec{R}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

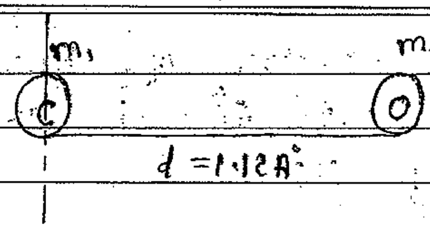
$$= \frac{6(6\hat{i}-7\hat{j}) + 2(2\hat{i}+10\hat{j}-8\hat{k})}{6+2}$$

$$= \frac{36\hat{i}-42\hat{j}+4\hat{i}+20\hat{j}-16\hat{k}}{8}$$

$$\boxed{\vec{R}_{cm} = \frac{40\hat{i}-22\hat{j}-16\hat{k}}{8}}$$

Q.4. The distance b/w centre of atoms of Oxygen (O=16) and carbon (C=12 unit) in CO molecule is 1.12 Å. Determine position of COM of CO molecule from C-atom

Sol^n



C = 12 unit
O = 16 unit

Note:
$\frac{35}{15}$ a.m.u $\rightarrow \frac{35}{15} \times 1.67 \times 10^{-27}$ kg

$\because$ We know,

$$\vec{R}_{CM} = \vec{x}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

w.r.t. 'C'

$$x_{CM} = \frac{12 \times 0 + 16 \times 1.12 \times 10^{-10}}{12 + 16}$$

$$\boxed{x_{CM} = \frac{16 \times 1.12\,Å}{28}}$$ w.r.t 'carbon'

$x_{CM} =$

Q.5. Co-ordinate of 3 particle of masses 7, 4, 10 gram are (1, 5, −3), (2, 5, 7), (3, 3, −1) cm respectively. Determine the position of COM of system.

Sol^n Given, $m_1 = 7g$; $\vec{r_1} = \hat{i} + 5\hat{j} - 3\hat{k}$
$m_2 = 4g$; $\vec{r_2} = 2\hat{i} + 5\hat{j} + 7\hat{k}$
$m_3 = 10g$; $\vec{r_3} = 3\hat{i} + 3\hat{j} - \hat{k}$

To find, $\vec{R}_{CM} = ?$

Sol^n

We know that, for 3 particles

$$\vec{R}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3}$$

$$\vec{R}_{CM} = \frac{7(\hat{i}+5\hat{j}-3\hat{k}) + 4(2\hat{i}+5\hat{j}+7\hat{k}) + 10(3\hat{i}+3\hat{j}-\hat{k})}{7+4+10}$$

$$\vec{R}_{CM} = \frac{7\hat{i}+35\hat{j}-21\hat{k}+8\hat{i}+20\hat{j}+28\hat{k}+30\hat{i}+30\hat{j}-10\hat{k}}{21}$$

$$\vec{R}_{CM} = \frac{45\hat{i}+85\hat{j}-3\hat{k}}{21}$$

$$\boxed{R_{CM} = \left(\frac{45}{21}, \frac{85}{21}, \frac{-3}{21}\right)}$$

Q.6. Centre of mass of 3 particles having masses 2, 4, 6g are at co-ordinate (1,1,1). Find the position of 4th particle of mass 40g, so that centre of mass of all particle is at (0,0,0)

Sol^n,
$m_1 = 2g$; $\vec{r_1} \rightarrow \vec{R}_{CM} = \hat{i}+\hat{j}+\hat{k}$
$m_2 = 4g$;
$m_3 = 6g$;
$m_4 = 60g$, $\vec{R'}_{CM} = (0\hat{i}+0\hat{j}+0\hat{k})$

for 3-particle

$$R_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3}$$

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$$\hat{i}+\hat{j}+\hat{k} = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3}{2+4+6}$$

$$m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} = 12(\hat{i}+\hat{j}+\hat{k}) \quad -(1)$$

For 4-particle system

$$\vec{R'}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + m_4\vec{r_4}}{m_1+m_2+m_3+m_4}$$

by eq (1)

$$0 = \frac{12(\hat{i}+\hat{j}+\hat{k}) + 40\vec{r_4}}{12+40}$$

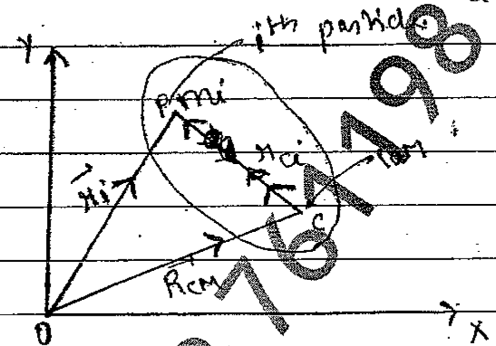
$$40\vec{r_4} = -12(\hat{i}+\hat{j}+\hat{k})$$

$$\vec{r_4} = -\frac{12}{40}(\hat{i}+\hat{j}+\hat{k})$$

$$r_4 = \left(\frac{-3}{10}, \frac{-3}{10}, \frac{-3}{10}\right)$$

Kinetic Energy of many particle system :-

let us consider a many particle system having 'n' no. of particles. Distribution of particle is 'discontinuous' in this system. Position vector COM w.r.t. 'origin' is $\vec{R}_{cm}$ and position vector of m_i mass particle w.r.t. origin is $\vec{r}_i$ and similar position vector of m_i mass particle w.r.t. to COM is $\vec{r}_{ci}$



Velocity of m_i mass particle depend on external force $\vec{F}_i$

let us consider velocity of m_i particle is $\vec{V}_i$

By vector triangle law

$$\cancel{\vec{OC} = \vec{OP} + \vec{PC}}$$

$$\cancel{\vec{OC} = \vec{OP} +} \quad \cancel{\vec{OP} = \vec{R}_{cm} = \vec{r}_i + \vec{r}_{ci}}$$

$$\vec{OP} = \vec{OC} + \vec{CP}$$

$$\vec{r}_i = \vec{R}_{cm} + \vec{r}_{ci} \quad \text{---(1)}$$

D. (1) w.r.t. 't'

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{R}_{cm}}{dt} + \frac{d\vec{r}_{ci}}{dt}$$

$$\vec{V}_i = \vec{V}_{cm} + \vec{V}_{ci} \quad \text{---(2)}$$

Where :

$\vec{V}_{cm}$ = Velocity of centre of mass

$\vec{V}_i$ = Velocity m_i mass particle

$\vec{V}_{ci}$ = Velocity of m_i mass particle w.r.t. COM

Kinetic energy

$$K.E = \frac{1}{2} m v^2$$

$$K.E = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

K.E. of i^{th} particle

$$(K.E)_i = \frac{1}{2} m_i (\vec{V}_i \cdot \vec{V}_i)$$

by eq. (1)

$$(K.E)_i = \frac{1}{2} m_i (\vec{V}_{cm} + \vec{V}_{ci})^2$$

$$(K.E)_i = \frac{1}{2} m_i [V_{cm}^2 + 2\vec{V}_{cm} \cdot \vec{V}_{ci} + V_{ci}^2]$$

K.E. of whole system

$$\sum_{i=1}^{n} (K.E)_i = \frac{1}{2} \sum_{i=1}^{n} m_i V_{cm}^2 + \sum_{i=1}^{n} \vec{V}_{cm} \cdot \vec{V}_{ci} \, m_i + \frac{1}{2} \sum_{i=1}^{n} V_{ci}^2 m_i$$

$$\Rightarrow \sum_{i=1}^{n} (K.E)_i = \frac{1}{2} \sum_{i=1}^{n} m_i V_{cm}^2 + \vec{V}_{cm} \cdot \sum_{i=1}^{n} m_i \vec{V}_{ci} + \frac{1}{2} \sum_{i=1}^{n} V_{ci}^2 m_i$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i V_{cm}^2 + \vec{V}_{cm} \cdot \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r}_{ci} + \frac{1}{2} \sum_{i=1}^{n} V_{ci}^2 m_i$$

$\because \sum m_i \vec{r}_{ci} = 0$

$$\sum_{i=1}^{n} (K.E)_i = \frac{1}{2}\sum_{i=1}^{n} m_i v_{cm}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_{ci}'^2$$

$$\boxed{E_K = E_{K(cm)} + E'_{Ki}}$$

Where

E_K = K.E. of whole system
$E_{K(cm)}$ = K.E. of centre of mass
E'_{Ki} = Kinetic energy of whole system w.r.t i^{th} particle

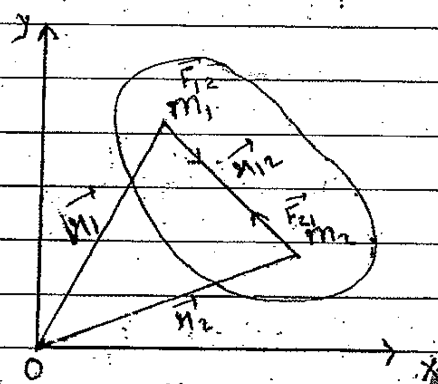
Reduced mass

Let's consider a system in which two particle of mass m_1 and m_2 and position vector are $\vec{r_1}$ & $\vec{r_2}$ respectively. Here mass m_1 apply a force $\vec{F_{12}}$ on mass m_2. and Similarly mass m_2 apply a force $\vec{F_{21}}$ on mass m_1.

Reduced mass concept arise when particle doing relative motion with each other.

Reduced mass concept of 2 particle system problem convert into one particle system.

→ Reduced mass concept used when problem is difficult



mass m_1 apply a force $\vec{F_{12}}$ on 'm_2'

$$\vec{F_{12}} = m_2 \frac{d^2\vec{r_2}}{dt^2} \quad \text{——(I)}$$

m_2 apply a force $\vec{F_{21}}$ on m_1

$$\vec{F_{21}} = m_1 \frac{d^2\vec{r_1}}{dt^2} \quad \text{——(II)}$$

Displacement vector $\vec{r_{12}} = \vec{r_2} - \vec{r_1}$ ——(III)

D. (III) w.r.t 't' (double differentiation)

$$\frac{d\vec{r_{12}}}{dt} = \frac{d\vec{r_2}}{dt} - \frac{d\vec{r_1}}{dt}$$

$$\frac{d^2\vec{r_{12}}}{dt^2} = \frac{d^2\vec{r_2}}{dt} - \frac{d^2\vec{r_1}}{dt}$$

by eq. (I) and (II)

$$\frac{d^2\vec{r_{12}}}{dt^2} = \frac{\vec{F_{12}}}{m_2} - \frac{\vec{F_{21}}}{m_1}$$

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By Newton's 3rd law

$$\vec{F_{12}} = -\vec{F_{21}} \Leftrightarrow \vec{F_{21}} = -\vec{F_{12}}$$

$$\frac{d^2\vec{r_{12}}}{dt^2} = \vec{F_{12}}\left(\frac{1}{m_2} + \frac{1}{m_1}\right)$$

$$\frac{d^2\vec{r_{12}}}{dt^2} = \vec{F_{12}}\frac{(m_1 + m_2)}{m_1 \cdot m_2}$$

$$\boxed{\vec{F_{12}} = \left(\frac{m_1 \cdot m_2}{m_1 + m_2}\right)\frac{d^2\vec{r_{12}}}{dt^2}}$$

$$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2} \quad \text{---(4)}$$

$$\boxed{\vec{F_{12}} = \mu\frac{d^2\vec{r_{12}}}{dt^2}}$$

eq.(4) represent Reduced mass.

Case I: $m_1 = m_2 = m$

$$\therefore \mu = \frac{m_1 \cdot m_2}{m_1 + m_2} = \frac{m \cdot m}{m + m}$$

$$\mu = \frac{m}{2} = \frac{m_1}{2} = \frac{m_2}{2}$$

If mass of both particle are equal, then reduced mass will be Half of one of the mass.

Case II. $m_1 >>>> m_2$

$$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$$

$$\mu = \frac{m_1 \cdot m_2}{m_1\left(1 + \frac{m_2}{m_1}\right)}$$

$$\mu = m_2\left(1 + \frac{m_2}{m_1}\right)^{-1}$$

$$\boxed{\mu \approx m_2}$$

Case III. $m_2 >>>> m_1$

$$\mu = \frac{m_1 \cdot m_2}{m_2\left(1 + \frac{m_1}{m_2}\right)}$$

$$\mu = m_1\left(1 + \frac{m_1}{m_2}\right)^{-1}$$

$$\boxed{\mu \approx m_1}$$

Reduced mass shift toward the particle of less mass

Application of Reduced mass system :-

(1) Reduced mass of Hydrogen :-

There is $1e^-$ and $1p^+$ in Hydrogen

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Due to coulombic force e^- revolves in an orbit.

let mass of e^- = m_e
mass of p = m_p

By reduced mass

$$\mu = \frac{m_e \cdot m_p}{m_e + m_p}$$

Note: $\frac{m_e}{m_p} = \frac{1}{1836}$

$m_p = 1836\, m_e$

$m_e = 9.1 \times 10^{-31}$ Kg

$$\mu = \frac{m_e \cdot m_p}{m_p\left(1 + \frac{m_e}{m_p}\right)}$$

$$\mu = m_e\left(1 + \frac{m_e}{m_p}\right)^{-1}$$

$$\boxed{\mu \approx m_e}$$

(2) Reduced mass of Deutron :- ($_1^2H$)

There is 1p and 1n in $_1^2H$

let mass of proton = m_p
let mass of neutron = m_n

By reduced mass

$$\mu = \frac{m_p \cdot m_n}{m_p + m_n}$$

$\because m_p \approx m_n$

$$\boxed{\mu = \frac{m_p}{2} = \frac{m_n}{2} = \frac{m}{2}}$$

(3) Reduced mass of Earth & Satellite :-

Satellite revolve around earth due to gravitional force. let mass of earth & satellite be m_e and m_s respectively.

By definition of reduced mass

$$\mu = \frac{m_e \cdot m_s}{m_e + m_s}$$

$$\vec{F_s} = \mu \frac{d^2 \vec{r_{es}}}{dt^2}$$

$$\vec{F_s} = \frac{m_e \cdot m_s}{m_e + m_s} \cdot \frac{d^2 \vec{r_{es}}}{dt^2}$$

$$\vec{F_s} = \frac{m_e \cdot m_s}{m_e\left(1 + \frac{m_s}{m_e}\right)} \cdot \frac{d^2 \vec{r_{es}}}{dt^2}$$

$$F_s = m_s\left(1 + \frac{m_s}{m_e}\right)^{-1} \cdot \frac{d^2 \vec{r_{es}}}{dt^2}$$

$(1+x)^{-1} = 1-x$

$$\boxed{\mu \approx m_s}$$

$$\boxed{F_s = m_s \frac{d^2 \vec{r_{es}}}{dt^2}}$$

Here $\frac{d^2 \vec{r_{es}}}{dt^2}$ = displacement of satelite w.r.t earth

(4) Reduced mass of earth & moon :-

$$\vec{F_M} = \mu \frac{d^2 \vec{r}_{EM}}{dt^2}$$

$$\vec{F_M} = \frac{M_m \cdot M_E}{M_m + M_E} \cdot \frac{d^2 \vec{r}_{EM}}{dt^2}$$

$$\vec{F_M} = \frac{M_m \cdot M_E}{M_E\left(1 + \frac{M_m}{M_E}\right)} \cdot \frac{d^2 \vec{r}_{EM}}{dt^2}$$

$$\vec{F_M} = M_m \left(1 + \frac{M_m}{M_E}\right)^{-1} \frac{d^2 \vec{r}_{EM}}{dt^2}$$

$$\boxed{\mu \approx M_m}$$

Momentum and Momentum Conservation law :-

Momentum is a vector qty.
It is defined as the product of mass and velocity.
It's dirn in dirn of velocity.

$\vec{P} = m\vec{v}$ (Mathematical form)

unit = kg. m sec^{-1}

It is linear momentum

let a body of many particle of mass $m_1, m_2, m_3 \cdots m_n$.

then momentum

$$\vec{p} = \vec{p_1} + \vec{p_2} + \vec{p_3} + \cdots + \vec{p_n}$$

$$\vec{p} = m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \cdots + m_n\vec{v_n}$$

$$\because \vec{v}_{CM} = \frac{\sum_{i=1}^{n} m_i\vec{v_i}}{M}$$

$$\vec{v}_{CM} = \frac{m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \cdots + m_n\vec{v_n}}{M}$$

$$M\vec{v}_{CM} = m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \cdots + m_n\vec{v_n}$$

Hence $$\boxed{\vec{p} = M\vec{v}_{CM}} \quad \text{---} (1)$$

Here, momentum of whole body is equal to product of total mass of system and velocity of COM.

D. (1) w.r.t. 't'

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}_{cm}}{dt} \quad \text{---} (2)$$

$$\boxed{\vec{F}_{ext} = m\vec{a}_{cm}}$$

Case:- If $\vec{F}_{ext} = 0$

$$0 = M\frac{d\vec{v}_{cm}}{dt}$$

$$\frac{d\vec{v}_{cm}}{dt} = 0$$

$$\boxed{\vec{v}_{cm} = \text{constant}}$$

If com of any system is moving $\bar{c}$ constant velocity. This phenomena is called momentum conservation law.

Case II: $\vec{F}_{ext} = 0$

$$\frac{d\vec{P}'}{dt} = 0$$

$$\vec{P}' = \text{constant}$$

$$\vec{P}' = \vec{P_1} + \vec{P_2} + \vec{P_3} + \dots + \vec{P_n} = 0 = \text{constant}$$

In component form
x - dirⁿ

$$P_{x_1} + P_{x_2} + \dots + P_{xn} = \text{constant}$$

y - dirⁿ

$$P_{y_1} + P_{y_2} + \dots + P_{yn} = \text{constant}$$

z - dirⁿ

$$P_{z_1} + P_{z_2} + \dots + P_{zn} = \text{constant}$$

So we can say that component of momentum x, y, z will also be conserved.

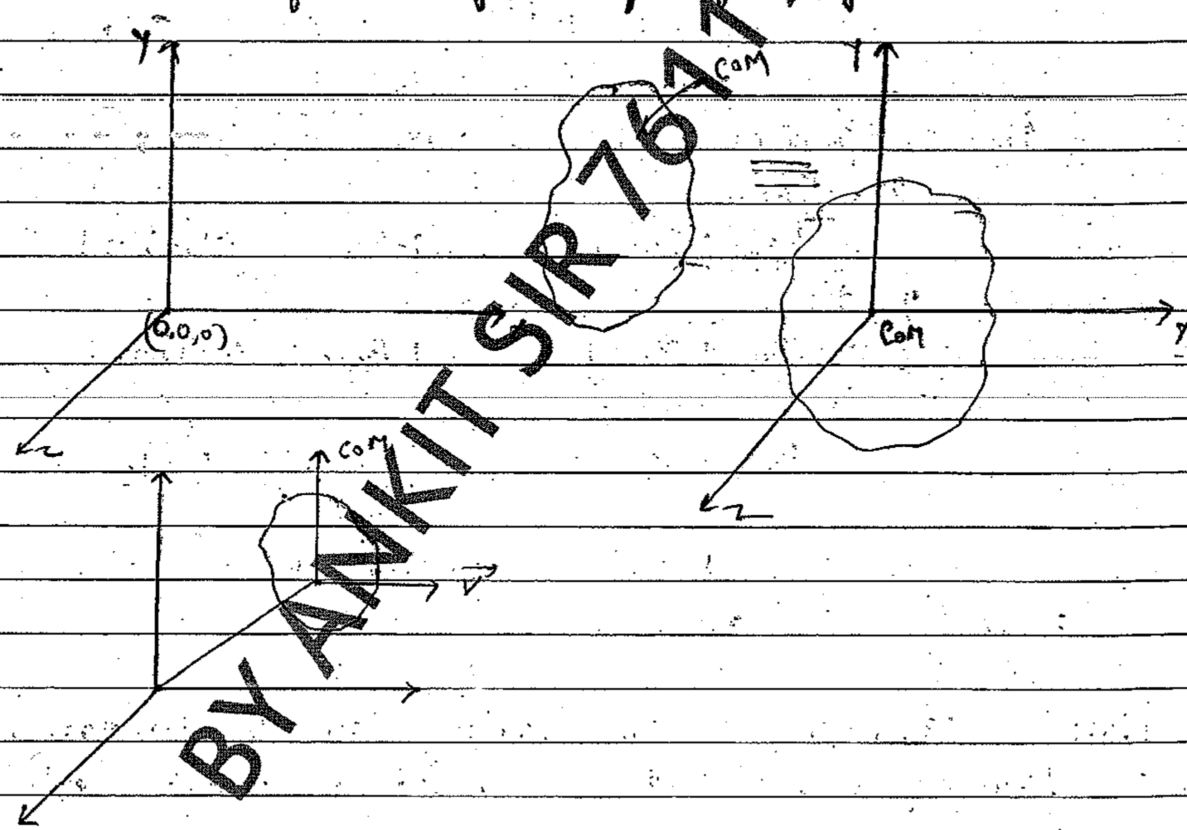
Note :- Momentum conservation law is fundamental law of nature. It is based on Newton eqⁿ of motion.

L-fram (laboratory frame)

let a frame 'S' is rest and object is moving with velocity $\vec{v}$ w.r.t 'S' then this frame of object is called laboratory ~~refrence~~ frame.

It is real frame of ref.

Centre of mass frame of ref (C-frame)

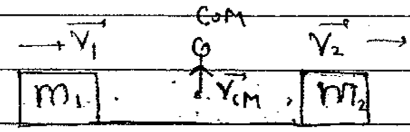


~~If centre~~ of let an object moving with velocity $\vec{v}$

If centre of mass of object is attached to origin then origin of frame start to move with same velocity. This frame is called C-frame.

In this frame total ~~reduce~~ momentum of body is Zero so this frame is also known as Zero Momentum frame of ref.

e.g.



Velocity of m_1 mass w.r.t to COM

$$(\vec{V}_{m_1})_{CM} = \vec{V_1} - \vec{V}_{CM}$$

$$\because \vec{V}_{CM} = \frac{m_1\vec{V_1} + m_2\vec{V_2}}{m_1 + m_2}$$

$$(\vec{V}_{m_1})_{CM} = \vec{V_1} - \left(\frac{m_1\vec{V_1} + m_2\vec{V_2}}{m_1 + m_2}\right)$$

$$(\vec{V}_{m_1})_{CM} = \frac{m_1\vec{V_1} + m_2\vec{V_1} - m_1\vec{V_1} - m_2\vec{V_2}}{m_1 + m_2}$$

$$\boxed{(\vec{V}_{m_1})_{CM} = \frac{m_2(\vec{V_1} - \vec{V_2})}{m_1 + m_2}}$$

Similarly

Velocity of 'm_2' mass w.r.t COM

$$(\vec{V}_{m_2})_{CM} = \vec{V_2} - \vec{V}_{CM}$$

$$(\vec{V}_{m_2})_{CM} = \vec{V_2} - \left(\frac{m_1\vec{V_1} + m_2\vec{V_2}}{m_1 + m_2}\right)$$

$$(\vec{V}_{m_2})_{CM} = \frac{m_1(\vec{V_2} - \vec{V_1})}{m_1 + m_2}$$

$$(\vec{P}_{CM})_{system} = (V_{m_1})_{CM} \cdot m_1 + (V_{m_2})_{CM} \cdot m_2$$

$$= \frac{m_1 m_2 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2} + \frac{m_1 m_2 (\vec{v}_2 - \vec{v}_1)}{m_1 + m_2}$$

$$= \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2 + \vec{v}_2 - \vec{v}_1)$$

$$\boxed{(\vec{P}_{CM})_{system} = 0}$$

Collision

Collision is a interaxn in which particle exchange our momentum. That interaxn is called collision.

For collision, it is not necessary that particle should interact with each other.

On basis of kinetic Energy conservation law, collision is divided into 2 parts-

Collision

- Completely elastic collision
- Completely Inelastic collision

~~perfectly~~ Completely Elastic Collision :-

Collision in which momentum conservation and energy conservation is vallid, This type collision is called perfectly elastic collision.

e.g. Nuclear Rxn.

Perfectly Inelastic Collision :-

Collision in which momentum conservation law is valid but kinetic energy conservation law is not valid. this type collision is called perfectly inelastic collision.

Head-on-collision :- (L-frame)

The collision in which particles by and after collision move in straight line is called Head-on elastic collision

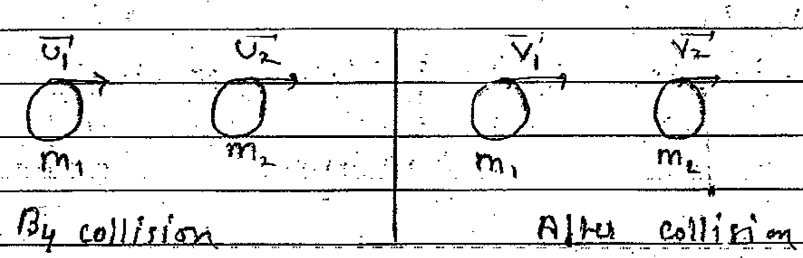
let's consider two particle of mass are m_1 & m_2

By collision

Velocity of particle $= \vec{U_1}, \vec{U_2}$

after collision

Velocity of particle $= \vec{V_1}$ &, $\vec{V_2}$



By collision | After collision

By momentum conservation law :-

$$P_{initial} = P_{final}$$

$$m_1\vec{u_1} + m_2\vec{u_2} = m_1\vec{v_1} + m_2\vec{v_2} \quad \text{—— (1)}$$

$$m_1(\vec{u_1} - \vec{v_1}) + m_2(\vec{u_2} - \vec{v_2}) = 0 \quad \text{—— (2)}$$

By kinetic energy conservation law

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$$

$$m_1(u_1^2 - v_1^2) + m_2(u_2^2 - v_2^2) \quad \text{—— (3)}$$

eq(2) ÷ eq. (3)

$$\Rightarrow \frac{m_1(u_1^2 - v_1^2)}{m_1(u_1 - v_1)} = \frac{-m_2(u_2^2 - v_2^2)}{m_2(u_2 - v_2)}$$

$$\Rightarrow \quad \vec{u_1} + \vec{v_1} = \vec{u_2} + \vec{v_2} \quad \text{—— (4)}$$

$$\Rightarrow \quad \vec{v_2} = \vec{u_1} \mp \vec{u_2} + \vec{v_1} \quad \text{—— (5)}$$

put value of (5) in (1)

$$m_1\vec{u_1} + m_2\vec{u_2} = m_1\vec{v_1} + m_2(\vec{u_1} + \vec{u_2} + \vec{v_1})$$

$$m_1\vec{u_1} + \cancel{m_2\vec{u_2}} = m_1\vec{v_1} + m_2\vec{u_1} \mp \cancel{m_2\vec{u_2}} + m_2\vec{v_1}$$

$$m_1\vec{u_1} = m_1\vec{v_1} + m_2\vec{u_1} + m_2\vec{v_1} \mp 2m_2\vec{u_2}$$

$$\vec{u_1}(m_1 - m_2) = \vec{v_1}(m_1 + m_2) \mp 2m_2\vec{u_2}$$

$$\boxed{\vec{v_1} = \frac{\vec{u_1}(m_1 - m_2)}{m_1 + m_2} + \frac{2m_2\vec{u_2}}{m_1 + m_2}} \quad \text{—— (6)}$$

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by eq. (4)

$$\vec{V_1'} = \vec{V_2'} - \vec{U_1} + \vec{U_2} \quad \text{--- (7)}$$

$$m_1\vec{U_1} + m_2\vec{U_2} = m_1\vec{V_2'} - m_1\vec{U_1} + m_1\vec{U_2} + m_2\vec{V_2'}$$

$$2m_1\vec{U_1} + \vec{U_2}(m_2 - m_1) = \vec{V_2'}(m_1 + m_2)$$

$$\boxed{\vec{V_2'} = \frac{2m_1\vec{U_1}}{m_1+m_2} + \frac{\vec{U_2}(m_2-m_1)}{m_1+m_2}} \quad \text{--- (8)}$$

eq^n 6 and 8 represent velocity of particle after collision

Case I. If $m_1 = m_2$

$$\vec{V_1'} = \frac{2m_2\vec{U_2}}{m_1+m_2} + \frac{\vec{U_1}(m_1-m_2)}{m_1+m_2}$$

$$\boxed{\vec{V_1'} = \vec{U_2}}$$

$$\vec{V_2'} = \frac{2m_1\vec{U_1}}{m_1+m_2} + \frac{\vec{U_2}(m_2-m_1)}{m_1+m_2}$$

$$\boxed{\vec{V_2'} = \vec{U_1}}$$

velocity of particle after collision exchange and exchange max^m momentum

e.g. Nuclear Reactors

Case II If $m_1 >>>> m_2$

$$\vec{V_1'} = \frac{2m_2\vec{U_2}}{m_1+m_2} + \frac{\vec{U_1}(m_1-m_2)}{m_1+m_2}$$

$$\vec{V_1'} = \frac{2m_2\vec{U_2}}{m_1\left(1+\frac{m_2}{m_1}\right)} + \frac{u_1(m_1-m_2)}{m_1+m_2}$$

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$m_2 \approx 0$

$$\boxed{\vec{v_1'} = \vec{u_1}}$$

$$\vec{v_2'} = \frac{2m_1\vec{u_1}}{m_1+m_2} + \frac{u_2(\vec{m_2}-\vec{m_1})}{(m_1+m_2)}$$

$m_2 \approx 0$

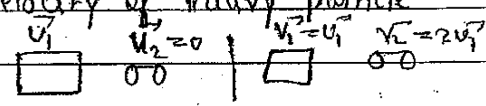
$$\vec{v_2} = -\vec{u_2} + 2\vec{u_1}$$

$$\boxed{\vec{v_2'} = 2\vec{u_1} - \vec{u_2}} \quad \text{— (9)}$$

If second particle is in rest $\vec{u_2} = 0$

$$\boxed{\vec{v_1'} = \vec{u_1}} \qquad \boxed{\vec{v_2'} = 2\vec{u_1}} \quad \text{— (10)}$$

If a Heavy particle collide to a less mass particle then after collision, velocity of Heavier particle would be same and less mass particle move with double velocity of heavy particle



Case III. $m_2 >>>> m_1$

$$\vec{v_1'} = \frac{2m_2\vec{u_2}}{m_1+m_2} + \frac{\vec{u_1}(m_1-m_2)}{m_1+m_2}$$

$m_1 \approx 0$

$$\boxed{\vec{v_1'} = 2\vec{u_2} - \vec{u_1}}$$

$$\vec{v_2'} = \frac{2m_1\vec{u_1}}{m_1+m_2} + \frac{\vec{u_2}(m_2-m_1)}{m_1+m_2}$$

$$\boxed{\vec{v_2'} = \vec{u_2}}$$

If $\vec{u_2} = 0$, $\boxed{\vec{v_1'} = -\vec{u_1}}$ and $\boxed{\vec{v_2'} = 0}$

e.g. When ball fall from certain Height on smooth suface, this case will be similar to this case.

If a less mass particle collide to heavy mass particle then after collision, less mass particle return back with same velocity & heavier mass particle will be zero

Head-on-elastic collision (In C-frame) of 2 particles :-

Let velocity of particle By collision are $\vec{U_1'}$ and $\vec{U_2'}$ and after collision $\vec{V_1'}$ and $\vec{V_2'}$

We know

$$\vec{R}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{M} \quad \text{—(1)}$$

$$\vec{V}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{v_i}}{M} \quad \text{—(2)}$$

By velocity transformation eq^n.

$$\vec{V_1'} = \vec{V_1} - \vec{V}_{CM}$$

$$\vec{U_1'} = \vec{U_1} - \vec{V}_{CM}$$

$$\vec{U_1'} = \vec{U_1} - \left(\frac{m_1\vec{v_1} + m_2\vec{v_2}}{m_1+m_2}\right)$$

$$\vec{U_1'} = \frac{m_2\vec{U_1} - m_2\vec{U_2}}{m_1+m_2}$$

$$\vec{U_1'} = \frac{m_2(\vec{U_1} - \vec{U_2})}{m_1+m_2} \quad \text{—(3)}$$

Similarly

$$\vec{U_2'} = \vec{U_2} - \vec{V}_{cm}$$

$$\vec{U_2'} = \vec{U_2} - \left(\frac{m_1\vec{U_1} + m_2\vec{U_2}}{m_1+m_2}\right)$$

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$$\vec{U_2'} = \frac{m_1(\vec{U_2} - \vec{U_1})}{m_1+m_2} \quad \text{—(4)}$$

Total momentum By collision

$$m_1\vec{U_1'} + m_2\vec{U_2'} = 0$$

$$\frac{m_1 m_2(\vec{U_1} - \vec{U_2})}{m_1+m_2} + \frac{m_1 m_2(\vec{U_2} - \vec{U_1})}{m_1+m_2} = 0$$

$$m_1\vec{U_1'} + m_2\vec{U_2'} = 0$$

$$m_1\vec{U_1'} = -m_2\vec{U_2'}$$

$$\boxed{\vec{U_1'} = \frac{-m_2}{m_1}\vec{U_2'}}$$

Taking magnitude

$$|\vec{U_1'}| = \left|\frac{m_2}{m_1}\right| \vec{U_2'} \quad \text{—— (5)}$$

Similarly after collision

$$m_1\vec{V_1'} + m_2\vec{V_2'} = 0$$

$$\vec{V_1'} = \frac{-m_2}{m_1}\vec{V_2'}$$

$$|\vec{V_1'}| = \frac{m_2}{m_1}\vec{V_2'} \quad \text{—— (6)}$$

By K.E. conservation law

$$\frac{1}{2}m_1U_1'^2 + \frac{1}{2}m_2U_2'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \quad \text{—(7)}$$

~~value of~~ eq. 5 and 6 in eq. (7)

$$\frac{1}{2}m_1\left[\left(\frac{m_2}{m_1}\right)U_2'\right]^2 + \frac{1}{2}m_2U_2'^2 = \frac{1}{2}m_1\left[\frac{m_2}{m_1}V_2'\right]^2 + \frac{1}{2}m_2V_2'^2$$

$$\frac{1}{2}\frac{m_2^2 \cdot U_2'^2}{m_1} + \frac{1}{2}m_2 \cdot U_2'^2 = \frac{1}{2}\frac{m_2^2 \cdot V_2'^2}{m_1} + \frac{1}{2}m_2V_2'^2$$

$$m_2 U_2'^2\left[\frac{m_2}{m_1}+1\right] = m_2 V_2'^2\left[1+\frac{M_2}{M_1}\right]$$

$$U_2'^2 = V_2'^2$$

$$\boxed{|U_2'| = |V_2'|}$$

Similarly $|U_1'| = |V_1'|$

Hence, In COM frame of ref. velocity of particle B4 and ~~col~~ After collision remain same

Numerical

Q.7. The position vector of 2 particle at any instant time 't' are $2\hat{i}+5\hat{j}+13\hat{k}$ and $-6\hat{i}+4\hat{j}-2\hat{k}$ having mass 100 and 300 g. These particle move with velocity $10\hat{i}-7\hat{j}-3\hat{k}$ and $7\hat{i}-9\hat{j}+6\hat{k}$ m/s Find Position of COM.

(i) Position of COM
(ii) Velocity of 2nd particle w.r.t. COM

Soln

$m_1 = 100g \;:\; \vec{r_1} = 2\hat{i}+5\hat{j}+13\hat{k}$

$m_2 = 300g \;:\; \vec{r_2} = -6\hat{i}+4\hat{j}-2\hat{k}$

$\vec{v_1} = 10\hat{i}-7\hat{j}-3\hat{k}$

$\vec{v_2} = 7\hat{i}-9\hat{j}+6\hat{k}$

(i) $$\vec{R}_{cm} = \frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2}$$

$$= \frac{100(2\hat{i}+5\hat{j}+13\hat{k})+300(-6\hat{i}+4\hat{j}-2\hat{k})}{400}$$

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Date

$$\vec{R}_{CM} = \frac{1}{4}\left[2\hat{i} + 5\hat{j} + 13\hat{k} + \cancel{1} - 18\hat{i} + 12\hat{j} - 6\hat{k}\right]$$

$$\vec{R}_{CM} = \frac{1}{4}\left[-16\hat{i} + 17\hat{j} + 7\hat{k}\right]$$

(ii) $\vec{V_2'} = \vec{V_2} - \vec{V}_{CM}$

$$\vec{V_2'} = V_2 - \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

$\Rightarrow$ $|U_1'| = |V_1'|$

$\Rightarrow$ $|U_2'| = |V_2'|$

$\because U_1' = -\frac{m_2}{m_1} V_2' \quad \Leftrightarrow \quad \vec{U_2'} = -\frac{m_1}{m_2}\vec{U_1'}$

$$\vec{V_2'} = -\frac{m_1}{m_2}\vec{V_1'}$$

$\cancel{|V_2'| = -\frac{m_1}{m_2}\vec{U_1'}} \quad \vec{U_2'} = -\frac{m_1}{m_2}\vec{U_1'}$

$\cancel{U_2' =} \quad \vec{U_2'} = -\frac{m_1}{m_2}|\vec{V_1'}|$

$$\vec{U_2'} = -\frac{\cancel{m_1}}{\cancel{m_2}}\left[+\frac{\cancel{m_2}}{\cancel{m_1}}\right]\cdot|\vec{V_2'}|$$

$$\boxed{\vec{U_2'} = -V_2'}$$

$$\boxed{\vec{U_1'} = -\vec{V_1'}}$$

Hence, In COM frame, magnitude velocity B4 & after collision remain same but dirn is opposite. It is necessary for it that collision should 1-D.

$$\boxed{u_2' = -v_2' = \frac{-m_1(\vec{u_2}-\vec{u_1})}{m_1+m_2}}$$

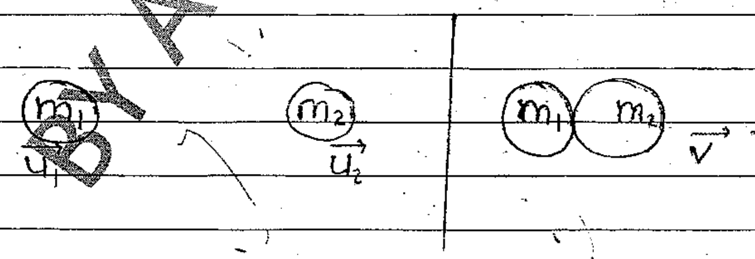
$$\boxed{u_1' = -v_1' = \frac{-m_2(\vec{u_1}-\vec{u_2})}{m_1+m_2}}$$

Q. prove that in 1-D collision magnitude of velocity remain same but dirn is opposite (In C-frame)

Head-on collision of two particles which stick together :-

(a) In L-Frame of ref. :-

let velocity of particles B4 collision are $\vec{u_1}$ and $\vec{u_2}$ while after collision, both particle stick and move with combined velocity $\vec{V}$.



By Momentum conservation

$$m_1\vec{u_1} + m_2\vec{u_2} = (m_1+m_2)\vec{V}$$

$$\vec{V} = \frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1+m_2}$$

Special condⁿ : if $\vec{u_2} = 0$

$$\vec{V} = \frac{m_1 \vec{u_1}}{m_1 + m_2} \longrightarrow (1)$$

by eqⁿ (1), we can say that $\vec{u_1} > \vec{V}$

K.E. after collision

$$K_f = K_2 = \frac{1}{2} m V^2$$

$$K_2 = \frac{1}{2} m \left(\frac{m_1 u_1}{m_1 + m_2}\right)^2$$

$$K_2 = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1^2 \cdot u_1^2}{(m_1 + m_2)^2}\right)$$

$$K_2 = \frac{1}{2}\left(\frac{m_1^2 u_1^2}{m_1 + m_2}\right)$$

B₄ collision K.E

$$K_i = K_1 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$u_2 = 0$$

$$K_1 = \frac{1}{2} m_1 u_1^2$$

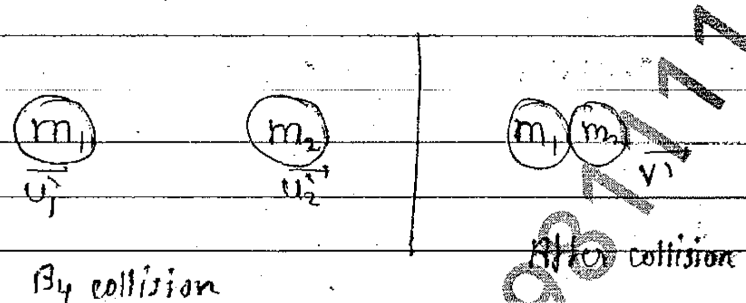
$$\frac{K_2}{K_1} = \frac{\frac{1}{2}\frac{m_1^2 u_1^2}{(m_1 + m_2)}}{\frac{1}{2} m_1 u_1^2}$$

$$\boxed{\frac{K_2}{K_1} = \frac{m_1}{m_1 + m_2}} \longrightarrow (2)$$

Hence, $\boxed{K_2 < K_1}$ after collision K.E. get reduced. This reduction is in other form

(b) **In C- Frame :**

Let velocity of particle by collision are $\vec{u_1}$ and $\vec{u_2}$ and after collision combined velocity $\vec{v'}$.



$$\vec{u_1'} = \vec{u_1} - \vec{v_{cm}}$$

$$\vec{u_1'} = \vec{u_1} - \left(\frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1 + m_2}\right)$$

@ $u_2 = 0$

$$\vec{u_1'} = \frac{m_1\vec{u_1} - m_1\vec{u_1} + m_2\vec{u_1}}{m_1 + m_2}$$

$$\boxed{\vec{u_1'} = \frac{m_2\vec{u_1}}{m_1 + m_2}} \quad \text{——(I)}$$

Similarly $\quad \vec{u_2'} = \vec{u_2} - v_{cm} \quad \because u_2 = 0$

$$\vec{u_2'} = \frac{-m_1\vec{u_1}}{m_1 + m_2} \quad \text{——(II)}$$

By momentum conservation

$$m_1\vec{u_1'} + m_2\vec{u_2'} = (m_1 + m_2)\vec{v'}$$

$$\frac{m_1 m_2\vec{u_1}}{m_1 + m_2} - \frac{m_1 m_2\vec{u_1}}{m_1 + m_2} = (m_1 + m_2)\vec{v'}$$

$$(m_1 + m_2)\vec{v'} = 0$$

$(m_1 + m_2) \neq 0,\quad \boxed{\vec{v'} = 0}$

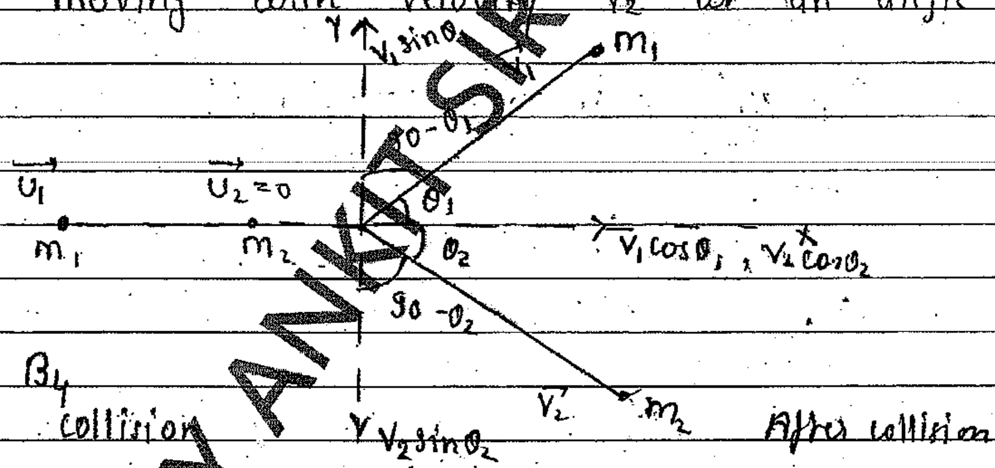
Thus velocity of composite particle in C-frame is '0' so we can say that composite particle remain stationary after collision

Elastic Collision In 2-D :-

(a) In L-frame :-

let a particle of mass m_1 moving with constant velocity $\vec{U_1}$ B4 collision and another particle is stationary.

After collision, particle of mass 'm_1' moving with velocity $\vec{V_1}$ at an angle θ_1 and particle of mass $\vec{m_2}$ moving with velocity $\vec{V_2}$ at an angle 'θ_2'



Momentum conservation in X-dirnn

$$P_i = P_f$$

$$m_1\vec{U_1} + m_2\vec{U_2} = m_1V_1\cos\theta_1 + m_2V_2\cos\theta_2$$

$$m_1\vec{U_1} = m_1V_1\cos\theta_1 + m_2V_2\cos\theta_2 \quad -(1)$$

In y-dirn, Momentum will be zero, particle are in x-dirn

After collision, momentum in y-dirn

$$m_1 v_1 \sin\theta_1 = + m_2 v_2 \sin\theta_2 \quad \text{—(II)}$$

Energy conservation

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \text{—(III)}$$

There are four unknown qty (θ_1, θ_2, $\vec{v_1}$, $\vec{v_2}$) and eq^n are only three. so we can not find value in L-frame

(b) <u>In C-frame of ref.</u> :-

COM frame of ref. is a pseudo frame of ref. Velocity of particle B4 collision are $\vec{u_1'}$, $\vec{u_2'}$ and after collision $\vec{v_1'}$ and $\vec{v_2'}$

Position vector in C-frame of ref.

$$\vec{R}_{CM} = \sum_{i=1}^{n} m_i \vec{r_i}$$

$$M = \sum_{i=1}^{n} m_i$$

$$\vec{R}_{CM} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} \quad (\text{for 2 particle})$$

D. w.r.t. t

$$\vec{V}_{CM} = \frac{m_1 \vec{v_1} + m_2 \vec{v_2}}{m_1 + m_2}$$

If $u_2 = 0$

$$\vec{V}_{cm} = \frac{m_1 u_1}{m_1 + m_2} \quad \text{—(I)}$$

In C-frame, velocity of Particle

$$\vec{u_1'} = \vec{u_1} - \vec{V}_{cm} \quad \text{—(II)}$$

$$\vec{u_1'} = \vec{u_1} - \frac{m_1 \vec{u_1}}{m_1 + m_2}$$

$$\vec{u_1'} = \frac{m_1 \vec{u_1} + m_2 \vec{u_1} - m_1 \vec{u_1}}{m_1 + m_2}$$

$$\vec{u_1'} = \frac{m_2 \vec{u_1}}{m_1 + m_2} \quad \text{—(III)}$$

$$\vec{u_2'} = \vec{u_2} - \vec{V}_{cm}$$

$$\vec{u_2} = 0$$

$$\vec{u_2'} = -\frac{m_1 \vec{u_1}}{m_1 + m_2} \quad \text{—(IV)}$$

(III) $\times m_1$ + (IV) $\times m_2$

$$m_1 \vec{u_1'} + m_2 \vec{u_2'} = \frac{m_1 m_2 \vec{u_1}}{m_1 + m_2} - \frac{m_1 m_2 \vec{u_1}}{m_1 + m_2}$$

$$m_1 \vec{u_1'} + m_2 \vec{u_2'} = 0 \quad \text{—(V)}$$

By Momentum conservation

$$m_1 \vec{u_1'} + m_2 \vec{u_2'} = m_1 \vec{v_1'} + m_2 \vec{v_2'} \quad \text{—(VI)}$$

by (V) & (VI)

$$m_1 \vec{v_1'} + m_2 \vec{v_2'} = 0 \quad \text{—(VII)}$$

by eq^n. (5) and (7)

$$\left.\begin{aligned}\vec{U_2'} &= -\frac{m_1}{m_2}\vec{U_1'} \\ \vec{V_2'} &= -\frac{m_1}{m_2}\vec{V_1'}\end{aligned}\right\} \text{—(8)}$$

By K.E. conservation law

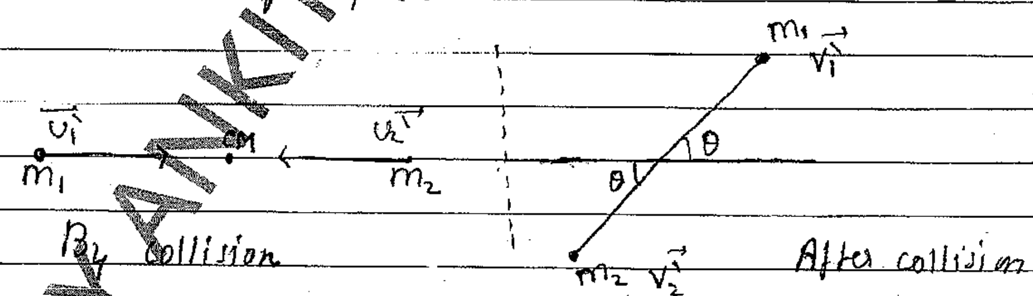
$$\frac{1}{2}m_1U_1'^2 + \frac{1}{2}m_2U_2'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2$$

$$\frac{1}{2}m_1U_1'^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_2}\right)^2U_1'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_2}\right)^2(V_1'^2)$$

$$\frac{1}{2}m_1U_1'^2 + \frac{1}{2}\frac{m_1^2}{m_2}\cdot U_1'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}\frac{m_1^2}{m_2}\cdot V_1'^2$$

$$\because \left.\begin{aligned}|U_1'| &= |V_1'| \\ |U_2'| &= |V_2'|\end{aligned}\right\} \text{—(9)}$$

On basis of eq. (8) & (9)



So explaination of 2-D collision given by C-frame of ref.

Scattering Angle :-

To find the value of Scattering angle, centre of mass frame of ref. is used.

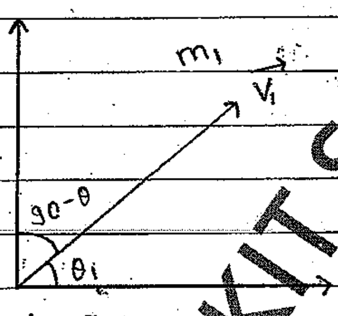
In CM frame of ref., velocity of particle are $\vec{v_1'}$ and $\vec{v_2'}$ respectively.

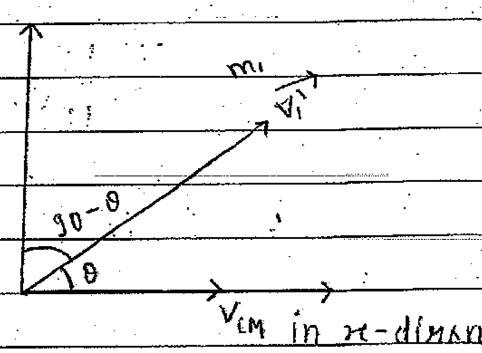
By transformation eqn.,

$$\vec{v_1'} = \vec{v_1} - \vec{v_{CM}}$$

$$\vec{v_1} = \vec{v_1'} + \vec{v_{CM}} \quad \text{—(10)}$$

Divide eqn 10 in component form





Note :- COM frame of ref. always move in 1-D (linear motion)

In x-dirxn,

$$v_1 \cos\theta_1 = v_1' \cos\theta + \vec{v_{CM}} \quad \text{—(11)}$$

In y-dirxn,

$$v_1 \sin\theta_1 = v_1' \sin\theta + 0 \quad \text{—(12)}$$

eq^n (12) $\div$ by eq. (11)

$$\tan\theta_1 = \frac{V_1' \sin\theta}{V_1' \cos\theta + \vec{V}_{CM}}$$

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + \frac{\vec{V}_{cm}}{V_1'}} \quad \text{---(13)}$$

by eq. (11)

$$\vec{U_1'} = \vec{U_1} - \vec{V}_{CM}$$

$$\vec{V}_{CM} = \frac{m_1 U_1}{m_1 + m_2}$$

$$\vec{U_1'} = \frac{m_1 + m_2}{m_1} \vec{V}_{CM} - \vec{V}_{CM}$$

$$\vec{U_1'} = \left(\frac{m_1 + m_2 - m_1}{m_1}\right) \vec{V}_{CM}$$

$$\vec{U_1'} = \frac{m_2}{m_1} \vec{V}_{CM}$$

$$\frac{\vec{V}_{CM}}{\vec{U_1'}} = \frac{m_1}{m_2}$$

$$|\vec{U_1'}| = |\vec{V_1'}|$$

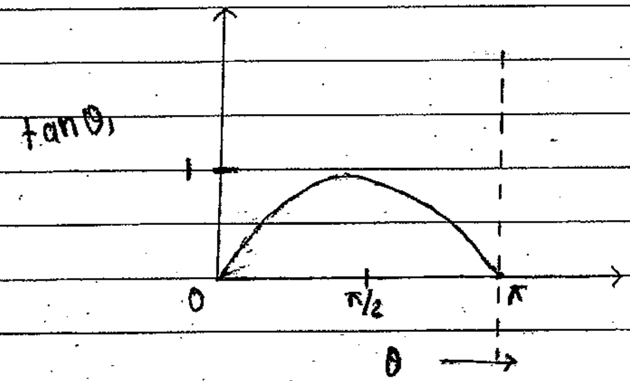
so

$$\frac{\vec{V}_{CM}}{\vec{U_1'}} = \frac{\vec{V}_{CM}}{\vec{V_1'}} = \frac{m_1}{m_2} \quad \text{---(14)}$$

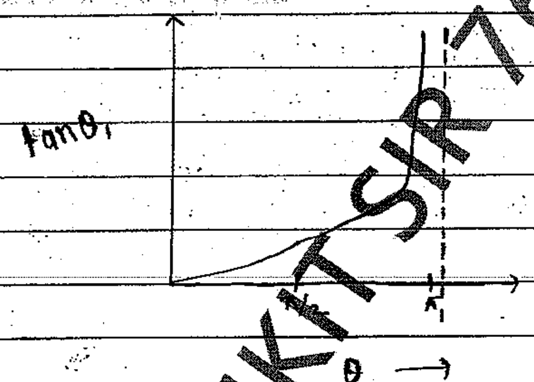
by eq. 13 and 14

$$\boxed{\tan\theta_1 = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}}$$

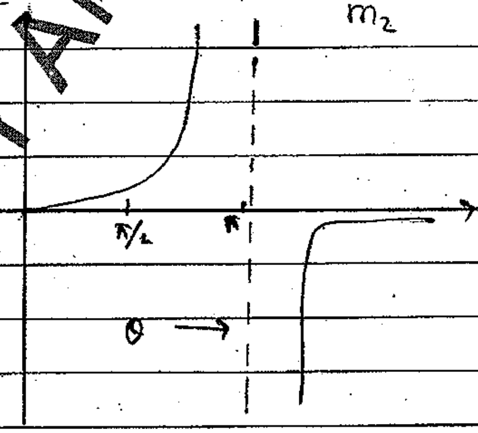
Case I   If $m_1 > m_2$, $\frac{m_1}{m_2} > 1$



Case II   If $m_1 = m_2$, $\frac{m_1}{m_2} = 1$



Case III   If $m_2 > m_1$, $\frac{m_1}{m_2} < 1$



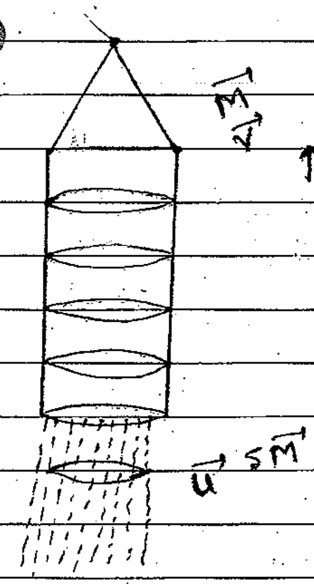
Que. Explain 2-D collision in C-frame of ref.

Rocket Motion (Motion of changing mass) :-

When rocket moves upward, its mass get reduced with time that's why rocket is system of variable mass.

let @ time 't' velocity of rocket is $\vec{V}$ and mass M. Here mass of gas coming from rocket is δm and velocity is $\vec{u}$.

@ $t + \delta t$, mass of rocket become $M - \delta M$ and velocity become $V + \delta V$.



Rocket

Due to change in momentum of rocket, velocity increase.

By newton's 2nd law, rate of change in momentum is equal to external force.

$$\vec{F}_{net} = \frac{\text{Change in momentum}}{\text{change in time}}$$

$$\vec{F}_{ext.} = \frac{P_{Final} - P_{initial}}{\text{change in time}}$$

$$\vec{F}_{ext.} = \frac{(M - \delta M)(\vec{V} + \delta\vec{V}) + \delta M\vec{u} - M\vec{V}}{t + \delta t - t}$$

$$\vec{F}_{ext.} = \frac{M\vec{V} + M\delta\vec{V} - \delta M\vec{V} - \delta M\delta\vec{V} + \delta M\vec{U} - M\vec{V}}{\delta t}$$

$$\vec{F}_{ext} = M\cdot\frac{\delta\vec{v}}{\delta t} - \frac{\delta M}{\delta t}\cdot\vec{v} - \frac{\delta M}{\delta t}\cdot\delta\vec{v} + \frac{\delta M}{\delta t}\cdot\vec{u}$$

Special condition, $\delta t \to 0$; $\frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt}$

$$\frac{\delta M}{\delta t} = -\frac{dM}{dt}, \quad \delta\vec{v} \to 0$$

$$\vec{F}_{ext} = M\cdot\frac{d\vec{v}}{dt} + \vec{v}\frac{dM}{dt} - 0 - \vec{u}\cdot\frac{dM}{dt}$$

$$\vec{F}_{ext} = M\cdot\frac{d\vec{v}}{dt} + (\vec{v}-\vec{u})\frac{dM}{dt}$$

$\because$ $\vec{u} - \vec{v}$ = Relative velocity = $\vec{v}_r$
(Velocity of rocket w.r.t gas)

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} - \vec{v}_r\cdot\frac{dM}{dt}$$

$$\boxed{M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{v}_r\cdot\frac{dM}{dt}}$$ ← Rocket eq^n

Here,

$M\frac{d\vec{v}}{dt}$ = force acting on rocket

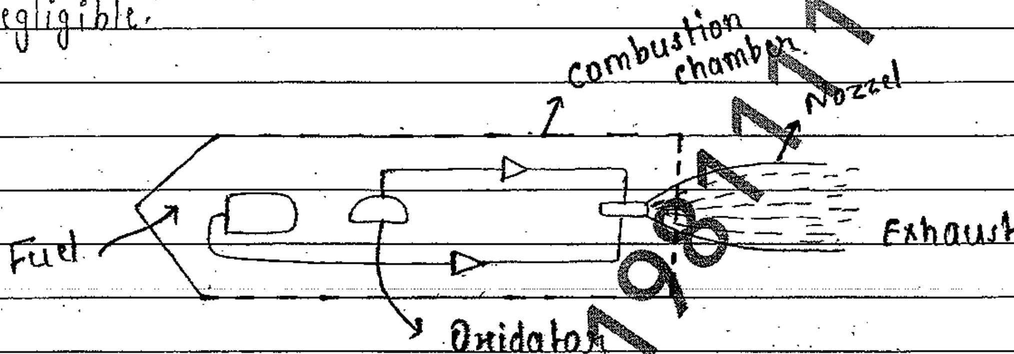
$\vec{F}_{ext}$ = external force

$\vec{v}_r\frac{dM}{dt}$ = Force due to changing mass of rocket.

If rate of change of mass of rocket increases, then velocity of rocket also increase.

Final Velocity of Rocket

External force on rocket is due to gravitational force. Here fricitional force is negligible.



Oxidants react with propellent (fuel) and gas @ high pressure is reduced. From the nozzel gas comes out - side with very high speed.
So according to law of conservation of momentum rocket goes upward with same momentum as propellent leaves the rocket. so gases apply force downward as action so reaction force applied upward.

Rocket follows action- reaction law and rocket aleso work on zet principle.

By rocket eqn

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext.} + \vec{V}_r \frac{dM}{dt}$$

Here,

$$\vec{F}_{ext} = \text{Gravitational} + \text{Frictional force}$$

taking fricitional force as negligible

$$F_{ext} = -mg$$

$$V_r \frac{dM}{dt} = -V_r \frac{dM}{dt} \quad (\text{reduction in mass})$$

$$M\frac{d\vec{v}}{dt} = -mg - \vec{V_r}\frac{dM}{dt}$$

$$\frac{d\vec{v}}{dt} = -g - \frac{\vec{V_r}}{M}\frac{dM}{dt}$$

$$d\vec{v} = -g\,dt - \vec{V_r}\cdot\frac{1}{M}\,dM$$

Integerate above eq^n

$$\int d\vec{v} = \int -g\,dt - \vec{V_r}\cdot\int\frac{1}{M}\,dM$$

$$\vec{v} = -gt - \vec{V_r}\cdot\log_e M + C$$

In Initial state,

at, $t=0$, $M=M_0$, $\vec{v}=\vec{V_0}$

$$\vec{V_0} = 0 - \vec{V_r}\log_e M_0 + C$$

$$C = \vec{V_0} + \vec{V_r}\log_e M_0$$

then $\vec{v} = -gt - V_r\log_e M + \vec{V_0} + \vec{V_r}\cdot\log_e M_0$

$$\boxed{\vec{v} = \vec{V_0} - gt + \vec{V_r}\cdot\log\frac{M_0}{M}}$$

If $t \to 0$

$\vec{V_0} = 0$

$$\boxed{V = V_r \log_e \frac{M_0}{M}} \quad OR \quad \boxed{V = 2.3 \log_{10} \frac{M_0}{M}}$$

This is final velocity of rocket

Where,

$\vec{V_0}$ = Initial Velocity

$\vec{V_r}$ = Relative Velocity

M_0 = Initial mass

M = Final mass

g = gravitational acceleration

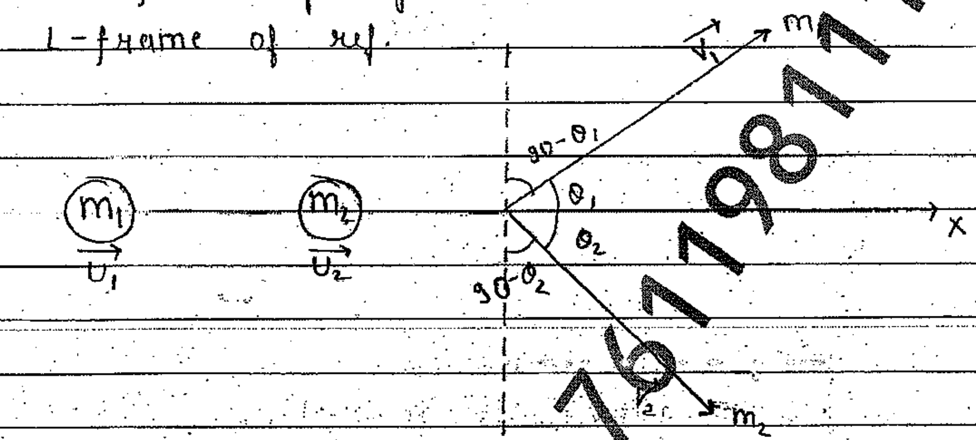
Multi Stage Rocket :-

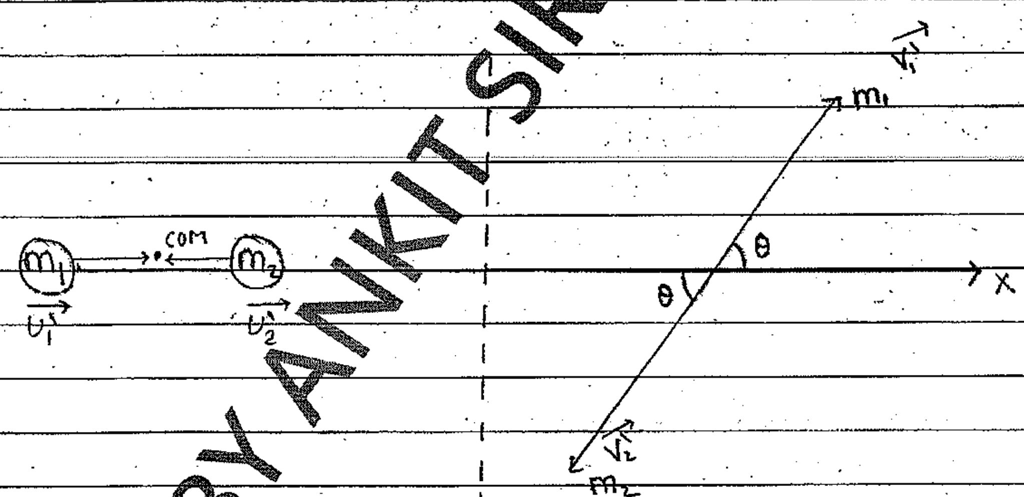
In Single stage rocket, velocity of burnt propellent coming out from rocket can't exceed 2.5 km/sec. otherwise cooling problem occurs. and we can't exceed $\frac{M_0}{M}$ more than 4 otherwise we have to take internal cover of more thickness.

Slow Down Of Moving Neutron In Moderator

Velocity of neutron obtained by Uranium fission is very high, so to control the speed, moderator is used. We can explain it in COM frame of ref. and L-frame of refrence.

L-frame of ref.





Velocity of COM $\vec{V}_{CM} = \dfrac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1 + m_2}$

If $\vec{u_2} = 0$

$$\vec{V}_{CM} = \frac{m_1\vec{u_1}}{m_1+m_2}$$

$$\vec{V}_{CM} = \frac{\vec{U}_1}{\frac{m_2}{m_1} + 1}$$

let $\frac{m_2}{m_1} = A$

$$\vec{V}_{CM} = \frac{\vec{U}_1}{1+A} \quad \text{——— (I)}$$

Velocity of particle By collision

$$\vec{U_1'} = \vec{U}_1 - \vec{V}_{CM}$$

$$\vec{U_1'} = \vec{U}_1 - \frac{\vec{U}_1}{1+A}$$

$$\vec{U_1'} = \frac{A\vec{U}_1}{1+A} \quad \text{——— (II)}$$

Similarly, $\vec{U_2'} = \vec{U}_2 - \vec{V}_{CM}$

$\because U_2 = 0$

$$\vec{U_2'} = -\vec{V}_{CM}$$

$$\vec{U_2'} = -\frac{\vec{U}_1}{1+A} \quad \text{——— (III)}$$

Velocity of particle After collision

$$\vec{V_1'} = \vec{V}_1 - \vec{V}_{CM}$$

$$\vec{V}_1 = \vec{V_1'} + \vec{V}_{CM} \quad \text{——— (IV)}$$

Similarly,

$$\vec{V}_2 = \vec{V_2'} + \vec{V}_{CM} \quad \text{—— (V)}$$

squaring both side of eq. (iv)

$$(\vec{V}_1)^2 = (\vec{V_1'} + \vec{V}_{CM})^2$$

$$|\vec{V}_1|^2 = |\vec{V_1'}|^2 + 2\vec{V_1'}\cdot\vec{V}_{CM} + V_{CM}^2$$

$$\because |\vec{U_1'}| = |\vec{V_1'}|$$

$$|\vec{U_2'}| = |\vec{V_2'}|$$

$$V_1^2 = U_1'^2 + 2\vec{V}_{CM}\cdot\vec{U_1'} + V_{CM}^2$$

$$V_1^2 = \left(\frac{A\vec{U_1}}{1+A}\right)^2 + 2\left(\frac{\vec{U_1}}{1+A}\right)\left(\frac{A\vec{U_1}}{1+A}\right)\cos\theta + \left(\frac{\vec{U_1}}{1+A}\right)^2$$

$$V_1^2 = \frac{U_1^2}{(1+A)^2}\left[A^2 + 2A\cos\theta + 1\right]$$

$$V_1^2 = \frac{U_1^2}{(1+A)^2}\left[1 + A^2 + 2A\cos\theta\right] \quad \text{—— (iii)}$$

Kinetic energy of neutron b4 collision $= E_K$
" " " " after collision $= E_K'$

By collision

$$E_K = \frac{1}{2}m_1U_1^2$$

After collision,

$$E_K' = \frac{1}{2}m_1V_1^2$$

$$\frac{E_K'}{E_K} = \frac{\frac{1}{2}m_1V_1^2}{\frac{1}{2}m_1U_1^2}$$

$$\frac{E_K'}{E_K} = \frac{U_1^2(1+A^2+2A\cos\theta)}{U_1^2(1+A)^2}$$

$$\boxed{\frac{E_K'}{E_K} = \frac{1+A^2+2A\cos\theta}{(1+A)^2}}$$

Case I. If $\theta = 0$, (In this state, energy is not lost)

$$\frac{E_K'}{E_K} = \frac{1+A^2+2A}{(1+A)^2}$$

$$\frac{E_K'}{E_K} = 1$$

$$\boxed{E_K' = E_K}$$

Case II $\quad \theta = \pi$ (loss of energy is max^m in this state)

$$\frac{\epsilon_k'}{\epsilon_k} = \frac{1 + A^2 - 2A}{(1+A)^2}$$

$$\frac{\epsilon_k'}{\epsilon_k} = \frac{(1-A)^2}{(1+A)^2}$$

$$\boxed{\epsilon_k' < \epsilon_k}$$

Per unit energy :-

$$1 - \frac{\epsilon_k'}{\epsilon_k} = 1 - \left(\frac{1-A}{1+A}\right)^2$$

$$\frac{\epsilon_k - \epsilon_k'}{\epsilon_k} = \frac{1 + A^2 + 2A - 1 - A^2 + 2A}{(1+A)^2}$$

$$\frac{\epsilon_k - \epsilon_k'}{\epsilon_k} = \frac{4A}{(1+A)^2}$$

If $m_1 = m_2 =$

$$A = \frac{m_2}{m_1} = 1$$

$$\frac{\epsilon_k - \epsilon_k'}{\epsilon_k} = \frac{4(1)}{(1+1)^2}$$

$$\boxed{\frac{\epsilon_k - \epsilon_k'}{\epsilon_k} = 1}$$

In this per unit energy is max^m

Unit - II

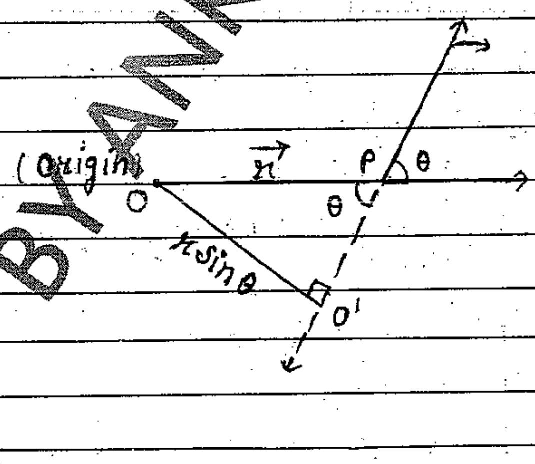
Note :-

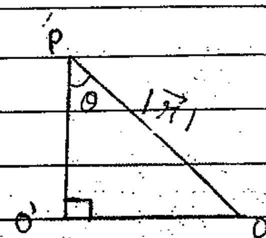
	Linear	Angular
Displacement	x	θ
~~Vector~~ Velocity	$\vec{v}$	$\vec{\omega}$
Acceleration	$\vec{a}$	$\vec{\alpha}$
Force	$\vec{F} = m\vec{a}$	$\tau = I\alpha$ Torque

Torque / Turning Effect Force :-

If moment is taken about the origin for any force applied on any point is called Torque.

Consider a point 'P', position vector about to origin is $\vec{r}$. Force $\vec{F}$ is acting on this point.





by $\Delta OO'P$

$\sin\theta = \frac{OO'}{OP} \Rightarrow |\vec{r}|\sin\theta = OO'$

τ = Force x prependicular distance from force line

$\Rightarrow \quad \tau = F \times OO'$

$\Rightarrow \quad \tau = F|\vec{r}|\sin\theta$

$$\boxed{\tau = \vec{r} \times \vec{F}}$$

Case I :- If $\theta = 0^\circ$

$\tau = F \cdot r \sin 0^\circ$

$\tau = F \cdot r(0)$

$\boxed{\tau = 0}$ minimum

Case II :- If $\theta = 90^\circ$

$\tau = F \cdot r \sin 90^\circ$

$\tau = F \cdot r(1)$

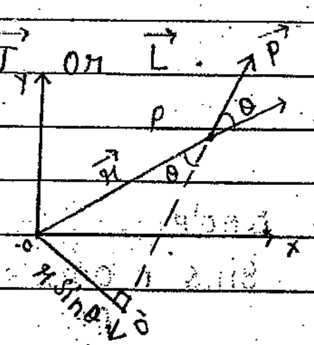
$\boxed{\tau = F \cdot r}$ maximum

Angular Momentum ($\vec{J}$ / $\vec{L}$) :-

Let momentum of any particle at any point is $\vec{p}$ and position vector of particle about origin is $\vec{r}$.

If we take moment of momentum acting at any point then it is called Angular momentum.

It is denoted by $\vec{J}$ or $\vec{L}$.



Angular momentum = momentum × prependicular distance from momentum line.

$$\vec{J} = p \cdot r\sin\theta$$

$$\Rightarrow \boxed{\vec{J} = \vec{r} \times \vec{p}}$$

$$\because \vec{p} = m\vec{v}$$

$$\Rightarrow \vec{J} = \vec{r} \times m\vec{v}$$

$$\Rightarrow \vec{J} = m(\vec{r} \times \vec{v})$$

$$\because \vec{v} = \vec{r} \times \vec{\omega}$$

$$\Rightarrow \vec{J} = m(\vec{r} \times (\vec{r} \times \vec{\omega}))$$

$$\because A \times (B \times C) = 1$$

$$\Rightarrow \vec{J} = m[(\vec{r} \cdot \vec{r})\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}]$$

$$\vec{r} \cdot \vec{\omega} = r\omega\cos\theta, \ \theta = 90'$$
$$= 0$$

$$\Rightarrow \vec{J} = m(r^2 \cdot \vec{\omega})$$

$$\Rightarrow \vec{J} = mr^2 \cdot \vec{\omega}$$

$$\boxed{\vec{J} = I \cdot \vec{\omega}}$$

Rotational Eqn of motion / Relation b/w Angular momentum and torque.

By definition of Angular Momentum

$$\vec{J} = \vec{r} \times \vec{p}$$

$$\because \vec{p} = m\vec{v}$$

$$\Rightarrow \vec{J} = \vec{r} \times m\vec{v}$$

$$\Rightarrow \vec{J} = m(\vec{r} \times \vec{v}) \quad \text{——— (i)}$$

Differentiate (i) w.r.t. 't'

$$\Rightarrow \frac{d\vec{J}}{dt} = m\left[\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right]$$

$$\Rightarrow \frac{d\vec{J}}{dt} = m\left(\vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) \qquad \vec{v} \times \vec{v} = 0$$

$$\Rightarrow \frac{d\vec{J}}{dt} = m\,\vec{r} \times \frac{d\vec{v}}{dt}$$

$$\Rightarrow \frac{d\vec{J}}{dt} = \vec{r} \times \frac{d m\vec{v}}{dt}$$

$$\Rightarrow \frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

By 2^{nd} eqn of motion

$$\Rightarrow F_{ext.} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$$

$$\boxed{\frac{d\vec{J}}{dt} = \vec{\tau}}$$

Angular Momentum Conservation law :-

If a body is doing rotational motion, then angular momentum is conserved ~~when~~ external torque will be zero.

We know that,

$$\frac{d\vec{J}}{dt} = \vec{\tau}$$

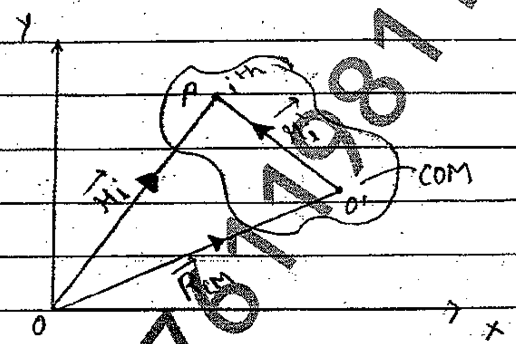
$\vec{\tau} = 0$ (without ext. force)

$$\frac{d\vec{J}}{dt} = 0$$

$$\vec{J} = \text{constant}$$

Angular Momentum of many particle System :-

Let a System having 'n' no of particles, distribution of particle is discontinuous in system. The position vector of centre of mass w.r.t. origin is $\vec{R}_{CM}$.



By triangle law of addition

$$\vec{OP} = \vec{OO'} + \vec{O'P}$$

$$\vec{r_i} = \vec{R}_{CM} + \vec{r_i}'$$

$$\vec{r_i}' = \vec{r_i} - \vec{R}_{CM} \quad \text{——(i)}$$

If mass and velocity of i^{th} particle are m_i and $\vec{v_i}$ respectively then angular momentum will be $\vec{J_i}$.

$$\vec{J_i} = \vec{r_i} \times \vec{p_i}$$

$$\vec{J_i} = \vec{r_i} \times m_i\vec{v_i}$$

$$\vec{J_i} = m_i(\vec{r_i} \times \vec{v_i}) \quad \text{——(ii)}$$

Angular momentum w.r.t. origin for whole system be $\vec{J_0}$.

so, $$\vec{J_0} = \sum_{i=1}^{n} \vec{J_i}$$

$$\vec{J_o} = \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{V_i}) \quad \text{——(III)}$$

Angular momentum w.r.t. COM is $\vec{J}_{CM}$

$$\vec{J}_{CM} = \sum_{i=1}^{n} m_i (\vec{r_i'} \times \vec{V_i'})$$

by eq. (I)

$$\vec{r_i'} = \vec{r_i} - \vec{R}_{CM}$$
$$\vec{V_i'} = \vec{V_i} - \vec{V}_{CM}$$

$$\vec{J}_{CM} = \sum_{i=1}^{n} m_i [(\vec{r_i} - \vec{R}_{CM}) \times (\vec{V_i} - \vec{V}_{CM})]$$

$$\vec{J}_{CM} = \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{V_i}) - \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{V}_{CM}) - \sum_{i=1}^{n} m_i (\vec{R}_{CM} \times \vec{V_i}) + \sum_{i=1}^{n} m_i (\vec{R}_{CM} \times \vec{V}_{CM}) \quad \text{--(IV)}$$

1st part of eq. (IV)

$$\Rightarrow \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{V_i}) = \vec{J_o} \quad \text{by eq}^n \text{ (III)}$$

2nd part

$$\Rightarrow \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{V}_{CM}) = \sum_{i=1}^{n} m_i \vec{r_i} \times \vec{V}_{CM}$$

$$\because \quad R_{CM} = \frac{\sum_{i=1}^{n} m_i r_i}{M}$$

$$= \sum_{i=1}^{n} M \vec{R}_{CM} \times \vec{V}_{CM} = M R_{CM} \times \vec{V}_{CM}$$

3rd part.

$$\Rightarrow \sum_{i=1}^{n} m_i (\vec{R}_{CM} \times \vec{V_i}) = \sum_{i=1}^{n} R_{CM} \times m_i V_i$$

$$\because \sum_{i=1}^{n} m_i V_i = M\vec{V}_{CM}$$

$$\Rightarrow R_{CM} \times M\vec{V}_{CM}$$

put all values in eq^n (IV)

$$\vec{J}_{CM} = \vec{J}_o - M\vec{R}_{CM} \times \vec{V}_{CM} - \vec{R}_{CM} \times M\vec{V}_{CM} + M\vec{R}_{CM} \times \vec{V}_{CM}$$

$$\vec{J}_{CM} = \vec{J}_o - \vec{R}_{CM} \times M\vec{V}_{CM}$$

$$\vec{J}_{CM} = \vec{J}_o - \vec{J}_{CMO}$$

$$\boxed{\vec{J}_o = \vec{J}_{CM} + \vec{J}_{CMO}}$$

Here

$\vec{J}_o$ = Angular momentum of i^{th} particle w.r.t origin

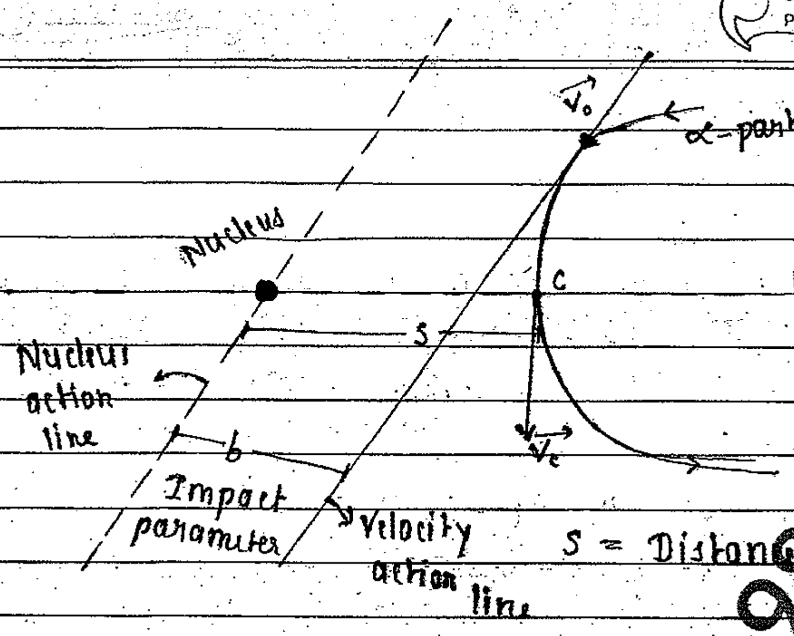
$\vec{J}_{CM}$ = Angular momentum of i^{th} particle w.r.t Centre of mass

$\vec{J}_{CMO}$ = Angular momentum of com w.r.t origin

Examples of Conservation of Angular Momentum

α - Scattering by Nucleus :-

Let us consider a nucleus whose atomic no. is 'Z' and charge Ze. A (+)ive charge particle comes nearly to nucleus from infinite. Hence repulsion force b/w nucleus and charge particle is Coulombic repulsion force.



S = Distance of closest approach

let when charge particle is at infinite distance, then its velocity is V_0 and kinetic energy $\frac{1}{2}mV_0^2$ and prependicular distance from axial line 'b' and angular momentum $mV_0 b$.

$\vec{V_c}$ is a velocity at 'c' and 's' is distance of ~~closed~~ closest approach.

By law of Angular momentum conservation

$$\cancel{m} b \vec{V_0} = \cancel{m} s \vec{V_c}$$

$$b\vec{V_0} = s\vec{V_c}$$

$$\boxed{\vec{V_0} = \frac{S}{b}\vec{V_c} \quad OR \quad \vec{V_c} = \frac{b}{S}\vec{V_0}} \quad \text{—(i)}$$

Using energy conservation law :-

$$E_{initial} = E_{final}$$

$$(K.E)_i + U_i = (K.E)_f + U_f$$

$$\frac{1}{2}mV_0^2 + 0 = \frac{1}{2}mV_c^2 + \frac{Kze^2}{S}$$

∵ potential energy at $\infty = 0$

near, $U = \frac{K q_1 q_2}{r} = \frac{K(Ze)(e)}{S}$; atomic no. of ${}_2He^4 = 2$

$$\Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} m v_c^2 + \frac{2KZe^2}{S}$$

$$\Rightarrow \frac{1}{2} m (v_0^2 - v_c^2) = \frac{2KZe^2}{S}$$

put value of V_c from eq^n (i)

$$\Rightarrow \frac{1}{2} m \left(v_0^2 - \frac{b^2}{S^2} v_0^2 \right) = \frac{2KZe^2}{S}$$

$$\Rightarrow \boxed{\frac{1}{2} m v_0^2 \left(1 - \frac{b^2}{S^2} \right) = \frac{2KZe^2}{S}}$$

From this eq^n, the distance of closest approach 'S' can be determined.

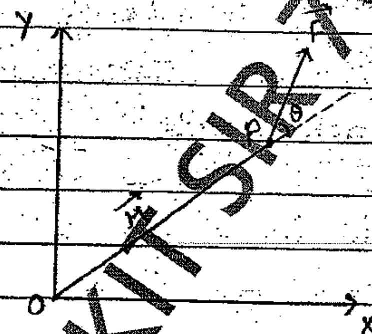
Numerical

1. A Force $\vec{F} = (3\hat{i} + 2\hat{j} - 4\hat{k})$ N is acting on a particle whose position vector about origin is $\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k})$ m. Determine Torque about origin and its magnitude.

Ans. Given: $\vec{F} = (3\hat{i} + 2\hat{j} - 4\hat{k})$ N
$\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k})$ m

To find: Torque $(\vec{\tau}) = ?$
magnitude of Torque $(|\vec{\tau}|) = ?$

Soln



We know that

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= (2\hat{i} - 4\hat{j} + 2\hat{k}) \times (3\hat{i} + 2\hat{j} - 4\hat{k})$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 2 \\ 3 & 2 & -4 \end{vmatrix}$$

$$\vec{\tau} = \hat{i}(16-4) - \hat{j}(-8-6) + \hat{k}(4+12)$$

$$\boxed{\vec{\tau} = 12\hat{i} + 14\hat{j} + 16\hat{k}}$$

Magnitude of Torque

$$|\vec{\tau}| = \sqrt{(12)^2 + (14)^2 + (16)^2}$$

$$\boxed{|\vec{\tau}| = \sqrt{596}}$$

Date ______
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R4 * * *

Q. 2. Mass of empty rocket is 5000 kg in which fuel of mass 40,000 kg is filled up. If exhaust velocity is 2.5 km/sec. Find max^m velocity achieved by rocket.

Ans. Given,

Mass of empty rocket = 5000 kg
Mass of fuel = 40,000 kg
Exhaust velocity (V_H) = 2.5 km/sec.

To find,

Velocity achieved by rocket (V) = ?

Sol^n.

We know that

$$|\vec{V}| = |\vec{V_0}| + |\vec{V_H}| \log_e \frac{M_0}{M}$$

at $t=0$, $\vec{V_0} = 0$

$$|\vec{V}| = |\vec{V_H}| \log_e \frac{M_0}{M}$$

$$V = V_H \log_e \frac{M_0}{M}.$$

$$V = 2.5 \log_e \frac{40,000}{5000}$$

$$V = 2.5 \log_e 8$$

$$V = 2.5 \log_e 2^3$$

$$\boxed{V = 2.5 \times 3 \log_e 2} \text{ km/sec.}$$

$$\boxed{V = 2.5 \times 3 \times 2.3 \log_{10} 2} \text{ km/sec.}$$

Q.3. The distance of closest approach of neutron in presence of proton is 4×10^{-14} m and its angular momentum is 10.11×10^{-34} J. sec. Determine energy of neutron. ($m = 1.6 \times 10^{-27}$ Kg)

Soln Given,

distance of closest approach (S) = 4×10^{-14} m

Angular momentum (J) = 10.11×10^{-34} J. sec.

$m = 1.6 \times 10^{-27}$ Kg

To find,

Energy of neutron = ?

Soln

We know, $J = rp$

$J = mvs$ —(i) ∵ $p = mv$, $r = s$

$K.E = \frac{1}{2} mv^2$ —(ii)

put value of v from (i)

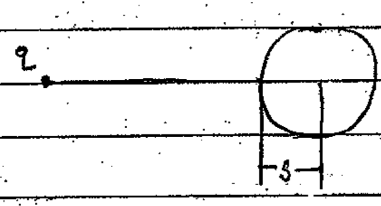
$$K.E = \frac{1}{2} m \left(\frac{J}{ms} \right)^2$$

$$K.E = \frac{1}{2} \frac{J^2}{ms^2}$$

$$K.E = \frac{1}{2} \times \frac{(10.11 \times 10^{-34})^2}{1.6 \times 10^{-27} \times (4 \times 10^{-14})^2}$$

Note :-

Distance of closest approach



By energy conservation

$$0 + \frac{1}{2}mv^2 = 0 + \frac{Kq(Ze)}{S}$$

$$\boxed{S = \frac{Kq(Ze)}{K.E}}$$

Inertia coefficient :-

Ext. torque is needed to change the velocity of any object which is either at rest or rotating with uniform angular velocity. This qty. is measured by Inertial coefficient in rotational motion.

Inertial coefficient have some significant features as inertia have in linear motion ~~mot~~ moment of Inertia depend on mass as well as axis of rotation.

let an object is made up of 'n' no. of particle and mass of i^{th} particle is m_i and velocity is $\vec{V_i}$ and this object is rotating about any fixed axis.

Angular momentum of i^{th} particle

$$\vec{J_i} = \vec{r_i} \times \vec{p_i}$$

for whole system.

$$\sum_{i=1}^{n} \vec{J_i} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i}$$

$$\vec{J} = \sum_{i=1}^{n} (\vec{r_i} \times \vec{p_i})$$

$$\vec{J} = \sum_{i=1}^{n} m_i (\vec{r_i} \times \vec{v_i}) \quad \text{---(1)}$$

$$\vec{v_i} = \vec{\omega_i} \times \vec{r_i}$$

$$\vec{J} = \sum_{i=1}^{n} m_i [\vec{r_i} \times (\vec{\omega_i} \times \vec{r_i})]$$

$$A \times B \times C = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{J} = \sum_{i=1}^{n} m_i [\vec{\omega_i} r_i^2 - \vec{r_i}(\vec{r_i} \cdot \vec{\omega_i})]$$

$$\because \vec{r_i} = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$

$$\vec{\omega_i} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\vec{r_i} \cdot \vec{\omega_i} = x_i \omega_x + y_i \omega_y + z_i \omega_z$$

$$\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k}$$

$$J_x \hat{i} + J_y \hat{j} + J_z \hat{k} = \sum_{i=1}^{n} m_i [\omega_i \cdot r_i^2 - (x\hat{i} + y\hat{j} + z\hat{k})\{\omega_x x_i + \omega_y y_i + \omega_z \cdot z_i\}]$$

Comparing coefficient of $\hat{i}, \hat{j}, \hat{k}$

$$J_x = \sum_{i=1}^{n} m_i [r_i^2 \cdot \omega_x - \{x_i^2 \cdot \omega_x + \omega_y \cdot x_i y_i + \omega_z x_i z_i\}]$$

$$J_x = \sum_{i=1}^{n} m_i (r_i^2 - x_i^2)\omega_x + \sum_{i=1}^{n} (-m_i x_i y_i \omega_y) + \sum_{i=1}^{n} (-m_i x_i z_i \omega_z)$$

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \quad \text{---(II)}$$

Similarly

$$J_y = \sum_{i=1}^{n} -m_i(x_i y_i)\omega_x + \sum_{i=1}^{n} m_i(r_i^2 - y_i^2)\omega_y + \sum_{i=1}^{n}(-m_i y_i z_i)\omega_z$$

$$J_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \quad \text{—(III)}$$

$$J_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \quad \text{—(IV)}$$

On writing eq. (II) (III) and (IV) in matrix method

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

principal axis

$$[J] = [I][\omega]$$

Here

$$I_{xx} = \sum_{i=1}^{n} m_i(r_i^2 - x_i^2)$$

$$I_{xx} = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2 + z_i^2 - x_i^2)$$

$$I_{xx} = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2) \quad \text{(Inertial coefficient w.r.t. (x-axis))}$$

Similarly

$$I_{yy} = \sum_{i=1}^{n} m_i(x_i^2 + z_i^2) \quad \text{(w.r.t. (y-axis))}$$

$$I_{zz} = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2) \quad \text{(w.r.t. z-axis)}$$

$$I_{xy} = I_{yx} = \sum_{i=1}^{n}(-m_i x_i y_i) \quad \text{(w.r.t. x-y axis)}$$

$$I_{xz} = I_{zx} = \sum_{i=1}^{n}(-m_i x_i z_i) \quad \text{(w.r.t. x-z axis)}$$

$$I_{yz} = I_{zy} = \sum_{i=1}^{n}(-m_i y_i z_i) \quad \text{(w.r.t. y-z axis)}$$

$\because$ $[J]$ = matrix of Angular momentum
$[I]$ = " " Inertial coefficient
$[\omega]$ = " " Angular velocity

eq. (II), (III) and (IV), 9 coefficients are known as Inertial coefficients.

Angular momentum is parallel to three prependicular axis then axis is called principal axis.

For principal axis $[\, I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0 \,]$

$$J_x = I_{xx}\,\omega_x$$
$$J_y = I_{yy}\,\omega_y$$
$$J_z = I_{zz}\,\omega_z$$

Imp. Position of $\vec{J}$ and $\vec{\omega}$ is not parallel :-

If a body is doing rotational motion, then having angular momentum $\vec{J}$ and angular velocity $\vec{\omega}$

let body is doing motion about x-axis then

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\left.\begin{array}{l}\omega_y = \omega_z = 0 \\ \omega_x \neq 0\end{array}\right\} \text{body moving along } x\text{-axis}$$

$$\vec{\omega} = \omega_x \hat{i} \quad \text{---(I)}$$

Angular momentum $\vec{J}$

$$J_x = I_{xx} \cdot \omega_x$$
$$J_y = I_{yx} \cdot \omega_x$$
$$J_z = I_{zx} \cdot \omega_x$$

$$\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k}$$
$$\vec{J} = I_{xx} \omega_x \hat{i} + I_{yx} \cdot \omega_x \hat{j} + I_{zx} \cdot \omega_x \hat{k} \quad \text{---(II)}$$

Taking vector product of (I) and (II)

$$\vec{\omega} \times \vec{J} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & 0 & 0 \\ I_{xx}\omega_x & I_{xy}\omega_x & I_{zx}\omega_x \end{vmatrix}$$

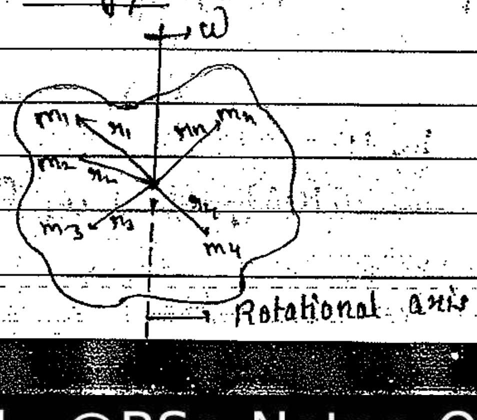
$$\vec{\omega} \times \vec{J} = \omega_x \hat{i} [(I_{zx} \cdot \omega_x)\hat{j} - \hat{k}(I_{xy} \cdot \omega_x)]$$

$$\boxed{\vec{\omega} \times \vec{J} \neq 0}$$

mean $\vec{\omega}$ and $\vec{J}$ are not parallel
becz if it is parallel then $\vec{\omega} \times \vec{J}$ should be zero.

Hence $\vec{\omega}$ and $\vec{J}$ are not $\parallel$.

Rotational kinetic energy :-



let us consider a system having 'n' no. of particles whose distribution is discontinuous. This body is rotating with angular velocity

let mass mass of i^{th} particle is m_i and velocity be v_i respectively. then K.E will be

$$(K.E)_i = \frac{1}{2} m_i v_i^2$$

Note :- In angular motion, linear velocity of particle remain same while angular velocity is different.

$$(K.E)_i = \frac{1}{2} m_i (\vec{v_i} \cdot \vec{v_i})$$

$$\because v_i = \vec{\omega_i} \times \vec{r_i}$$

$$(K.E)_i = \frac{1}{2} m_i [(\vec{\omega_i} \times \vec{r_i}) \cdot (\vec{\omega_i} \times \vec{r_i})]$$

By algebric

$$[(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\vec{A} \times (\vec{A} \times \vec{B}))]$$

$$(K.E)_i = \frac{1}{2} m_i [\vec{\omega_i} \cdot (\vec{r_i} \times (\vec{\omega_i} \times \vec{r_i}))] \quad \text{--- (1)}$$

$$\because \vec{A} \times \vec{B} \times \vec{C} = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

Numerical

$$(K.E)_i = \frac{1}{2} m_i [\vec{\omega_i} \cdot \{ \vec{\omega_i} r_i^2 - \vec{r_i} (\vec{\omega_i} \cdot \vec{r_i})] \quad \text{--- (2)}$$

$$\because v_i = \omega_i \times r_i$$

by (1)

$$(K.E)_i = \frac{1}{2} m_i [\vec{\omega_i} \cdot (\vec{r_i} \times \vec{v_i})]$$

$$(K\cdot E)_i = \frac{1}{2}(\vec{\omega})(\vec{r_i} \times \vec{p_i})$$

$$(K\cdot E)_i = \frac{1}{2}\vec{\omega}\cdot\vec{J_i} \quad \text{——(3)}$$

For whole System

$$E_K = \sum_{i=1}^{n} (K\cdot E)_i$$

$$E_K = \sum_{i=1}^{n} \frac{1}{2}\vec{\omega}\cdot\vec{J_i}$$

$$E_K = \frac{1}{2}\vec{\omega}\sum_{i=1}^{n}\vec{J_i}$$

$$\because \sum_{i=1}^{n}\vec{J_i} = \vec{J}$$

$$E_K = \frac{1}{2}\vec{\omega}\cdot\vec{J} \quad \text{——(4)}$$

Inertial coefficient is a tensor qty.

$$E_K = \sum_{\mu} \frac{1}{2}\omega_\mu \cdot J_\mu$$

$$J = \sum_{\nu} I_{\mu\nu}\,\omega_{\mu\nu}$$

$$J_\mu = I_{\mu\nu}\,\omega_{\mu\nu}$$

$$E_K = \sum_{\mu,\nu} \frac{1}{2}\omega_\mu I_{\mu\nu}\cdot\omega_\nu$$

$$E_K = \frac{1}{2}\omega_x^2 I_{xx} + \frac{1}{2}\omega_x\cdot I_{xy}\cdot\omega_y + \frac{1}{2}\omega_x\cdot I_{xz}\omega_z +$$
$$\frac{1}{2}\omega_y^2 I_{yy} + \frac{1}{2}I_{yx}\,\omega_x\cdot\omega_y + \frac{1}{2}I_{zx}\cdot\omega_z\cdot\omega_y +$$
$$\frac{1}{2}\omega_z^2 I_{zz} + \frac{1}{2}I_{zx}\,\omega_z\cdot\omega_x + \frac{1}{2}I_{zy}\,\omega_z\cdot\omega_y$$

$$E_K = \frac{1}{2}\omega_x^2 \cdot I_{xx} + \frac{1}{2}\omega_y^2 \cdot I_{yy} + \frac{1}{2}\omega_z^2 \cdot I_{zz} +$$
$$I_{xy} \cdot \omega_x \cdot \omega_y + I_{xz}\,\omega_x \cdot \omega_z + I_{yz}\,\omega_z \cdot \omega_y \quad \text{—(5)}$$

If body rotates about (x, y, z) then

$$I_{xy} = I_{zx} = I_{xz} = I_{yx} = I_{yz} = I_{zy} = 0$$

$$E_K = \frac{1}{2}\omega_x^2 \cdot I_{xx} + \frac{1}{2}\omega_y^2 \cdot I_{yy} + \frac{1}{2}\omega_z^2 \cdot I_{zz}$$

If body is symmetric about x, y, z

$$I_{xx} = I_{yy} = I_{zz} = I$$

$$E_K = \frac{1}{2} I (\omega_x^2 + \omega_y^2 + \omega_z^2)$$

$\because \vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

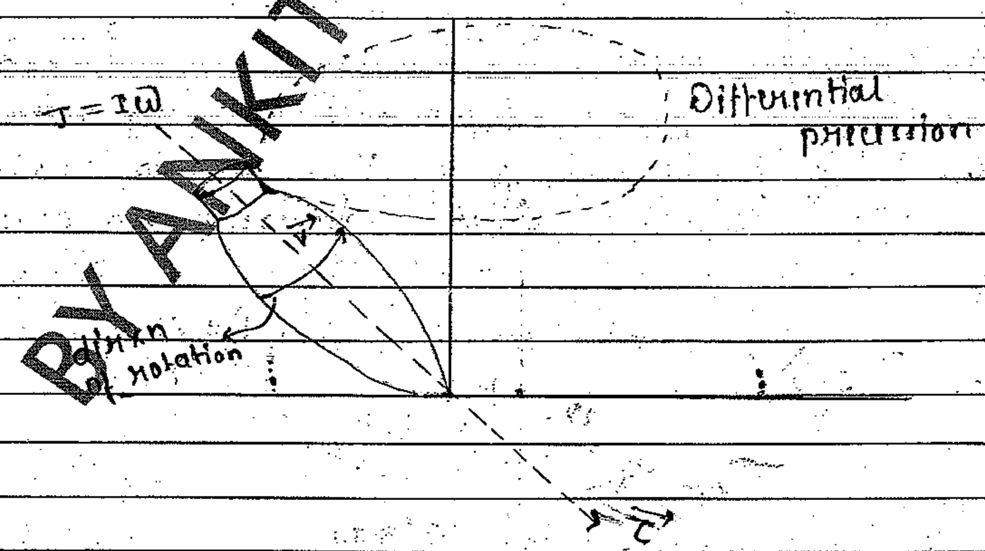
$$\boxed{E_K = \frac{1}{2} I \omega^2}$$

Rotational K.E. energy of body

Precession motion of Spining top :-

When a rotating body is acted upon by a torque whose axis is prependicular to axis of rotation of body then rate of rotating body remains constant (ω/ J = constant) but dirxn of axis of rotation change i.e. axis of rotation itself rotate.

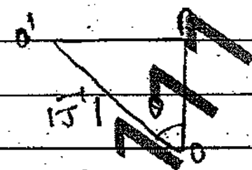
The rotation of axis of rotation is called as precession. The axis about which dirxn of body precerise is called axis of precessional



Precessional rate :- The rate of precessional motion is called precessional rate.

$$\tau = r mg \sin\theta \quad —(I)$$

$$\because \omega_p = \frac{d\phi}{dt}$$



$$\text{angle} = \frac{\text{arc}}{\text{radius}}$$

$$d\phi = \frac{dJ}{O'B} = \frac{dJ}{J\sin\theta} \quad —(II)$$

$$\frac{d\phi}{dt} = \frac{dJ}{dt \cdot J\sin\theta} \Rightarrow \frac{d\phi}{dt} = \frac{\tau}{J\sin\theta} \quad —(III) \quad \because \tau = \frac{dJ}{dt}$$

$$\Rightarrow \omega_p = \frac{\tau}{J\sin\theta} \quad —(IV)$$

by (I)

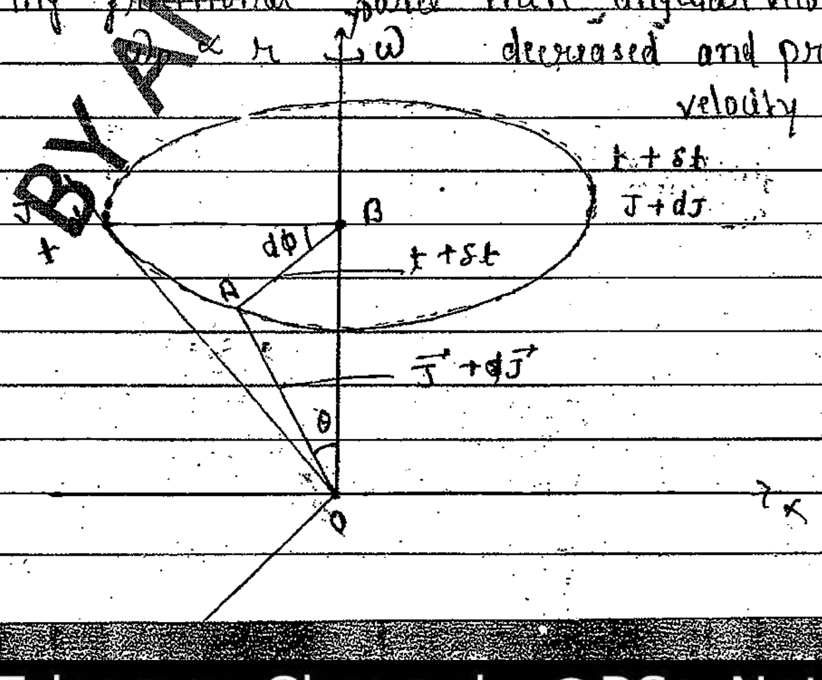
$$\Rightarrow \omega_p = \frac{r mg \sin\theta}{J\sin\theta} = \frac{mgr}{J} \quad —(V)$$

eq. (V) represent algebric expression of precessional rate

(I) $\omega_p \propto \frac{1}{J}$ | ω_p ↓-es on ↑-ing value of J

(II) On ↑-ing frictional force then angular velocity will be decreased and precessional velocity will be ↓-ed

(III) $\omega_p \propto r$



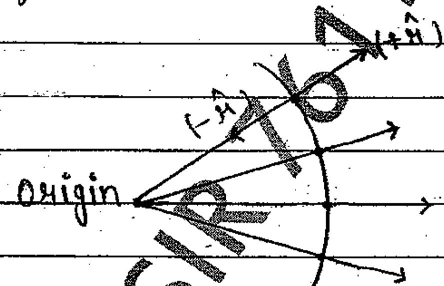
Unit - 3

Motion Under Central forces

Central force :-

If the force on a body is always towards a fixed ~~direction~~ point, it is called Central force. Take the fixed point be origin.

Central force is conservative force



Mathematical expression :

$$\vec{F}(\vec{r}) = F(r)(\pm\hat{r})$$

$$\vec{F}(\vec{r}) = \frac{K}{r^n}(\pm\hat{r})$$

Examples :-

(1) Gravitational force $\vec{F_G} = -\frac{Gm_1m_2}{r^2}(\hat{r})$

$$\vec{F_G} = \frac{Gm_1m_2}{r^2}(-\hat{r})$$

(2) Coulombic force

$$\vec{F} = \frac{Kq_1q_2}{r^2}(+\hat{r})$$

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Imp

Prove that under central forces, motion of particle is always in a plane.

Proof :-

By definition of Central force

$$\vec{F}(\vec{r}) = F(r)(\pm\hat{r}) \quad \text{--- (I)}$$

$\Rightarrow$ $\vec{F}(\vec{r}) = F(r)(+\hat{r})$ (away from origin)

cross product of $\vec{r}$

$\Rightarrow$ $\vec{r} \times \vec{F}(\vec{r}) = F(r)(\vec{r} \times \hat{r})$

$\hat{r}$ = unit vector $= \dfrac{\vec{r}}{|\vec{r}|}$

$\Rightarrow$ $\vec{r} \times \vec{F}(\vec{r}) = F(r)\left(\vec{r} \times \dfrac{\vec{r}}{|\vec{r}|}\right)$

$\because$ $\hat{i} \times \hat{i} = 0 \quad (\theta = 0^\circ)$

$\vec{r} \times \vec{r} = 0$

$\Rightarrow$ $\vec{r} \times \vec{F}(\vec{r}) = 0 \quad \text{--- (II)}$

According to Newton

$\Rightarrow$ $\vec{F}(\vec{r}) = m\dfrac{d^2\vec{r}}{dt^2}$

$\Rightarrow$ $\vec{r} \times \vec{F}(\vec{r}) = m\dfrac{d^2\vec{r}}{dt^2} \times \vec{r} = 0$

$m \neq 0$

$\Rightarrow$ $\vec{r} \times \dfrac{d^2\vec{r}}{dt^2} = 0 \quad \text{--- (III)}$

$\Rightarrow$ $\vec{r} \times \vec{v} = \vec{r} \times \dfrac{d\vec{r}}{dt} \quad \text{--- (IV)}$

D. eq. (IV) w.r.t.

$\Rightarrow$ $\dfrac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \dfrac{d\vec{v}}{dt} = \dfrac{d\vec{r}}{dt} \times \dfrac{d\vec{r}}{dt} + \vec{r} \times \dfrac{d^2\vec{r}}{dt^2}$

$\Rightarrow \quad \frac{d}{dt}(\vec{r} \times \vec{v}) = 0 \qquad$ [by eq. (III)]

$\Rightarrow \quad \vec{r} \times \vec{v} = \text{constant} \qquad \text{--- (V)}$

From eq. (5) we can say that $\vec{r} \times \vec{v}$ = constant is only possible when particle does motion in a plane.

$\Rightarrow \quad \vec{\tau} = \frac{d\vec{J}}{dt}$

$\Rightarrow \quad \vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}(\vec{r})$

$\Rightarrow \quad \frac{d\vec{J}}{dt} = 0 \qquad$ by eq. (II)

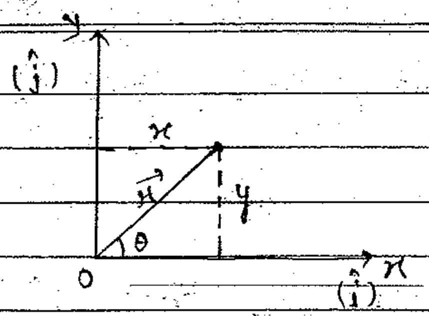
$\because$ Angular momentum $\vec{J} = \vec{r} \times \vec{p}$

$\Rightarrow \quad \frac{d}{dt}(\vec{r} \times \vec{p}) = 0$

$\boxed{\vec{r} \times \vec{p} = \text{constant}}$ Hence proved

Eq^n of motion of particle under central force :-

Let a particle having mass 'm' whose co-ordinate in cartesian coordinate system (x, y) and position vector is $\vec{r}$. Co-ordinate of this particle in polar co-ordinate system are (r, θ)



From Geometry

$\Rightarrow \quad x = r\cos\theta \quad \text{---} (1)$

$\Rightarrow \quad y = r\sin\theta \quad \text{---} (2)$

Acceleration of particle in cartesion co-ordinate system

$\Rightarrow \quad \vec{a} = a_x\hat{i} + a_y\hat{j} \quad \text{---} (3)$

We know,

$\Rightarrow \quad a_x = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$\Rightarrow \quad \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - \sin\theta\cdot r\frac{d\theta}{dt}$

$a_x = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$\Rightarrow \quad = \frac{d^2r}{dt^2}\cos\theta - \sin\theta\cdot\frac{d\theta}{dt}\cdot\frac{dr}{dt} - \left[\cos\theta\left(\frac{d\theta}{dt}\right)^2\cdot r + \sin\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} + \sin\theta\cdot r\frac{d^2\theta}{dt^2}\right]$

$\Rightarrow a_x = \frac{d^2r}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)\sin\theta - r\left(\frac{d\theta}{dt}\right)^2\cos\theta - \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\sin\theta - r\frac{d^2\theta}{dt^2}\cdot\sin\theta$

$\Rightarrow a_x = \cos\theta\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) - \sin\theta\left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right) \quad \text{---} (4)$

Similarly

$$\Rightarrow a_y = \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{dr}{dt}\cdot\sin\theta + \cos\theta\cdot r\frac{d\theta}{dt}$$

$$a_y = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2}\cdot\sin\theta + \cos\theta\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + \cos\theta\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}$$

$$\Rightarrow a_y = \sin\theta\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + \cos\theta\left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right) \quad -(5)$$

put value of eq. (4) and (5) in eq. (3)

$$\Rightarrow \vec{a} = \left[\cos\theta\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} - \sin\theta\left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right)\right]\hat{i} + \left[\sin\theta\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} + \cos\theta\left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right)\right]\hat{j}$$

$$\Rightarrow \vec{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}(\hat{i}\cos\theta + \hat{j}\sin\theta) + \left\{2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right\}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \quad -(6)$$

Here,

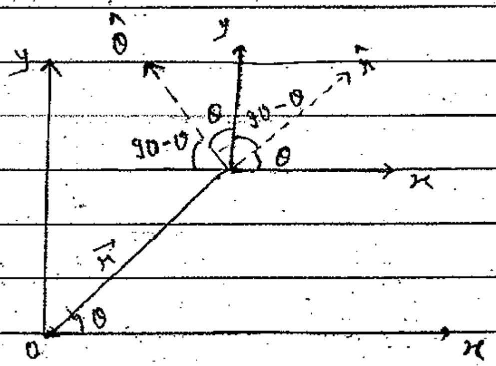
radial acceleration $(a_r) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = (r'' - r\theta'^2)$ $-(7)$

tangential acceleration $(a_\theta) = \left[2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right] = [2r'\theta' + r\theta'']$ $-(8)$

put value of eq. (7) and (8) in eq. (6)

$$\Rightarrow \vec{a} = a_r(\hat{i}\cos\theta + \hat{j}\sin\theta) + a_\theta(-\hat{i}\sin\theta + \hat{j}\cos\theta) \quad -(9)$$

Relation b/w Cartisian co-ordinate & polar co-ordinate system



$$\hat{r} = |\hat{r}|\cos\theta\ \hat{i} + |\hat{r}|\cos(90-\theta)\hat{j}$$

$$\Rightarrow \hat{r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j} \quad \text{---}(10)$$

$$\hat{\theta} = |\hat{\theta}|\cos\theta\hat{j} + |\hat{\theta}|\cos(90-\theta)(-\hat{i})$$

$$\Rightarrow \hat{\theta} = \cos\theta\hat{j} - \sin\theta\hat{i}$$

$$\Rightarrow \hat{\theta} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j} \quad \text{---}(11)$$

put value of eq^n (10) and (11) in eq. (9)

$$\Rightarrow \boxed{\vec{a} = a_r\cdot\hat{r} + a_\theta\cdot\hat{\theta}} \quad \text{---}(12)$$

eq^n (12) represent acceleration in polar co-ordinate system.

Force

$$\vec{F} = m\vec{a}$$

multiply eq. (12) by m

$$\Rightarrow \vec{F} = ma_r\cdot\hat{r} + m\cdot a_\theta\cdot\hat{\theta}$$

$$\Rightarrow \vec{F}(\vec{r}) = ma_r\cdot\hat{r} + ma_\theta\cdot\hat{\theta} \quad \text{---}(13)$$

By definition of central force

$$\Rightarrow \quad \vec{F}(\vec{r}) = F(r)(\hat{r}) \quad \text{——(14)}$$

Comparing eq^n (13) and (14)

$$\Rightarrow \quad F(r) = m a_r$$

$$\Rightarrow \quad F(r) = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \quad \text{——(15)}$$

$$\Rightarrow \quad m a_\theta = 0$$

$$\Rightarrow \quad m\left[2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\right] \quad \text{——(16)}$$

eq^n 15 and 16 are known as equation of motion of a particle under central force.

Que. Prove that angular momentum is a conserved qty. under central forces.

→ We know that,

eq^n of motion under central forces

by multiply eq^n (16) by 'r'

$$\Rightarrow \quad m\left[2r\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r^2\frac{d^2\theta}{dt^2}\right] = 0 \quad \text{——(I)}$$

$$\Rightarrow \quad \frac{d}{dt}m\left[r^2\frac{d\theta}{dt}\right] = 0$$

$$\Rightarrow \quad m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad \text{——(II)}$$

$$\because \quad \omega = \frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{d}{dt}(mr^2\omega) = 0$$

$$\Rightarrow \quad \frac{d}{dt}(I\omega) = 0$$

$$\Rightarrow \quad \frac{d}{dt}(\vec{J}) = \text{constant}$$

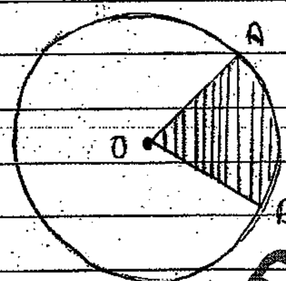
$\Rightarrow \boxed{\vec{J} = \text{constant}}$ Hence proved

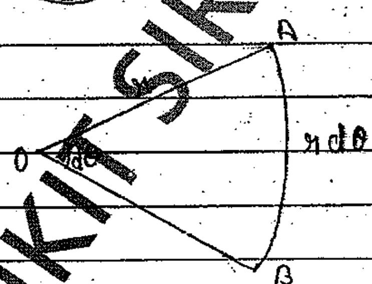
divide eq^n (II) by 2m

$$\Rightarrow m\frac{d}{dt}\left(\frac{r^2}{2m}\frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{r^2}{2}\frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \frac{r^2}{2}\frac{d\theta}{dt} = \text{constant} \quad \text{—(III)}$$





$\because$ angle = $\frac{\text{Arc}}{\text{radius}}$

$$\Rightarrow \theta = \frac{AB}{OA}$$

$$\Rightarrow AB = d\theta \cdot OA$$

$$\Rightarrow \boxed{AB = d\theta \cdot r}$$

Note :-

circumference = Area

$2\pi = \pi r^2$

$1 = \frac{\pi r^2}{2\pi}$

$1 = \frac{r^2}{2}$

$dA(d\theta) = \frac{r^2}{2}d\theta$

$$\Rightarrow dA = \frac{r^2}{2}d\theta \quad \text{—(IV)}$$

divide by 'dt'

$$\Rightarrow \frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} \quad \text{—(V)}$$

by (III) and (V)

$$\Rightarrow \frac{r^2}{2}\frac{d\theta}{dt} = \text{constant}$$

then, $\frac{dA}{dt} = \text{constant}$

According to Kepler's 2^{nd} law $\frac{1}{2} r^2 \frac{d\theta}{dt}$ define areal velocity of particle rotating under central force.

Q. Prove that total energy of particle is a constant qty. under central forces.

Sol^n By 1^{st} eq^n of motion,

$$\Rightarrow m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = F(r) \quad -(1)$$

For central forces,

$$\Rightarrow F(r) = -\frac{dU}{dr}$$

so

$$\Rightarrow -\frac{dU}{dr} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

Angular momentum

$$\vec{J} = I\vec{\omega}$$

$$\vec{J} = mr^2\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{\vec{J}}{mr^2}$$

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$$\Rightarrow \quad m\left[\frac{d^2r}{dt^2} - r\left(\frac{\vec{J}}{mr^2}\right)^2\right] = -\frac{dU}{dr}$$

$$\Rightarrow \quad m\frac{d^2r}{dt^2} - \frac{J^2}{mr^3} = -\frac{dU}{dr}$$

$$\because \quad \frac{d}{dr}\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}$$

$$\frac{-1}{r^3} = \frac{1}{2}\frac{d}{dr}\left(\frac{1}{r^2}\right)$$

$$\Rightarrow \quad m\frac{d^2r}{dt^2} + \frac{J^2}{m}\left(\frac{1}{2}\frac{d}{dr}\left(\frac{1}{r^2}\right)\right) = -\frac{dU}{dr}$$

$$\Rightarrow \quad m\frac{d^2r}{dt^2} = -\frac{dU}{dr} - \frac{d}{dr}\left(\frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad m\frac{d^2r}{dt^2} = -\frac{d}{dr}\left(U + \frac{J^2}{2mr^2}\right)$$

multiply above eqn. by $\frac{dr}{dt}$

$$\Rightarrow \quad m\left(\frac{d^2r}{dt^2}\right)\left(\frac{dr}{dt}\right) = -\frac{d}{dr} \times \frac{dr}{dt}\left(U + \frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad m\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2\right] = -\frac{d}{dt}\left[U + \frac{J^2}{2mr^2}\right]$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2\right] = -\frac{d}{dt}\left(U + \frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{J^2}{2mr^2}\right] = 0$$

Total energy $\quad \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{J^2}{2mr^2} = \text{constant}$

Where

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = \text{kinetic energy}$$

$$U = \text{potential energy}$$

$$\frac{J^2}{2mr^2} = \text{rotational kinetic energy}$$

$$U + \frac{J^2}{2mr^2} = V_{eff} = \text{effective potential}$$

Rotational kinetic energy

$$\because \frac{J^2}{2mr^2}$$

$$\because J = I\omega$$

$$J^2 = I^2\omega^2$$

$$I = mr^2$$

$$\Rightarrow \frac{I^2\omega^2}{2I}$$

$$\Rightarrow \boxed{E_r = \frac{1}{2}I\omega^2}$$

Hence we can say that total energy of particle is ~~contel~~ constant quantity under central forces.

R.U.

Que. Find the path eq^n of particle under central forces motion.

OR

Find the force eq^n

Under central forces first eq^n of motion

$$\Rightarrow \quad m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = f(r) \quad \text{---(I)}$$

Angular momentum

$$\Rightarrow \quad J = mr^2\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{J}{mr^2}$$

$$\frac{d\theta}{dt} = \frac{J}{mr^2}\cdot\frac{d\theta}{d\theta}$$

operator.

$$\frac{d}{dx}(\text{function}) = \frac{d}{dx}(\text{function})$$

$$\frac{d}{dx} = \frac{d}{dx}$$

$$\frac{d}{dt} = \frac{J}{mr^2}\frac{d}{d\theta} \quad \text{---(II)}$$

$$\Rightarrow \quad m\left[\frac{d^2r}{dt^2} - \cancel{m}r\left(\frac{J}{mr^2}\right)^2\right] = f(r)$$

$$\Rightarrow \quad m\left[\frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{\cancel{m}rJ^2}{m^2r^4}\right] = f(r)$$

by (II)

$$\Rightarrow \quad m\left(\frac{J}{mr^2}\frac{d}{d\theta}\right)\left(\frac{J}{mr^2}\frac{dr}{d\theta}\right) - mr\frac{J^2}{m^2r^4} = f(r)$$

$$\Rightarrow \quad \frac{J^2}{mr^4}\,\cancel{d\theta}\,\frac{d}{d\theta}\cdot\frac{dr}{d\theta} - \frac{J^2}{mr^3} = f(r)$$

$$\Rightarrow \text{Taking} \quad \frac{J^2}{mr^2}\left[\frac{d}{d\theta}\,\frac{1}{r^2}\frac{dr}{d\theta}\right] - \frac{J^2}{mr^3} = f(r)$$

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J = constant

$$\frac{-1}{r}\frac{d}{d\theta} = \frac{\pm 1}{r^2}\frac{dr}{d\theta}$$

$$\Rightarrow \frac{J^2}{mr^2}\left[\frac{d}{d\theta}\left(\frac{-1}{r}\frac{d}{d\theta}\right)\right] - \frac{J^2}{mr^3} = f(r)$$

$$\Rightarrow \frac{-J^2}{mr^2}\left\{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right\} = f(r)$$

putting $\frac{1}{r} = u$

$$\Rightarrow \boxed{\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right)} \quad \text{--- (3)}$$

eq.(3) represent path eq^n of motion of moving particle under central force motion.

Q.1 $r = 2a\cos\theta$

(i) $F(r) \propto \frac{1}{r^3}$ (ii) $F(r) \propto \frac{1}{r^2}$

(iii) $F(r) \propto \frac{1}{r^4}$ (iv) $F(r) \propto \frac{1}{r^5}$

Sol^n (i) $u = \frac{1}{r}$

$$u = \frac{1}{2a\cos\theta}$$

$$u = \frac{1}{2a}\sec\theta$$

$$\Rightarrow \frac{dU}{d\theta} = \frac{1}{2a} \sec\theta \cdot \tan\theta$$

$$\Rightarrow \frac{d^2U}{d\theta^2} = \frac{1}{2a}\left(\sec\theta \cdot \sec^2\theta + \sec\theta \cdot \tan^2\theta\right)$$

$$\Rightarrow \frac{d^2U}{d\theta^2} = \frac{1}{2a}\left[\sec^3\theta + \sec\theta \cdot \tan^2\theta\right]$$

$$\because \quad -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m}\left[\frac{d^2U}{d\theta^2} + U\right]$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m}\left[\frac{1}{2a}\left(\sec^3\theta + \sec\theta \cdot \tan^2\theta\right) + \frac{1}{2a}\sec\theta\right]$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m} \cdot \frac{1}{2a}\left[\sec^3\theta + \sec\theta \cdot \tan^2\theta + \sec\theta\right]$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m} \cdot \frac{1}{2a}\left[\sec\theta\left(\sec^2\theta + \tan^2\theta + 1\right)\right]$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m} \cdot \frac{1}{2a}\left[\sec\theta \cdot \left(\sec^2\theta + \sec^2\theta\right)\right]$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{J^2U^2}{m} \cdot \frac{1}{2a} \cdot 2\sec^3\theta$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = \frac{2J^2U^2 \cdot (2a)^2}{m} \cdot \frac{1}{(2a)(2a)^2} \cdot \sec^3\theta$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = 2\cdot(2a)^2 \cdot \frac{J^2U^2}{m} \cdot U^3$$

$$\Rightarrow -F\left(\frac{1}{U}\right) = 2\cdot(2a)^2 \cdot \frac{J^2U^5}{m}$$

$$\Rightarrow \boxed{F(r) \propto \frac{1}{r^5}}$$

Note :

(i) $r^n = K \cdot \cos n\theta$

$r^n = K \cdot \sin n\theta$

$F(r) \propto \frac{1}{r^{2n+3}}$

Q. $r = a \cos^2\theta$

Solution →

solution

Some Inner view of nature of motion under central forces :-

We know that under central forces, total energy of particle is conserved as constant qty.

$$\Rightarrow \quad E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \underbrace{U(r) + \frac{J^2}{2mr^2}}_{V_{eff.}}$$

$$\Rightarrow \quad \boxed{E = \frac{1}{2}m(\dot{r})^2 + V_{eff.}}$$

$$\because \quad m\frac{d^2r}{dt^2} = -\frac{d}{dr}\left(U + \frac{J^2}{2mr^2}\right)$$

$$m\frac{d^2r}{dt^2} = -\frac{d}{dr}V_{eff.}$$

By definition of central force

$$\Rightarrow \quad F'(r) = -\frac{d}{dr}V_{eff.}$$

$$\Rightarrow \quad m\frac{d^2r}{dt^2} = -\frac{d}{dr}\left(U + \frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad F'(r) = -\frac{d}{dr}\left(U + \frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad F'(r) = -\frac{dU}{dr} - \frac{d}{dr}\left(\frac{J^2}{2mr^2}\right)$$

$$\Rightarrow \quad F'(r) = -\frac{dU}{dr} - \frac{J^2}{2m}\frac{d}{dr}\left(\frac{1}{r^2}\right)$$

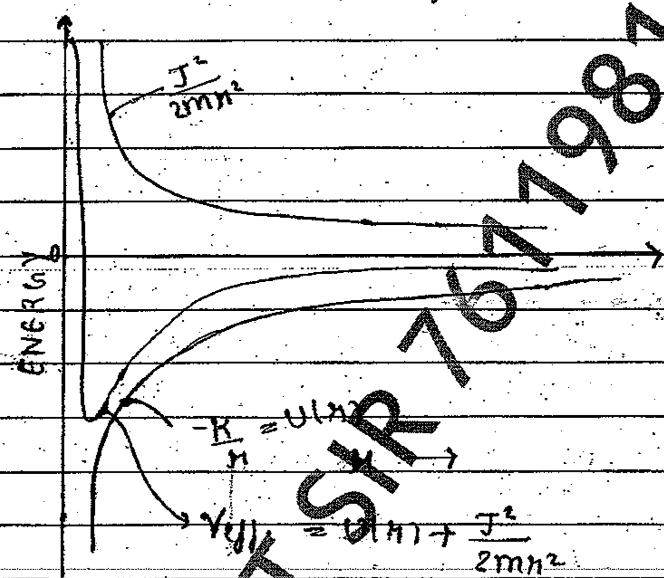
$$\Rightarrow \quad F'(r) = -\frac{dU}{dr} + \frac{2}{r^3}\cdot\frac{J^2}{2m}$$

$$\Rightarrow \quad F'(r) = -\frac{dU}{dr} + \frac{J^2}{mr^3}$$

$$F(r) = -\frac{dU}{dr}$$

$$\boxed{F'(r) = F(r) + \frac{J^2}{mr^3}} \quad \text{——(2)}$$

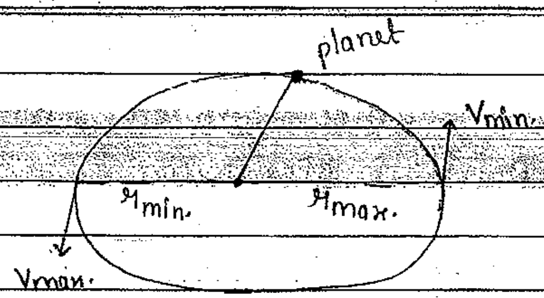
eqⁿ (2) represent inner view of motion of particle under central force motion.



Kepler's law of planetary motion :-

Planetary motion of planet, kepler gave three law, this law is known as kepler's planetary motion law.

1. First law of kepler :-
According to this law, every planet revolving around sun have elliptical orbit and sun is at focus.

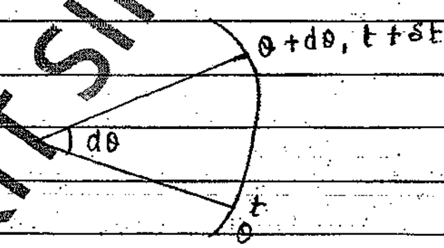


(2) Kepler's 2nd law :-

According to this law, areal velocity remain constant.

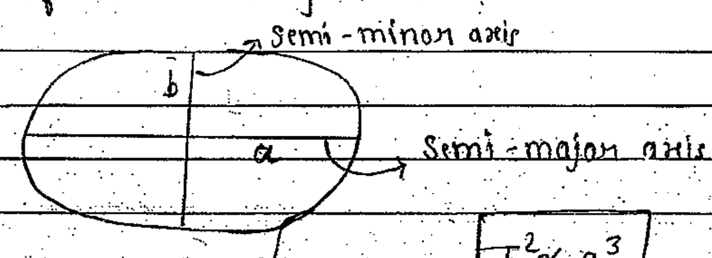
Area swept per second by line joining planet to sun is always constant.

$$\frac{dA}{dt} = \frac{J}{2m} = \frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right) = \text{constant}$$



(3) Kepler 3rd law :-

According to this square of time period of planet is directly proportional to cube of semi-major axis.

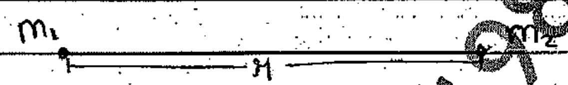


$$\boxed{T^2 \propto a^3}$$

Gravitational Force :-

Force between two mass object is called gravitational force.

Gravitational force b/w two object be directly proportional to product of their mass and inversely proportional to square of separation of distance.



$$\Rightarrow \quad F_G = -\frac{G m_1 m_2}{r^2}\hat{r}$$

$$\Rightarrow \quad F_G = \frac{G m_1 m_2}{r^2}(-\hat{r})$$

$$\because \quad G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$$

Inertial mass :-

Required force to accelerated any object by $1\,m/sec^2$ is called Inertial mass.

$$\Rightarrow \quad F = ma$$

$$\Rightarrow \quad m = \frac{F}{a}$$

If $a = 1\,m/sec^2$

$$\Rightarrow \quad \boxed{m = |F|}$$

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Gravitational mass :-

If two objects of different masses are dropped from same height, then acceleration of both object will be same. Hence in this condition, mass of object is called gravitational mass.

Eq^n of motion of particle in gravitational attaraction :-

Gravitational force is also a central force so we can say that eq^n of motion under central forces is given below

$$\Rightarrow \quad \frac{J^2U^2}{m}\left[\frac{d^2U}{d\theta^2}+U\right] = -f\left[\frac{1}{U}\right] \quad \text{——(1)}$$

Gravitational force, $F_G = -\frac{Gm_1m_2}{r^2}$

$$\because \quad F_G = \frac{-K}{r^2}$$

(potential energy) $U = \frac{-K}{r}$

by (1)

$$\Rightarrow \quad \frac{\partial^2 U}{\partial\theta^2} + U = -\frac{f(r)\cdot m}{J^2U^2} \qquad \because U = \frac{1}{r}$$

$$\Rightarrow \quad \frac{\partial^2 U}{\partial\theta^2} + U = -\frac{f(r)\,m}{J^2} \cdot \frac{K}{r^2}\cdot\frac{m}{J^2U^2} \qquad \because f(r) = \frac{-K}{r^2}$$

$$\Rightarrow \frac{\partial^2 U}{\partial \theta^2} + U = \frac{K}{r^2} \cdot \frac{m}{J^2} \cdot r^2 \qquad \left[U = \frac{1}{r} \right]$$

$$\Rightarrow \frac{\partial^2 U}{\partial \theta^2} + U = \frac{km}{J^2}$$

$$\Rightarrow \frac{\partial^2 U}{\partial \theta^2} + U - \frac{km}{J^2} = 0$$

let $\because U - \frac{k \cdot m}{J^2} = \Psi$

J, m, K constant

$$\because \frac{d^2\Psi}{d\theta^2} = \frac{d^2 U}{d\theta^2}$$

$$\Rightarrow \frac{d^2 U}{d\theta^2} + \Psi = 0$$

$$\Rightarrow \boxed{\frac{d^2\Psi}{d\theta^2} + \Psi = 0} \quad \text{——— (2)}$$

eq^n (2) is differential eq^n so having a particular sol^n

solution of eq^n (2) is given below

$$\Rightarrow \Psi = \Psi_0 \cos(\theta - \theta_0) \quad \text{——— (3)}$$

$$\Rightarrow \because \Psi = U - \frac{k \cdot m}{J^2} = \Psi_0 \cos(\theta - \theta_0)$$

$$\Rightarrow U = \Psi_0 \cos(\theta - \theta_0) + \frac{k \cdot m}{J^2}$$

$$\Rightarrow U = \frac{m \cdot k}{J^2} \left[1 + \frac{J^2}{mk} \Psi_0 \cos(\theta - \theta_0) \right] \quad \text{—(4)}$$

let $p = \frac{J^2}{mk}$

$$\mathcal{E} = \frac{J^2 \Psi_0}{mk}$$

$$\Rightarrow \quad U = \frac{1}{p}\left[1 + e\cos(\theta - \theta_0)\right]$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{p}\left[1 + e\cos(\theta - \theta_0)\right]$$

$$\Rightarrow \quad \boxed{r = \frac{p}{1 + e\cos(\theta - \theta_0)}} \quad \text{——— (5)}$$

Case I. If $\theta = \theta_0$

$$r = \frac{p}{1 + e\cos 0}$$

$$\boxed{r = \frac{p}{1+e}} \quad \text{——— (6)}$$

Case II. If $\theta = \theta_0 + \pi$

$$r = \frac{p}{1 + e\cos\pi}$$

$$\boxed{r = \frac{p}{1-e}} \quad \text{——— (7)}$$ maximum

Case III. If $\theta_0 = 0$

$$\boxed{r = \frac{p}{1 + e\cos\theta}} \quad \text{——— (8)}$$

eq. (8) represent conic equation whose eccentricity is e.

Eccentricity :-

Total energy is conserved under central forces.

$$\Rightarrow \quad E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{J^2}{2mr^2}$$

$$\because r_{min.} = \frac{p}{1+\varepsilon} \qquad \because p = \frac{J^2}{mk}$$

$$\because \frac{dr_{min.}}{dt} = 0$$

$$\Rightarrow \quad E = U + \frac{J^2}{2mr^2_{min.}}$$

$$\Rightarrow \quad E = U + \frac{J^2}{2m \cdot p^2}(1+\varepsilon)^2 \quad \text{(putting value of } r_{min}\text{)}$$

$$\because p^2 = \frac{J^4}{m^2k^2}$$

$$\Rightarrow \quad E = U + \frac{J^2}{2m \cdot J^4} m^2k^2 (1+\varepsilon)^2$$

$$\Rightarrow \quad E = U + \frac{mk^2}{2J^2}(1+\varepsilon)^2$$

$$\Rightarrow \quad E = \frac{-k}{r_{min.}} + \frac{mk^2}{2J^2}(1+\varepsilon)^2$$

$$\Rightarrow \quad E = \frac{-k(1+\varepsilon)}{p} + \frac{mk^2}{2J^2}(1+\varepsilon)^2$$

$$\Rightarrow \quad E = \frac{(1+\varepsilon)^2 mk^2}{2J^2} - \frac{k}{J^2}(1+\varepsilon) \cdot mk$$

$$\Rightarrow \quad E = \frac{(1+\varepsilon) mk^2}{J^2}\left(\frac{1+\varepsilon}{2} - 1\right)$$

$$\Rightarrow \quad E = \frac{(\varepsilon+1)(\varepsilon-1)mk^2}{2J^2}$$

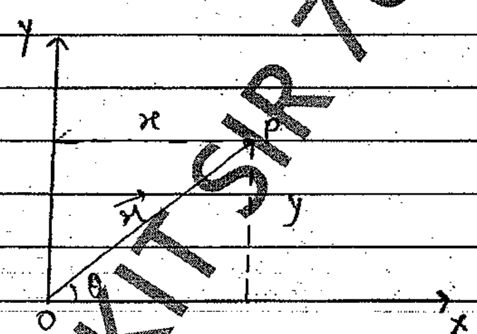
$$\Rightarrow \quad E = \frac{(\epsilon^2 - 1)\, mk^2}{2J^2}$$

$$\Rightarrow \quad \frac{2J^2 E}{mk^2} + 1 = \epsilon^2$$

$$\Rightarrow \quad \boxed{\epsilon = \sqrt{\frac{2EJ^2}{mk^2} + 1}}$$

Eq^n of motion of particle in cartesian co-ordinate system.

Let's consider a particle having mass 'm' which is situated in cartesian co-ordinate system at point 'P' at angle 'θ'



By geometry,

$$\Rightarrow \quad x = r\cos\theta \qquad \text{—— (1)}$$

$$\Rightarrow \quad y = r\sin\theta \qquad \text{—— (2)}$$

$$eq.(1)^2 + eq.(2)^2$$

$$\Rightarrow \quad x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow \quad x^2 + y^2 = r^2$$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2} \qquad \text{—— (3)}$$

We know that,

$\Rightarrow \quad r = \dfrac{p}{1+\varepsilon\cos\theta}$ @ $\theta_0 = 0°$ —(4)

$\Rightarrow \quad r(1+\varepsilon\cos\theta) = p$

$\Rightarrow \quad r + r\varepsilon\cos\theta = p$

$\Rightarrow \quad r + r\cos\theta\cdot\varepsilon = p$

$\Rightarrow \quad \sqrt{x^2+y^2} + x\varepsilon = p$ —— (5)

$\Rightarrow \quad \sqrt{x^2+y^2} = p - \varepsilon x$

squaring both side

$\Rightarrow \quad x^2+y^2 = (p-\varepsilon x)^2$

$\Rightarrow \quad x^2+y^2 = p^2 + \varepsilon^2 x^2 - 2\varepsilon x\cdot p$

$\Rightarrow \quad x^2 - \varepsilon^2 x^2 + y^2 = p^2 - 2\varepsilon x\cdot p$

$\Rightarrow \quad x^2(1-\varepsilon^2) + y^2 = p^2 - 2\varepsilon x\cdot p$

$\Rightarrow \quad x^2(1-\varepsilon^2) + y^2 + 2p\varepsilon x = p^2$ —— (6)

Case I, If $\varepsilon = 1$

$\Rightarrow \quad (0)x^2 + y^2 + 2px = p^2$

$\Rightarrow \quad y^2 + 2px = p^2$ —— (7)

$\Rightarrow \quad y^2 = p^2 - 2px$

This is a eq of parabola

Case II. If $\varepsilon < 1$

add $\dfrac{p^2\varepsilon^2}{1-\varepsilon^2}$ both side of eq.(VI)

$\Rightarrow x^2(1-\varepsilon^2) + y^2 + 2p\varepsilon x + \dfrac{p^2\varepsilon^2}{1-\varepsilon^2} = p^2 + \dfrac{p^2\varepsilon^2}{1-\varepsilon^2}$

$\Rightarrow x^2(1-\varepsilon^2) + y^2 + 2p\varepsilon x + \dfrac{p^2\varepsilon^2}{1-\varepsilon^2} = \dfrac{p^2 - p^2\varepsilon^2 + p^2\varepsilon^2}{1-\varepsilon^2}$

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$$\Rightarrow x^2(1-\varepsilon^2) + y^2 + 2p\varepsilon x + \frac{p^2\varepsilon^2}{1-\varepsilon^2} = \frac{p^2}{1-\varepsilon^2}$$

Divide above eq^n by $(1-\varepsilon^2)$

$$\Rightarrow x^2 + \frac{y^2}{1-\varepsilon^2} + \frac{2p\varepsilon x}{1-\varepsilon^2} + \frac{p^2\varepsilon^2}{(1-\varepsilon^2)^2} = \frac{p^2}{(1-\varepsilon^2)^2}$$

$$\Rightarrow (x^2 + 2xy + y^2) = (x+y)^2$$

$$\Rightarrow \left(x + \frac{p\varepsilon}{1-\varepsilon^2}\right)^2 + \left(\frac{y^2}{1-\varepsilon^2}\right) = \frac{p^2}{(1-\varepsilon^2)^2}$$

let

$$x_0 = \frac{p\varepsilon}{1-\varepsilon^2}$$

$$\Rightarrow (x + x_0)^2 + \left(\frac{y^2}{1-\varepsilon^2}\right) = \frac{p^2}{(1-\varepsilon^2)^2}$$

$$\Rightarrow \frac{(x+x_0)^2}{\frac{p^2}{(1-\varepsilon^2)^2}} + \frac{y^2}{\frac{\cancel{(1-\varepsilon^2)}\,p^2}{(1-\varepsilon^2)^2}} = 1$$

$$\Rightarrow \frac{(x+x_0)^2}{\left(\frac{p}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{p}{\sqrt{1-\varepsilon^2}}\right)^2} = 1 \quad \text{---}(8)$$

let $\frac{p}{1-\varepsilon^2} = a$ and $\frac{p}{\sqrt{1-\varepsilon^2}} = b$

$$\Rightarrow \frac{(x+x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{---}(9)$$

eq^n (9) is similar to equation of ellipse.

Case III If $\varepsilon > 1$

by eq. (8)

$$\Rightarrow \frac{(x+x_0)^2}{\left(\frac{p}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{p}{\sqrt{1-\varepsilon^2}}\right)^2} = 1$$

$$\Rightarrow \frac{(x+x_0)^2}{\left(\frac{p}{1-\varepsilon^2}\right)^2} - \frac{y^2}{\frac{p^2}{\varepsilon^2-1}} = 1 \quad \text{---} (10)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

eq^n (10) is eq^n of Hyperbola

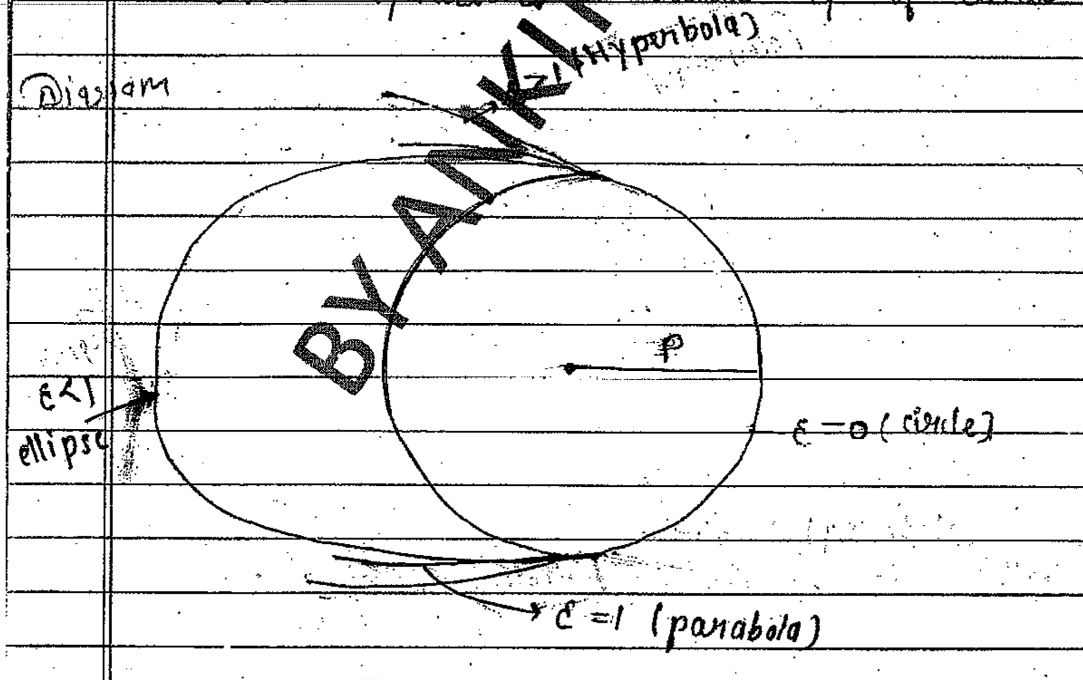
Case IV If $\varepsilon = 0$

by eq. (6)

$$\Rightarrow x^2 + y^2 = p^2 \quad \text{---} (11)$$

above equation is called eq^n of circle.

Diagram



Mainly two motion of particle under central force.

(I) Circular Orbit Motion

(II) Elliptical Orbit Motion

(I) Circular Orbit Motion :-

$\Rightarrow \quad x^2 + y^2 = p^2$

$\Rightarrow \quad x^2 + y^2 = r^2$

then,

$\Rightarrow \quad p^2 = r^2$

$\Rightarrow \quad p = r \quad \text{---} (1)$

$\because \; p = \frac{J^2}{mk}$ (we know)

$\because \; J = mvr \qquad \because v = r\omega$

$\because \; J = mr^2\omega$

$\because \; J^2 = m^2 r^4 \omega^2$

$\Rightarrow \quad p = \frac{m^2 r^4 \omega^2}{mk}$

by eq^n (1) $\quad p = r$

$\Rightarrow \quad \frac{m^2 r^4 \omega^2}{mk} = r$

$\Rightarrow \quad \frac{m r^3 \omega^2}{K} = 1$

$\Rightarrow \quad m\omega^2 r^3 = K$

$\Rightarrow \quad \frac{K}{m} = \omega^2 r^3 \quad \text{---} (2)$

(a) Time period :-

Time required for complete one cycle.

by eq. 2

$$\Rightarrow \frac{K}{m} = \omega^2 r^3$$

$$\Rightarrow \omega^2 = \frac{K}{mr^3}$$

$$\Rightarrow \left(\frac{2\pi}{T}\right)^2 = \frac{K}{mr^3}$$

$$\Rightarrow \frac{4\pi^2 mr^3}{K} = T^2$$

$$\Rightarrow T^2 = \frac{4\pi^2 mr^3}{K}$$

$$\Rightarrow \boxed{T^2 \propto r^3} \quad \text{--- (3)}$$

eq^n (3) is known as kepler's third law for circular orbit.

(b) Velocity :-

$$\Rightarrow \frac{K}{m} = \omega^2 r^3$$

$$\because v = r\omega$$

$$\Rightarrow \frac{K}{m} = (r\omega)^2 \cdot r$$

$$\Rightarrow \frac{K}{m} = v^2 \cdot r$$

$$\Rightarrow v^2 = \frac{K}{mr}$$

$$\Rightarrow v = \sqrt{\frac{K}{mr}}$$

$$\Rightarrow v \propto \frac{1}{\sqrt{r}}$$

(ii) Elliptical Orbit Motion :- $\varepsilon < 1$

$$\because \frac{(x+x_0)^2}{\left(\frac{p}{(1-\varepsilon^2)}\right)^2} + \frac{y^2}{\left(\frac{p}{\sqrt{1-\varepsilon^2}}\right)^2} = 1$$

$$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = \frac{p}{1-\varepsilon^2}$ and $b = \frac{p}{\sqrt{1-\varepsilon^2}}$

$$\because \varepsilon^2 = \frac{2EJ^2}{mk^2}$$

$$\Rightarrow a = \frac{p}{1-1-\frac{2EJ^2}{mk^2}}$$

$$\Rightarrow a = \frac{p}{-\frac{2EJ^2}{mk^2}}$$

$$\Rightarrow a = -\frac{p\,mk^2}{2EJ^2} \qquad \because p = \frac{J^2}{mk}$$

$$\Rightarrow a = -\frac{J^2}{mk}\cdot\frac{mk^2}{2EJ^2}$$

$$\Rightarrow \boxed{a = \frac{-k}{2E}} \quad — 4$$

Here

(−)ive sign represent bounded state

$$\Rightarrow \quad b = \frac{p}{\sqrt{1-\varepsilon^2}}$$

$$\Rightarrow \quad b = \frac{p}{\sqrt{\frac{-2EJ^2}{mk^2}}}$$

$$\Rightarrow \quad b = \sqrt{\frac{p^2 mk^2}{-2EJ^2}}$$

$$\Rightarrow \quad b = \sqrt{\frac{-J^4}{m^2k^2} \cdot \frac{mk^2}{2EJ^2}}$$

$$\Rightarrow \quad b = \sqrt{\frac{-J^2}{2mE}} \quad \text{—— (5)}$$

Here (-)ive sign represent bounded motion.

(II) Time Period :-

By Kepler's second law

$$\Rightarrow \quad \frac{dA}{dt} = \frac{J}{2m}$$

$$\Rightarrow \quad dA = \frac{J}{2m} dt$$

on Integrating

$$\Rightarrow \quad A = \frac{J}{2m} \int_0^T dt$$

$\because$ A = area of ellipse = πab

$$\Rightarrow \quad \pi ab = \frac{J}{2m} \cdot T$$

$$\Rightarrow \quad T = \frac{2m}{J} \cdot \pi ab$$

put value of a and b

$$\Rightarrow \quad T = \frac{\cancel{2}m}{J}\cdot\pi\left(\frac{k}{\cancel{2}E}\right)\sqrt{\frac{J^2}{2Em}}$$

$$\Rightarrow \quad T = \frac{m\pi k}{E}\sqrt{\frac{1}{2Em}}$$

$$\Rightarrow \quad T = \pi\sqrt{\frac{m^2k^2}{2E\cancel{m}\cdot E^2}}$$

$$\Rightarrow \quad T = \pi\sqrt{\frac{mk^2}{2E^3}}$$

$$\Rightarrow \quad T = \pi\sqrt{\frac{mk^2\times 4\cdot k}{2E^3\ 4\ k}}$$

$$\Rightarrow \quad T = \pi\sqrt{\frac{mk^3\cdot 4}{(2E)^3\ k}}$$

$$\Rightarrow \quad T = \pi\sqrt{\frac{4ma^3}{k}}$$

$$\Rightarrow \quad T \propto \sqrt{a^3} \quad — 6$$

eqⁿ (6) represent time period for elliptical orbit.

(b) Velocity :-

Total energy of particle under central force is always constant.

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{J^2}{2mr^2}$$

$$\because J = mr^2\frac{d\theta}{dt}$$

$$\Rightarrow E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \left(mr^2\frac{d\theta}{dt}\right)^2 \cdot \frac{1}{2mr^2}$$

$$\Rightarrow E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{mr^2}{2}\left(\frac{d\theta}{dt}\right)^2$$

$$\Rightarrow E = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + U$$

$$\because V_r = \frac{dr}{dt}$$

$$V_\theta = \frac{r\,d\theta}{dt}$$

$$\therefore V^2 = V_r^2 + V_\theta^2$$

$$\Rightarrow E = \frac{1}{2}m\left[V_r^2 + V_\theta^2\right] + U$$

$$\Rightarrow E = \frac{1}{2}mv^2 + U$$

$$\Rightarrow E - U = \frac{1}{2}mv^2$$

$$\Rightarrow v^2 = \frac{2}{m}(E-U)$$

$$\Rightarrow v = \sqrt{\frac{2}{m}(E-U)} \quad \text{--- (7)}$$

eq^n (7) represent velocity of elliptical orbital.

Important Result of Kepler's law :

Kepler's 3rd law for circular orbit

$$T^2 \propto r^3$$

$$T^2 = K r^3$$

$$\left(\frac{2\pi}{\omega}\right)^2 = K r^3$$

$$\frac{4\pi^2}{\omega^2} = K r^3$$

$$\omega^2 = \frac{4\pi^2}{K r^3}$$

$$\frac{v^2}{r^2} = \frac{4\pi^2}{K r^3} \qquad \because v = r\omega, \quad \omega = \frac{v}{r}$$

$$v^2 = \frac{4\pi^2}{K r}$$

Centripetal force :-

$$F = \frac{mv^2}{r}$$

$$F = \frac{m\omega^2 r^2}{r} \qquad \because v = r\omega$$

$$F = m\omega^2 r$$

$$F = m\,\frac{4\pi^2}{K r^3}\cdot r \qquad \omega^2 = \frac{4\pi^2}{K r^3}$$

$$\boxed{F = \frac{4\pi^2 m}{K r^2}}$$

Dirxn of centripetal force is always toward sun.

Centripetal force is directly proportional to mass of planet.

Centripetal force is inversely proportional to to square of distance

$$\boxed{F \propto m}$$

$$\boxed{F \propto \frac{1}{r^2}}$$

Numerical

Example. A body is moving under influence of following central force

$$\vec{F} = -\frac{K}{r^4} \vec{r}$$

Prove that if the energy of body is E then velocity of Body will be

$$v = \sqrt{\frac{K}{mr^2} + \frac{2E}{m}}$$

Sol^n. By definition of central force

$$\vec{F}(\vec{r}) = F(r) \cdot \hat{r}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{F}(\vec{r}) = F(r) \cdot \frac{\vec{r}}{|\vec{r}|}$$

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$$F = \frac{-K}{r^4}\vec{r} \quad -(11)$$

$$\frac{F(r)}{r} = \frac{-K}{r^4}$$

$$\vec{F}(\vec{r}) = \frac{-K}{r^3}$$

$$\because U = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r}$$

$$U = \int_{\infty}^{r} \frac{K}{r^3}\cdot d\vec{r}$$

$$U = \left[\frac{-K}{2r^2}\right]_{\infty}^{r}$$

$$U = \frac{K}{r^2}$$

$$E = K.E + P.E$$

$$E = \frac{1}{2}mv^2 - \frac{K}{2r^2}$$

$$\frac{1}{2}mv^2 = E + \frac{K}{2r^2}$$

$$v^2 = \frac{2E}{m} + \frac{K}{2mr^2}$$

$$\boxed{v = \sqrt{\frac{2E}{m} + \frac{K}{mr^2}}}$$

Note: Circular Orbit

$$F(r) = ?$$

when other asked time, velocity etc.

Soln.

Step I $\quad F(r) = \frac{mv^2}{r}$

$$v = \sqrt{\frac{rF(r)}{m}}$$

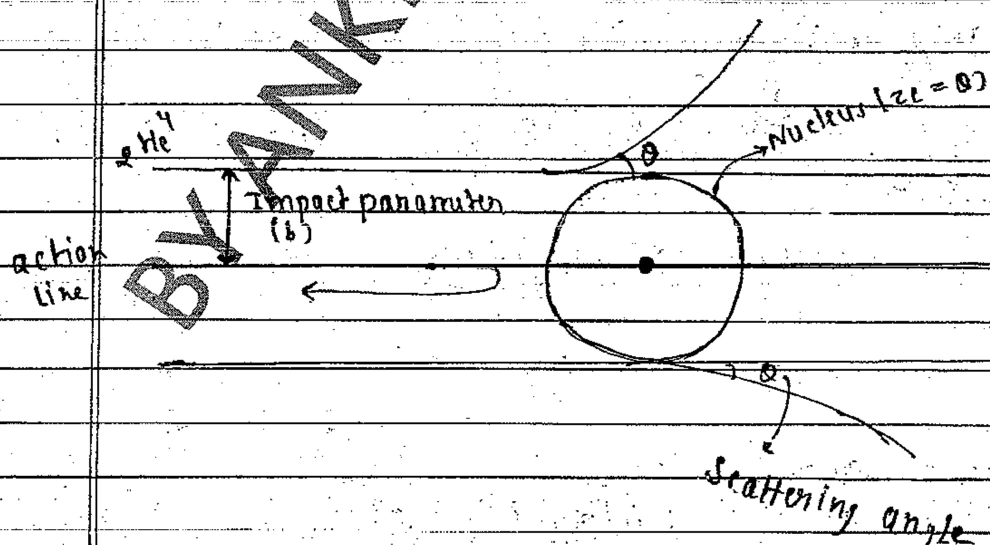
$$T = \frac{2\pi r}{v}$$

Rutherford Scattering :-

In 1911, Rutherford did an experiment on nucleus which is called Rutherford or α-particle scattering.

α- particle are (+)ive particles of Helium nucleus which is represented by $_2He^4$. First of all particles are bombarded on a thin gold foil and observed, that most of particles go straight without deviation and out of 800 particles, one α-particle deviat an angle greater than 90° and out of 20,000 particles one α-particle get back to its path i.e. repeat its own path.

On the basis of this result, Rutherford assumed nucleus a most density point.



Here a repulsive coulombic force acts under which α-particle repeat a conical equation which can be shown as below, we know that coulombic repulsion is a central force and under central force, total energy of particle remain constant.

Total energy of particle:

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U + \frac{J^2}{2mr^2} \qquad \text{—(1)}$$

$$U = E - \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{J^2}{2mr^2} \qquad \text{—(2)}$$

By definition of angular momentum

$$J = I\omega = mr^2\frac{d\theta}{dt} \qquad \text{—(3)}$$

By Force eq^n

$$u = \frac{1}{r} \qquad \text{—(4)}$$

Differentiate eq. (4) w.r.t. 'θ'

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

by eq^n (3)

$$\frac{du}{d\theta} = -\frac{m}{J}\cdot\frac{d\theta}{dt}\cdot\frac{dr}{d\theta}$$

$$\frac{du}{d\theta} = -\frac{m}{J}\frac{dr}{dt} \qquad \text{—(5)}$$

by eq. (2) and (5)

$$U = E - \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{J^2}{2mr^2}$$

$$U = E - \frac{1}{2}m\left(-\frac{J}{m}\frac{du}{d\theta}\right)^2 - \frac{J^2}{2mr^2}$$

$$U = E - \frac{1}{2}m\left(\frac{J^2}{m^2}\right)\left(\frac{du}{d\theta}\right)^2 - \frac{J^2u^2}{2m} \quad -(6) \because u = \frac{1}{r}$$

By definition of Coulombic force

$$F_c = \frac{Kq_1q_2}{r^2}$$

$\because$ $q_1 = Ze$ (charge on nucleus)
$q_2 = 2e$ (" " α-particle)

$$F_c = \frac{2KZe^2}{r^2}$$

$\because 2KZe^2 = K$ (constant)

$$F_c = \frac{K}{r^2}$$

Potential energy,

$$U = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r}$$

$$U = -\int_{\infty}^{r} \frac{K}{r^2}\,dr$$

$$U = K\left[\frac{1}{r}\right]_{\infty}^{r}$$

$$U = \frac{K}{r}$$

$$U = Ku \quad -(7)$$

by eq^n 6 and 7

$$Ku = E - \frac{1}{2} m \frac{J^2}{m^2}\left(\frac{du}{d\theta}\right)^2 - \frac{J^2u^2}{2m}$$

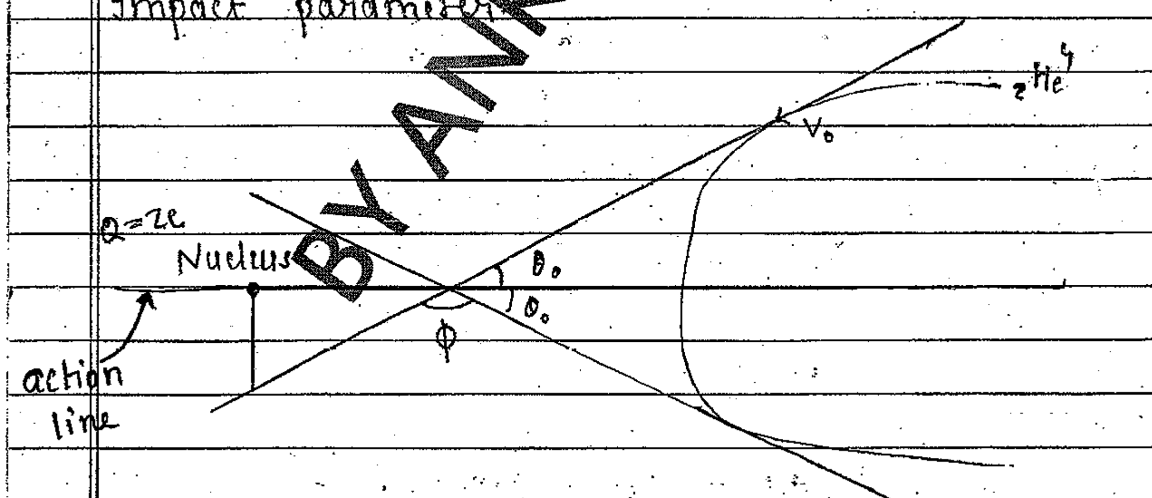
$$Ku = E - \frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 - \frac{J^2u^2}{2m}$$

$$Ku = E - \frac{J^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$E - Ku = \frac{J^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = (E - Ku)\frac{2m}{J^2} \quad \text{—(VIII)}$$

Impact parameter (b) – Perpendicular distance b/w action line passing through nuclei and wave vector of nuclei is called Impact parameter.



$$J = mV_0 b \quad \text{--- } 9 \quad \text{(Angular momentum w.r.t. nucleus)}$$

$$(K.E)_i = \frac{1}{2} m V_0^2$$

$$E = \frac{p^2}{2m} \qquad \because p = mV_0$$

$$p = \sqrt{2mE}$$

$$J = \sqrt{2mE} \cdot b \quad \text{--- } 10$$

by eq. 8 and 10

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{2mE \cdot b^2}\left[E - Ku\right]$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{b^2} - \frac{Ku}{E \cdot b^2}$$

$$\text{let } P = \frac{2E \cdot b^2}{K}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{b^2} - \frac{2u}{P}$$

On ~~adding~~ and subtracting $\frac{1}{P^2}$ in above eq^n.

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{b^2} - \frac{2u}{P} + \frac{1}{P^2} - \frac{1}{P^2}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{b^2} + \frac{1}{P^2} - \frac{2u}{P} - \frac{1}{P^2}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{b^2} + \frac{1}{P^2} - \frac{2u}{P} - \frac{1}{P^2} - u^2$$

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$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{b^2} + \frac{1}{p^2} - \left(\frac{2u}{p} + \frac{1}{p^2} + u^2\right)$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{b^2} + \frac{1}{p^2} - \left(\frac{1}{p} + u\right)^2$$

$$\frac{du}{d\theta} = \sqrt{\left(\frac{1}{b^2} + \frac{1}{p^2}\right) - \left(\frac{1}{p} + u\right)^2}$$

by variable seperation method

$$\int \frac{1}{\sqrt{\left(\frac{1}{b^2} + \frac{1}{p^2}\right) - \left(\frac{1}{p} + u\right)^2}}\, du = \int d\theta$$

$$\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} x = -\cos^{-1} x$$

$$= -\cos^{-1}\left(\frac{\frac{1}{p} + u}{\sqrt{\frac{1}{b^2} + \frac{1}{p^2}}}\right) = \theta$$

$$\cos^{-1}\left(\frac{1 + up}{p\sqrt{\frac{1}{b^2} + \frac{1}{p^2}}}\right) = -\theta$$

$$\frac{up + 1}{p\sqrt{\frac{p^2 + 1}{b^2 \quad p^2}}} = \cos(-\theta)$$

$$\because \cos(-\theta) = \cos\theta \quad \text{even}$$

$$\sin(-\theta) = -\sin\theta \quad \text{odd}$$

$$\frac{up + 1}{\sqrt{\frac{p^2}{b^2} + 1}} = \cos\theta$$

$$\text{let } \varepsilon = \sqrt{\frac{p^2}{b^2}+1}$$

$$\varepsilon^2 = \frac{p^2}{b^2}+1$$

$$p^2 = (\varepsilon^2-1)b^2$$

$$p = b(\sqrt{\varepsilon^2-1}) \quad \text{---(12)}$$

$$\frac{up+1}{(\varepsilon^2)^{1/2}} = \cos\theta$$

$$up+1 = \varepsilon\cos\theta$$

$$up = \varepsilon\cos\theta - 1$$

$$u = \frac{\varepsilon\cos\theta-1}{p}$$

$$\frac{1}{r} = \frac{\varepsilon\cos\theta-1}{p}$$

$$\boxed{r = \frac{p}{\varepsilon\cos\theta-1}} \quad \text{---(13)}$$

eq^n (13) represent conic eq^n.
If $\varepsilon > 1$ then this eq^n is reduced in Hyperbola eq^n.

Relation b/w Impact parameter and Scattering angle :=

If $\varepsilon\cos\theta = 1$, $r = \infty$, then in this cond^n, radial vector is known as Asymptoes.

$$\varepsilon\cos\theta = 1$$

or $\cos\theta = \frac{1}{\epsilon}$

$$\theta = \cos^{-1}\left(\frac{1}{\epsilon}\right)$$

$$\theta = \theta_0 = \pm\cos^{-1}\left(\frac{1}{\epsilon}\right) \quad -(14)$$

Scattering angle, (ϕ)

$$\because \phi = \pi - 2\theta_0$$

$$\frac{\phi}{2} = \frac{\pi}{2} - \theta_0$$

$$\tan\frac{\phi}{2} = \tan\left(\frac{\pi}{2} - \theta_0\right)$$

$$\tan\frac{\phi}{2} = \cot\theta_0$$

$$\tan\frac{\phi}{2} = \frac{\cos\theta_0}{\sin\theta_0}$$

$$\tan\frac{\phi}{2} = \frac{\cos\theta_0}{\sqrt{1-\cos^2\theta_0}}$$

$\because$ by eq. 14

$$\tan\frac{\phi}{2} = \frac{\frac{1}{\epsilon}}{\sqrt{1-\cos^2\theta_0}}$$

$$\tan\phi/2 = \frac{1}{\sqrt{\epsilon^2 - \epsilon^2\cos^2\theta_0}}$$

$$\tan\phi/2 = \frac{1}{\sqrt{\epsilon^2 - 1}} \quad \text{by (12)}$$

$$\because p = b\sqrt{\epsilon^2 - 1}$$

$$\sqrt{\epsilon^2 - 1} = \frac{p}{b}$$

$$\tan \phi/2 = \frac{b}{p}$$

$$\boxed{b = p \tan \phi/2}$$

$$\because \quad p = \frac{2Eb^2}{K}$$

$$\cancel{b} = \frac{2Eb^{\cancel{2}}}{K} \tan\phi/2$$

$$b = \frac{K}{2E} \cot \phi/2$$

$$b = \frac{2Kze^2}{2E} \cot\phi/2$$

$$\boxed{b = \frac{D}{2} \cot\phi/2} \quad \text{—(15)}$$

b = Impact parameter
D = distance of closest approach
ϕ = scattering angle

$$b \propto \cot \phi/2$$

eqⁿ 15 represent relation b/w Impact parameter & scattering angle.

$$\text{Here } D = \frac{2Kze^2}{E}$$

$$\because \quad E = \frac{1}{2} m V_0^2$$

$$D = \frac{2Kze^2}{\frac{1}{2} m v_0^2}$$

Date
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Unit - III

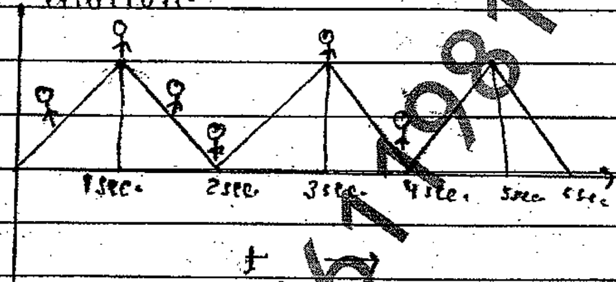
part - II

Wave & Oscillation

(1) Periodic motion :-

Motion that ~~repeat~~ repeat on a regular cycle in equal time interval is called periodic motion.

e.g. (1)



$h = ut + \frac{1}{2}gt^2$ (coming downward)

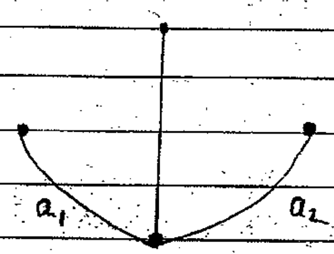
$h = ut - \frac{1}{2}gt^2$ (going upward

(11) motion ~~of~~ a moon around earth.

(11) Oscillatory motion :-

~~When a~~ It is motion of oscillatory body around it's rest point, where the motion is repeated in equal period of ~~time~~ time.

e.g.



oscillatory motion

$a_1 \neq a_2$

Simple Harmonic Motion :-

When a particle does oscillatory motion under