Unit - III

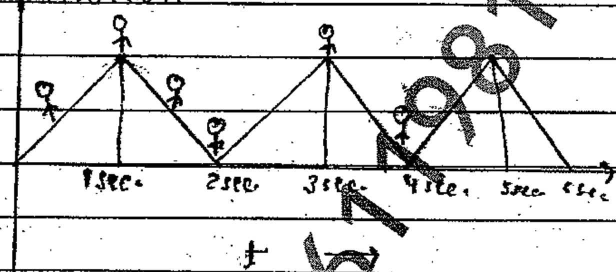
part - II

Wave & Oscillation

(1) **Periodic motion :-**

Motion that ~~repeat~~ on a regular cycle in equal time interval is called periodic motion.

e.g. (1)



$h = ut + \frac{1}{2} g t^2$ (coming downward)

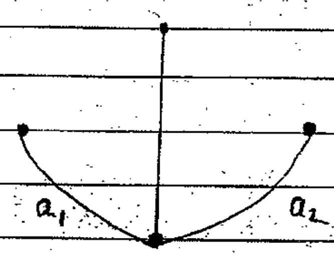
$h = ut - \frac{1}{2} g t^2$ (going upward

(11) motion of a moon around earth.

(11) **Oscillatory motion :-**

~~When a~~ It is motion of oscillatory body, around it's rest point, where the motion is repeated in equal period of time.

e.g.



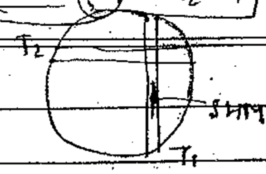
oscillatory motion

$a_1 \neq a_2$

Simple Harmonic Motion :-

When a particle does oscillatory motion under

$T_1 = T_2 = T$



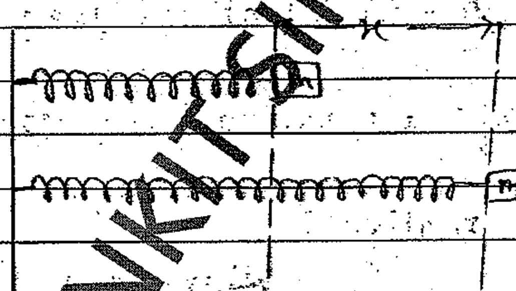
restoring force. e.g. Simple pendulum.
Restoring force :-

Any motion where restoring force is directly proportional to distance away from equil^m position.

$F \propto -x$ (In x-dirⁿ)
$F \propto -y$ (In y-dirⁿ)
$F \propto -z$ (In z-dirⁿ)

$F \propto -x$
$$\boxed{F = -kx}$$
where K is constant

e.g. Spring force is restoring force.



$F = -kx$

Differential eq^n of SHM :-

By definition of restoring force
$F = -kx$ ---(i)
-ive sign represent force in opposite dirⁿ to motion of displacement.

Acc. to Newton

$$F = m\frac{d^2x}{dt^2} \quad \text{---(II)}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

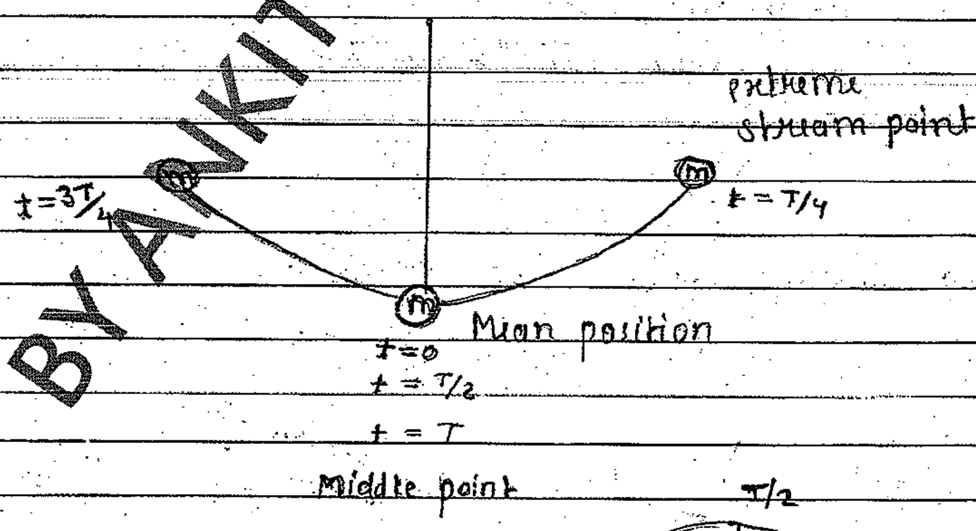
$$\because \omega^2 = \frac{k}{m} \Leftrightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

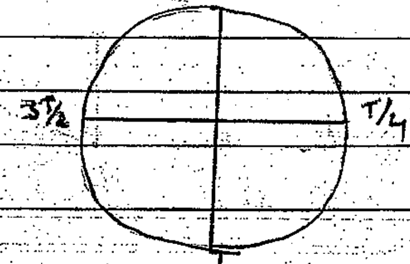
$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{---(III)}$$

eq. (III) represent differential form of SHM



Note :-



Equation of Velocity :-

Rate of change of position with respect to time is called Velocity

By Differential eq^n of SHM

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

multiply by $\frac{dx}{dt}$

$$\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + \omega^2 x \frac{dx}{dt} = 0$$

$$\frac{dx}{dt}\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right] + \omega^2 x \cdot \frac{dx}{dt} = 0$$

$$v \cdot \frac{dv}{dt} + \omega^2 x \frac{dx}{dt} = 0$$

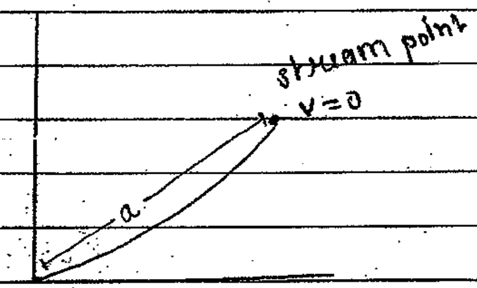
$$v\,dv + \omega^2 x\,dx = 0$$

Integerate both side

$$\int v\,dv + \omega^2 \int x\,dx = 0$$

$$\frac{v^2}{2} + \frac{\omega^2 x^2}{2} = 0 + c$$

@ stream point, $x = a$, $v = 0$



$$C = \frac{\omega^2 a^2}{2}$$

$$\frac{v^2}{2} + \frac{\omega^2 x^2}{2} = \frac{\omega^2 a^2}{2}$$

$$v^2 = \omega^2 (a^2 - x^2)$$

$$\boxed{v = \omega \sqrt{a^2 - x^2}} \quad \text{——(iv)}$$

eq^n (iv) represent velocity eq^n of SHM

Displacement equation :-

We know that

$$v = \frac{dx}{dt} = \omega \sqrt{a^2 - x^2}$$

By variable seperation

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega \, dt$$

On integerating

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \omega dt$$

$$\sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi \qquad \because \phi = \text{constant}$$

$$\frac{x}{a} = \sin(\omega t + \phi)$$

$$\boxed{x = a \sin(\omega t + \phi)} \quad \text{——(v)}$$

where,

a = amplitude
ω = frequency
t = time
ϕ = phase

eq^n (V) represent displacement eq^n in SHM

Curve b/w x and t

$$x = a\sin(\omega t + \phi)$$

Case I   If $\omega t + \phi = 0$

$$x = a\sin 0$$

$$\boxed{x = 0}$$

II   If $\omega t + \phi = \frac{\pi}{2}$

$$x = a\sin\frac{\pi}{2}$$

$$\boxed{x = a}$$

III   If $\omega t + \phi = \pi$

$$x = a\sin\pi$$

$$\boxed{x = 0}$$

IV   If $\omega t + \phi = \frac{3\pi}{2}$

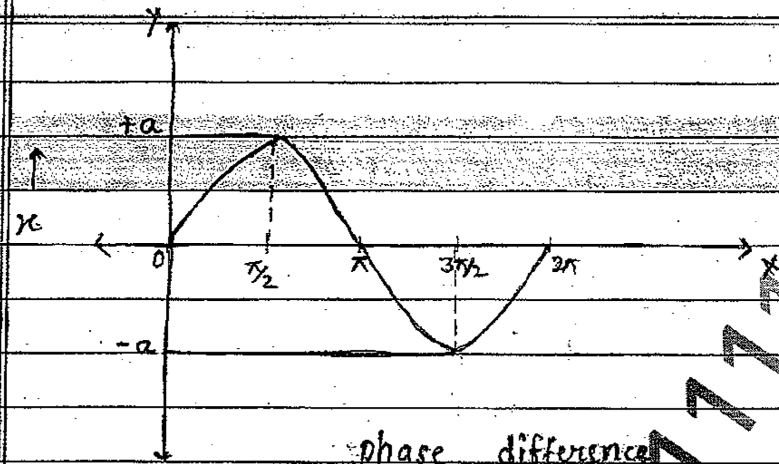
$$x = a\sin\frac{3\pi}{2}$$

$$\boxed{x = -a}$$

V   If $\omega t + \phi = 2\pi$

$$x = a\sin 2\pi$$

$$\boxed{x = 0}$$



phase difference

Graphical Representation of Velocity :-

$$v = \omega\sqrt{a^2 - x^2}$$

$\because \quad x = a\sin(\omega t + \phi)$

$$v = \omega\sqrt{a^2 - a^2\sin^2(\omega t + \phi)}$$

$$v = a\omega\sqrt{1 - \sin^2(\omega t + \phi)}$$

$$\boxed{v = a\omega\cos(\omega t + \phi)}$$

Case I if $\omega t + \phi = 0$

$v = a\omega\cos 0$

$v = a\omega$

if $\omega t + \phi = \frac{\pi}{2}$

$v = a\omega\cos\frac{\pi}{2}$

$v = 0$

if $\omega t + \phi = \pi$

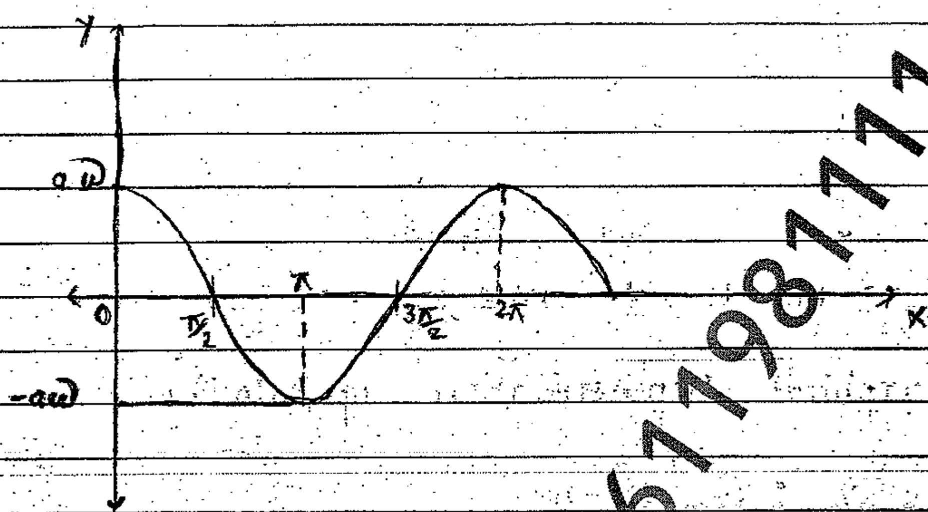
$v = -a\omega$

if $\omega t + \phi = \frac{3\pi}{2}$

$v = 0$

If $\omega t + \phi = 2\pi$

$v = a\omega$



Note, $\frac{dx}{dt} = a\omega \cos(\omega t + \phi)$

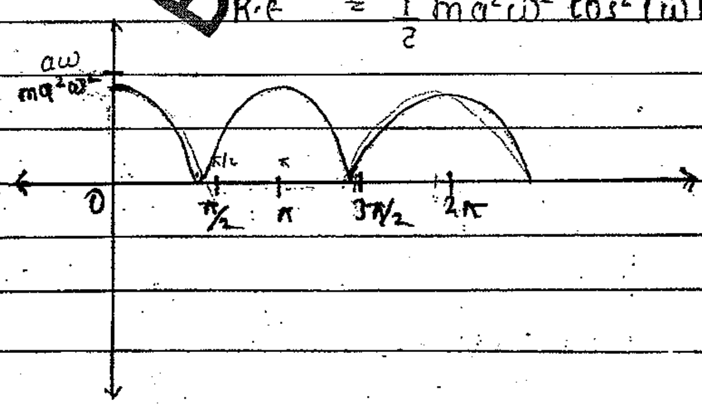
Kinetic energy :-

$$K.E = \frac{1}{2} m v^2$$

$$v = a\omega \cos(\omega t + \phi)$$

$$K.E. = \frac{1}{2} m (a\omega \cos(\omega t + \phi))^2$$

$$K.E = \frac{1}{2} m a^2 \omega^2 \cos^2(\omega t + \phi) \qquad \text{—(vi)}$$



Potential energy :- (U)

Work done in displacing the particle from its mean position against restoring force is equal to gain in potential energy of oscillator.

Restoring force $F_R = -kx$ — (1)

By definition of potential energy

$$\Rightarrow \quad U = -\int_0^x F_R \, dx$$

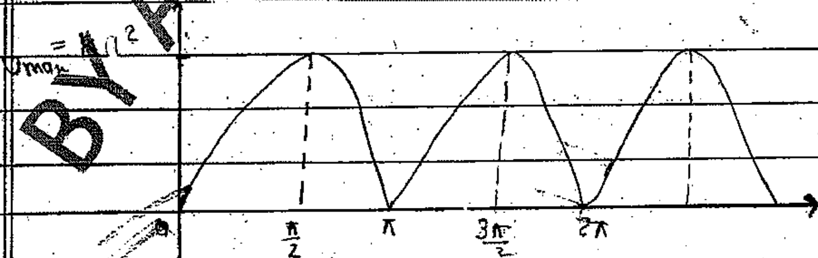
$$\Rightarrow \quad U = \int_0^x kx \, dx$$

$$\Rightarrow \quad U = \frac{1}{2}kx^2 \quad \text{(Spring potential energy)}$$

$$\Rightarrow \quad U = \frac{1}{2}k[a\sin(\omega t + \phi)]^2$$

$$\Rightarrow \quad U = \frac{1}{2}ka^2\sin^2(\omega t + \phi)$$

$$\Rightarrow \quad U = \frac{1}{2}k(\text{elongation / compression})^2$$



phase

Average potential energy for one Time Period:-

$$\Rightarrow \langle U \rangle = \frac{\int_0^T U\,dt}{\int_0^T dt}$$

U = potential energy

$= \frac{1}{2} K a^2 \sin^2(\omega t + \phi)$

$$\Rightarrow \langle U \rangle = \frac{\int_0^T \frac{1}{2} K a^2 \sin^2(\omega t + \phi)\,dt}{\int_0^T dt}$$

$$\Rightarrow \langle U \rangle = \frac{1}{2T} \cdot K a^2 \int_0^T \sin^2(\omega t + \phi)\,dt$$

Note:-

$$\int_0^T \sin^2(\omega t + \phi)\,dt = \frac{T}{2}$$

$$\int_0^T \cos^2(\omega t + \phi)\,dt = \frac{T}{2}$$

$$\Rightarrow \langle U \rangle = \frac{K a^2}{2T} \cdot \frac{T}{2}$$

$$\Rightarrow \langle U \rangle = \frac{1}{4} K a^2$$

Average kinetic energy

$$\Rightarrow \langle K.E \rangle = \frac{\int_0^T (K.E)\,dt}{\int_0^T dt}$$

$$\because K.E = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\phi)$$

$$\Rightarrow \langle K.E \rangle = \frac{\int_0^T \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\phi)}{\int_0^T dt}$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{2T}ma^2\omega^2\int_0^T \cos^2(\omega t+\phi)$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{2T}\cdot ma^2\omega^2\cdot\frac{T}{2}$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{4}m\omega^2a^2$$

By SHM, $\frac{K}{m} = \omega^2$

$$K = m\omega^2$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{4}Ka^2$$

Time period :- (T)

From displacement eq^n

$\Rightarrow$ $x = a\sin(\omega t + \phi)$ ——(1)

We know

$\sin(2\pi + \theta) = \sin\theta$

from eq. (1)

$\Rightarrow$ $x = a\sin\{2\pi + (\omega t + \phi)\}$

$\Rightarrow$ $x = a\sin\{(2\pi + \omega t) + \phi\}$

$\Rightarrow$ $x = a\sin\{\omega(t + \frac{2\pi}{\omega}) + \phi\}$

$\because$ let $t' = t + \frac{2\pi}{\omega}$

$t' - t = \frac{2\pi}{\omega}$

$\Rightarrow$ $x = a\sin(\omega t' + \phi)$

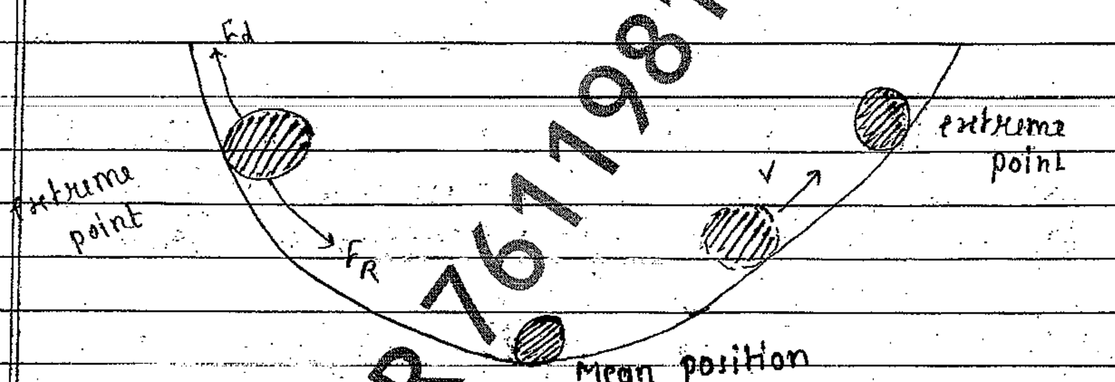
$\Rightarrow$ $T = \frac{2\pi}{\omega}$

$\Rightarrow$ By SHM, $\omega = \sqrt{\frac{K}{m}}$

$\Rightarrow$ $\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$

Damped force (f_d)

When a body doing SHM, In this condition some forces (fricitional, resistance force) work on particle which decrease the value of velocity of particle, that forces is called Drag force.



$\Rightarrow \quad F_d \propto -V$

$\Rightarrow \quad F_d = -\gamma V \qquad \because \gamma = \text{damped coefficient}$

Acc. to Newton,

$\Rightarrow \quad F_d = m\dfrac{dv}{dt}$

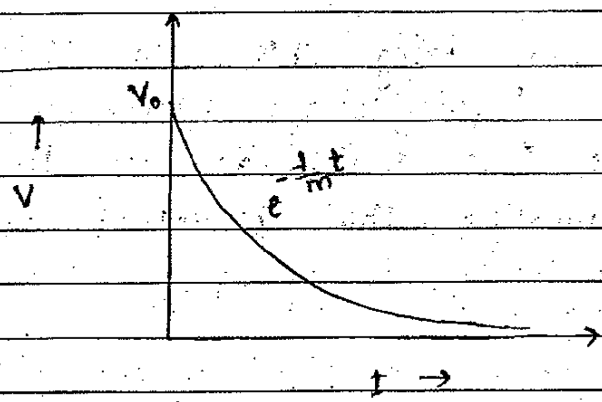
$\Rightarrow \quad m\dfrac{dv}{dt} = -\gamma V$

$\Rightarrow \quad \dfrac{dv}{V} = -\dfrac{\gamma}{m}\,dt$

$\Rightarrow \quad \log_e V = -\dfrac{\gamma}{m}t \quad \Leftrightarrow \quad V = e^{-\gamma/m\,t}$

When time increases, then velocity decreases exponentially.

Relaxation Time :-



The time in which velocity of the body decreases from its maximum velocity (V_0) to $\frac{1}{e}$ or 36.8% is called Relaxation Time.

$\Rightarrow \quad t = \frac{m}{\lambda}$ (Relaxation time)

$\Rightarrow \quad V = V_0 e^{-\frac{\lambda}{m}\frac{m}{\lambda}}$

$\Rightarrow \quad \boxed{V = \frac{V_0}{e}}$ OR $V = 36.8\% \; V_0$

Damped Simple Harmonic Motion (SHM)

When a particle doing SHM, in this condition, mainly two forces are working

(i) Restoring Force
(ii) Damped force

Total force, $F = F_R + F_d$

$\Rightarrow F = -kx + (-\lambda v)$

$\Rightarrow F = -kx - \lambda v$

Acc. to Newton

$\because F = m\frac{d^2x}{dt^2}$

$\Rightarrow m\frac{d^2x}{dt^2} = -kx - \lambda v$

$\Rightarrow m\frac{d^2x}{dt^2} + kx + \lambda v = 0$

$\because$ let $\frac{k}{m} = \omega_0^2$ and $\frac{\lambda}{m} = 2r$

$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{\lambda}{m}v = 0$

$\Rightarrow \frac{d^2x}{dt^2} + \omega_0^2 x + 2rv = 0$

$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega_0^2 x + 2r\frac{dx}{dt} = 0}$ —— (1)

eq^n (1) is known as differential eq^n of damped SHM

let $x = e^{\alpha t}$ is a particular sol^n of above eq which satify eq. (1)

$\Rightarrow \frac{d^2}{dt}(e^{\alpha t}) + \omega_0^2 e^{\alpha t} + 2r\frac{d}{dt}e^{\alpha t} = 0$

$\Rightarrow \alpha^2 \cdot e^{\alpha t} + \omega_0^2 \cdot e^{\alpha t} + 2r\alpha \cdot e^{\alpha t} = 0$

$\Rightarrow \quad e^{\alpha t}\,[\,\alpha^2 + 2r\alpha + \omega_0^2\,] = 0$

so,

$\Rightarrow \quad e^{\alpha t} \neq 0$

and

$\Rightarrow \quad \alpha^2 + 2r\alpha + \omega_0^2 = 0 \quad \text{--- (II)}$

eq^n (II) is a quadratic eq^n whose sol^n can be found with the help of Shri Dharacharya method.

$$\because \quad \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad \alpha = \frac{-2r \pm \sqrt{4r^2 - 4\omega_0^2}}{2}$$

$$\Rightarrow \quad \boxed{\alpha = -r \pm \sqrt{r^2 - \omega_0^2}} \quad \text{--- (3)}$$

$\because \quad x = e^{\alpha t}$

$$\Rightarrow \quad x = A\cdot e^{(-r + \sqrt{r^2 - \omega_0^2})t} + B e^{(-r - \sqrt{r^2 - \omega_0^2})t}$$

Case I. Low Damping Condition ($r <<<< \omega_0$)

$$\Rightarrow \quad \sqrt{r^2 - \omega_0^2} = \sqrt{-1(\omega_0^2 - r^2)} = \sqrt{-1}\,\sqrt{\omega_0^2 - r^2}$$
$$= \sqrt{-1}\,\omega = i\omega$$

$$\Rightarrow \quad x = A\cdot e^{-rt}\,[A\cdot e^{i\omega t} + B e^{-i\omega t}] \quad \text{--- (IV)}$$

Note: $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$; $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

$$\Rightarrow x = e^{-rt}\left[A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)\right]$$

$$\Rightarrow x = e^{-rt}\left[\cos\omega t\,(A+B) + i\sin\omega t\,(A-B)\right]$$

$\because$ let $A+B = \sin\phi \cdot x_0$

$i(A-B) = x_0\cos\phi$

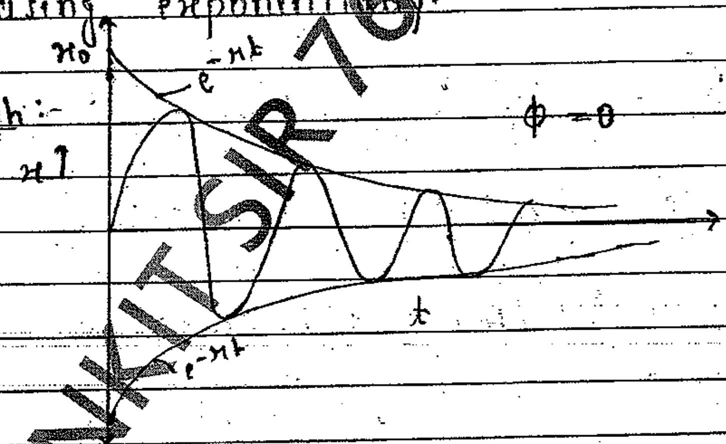
$$\Rightarrow x = e^{-rt}\left[x_0\cos\omega t\cdot\sin\phi + x_0\sin\omega t\cdot\cos\phi\right]$$

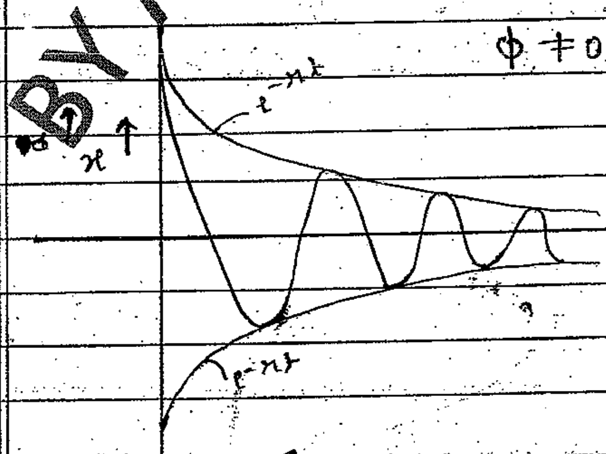
$$\Rightarrow x = e^{-rt}\cdot x_0\left[\cos\omega t\cdot\sin\phi + \sin\omega t\cdot\cos\phi\right]$$

$$\Rightarrow \boxed{x = x_0 e^{-rt}\sin(\omega t+\phi)} \quad \text{—— (v)}$$

Under low damping cond^n, amplitude decreasing exponentially.

Graph:-





Case II Critical Damping cond^n $(r \approx \omega_0)$

$\Rightarrow \quad \sqrt{r^2 - \omega_0^2} = h$ (very low)

$\Rightarrow \quad x = e^{-rt}\left[Ae^{ht} + Be^{-ht}\right]$

$\because \quad e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} + \cdots$

$\Rightarrow \quad x = e^{-rt}\left[A\left(1 + ht + \frac{h^2t^2}{2!} + \cdots\right) + B\left(1 - ht + \frac{h^2t^2}{2!} - \cdots\right)\right]$

h is very low then $h^2 \approx$ negligible

$\Rightarrow \quad x = e^{-rt}\left[A(1+ht) + B(1-ht)\right]$

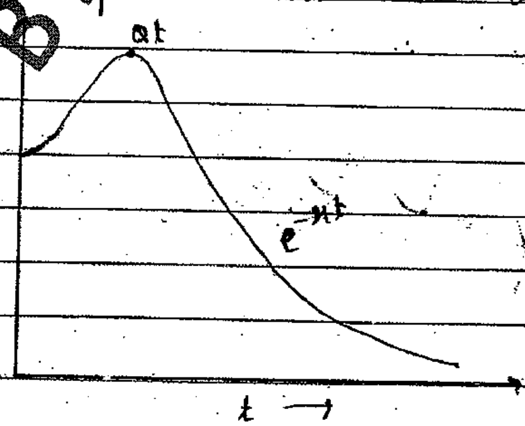
$\Rightarrow \quad x = e^{-rt}\left[(A+B) + ht(A-B)\right]$

let $A + B = P$

$h(A-B) = Q$

$\Rightarrow \quad x = e^{-rt}(P + Qt) \quad \text{—— (VI)}$

eq^n (VI) represent sol^n in critical cond^n



Note: $\psi = x \cdot e^{-x}$

x dominate for smaller values

e^{-x} dominate for larger value of 'x'

Case III: High Damping Cond^n :- $(r >>> \omega_0)$

By eq^n (IV)

$$\Rightarrow \sqrt{r^2 - \omega_0^2} = \gamma$$

$$\Rightarrow x = e^{-rt}[A \cdot e^{\gamma t} + Be^{-\gamma t}]$$

$$\because e^{\gamma t} = \cosh\gamma t + \sinh\gamma t$$

$$e^{-\gamma t} = \cosh\gamma t - \sinh\gamma t$$

$$\Rightarrow x = e^{-rt}[A(\cosh\gamma t + \sinh\gamma t) + B(\cosh\gamma t - \sinh\gamma t)]$$

$$\Rightarrow x = e^{-rt}[\cosh\gamma t(A+B) + \sinh\gamma t(A-B)]$$

let $A+B = x_0 \sinh\psi$

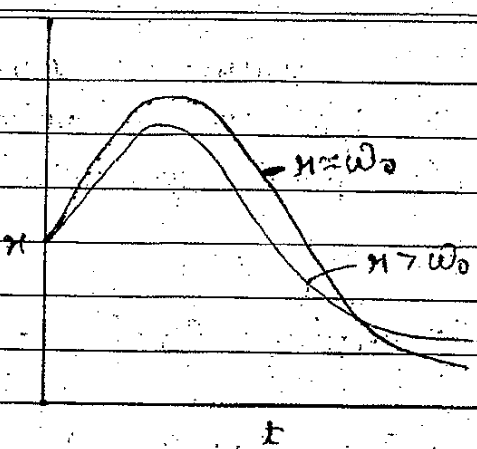
$A-B = x_0 \cosh\psi$

$$\Rightarrow x = e^{-rt} \cdot x_0 [\cosh\gamma t \cdot \sinh\psi + \sinh\gamma t \cosh\psi]$$

$$\Rightarrow \boxed{x = x_0 e^{-rt} \sinh(\gamma t + \psi)} \text{ --- (VII)}$$

eq^n (VII) represent displacement eq^n for high Damping cond^n

graph :-



Amplitude of oscillator is decreasing w.r.t. time in high damping condition. But motion is not oscillator i.e. motion is non-oscillatory or aperiodic motion.

Energy of a damped Harmonic Oscillator :-

(a) Kinetic energy (K.E) :-

$\Rightarrow$ $$K.E = \frac{1}{2} m v^2$$

$\Rightarrow$ $$K.E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \quad \text{---- (I)}$$

Displacement for low damping condⁿ.

$\Rightarrow$ $$x = x_0 e^{-rt} \sin(\omega t + \phi) \quad \text{---- (II)}$$

D. (II) w.r.t. 't'

$\Rightarrow$ $$\frac{dx}{dt} = x_0 \left[-r e^{-rt} \sin(\omega t + \phi) + e^{-rt} \omega \cos(\omega t + \phi) \right]$$

$\Rightarrow$ $$\frac{dx}{dt} = x_0 e^{-rt} \left[-r \sin(\omega t + \phi) + \omega \cos(\omega t + \phi) \right] \quad \text{---- (III)}$$

$$\Rightarrow K.E = \frac{1}{2} m \left[x_0 e^{-rt} \left(-r\sin(\omega t+\phi) + \omega\cos(\omega t+\phi) \right) \right]^2$$

$$\Rightarrow K.E = \frac{1}{2} m x_0^2 e^{-2rt} \left[r^2 \sin^2(\omega t+\phi) + \omega^2 \cos^2(\omega t+\phi) - 2r\omega \sin(\omega t+\phi)\cdot\cos(\omega t+\phi) \right]$$

Average kinetic energy

$$\Rightarrow \langle K.E \rangle = \frac{1}{2} m x_0^2 e^{-2rt} \left[r^2 \langle \sin^2(\omega t+\phi) \rangle + \omega^2 \langle \cos^2(\omega t+\phi) \rangle - 2r\omega \langle \sin(\omega t+\phi) \rangle \cdot \langle \cos(\omega t+\phi) \rangle \right]$$

Note:

$$\langle \sin^2(\omega t+\phi) \rangle = \frac{1}{2}$$

$$\langle \cos^2(\omega t+\phi) \rangle = \frac{1}{2}$$

$$\langle \sin(\omega t+\phi) \rangle = \langle \cos(\omega t+\phi) \rangle = 0$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{2} m x_0^2 \cdot e^{-2rt} \left[\frac{r^2}{2} + \frac{\omega^2}{2} \right]$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{2} m x_0^2 e^{-2rt} \cdot \frac{1}{2} (r^2 + \omega^2)$$

$$\because \sqrt{\omega_0^2 - r^2} = \omega$$

$$\omega_0^2 - r^2 = \omega^2 \Leftrightarrow \omega_0^2 = \omega^2 + r^2$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{2} m x_0^2 \cdot e^{-2rt} \cdot \frac{1}{2} \omega_0^2$$

$$\Rightarrow \langle K.E \rangle = \frac{1}{4} m \omega_0^2 x_0^2 e^{-2rt} \quad \text{—— (iv)}$$

(b) Potential Energy :-

$$\Rightarrow \quad U = \frac{1}{2} K x^2$$

Displacement eq^n for low damping cond^n

$$\Rightarrow \quad x = x_0 e^{-rt} \sin(\omega t + \phi)$$

$$\Rightarrow \quad U = \frac{1}{2} K [x_0 e^{-rt} \cdot \sin(\omega t + \phi)]^2$$

$$\Rightarrow \quad U = \frac{1}{2} K \cdot x_0^2 \cdot e^{-2rt} \cdot \sin^2(\omega t + \phi)$$

$$\Rightarrow \quad \langle U \rangle = \frac{1}{2} K \cdot x_0^2 \cdot e^{-2rt} \langle \sin^2(\omega t + \phi) \rangle$$

$$\Rightarrow \quad \langle U \rangle = \frac{1}{2} K \cdot x_0^2 \cdot e^{-2rt} \cdot \frac{1}{2}$$

$$\Rightarrow \quad \langle U \rangle = \frac{1}{4} K x_0^2 e^{-2rt}$$

$$\because \omega_0 = \sqrt{\frac{K}{m}} \Leftrightarrow K = m\omega_0^2$$

$$\Rightarrow \quad \langle U \rangle = \frac{1}{4} m\omega_0^2 \cdot x_0^2 \cdot e^{-2rt} \quad \text{——— (5)}$$

eq^n (5) represent potential energy

Total Energy :-

Value of Total energy is equal to sum of potential & kinetic energy.

Total Energy (E) = K.E + U

by (IV) and (V)

$\Rightarrow \langle E \rangle = \langle K.E \rangle + \langle U \rangle$

$\Rightarrow \langle E \rangle = \frac{1}{4} m\omega_0^2 \cdot x_0^2 \cdot e^{-2rt} + \frac{1}{2} m\omega_0^2 \cdot x_0^2 \cdot e^{-2rt}$

$\Rightarrow \langle E \rangle = \frac{1}{2} m\omega_0^2 \cdot x_0^2 \cdot e^{-2rt}$ ——(VI)

eq. (VI) represent total average energy of Damped Harmonic Oscillator.

Relaxation Time :- (τ)

$\Rightarrow t = \frac{1}{2r} = \tau$

$\Rightarrow \langle E \rangle = \frac{1}{2} m\omega_0^2 \cdot x_0^2 \cdot e^{-2r \cdot \frac{1}{2r}}$

$\Rightarrow \langle E \rangle = \frac{\frac{1}{2} m\omega_0^2 \cdot x_0^2}{e}$

$\Rightarrow \boxed{\langle E \rangle = \frac{\langle E \rangle_{max}}{e}}$

Power Dissipation :-

Mean power dissipation of oscillator is equal to ratio of loss of energy of an oscillator is called mean power dissipation

$$\Rightarrow \langle P \rangle = -\frac{d\langle E \rangle}{dt}$$

Here (-)ive sign indicate loss of energy.

$$\Rightarrow \langle E \rangle = \frac{1}{2} m\omega_0^2 x_0^2 \cdot e^{-2rt}$$

$$\Rightarrow \langle P \rangle = -\frac{d}{dt}\left(\frac{1}{2} m\omega_0^2 x_0^2 \cdot e^{-2rt}\right)$$

$$\Rightarrow \langle P \rangle = -\frac{1}{2} m\omega_0^2 x_0^2 \cdot (-2r) e^{-2rt}$$

$$\Rightarrow \langle P \rangle = 2r \langle E \rangle$$

$$\Rightarrow \langle P \rangle = \frac{\langle E \rangle}{\frac{1}{2r}} = \frac{\langle E \rangle}{\tau}$$

Quality factor

The ratio of stored energy to the energy loss in one time period is called Quality factor.

$$\Rightarrow Q = 2\pi \times \frac{\text{Stored energy}}{\text{Energy loss in one time period}}$$

$$\Rightarrow Q = 2\pi \times \frac{\langle E \rangle}{T \langle P \rangle}$$

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$$\Rightarrow \quad Q = 2\pi \times \frac{\langle E \rangle}{T \frac{\langle E \rangle}{\tau}}$$

$$\Rightarrow \quad Q = \frac{2\pi \tau}{T}$$

$$\Rightarrow \quad \boxed{Q = \omega \tau}$$

$$\Rightarrow \quad \boxed{Q = \frac{\omega}{2r}}$$

Case I: If value of damping force is increased then value of Quality factor decreases.

Case II: If value of damping force is decreased then value of Quality factor d Increases.

Case III: If $r = 0$ means $Q \approx \infty$, In this condition oscillator behave like free oscillator.

Imp 2021

Frequency of low Damped Oscillator

We know that,

$$\omega = \sqrt{\omega_0^2 - r^2} \quad \text{---} (1)$$

$$\Rightarrow \quad \omega = \omega_0 \sqrt{1 - \frac{r^2}{\omega_0^2}}$$

$$\because \tau = \frac{1}{2r}$$

$$r = \frac{1}{2\tau}$$

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$$\Rightarrow \omega = \omega_0 \sqrt{1 - \frac{1}{4\tau^2\omega_0^2}}$$

$$\Rightarrow \frac{\omega}{\omega_0} = \sqrt{1 - \frac{1}{4\tau^2\omega_0^2}}$$

$$\because Q = \omega_0 \tau$$

$$\Rightarrow \boxed{\frac{\omega}{\omega_0} = \sqrt{1 - \frac{1}{4Q^2}}} \quad \text{(ii)}$$

$$\Rightarrow \frac{\omega^2}{\omega_0^2} = 1 - \frac{1}{4Q^2}$$

$$\Rightarrow \frac{1}{4Q^2} = 1 - \frac{\omega^2}{\omega_0^2}$$

$$\Rightarrow \frac{1}{4Q^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2}$$

$$\Rightarrow \boxed{Q^2 = \frac{\omega_0^2}{4(\omega_0^2 - \omega^2)}}$$

By eq. (ii)

$$\Rightarrow \frac{\omega}{\omega_0} = \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

$$\Rightarrow \frac{2\pi}{T} \cdot \frac{T_0}{2\pi} = \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

$$\Rightarrow \frac{T_0}{T} = \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

By Binomial expression

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

$$\Rightarrow \frac{T_0}{T} = 1 - \frac{1}{2}\left(\frac{1}{4Q^2}\right)$$

$$\Rightarrow \frac{T_0}{T} = 1 - \frac{1}{8Q^2}$$

$$\Rightarrow \frac{1}{8Q^2} = 1 - \frac{T_0}{T}$$

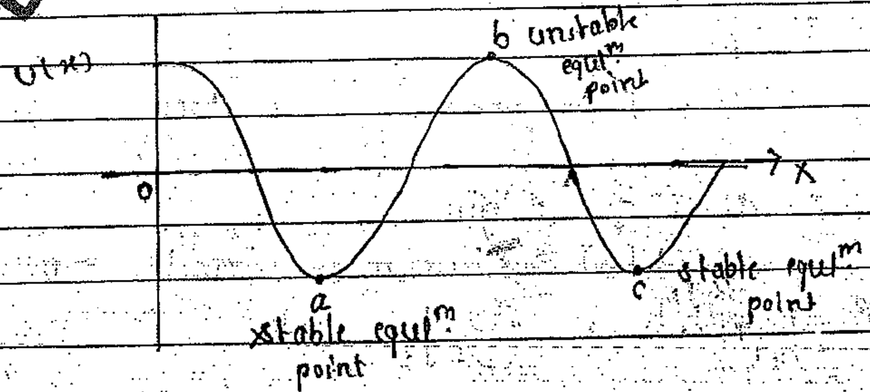
$$\Rightarrow \frac{1}{8Q^2} = \frac{T - T_0}{T}$$

$$\Rightarrow \boxed{Q^2 = \frac{T}{8(T-T_0)}}$$

$$\boxed{Q^2 = 0.125\,\frac{T}{T-T_0}}$$

Oscillator motion under arbitrary potential

Let a particle doing motion under a arbitrary potential then curve b/w potential energy $U(x)$ and position 'x' is given below



Under conservative force

$\Rightarrow \quad \vec{F} = -\nabla U \quad \text{---(1)}$

$\Rightarrow \quad \vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$

If motion is only in x-dirsn

$\Rightarrow \quad F_x = -\frac{\partial U}{\partial x}$

Point a and c are called stable equilibrium point becz in this particle displaced then particle get back from initial condition.

Here particle at point 'b' then point 'b' is known as unstable equil^m point becz in this cond^n particle small displaced then particle not get back from initial condition.

Mathematical Analysis:-

If particle is displaced at point 'a' (stable equil^m point) then change in potential energy is given on the basis of Taylor's series.

Taylor Series:-

$$\Rightarrow U(x) = U(x_0) + \frac{(x-x_0)}{1!}\frac{\partial U}{\partial x}\bigg|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{\partial^2 U}{\partial x^2}\bigg|_{x=x_0}$$

$\because$ let, x_0 is @ origin.

$$U(x) = U(x)\,\text{min}, \quad \frac{\partial U}{\partial x}\bigg|_{x=x_0} = 0$$

$$\frac{\partial^2 U}{\partial x^2}\bigg|_{x=x_0} = k, \quad \frac{\partial^3 U}{\partial x^3}\bigg|_{x=x_0} = k_1$$

$$\Rightarrow U(x) = U(x_0) + \frac{(x-x_0)^2}{2!}K + \frac{(x-x_0)^3}{3!}K_1$$

$$\Rightarrow U(x) = U(x_0) + \frac{(x-x_0)^2}{2}K + \frac{(x-x_0)^3}{6}K_1 \quad \leftarrow (3)$$

Case I If $x_0 = 0$ and $U(x_0) = 0$
and higher term will be negligible for small displacement.

$$\Rightarrow U = \frac{1}{2}Kx^2$$

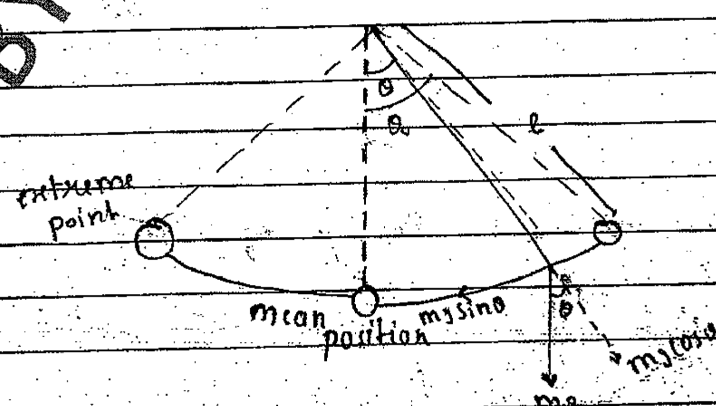
In this condition, particle show SHM at stable equil^m point.

Imp.

Simple Pendulum

A point mass attached to a light inextensible string and suspended from a fixed support is called a simple pendulum

The vertical line passing through fix support is mean position of simple pendulum



Here maximum angle is θ_0

$\Rightarrow$ Torque τ = Force $\times$ $\perp$ distance to force.

Here

$\because F = -mg\sin\theta$

(-)ive sign represent opposite direction

$\Rightarrow \quad \tau = -mg\sin\theta \times l$ —— (1)

$\because \tau = I\alpha$

so,

$\Rightarrow \quad I\alpha = -mgl\sin\theta$

$\Rightarrow \quad \alpha = \dfrac{-mgl\sin\theta}{I}$

$\because I$ = moment of inertia = ml^2 $\because r = l$

$\Rightarrow \quad \alpha = \dfrac{-mgl\sin\theta}{ml^2}$

$\Rightarrow \quad \alpha = -\dfrac{g}{l}\sin\theta$

$\Rightarrow \quad \alpha + \dfrac{g}{l}\sin\theta = 0$

Here α = angular acceleration = $\dfrac{d^2\theta}{dt^2}$

$\Rightarrow \quad \dfrac{d^2\theta}{dt^2} + \dfrac{g}{l}\sin\theta = 0$ —— (2)

eq. (2) is similar to eq. of SHM

so,

$$\Rightarrow \quad \omega_0^2 = \frac{g}{l}$$

$$\Rightarrow \quad \omega_0 = \sqrt{\frac{g}{l}} \quad \text{(angular frequency)}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0$$

multiply by $\frac{d\theta}{dt}$

$$\Rightarrow \quad \frac{d\theta}{dt}\left(\frac{d}{dt}\cdot\frac{d\theta}{dt}\right) + \omega_0^2 \sin\theta \frac{d\theta}{dt} = 0$$

$$\Rightarrow \quad \omega \frac{d\omega}{dt} + \omega_0^2 \sin\theta \frac{d\theta}{dt} = 0$$

$$\Rightarrow \quad \omega\, d\omega + \omega_0^2 \sin\theta \cdot d\theta = 0$$

$$\Rightarrow \quad \int \omega\, d\omega + \int \omega_0^2 \sin\theta \cdot d\theta = 0$$

$$\Rightarrow \quad \frac{\omega^2}{2} + \omega_0^2(-\cos\theta) = A$$

$$\Rightarrow \quad \frac{\omega^2}{2} - \omega_0^2 \cos\theta = A$$

@ extreme point, $\theta = \theta_0$, $\omega = 0$, $t = \frac{T}{4}$

$\Rightarrow$ so, $A = -\omega_0^2 \cos\theta_0$

$$\Rightarrow \quad \frac{\omega^2}{2} - \omega_0^2 \cos\theta = -\omega_0^2 \cos\theta_0$$

$$\Rightarrow \quad \frac{\omega^2}{2} - \omega_0^2(\cos\theta - \cos\theta_0) = 0$$

$$\Rightarrow \quad \omega^2 = 2\omega_0^2(\cos\theta - \cos\theta_0)$$

$\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2}$

$\Rightarrow \omega^2 = 2\omega_0^2\left(1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\theta_0}{2}\right)$

$\Rightarrow \omega^2 = 2\omega_0^2\left(2\sin^2\frac{\theta_0}{2} - 2\sin^2\frac{\theta}{2}\right)$

$\Rightarrow \omega^2 = 4\omega_0^2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$

$\Rightarrow \omega = 2\omega_0\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}$

$\because \omega = \frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt} = 2\omega_0\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}$

$\Rightarrow \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} = 2\omega_0\, dt$

On Integrating

$\Rightarrow \int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} = 2\omega_0\int_0^{T/4} dt$

On solving,

$\Rightarrow \omega_0\frac{T}{4} = \frac{\pi}{2}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}\right]$

$\Rightarrow \omega_0 = \frac{2\pi}{T_0}$ (angular freq.)

$$\Rightarrow \frac{\cancel{2\pi}}{T_0} \cdot \frac{T}{\cancel{4}} = \frac{\cancel{\pi}}{2}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}\right]$$

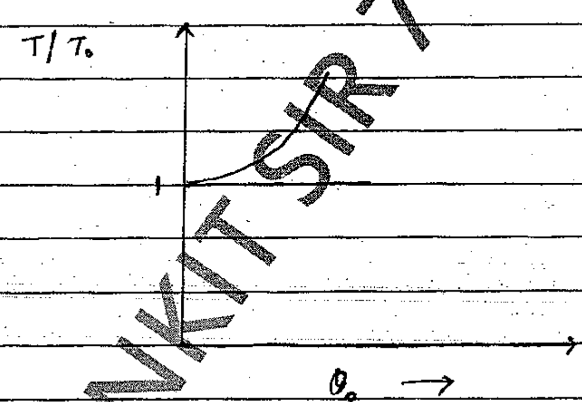
$$\Rightarrow \boxed{\frac{T}{T_0} = 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}}$$

Case I If θ_0 is very less

$\sin\theta_0 \approx \theta_0$

$$\Rightarrow \frac{T}{T_0} = 1 + \frac{1}{4} \times \frac{\theta_0^2}{2}$$

$$\Rightarrow \boxed{\frac{T}{T_0} = 1 + \frac{\theta_0^2}{8}}$$



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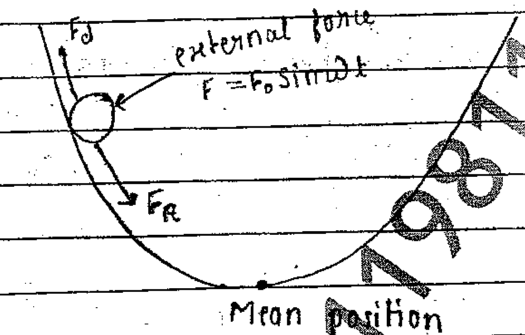
Unit - IV

Forced Oscillator / Driven Oscillator :-



When a body oscillates, then its energy and amplitude will decrease with respect to time 't' due to damped force.
In this state if amplitude of oscillator become constant applying force $F = F_0 \sin\omega t$ on damped oscillator that oscillator is called driven or forced oscillator.

Mainly three forces are working on driven oscillator.

(i) Restoring force ; $F_R = -kx$
(ii) Damped force ; $F_d = -rv$
(iii) external force ; $F_e = F_0 \sin\omega t$

Total force $F = F_R + F_d + F_e$

According to Newtom

$$F = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -kx - \gamma v + F_0 \sin\omega t$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + kx + \gamma v = F_0 \sin\omega t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{\gamma}{m}v = \frac{F_0}{m}\sin\omega t$$

let $\frac{k}{m} = \omega_0^2$; $\frac{\gamma}{m} = 2r$

$$\frac{F_0}{m} = f_0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x + 2rv = f_0 \sin\omega t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x + 2r\frac{dx}{dt} = f_0 \sin\omega t \quad \text{--- (1)}$$

Obtained eq^n is differential eq^n of forced oscillator ~~or~~ driven oscillator.
In this natural frequency ω_0 is constant
If frequency of applied ext. force ω is constant, then we will get stable sol^n of above eq^n

let, Stable solⁿ be

$\Rightarrow \quad x = x_0 \sin(\omega t - \phi) \quad \text{—— (II)}$

$\Rightarrow \quad \frac{dx}{dt} = x_0 \omega \cos(\omega t - \phi) \quad \text{—— (III)}$

$\Rightarrow \quad \frac{d^2x}{dt^2} = -x_0 \omega^2 \sin(\omega t - \phi) \quad \text{—— (IV)}$

put value of eq. (II), (III), (IV) in (I)

$\Rightarrow \quad -x_0\omega^2 \sin(\omega t - \phi) + \omega_0^2 \cdot x_0 \sin(\omega t - \phi) + 2r x_0 \omega \cos(\omega t - \phi) = f_0 \sin(\omega t)$

$\Rightarrow \quad -x_0\omega^2 \sin(\omega t - \phi) + \omega_0^2 x_0 \sin(\omega t - \phi) + 2r\omega x_0 \cos(\omega t - \phi) = f_0 \sin(\omega t - \phi + \phi)$

$\because$ on adding and subtracting ϕ

$\because \sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$\Rightarrow \quad -x_0\omega^2 \sin(\omega t - \phi) + x_0 \omega_0^2 \sin(\omega t - \phi) + 2r x_0 \omega \cos(\omega t - \phi) = f_0 [\sin(\omega t - \phi) \cdot \cos\phi + \cos(\omega t - \phi) \cdot \sin\phi]$

On comparing coefficient of $\sin(\omega t - \phi)$ and

$\Rightarrow$ $\cos(\omega t - \phi)$ in L.H.S. & R.H.S. of above eqⁿ

$\Rightarrow \quad -x_0\omega^2 + x_0\omega_0^2 = f_0 \cos\phi \quad \text{—— (V)}$

$\Rightarrow \quad 2r x_0 \omega = f_0 \sin\phi \quad \text{—— (VI)}$

$(V)^2 + (VI)^2$

$\Rightarrow \quad f_0^2 (\sin^2\phi + \cos^2\phi) = x_0^2(-\omega^2 + \omega_0^2)^2 + (2r x_0 \omega)^2$

$\Rightarrow \quad f_0^2 = x_0^2(-\omega^2 + \omega_0^2)^2 + 4r^2 x_0^2 \omega^2$

$\Rightarrow \quad f_0^2 = x_0^2(\omega_0^2 - \omega^2) + 4r^2\omega^2 x_0^2$

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$$\Rightarrow \quad x_0 = \frac{f_0}{[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2]^{1/2}} \quad \text{——(VII)}$$

eq^n (VII) represent maximum amplitude.

eq^n (VI) $\div$ eq^n (V)

$$\Rightarrow \quad \tan\phi = \frac{2r\omega}{\omega_0^2-\omega^2} \quad \text{—— (VIII)}$$

eq^n (VIII) represent phase.

$$\Rightarrow \quad x = x_0 \sin(\omega t - \phi) \quad (\text{by eq. (II)})$$

$$\Rightarrow \quad x = \frac{f_0}{[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2]^{1/2}} \sin\left[\omega t - \tan^{-1}\left(\frac{2r\omega}{\omega_0^2-\omega^2}\right)\right] \quad \text{——(IX)}$$

Here,

x = Amplitude

x_0 = maximum amplitude

Phase :- The phase difference b/w displacement and driven force can be described by

$$\Rightarrow \quad \tan\phi = \frac{2r\omega}{(\omega_0^2-\omega^2)}$$

Case I, Low Driven frequency

$$\omega <<<< \omega_0$$

$$\Rightarrow \quad \tan\phi = 0$$

$$\phi = 0$$

Here force and displacement are in same phase.

Case II Critical Driven frequency

$\Rightarrow \omega \approx \omega_0$

$\Rightarrow \tan\phi = \infty$

$\Rightarrow \phi = \frac{\pi}{2}$

$F = F_0 \sin \omega t$

$\Rightarrow x = x_0 \sin(\omega t - \frac{\pi}{2})$

In this condn, phase of displacement of forced oscillator lags behind that of driven force by $\frac{\pi}{2}$

Case III High Driven frequency

$\omega >>>> \omega_0$

$\Rightarrow \tan\phi = \frac{2r\omega}{-\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)}$

$\Rightarrow \tan\phi = \frac{-2r}{\omega}$

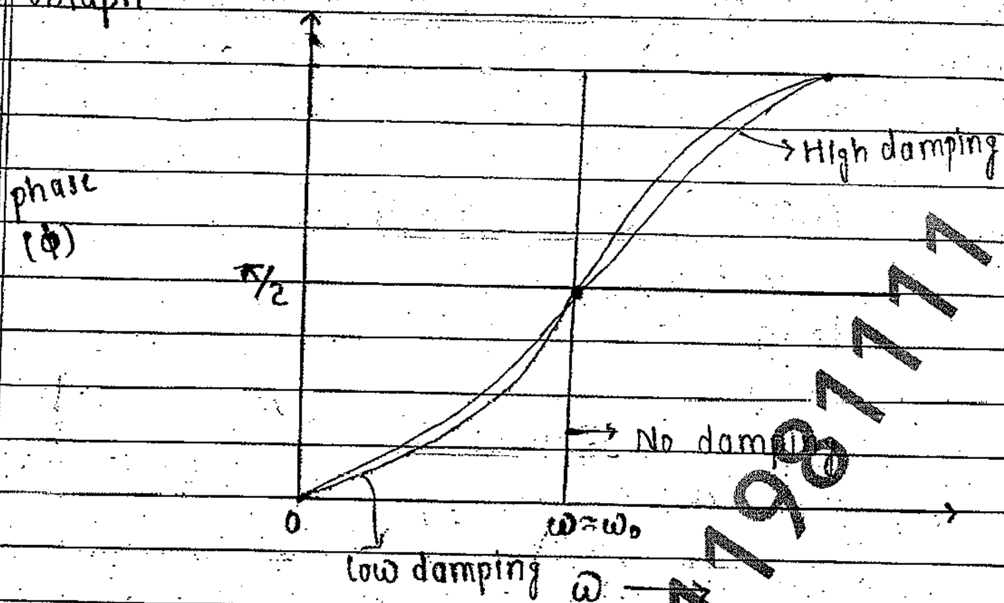
$\phi = \pi$

$\Rightarrow F_t = F_0 \sin(\omega t)$

$\Rightarrow x = x_0 \sin(\omega t - \pi)$

In this condn, phase of displacement of forced oscillator lags behind that of driven force by π.

Graph



Amplitude of forced Oscillator :-

$$\Rightarrow \quad x_0 = \frac{f_0}{\left[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2\right]^{1/2}}$$

Case I. Low Driven frequency $(\omega <<< \omega_0)$

$$\Rightarrow \quad x_0 = \frac{f_0}{\left[(\omega_0^2)^2\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + 4r^2\omega^2\right]^{1/2}}$$

$$\Rightarrow \quad x_0 = \frac{f_0}{(\omega_0^4)^{1/2}\left[\left(1-\frac{\omega^2}{\omega_0^2}\right) + \frac{4r^2\omega^2}{\omega_0^4}\right]^{1/2}}$$

$\Rightarrow \quad \because \omega_0 >>> \omega$

$$\frac{\omega^2}{\omega_0^2} \ \& \ \frac{\omega^2}{\omega_0^4} \approx 0$$

$$\Rightarrow \quad x_0 = \frac{f_0}{\omega_0^2} \qquad \because \; f_0 = \frac{F_0}{m}$$

$$\Rightarrow \quad x_0 = \frac{F_0}{m\omega_0^2} \qquad K = m\omega_0^2$$

$$\Rightarrow \quad \boxed{x_0 = \frac{F_0}{K}}$$

Case II Critical Driven frequency

$$\omega \approx \omega_0$$

$$\Rightarrow \quad x_0 = \frac{f_0}{(4r^2\omega_0^2)^{1/2}}$$

$$\Rightarrow \quad x_0 = \frac{f_0}{2r\omega_0}$$

$$\Rightarrow \quad x_0 = \frac{F_0}{2rm\omega_0} =$$

$$x_0 = \frac{F_0 \tau}{m\omega} \qquad \because \; \frac{\lambda}{m} = 2r, \quad \lambda = 2mr$$

$$\Rightarrow \quad \boxed{x_0 = \frac{F_0}{\lambda\omega_0}}$$

Case III High Driven frequency

$$\omega >>> \omega_0$$

$$\Rightarrow \quad x_0 = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

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$$\Rightarrow \quad x_0 = \frac{f_0}{\left[(-\omega^2)^2\left(1-\frac{\omega_0^2}{\omega^2}\right)^2 + 4r^2\omega^2\right]^{1/2}}$$

$$\Rightarrow \quad x_0 = \frac{f_0}{(\omega^2)^{1/2}\left[\omega^2 + 4r^2\right]^{1/2}}$$

$$\Rightarrow \quad x_0 = \frac{f_0}{\omega\left[\omega^2 + 4r^2\right]^{1/2}}$$

$$\Rightarrow \quad \boxed{x_0 = \frac{F_0}{m\omega\left[\omega^2 + 4r^2\right]^{1/2}}}$$

Maximum Amplitude

For maximum value of Amplitude $\left[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2\right]^{1/2}$ should be minimum — (I)

$$\Rightarrow \quad x_0 = \frac{f_0}{\left[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2\right]^{1/2}} \quad \text{—— (II)}$$

by (I) and (II)

(x_0) is maximum only when $\left[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2\right]^{1/2}$ is minimum.

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D. (1) w.r.t. ω

$\Rightarrow \left[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2\right]^{1/2} = \text{min.}$

$\Rightarrow (\omega_0^2-\omega^2)^2 + 4r^2\omega^2 = (\text{min})^2$

$\Rightarrow -2(\omega_0^2-\omega^2)(2\omega) + 8r^2\omega = 0$

$\Rightarrow -4\omega(\omega_0^2-\omega^2) + 8r^2\omega = 0$

$\Rightarrow -4\omega_0^2 + 4\omega^2 + 8r^2 = 0$

$\Rightarrow 4\omega^2 = 4\omega_0^2 - 8r^2$

$\Rightarrow \omega^2 = \omega_0^2 - 2r^2$

$\Rightarrow \omega = \omega_R = \sqrt{\omega_0^2 - 2r^2} \quad \text{—— (III)}$

eq^n (3) is known as Resonant frequency

from eq^n (II)

$\Rightarrow (x_0)_{max} = \dfrac{f_0}{\left[(\omega_0^2 - \omega_0^2 + 2r^2)^2 + 4r^2(\omega_0^2 - 2r^2)\right]^{1/2}}$

$\Rightarrow (x_0)_{max} = \dfrac{f_0}{(4r^4 + 4r^2\omega_0^2 - 8r^4)^{1/2}}$

$\Rightarrow (x_0)_{max} = \dfrac{f_0}{(4r^2\omega_0^2 - 4r^4)^{1/2}}$

$\Rightarrow (x_0)_{max} = \dfrac{f_0}{2r(\omega_0^2 - r^2)^{1/2}}$

$\because f_0 = \dfrac{F_0}{m} \;\&\; \dfrac{1}{2r} = \tau$

$\Rightarrow (x_0)_{max} = \dfrac{\tau F_0}{m(\omega_0^2 - r^2)^{1/2}}$

$$\Rightarrow (x_0)_{max.} = \frac{F_0 \tau}{m\omega_0 \left(1 - \frac{r^2}{\omega_0^2}\right)^{1/2}}$$

$$\Rightarrow (x_0)_{max.} = \frac{\tau \cdot F_0}{m\omega_0} \left(1 - \frac{r^2}{\omega_0^2}\right)^{-1/2}$$

By Binomial theorum

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2$$

$$\Rightarrow (x_0)_{max.} = \frac{\tau \cdot F_0}{m\omega_0} \left[1 + \frac{1}{2}\frac{r^2}{\omega_0^2} + \cdots\right]$$

Higher term will be negligible

$$\Rightarrow (x_0)_{max.} = \frac{\tau \cdot F_0}{m\omega_0} \left[1 + \frac{r^2}{2\omega_0^2}\right]$$

$$\Rightarrow (x_0)_{max.} = \frac{\tau \cdot F_0}{m\omega_0} \left[1 + \frac{4r^2}{8\omega_0^2}\right]$$

$$\Rightarrow (x_0)_{max.} = \frac{\tau \cdot F_0}{m\omega_0} \left[1 + \frac{1}{8\left(\frac{\omega_0}{2r}\right)^2}\right]$$

∴ Quality factor

$$\Rightarrow Q = \frac{x_0(\omega \simeq \omega_0)}{x_0(\omega_0 >> \omega)}$$

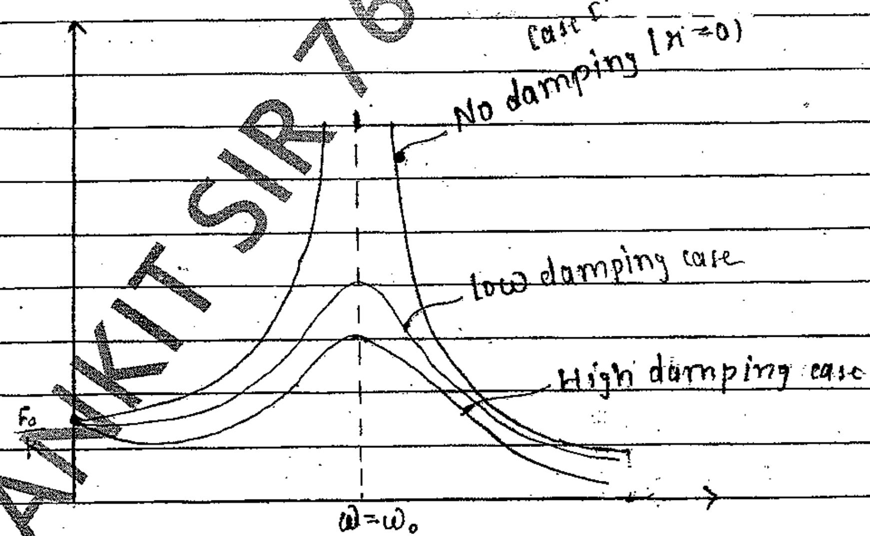
$$Q = \frac{\omega_0}{2r}$$

$$\Rightarrow (x_0)_{max} = \frac{\tau \cdot F_0}{m\omega_0}\left[1 + \frac{1}{8Q^2}\right]$$

$$\Rightarrow (x_0)_{max} = \frac{\omega_0 \tau \cdot F_0}{m\omega_0^2}\left[1 + \frac{1}{8Q^2}\right]$$

$$Q = \omega_0 \tau$$

$$\Rightarrow \boxed{(x_0)_{max} = \frac{Q F_0}{m\omega_0^2}\left[1 + \frac{1}{8Q^2}\right]}$$



Velocity of Driven Oscillator :-

Rate of change of displacement is called velocity of Oscillator.

$$x = x_0 \sin(\omega t - \phi) \quad \text{—— (I)}$$

$$\because v = \frac{dx}{dt}$$

$$v = \frac{d}{dt}[x_0 \sin(\omega t - \phi)]$$

$$v = x_0 \omega \cos(\omega t - \phi)$$

$$v = x_0 \omega \sin\left[\frac{\pi}{2} + (\omega t - \phi)\right] \quad \because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$v = x_0 \omega \sin\left[\omega t - \left(\phi - \frac{\pi}{2}\right)\right]$$

let $\phi - \frac{\pi}{2} = \delta$ (phase)

$$\boxed{v = x_0 \omega \sin(\omega t - \delta)} \quad \text{—— (II)}$$

eq^n (II) is known as velocity of Oscillator

Maximum Velocity of Oscillator :-

$$v = x_0 \omega \sin(\omega t - \delta)$$

$$v_0 = x_0 \omega$$

$$v_0 = \frac{f_0 \, \omega}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$v_0 = \frac{f_0}{\frac{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}{\omega}}$$

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$$V_0 = \frac{f_0}{\left[\frac{(\omega_0^2-\omega^2)^2}{\omega^2} + 4r^2\right]^{1/2}}$$

$$V_0 = \frac{f_0}{\left[\left(\frac{(\omega_0^2-\omega^2)}{\omega^2}\right)^2 + 4r^2\right]^{1/2}}$$

$$V_0 = \frac{f_0}{\left[\left(\frac{\omega_0^2}{\omega}-\omega\right)^2 + 4r^2\right]^{1/2}}$$

For maximum V_0

$\left[\left(\frac{\omega_0^2}{\omega}-\omega\right)^2 + 4r^2\right]^{1/2}$ should be min.

Differentiate w.r.t. ω

$$2\left[\frac{\omega_0^2}{\omega}-\omega\right]\left(-\frac{\omega_0^2}{\omega^2}-1\right) = 0$$

$$(\omega_0^2-\omega^2)(\omega_0^2+\omega^2) = 0$$

$$\boxed{\omega_0 = \omega}$$

$\Uparrow$

Resonance condition of frequency

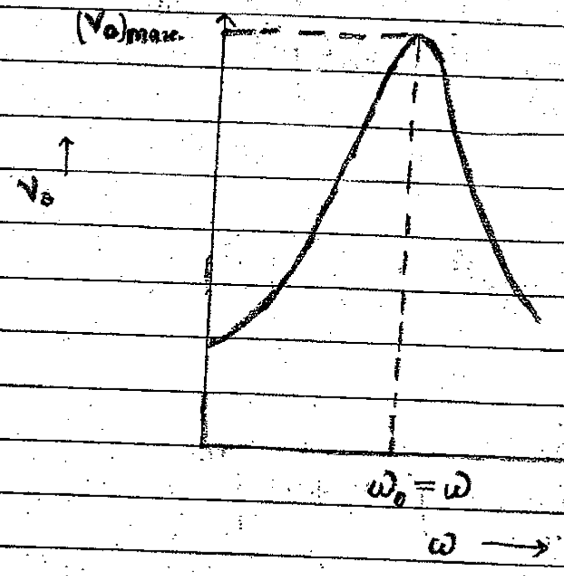
$$V_0 = \frac{f_0}{(4r^2)^{1/2}}$$

$\because f_0 = \frac{F_0}{m}$

$\tau = \frac{1}{2r}$

$$(V_0)_{max.} = \frac{f_0}{2r}$$

$$\boxed{(V_0)_{max.} = \frac{F_0}{m \cdot 2r} = \frac{F_0\tau}{m}}$$



Kinetic energy of driven Oscillator :-

$$K.E = \frac{1}{2} m v^2$$

$$\because v = V_0 \sin(\omega t - \delta)$$

$$K.E = \frac{1}{2} m V_0^2 \sin^2(\omega t - \delta)$$

Average K.E

$$\langle K.E \rangle = \frac{1}{2} m V_0^2 \langle \sin^2(\omega t - \delta) \rangle$$

$$\langle K.E \rangle = \frac{1}{2} m V_0^2 \cdot \frac{1}{2}$$

$$\langle K.E \rangle = \frac{1}{4} m V_0^2$$

$$\because V_0 = x_0 \omega$$

$$\langle K.E \rangle = \frac{1}{4} m x_0^2 \omega^2$$

$$\because \quad x_0 = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$\langle K.E \rangle = \frac{1}{4} \frac{f_0^2 \cdot m\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]}$$

Max^m kinetic energy :-

for maximum K.E. $[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2] = \text{min.}$

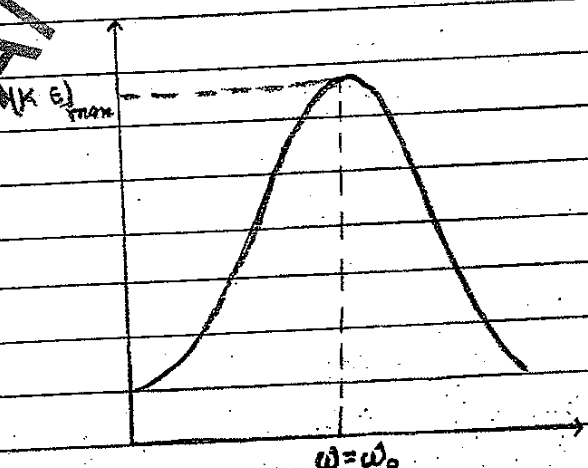
on D. w.r.t. ω

$$\omega_0 = \omega$$

$$(K.E)_{max} = \frac{1}{4} \frac{f_0^2 \cdot \omega^2 m}{4r^2\omega^2}$$

$$= \frac{1}{16} \cdot \frac{m f_0^2}{r^2}$$

$$(K.E)_{max} = m\left(\frac{f_0}{4r}\right)^2$$



Power of Oscillator :-

When object is allowed to oscillate, its amplitude decreases with respect to time. If in this state external force is applied then oscillator absorbs energy and keeps its amplitude constant.

The rate of energy absorption is called power of Oscillator.

$$P = F_t \cdot V$$

$\because$ $V = x_0 \omega \cos(\omega t - \phi)$ = velocity of oscillator

$F_t = F_0 \sin \omega t$ = external force.

$$P = (F_0 \sin\omega t) \cdot (x_0 \omega \cos(\omega t - \phi))$$

$$P = F_0 x_0 \omega \sin\omega t \cdot [\cos\omega t \cdot \cos\phi + \sin\omega t \cdot \sin\phi)$$

$$P = F_0 x_0 \omega [\sin\omega t \cdot \cos\omega t \cdot \cos\phi + \sin^2\omega t \cdot \sin\phi)$$

$$\langle P \rangle = F_0 x_0 \omega [\langle \sin\omega t \rangle \langle \cos\omega t \rangle \cos\phi + \langle \sin^2 \omega t \rangle \cdot \sin\phi]$$

$$\langle P \rangle = F_0 x_0 \omega \cdot \sin\phi \quad \text{---(1)}$$

$\because$ $\langle \sin\omega t \rangle = \langle \cos\omega t \rangle = 0$

$\langle \sin^2 \omega t \rangle = \frac{1}{2}$

$$x_0 = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4 r^2 \omega^2]^{1/2}} \quad \text{---(II)}$$

$$\because \sin\phi = \frac{\tan\phi}{\sqrt{1 + \tan^2\phi}}$$

$$\because \tan\phi = \frac{2r\omega}{\omega_0^2 - \omega^2}$$

$$\sin\phi = \frac{2r\omega}{(\omega_0^2-\omega^2)\sqrt{1+\frac{4r^2\omega^2}{(\omega_0^2-\omega^2)^2}}}$$

$$\sin\phi = \frac{2r\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}} \quad \text{---(III)}$$

by (I), (II), (III)

$$\langle P\rangle = \frac{F_0 x_0 \omega \cdot 2r\omega}{2\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}}$$

$$\langle P\rangle = \frac{1}{2}\cdot\frac{F_0\, 2r\omega^2 \cdot f_0}{\left(\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}\right)\left(\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}\right)}$$

$$\langle P\rangle = \frac{1}{2}\cdot\frac{F_0\omega^2\, 2r f_0}{\left[(\omega_0^2-\omega^2)^2+4r^2\omega^2\right]}$$

$$\langle P\rangle = \frac{1}{2}\cdot\frac{F_0 \cancel{\omega^2}\cdot 2r f_0}{\cancel{\omega^2}\left[\left[\frac{\omega_0^2}{\omega}-\omega\right]^2+4r^2\right]}$$

$$\langle P\rangle = \frac{1}{2}\frac{F_0 f_0 \cdot 2r}{\left[\left(\frac{\omega_0^2}{\omega}-\omega\right]^2+4r^2\right]} \quad \text{---(IV)}$$

For maximum power

$$\left(\frac{\omega_0^2}{\omega}-\omega\right]^2+4r^2 = \text{min.}$$

On D. w.r.t. 'ω'

$$\omega_0 = \omega$$

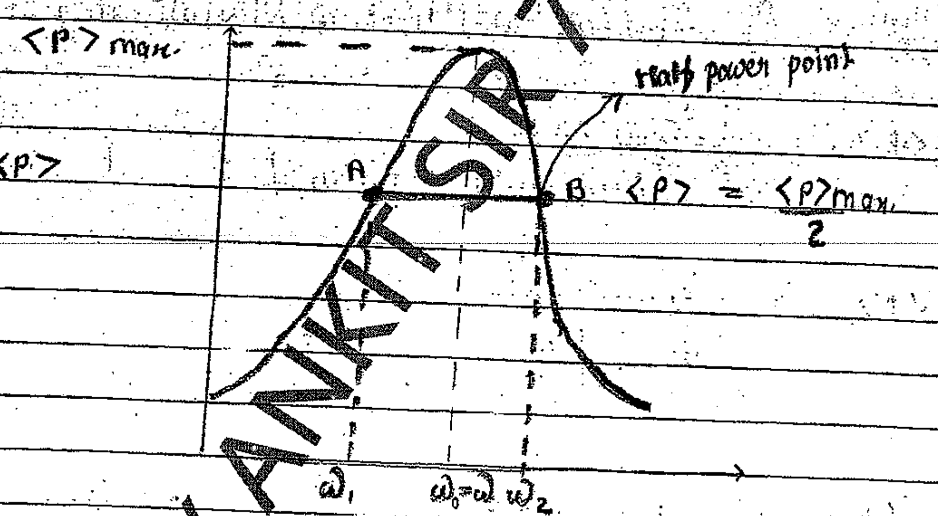
$$\langle P \rangle_{max} = \frac{1}{2}\frac{F_0 f_0 \, 2r}{(4r^2)}$$

$$\langle P \rangle_{max} = \frac{1}{2}\left(\frac{1}{2r}\right) \cdot F_0 f_0$$

$$\boxed{\langle P \rangle_{max} = \frac{1}{2} \tau \cdot F_0 f_0}$$

$$\langle P \rangle_{max} = \frac{1}{2} \tau \cdot F_0 \cdot \frac{F_0}{m} \qquad \tau = \frac{1}{2r}$$

$$\boxed{\langle P \rangle_{max} = \frac{1}{4r}\frac{F_0^2}{m}}$$



If curve of average power with frequency then we get two points where value of average power will be half is called Half power point and frequency is called Half power frequency.

These points are denoted by A & B

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@ Half power point

$$\langle P \rangle = \frac{\langle P \rangle_{max}}{2}$$

$$\because \langle P \rangle = \frac{1}{2} \frac{F_0 f_0 \cdot 2r}{\left[\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4r^2\right]}$$

$$\langle P \rangle_{max} = \frac{1}{4} \frac{F_0^2}{rm}$$

$$\frac{\cancel{1}}{\cancel{2}} \frac{\cancel{F_0} \cancel{f_0} \cdot 2r}{\left[\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4r^2\right]} = \frac{\cancel{1}}{\cancel{2}} \cdot \frac{1}{4} \frac{F_0^2}{r\cancel{m}}$$

$$8r^2 = \left[\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4r^2\right]$$

$$\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 4r^2$$

$$\frac{\omega_0^2}{\omega} - \omega = \pm 2r$$

$$\omega_0^2 - \omega^2 = \pm 2r\omega$$

@ point A, $\omega = \omega_1$; $\omega_0 > \omega_1$

$$\omega_0^2 - \omega_1^2 = +2r\omega_1$$

@ point B, $\omega = \omega_2$; $\omega_0 < \omega_2$

$$\omega_0^2 - \omega_2^2 = -2r\omega_2$$

Case

$$(\omega_0 + \omega)(\omega_0 - \omega) = \pm 2r\omega$$

$$\omega_0 = \omega$$

$$2\cancel{\omega_0}(\omega_0 - \omega) = \pm 2r\cancel{\omega_0}$$

$$\omega_0 - \omega = \pm r$$

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$$\omega_0 - \omega_1 = r$$
$$\omega_0 - \omega_2 = -r$$

$$\omega_0 = \omega_1 + r$$
$$\omega_0 = \omega_2 - r$$

$$\omega_2 - r - \omega_1 - r = 0$$
$$\boxed{\omega_2 - \omega_1 = 2r}$$

Band width

Quality Factor :-

$$Q = \frac{\omega_0}{2r}$$

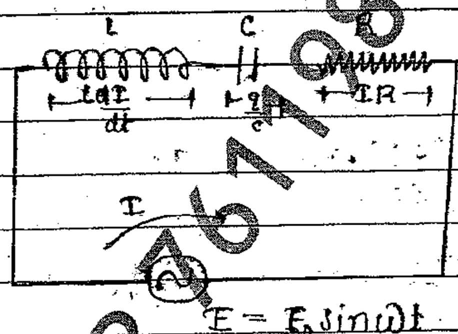
$$\boxed{Q = \frac{\omega_0}{\omega_2 - \omega_1}}$$

Quality factor of driven force oscillator

Completed Unit-IV

L.C.R. Series Circuit

The electric circuit in which coil (L), capacitor (C) and resistance (R) connected in series is known as L.C.R. Series circuit. Let in L.C.R. Series circuit, an A.C. Source connected is $E = E_0 \sin\omega t$ due to which current (I) is flowing in the circuit.



According to Kirchoff 2nd law

$$L\frac{dI}{dt} + \frac{q}{C} + IR = E_0 \sin\omega t \quad \text{--- (1)}$$

$$\because I = \frac{dq}{dt}$$

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{q}{C} + \left(\frac{dq}{dt}\right)R = E_0 \sin\omega t$$

$$\frac{d^2q}{dt^2} + \frac{dq}{dt}\cdot\frac{R}{L} + \frac{q}{LC} = \frac{E_0}{L}\sin\omega t \quad \text{--- (II)}$$

Let

$$\frac{R}{L} = 2r \quad ; \quad \frac{1}{LC} = \omega_0^2 \quad ; \quad \frac{E_0}{L} = \alpha_0$$

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From differential eq. of driven oscillator

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f_0 \sin\omega t \quad \text{—(III)}$$

$$x = x_0 \sin\omega t$$
$$v = v_0 \cos\omega t \qquad (v_0 = x_0\omega)$$

$$x_0 = \frac{f_0}{[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

by eq. (II)

$$\frac{d^2q}{dt^2} + \left(\frac{dq}{dt}\right)\cdot 2r + q\omega_0^2 = \alpha_0 \sin\omega t \quad \text{—(IV)}$$

eq. (IV) is known as differential eq. of L.C.R. Series.

On comparing eq. (III) and (IV)

$$q = q_0 \sin\omega t$$

$$\frac{dq}{dt} = I = I_0 \cos\omega t \qquad (I_0 = q_0\omega)$$

$$q_0 = \frac{\alpha_0}{[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$I_0 = q_0\omega = \frac{\alpha_0\omega}{[(\omega_0^2-\omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$I_0 = \frac{\frac{E_0}{L}\omega}{\left[\left(\left(\frac{1}{LC}\right)^2 - \omega^2\right)^2 + \left(\frac{R}{L}\right)^2\omega^2\right]^{1/2}}$$

$$\because \alpha_0 = \frac{E_0}{L} \;;\; \omega_0^2 = \frac{1}{LC} \;;\; 2r = \frac{R}{L}$$

$$I_0 = \frac{\frac{\epsilon_0}{L}\omega}{\left[\left(\frac{\omega}{L}\right)^2\left(\frac{1}{\omega C} - \omega L\right)^2 + \frac{R^2}{L^2}\omega^2\right]^{1/2}}$$

$$I_0 = \frac{\frac{\epsilon_0}{L}\omega}{\frac{\omega}{L}\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}$$

$$I_0 = \frac{\epsilon_0}{\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}$$

For I_0 = maximum, $\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}$ = min.

Differentiate w.r.t. ω

$$2\left(\frac{1}{\omega C} - \omega L\right)\left(\frac{-1}{\omega^2 C} - L\right) = 0$$

$$\left(\frac{1}{\omega C} - \omega L\right)\left(L + \frac{1}{\omega^2 C}\right) = 0$$

$$\frac{1}{\omega^2 C} + L \neq 0$$

so, $$\frac{1}{\omega C} - \omega L = 0$$

$$\omega L = \frac{1}{\omega C}$$

$$\omega^2 = \frac{1}{LC}$$

$$\boxed{\omega_R^2 = \omega_0^2 = \frac{1}{LC}} \quad \text{(Resonance frequency)} \quad \text{— (V)}$$

eq. (V) is known as Resonance condition and in this condition, Resonance frequency.

$$I_0 = \frac{\mathcal{E}_0}{\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}$$

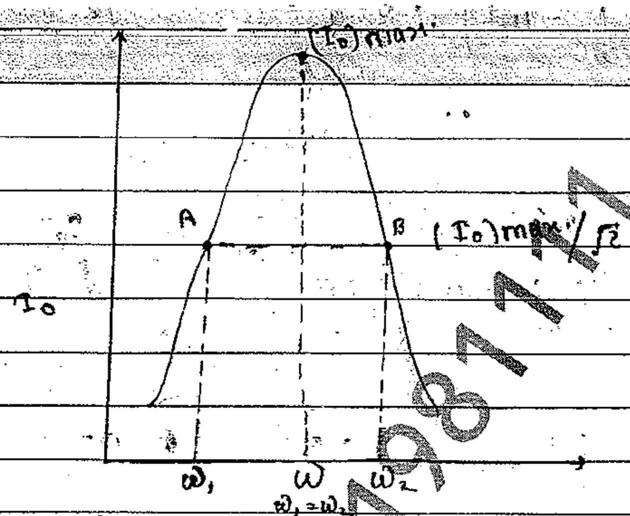
$$(I_0)_{max.} = \frac{\mathcal{E}_0}{\left[\left(\frac{1}{\omega_R C} - \omega_R L\right)^2 + R^2\right]^{1/2}}$$

$$(I_0)_{max.} = \frac{\mathcal{E}_0}{\left[\left(\frac{1}{C}\sqrt{LC} - \frac{1}{\sqrt{LC}} \cdot L\right)^2 + R^2\right]^{1/2}}$$

$$(I_0)_{max.} = \frac{\mathcal{E}_0}{\left[\left(\sqrt{\frac{L}{C}} - \sqrt{\frac{L}{C}}\right)^2 + R^2\right]^{1/2}}$$

$$\boxed{(I_0)_{max.} = \frac{\mathcal{E}_0}{R}}$$

Half Power Point :-



Note :-

Total Impedence

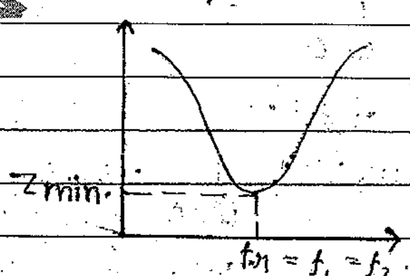
$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (V_L > V_C)$$

$$\because X_L = \omega L = 2\pi f L$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

$$Z = \sqrt{R^2 + (X_C - X_L)^2} \quad (V_C > V_L)$$

angular frequency $\omega = 2\pi f$



@ Half power point

$$I_0 = \frac{(I_0)_{max}}{\sqrt{2}} \quad \text{—— (VII)}$$

$$\frac{\mathcal{E}_0}{\left[\left(\frac{1}{\omega C}-\omega L\right)^2+R^2\right]^{1/2}} = \frac{(I_0)_{max}}{\sqrt{2}}$$

$$\because \quad (I_0)_{max} = \frac{\mathcal{E}_0}{R}$$

$$\frac{\cancel{\mathcal{E}_0}}{\left[\left(\frac{1}{\omega C}-\omega L\right)^2+R^2\right]^{1/2}} = \frac{\cancel{\mathcal{E}_0}}{\sqrt{2}\,R}$$

$$\left(\frac{1}{\omega C}-\omega L\right)^2+R^2 = 2R^2$$

$$\left(\frac{1}{\omega C}-\omega L\right)^2 = R^2$$

taking Square root

$$\frac{1}{\omega C}-\omega L = \pm R \quad \text{—(VIII)}$$

taking (+)ive sign, $\omega = \omega_1$

$$\frac{1}{\omega_1 C}-\omega_1 L = R \quad \text{—(IX)}$$

taking (−)ive sign, $\omega = \omega_2$

$$\frac{1}{\omega_2 C}-\omega_2 L = -R \quad \text{—(X)}$$

add eq. (IX) and (X)

$$\frac{1}{\omega_1 C}-\omega_1 L+\frac{1}{\omega_2 C}-\omega_2 L = 0$$

$$\frac{1}{C}\left(\frac{\omega_1+\omega_2}{\omega_1\cdot\omega_2}\right) - L(\omega_1+\omega_2) = 0$$

$$\frac{1}{C(\omega_1\cdot\omega_2)} = L$$

$$\omega_1\cdot\omega_2 = \frac{1}{LC} \quad \text{—(XI)}$$

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Subtract eq. (X) from eq. (IX)

$$\frac{1}{\omega_1 C} - \omega_1 L - \frac{1}{\omega_2 C} + \omega_2 L = 2R$$

$$\frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1 \cdot \omega_2}\right) + L(\omega_2 - \omega_1) = 2R$$

$$\frac{1}{C}\left(LC\right)(\omega_2 - \omega_1) + L(\omega_2 - \omega_1) = 2R$$

$$2L(\omega_2 - \omega_1) = 2R$$

$$\omega_2 - \omega_1 = \frac{R}{L} \quad \text{—— (XII)}$$

eq. (XII) represent Band Width

Quality Factor :-

The ratio of stored energy to energy loss in one time period is multiplied by 2π then we get Quality factor.

In L.C.R. circuit, energy lost only due to resistance.

Energy stored in coil and capacitor.

If maximum current flow in coil then stored energy $= \frac{1}{2} L (I_0)^2_{max}$. In this condition charge on capacitor is zero.

If maximum stored energy @ capacitor is $\frac{q^2}{2C}$ then in this condn zero current flow in circuit.

$$Q = 2\pi \times \frac{\text{Stored energy}}{\text{energy loss in one time period}} \quad \text{---(XIII)}$$

energy loss $= I^2 R$

@ Half power point $\left(\frac{I_0}{\sqrt{2}}\right)^2 R = \frac{I_0^2}{2} R$

(a) Quality Factor of coil

$$Q = \frac{2\pi}{T} \times \frac{\frac{1}{2} L I_0^2}{\frac{I_0^2}{2} R}$$

$$Q = \frac{2\pi}{T} \times \frac{L}{R} \qquad \omega_0 = \omega = \frac{2\pi}{T}$$

$$Q = \omega_0 \times \frac{1}{\omega_2 - \omega_1}$$

$$\boxed{Q = \frac{\omega_0}{\omega_2 - \omega_1}} \quad \text{---(XIV)}$$

(b) Quality factor of Capacitor

$$Q = \frac{2\pi}{T} \times \frac{q_0^2 / 2C}{I_0^2 R / 2}$$

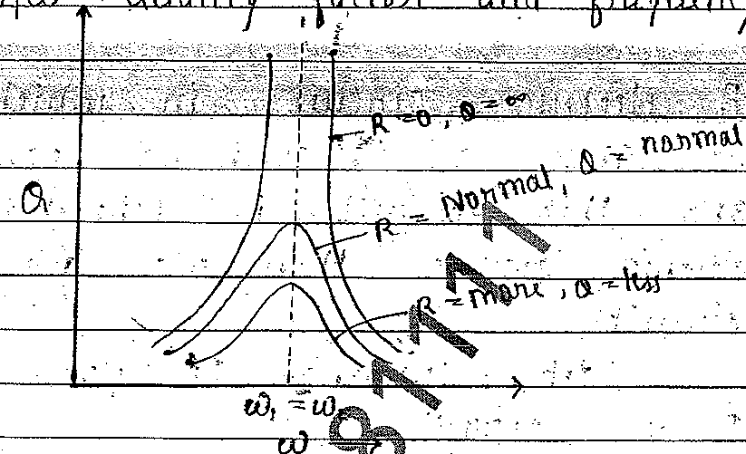
$$Q = \frac{2\pi}{T} \times \frac{q_0^2}{I_0^2 R C} \qquad \because I_0 = q_0 \omega_0$$

$$Q = \frac{2\pi}{T} \times \frac{q_0^2}{q_0^2 \cdot \omega_0^2 C R}$$

$$Q = \omega_0 \times \frac{1}{\omega_0^2 C R}$$

$$\boxed{Q = \frac{1}{\omega_0 C R}}$$

Curve b/w Quality factor and frequency



If value of $R=0$, then Quality factor $=\infty$ this condition also known as zero damped condition.

Voltage Amplification

In the condition of resonance, current will be maximum.

$$I_0 = \frac{E_0}{R}$$

In condition of resonance, voltage amplification

$$E_H = X_L I_0$$

$$= \omega L \cdot \frac{E_0}{R}$$

$$= \omega_0 E_0 \frac{L}{R}$$

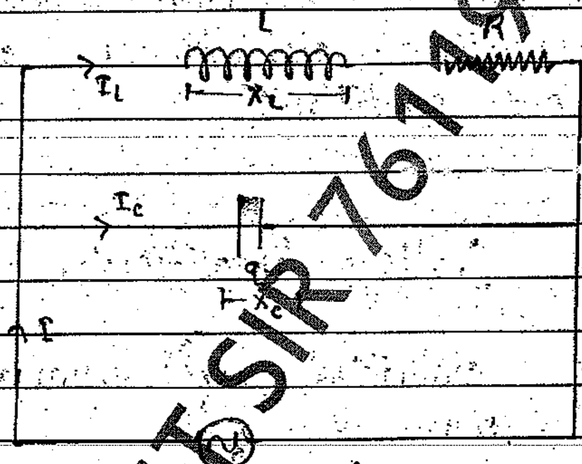
$$\because Q = \omega_0 \frac{L}{R}$$

$$E_H = Q E_0$$

L-C-R Parallel Circuit

A electric circuit in which coil and resistance are connected parallel to capacitor and this circuit is connected with a electric source, then we get parallel L-C-R circuit.

Let current I_L flow in coil and I_C in capacitor.



$e = e_0 \sin\omega t$

Total current

$$I = I_L + I_C \quad \text{--- (I)}$$

Acc. to ohm

$$V = IR = IZ$$

$$I = \frac{V}{Z} \quad \text{--- (II)}$$

Voltage remains constant in parallel series

$$I_L = \frac{V}{Z} = \frac{V}{(R + j\omega L)}$$

$$I_L = \frac{V}{Z} = \frac{V}{R + X_L}$$

$$\because X_L = j\omega L$$

$$I_L = V/(j\omega L + R) \quad \text{—(III)} \qquad \because j^2 = -1$$

$$I_C = \frac{V}{Z} = \frac{V}{X_C}$$

$$\because X_C = \frac{1}{j\omega C}$$

$$I_C = V \cdot j\omega C \quad \text{—(IV)}$$

put eq. 2, 3, 4 in eq.(1)

$$\frac{V}{Z} = \frac{V}{R + j\omega L} + V \cdot j\omega C$$

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + j\omega C \quad \text{—(V)}$$

~~Impeden~~
admittance (Y) :- inverse of impedence

$$Y = \frac{1}{Z}$$

taking conjugate

$$Y = \frac{1}{Z} = \frac{1}{R + j\omega L} \times \frac{R - j\omega L}{R - j\omega L} + j\omega C$$

$$Y = \frac{R - j\omega L}{R^2 - (j\omega L)^2} + j\omega C$$

$$Y = \frac{1}{Z} = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C \qquad \because j^2 = -1$$

$$Y = \frac{1}{Z} = \frac{R - j\omega L + j\omega C(R^2 + \omega^2 L^2)}{(R^2 + \omega^2 L^2)}$$

$$Y = \frac{1}{Z} = \frac{R - j\omega[L - (CR^2 + C\omega^2 L^2)]}{R^2 + \omega^2 L^2} \quad \text{—(VI)}$$

eq. (VI) represent ~~an~~ admittance.

$$Z = \frac{R^2 + \omega^2 L^2}{R - j\omega[L - (CR^2 + C\omega^2 L^2)]} \quad \text{— (VII)}$$

eq. (VII) represent Impedence

@ resonance condition

(I) $X_L = X_C$

(II) Imaginary part will be zero

$$Z = x + iy$$

$$\omega[L - (CR^2 + \omega^2 L^2 C)] = 0$$

$$L = CR^2 + \omega^2 L^2 C$$

$$L - CR^2 = \omega^2 L^2 C$$

$$\frac{1}{LC} - \frac{R^2}{L^2} = \omega^2$$

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)^{1/2}$$

$$\omega = \omega_r$$

$$\omega_r = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)^{1/2} \quad \text{— (VIII)}$$

eq. (VIII) represent frequency @ resonance.

In resonance condition,

$$(Z)_{max.} = \frac{R^2 + \omega_r^2 L^2}{R} \quad \text{— (IX)}$$

$$(Y)_{min.} = \frac{R}{R^2 + \omega_r^2 L^2} \quad \text{— (X)}$$

Put value of eq. (VIII) in eq.(X)

$$(Y)_{min.} = \frac{R}{R^2 + \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)L^2}$$

$$(Y)_{min.} = \frac{R}{R^2 + \frac{L}{C} - R^2}$$

$$(Y)_{min.} = \frac{R}{L/C}$$

$$(Y)_{min.} = \frac{RC}{L}$$

$$(Z)_{max.} = \frac{1}{(Y)_{min}} = \frac{L}{RC} \quad \text{---(XI)}$$

Antiresonance :- Cond^n

In L-C-R parallel circuit, total current and e.m.f. will be in same phase and ~~for~~ required cond^n for same ~~resonance~~ phase is greater than zero and impedence also rare.

$$\frac{1}{LC} - \frac{R^2}{L^2} > 0$$

$$\frac{1}{LC} > \frac{R^2}{L^2}$$

$$L > R^2C \quad \text{(Antiresonance condition)}$$

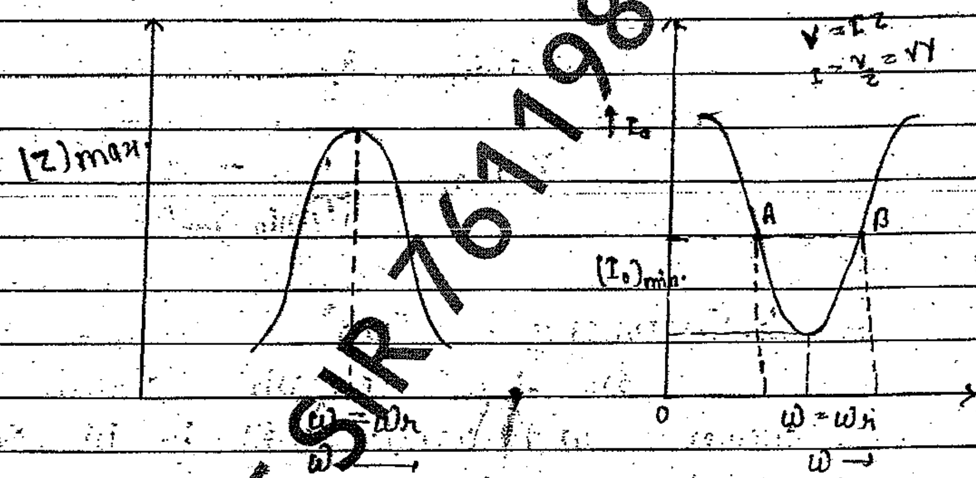
In L-C-R Series circuit, current obtained maximum @ resonance condition but in L-C-R parallel ckt, current obtained minimum, that's why, resonance is called Antiresonance.

$$(I)_{min} = \frac{E_0}{(Z)_{max}}$$

by eq-(XI)

$$\boxed{(I)_{min} = \frac{E_0 RC}{L}} \quad —12$$

Curve.



Half Power Point :-

Curve b/w Impedance and frequency in parallel L-C-R ckt. then we obtain maximum impedance and minimum admidance at resonance condⁿ.

Point at value of admidance is root two ($\sqrt{2}$) times of minimum admidance is called Half power point.

$$|Y| = Y_{min}\sqrt{2} \quad —(13)$$

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$$Y = \frac{1}{Z} = \frac{1}{R+j\omega L} + j\omega C \quad | \text{ Total impedence}$$

$$Y = \frac{1}{Z} = \frac{1 + j\omega C(R+j\omega L)}{R+j\omega L}$$

$$\cancel{Y = EZ}$$

$$\cancel{I = 1} \qquad \because j^2 = -1$$

$$Y = \frac{1 + j\omega CR - \omega^2 CL}{R+j\omega L}$$

$$Y = \frac{(1-\omega^2 CL) + j\omega CR}{R+j\omega L}$$

$$Z = x + iy$$

$$|Z| = \sqrt{(R+L)^2 +}$$

$$|Y| = \left[\frac{(1-\omega^2 CL)^2 + \omega^2 C^2 R^2}{R^2 + \omega^2 L^2}\right]^{1/2} \quad \text{---} (14)$$

by eq. (13) and (14)

$$\left[\frac{(1-\omega^2 CL)^2 + \omega^2 C^2 R^2}{R^2 + \omega^2 L^2}\right]^{1/2} = \frac{\sqrt{2}\, RC}{L}$$

Squaring both side

$$\frac{(1-\omega^2 CL)^2 + \omega^2 C^2 R^2}{R^2 + \omega^2 L^2} = \frac{2R^2 C^2}{L^2}$$

$$(1-\omega^2 CL)^2 + \omega^2 C^2 R^2 = \frac{2R^2 C^2 (R^2 + \omega^2 L^2)}{L^2}$$

$$(1-\omega^2 CL)^2 + \omega^2 C^2 R^2 = \frac{2R^4 C^2}{L^2} + \frac{2R^2 C^2 \omega^2 \cancel{L^2}}{\cancel{L^2}}$$

$$(1-\omega^2 CL)^2 - \omega^2 C^2 R^2 = \frac{2R^4 C^2}{L^2}$$

$$(1-\omega^2 CL)^2 = \frac{2R^4C^2}{L^2} + R^2C^2\omega^2$$

Resonance $cond^n$

$$L > R^2 C$$

$$L^2 > R^2 C$$

then $\frac{R^4C^2}{L^2} \approx 0$

$$(1-\omega^2 CL)^2 = R^2C^2\omega^2$$

For resistance, angular frequency

$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$(1-\omega^2 CL)^2 = R^2C^2\omega^2$$

$$\left(1-\omega_r^2 \cdot \frac{1}{\omega_r^2}\right) = R^2C^2\omega^2$$

Taking square root

$$1-\frac{\omega^2}{\omega_r^2} = \pm RC\omega$$

Taking (+)ive sign, $\omega = \omega_1$

$$1-\frac{\omega_1^2}{\omega_r^2} = +R\omega_1 C \quad \text{---(15)}$$

Taking (-)ive sign, $\omega = \omega_2$

$$1-\frac{\omega_2^2}{\omega_r^2} = -R\omega_2 C \quad \text{---(16)}$$

add eq. (15) and (16)

$$2 - \frac{1}{\omega_r^2}(\omega_1^2 + \omega_2^2) = RC(\omega_1 - \omega_2)$$

$$2 - \frac{1}{\omega_r^2}(\omega_2^2 + \omega_1^2) = 0$$

subtract eq. (15) from (16)

$$1 - \frac{\omega_1^2}{\omega_r^2} - 1 + \frac{\omega_2^2}{\omega_r^2} = RC(\omega_1 + \omega_2)$$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_r^2} = RC(\omega_1 + \omega_2)$$

$$\frac{\omega_2 - \omega_1}{\omega_r^2} = RC$$

$$\boxed{\omega_2 - \omega_1 = \omega_r^2 RC} \quad -(17)$$

Band width

Q. Write the difference b/w L-C-R parallel and L-C-R Series circuit.

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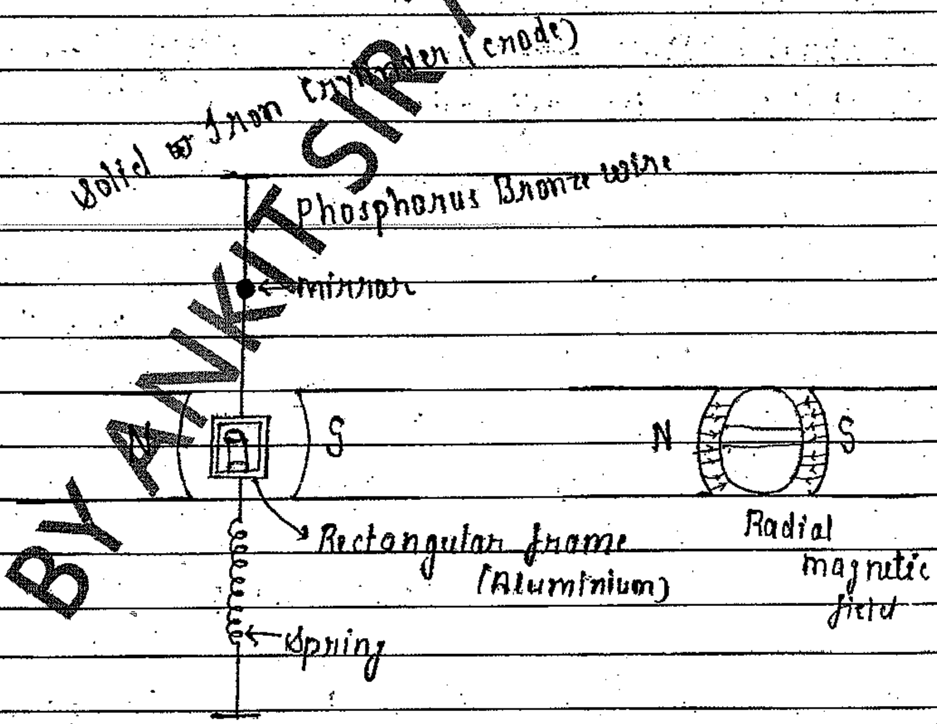
Ballistic Galvanometer

Ballistic galvanometer consist of a rectangular coil of wire of many turns and suspended by phosphor bronze wire between two strong permanent magnetic poles when charge is passed through the conducting coil, it gets rotated from a moment as a result Torasil couple is produced in the Suspension wire opposite to the rotation. This deflecting couple is produced in the wire is called Restoring Couple.

If Toransil rigidity of Suspension wire is C



Then torque developed due to twisting of wire by an angle θ will be $-C\theta$.

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where $C = \frac{\pi \eta r^4}{2l}$

r = radius
l = length
η = rigidity constant

Here two torque also work

(i) Damping of the motion of coil, a open circuit due to couple is produced by the mechanical force such that here resistance force and viscous force. The torque of this couple acting on coil is directly proportional to angular velocity of coil.

Hence

Mechanical damping Torque $\tau \propto \frac{d\theta}{dt} \propto \omega$

$$= -\gamma \frac{d\theta}{dt} \quad -(1)$$

γ = damping constant

(ii) If current is flowing in coil, then a torque is produced. This torque is known as electromagnetic torque.

$$\text{electromagnetic torque} = \frac{m}{R}\frac{d\theta}{dt} \quad -(2)$$

where R = resistance of coil
m = constant

where m depends on area of coil, magnetic field of coil and no. of turns of coil.

Resultant Torque represent by following types :-

$$\tau = -C\theta - r\frac{d\theta}{dt} - \frac{m}{R}\frac{d\theta}{dt}$$

$$\because \tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta - \frac{r}{I}\frac{d\theta}{dt} - \frac{m}{IR}\frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\left(\frac{r}{I} + \frac{m}{IR}\right) + \frac{C}{I}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{1}{I}\frac{d\theta}{dt}\left(r + \frac{m}{R}\right) + \frac{C}{I}\theta = 0 \quad \text{—(III)}$$

let, $\frac{1}{I}\left(r + \frac{m}{R}\right) = 2K$

$$\frac{C}{I} = \omega_0^2$$

$$\frac{d^2\theta}{dt^2} + 2K\frac{d\theta}{dt} + \omega_0^2\theta = 0 \quad \text{—(IV)}$$

eq. (IV) represent differential eq. of Galvastic Galvanometer.

Solution of this eq. is represent by following type

$$\theta = e^{-Kt}\left[Ae^{\sqrt{K^2-\omega^2}\,t} + Be^{-\sqrt{K^2-\omega^2}\,t}\right] \quad \text{—(V)}$$

Ini. @ $t=0,\ \theta=0$

$$0 = A + B$$

$$A = -B \quad \text{—(VI)}$$

by eq. (V) and (VI)

$$\theta = Ae^{-Kt}\left[e^{\sqrt{K^2-\omega^2}\,t} - e^{-\sqrt{K^2-\omega^2}\,t}\right] \quad \text{—(VII)}$$

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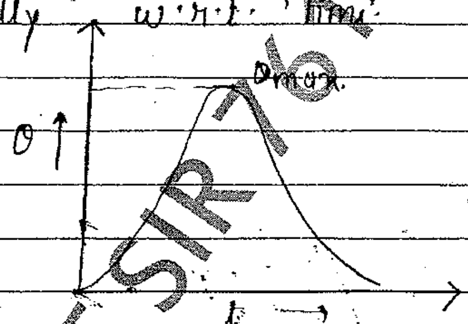
Case I High damping case ($n >>> \omega_0$)
means

$$\frac{1}{2I}\left(r + \frac{m}{R}\right) >> \sqrt{\frac{c}{I}}$$

on putting $\omega_0^2 \approx 0$ in eq.-(VII)

$$\theta = Ae^{-nt}\left[e^{nt} - e^{-nt}\right]$$
$$\theta = A\left[1 - e^{-2nt}\right] \quad \text{—(VIII)}$$

Here, first of time is increased and θ is also increased but at some time, θ is decreased exponentially w.r.t. 'time'.



Case II Critical Damping condition :- ($n \approx \omega_0$)
means

$$\frac{1}{2I}\left(r + \frac{m}{R}\right) \approx \sqrt{\frac{c}{I}}$$

by eq. (7)

In this case coil is deflected upto maximum value and then decrease to zero without oscillating in minimum time.

Case III low damping condition ($n << \omega_0$)

$$\frac{1}{2I}\left(r + \frac{m}{R}\right) << \sqrt{\frac{c}{I}}$$

by eq. (vii)

$$\sqrt{\mu^2 - \omega_0^2} = \sqrt{-1}\,\sqrt{\omega_0^2 - \mu^2} = i\omega$$

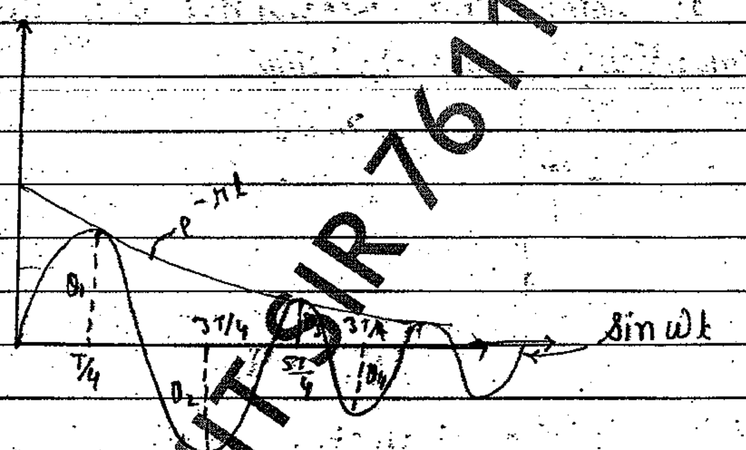
$$\theta = Ae^{-\mu t}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

$$\because\ e^{i\omega t} + e^{-i\omega t} = 2i\sin\omega t$$

$$\theta = 2iAe^{-\mu t}\sin\omega t$$

let $2iA = \theta_0$

$$\theta = \theta_0 e^{-\mu t}\sin\omega t \quad \text{—— (VIII)}$$



Decrement Coefficient :-

In eq. (VIII) @ $t = \frac{T}{4}$

then $\theta = \theta_1$

$$\theta_1 = \theta_0 e^{-\mu T/4} \cdot \sin\omega \frac{T}{4} \qquad \because\ \frac{\omega T}{4} = \frac{2\pi}{T}\cdot\frac{T}{4} = \frac{\pi}{2}$$

$$\theta_1 = \theta_0 e^{-\mu T/4} \quad \text{—— (IX)}$$

Similarly

$$\theta_2 = \theta_0 e^{-\lambda 3T/4} \quad \text{---(X)}$$

$$\theta_3 = \theta_0 e^{-\lambda 5T/4} \quad \text{---(XI)}$$

$$\theta_4 = \theta_0 e^{-\lambda 7T/4} \quad \text{---(XII)}$$

From eq. (XI) and (X)

$$\frac{\theta_1}{\theta_2} = \frac{\cancel{\theta_0} e^{-\lambda T/4}}{\cancel{\theta_0} e^{-\lambda 3T/4}}$$

$$\frac{\theta_1}{\theta_2} = e^{\lambda T/2} \quad \text{---(XIII)}$$

$$\tau = \frac{1}{2\lambda}$$

$$\lambda = \frac{1}{2\tau}$$

$$\frac{\theta_1}{\theta_2} = e^{\frac{T}{4\tau}} \quad \text{---(XIV)}$$

by eq. (X) and (XI)

$$\frac{\theta_2}{\theta_3} = \frac{\theta_0 e^{-\lambda 3T/4}}{\theta_0 e^{-\lambda 5T/4}}$$

$$\frac{\theta_2}{\theta_3} = e^{\lambda T/2}$$

$$\frac{\theta_2}{\theta_3} = e^{T/4\tau} \quad \text{---(XV)}$$

Similarly,

$$\frac{\theta_3}{\theta_4} = e^{T/4\tau} \quad \text{---(XVI)}$$

Hence we can say that,

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} \text{ ------ } = e^{T/4\tau} = d \text{ (decrement coefficient)}$$

logarithm log is called logarithm decrement it is generally represent by λ

$$\log_e d = \log_e e^{T/4\tau}$$

$$\boxed{\lambda = \frac{T}{4\tau} = \log_e d}$$ logarithm decrement coefficient

Taking ratio of angular displacement

$$\frac{\theta_1}{\theta_n} = \frac{\theta_1}{\theta_2}\,\frac{\theta_2}{\theta_3}\,\frac{\theta_3}{\theta_4} \cdots \frac{\theta_{n-1}}{\theta_n}$$

$$\frac{\theta_1}{\theta_n} = e^{(n-1)T/4\tau} = e^{(n-1)\lambda} \quad \text{——(XVIII)}$$

taking $\log_{10}$

$$\lambda = \frac{2.303}{n-1}\log_{10}\frac{\theta_1}{\theta_n} \quad \text{——(XIX)}$$

Amplitude of charge with Help of Ballistic Galvanometer

In the case of low damping, current I flow in coil and angular velocity ω. so angular momentum is represent by J.

$$J = nAB\int_0^t i\,dt \qquad \because \int_0^t i\,dt = q \text{ (charge)}$$

$$J = nABq \quad \text{---(I)}$$

where n = no. of turns
A = Area of coil
B = magnetic field
q = charge

$$\because \quad J = I\omega'$$

$$I\omega' = nABq \quad \text{---(A)}$$

~~$E_r = I\omega'^2$~~

If rotation kinetic energy of coil is E_r
then

$$E_r = \frac{1}{2} I\omega'^2 \quad \text{---(II)}$$

If damping is absent then this energy will be utilised in doing work against a Torsional torque.

let maximum angular displacement $= \theta_0$
then

$$\text{work } W = \int_0^{\theta_0} |\tau| d\theta$$

$$W = \int_0^{\theta_0} C\theta d\theta \qquad \because |\tau| = C\theta$$

$$W = \left[\frac{C\theta^2}{2}\right]_0^{\theta_0}$$

$$W = \frac{C\theta_0^2}{2}$$

$$W = E_r$$

$$\frac{C\theta_0^2}{\cancel{2}} = \frac{1}{\cancel{2}} I\omega'^2$$

$$\omega'^2 = \frac{C\theta_0^2}{I} \quad \text{---(3)}$$

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by eq. A

$$\omega' = \frac{nABq}{I}$$

from eq. (3)

$$\omega' = \sqrt{\frac{C}{I}}\,\theta_0$$

$$\frac{nABq}{I} = \sqrt{\frac{C}{I}}\,\theta_0$$

$$q = \frac{\sqrt{IC}\,\theta_0}{nAB} \quad (4)$$

We know that

$$\omega_0^2 = \frac{C}{I}$$

$$\omega_0 = \sqrt{\frac{C}{I}}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\frac{2\pi}{T} = \sqrt{\frac{C}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{C}}$$

by eq. (IV)

$$q = \frac{\sqrt{IC}}{C}\,\frac{\theta_0 C}{nAB} \quad \text{[multiply N and D by C]}$$

$$q = \sqrt{\frac{I}{C}}\,\frac{\theta_0 C}{nAB} \longrightarrow (VI)$$

angular displacement in low damping condⁿ

$$\theta = \theta_0 e^{-\lambda t}\sin\omega t$$

$$@\ t = \frac{T}{4},\quad \theta = \theta_1$$

$$\theta_1 = \theta_0 e^{-\lambda T/4} \sin\left(\frac{2\pi}{T} \cdot \frac{T}{4}\right)$$

$$\theta_1 = \theta_0 e^{-\lambda T/4}$$

$$\lambda = \frac{1}{2\tau}$$

$$\theta_1 = \theta_0 e^{-T/8\tau}$$

$\because \frac{T}{4\tau} = \lambda$ (Decrement constant)

$$\theta_1 = \theta_0 e^{-\lambda/2}$$

$$\frac{\theta_0}{\theta_1} = e^{\lambda/2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\frac{\theta_0}{\theta_1} = 1 + \frac{\lambda}{2} + \frac{\lambda^2}{8} + \cdots$$

Taking higher term of λ negligible

$$\frac{\theta_0}{\theta_1} = 1 + \frac{\lambda}{2}$$

$$\theta_0 = \theta_1\left(1 + \frac{\lambda}{2}\right) \longrightarrow (7)$$

put value of eq. (5) and (7) in eq. (vi)

$$q = \frac{T}{2\pi} \cdot \frac{c}{nAB}\, \theta_1\left(1 + \frac{\lambda}{2}\right)$$

$\because K = \frac{T}{2\pi} \cdot \frac{c}{nAB}$ (Galvanic constant) dimension c

$$\boxed{q = K\theta_1\left(1 + \frac{\lambda}{2}\right)}$$

Coupled Oscillator

If motion ~~dey~~ of a oscillator depends on motion of other oscillator, that type ~~mo~~ oscillator is called coupled oscillator.

Here oscillator exchange energy with each other.

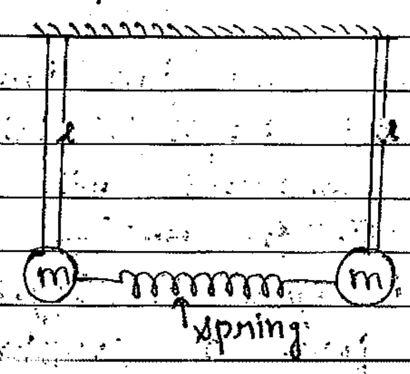
If motion of two oscillator depend on each other then it is called Two Coupled Oscillator.

(I) Mechanical Two Coupled Oscillator :-

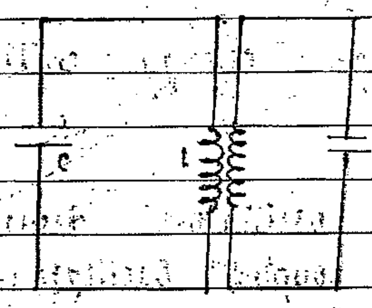
In this, two same mass bobs and same length are suspended and both are connected with Spring.

If one bob oscillates then other bob also ~~oscillator~~ oscillates. This type is called Mechanical two coupled Oscillator.

~~(II) Electric Two coupled Oscillator :-~~



(II) Electric Two coupled Oscillator :-



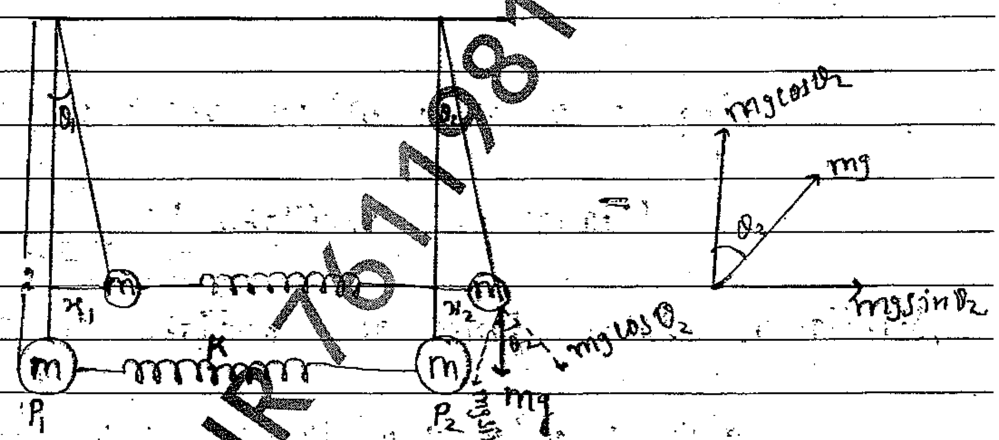
Atomic Coupled Oscillator :-

v.v. Imp.

Motion of Two Coupled Simple Harmonic Oscillator :-

let us consider there are two identical pendulum each of mass 'm' and length 'l' bob of these pendulum are connected with spring of force constant K. In equi^m, the distance b/w bobs is equal to length of spring. This arrangement form a coupled oscillator. If bobs are displaced slightly from its equil^m.

position and then released both pendulums ~~whose~~ would start oscillating. Here angular displacement θ_1 and θ_2 respectively and linear displacement x_1 and x_2 respectively.



Total force acting on oscillator $P_2 = F_2$

$$F_2 = -mg\sin\theta_2 + \text{Restoring force}$$
$$F_2 = -mg\sin\theta_2 - K(\text{elongation})$$
$$F_2 = -mg\sin\theta_2 - K(x_2 - x_1)$$

$\because$ acc. to Newton

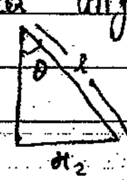
$$m\frac{d^2x}{dt^2} = F$$

So,

$$m\frac{d^2x_2}{dt^2} + mg\sin\theta_2 + K(x_2 - x_1) = 0 \quad \text{—(4)}$$

$$\frac{d^2x_2}{dt^2} + g\sin\theta_2 + \frac{K}{m}(x_2 - x_1) = 0 \quad \text{—(1)}$$

for small angle, $\sin\theta \simeq \tan\theta \simeq \theta$



$$\tan\theta = \frac{x_2}{l}$$

$$\sin\theta_2 = \frac{x_2}{l}$$

$$\frac{d^2x_2}{dt^2} + \frac{g}{l}x_2 + \frac{K}{m}(x_2 - x_1) = 0$$

let $\because \frac{g}{l} = \omega_0^2$;

$\frac{K}{m} = \omega_c^2$ (coupled frequency)

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 + \omega_c^2(x_2 - x_1) = 0 \quad \text{—(II)}$$

(Force on oscillator P_2)

Similarly, total force on oscillator $P_1 = F_1$

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 + \omega_c^2(x_1 - x_2) = 0 \quad \text{—(III)}$$

general co-ordinate for two coupled oscillator which is represent by following type

let $X = x_1 + x_2$ —(IV)

$Y = x_1 - x_2$ —(V)

~~eq. (4) and (5)~~

on adding eq. (II) and (III)

$$\frac{d^2}{dt^2}(x_2 + x_1) + \omega_0^2(x_1 + x_2) + \omega_c^2(x_2 - x_1 + x_1 - x_2) = 0$$

$$\frac{d^2X}{dt^2} + \omega_0^2 X + 0 = 0$$

$$\frac{d^2X}{dt^2} + \omega_0^2 X = 0 \quad \text{—(VI)}$$

let $\omega_0^2 = \omega_1^2$ or $\omega_0 = \omega_1$

$$\frac{d^2x}{dt^2} + \omega_1^2 x = 0 \quad \text{—— (VI)}$$

subtract eq. (II) from eq. (III)

$$\frac{d^2}{dt^2}(x_1 - x_2) + \omega_0^2 (x_1 - x_2) + \omega_c^2 (x_1 - x_2 - x_2 + x_1) = 0$$

$$\frac{d^2y}{dt^2} + \omega_0^2 y + 2\omega_c^2 (y) = 0$$

$$\frac{d^2y}{dt^2} + y(\omega_0^2 + 2\omega_c^2) = 0$$

let $\omega_2^2 = \omega_0^2 + 2\omega_c^2$

$$\omega_2 = (\omega_0^2 + 2\omega_c^2)^{1/2}$$

$$\frac{d^2y}{dt^2} + y\omega_2^2 = 0 \quad \text{—— (VII)}$$

Here eq. (VI) and (VII) are known as differential equation of 2 coupled oscillator.

Normal Vibrational Mode

Case I If $y = 0$, then $x_1 - x_2 = 0$ means $x_1 = x_2$

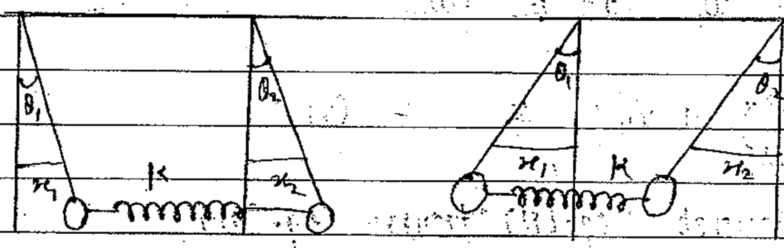
then ~~from eq~~ from eq. (VI) and (VII)

$$\frac{d^2x}{dt^2} + \omega_1^2 x = 0$$

$$\omega_1 = \omega_0$$

This is known as first Vibrational mode.

In this condn, displacement of both bobs are equal from equilm position and bobs are displaced in same dirxn.



Here $x_1 = x_2$

Case II If $X = 0$, then $x_1 + x_2 = 0$

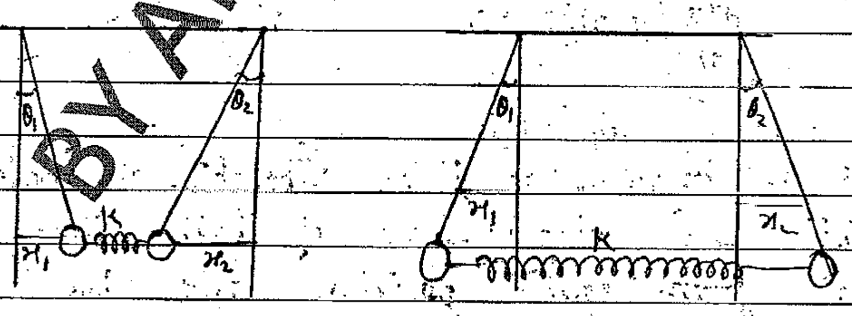
$$x_1 = -x_2$$

by eq. (6) and (7)

$$\frac{d^2Y}{dt^2} + \omega_2^2 Y = 0$$

(where $\omega_2 = (\omega_0^2 + 2\omega_c^2)^{1/2}$)

This is known as 2^{nd} vibrational mode. Here displacement of bobs from equilibrium position are opposite. But displacement is equal in magnitude.



$x_1 = -x_2$ $x_1 = -x_2$

Motion In the Mixed Mode and Transient Behaviour (energy exchange)

Displacement from mean position x_1, x_2 respectively in the two coupled oscillator so related differential eq^n from displacement is represent as follows.

$$\frac{d^2x}{dt^2} + \omega_1^2 x = 0 \quad \text{---(I)}$$

$\omega_1 = \omega_0$ (First Normal Vibrational mode)

$$\frac{d^2y}{dt^2} + \omega_2^2 y = 0 \quad \text{---(II)}$$

$$\omega_2^2 = \omega_0^2 + 2\omega_c^2$$

$\omega_2 = (\omega_0^2 + 2\omega_c^2)^{1/2}$ (2nd Normal Vibrational mode)

If eq. (1) and (2) are superimposed to each other then we get mixed normal modes.

Solution of (1) and (2) is represented as given below.

$$X = x_0 \cos(\omega_1 t + \phi_1) \quad \text{---(III)}$$
$$Y = Y_0 \cos(\omega_2 t + \phi_2) \quad \text{---(IV)}$$

Here,

ϕ_1, ϕ_2 = Initial phase difference

x_0, Y_0 = amplitude

let $X_0 = Y_0 = 2a$

$\phi_1 = \phi_2 = 0$

from eq. (III) and (IV)

$$X = 2a \cos(\omega_1 t) \quad \text{---(V)}$$
$$Y = 2a \cos(\omega_2 t) \quad \text{---(VI)}$$

We know that

$$X = x_1 + x_2 \quad -(VII)$$

$$Y = x_1 - x_2 \quad -(VIII)$$

add eq. (VII) and (VIII)

$$X + Y = 2x_1$$

$$x_1 = \frac{X+Y}{2} \quad -(IX)$$

subtract eq. (VIII) from eq. (VII)

$$\frac{X-Y}{2} = x_2 \quad (X)$$

put value of X and Y in eq. (IX)

$$x_1 = \frac{2a[\cos\omega_1 t + \cos\omega_2 t]}{2}$$

$$x_1 = a[\cos\omega_1 t + \cos\omega_2 t]$$

$$\cos C + \cos D = 2\cos\frac{C-D}{2}\cos\frac{C+D}{2}$$

$$x_1 = 2a\cos\frac{(\omega_1 - \omega_2)}{2}t \cos\frac{(\omega_1 + \omega_2)}{2}t$$

$\because \omega_2 > \omega_1$

$$x_1 = 2a\cos\frac{(\omega_2 - \omega_1)}{2}t \cos\frac{(\omega_1 + \omega_2)}{2}t \quad -(11)$$

put value of X and Y in eq. (X)

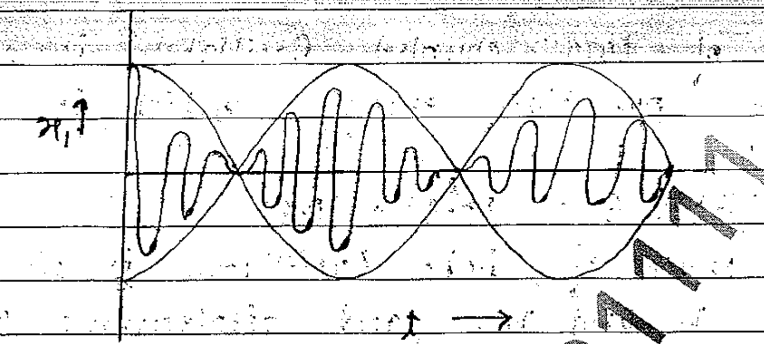
$$x_2 = \frac{2a[\cos\omega_1 t - \cos\omega_2 t]}{2}$$

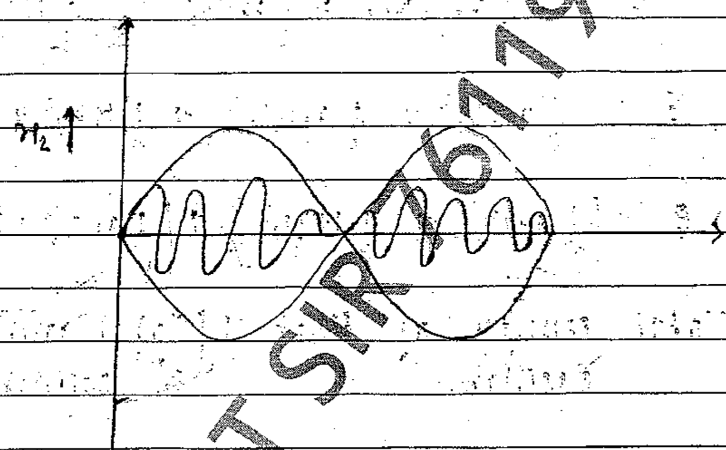
$$x_2 = 2a\sin\frac{(\omega_2 - \omega_1)}{2}t \sin\frac{(\omega_1 + \omega_2)}{2}t \quad -(12)$$

$2a\cos\frac{(\omega_2 - \omega_1)}{2}t$ and $2a\sin\frac{(\omega_2 - \omega_1)}{2}t$

(View)

Graphical representation of eq.(xi) and (xii)





Here phase difference b/w both amplitude is $\frac{\pi}{2}$ so we can say that

(i) When amplitude of first oscillator is maximum in this condition amplitude of second oscillator is min. Similarly if amplitude of second oscillator is max^m then amplitude of first oscillator is min.

so we can say they exchange energy.

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Total Energy of 2 Coupled Oscillator

Total energy of two coupled Oscillator is sum of kinetic and potential energy which is represent following as :-

$$\text{Total } E = (K.E) + (P.E)$$

Here mass of bobs (oscillator) is m and velocity is v_1 and v_2 and displacement x_1 and x_2.

$$E = \frac{1}{2}m(v_1)^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}K(x_1 - x_2)^2$$

$$E = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}Kx_1^2 + \frac{1}{2}Kx_2^2 - Kx_1x_2$$

$$E = \left[\frac{1}{2}mv_1^2 + \frac{1}{2}Kx_1^2\right] + \left[\frac{1}{2}mv_2^2 + \frac{1}{2}Kx_2^2\right] - Kx_1x_2$$

E = (Total energy of first Oscillator) + [Total energy of 2^{nd} oscillator] + pairing energy

Here pairing energy term represent energy exchange b/w oscillator

(I) If pairing energy term is not present then total energy of each oscillator be remain constant

(II) If pairing energy term is present then also total energy remain constant.

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Energy Exchange & Couplining Effect

Displacement from mean position are x_1 and x_2 in two coupled oscillator.

$$x_1 = 2a\cos\left(\frac{\omega_2-\omega_1}{2}\right)t \, \cos\left(\frac{\omega_1+\omega_2}{2}\right)t \quad \text{—(I)}$$

$$x_2 = 2a\sin\left(\frac{\omega_2-\omega_1}{2}\right)t \, \sin\left(\frac{\omega_1+\omega_2}{2}\right)t \quad \text{—(II)}$$

let $2a = A_m$

$$\frac{\omega_1+\omega_2}{2} = \omega$$

$$\frac{\omega_2-\omega_1}{2} = \omega_m$$

by eq.(I) and eq.(II)

$$x_1 = A_m \cos\omega_m t \cos\omega t \quad \text{—(III)}$$
$$x_2 = A_m \sin\omega_m t \sin\omega t \quad \text{—(IV)}$$

let $A_m \cos\omega_m t = a_1$
$A_m \sin\omega_m t = a_2$

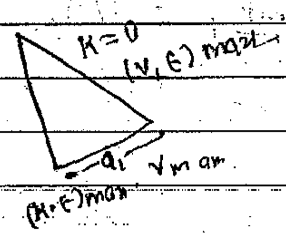
by (III) and (IV)

$$x_1 = a_1 \cos\omega t \quad \text{—(V)}$$
$$x_2 = a_2 \sin\omega t \quad \text{—(VI)}$$

let energy of oscillators E_1 and E_2

Total energy $= E$

Note:



$$E_1 = \frac{1}{2} K a_1^2 \quad \text{---(VII)}$$

$\because K = m\omega^2$ (SHM)

$$E_1 = \frac{1}{2} m\omega^2 a_1^2$$

$$E_1 = \frac{1}{2} m\omega^2 A_m^2 \cos^2\omega_m t \quad \text{---(VIII)}$$

Similarly

$$E_2 = \frac{1}{2} K a_2^2$$

$$E_2 = \frac{1}{2} m\omega^2 a_2^2$$

$$E_2 = \frac{1}{2} m\omega^2 A_m^2 \sin^2\omega_m t \quad \text{---(IX)}$$

Now

$$E = E_1 + E_2$$

$$E = \frac{1}{2} m\omega^2 [A_m^2] [\cos^2\omega_m t + \sin^2\omega_m t]$$

$$E = \frac{1}{2} m\omega^2 A_m^2 \quad \text{---(X)}$$

put value of eq.(X) in (VIII) and (IX)

$$E_1 = E\cos^2\omega_m t$$

$$E_1 = E\left[\frac{1+\cos 2\omega_m t}{2}\right] \quad \text{---(XI)}$$

Similarly

$$E_2 = E\sin^2\omega_m t$$

$$E_2 = E\left[\frac{1-\cos 2\omega_m t}{2}\right] \quad \text{---(XII)}$$

Case I, If $t=0$, then by eq. (XI)

$$\epsilon_1 = \frac{\epsilon}{2}\left[1+\cos 2\omega_m t\right]$$

$$\boxed{\epsilon_1 = \epsilon}$$

by eq. (XII)

$$\epsilon_2 = \frac{\epsilon}{2}\left[1-\cos 2\omega_m t\right]$$

$$\boxed{\epsilon_2 = 0}$$

Case II

@ $t = \frac{T}{4}$,

by eq. (XI) and (XII)

$$\epsilon_1 = \frac{\epsilon}{2}\left[1+\cos 2\omega_m \frac{T}{4}\right]$$

$$\epsilon_1 = \frac{\epsilon}{2}\left[1+\cos \cdot \frac{2\pi}{T} \cdot \frac{T}{2}\right]$$

$$\boxed{\epsilon_1 = 0}$$

$$\epsilon_2 = \frac{\epsilon}{2}\left[1-\cos 2\omega_m \frac{T}{4}\right]$$

$$\boxed{\epsilon_2 = \epsilon}$$

Hence we can say that oscillator exchange energy.