Uniform estimates for fourier restriction to complex polynomial curves in $\mathbb{R}$

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Abstract

This is the first instance of a uniform optimal estimate (not including the endpoint) for families of surfaces of dimension¿1. On the other hand, they are still on the specific $d|cod$, which seems to be easier.

1 Introduction

Theorem 1. *For each N, d and (p, q) satisfying:*

$$p' = \frac{d(d+1)}{2} q, \quad q > \frac{d^2 + d + 2}{d^2 + d} \tag{1}$$

there is a constant $C_{N,d,p}$ such that for all polynomials $\gamma : \mathbb{C} \to \mathbb{C}^d$ of degree up to N we have:

$$\|\hat{f}\|_{L^q(d\lambda_\gamma)} \leq C_{N,d,p} \|f\|_{L^q(dx)} \tag{2}$$

for all Schwartz functions f.

We will instead prove the dual problem, showing boundedness of the adjoint (extension) operator $\mathcal{E}_\gamma$.

2 Uniform Local restriction

Definition 1. *For a polynomial $\gamma : \mathbb{C} \to \mathbb{C}^d$ we define*

$$L_\gamma(z) := \det(\gamma'(z), \gamma^{(2)}(z), \dots, \gamma^d(z))$$

$$J_\gamma(z_1, \dots, z_d) = \det(\gamma'(z_1), \dots, \gamma'(z_d))$$

In this section we will prove the following result:

Theorem 2. *Fix $d > 2$, N, and a feasible (p, q) pair. For every triangular set $S \subset \mathbb{C}^d$ and every degree N polynomial $\gamma : \mathbb{C} \to \mathbb{C}^d$ satisfying:*

$$0 < C_1 \leq \Re L_\gamma(z) \tag{3}$$

$$|L_\gamma(z)| \leq C_2 < \infty \tag{4}$$

in S we have the extension estimate:

$$\|\mathcal{E}_\gamma\|_{L^q} \leq C_{d,N,\frac{C_1}{C_2}} \|f\|_{L^p(d\lambda_\gamma)} \tag{5}$$

Without loss of generality, we can split in a controlled number of triangles, and assume $C_1 = \frac{1}{2}$, $C_2 = 2$.

3 $\epsilon-$aligned sets, $\epsilon-$aligned functions

The main idea of the paper is to uniformly partition

Definition 2 ($\epsilon-$aligned set)**.** *We say that a list $S = (z_1, ..z_n)$ is $\epsilon-$aligned in a direction $s \in C \setminus 0$ if $\arg(s^{-1}(z_i - z_j)) \in [-\pi + \epsilon, \pi - \epsilon]$ whenever*[1] *$i < j$. (or, equivalently, whenever $j = i + 1$). Given a list s let $A_\epsilon(S)$ be the set of s such that S is ϵ-aligned with respect to s*

Definition 3. *() Given two lists $S = (z_1, \ldots z_n)$, $T = (w_1, \ldots w_{n+1})$ we say $T > S$ if w_i is a real convex combination of $z_i, z_i + 1$. We extend the relationship by transitivity to arbitrary tuples.*

Now, it is an easy to see that that $S > T$ implies that $A_\epsilon(S) \subseteq A_\epsilon(T)$.

Decomposition procedure D1'

There exists a decomposition of a rational function $m(z)$onto $O_{\epsilon,N}(1)$ convex sets C_i, such that there exists $w_i, c_i \in \mathbb{C}$, $n_i \in \mathbb{N}$, such that on each c_i:

$$\frac{p(z)}{w_i(z - c_i)^n} \in B(1, \epsilon) \tag{6}$$

Using this lemma,

Partition for injectivity:

Let $P(x) = (x - x_i)_i^k$. Do ϵ partitions for the jacobian and all the involved polynomials and their derivatives. Assume you are in one of the $\epsilon-$regions where the Jacobian does not vanish first. You can write

4 Uncoordinated polynomials

In this section we will study functions that $\epsilon-$look like polynomials (in the sense above), and whose derivatives also $\epsilon-$look like polynomials.

Note that $P =_\epsilon Q$ in D does not imply $P' =_\epsilon Q$ in D

Want to solve: $\sigma_M(\vec{x}) = \sum M(x_i) = 0$

$$d\Sigma_M = \frac{1}{K!} V(x_i)$$

[1] we use the convention $\arg(0) = 0$