A geometric lemma for complex polynomial curves with applications in harmonic analysis.

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Abstract

ABSTRACT TO BE DONE

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Check what's the most natural way to write a ball of radius r

1 Introduction

Multiple results in harmonic analysis involving integrals of functions over curves [xx,yy] depend strongly on the non-vanishing of the torsion of curves. A program initiated by XXX aims to extend these results to the degenerate case where the torsion vanishes at a finite number of points, by using the affine arc-length as a measure. As a model case, multiple results have been proven in which the implicit constants do not depend on the polynomial curve, but only on its degree and the dimension of the ambient place.

A key part of this developments have been based on a geometric lemma by Dendrinos and Wright [DW14] that provides very precise bounds to the torsion in the case of real polynomial curves. At an intuitive level, the geometric Lemma states that a polynomial curve can be split into a finite (depending only on the dimension and the degree of the polynomial) number of open sets so that, in each open set the curve behaves like a *model curve* that is easier to study. This paper extends the result for complex variables using compactness techniques.

As a consequence this paper extends two theorems by Stovall to the complex case. First proving the optimal restriction theorem [Sto16] for a complex polynomial curves $\gamma : \mathbb{C} \to \mathbb{C}^d$ under the isomorphism $\mathbb{C} \sim \mathbb{R}^2$. We show, following the proof in [Sto16] that the restriction is uniform over all complex curves, a conjecture by Bak and Ham. [BakHam]

Using the techniques by Stovall and Christ [Sto18,Chr17] we [hopefully will] prove the optimal range for the convolution operator for polynomial curves, extending the result of [Chung, Bak] to the full polynomial case, and recovering a full analogue of theorem 1 in [Sto18].

1.1 The affine measure on a complex curve

Here we will explain the $\Lambda^{(d)}$ notation

In this article we will consider the extension/restriction operators with respect to the complex affine *arclength* measure. Inspired by the real affine arclength measure, it is defined for a complex curve $\gamma(z) : \mathbb{C} \to \mathbb{C}^d$ as the weighted push-forward

$$d\lambda_\gamma = \frac{1}{d!}\gamma_* \left(\det[\gamma'(z), \gamma''(z), \dots, \gamma^{(d)}(z)]^{\frac{4}{d^2+d}} |dz| \right) \tag{1}$$

this measure has been considered in the literature [][] for two of its properties:

- The measure λ is co-variant both under re-parametrization of z ($\gamma \circ \phi(z)$ represents the same measure for ϕ a a conformal map) and affine maps applied on $\mathbb{C}^d$ (that is, if $L \in GL(\mathbb{C}; d)$, then $d\lambda_{L\circ\gamma} = L_* d\lambda_\gamma$)
- The measure λ vanishes at the points where the torsion of γ vanishes. The relevance of this property comes from the fact that the restriction theorem in the full range fails for the arc-length measure at neighbored of a point where the torsion of γ vanishes.

moreover, [] shows that this measure is optimal, in the sense that any measure for which theorem ZZZ holds must be absolutely continuous with respect to $d\lambda$.

To make things more notationally convenient further down, we will not only consider the affine measure, but a set of related differential forms, for $0 < k \leq d$:

$$\Lambda_\gamma^{(k)}(z) := \gamma(z) \wedge \dots \wedge \gamma^{(d)}(z) \tag{2}$$

$$\Lambda_\gamma(z_1, \dots, z_k) := \gamma(z_1) \wedge \dots \wedge \gamma(z_k) \tag{3}$$

Note that Λ_γ is a function with variable arity (which will be clear by the context) that has an element of $\mathbb{C}^k$ as an input and returns a $k-$form as an output.

$$v(z_1, \dots z_k) := \prod_{i<j} (z_i - z_j) \tag{4}$$

and, to be consistent with the previous notation in XXXX, we will define

$$J_\gamma(z) := \frac{1}{d!}\Lambda_\gamma^{(d)}(z) = \frac{1}{d!}\det[\gamma'(z), \gamma''(z), \ldots, \gamma^{(d)}(z)] \tag{5}$$

and $L_\gamma = J_\gamma^{4/d^2+d}$

1.2 Restriction Problem

In section XXX we will prove the following theorem:

Theorem 1. *For each N, d and (p, q) satisfying:*

$$p' = \frac{d(d+1)}{2}q, \quad q > \frac{d^2+d+2}{d^2+d} \tag{6}$$

there is a constant $C_{N,d,p}$ such that for all polynomials $\gamma : \mathbb{C} \to \mathbb{C}^d$ of degree up to N we have:

$$\|\hat{f}\|_{L^q(d\lambda_\gamma)} \le C_{N,d,p}\|f\|_{L^p(dx)} \tag{7}$$

for all Schwartz functions f, where the Fourier transform is the $\mathbb{R}^{2n}$-dimensional Fourier transform.

The real polynomial counterpart to this theorem was proven originally in [XXX Stovall], and [Bak, Ham] provides a partial answer to the theorem above for particular complex curves, and shows the optimality of the measure λ, that is, that any other measure supported on γ for which Theorem XXX holds in the given range must be absolutely continuous with respect to γ. This also follows from the fact [CCCCC] that the measure λ corresponds to the measure described in [Dury's paper] for γ.

This paper follows the proof in [XXX Stovall]. The main challenge is finding a complex substitute for Lemma XXX (a modification of Theorem YYY). Once this matter is resolved, section XXX follows [XXXX Stovall] closely, with small modifications whenever necessary.

A more amenable version of the problem above is the extension problem, instead of bounding the restriction operator defined above, we bound the dual extension operator $\mathcal{E}_\gamma : L^p(d\lambda_\gamma) \to L^q(\mathbb{R}^n)$ defined as

$$\mathcal{E}_\gamma(f) = \mathcal{F}^{-1}(f d\lambda_\gamma). \tag{8}$$

Theorem YYYY now becomes

Theorem 1 (Dual version)**.** *For each N, d and (p, q) satisfying:*

$$p' = \frac{d(d+1)}{2}q, \quad q > \frac{d^2+d+2}{d^2+d} \tag{9}$$

there is a constant $C_{N,d,p}$ such that for all polynomials $\gamma : \mathbb{C} \to \mathbb{C}^d$ of degree up to N we have:

$$\|f\|_{L^{p'}(dx)} \le C_{N,d,p}\|f\|_{L^{q'}(d\lambda_\gamma)} \tag{10}$$

for all Schwartz functions f, where the Fourier transform is the $\mathbb{R}^{2n}$-dimensional Fourier transform.

1.3 Convolution against complex curves

1.4 Outline of the work

In both Theorem XXXX and Theorem YYY the strategy to show uniformity of the associated operator norm goes as follows:

1. The complex plane is partitioned into sets $\mathbb{C} = \bigsqcup_{i=0}^{N(n,d)} U_i$, so that for each U_i there is a moment curve γ_i so that $\gamma_i \sim_{n,d} \gamma$ in U_i. The meaning of $\sim$ will become clear in the following sections). This is the goal of Section XXXX.
2. Then it suffices to prove the respective theorem for perturbations of generalized moment curves $\gamma(z) \sim (z^{n_1}, \ldots z^{n_d})$. In the case of the Restriction theorem (Theorem XXXX), this corresponds to section WWWW. In the case of the Convolution theorem (Theorem YYYY) this corresponds to section WWWW.
3. The theorem now follows by the triangle inequality.

2 The partitioning lemma

Theorem 2 (Lemma 3.1 in [Stovall], originally due to [Dendrinos&Wright], complexified)**.** *Let $\gamma : \mathbb{C} \to \mathbb{C}^d$ be a polynomial curve of degree N, and assume $\Lambda^{(d)\gamma} \not\equiv 0$, then we can split $\mathbb{C} \cup \{\infty\}$ into $M = O_N(1)$ non-overlapping triangles $\{T_j\}_{j=1}^M$ so that on each triangle T_j:*

$$|\Lambda^{(d)}_{\gamma'}(z)| \sim A_j |z - b_j|^{k_j}, \text{ and } |\gamma_1'(t)| \sim B_j |z - b_j|^{l_j} \tag{11}$$

and, for $z := (z_1, \dots z_d) \in T_j^d$:

$$\left| \frac{J_{\gamma'}(z)}{v(z)} \right| \gtrsim_N \prod_{i=1}^{d} \Lambda^{(d)}_{\gamma'}(z_i)^{1/d} \tag{DW}$$

Moreover, for each triangle T_j there is a closed, zero-measure set $R_j \subseteq T_j^d$ so that the sum map $\Sigma(z) := \sum_{i=1}^d \gamma(z_i)$ is $O_N(1)$-to-one in $T_j^d \setminus R_j$.

Schur polynomials are strictly column increasing weak row increasing.

To simplify notation, and since the inequality we want to prove above concerns γ' only and not γ, we will prove inequality (DW) for a generic curve γ that will end up being the γ' above.

The strategy to prove the theorem will be the following: First inequality (DW) will be shown for the moment curve, or for generalized moment curves (that is, curves that are affine equivalent to a curve of the form $(z_1^{\delta_1}, \dots z_1{}^{\delta_d} x)$). Then, we will show that the result is in fact stable to suitable small perturbations of the polynomial. This, together with a compactness argument on $\mathbb{C} \cup \{\infty\}$, will give the non-uniform estimate (where the constant could depend on the polynomial). The only potential source of non-uniformity at this point will be the number of open sets, and finally we will show a stability on the number of open sets to use, and a compactness argument on the set of polynomials with coefficients $\lesssim 1$ will finish the proof.

2.1 Preliminaries

In this section we will define a systematic way of changing co-ordinates to polynomial curves to understand the behavior near a point, which we refer to as *the zoom-in method* from now on.

Definition 1 (Canonical form for a curve)**.** *Let $\gamma = (\gamma_1, \dots \gamma_d)$ be a polynomial curve. Let δ_i be the lowest degree of a non-zero monomial in γ_i. Then γ is in canonical form at zero if:*

- $\delta_1 < \cdots < \delta_d$
- *The coefficient of degree δ_i in γ_i is* 1.

similarly, that γ is in canonical form at $c \in \mathbb{C}$ if $\gamma(z - c)$ is in canonical form at zero. The curve γ is in canonical form at infinity if $z^D \gamma(z^{-1})$ is in canonical form at zero, where D is the maximum of the degrees of γ_i.

A polynomial admits a canonical form for every point if and only if the Jacobian for the curve is not the zero polynomial (that is, as long as the curves are linearly independent as polynomials). Given a polynomial γ, and a linear transformation $L \in GL(d; \mathbb{C})$ so that $L\gamma$ is in canonical form, we define a **zoom in at zero at scale** λ as the (normalized) zoom-in:

Definition 2. *The zoom-in of γ at scale λ is the polynomial curve $\mathcal{B}_\lambda[\gamma](z) := diag(\lambda^{-\delta_1}, \dots, \lambda^{-\delta_d}) L\gamma(\lambda z)$. Note that the coefficents of $\mathcal{B}_\lambda[\gamma](z)$ converge to the polynomial $(z^{\delta_1}, \dots z^{\delta_d})$ as λ goes to zero.*

2.2 Model Case: Generealized moment curve

For a generalized moment curve γ with exponents $\mathbf{n} := (n_1, \dots n_d)$ (that is $\gamma(z) := (z^{n_1}, \dots z^{n_d})$), and for $\mathbf{z} := (z_1, \dots z_d)$ the following holds:

$$\frac{J(\mathbf{z})}{v(\mathbf{z})} = S_{\mathbf{n}}(\mathbf{z}) \tag{12}$$

where $S_{\mathbf{n}}$ is the Schur polynomial of degree $\mathbf{n}$, defined as

$$S_{\mathbf{n}}(z_1, \dots z_d) = \sum_{(t_i) \in T_{\mathbf{n}}} z_1^{t_1} \dots z_d^{t_d} \tag{13}$$

where $T_{\mathbf{n}}$ is the set of semistandard Young Tableaux of shape $\mathbf{n}$. Now, to compare $J(z_i)$ with $\Lambda(z_1, \dots, z_d)$, the following fact is useful:

Lemma 1. *Let $\mathbf{z} \in \mathbb{C}^d$, with $z_i \neq z_j$ for $i \neq z$, let $s \in \mathbb{C}$, and $\gamma : \mathbb{C} \to \mathbb{C}^d$ a polynomial curve, then:*

$$J_\gamma(s) = \lim_{\lambda \to 0} \frac{\Lambda_\gamma(\lambda \mathbf{z} + s)}{v(\lambda \mathbf{z})} \tag{14}$$

and, in particular, in the case when γ is a moment curve of exponent $\mathbf{n}$,

$$\Lambda_\gamma^{(d)}(s) = S_{\mathbf{n}}(s, \dots s) \tag{15}$$

Proof. By Taylor expansion we have:

$$\gamma_i(s + \lambda \mathbf{z}_j) = \sum_{k=0}^{d-1} \frac{1}{k!} \gamma_i^{(k)}(s) \lambda^k \mathbf{z}_j^k + O(\lambda^d) \tag{16}$$

now, defining the matrices $\Gamma_{ij} = \gamma_i(s + \lambda \mathbf{z}_j)$, $Z_{kj} = (\lambda \mathbf{z}_j)^k$, and $(T_\gamma)_{ik} = \frac{1}{k!}\gamma_i^{(k)}(s)$ the equation above can be rewritten as:

$$\Gamma = T_\gamma Z + O(\lambda^d) \tag{17}$$

since the determinant of Z is $v(\mathbf{z})$, the lemma follows from the multiplicative property of the determinant:

$$\frac{\Lambda_\gamma(\lambda \mathbf{z} + s)}{v(\lambda \mathbf{z})} = \frac{\det \Gamma}{\det Z} = \det[T_\gamma + Z^{-1} O(\lambda^d)] \to_{\lambda \to 0} \det T_\gamma = \Lambda_\gamma^{(d)}(s) \tag{18}$$

The fact that $Z^{-1} = o(\lambda^{-d})$ (and thus we can eliminate the term as $\lambda \to 0$) is a quick computation from the adjoint forumla for the inverse. □

Remark. *The same argument works as well for $\Lambda_\gamma^{(k)}$, $1 \leq k < d$, since each component of $\Lambda_\gamma^{(k)}$ is a determinant of components of the polynomial.*

Lemma 1 is (together with Schur positivity) all we need to show the theorem for the generalized moment curve:

Proof (of (DW)*, moment curve case).* Let μ be a generalized moment curve of exponents $\mathbf{n}$. Decompose $\mathbb{C} = \bigcup W_i$ into finitely many sectors $W_i = \{z : |\arg z - \theta_i| < \epsilon\}$ of angle ϵ small enough (depending on the exponents). Now, for $\mathbf{z} = (\mathbf{z}_1, \dots \mathbf{z}_d) \in W_i^d$

$$C_n |S_{\mathbf{n}}(\mathbf{z})| \geq |\mathbf{z}_1 \cdot \mathbf{z}_2 \dots \mathbf{z}_d|^{\frac{\deg S_{\mathbf{n}}}{d}} = C_{\mathbf{n}} \left| \prod_{i=1}^d S_{\mathbf{n}}(\mathbf{z}_i, \dots, \mathbf{z}_i) \right|^{1/d} \tag{19}$$

the first inequality is AM-GM inequality for all the monomials of $S_{\mathbf{n}}(\mathbf{z})$. The second equality follows from the fact that $S_{\mathbf{n}}(\mathbf{z}_i, \dots, \mathbf{z}_i) = C_n \mathbf{z}_i^{\deg S_{\mathbf{n}}/d}$. Now the result follows from equation (15) on Lemma 1. □

Lemma 1 leads to the definition of a new differential form that corrects for the Vandermonde factor:

Definition 3 (Corrected multilinear form)**.** *For $\gamma : \mathbb{C} \to \mathbb{C}^n$ and $\mathbf{z} \in \mathbb{C}^n$ we define:*

$$\tilde{\Lambda}_\gamma(\mathbf{z}) = \frac{\Lambda_\gamma(\mathbf{z})}{v(\mathbf{z})} \tag{20}$$

moreover, (as we shall see in the following section) the map $\tilde{\Lambda}_\cdot(\cdot)$ is continuous in its domain $\mathbb{C}^d \times P_N(\mathbb{C})^d$.

there is an extra property of the generalized moment curve that will be used later in this section, that we will prove now:

Is this lemma really clearer in sequence form?

Lemma 2 (Transversality of the corrected multilinear form for moment curves)**.** *Let μ be a moment curve, and W a wedge of $\mathbb{C}$ of angle ϵ (depending on μ) small enough. Let $\{\mathbf{w^{(k)}}\}_{\mathbf{k=1}}^{\infty}$ be a sequence of elements in W^s, $\{\mathbf{z}^{(k)}\}_{k=1}^{\infty}$ a sequence in W^t, with $k := c + s \leq d$, assume $|\mathbf{z}_i^{(k)}| = O(1)$, and $\mathbf{w}^{(k)} \to 0$.*

$$\|\tilde{\Lambda}_\mu(\mathbf{z}_1^{(k)} \dots \mathbf{z}_t^{(k)}, \mathbf{w}_1^{(k)} \dots \mathbf{w}_s^{(k)})\| \approx_\mu \|\tilde{\Lambda}_\mu(\mathbf{z}_1^{(k)} \dots \mathbf{z}_t^{(k)})\| \|\tilde{\Lambda}_\mu(\mathbf{w}_1^{(k)} \dots \mathbf{w}_s^{(k)})\| \tag{21}$$

Proof. The $\lesssim$ direction is the fact that, for forms $\|a \wedge b\| \leq \|a\|\|b\|$.

For the converse, there is one co-ordinate $e_{n_1} \wedge e_{n_2} \dots \wedge e_{n_k}$ on the LHS that dominates the norm of the form. By restricting to that co-ordinate, it can be assumed that $k = d$. Note also that the term $\|\tilde{\Lambda}_\mu(z_1^{(k)} \dots z_t^{(k)})\|$ is a the absolute value of a Schur polynomial (and therefore is $O(1)$), so we can omit it in the estimates.

By using the Young tableau decomposition of the Schur polynomials again, it suffices to show that each monomial in $\|\tilde{\Lambda}_\mu(w_1^{(k)} \dots w_s^{(k)})\|$ is dominated by a monomial in $\|\tilde{\Lambda}_\mu(z_1^{(k)} \dots z_t^{(k)}, w_1^{(k)} \dots w_s^{(k)})\|$. This can be done as follows:

[DRAW PICTURE OF YOUNG TABLEAUX] ☐

[DO WE NEED WHAT WAS LEMMA 2.5 ANYWHERE]

2.3 Fixed polynomial case

This section shows that locally any polynomial can be approximated by a moment curve in such a way that the estimates can be transfered from the moment curve to the polynomial. By compactness, this will allow us to conclude Lemma (DW) for single polynomials, but with a number of open sets that might depend on the polynomial.

Lemma 3 (Convergence to the model in the non-degenerate set-up)**.** *The function $\tilde{\Lambda}_\mu(\mathbf{z})$ is continuous in $(\mathbf{z}, \mu) \in \mathbb{C}^k \times P_n(\mathbb{C})^d$, where $P_n(\mathbb{C})$ denotes the set of polynomials of degree n.*

Proof. Consider both the numerator and denominator of $\tilde{\Lambda}_\mu(\mathbf{z})$ as a polynomial in the components of μ and z. The polynomial $\Lambda_\mu(\mathbf{z})$ on the numerator vanishes on the zero set set $\mathcal{Z}(v(z_1, \dots z_k))$, and since this polynomial does not have repeated factors, $v(z_1, \dots z_k)$ divides the numerator by the Nullstellensatz and the result follows. ☐

This lemma implies the local version of the theorem around points where the Jacobian does not degenerate:

Proposition 1. *Let γ be a polynomial curve in $\mathbb{C}^d$, such that $\Lambda^{(d)}(z) \neq 0$. Then there is a neighborhood $B_\epsilon(0)$, with $\epsilon = \epsilon(\gamma)$ where* (DW) *holds with constant depending only on the dimension.*

Proof. By the affine invarance of (DW), we can consider a sequence of zoom-ins in the canonical form parametrized by λ that converge to the moment curve (the sequence cannot converge to any other generalized moment curve as the determinant does not vanish at zero). Therefore, we have to show that, for λ small enough, the lemma is true for $\mathcal{B}_\lambda[\gamma]$, that is:

$$\left| \frac{J_{\mathcal{B}_\lambda[\gamma]}(z)}{v(z)} \right| \gtrsim_N \prod_{i=1}^{d} \Lambda_{\mathcal{B}_\lambda[\gamma]}^{(d)}(z_i)^{1/d} \tag{22}$$

For the moment curve (the case $\lambda = 0$) inequality (DW) is true, and reads:

$$\tilde{\Lambda}_\mu(z_1, \dots z_d) \gtrsim 1 \tag{23}$$

and since both sides of the inequality converge locally uniformly as $\lambda \to 0$ (the LHS by Lemma 3 and the RHS because it is the $d-$th root of a sequence of converging polynomials), the inequality is true for λ small enough in the zoom-in. ☐

For the degenerate points where the Jacobian vanishes a similar, but slightly more technical approach gives the same result.

Lemma 4 (Convergence to the model case in the degenerate set-up with zoom-in)**.** *Let $W \subseteq C$ be a sector of small amplitude $\epsilon(N,d)$ to be determined. Let $\mathbf{z}_j$ be a sequence of points in W^k that have norm $\lesssim 1$. Let $\gamma_j := \mathcal{B}_{\lambda_j}[\gamma]$, $\lambda_j \to 0$, be a sequence of polynomial curves of degree N that converges to μ, a generalized moment curve of exponents $\mathbf{n} = (n_1 \dots n_d)$. Then:*

$$\lim_{j\to\infty} \frac{\Lambda_{\gamma_j}(\mathbf{z}_j)}{|\Lambda_{\gamma_j}(\mathbf{z}_j)|} - \frac{\Lambda_{\mu}(\mathbf{z}_j)}{|\Lambda_{\mu}(\mathbf{z}_j)|} = 0 \tag{24}$$

Proof. First note that it suffices to prove that the lemma is true for a subsequence of the $(\lambda_j, \mathbf{z}_j)$. The lemma will follow if we can prove that, for any fixed coordinate $e = e_{l_1} \wedge \cdots \wedge e_{l_k}$ we have:

$$\lim_{j\to\infty} \frac{\Lambda_{\gamma_j}(\mathbf{z}_j)|_e}{\Lambda_{\mu}(\mathbf{z}_j)|_e} = 1 \tag{25}$$

using the notation $w|_e$ to denote the $e-$th co-ordinate of the form w. By restricting the problem to the co-ordinates $(e_{l_1}, \dots e_{l_k})$ we may assume $k = d$, and then it suffices to show, in the same set-up of the lemma, that:

$$\lim_{j\to\infty} \frac{\tilde{\Lambda}_{\gamma_j}(\mathbf{z}_j)}{\tilde{\Lambda}_{\mu}(\mathbf{z}_j)} = 1 \tag{26}$$

We will prove this by induction. By passing to a subsequence if necessary, assume WLOG that $\mathbf{z}_j$ has a limit. In the base case none of the components of $\mathbf{z}_j$ has limit zero. In that case, the denominator converges to a non-zero number (since the denominator is a Schur polynomial in the components of $\mathbf{z}_j$) and the result follows.

Our first induction case is when all the components went to zero. In this case, by doing a further zoom-in and passing to a further subsequence if necessary, one can reduce to the case where not all the components of $\mathbf{z}_j$ go to zero. Thus, assume some, but not all the components of $\mathbf{z}_j$ go to zero.

Without loss of generality assume it's the first $0 < k' < k$ components that go to zero. Let $\mathbf{z'}_j := ((\mathbf{z}_j)_1, \dots (\mathbf{z}_j)_{k'})$ be the sequence made by the first k' components of each $\mathbf{z}_j$, and $\mathbf{z}''_j$ the sequence made by the remaining components. Then,

$$\frac{\tilde{\Lambda}_{\gamma_j}(\mathbf{z}_j)}{\tilde{\Lambda}_{\mu}(\mathbf{z}_j)} = \frac{\sum_{e'\wedge e''=e} \tilde{\Lambda}_{\gamma_j}(\mathbf{z}'_j)|_{e'} \cdot \tilde{\Lambda}_{\gamma_j}(\mathbf{z}''_j)|_{e''}}{\sum_{e'\wedge e''=e} \tilde{\Lambda}_{\mu}(\mathbf{z}'_j)|_{e'} \cdot \tilde{\Lambda}_{\mu}(\mathbf{z}''_j)|_{e''}} \tag{27}$$

we know by the induction hypothesis that each of the terms in the sum in the numerator converges to the corresponding term in the denominator (in the sense that their quotient goes to 1). So the result will follow if we can prove there is not much cancellation going on on the denominator, that is:

$$\limsup_{j\to\infty} \frac{\sum_{e'\wedge e''=e} |\tilde{\Lambda}_{\mu}(\mathbf{z'}_j)|_{e'} \cdot \tilde{\Lambda}_{\mu}(\mathbf{z''}_j)|_{e''}|}{|\tilde{\Lambda}_{\mu}(\mathbf{z}_j)|_e|} < \infty \tag{28}$$

but this is a consequence of Lemma 2, because we can bound each of the elements in the sum by $|\tilde{\Lambda}_{\mu}(\mathbf{z'}_j)| \cdot |\tilde{\Lambda}_{\mu}(\mathbf{z''}_j)| \lesssim |\tilde{\Lambda}_{\mu}(\mathbf{z}_j)|$, by equation (21).

□

We can also see this lemma in the continuity set-up. Following the same proof as in Proposition 1, we can prove

Proposition 2. *Let γ be a polynomial curve in $\mathbb{C}^d$, such that $\Lambda^{(d)}(0) = 0$. Then there is a neighborhood $B_\epsilon(0)$, with $\epsilon = \epsilon(\gamma)$ where* (DW) *holds with constant depending only on the dimension.*

Proof. We have to show that there exists a zoom-in $\mathcal{B}_\lambda[\gamma]$ for λ small enough so that the inequality

$$\left|\frac{J_{\mathcal{B}_\lambda[\gamma]}(z)}{v(z)}\right| \prod_{i=1}^{d} \Lambda^{(d)}_{\mathcal{B}_\lambda[\gamma]}(z_i)^{-1/d} \gtrsim_N 1$$

holds in the unit ball, but lemma 4 implies that

$$\lim_{\lambda\to 0}\left|\frac{J_{\mathcal{B}_\lambda[\gamma]}(z)}{v(z)}\right|\prod_{i=1}^{d}\Lambda^{(d)}_{\mathcal{B}_\lambda[\gamma]}(z_i)^{-1/d}=\left|\frac{J_\mu(z)}{v(z)}\right|\prod_{i=1}^{d}\Lambda^{(d)}_{\mu}(z_i)^{-1/d}\gtrsim 1$$

where the convergence is locally uniform, and the second inequality is inequality DW for the moment curve.

□

This finishes the proof that (DW) if we split compact sets in a finite number of sets that may depend on the polynomial. A small variation of Lemma 2 can be used at a neighborhood of infinity. In this exposition infinity will be considered simultaneously with the uniform case instead.

2.4 Uniformity for polynomials

The aim of this section is to show that the number of open sets in the geometric Lemma 2 does not depend on the polynomial. In order to do so, we will show that given a sequence of polynomial curves there exists a subsequence of curves for which (DW) holds with a uniformly bounded amount of subsets, and thus there must be a uniform bound for all polynomial curves.

The main challenge in the proof of the uniformity of the number of open sets in Lemma 2 is the case in which the zeros of the Jacobian *merge*, that is, the curves γ_n converge to a curve γ such that J_γ has less zeros than γ (without counting multiplicity). We will use zoom-ins near the zeros of γ to keep track of this cases. The following lemma is a key tool to do the zoom in:

Lemma 5. *Let γ be a non-degenerate polynomial curve in $\mathbb{C}^d$ of degree N such that*

$$\gamma_i=\prod_{k=1}^{n_j}(z-w_{i,k})\prod_{l=1}^{m_j}\left(1-\frac{z}{v_{i,l}}\right)$$

, $v_{i,l},w_{j,k}\in B_R\setminus B_r$, $n_1<n_2<\cdots<n_d+m_d$ Then there is a constant $C:=C(N,d)$ such that (PI) holds on $W\cap(B_{C^{-1}R}\setminus B_{Cr})$ for any wedge W of angle $\leq\epsilon(N,d)$.

The lemma (and its proof) can be informally stated as: "If all the zeros of the components of γ are far from an annuli, then γ behaves like the corresponding moment curve in the annuli". To prove the lemma we will have to re-write it into an equivalent form, more suitable for compactness arguments:

Lemma 6 (Lemma 4, annuli verison)**.** *Let γ_n be a sequence of polynomial curves for which the coefficients coefficients $w_{(i,k),n}\to_{n\to\infty}0$, $v_{(i,l),n}\to_{n\to\infty}\infty$, ($v,w$ defined using the notation of the previous lemma, $\gamma_n\to\mu$, a non-degenerate moment curve). Let r_n define a sequence of annuli $A_n=B_0(1)\setminus B_0(r_n)$, so that $\max_{i,k}w_{(i,k),n}=o(r_n)$. Let $\mathbf{z}_n\in(A_n\cap W)^k$, where W is a sector of small enough angle depending of n,d only. Then:*

$$\lim_{j\to\infty}\frac{\Lambda_{\gamma_j}(\mathbf{z}_j)}{|\Lambda_{\gamma_j}(\mathbf{z}_j)|}-\frac{\Lambda_{\mu}(\mathbf{z}_j)}{|\Lambda_{\mu}(\mathbf{z}_j)|}=0 \tag{29}$$

Proof. The proof is the same as the proof in Lemma 4. The key difference being the reason why we can zoom in again. In Lemma 4 the γ_n were themselves blow-ups, so blowing up did not change the hypothesis of the lemma. Here, the control of r_n ensures the blow-up will be always at a smaller scale than the scale at which the zeros of γ_n are. □

Remark. *Note that in the particular case in which there are no $v_{(i,l),n}$ (that is, all the zeros are going to zero) the annuli can be taken to have exterior radius equal to infinity (that is, the annuli can degenerate to the complement of a disk) or, in the case where all the $w_{(i,k),n}$ are exactly equal to zero, it can be taken to have interior radius equal to 0.*

Lemma 5 that we just proved (in its convergence form) lets us control J_γ far from the zeros of the components of γ. The zeros of the components, however, depend on the co-ordinates we take. In order to solve this, we will show that there is one *honest* co-ordinate system in which, if we have a zero of a co-ordinate of γ that has size $O(1)$ then there is also a zero of $\mathcal{J}_\gamma$ that has size $O(1)$.

Lemma 7 (Honest zeros lemma)**.** *For a non-degenerate polynomial curve γ, let $R(\gamma)$ be the (absolute value of the) supremmum of the zeros of J_γ that has absolute value smaller than 1. Let r_γ be the supremmum of the zeros of the co-ordinates of γ (again in absolute value, and counting only the zeros that have absolute value less than 1).*

Then, for any sequence of polynomial curves $\gamma_n \to \mu$, a non-degenerate generalized moment curve, there is a constant $k := k(\gamma_n)$, a sequence of linear operators $L_n \in GL(n;\mathbb{C})$ converging to the identity and a sequence of constants $c_n \to 0$ so that:

$$R(L_n\gamma_n(z-c_n)) \geq kr(L_n\gamma_n(z-c_n)) \tag{30}$$

In other words, after a suitable change of co-ordinates, controlling the zeros of a sequence of polynomial curves allows us to control the zeros of its Jacobian without significant losses.

Proof. We will assume that μ is not the standard moment curve, since otherwise the result is trivial because $J_\mu = 1$. We choose the $c_n \to 0$ to re-center the γ_n so that J_{γ_n} always has a zero at zero. Let n_i be the degree of the $i-$th co-ordinate of μ. By composing with suitable $L_n \to Id$ we can assume that the degree n_i component of the $i-th$ co-ordinate of γ_n is always 1, and the degree n_j of the $i-$th component (for $i \neq 0$) is 0 for all γ_n. Let $\tilde{\gamma}_n = L_n\gamma_n(z-c_n)$, then, the following holds:

Lemma 8. *Let $\hat{\gamma}_n$ be a sequence of zoom-ins to $L_n\gamma_n(z-c_n)$ at the scale where the zeros of the co-ordinates appear, and assume $\hat{\gamma}_n \to \gamma$. Then the multiplicity of the zero of J_γ at zero is strictly smaller than the multiplicity of J_μ at zero.*

using the lemma above, we can finish the proof by contradiction. Assume γ_n is such that $\tilde{\gamma}n = L_n\gamma_n(z-c_n)$ contradicts the lemma. Pick a subsequence for which $R(\tilde{\gamma}_n)/r(\tilde{\gamma}_n)$ goes to zero. Let $\hat{\gamma}_n$ be a zoom-in at the scale at which the first zeros of the components of $\tilde{\gamma}_n$ appear.Assume, by passing to a subsequence if necesary, that $\hat{\gamma}_n$ converges. By the hypothesis of $R(\tilde{\gamma}_n)/r(\tilde{\gamma}_n)$, it must be that all the zeros of $J : \hat{\gamma}_n$ concentrate back at zero, but this contradicts lemma 8. □

Proof (of Lemma 8). Define a matrix $M_{i,k} \in \mathcal{M}_{(d,N)}(\mathbb{C})$ so that $M_{i,k}$ is the coefficient of degree k of the i component. By the transformation we have done, we know the matrix $M_{i,k}$ has rank d, and that for the column i, the element M_{i,n_i} is equal to 1 and for $j > n_i$, M_{i,n_i} is equal to 0. The multiplicity of $J\mu$ at zero is $\sum n_i - \frac{d^2+d}{2}$. To compute the multiplicity of J_γ at zero, we do the following procedure:

Pick the first column (smallest k index) that is non-zero. Pick the first element of this column (smallest i) index that is non-zero. Define $\tilde{n}_i := k$, where i is the index that is not zero. Row-reduce $M_{i,k}$ so that the $k-$th column is e_i. Set all the elements on the right of (i, n_i) to zero. Repeat this process d times.

This procedure is a row-reduction and blow-up procedure at the origin, that shows that, at the origin, the polynomial curve looks like a generalized moment cuve of degrees $\tilde{n}_i$, where $\tilde{n}_i \leq n_i$ with at least one of the $\tilde{n}_i < n_i$. Therefore, the degree at the origin is strictly smaller. □

Now we have all the necessary tools to prove (DW) in Lemma 2 in the full generality case. The proof is as follows:

- The proof is a proof by contradiction. Assume there is a sequence of γ_n for which the minimum number of sets need for the geometric Lemma 2 to hold grows to infinity. The contradiction will come from showing that a certain subsequence of the γ_n can be covered by a bounded number of subsets.
- By passing to a subsequence if necessary, and re-parametrizing, we will assume that the γ_n converge to a non-degenerate generalized moment curve $\hat{\gamma}$, and that all the zeros of J_{γ_n} converge to the origin, with one zero of J_{γ_n} being exactly at the origin.
- By Lemma 5 [convergence of the Jacobian on annuli with possibly infinite radius], after a suitable reparametrization if necessary, we can cover uniformly $\mathbb{C} \setminus B_{r_n}$ with wedges so that property (DW) holds, where r_n is proportional (with a constant depending on d, N) to the size of the biggest zero of J_{γ_n}. After a suitable change of coordinates by Lemma 7, this is equivalent to consider r_n to be of the size of the biggest zero of a component of γ_n.

- Zoom in to the polynomials γ_n at scale r_n to obtain the polynomials γ'_n. We have to show now that the theorem holds for γ'_n on the unit ball. By passing to a subsequence, assume WLOG that the polynomials converge to a non-degenerate polynomial curve γ'. Note that the zeros of $J_{\gamma'}$ cannot all converge to the origin (because for each γ'_n there is a zero with size $O(1)$).
- By Lemma 5 again, we can find a sequence of annuli of outer radius $O(1)$, centered at the zeros of γ'_n so that the condition (DW) holds after splitting the annuli into wedges.
- On the intersection of all the exteriors of all the Annuli (by the exterior meaning the connected component containing infinity of the complement of the annuli), property (DW) holds for n big enough after splitting into $O(1)$ sets, by compactness and Proposition 1 [Local version of (DW) in the non-degenerate case].
- Therefore, it suffices to prove that property (DW) wolds in the interior component of the complement of the annuli. But this can be done by induction: Zoom in into each of those components (which, by hypothesis have lower degree than the original one), and repeat the argument.

2.5 Injectivity of the Σ map

The goal of this section is the last part of Lemma 2, which we re-state:

Lemma 9. *For each triangle T_j described in the proof of Lemma 2 there is a closed, zero-measure set $R_j \subseteq T_j^d$ so that the sum map $\Sigma(z) := \sum_{i=1}^d \gamma(z_i)$ is $O_N(1)$-to-one in $T_j^d \setminus R_j$.*

Proof. Our set R_j is the set where there is $i \neq j$ where $z_i = z_j$. The fact that $\Lambda'_\gamma(z_1, \ldots z_d)$ does not vanish in $T_j \setminus R_j$ (a consequence of (DW)) tells us that $(z_1, \ldots z_d)$ does not belong to an irreducible variety of dimension greater than zero of the variety defined by $\{(x_1, \ldots x_d) \in \mathbb{C}^d | \Lambda'_\gamma(x_1, \ldots x_d) = \Lambda'_\gamma(z_1, \ldots z_d)\}$. Therefore, the result follows by Bezout's theorem. $\square$

3 Uniform restriction for polynomial curves

This section outlines the modifications that must be done to the argument [Stovall] to extend it to the complex case. The paper reduces the analytic result (whether the operator is bounded from a certain L^p to a certain L^q) to a geometric result, previously proven by Dendrinos and Wright in the real case, proven in section 2 for the complex scenario.

3.1 Uniform Local Restriction

The first step in the proof is the local result:

Theorem 3 (Theorem 2.1 in [Stovall])**.** *Fix $d \geq 2$, N, and (p, q) satisfying ZZZZ. Then, for every ball $B \subseteq \mathbb{R}^d$ and every degree N polynomial $\gamma : \mathbb{C} \to \mathbb{C}^d$ satisfying*

$$0 < C_1 \leq J_\gamma(z) \leq C_2, \;\; z \in B \tag{31}$$

we have the extension estimate

$$\|\mathcal{E}_\gamma(\chi_B f)\|_q \leq C_{d,N,\log \frac{C_2}{C_1}} \|f\|_p \tag{32}$$

An important preliminary fact, proven in [Stovall, lemma XXX] is that whenever $J_\gamma \lesssim 1$ on a compact, convex set, there is a change of co-ordinates $L \in SU(d;\mathbb{C})$ so that all the coefficients in $L \circ J_\gamma$ are $O_{N,d}(1)$. The proof is done through a compactness argument, and transfers without any significant modification to the complex case for convex domains. The exact formulation that we will use is:

Lemma 10. *Fix $N, d \geq 2$, $\epsilon > 0$. Then for any polynomial curve γ and triangle T with $B_\epsilon \subset T \subset B(0,1)$ obeying that $|J_\gamma(T)| \subseteq [1/2, 2]$ there exists a transformation $A \in SU(n;\mathbb{C})$ so that $\|A\gamma\|_{C^N(K)} \lesssim_\epsilon 1$.*

Proof. Define $\gamma_\epsilon = \epsilon^{\frac{d^2+d}{2}} \gamma(\epsilon^{-1} z)$. Now γ_ϵ has the property that $|J_{\gamma_\epsilon}(B(0,1))| \subseteq [1/2, 2]$. This reduces the problem to the situation in Lemma XXXX in [Sto] □

The second preliminary is a statement about offspring curves. Given $\mathbf{h} := (\mathbf{h}_1, \ldots, \mathbf{h}_k)$, the offspring curve $\gamma_h(z)$ is defined as $\gamma_h := \frac{1}{K}\sum_{i=1}^{K} \gamma(z + h_i)$. Lemma 2.3 in [Stovall] states:

Lemma 11. *Fix $N, d \geq 2$ and $\epsilon > 0$. There exists a constant $c_d > 0$ and a radius $\delta := \delta(\epsilon, N, d)$ so that for any triangle $B_\epsilon(0) \subseteq T \subseteq B_1$ and so that the following conditions hold for any polynomial curve γ satisfying*

$$|J_\gamma(T)| \subseteq [1/2, 2] \tag{33}$$

For any ball B of radius δ centered at a point in T, and any $\mathbf{h} := (\mathbf{h}_1, \ldots, \mathbf{h}_k) \in \mathbb{C}^k$, the curve γ_h satifies the following inequalities $\tilde{B} = \bigcap_{i=1}^{k}(B - h_j)$:

$$|J_{\gamma_h}(\mathbf{z})| \gtrsim_{N,d,\epsilon} |v(\mathbf{z})| \prod_{i=1}^{d} |\Lambda_{\gamma_h}^{(d)}(\mathbf{z}_i)|, \; \textit{for any } \mathbf{z} \in \tilde{B}^d \tag{34}$$

$$|L_{\gamma_h}(z)| \approx_{N,d,\epsilon} \; \textit{for any } 1 \in \tilde{B} \tag{35}$$

in particular, one can cover any such T by $O_{N,d,\epsilon}(1)$ open sets so that the conclusions (34) *and* (35) *hold for any polynomial that follows* (33)*.*

Proof. First note, that by choosing δ small enough and the Lemma [THE PREVIOUS LEMMA], we have that

$$|J_\gamma(T + B_\delta)| \subseteq [1/4, 4]$$

. This fact prevents any issues arising from the different triangle shapes. From here on, the result follows as the proof in [STOVALL, Lemma 2.3]. □

Using those preliminaries, we can proceed to the proof of Theorem 3. The proof follows exactly as in [xxxx], which is in itself a sketch of the proof in [That old paper that's super nice]. The following lemma is a simple computation, that takes the role of the equivalent necessary result in the real case.

Lemma 12. *The function $|v(0, z_1, \ldots z_{d-1})|^{-2(a-1)}$ belongs to $L_{B_1^d(0)}^{\frac{d}{2(a-1)} - \epsilon}$ for $\epsilon > 0$ small.*

other than this difference (and the factors of 2 that appear at the exponent everywhere where the Jacobian appears, that lead to the factor of 2 in Lemma [XXX the lemma above]), the proof of Lemma 2.4 in [Sto] applies mutatis mutandis to the complex case.

3.2 Almost orthogonality

The main result in this section is:

Lemma 13. *Let $\gamma : \mathbb{C} \to \mathbb{C}^d$ be a complex polynomial curve, and let T_j be one of the sets in Lemma 2. For $n \in \mathbb{Z}$ define the dyadic partition*

$$T_{j,n} = \{z \in T_j, |z_j - b_j| \sim 2^n\} \tag{36}$$

Then for each (p, q) satisfying $q = \frac{d(d+1)}{2}p'$ and $\infty > q > \frac{d^2+d+2}{2}$ and $f \in L^p(d\lambda_\gamma)$ we have:

$$\|\mathcal{E}_\gamma(\chi T_j f)\|_{L^q(\mathbb{R}^{2d})} \lesssim \left\| \left(\sum_n |\mathcal{E}_\gamma(\chi T_{j,n} f)|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^{2d})} \tag{37}$$

Proof. The proof is a standard Littlewood-Paley argument, using property ZZZ for the support of the Littlewood-Paley blocks, see Stovall again. □

The goal is now to show that we can sum the pieces in the Littlewood-Paley decomposition above. The main proposition to do so (essentially Lemma 4.1 in [Stovall]) is:

Lemma 14. *There exists $\epsilon(N, d, p) > 0$ such that, if $n_1 \leq \cdots \leq n_d$, and f_i is Schwartz and supported in T_{j,n_i} we have:*

$$\left\| \prod_{i=1}^{d} \mathcal{E}_\gamma[f_i] \right\|_{L^{d+1}} \lesssim 2^{n_D - n_1} \prod_{i=1}^{d} \|f_i\|_{L^2(d\lambda)} \tag{38}$$

before proving Lemma XXX, we will prove an auxiliary result (the complex version of a lemma by Christ in [STOXXX12]) that will be needed during the interpolation. We provide an outline here:

Lemma 15.

$$\int_{\mathbb{C}^l \sim \mathbb{R}^{2l}} \prod_{1 \leq i \leq l} f_i(z_i) \prod_{1 \leq i < j \leq l} g_{i,j}(z_i - z_j) dz \lesssim \prod_{i=1}^{l} \|f_i\|_p \prod_{1 \leq i < j \leq l} \|g_{i,j}\|_{q,\infty} \tag{39}$$

whenever $2l = 2lp^{-1} + \frac{2l(l-1)}{2}q^{-1}$

Proof. The proof starts by studying the set of powers $p_i, q_{i,j}$ with $\sum_i p_i^{-1} + \sum_{i,j} q_i^{-1} = l$ for which the estimate

$$\int_{\mathbb{C}^l \sim \mathbb{R}^{2l}} \prod_{1 \leq i \leq l} f_i(z_i) \prod_{1 \leq i < j \leq l} g_{i,j}(z_i - z_j) dz \lesssim \prod_{i=1}^{l} \|f_i\|_{p_i} \prod_{1 \leq i < j \leq l} \|g_{i,j}\|_{q_{i,j}} \tag{40}$$

holds. We will denote by capital letters the vectors $(p_1, \ldots p_l, p_{1,2}, \ldots p_{l-1,l})$, which we will think of as elements of the affine subspace $H := \{\sum_i p_i^{-1} + \sum_{i,j} q_i^{-1} = l\}$.

The base cases are $A := (p_i^{-1} = 1, q_{i,j}^{-1} = 0)$ and $B := (p_i^{-1} = \delta_{i,1}, q_{i,j}^{-1} = \delta_{i+1,j})$, which are Fubini's theorem. Now, the result is invariant over permutations over all the indices (i, j). This allows us to extend the second base B case to all the cases B_σ permutations obtained from those permutations.

By Riesz-Torin, the result is then true for $A' := \frac{1}{n!} \sum_{\sigma \in S_n} B_\sigma = (p_i = \frac{1}{n}, q_i = \frac{1}{2n})$. By Riesz-Torin again, the result is true for all the points interpolating A and A'. This proves the strong version of the theorem.

To get the strong case, it suffices to show that all the points joining A and A' lie on the interior of the interpolation polytope (interior with the affine topology on H). By convexity again, it suffices to show that A' does. The geometric argument can be seen in Christ (it is exactly the same as in dimension 2). □

4 Convolution estimates for measures on complex curves