The following definitions will be used through the section:

Definition 0.1 (Alternating forms on a polynomial). *If $p : \mathbb{C} \to \mathbb{C}^n$ is a polynomial we define:*

$$\Lambda_p(z_1, \dots, z_k) = p(z_1) \wedge \cdots \wedge p(z_n) \tag{0.1}$$

$$\Lambda_p^{(k)}(z) = p(z_1) \wedge \cdots \wedge p^{(k)}(z_n) \tag{0.2}$$

1 Going Back to Stovall's paper

- Section 2 in Stovall's paper (Uniform local restriction) transfers without any modification to the complex case.
- Section 3 is where most of the work happens. The paper uses a lemma by Dendrinos and Wright, which we will have to re-prove for the complex case. The lemma is as follows:

Lemma 1.1 (DW, Complexified). *Let $\gamma : \mathbb{C} \to \mathbb{C}^d$ be a polynomial of degree N, and assume $\Lambda_\gamma^{(d)} \neq 0$. We may decompose $\mathbb{C}$ as a disjoint union of $O_{N.d}(1)$ triangles T_j so that on each triangle:*

$$|\Lambda_{\gamma'}^{(d)}(z)| \sim A_j |z - b_j|^{k_j} \text{ , and } |\gamma_1'(t)| \sim B_j |t - b_j|^{l_j} \tag{1.1}$$

and for all $(z_1, \dots z_d) \in T_j^n$

$$\left| \frac{J_{p'}(z_1, \dots z_d)}{v(J(z_1, \dots z_d)} \right| \gtrsim_{N,d} \prod_{i=1}^{d} (\Lambda^{(d)}(z_i))^{1/d} \tag{1.2}$$

moreover, for each triangle there is a closed measure zero subset $R_i \subset T_i^d$ the sum map $\Sigma(z_1, \dots z_d) = \sum_i \gamma(z_i)$ is injective in $T_i^d \setminus R_i$.

Estimate (1.1) is no different from the proof in DW. Section 2 will show estimate (1.2), Section 3 will show injectivity.

Should I write a section talking about how to do the blow-ups (change co-ordinates so that you look like the moment curve you want at the origin, and zoom-in and rescale with the natural choice of exponents), or is it clear enough?

2 Uniform bounds on the Jacobian

The main goal of this section is to prove the following theorem

Theorem 2.1. *Given a degree N polynomial p in $\mathbb{C} \to \mathbb{C}^d$ we can split $\mathbb{C}$ into $O_{N,d}(1)$ convex regions so that in each region:*

$$\left|\frac{J_{p'}(z_1,\dots z_d)}{v(J(z_1,\dots z_d)}\right| \gtrsim_{N,d} \prod_{i=1}^{d} \left(\Lambda^{(d)}(z_i)\right)^{1/d} \tag{2.1}$$

The section will be structured in the following subsections:

2.1 First we will see that the theorem is true for *model* polynomials, namely the generalized moment curve.

2.2 Second, we will prove that the theorem is true for an arbitrary, fixed polynomial curve (with the implicit constant still independent of the polynomial curve). In order to do so, we will use that all non-trivial blow-ups have to converge to a generalized moment curve, so we will only have to understand how the Jacobian converges to the model Jacobian as we blow up.

2.3 The only step left for uniformity after step 2 will be to show that the number of sets in the partition is bounded for all polynomials. We will do so by a compactness argument again.

2.1 Model case

For a generalized moment curve γ with exponents $n = (n_1 < \cdots < n_d)$ we have that:

$$\frac{J(z_1,\dots,z_d)}{v(z_1,\dots,z_d)} = S_n(z_1\dots z_d) \tag{2.2}$$

where S_n is the Schur polynomial. Remember that:

$$S_n(z_1,\dots,z_d) = \sum_{(t_i)=T} z_1^{t_1}\cdot\dots\cdot z_d^{t_d} \tag{2.3}$$

where the sum is over all semistandard Young tableaux of size $(n_i - i + 1)_{i=1}^{d}$. To compare J with $\Lambda^{(d)}$ the following lemma is useful:

Lemma 2.2. *For fixed $z_i \neq z_j$, and an arbitrary polynomial curve $p = (p_1, p_2, \dots, p_d)$:*

$$\Lambda_p^{(d)}(s) = \lim_{\lambda\to 0} \frac{\Lambda_p(s+\lambda z_1,\dots,s+\lambda z_d)}{v(\lambda z_1,\dots,\lambda z_d)} \tag{2.4}$$

in the particular case when p is a moment curve with exponents n, we have:

$$\Lambda_p^{(d)}(s) = S_n \tag{2.5}$$

Proof. By Taylor expanding around s, we have:

$$p_i(s+\lambda z_j) = \sum_{k=1}^{d} p_i^{(k)}(s))\lambda^k z_j^k + \lambda^{d+1} O_p(1) \tag{2.6}$$

Define the matrices $(P)_{ij} = p_i(s+\lambda z_j)$, $(T_p)_{ik} = p_i^{(k)}(s))$, $(V)_{kj} = \lambda^k z_j^k$. Now the equation above reads:

$$P = T_p V + \lambda^{d+1} O(1) \tag{2.7}$$

Using the fact that V^{-1} has entries which are at most $O(\lambda^{-d})$ the result follows by the multiplicative property of the determinant. □

Remark: A similar argument works for $\Lambda^{(k)}$ with $k < d$ as well, since every coordinate of $\Lambda^{(k)}$ is some determinant generated by components of the polynomial.

We are now equipped to prove the lemma for the generalized moment curve:

Proof (of theorem 2.1 in the moment curve case). By our definition of Schur polynomial in (2.3), if we split the complex plane into sectors of angle ϵ small enough (depending on the exponents n_i) and choose z_i within that sector we have:

$$2|S_n(z_1, \dots z_n)| \gtrsim_{n,d} \left| \prod_{i=1}^{d} S_n(z_i, \dots, z_i)^{1/d} \right| \tag{2.8}$$

Now combining equation (2.2) with lemma 2.2 gives the inequality.

□

The lemmas above warrant a new definition, that is:

Definition 2.3 (Corrected multilinear form). *For $p : \mathbb{C} \to \mathbb{C}^n$ we define:*

$$\tilde{\Lambda}_p(z_1, \dots, z_k) = \frac{\Lambda_p(z_1, \dots, z_k)}{v(z_1, \dots, z_k)} \tag{2.9}$$

which is well defined and continuous for all $z_1, \dots z_k \in \mathbb{C}^d$ for all p (as we shall see later)

There is a couple extra properties of the moment curves that will be useful later on, and that we will prove now:

Lemma 2.4 (Transversality of the bilinear form on moment curves.). *Let $w_1, \dots w_s \in \mathbb{C}$, $\|w_i\| \to 0$, $z_1, \dots z_t \in \mathbb{C}$, $\|z_i\| = O(1)$, then:*

$$\|\tilde{\Lambda}_\mu(z_1 \dots z_t, w_1, \dots w_s)\| \approx_\mu \|\tilde{\Lambda}_\mu(w_1, \dots w_s)\| \tag{2.10}$$

Proof. The $\leq$ bound is just the standard $\|ab\| \leq \|a\|\|b\|$ for forms.

For the $\gtrsim$, first note that we can assume $k = d$. There will be one of the co-ordinates in the LHS that will dominate the norm (up to a constant), so we can restrict ourselves to that coordinate.

Now note that the LHS is a Schur polynomial that comes from the tableau $(\lambda_1, \dots \lambda_{t+s})$, and the RHS is the Schur polynomial that comes from the tableau $(\lambda_{t+1}, \dots \lambda_{t+s})$. If we are given a tableau for the (w_i), we can bound it by the tableaux obtained by adding constant rows of the (z_i) (because the z_i are all $O(1)$). Therefore, we can bound the polynomial on the RHS by the one on the LHS.

Add an image of the tableaux

□

Lemma 2.5. *For any generalized moment curve μ there is is a sum of Schur polynomials so that $\hat{\Lambda}_\mu(z_1, \dots z_k) = \frac{\tilde{\Lambda}_\mu(z_1, \dots z_k)}{\sum_{n_i} S_{n_i}(z_1, \dots z_k))}$ extends continuously to the wedge including the origin and so that*

$$\hat{\Lambda}_\mu(0, 0 \dots, 0) = C_\mu \cdot e_1 \wedge e_2 \wedge \dots \wedge e_k \tag{2.11}$$

Proof. WLOG we can assume $|z_i| \leq |z_{i+1}|$ by symmetry. Use the fact that every form coordinate is a Schur polynomial, dividing by the sum of all of them ensures boundedness. Now, near zero, the lowest degree (degree ordered by lexicographic order) Schur polynomial summand will dominate, which is the one corresponding to the first coordinates. $\square$

2.2 Fixed polynomial case

The idea in this section is to approximate the polynomial locally by the model polynomials by *blowing up* into each point, to show that for every point (including infinity) there is a neighborhood were theorem 2.1 holds. By compactness of $\mathbb{C} \cup \infty$ the theorem will hold on $\mathbb{C}^n$.

Lemma 2.6 (Convergence to the model case in the non-degenerate set-up). *The function* $f : \mathbb{C}^k \times P_N(\mathbb{C})^d$

$$f(z_1, z_2, \dots z_k; p) := \frac{\Lambda_p(z_1, \dots z_k)}{v(z_1, \dots z_k)} \tag{2.12}$$

is continuous in $\mathbb{C}^k \times (P_N)^d$, *where* P_N *is the vector space of complex polynomials of degree* N

Proof. Consider each component of the numerator as a polynomial both in the components of p and z_1, z_k. Do the same for the Vandermonde determinant. Since the Vandermonde denominator has no repeated factors and the zero set of the numerator includes that of the denominator, by Nullstellensatz we know that we can factor the denominator out of the numerator. $\square$

The first application of this lemma is the following:

Proposition 2.7. *Let* p *be a polynomial curve in* $\mathbb{C}^d$, *and* z *be a point such that* $\Lambda_p^{(d)}(z) \neq 0$, *then there is a neighbourhood* $B_\epsilon(z)$, *with* $\epsilon = \epsilon(d, p, z)$ *where theorem 2.1 holds with the implicit constant depending only on the dimension.*

Proof. By affine invariance of theorem 2.1, consider a sequence of blow-ups of the polynomial p near z. The polynomials p_n resulting from the blow-up will have to be a standard moment curve. Since the inequality is true for the moment curve and both sides are locally uniformly convergent, the result follows. $\square$

Now we have to look at the degenerate points in the jacobian. If P has a degenerate Jacobian at z, that means that a blow-up at z will give a non-trivial moment curve. We can understand this points with the following lemma:

Lemma 2.8. *(Convergence to the model case in the degenerate set-up with blow-up) In the same set-up as in the previous lemma, assume now that all the points* $z_{i,k}$ *have norm* $O(1)$, *and are in a sector with vertex at the origin, in the positive direction of angle* $< \epsilon$ *to be determined. Assume now the polynomial* $p_n(z) = L_n \circ p(\lambda^{-1} z)$ *converges to* $\tilde{\mu}$, *a generalized moment curve of exponents* $(n_1, \dots n_d)$. *Then:*

$$\lim_{j \to \infty} \frac{\tilde{\Lambda}_{p'_j}(z_{1,j}, \dots z_{k,j})}{\|\tilde{\Lambda}_\mu(z_{1,j}, \dots z_{k,j})\|} = \lim_{j \to \infty} \frac{\tilde{\Lambda}_\mu(z_{1,j}, \dots z_{k,j})}{\|\tilde{\Lambda}_\mu(z_{1,j}, \dots z_{k,j})\|} \tag{2.13}$$

Proof. To begin with, note that it suffices to show that the statement is true for a subsequence. If it weren't true in general, we could pick a subsequence of a counterexample such that the result didn't hold for any subsequence.

Now, we take coordinates: it suffices to prove that for any coordinate $e = e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_s}$ we have:

$$\lim_{j\to\infty} \frac{\tilde{\Lambda}_{p'_j}(z_{1,j}, \dots z_{k,j})|_e}{\tilde{\Lambda}_{\mu}(z_{1,j}, \dots z_{k,j})|_e} = 1 \tag{2.14}$$

where we denote by $w|_e$ the e-th component of w in the canonical basis. We will prove this by an induction argument. In our base case, none of the z_i is equal to zero, and since the denominator does not vanish far from zero in the sector (it is a Schur polynomial), we can compute the limit of the numerator separately, and the result is clear by locally uniform convergence of polynomials.

Now assume that some of the z_i (more precisely $z_1, \dots z_s$) go to zero. By yet another blow-up and induction, we may assume the theorem to be true for those $\{z_i\}_{i=1}^s$. Now note that:

$$\frac{\tilde{\Lambda}_{p'_j}(z_{1,j}, \dots z_{k,j})|_e}{\tilde{\Lambda}_{\mu}(z_{1,j}, \dots z_{k,j})|_e} = \frac{\sum_{e' \wedge e'' = e} \tilde{\Lambda}_{p'_j}(z_{1,j}, \dots z_{s,j})|_{e'} \cdot \tilde{\Lambda}_{p'_j}(z_{s+1,j}, \dots z_{s,j})|_{e''}}{\sum_{e' \wedge e'' = e} \tilde{\Lambda}_{\mu}(z_{1,j}, \dots z_{s,j})|_{e'} \cdot \tilde{\Lambda}_{\mu}(z_{s+1,j}, \dots z_{s,j})|_{e''}} \tag{2.15}$$

because the Vandermonde terms cancel out. Now note that there is not much cancellation happening in the numerator: by lemma 2.4, the numerator of the LHS cannot be much larger than the numerator of the RHS (each term on the RHS denominator has size $\approx \|\tilde{\Lambda}_{\mu}(z_{1,j}, \dots z_{s,j})|_{e'}\|$, and the LHS has size $\approx \sup_{e'} \|\tilde{\Lambda}_{\mu}(z_{1,j}, \dots z_{s,j})|_{e'}\|$ by lemma 2.4). Therefore, it suffices to show the convergence of each of the terms of the sum in the denominator converges to the counterpart in the numerator (in the sense that their quotient goes to 1). But after zooming in (and possibly passing to a subsequence) this is precisely the induction hypothesis, which closes the proof.

□

We can also see the lemma above in the *continuity* set-up:

Corollary 2.9. *Let $\mu \cdot P_N^d$ be the set of polynomials that look like the moment curve μ near the origin. Then we can split the unit ball into wedges so that in each wedge $\hat{\Lambda}_p(z_1, \dots, z_k)$ is a continuous map in $\Delta^n \times \mu \cdot P_N^d$ that extends to the boundary*

We used two key facts in the proof above (or two versions of the same fact):

- After the blow-up induction step, the model curve exponents did not change.
- After the blow up, the curve still converges to the original curves. Consider the case $p_n = z^2 - \frac{1}{n} z^4$: If the z_n go very slowly to the origin they will not *see* the oscillation even after blow-up. However, if they go *too fast* to zero, they will see it and the result will actually be false.

Proposition 2.10. *Let p be a polynomial curve in $\mathbb{C}^d$ with $\Lambda_p^{(d)}(z) \not\equiv 0$, and z be a point such that $\Lambda_p^{(d)}(z) = 0$, $p(z) \sim \sum \vec{v}_i z^{n_i}$, $\det(\vec{v}_i)_j \neq 0$. Then there is a neighbourhood $B_\epsilon(z)$, with $\epsilon = \epsilon(d, p, z)$ where theorem 2.1 holds when splitting the ball into wedges of size $O_{n,d}(1)$.*

Proof. By affine invariance of theorem 2.1, consider a sequence of blow-ups of the polynomial p near z. The polynomials p_n resulting from the blow-up will have to be a standard moment curve. Since the inequality is true for the generalized moment curve and both sides are locally uniformly convergent, the result follows. □

Now, using the previous two lemmas we get the non-uniform partition estimate for polynomials:

Proof (of theorem 2.1 for a fixed nondegenerate polynomial, in a compact set). In the compact ball, by the previous lemmas, the theorem is true in a neighborhood of every point, so it suffices to choose a finite cover. □

Note how the only source of non-uniformity is he number of sets we are partitioning on, so once we sort that out, we will have a uniform estimate. We are also only considering compact subsets of the domain so far. By the affine invariance property, if we show that the bound is uniform on compact sets, a compactness argument with a blow-in will show that we can extend it to all $\mathbb{C}$.

2.3 Uniformity for polynomials

Lemma 2.11. *Let P be a polynomial curve such that $P_i = \prod_{k=1}^{n_i}(z - w_{i,k}) \prod_{k=1}^{m_i}\left((1 - \frac{z}{v_{i,k}})\right)$, $n_{i+1} > n_i$ Then there is an absolute constant C only dependent on the degrees m_i, n_i and the dimension d such that if $|w_{i,k}| < \frac{r}{C}$ and $|v_{i,k}| > CR$, then the theorem holds on the annulus $B_0(R) \setminus B_0(r)$ intersected with a sector $\{z : |\arg z| < \epsilon\}$, where ϵ again only depends on the degree.*

The lemma can be rephrased in the following way, more suitable for a compactness argument:

Lemma 2.12. *Let $P_l = P_{(i),l}$ be a sequence of polynomial curves (indexed by l) for which the $w_{(i,k),l}$ go to zero, and the $v_{(i,k),l}$ to infinity. Let r_l define a sequence of annuli $B_0(1) \setminus B_0(r_l)$, so that $\max_{i,k} w_{(i,k),l} = o(r_l)$ and a sequence of k-tuples of points $z_{t,l} \in B_0(1) \setminus B_0(r_l)$. Then:*

$$\limsup_{l\to\infty} \frac{\Lambda_{P_l}(z_{(1),l}) \dots z_{(k),l})|_e}{\Lambda_\mu(z_{(1),l}) \dots z_{(k),l})|_e} = 1 \tag{2.16}$$

for any coordinate e of the $k-form$, where μ is the moment curve P_l converges to locally uniformly.

Proof. The proof is the same as the proof of lemma 2.6, as the $\max_{i,k} w_{(i,k),l} = o(r_l)$ ensures the induction hypothesis still apply as you blow-up, and lets us close the induction. □

Now the proof of lemma 2.11 follows by the standard contradiction annuli (if this wasn't the case, pick as sequence of annuli...).

There is one phenomenon that we have not ruled out, that could bring trouble: Consider the polynomial $P_n = L_n\mu + \epsilon_n$, where μ is a moment curve, ϵ_n is an error that converges very fast to zero, and L_n is a sequence of linear maps converging slowly to the identity. By affine invariance, we could apply L_n^{-1} and see that the 'bad radius' in the previous lemma is actually very small (because the error itself is small) but we would be grossly overestimating it if we do not apply L_n^{-1}. We have to find a way to normalize for the L_n, as shown in the following lemma:

Lemma 2.13 (Honest zeros lemma)**.** *Let $P_n \to P$ be a sequence of polynomial curves in $\mathbb{C}^n$. Let $\gamma_{i,n}$ be the set of zeros (with multiplicity) of $\Lambda^{(d)}P_n$, and γ_i the zeros of $\Lambda^{(d)}P_n$,$\gamma_{i,n} \to \gamma_i$ (with infinity as a possibility). WLOG assume $\gamma_{0,n} = 0$. Then there is a constant K (depending possibly on the sequence of P_n) and a bounded sequence $L_n \to Id$ in $GL(\mathbb{C}, n)$ so that if $\gamma_{i,n}$ does not belong to $A(r_n, 1)$ then the components of L_nP_n do not have zeros in $A(r_nK, K^{-1})$.*

Proof. We will prove it with a constant K depending on the sequence P_n. The standard diagonal argument will show that the constant does not depend on the sequence.

Let d be the smallest distance to zero of a component of P from zero (without counting the zeros at zero itself). Assume $K > 10/d$. Note that the zeros of L_nP_n (resp. $\Lambda^{(d)}L_nP_n$) converge to the zeros of P (or to infinity). Therefore, we will not have problems around the outer edge.

Now, by choosing a first round of L_n we may assume that the n_i coefficient of the j coordinate is $\delta_{i,j}$ vanishes. Now the result follows from the following lemma:

Lemma 2.14. *Let $P_n \to P$ be a sequence parametrized as in the previous lemma. Assume WLOG that P_n has a Jacobian zero at zero. Let $\tilde{P}_n$ be the zoom-in of P_n at the scale where the zeros of some co-ordinates are, and assume $\tilde{P}_n \to \tilde{P}$, then $\Lambda^{(d)}\tilde{P}$ has a zero at zero, of degree strictly smaller than the zero of $\Lambda^{(d)}\tilde{P}$.*

Using the lemma above, assume by contradiction the sequence P_n is such that L_nP_n contradicts the theorem. Pick a subsequence so that the minimum acceptable K for L_nP_n - call it K_n - goes to infinity. Zoom in at the scale where the first zeros appear, and by passing to a subsequence assume the $\tilde{P}_n$ converge. Since we are assuming K_n goes to infinity, the polynomial $\tilde{P}$ must have all the zeros concentrated back at zero again, which contradicts lemma 2.14. □

Proof (of lemma 2.14): Blow up at the scale where these zeros appear. New lower order term must appear on the 'blind spots' of the previous lower order terms, which means that the blow-up at the origin has smaller degree in one of the components, and therefore the Jacobian (which will have the degree of the appropriate Schur polynomial at zero) will have lower degree as well. □

We are now ready to make the proof of theorem 2.1 in full generality:

The proof below deserves a picture explaining what's going on

Proof (of theorem 2.1). :

Assume P_n is a sequence of polynomials such that the minimum number of connected components (which we know to be finite for each n by the previous section) goes to infinity as n goes to infinity. Without loss of generality, by a suitable affine transformation and by passing to a subsequence, we may assume that $P_n \to \mu$, a moment curve, and that the degree of each component of P_n is the same as the corresponding component on the moment curve.

This allows us, by lemma 2.11 setting $R = \infty$, to show that for n big enough, we can cover the complement of the unit ball with finitely many sets independently of n.

Now we will prove that the same is true for the interior of the ball. In order to do so we will assume that the $P_n \to P$ in the unit ball (not necessarily a moment curve anymore), and induct on the number of zeros (counting multiplicity) of $\Lambda^{(n)} P$ in the interior of the ball.

Our base cases are going to be:

- $\Lambda^{(d)} P$ has no zeros in the unit ball
- For any n, $\Lambda^{(d)} P_n$ only vanishes at zero in the unit ball.

the first base case follows from proposition 2.7 and compactness.

The second base case follows from the combination of lemma 2.11 and 2.13:

By 2.13, after the reparametrization (and re-centering) with the P_n, the components of $L_n P_n$ have no extraneous zeros in a neighborhood of zero, which means that 2.11 holds in the limit when we bring $r \to 0$ as well. Therefore, there is a neighborhood of zero that we can cover with an uniform number of open sets independent of n. The complement of that neighborhood in $\mathbb{C}$ is compact, and the polynomial is non-singular there, so we can fill it up uniformly for n big enough by proposition 2.7 again.

Now, for the case where there's multiple zeros, after possibly passing to a subsequence, we can group the zeros of $\Lambda^{(d)} P_n$ by the zero of $\Lambda^{(n)} P$ they converge to. For each of the groups, we can have two possiblities (again, after maybe passing to a subsequence):

1. All the zeros of $\Lambda^{(d)} P_n$ converging to the same zero are the same all the time (there is no *merging* of zeros).
2. All the subsequences have zeros *merging* to the zero of $\Lambda^{(d)} P$.

In the first case, by recentering a small $o(1)$ amount we can cover a neighbourhood of size $O(1)$ of the zero with $O(1)$ sets, by zooming in bit (independent of n) and using the second base case.

In the second case, center so that one of the zeros of $\Lambda^{(d)} P_n$ is exactly at zero. Recenter at that zero, modify the polynomials by L_n as in proposition 2.13 (which will only affect very close to the point of interest). Now, by 2.11 and the consequence of 2.13, we can cover an annuli of outer radius $O(1)$ and inner radius $o(1)$ with finitely many steps.

Now, the points outside the annuli (not counting the inner ball of the annuli) and the balls are non-singular, so can be covered with $O(1)$ sets, and to prove the theorem, we just have to prove it inside the balls we left inside the annuli.

By zooming into each ball and passing to a subsequence (with a diagonal argument for each ball), we can assume the zoomed-in versions converge. But by 2.13, inside each of the small ball all the zeros of the zoomed in cannot go to zero simultaneously (because the zeros of the components themselves do not). Therefore, this zoomed in limit must have the zeros separated (and thus each of them must have less degree), closing the induction. □

3 Injectivity

Lemma 3.1. *Let p be a polynomial curve such that p' is not degenerate (in the sense of the previous section). Let S be one of the sets in which we partitioned in the previous section. Then the map:*

$$\Sigma_p(z_1, \dots z_d) = \sum_{i=1}^{d} p(z_i) \tag{3.1}$$

is $O_{d,n}(1)-$to-1 modulo permutations in S^d excluding a possible bad set $B \subset \mathbb{S}^d$ of measure zero.

Proof. We can assume all the z_i are different (by the measure zero assumption). The fact that $\Lambda_{p'}(z_1, \dots z_d)$ does not vanish, tells us that the point does not belong to an irreducible variety of non-zero of $\Sigma_p^{-1}(\{\Sigma_p(z_1, \dots z_d)\})$. Therefore, only $O_{d,n}(1)-$ points on the set where all the z_i are different can have the same preimage. □

I conjecture that in fact Σ_p is injective in S (modulo permutations), but do not know how to prove it other than for the (regular) moment curve.