1 Study of the PDE

The 3-dimensional PDE we will study is:

$$\begin{cases} c_v T_t + \nabla \cdot q = 0 \\ q + \kappa \nabla T + l^2 (\nabla(\nabla \cdot q) + 2 * \triangle^{\otimes 3} q) = 0 \end{cases} \tag{1}$$

1.1 A one dimensional toy model

First, let's make some 1-dimensional back-of-the-envelope calculations. In 1 dimension, the equation reads:

$$\begin{cases} c_v T_t + q_x = 0 \\ q + \kappa T_x + 3l^2 q_{xx} = 0 \end{cases} \tag{2}$$

The ω-periodic solutions to this PDE can be found using the classical ansatz:

$$\begin{aligned} T &= \exp(i\omega t - kx) \\ q &= b \exp(i\omega t - kx) \end{aligned} \tag{3}$$

for possibly complex b,k. The parameter relationships then become:

$$\begin{cases} c_v i\omega - kb = 0 \\ b - \kappa k + 3l^2 bk^2 = 0 \end{cases} \tag{4}$$

Multiplying the second equation by k and substituting kb from the first one, this becomes:

$$c_v i\omega - \kappa k^2 + 3l^2 k^2 c_v i\omega = 0 \tag{5}$$

and therefore

$$k^2 = \frac{ic_v \omega}{\kappa} \left(1 - 3l^2 \frac{ic_v \omega}{\kappa}\right)^{-1} \tag{6}$$

note that (perturbatively) k has a complex phase angle of 45° and has a modulus of $\left(\frac{c_v \omega}{\kappa}\right)^{1/2}$, and therefore the penetration length λ of the heat wave is $\frac{1}{\Re(k)} = \frac{\sqrt{2}}{|k|}$. Therefore:

$$k^2 \approx k_0^2 \left(1 - 3il^2 |k|^2\right)^{-1} = k_0^2 \left(1 - 6i \left(\frac{l}{\lambda}\right)^2\right)^{-1} \tag{7}$$

By using the first equation on (4), we see that:

$$b^2 \approx b_0^2 \left(1 - 6i \left(\frac{l}{\lambda}\right)^2\right) \tag{8}$$

therefore, approximately

$$b \approx b_0 \left(1 - 3i \left(\frac{l}{\lambda}\right)^2\right) \tag{9}$$

Remark: A posteriori this computations turned out to be useless because we cannot simply relate on frequency related efects. This is because we have no clue of how do frequency-dependent issues work in the boundary between the metal and the silicon.

1.2 A surprising argument

Let (T_0, q_0) be the solution to a stationary fourier problem. That implies the following properties:

- $\triangle T = 0$

- $\triangle q = 0$ (because $Q = \nabla T$ and gradients and laplacians commute)

- $\nabla q = 0$

therefore $l^2(2\nabla(\nabla \cdot q) + \triangle q)$ is equal to zero, and the solution to the hydrodinamic problem shoud be the same. This contradicts previous results by COMSOL, so must be something fishy in my argument.

A troublesome way to bring those two results together would be to assume the solution to the pde is not unique.

1.3 Some simplification on the PDE

By simply plugging the second equation on the first one, and using the equality $\nabla \cdot \triangle^{\otimes 3} = \triangle \nabla \cdot$, we obtain the (simpler) pde:

$$c_v(1 + 3l^2\triangle)\dot{T} = \kappa\triangle T \tag{10}$$

In the case of $\omega-$periodic solutions, we can make the ansatz $\frac{d}{dt} = iw$ this becomes an eigenvalue problem:

$$T = -i\left(\frac{\kappa}{c_v\omega} - 3l^2\right)\triangle T \tag{11}$$

While this, a priori, gives a new understanding of the PDE it is not as deep as it might look at first time, because new boundary conditions are not trivial anymore (there is no way to directly, pointwise, recover q from T, one has to solve a new PDE). In problems where only Dirichlet conditions on T are given, however, this method would simplify the computations.

1.4 Perturbative analysis of the PDE

As an other approach, since l^2 is small, one can attempt to compute the derivative of the solutions with respect to l^2 (which will be denoted T_l, and use those as tools to see where to check for a solution. By doing so, one obtains:

$$\begin{cases} c_v\dot{T}_l + \nabla \cdot q_l = 0 \\ q_l + \kappa\nabla T_l + \nabla(\nabla \cdot q) + 2 * \triangle^{\otimes 3}q = 0 \end{cases} \tag{12}$$

Intuitively, we can understand this system as a system where there is an extra pumped heat, given by the term $-\nabla(\nabla \cdot q) - 2 * \triangle^{\otimes 3}q$ to the system. Therefore, in an attempt to solve this system, let $\tilde{q} = q + \nabla(\nabla \cdot q) + 2 * \triangle^{\otimes 3}q$ (the heat current minus the pumped heat). With this change of variables, the equation becomes:

$$\begin{cases} c_v\dot{T}_l + \nabla \cdot \tilde{q} = \nabla \cdot (\nabla(\nabla \cdot q) + 2 * \triangle^{\otimes 3}q) \\ \tilde{q} + \kappa\nabla T_l = 0 \end{cases} \tag{13}$$

If we use again the equality $\nabla \cdot \triangle^{\otimes 3} = \triangle \nabla \cdot$ and the inital equation $c_v T_t + \nabla \cdot q = 0$, one finds:

$$\begin{cases} c_v \dot{T}_l + \nabla \cdot \tilde{q} = -3c_v \dot{T} \\ \tilde{q} + \kappa \nabla T_l = 0 \end{cases} \tag{14}$$

This equation is akin to the original fourier equation, with the difference that now there is an external heat source inside the material.

$$\begin{cases} i\omega c_v T_l + \nabla \cdot \tilde{q} = -3i\omega c_v T \\ \tilde{q} + \kappa \nabla T_l = 0 \end{cases} \tag{15}$$

Using the same arguments as in the previous sections one can see that $\triangle q_0 = \nabla(\nabla \dot{q}) = \frac{i\omega c_v}{k} q$, and thus $q_1 - \tilde{q}$ has zero perpendicular component in the boundary. Since q_1, by Neumann conditions has the zero perpendicular component, the same must apply to $\tilde{q}$

$$\begin{cases} T = 0, & \text{On the heater boundary} \\ \tilde{q}_\perp = 0, & \text{Elsewhere} \end{cases} \tag{16}$$

So we can understand the phonon case as an extra heat term that appears.

2 The Experiment

We are in a situation where we do not (fully) know the physics inside the dectector, but we can assume it is some kind of linear-reacting object, at least near equilibrium.

Therefore we want to avoid using it in our computations. This leads to the use of the *Impedance method*.