Samedi 12 août 2023 **9** Relations

9.1 Relations and Their Properties

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

In this section we introduce the basic terminology used to describe binary relations. Later in this chapter we will use relations to solve problems involving
→ communications networks
→ project scheduling
→ and identifying elements in sets with common properties.

Definition 1. Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

$R \subseteq A \times B$

In other words, a binary relation from A to B is a set R of ordered pairs, where the first element of each ordered pair comes from A and the second element comes from B.
We use the notation aRb to denote that $(a,b) \in R$ and $a \not R b$ to denote that $(a,b) \notin R$. Moreover, when (a,b) belongs to R, a is said to be related to b by R.

Functions as Relations

A relation can be used to express a one-to-many relationship between the elements of the sets A and B, where an element of A may be related to more than one element of B.
A function represents a relation where exactly one element of B is related to each element of A. Relations are generalization of graphs of functions; they can be used to express a much wider class of relationships between sets. (Recall that the graph of the function f from A to B is the set of ordered pairs $(a, f(a))$ for $a \in A$.)

Relations on a set

Definition 2. A relation on a set A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a subset of $A \times A$.

Properties of Relations

There are several properties that are used to classify relations on a set.
In some relations an element is always related to itself. For instance, let $\mathcal{R}$ be the relation on the set of all people consisting of pairs (x,y) where x and y have the same mother and the same father. Then $x\mathcal{R}x$ for every person x.

Definition 3. A relation $\mathcal{R}$ on a set A is called reflexive if $(a,a) \in \mathcal{R}$ for every element $a \in A$.

Remark: Using quantifiers we see that the relation $\mathcal{R}$ on the set A is reflexive if $\forall a((a,a) \in \mathcal{R})$, where the universe of discourse is the set of all elements in A.

Definition 4. A relation $\mathcal{R}$ on a set A is called symmetric if $(b,a) \in \mathcal{R}$ whenever $(a,b) \in \mathcal{R}$, for all $a,b \in A$.
A relation $\mathcal{R}$ on a set A such that for all $a,b \in A$, if $(a,b) \in \mathcal{R}$ and $(b,a) \in \mathcal{R}$, then $a=b$ is called antisymmetric.

Remark: Using quantifiers, we see that the relation $\mathcal{R}$ on the set A is symmetric if $\forall a \forall b((a,b) \in \mathcal{R} \to (b,a) \in \mathcal{R})$.
Similarly, the relation $\mathcal{R}$ on the set A is antisymmetric if $\forall a \forall b(((a,b) \in \mathcal{R} \land (b,a) \in \mathcal{R}) \to (a=b))$.

In other words, a relation is symmetric iff a is related to b always implies that b is related to a.
For instance, the equality is symmetric because $a=b$ iff $b=a$.
A relation is antisymmetric iff there are no pairs of distinct elements a and b with a related to b and b related to a.
That is, the only way to have a related to b and b related to a is for a and b to be the same element.
For instance, the less than or equal to relation is antisymmetric. To see this, note that $a \le b$ and $b \le a$ implies that $a=b$.
The term symmetric and antisymmetric are not opposites, because a relation can have both of these properties or may lack both of them. A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a,b) in which $a=b$

Definition 5. A relation $\mathcal{R}$ on a set A is called transitive if whenever $(a,b) \in \mathcal{R}$ and $(b,c) \in \mathcal{R}$, then $(a,c) \in \mathcal{R}$ $\forall a,b,c \in A$.

Remark: Using quantifiers, $\forall a \forall b \forall c(((a,b) \in \mathcal{R} \land (b,c) \in \mathcal{R}) \to ((a,c) \in \mathcal{R}))$

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A and B can be combined in any way two sets can be combined.

Example 18 Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that $\mathcal{R}_1$ consist of all ordered pair (a,b), where a is a student who has taken course b, and $\mathcal{R}_2$ consist of all (a,b), where a is a student who requires course b to graduate. What are the relations $\mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}_1 - \mathcal{R}_2$, $\mathcal{R}_2 - \mathcal{R}_1$?

The relation $\mathcal{R}_1 \cup \mathcal{R}_2$: (a,b), where a is a student who either has taken course b or needs course b to graduate.

$\mathcal{R}_1 \cap \mathcal{R}_2$: (a,b), where a is a student who has taken course b and needs this course to graduate.

$\mathcal{R}_1 \oplus \mathcal{R}_2$: (a,b), where a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it.

$\mathcal{R}_1 - \mathcal{R}_2$: (a,b), where a taken course b but does not need it to graduate; that is, b is an elective course.

$\mathcal{R}_2 - \mathcal{R}_1$: (a,b), where b is a course that a needs to graduate but has not taken.

Example 19 Let $\mathcal{R}_1 = \{(x,y) \mid x<y\}$ and $\mathcal{R}_2 = \{(x,y) \mid x>y\}$. What are $\mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{R}_1 - \mathcal{R}_2$, $\mathcal{R}_2 - \mathcal{R}_1$ and $\mathcal{R}_1 \oplus \mathcal{R}_2$?

$\mathcal{R}_1 \cup \mathcal{R}_2$ iff $x<y$ or $x>y$. Because the condition $x<y$ or $x>y$ is the same as the condition $x \ne y$, it follows that $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(x,y) \mid x \ne y\}$.

Next, note it is impossible for a pair (x,y) to belong to both $\mathcal{R}_1$ and $\mathcal{R}_2$ because it is impossible that $x<y$ and $x>y$. It follows that $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$.

We also see that $\mathcal{R}_1 - \mathcal{R}_2 = \mathcal{R}_1$, $\mathcal{R}_2 - \mathcal{R}_1 = \mathcal{R}_2$, and $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}_2 \cup \mathcal{R}_2 - \underbrace{\mathcal{R}_1 \cap \mathcal{R}_2}_{=\emptyset} = \{(x,y) \mid x \ne y\}$

Composite relations $S \circ R$,
n-ary relations (used to represent computer databases)
SQL,
Data mining

} Skipped.
In this chapter, the word relation will always refer to binary relation.

Example 5 Consider these relations on the set of integers:

$\mathcal{R}_1 = \{(a,b) \mid a \le b\}$
$\mathcal{R}_2 = \{(a,b) \mid a > b\}$
$\mathcal{R}_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
$\mathcal{R}_4 = \{(a,b) \mid a = b\}$
$\mathcal{R}_5 = \{(a,b) \mid a = b+1\}$
$\mathcal{R}_6 = \{(a,b) \mid a+b \le 3\}$

Which of these relations contains each of the pairs $(1,1)$, $(1,2)$, $(2,1)$, $(1,-1)$, and $(2,2)$?

The pair $(1,1)$ is in $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_6$;
$(1,2)$ is in $\mathcal{R}_1$ and $\mathcal{R}_6$;
$(2,1)$ is in $\mathcal{R}_2, \mathcal{R}_5, \mathcal{R}_6$;
$(1,-1)$ is in $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_6$;
$(2,2)$ is in $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4$;

Example 8 Which of the relation from Ex 5 are reflexive?

The reflexive relations are $\mathcal{R}_1$ (because $a \le a$ for every integer a), $\mathcal{R}_3$ and $\mathcal{R}_4$.
For each of the other relations in this example it is easy to find a pair of the form (a,a) that is not in the relation.

Example 11 Which of the relation from Ex 5 are symmetric and which are antisymmetric?

$\mathcal{R}_3$ is symmetric, for if $a=b$ or $a=-b$, then $b=a$ or $b=-a$.
$\mathcal{R}_4$ is symmetric because $a=b$ implies that $b=a$.
$\mathcal{R}_6$ is symmetric because $a+b \le 3$ implies that $b+a \le 3$.

$\mathcal{R}_1$ is antisym. because the inequalities $a \le b$ and $b \le a$ imply that $a=b$.
$\mathcal{R}_2$ is antisym. because it is impossible that $a>b$ and $b>a$.
$\mathcal{R}_4$ is antisym. because two elements are related with respect to $\mathcal{R}_4$ iff they are equal.
$\mathcal{R}_5$ is antisym. because it is impossible that $a=b+1$ and $b=a+1$.

Example 14 Which of the relation in Ex 5 are transitive?

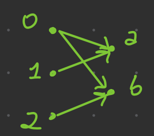
$\mathcal{R}_1$ is transitive because $a \le b$ and $b \le c$ imply that $a \le c$.
$\mathcal{R}_2$ is transitive because $a>b$ and $b>c$ imply that $a>c$.
$\mathcal{R}_3$ is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$.
$\mathcal{R}_4$ is clearly transitive.

$\mathcal{R}_5$ is not transitive because $(2,1)$ and $(1,0)$ belong to $\mathcal{R}_5$, but $(2,0)$ does not.
$\mathcal{R}_6$ is not transitive because $(2,1)$ and $(1,2)$ belong to $\mathcal{R}_6$, but $(2,2)$ does not.

Example 1 Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a,b), where a is a student enrolled in course b. For instance, if Jason and Deborah are enrolled in CS518, the pairs (Jason, CS518) and (Deborah, CS518) belong to R. If Jason is also enrolled in CS510, then the pair (Jason, CS510) is also in R. However, if Deborah is not enrolled in CS510, the the pair (Deborah, CS510) is not in R.
Note that if a students is not currently enrolled in any courses there will be no pairs in R that have this student as the first element. Similarly, if a course is not currently being offered there will be no pairs in R that have this course as their second element.

Example 2 Let A be the set of the cities in the USA, and let B be the set of the 50 states in the USA. Define the relation R by specifying that (a,b) belongs to R if a city with name a is in the state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R.

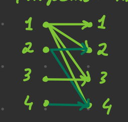
Example 3 Let $A = \{0,1,2\}$ and $B = \{a,b\}$. Then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation from A to B. This means, for instance, that $0Ra$, but that $1 \not R b$.
Relation can be represented graphically using arrows to represent ordered pairs. Another way to represent this relation is to use a table.



R	a	b
0	X	X
1	X	
2		X

Example 4 Let A be the set $\{1,2,3,4\}$. Which ordered pairs are in the relation $R = \{(a,b) \mid \overbrace{a \text{ divides } b}^{a \mid b}\}$?

Because (a,b) is in R iff a and b are positive integers not exceeding 4 such that a divides b, we see that $\mathcal{R} = \{(1,1), (1,2), (1,3), (1,3), (2,2), (2,4), (3,3), (4,4)\}$
The pairs in this relation can be displayed both graphically and in tabular form.



$\mathcal{R}$	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

Example 6 How many relations are there on a set with n elements?

A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements.
For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a,b,c\}$. $\} \mathcal{R} = \{\underbrace{\emptyset}_{1}, \underbrace{\{(a,a)\}}_{2}, \underbrace{\{(a,b)\}}_{3}, \underbrace{\{(a,c)\}}_{4}, \underbrace{\{(b,a)\}}_{5}, \ldots, \underbrace{\{(a,a), \ldots (c,c)\}}_{512}\}$

Example 7 Consider the following relations on $\{1,2,3,4\}$:

$\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,
$\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\}$,
$\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,
$\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
$\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,
$\mathcal{R}_6 = \{(3,4)\}$.

Which of these relations are reflexive?

The relation $\mathcal{R}_3$ and $\mathcal{R}_5$ are reflexive because they both contain all pairs of the form (a,a), namely, $(1,1),(2,2),(3,3)$ and $(4,4)$.
The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_4$ and $\mathcal{R}_6$ are not reflexive because $(3,3)$ is not in any of these relations.

Example 9 Is the "divides" relation on the set of positive integers reflexive?

Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive. (Note that if we replace the set of positive integers with the set of all integer the relation is not reflexive because by definition 0 does not divide 0.

Example 10 Which of the relations from Ex 7 are symmetric and which are antisymmetric?

The relation $\mathcal{R}_2$ and $\mathcal{R}_3$ are symmetric, because in each case (b,a) belongs to the relation whenever (a,b) does.
$\mathcal{R}_4, \mathcal{R}_5$ and $\mathcal{R}_6$ are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \ne b$ such that both (a,b) and (b,a) belong to the relation.

Example 12 Is the "divides" relation on the set of positive integers symmetric? Is it antisym.?

This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$.
However, it is antisym. To see this, note that if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a=b$.

Example 13 Which of the relation in Ex 7 are transitive?

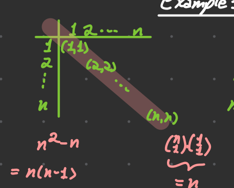
$\mathcal{R}_4, \mathcal{R}_5$ and $\mathcal{R}_6$ are transitive. For each of these relations, we can show that it is transitive by verifying that if (a,b) and (b,c) belong to this relation, then (a,c) also does.

$\mathcal{R}_1$ is not transitive because $(3,4)$ and $(4,1)$ belong to $\mathcal{R}_1$, but $(3,1)$ does not.
$\mathcal{R}_2$ is not transitive because $(2,1)$ and $(1,2)$ belong to $\mathcal{R}_2$, but $(2,2)$ does not.
$\mathcal{R}_3$ is not transitive because $(4,1)$ and $(1,2)$ belong to $\mathcal{R}_3$, but $(4,2)$ does not.

Example 15 Is the "divides" relation on the set of positive integers transitive?

Suppose that $a \mid b$ and $b \mid c$. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divise c.
It follows that this relation is transitive.

Example 16 How many reflexive relations are there on a set with n elements?



A relation $\mathcal{R}$ on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in $\mathcal{R}$.
However, if $\mathcal{R}$ is reflexive, each of the n ordered pairs (a,a) for $a \in A$ must be in $\mathcal{R}$. Each of the other $n(n-1)$ ordered pairs of the form (a,b), where $a \ne b$, may or may not be in $\mathcal{R}$. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations.

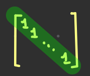
Formulas for the number of symmetric relations and the number of antisymmetric relations on a set with n elements can be found using similar reasoning.
However, no general formula is known that counts the transitive relations on a set with n elements.

9.3 Representing Relations

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the proprieties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix $M_{\mathcal{R}} = [m_{ij}]$, where $m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in \mathcal{R} \\ 0, & \text{if } (a_i, b_j) \notin \mathcal{R} \end{cases}$



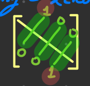
Reflexive

$m_{ii} = 1$, for $i = 1, 2, \ldots, n$

Symmetric

$m_{ij} = m_{ji}$

$M_{\mathcal{R}} = (M_{\mathcal{R}})^T$



Antisymmetric

$m_{ij} = 1$ with $i \neq j$

then $m_{ji} = 0$

$m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

The Boolean operations join and meet can be used to find the matrices representing the union and the intersection of two relations.

$$\begin{cases} M_{\mathcal{R}_1 \cup \mathcal{R}_2} = M_{\mathcal{R}_1} \vee M_{\mathcal{R}_2} \\ M_{\mathcal{R}_1 \cap \mathcal{R}_2} = M_{\mathcal{R}_1} \wedge M_{\mathcal{R}_2} \end{cases}$$

The matrix for the composite of relations can be found using the Boolean product of matrices

$$\underbrace{M_{S \circ \mathcal{R}}}_{\text{Composite of the relation } \mathcal{R} \text{ and } S.} = M_{\mathcal{R}} \odot M_S$$

$\mathcal{R} \subseteq \{1,2,3\} \times \{1,2,3,4\}$

with $\mathcal{R} = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$

$S \subseteq \{1,2,3,4\} \times \{0,1,2\}$

with $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$

$1 \to 1 \to 0$ $(1,0)$

$1 \to 4 \to 1$ $(1,1)$

$2 \to 3 \to 2$ $(2,1)$

$2 \to 3 \to 2$ $(2,2)$

$3 \to 1 \to 0$ $(3,0)$

$3 \to 4 \to 1$ $(3,1)$

Hence, $S \circ \mathcal{R} = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$

In general, $\mathcal{R}^1 = \mathcal{R}$ and $\mathcal{R}^{n+1} = \underbrace{\mathcal{R}^n}_{\mathcal{R}^n \subseteq \mathcal{R} \text{ for } n = 1,2,3,\ldots} \circ \mathcal{R}$

Representing Relations Using $\underbrace{\text{Digraphs}}_{\text{Directed graphs.}}$

Definition 1. A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs), edge (a, b) — initial vertex a, terminal vertex b.
An edge of the forme (a,a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.

The directed graph representing a relation can be used to determined whether the relation has various properties.
For instance, a relation is reflexive iff there is a loop at every vertex of the diagraph, so that every ordered pair of the form (x,x) occurs in the relation. A relation is symmetric iff for every distinct vertices in its diagraph there is an edge in the opposite direction, so that (y,x) is in the relation whenever (x,y) is in the relation.
Similarly, a relation is antisymmetric iff there are never two edges in opposite direction between distinct vertices.
Finally, a relation is transitive iff whenever there is an edge from vertex x to y and y to z, there is an edge from x to z. (Completing a triangle where each side is a directed edge with the correct direction).
Remark: Note that symmetric relation can be represented by an undirected graph, which is a graph where edges do not have directions.

9.4 Closures of Relations

Definition 1. If $\mathcal{R}$ is a relation on a set A, then the closure of $\mathcal{R}$ with respect to $\underbrace{P}_{\text{Reflexivity, symmetry, or transitivity.}}$, if it exist,
is the relation S on A with property P that contains $\mathcal{R}$ and is a subset of every subset of $A \times A$ containing $\mathcal{R}$ with property P.
If there is a relation S that is a subset of every relation containing $\mathcal{R}$ with property P, it must be unique.

(I.) The reflexive closure equals $\mathcal{R} \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation on A.

(II.) The symmetric closure of an relation can be constructed by taking the union of a relation with its inverse; that is, $\mathcal{R} \cup \mathcal{R}^{-1}$
where $\mathcal{R}^{-1} = \{(b,a) \mid (a,b) \in \mathcal{R}\}$

(III.) Constructing the transitive closure of a relation is more complicated. It can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Example 1 | What is the reflexive closure of the relation $\mathcal{R} = \{(a,b) \mid a < b\}$ on the set of integers?

The reflexive closure of $\mathcal{R}$ is $\mathcal{R} \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in \mathbb{Z}\} = \{(a,b) \mid a \leq b\}$

Example 2 | What is the symmetric closure of the relation $\mathcal{R} = \{(a,b) \mid a > b\}$ on the set of positive integers?

$\mathcal{R} \cup \mathcal{R}^{-1} = \{(a,b) \mid a > b\} \cup \{(a,b) \mid a < b\} = \{(a,b) \mid a \neq b\}$

Paths in Directed Graphs

Representing relations by directed graphs helps in the construction of transitive closure. We now introduce some terminology that we will use for this purpose.

Definition 2. A path from a to b in the directed graph G is a sequence of edges $(\underbrace{x_0}_{x_0 = a}, x_1), (x_1, x_2), \ldots, (x_{n-1}, \underbrace{x_n}_{x_n = b})$ in G.
The path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n.
We view the empty set of edges as a path of length zero from a to a.
A path of length $n \geq 1$ that begins and end at the same vertex is called a circuit or cycle
A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

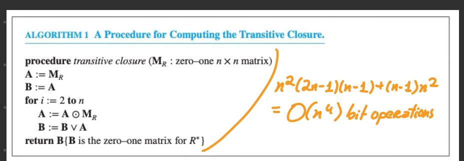
Theorem 1. Let $\mathcal{R}$ be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b iff $(a,b) \in \underbrace{\mathcal{R}^n}_{\mathcal{R}^n = \mathcal{R}^{n-1} \circ \mathcal{R}}$

Definition 3. Let $\mathcal{R}$ be a relation on a set A. The connectivity relation $\mathcal{R}^*$ consists of the pairs (a,b) such that there is a path of length at least one from a to b in $\mathcal{R}$.
Because $\mathcal{R}^n$ consists of the pairs (a,b) such that there is a path of length n from a to b, it follows that $\mathcal{R}^*$ is the union of all the sets $\mathcal{R}^n$.
In other words, $\underbrace{\mathcal{R}^*}_{(a,b) \in \mathcal{R}^*} = \bigcup_{n=1}^{\infty} \underbrace{\mathcal{R}^n}_{\substack{(a,c) \in \mathcal{R}^2 \\ (c,b) \in \mathcal{R}^2}}$ The connectivity relation is useful in many models

Theorem 2. The transitive closure of a relation $\mathcal{R}$ equals the connectivity relation $\mathcal{R}^*$

Lemma 1. Let A be a set with n elements, and let $\mathcal{R}$ be a relation on A. If there is a path of length at least one in $\mathcal{R}$ from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in $\mathcal{R}$ from a to b, then there is such a path with length $\underbrace{\text{not exceeding } n-1}_{\text{Trees (Chap. 10)}}$.

Theorem 3. Let $M_{\mathcal{R}}$ be the zero-one matrix of the relation $\mathcal{R}$ on a set with n elements.
Then the zero-one matrix of the transitive closure $\mathcal{R}^*$ is $M_{\mathcal{R}^*} = M_{\mathcal{R}} \vee M_{\mathcal{R}}^{[2]} \vee \ldots M_{\mathcal{R}}^{[n]}$



ALGORITHM 1 A Procedure for Computing the Transitive Closure.

```
procedure transitive closure (M_R : zero–one n × n matrix)
A := M_R
B := A
for i := 2 to n
    A := A ⊙ M_R
    B := B ∨ A
return B{B is the zero–one matrix for R*}
```

$n^2(2n-1)(n-1) + (n-1)n^2 = O(n^4)$ bit operations

Warshall's Algorithm (Roy-Warshaw algo.)

Warshall's algo. is an efficient method for computing the transitive closure of a relation using only $2n^3$ bit operations.

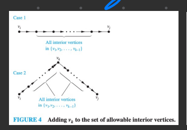
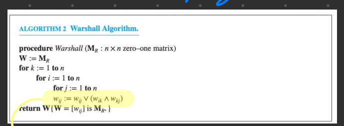


FIGURE 4 Adding v_k to the set of allowable interior vertices.



ALGORITHM 2 Warshall Algorithm.

```
procedure Warshall (M_R : n × n zero–one matrix)
W := M_R
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
            w_ij := w_ij ∨ (w_ik ∧ w_kj)
return W{W = [w_ij] is M_R*}
```

$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]})$

, where $w_{ij}^{(k)} = 1$, if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$

9.5 Equivalence Relations

In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. Consequently, the compiler considers strings longer than eight characters that agree in their first eight characters the same. Let $\mathcal{R}$ be the relation on the set of strings of characters such that $s\mathcal{R}t$, where s and t are two strings, if s and t are at least eight characters long and the first eight characters of s and t agree, or $s = t$. It is easy to see that $\mathcal{R}$ is reflexive, symmetric, and transitive.

Definition 1. A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition 2. Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements.

Example 1 | Let the relation on $\mathbb{Z}$ such that $a\mathcal{R}b$ iff $a = b$ or $a = -b$. In Section 9.1 we showed that $\mathcal{R}$ is reflexive, symmetric, and transitive. It follows that $\mathcal{R}$ is an equivalence relation.

Example 2 | Let $\mathcal{R}$ be the relation on $\mathbb{R}$ such that $a\mathcal{R}b$ iff $a - b$ is an integer. Is $\mathcal{R}$ an equivalence relation?

(I.) Reflexive: $a - a = 0$ is an integer $\forall a \in \mathbb{R}$. Hence $a\mathcal{R}a$.

(II.) Symmetric: Suppose $a\mathcal{R}b$, then $a - b$ is an integer, so $b - a$ is also an integer. Hence $a\mathcal{R}b \to b\mathcal{R}a$.

(III.) Transitive: $a - b$ and $b - c$ are integers, then $a - c = (a - c) + (b - c)$ is also an integer. Hence, $a\mathcal{R}b \wedge b\mathcal{R}c \to a\mathcal{R}b$

Example 3 | Let m be an integer with $m > 1$. Show that the relation $\mathcal{R} = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on $\mathbb{Z}$.
Recall that $a \equiv b \pmod{m}$ iff $m \mid (a - b)$.

(I.) Reflexivity: $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence $a \equiv a \pmod{m}$.

(II.) Symmetric: Suppose $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where $k \in \mathbb{Z}$. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.

(III.) Transitive: Suppose $a \equiv b[m]$ and $b \equiv c[m]$. Then m divides both $a - b$ and $b - c$. Therefore, $a - b = km$ and $b - c = \ell m$, where $k, \ell \in \mathbb{Z}$. Adding these two equations shows that $a - c = km + \ell m = (k + \ell)m$.
Thus $a \equiv c[m]$

Because $\mathcal{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation. ∎

Example 4 | Suppose that $\mathcal{R}$ is the relation on the set of strings of English letters such that $a\mathcal{R}b$ iff $\ell(a) = \ell(b)$, where $\ell(x)$ is the length of the string x. Is $\mathcal{R}$ an equivalence relation?

(I.) Reflexive: $\ell(a) = \ell(a)$, it follows that $a\mathcal{R}a$.

(II.) Symmetric: Suppose that $a\mathcal{R}b$, so that $\ell(a) = \ell(b)$. Then $b\mathcal{R}a$, because $\ell(b) = \ell(a)$.

(III.) Transitive: Suppose that $a\mathcal{R}b$ and $b\mathcal{R}c$, then $\ell(a) = \ell(b)$ and $\ell(b) = \ell(c)$. Hence, $\ell(a) = \ell(c)$, so $a\mathcal{R}c$.

Example 5 | Let n be a positive integer and S a set of strings. Suppose that $\mathcal{R}_n$ is the relation on S such that $s\mathcal{R}_n t$ iff $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same.
For instance, $01\,\mathcal{R}_3\,01$ and $00111\,\mathcal{R}_3\,00101$, but $01 \not\mathcal{R}_3\, 010$ and $01011 \not\mathcal{R}_3\, 01110$. Show that for every set S of strings and every positive integer n, $\mathcal{R}_n$ is an equivalence relation on S.

(I.) Reflexive: $s = s$, so that $s\mathcal{R}_n s$ whenever s is a string in S.

(II.) Symmetric: If $s\mathcal{R}_n t$, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that $t\mathcal{R}_n s$.

(III.) Transitive: Suppose that $s\mathcal{R}_n t$ and $t\mathcal{R}_n u$. Then either $s = t$ or ... and $t = u$ or ... From this, we can deduce that either $s = u$ or ...

Example 7 | Let $\mathcal{R}$ be the relation on the set of real numbers such that $x\mathcal{R}y$ iff x and y are real numbers that differ by less than 1. that is, $|x - y| < 1$. Show that $\mathcal{R}$ is not an equivalence relation.

(I.) Reflexive: $|x - x| = 0 < 1$, whenever $x \in \mathbb{R}$.

(II.) Symmetric: If $x\mathcal{R}y$, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that $y\mathcal{R}x$.

(III.) Transitive: Counterexample: $2.8\,\mathcal{R}\,1.9$ and $1.9\,\mathcal{R}\,1.1$, but $2.8 \not\mathcal{R}\, 1.1$.

Equivalence Classes

Definition 3. Let $\mathcal{R}$ be an equivalence relation on set A. The set of all elements that are related to an element a of A is called the equivalence class of a.
The equivalence class of a with respect to $\mathcal{R}$ is denoted by $[a]_{\mathcal{R}} = \{s \mid (a,s) \in \mathcal{R}\}$
If $b \in [a]_{\mathcal{R}}$, then b is called a representative of this equivalence class.

Equivalence Classes and Partitions

Th.1 shows that the equivalence classes of two elements of A are either identical or disjoint.

Theorem 1. Let $\mathcal{R}$ be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:
(I.) $a\mathcal{R}b$
(II.) $[a]=[b]$
(III.) $[a] \cap [b] \neq \emptyset$

$\Rightarrow \bigcup_{a \in A} [a]_{\mathcal{R}} = A$
$[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} = \emptyset$, when $[a]_{\mathcal{R}} \neq [b]_{\mathcal{R}}$

Theorem 2. Let $\mathcal{R}$ be an equivalence relation on a set S. Then the equivalence classes of $\mathcal{R}$ form a partition of S.
Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation $\mathcal{R}$ that has the sets A_i, $i \in I$, as its equivalence classes

9.6 Partial Orderings

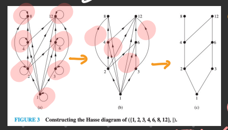
Definition 1. A relation $\mathcal{R}$ on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.
A set S together with a partial ordering $\mathcal{R}$ is called a partially ordered set, or poset, and is denoted by $(S,\mathcal{R})$
Member of S are called elements of the poset.

Definition 2. The elements a and b of a poset $(S, \preccurlyeq)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called incomparable.

Definition 3. If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Definition 4. $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Hasse Diagrams



(I.) (II.) (III.)

Many edges in the directed graph for a finite poset do not have to be shown because they must be present.
(I.) Remove all loops
(II.) Remove all edges (x,y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$
(III.) Remove all arrows.

Lattices...

Topological Sorting...

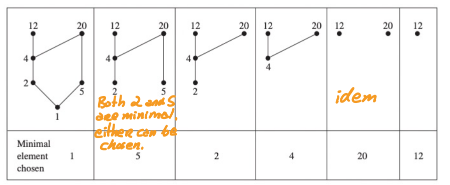


FIGURE 9 A topological sort of ({1, 2, 4, 5, 12, 20}, |).

This produce the total ordering
$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$
Topological sorting has an application to the scheduling of projects.

EXAMPLE 27 A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task $X \prec$ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure 10. Find an order in which these tasks can be carried out to complete the project.

Solution: An ordering of the seven tasks can be obtained by performing a topological sort. The steps of a sort are illustrated in Figure 11. The result of this sort, $A \prec C \prec B \prec E \prec F \prec D \prec G$, gives one possible order for the tasks. ◀

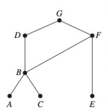


FIGURE 10 The Hasse diagram for seven tasks.

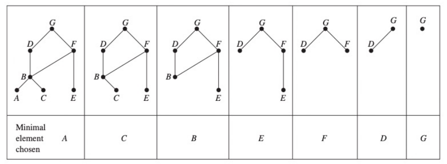


FIGURE 11 A topological sort of the tasks.

ALGORITHM 1 Topological Sorting.

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
  $S := S - \{a_k\}$
  $k := k + 1$
return $a_1, a_2, \ldots, a_n$ $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

Example 8 What is the equivalence class of an integer for the equivalence relation of Ex 1?

Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a=0$.

Example 10 What is the equivalence class of the string 0111 with respect to the equivalence relation $\mathcal{R}_3$ from Ex 5. on the set of all bit strings?

$[011]_{\mathcal{R}_3} = \{011, 0110, 0111, 01100, 01101, \ldots\}$
The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011.

Example 15 Let $\mathcal{R}_3$ be the relation from Ex 5. What are the sets in the partition of the set of all bit strings arising from the relation $\mathcal{R}_3$ on the set of all bit strings? (Recall that $s\mathcal{R}_3t$, where s and t are bit strings, if $s=t$ or s and t are bit strings with at least three bits that agree in their first three bits).

Note that every bit string of length less than three is equivalent only to itself.
Hence $[\lambda]_{\mathcal{R}_3} = \{\lambda\}$, $[0]_{\mathcal{R}_3} = \{0\}$, $[1]_{\mathcal{R}_3} = \{1\}$, $[00]_{\mathcal{R}_3} = \{00\}$, $[01]_{\mathcal{R}_3} = \{01\}$, $[10]_{\mathcal{R}_3} = \{10\}$, $[11]_{\mathcal{R}_3} = \{11\}$.
Note that every bit string of length three or more is equivalent to one of the eight bit strings 000, 100, 001, 101, 010, 110, 011, 111.
We have:

$[000]_{\mathcal{R}_3} = \{000, 0000, 0001, 00000, 00001, \ldots\}$
$[001]_{\mathcal{R}_3} = \{001, 0010, 0011, 00100, \ldots\}$
$[010]_{\mathcal{R}_3} = \{010, 0100, 0101, 01000, \ldots\}$
$[011]_{\mathcal{R}_3} = \{011, 0110, 0111, 01100, \ldots\}$
$[100]_{\mathcal{R}_3} = \{100, 1000, 1001, 10000, \ldots\}$
$[101]_{\mathcal{R}_3} = \{101, 1010, 1011, 10100, \ldots\}$
$[110]_{\mathcal{R}_3} = \{110, 1100, 1101, 11000, \ldots\}$
$[111]_{\mathcal{R}_3} = \{111, 1110, 1111, 11100, \ldots\}$

There are 15 equivalence classes are disjoint and every bit string is in exactly one of them.
As Th.2 tells us, these equivalence classe partition the set of all bit strings.

Example 5 In the poset $(\mathbb{Z}^+, |)$ are integers 3 and 9 comparable? Are 5 and 7 comparable?

The integers 3 and 9 are comparable, because $3 \mid 9$.
The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

Example 8 What is the equivalence class of 0, 1, 2, and 3 for congruence modulo 4?

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$
$[2] = \{\ldots -6, -2, 2, 6, 10, \ldots\}$
$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

The equivalence classes of the relation congruence modulo m are called the congruence classes modulo m. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots a-2m, a-m, a, a+m, a+2m, \ldots\}$
$= \{a + km \mid k \in \mathbb{Z}\}$

Example 14 What are the sets in the partition of the integers arising from congruence modulo 4?

In Ex 9 we found the four congruence classes: $[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
$[1]_m = \{\ldots -7, -3, 1, 5, 9, \ldots\}$
$[2]_m = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$
$[3]_m = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

These congruence classes are disjoint, and every integer is in exacty one of them.
In other words, Th.2 says, these congruence classes form a partition.

Example 1 Show that the greater than or equal to relation $(\geq)$ is a partial ordering on the set of integers.

(I.) Reflexive: $a \geq a$ for every integer a.
(II.) Antisymmetric: $a \geq b$ and $b \geq a$, then $a=b$. Hence $\geq$ is antisymmetric
(III.) Transitive: $a \geq b$ and $b \geq c$ imply that $a \geq c$.
It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset. ∎

Example 2 The divisibility relation $\mid$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive. We see that $(\mathbb{Z}^+, |)$ is a poset.

Example 3 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of S.

(I.) Reflexive: $A \subseteq A$ whenever A is a subset of S.
(II.) Antisymmetric: $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.
(III.) Transitive: $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$.

Hence, $\subseteq$ is a partial ordering on $\mathcal{P}(S)$, and $(\mathcal{P}(S), \subseteq)$ is a poset. ∎

Example 4 Let $\mathcal{R}$ be the relation on the set of people such that $x\mathcal{R}y$ if x and y are people and x is older than y. Show that $\mathcal{R}$ is not a partial ordering.

$\mathcal{R}$ is not reflexive, because no person is older than himself or herself. That is, $x \not\mathcal{R} x$ for all people x.
It follows that $\mathcal{R}$ is not a partial ordering.