Lundi 7 août 2023 **2** Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Many important discrete structures are built using sets, which are collection of objects.

Among the discrete structures built from sets are:
- (I.) Combinations: unordered collections of objects used extensively in counting (Chap. 7)
- (II.) Relations: sets of ordered pairs that represent relationships between objects (Chap. 3)
- (III.) Graphs: sets of vertices and edges that connect vertices (Chap. 9+10)
- (IV.) Finite state machines: used to model computing machines (Chap. 11+12)

A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct. Functions play important roles throughout discrete mathematics:
- (I.) They are used to represent the computational complexity of algorithms (Chap. 4)
- (II.) to study the size of sets, cardinality (Chap. 2)
- (III.) to count objects (Chap. 7)
- (IV.) Useful structures such as $\underbrace{\text{sequences and strings}}_{\text{Represents ordered lists of elements}}$ are special types of functions (Chap. 2)

Add consecutive terms of a sequence of numbers, with summation

Summation in the number of steps used by an algorithm to sort a list of numbers so that its terms are in increasing order. (Chap. 4)

Matrices are used in discrete mathematics to represent a variety of discrete structures.
We will review the basic material about matrices and matrix arithmetic needed to represent relations (Chap. 3) and graphs (Chap. 9+10)

2.1 Sets

Definition 1. A **sets** is an unordered collection of distinct objects, called elements or members of the set.
A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A.
The notation $a \notin A$ denotes that a is not an element of the set A.

There are several ways to describe a set: (I.) One way is to list all the members of a set, when this is possible.
This way of describing a set is know as the **roster method** (Ensemble par énumération: $V = \{a, e, i, o, u\}$)

(II.) Sometimes the roster method is used to describe a set without listing all its members.
Some members of the set are listed, and the ellipses (...) are used when the general pattern of the elements is obvious. (Ensemble par extension: $\mathbb{N} = \{0, 1, 2, \ldots\}$)

(III.) Another way to describe a set is to use **set builder notation**. (Ensemble par compréhension: $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$ $\Leftrightarrow O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$)
The general form of this notation is $\{x \mid x \text{ has property } P\}$ and is read "the set of all x such that x has property P".
We often use this type of notation to describe sets when it is impossible to list all the elements of the set.
For instance, the set $\mathbb{Q}^+$ of all positive rational numbers can be written as $\mathbb{Q}^+ = \{x \in \mathbb{R} \mid x = \frac{p}{q}$, for some positive integers p and $q\}$.

These set, each denoted using a boldface letter, play an important role in discrete mathematics:
- (I.) $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of all natural numbers.
- (II.) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers.
- (III.) $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all positive integers.
- (IV.) $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{and } q \neq 0\}$, the set of all rational numbers.
- (V.) $\mathbb{R}$, the set of all real numbers.
- (VI.) $\mathbb{R}^+$, the set of all positive real numbers.
- (VII.) $\mathbb{C}$, the set of all complex numbers.

Other set are **intervals**, set of all the real numbers between two numbers a and b, with or without a and b.
If a and b are real numbers with $a \leq b$, we denote these intervals by:

Contains b — Contains a
- (I.) $[a, b] = \{x \mid a \leq x \leq b\}$. Note that $[a, b]$ is called the closed interval from a to b
- (II.) $[a, b) = \{x \mid a \leq x < b\}$
- (III.) $(a, b] = \{x \mid a < x \leq b\}$
- (IV.) $(a, b) = \{x \mid a < x < b\}$ (a, b) is called the open interval

Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a datatype or type is the name of a set, together with a set of operations that can be performed on objects from that set.
For example, boolean is the name of the set $\{0, 1\}$, together with operators on one or more elements of this sets, such as AND, OR and NOT.

Definition 2. Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B) \equiv A = B$
We write $A = B$ if A and B are equal sets.

Example 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter.
Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

The Empty set

There is a special set that has no element, called the **empty set**, or **null set**, and is denoted by $\emptyset$, or $\{\}$
A set with one element is called a **singleton set**. A common error is to confuse the empty set $\emptyset$ with $\underbrace{\text{set } \{\emptyset\}}_{\text{which is a singleton set.}}$ or $\{\{\}\}$, the single element of the set $\{\emptyset\}$ is the empty set itself.

A useful analogy for remembering this difference is to think of folders in a computer file system.
The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Subsets

Definition 3.
The set A is a **subset** of B, and B is a **superset** of A, if and only if every element of A is also an element of B.
We use the notation $A \subseteq B$ to indicate that A is a subset of B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.
We see that $A \subseteq B$ iff the quantification $\forall x (x \in A \rightarrow x \in B)$ is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

Showing that A is a subset of B: To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.
Showing that A is not a subset of B: To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Example 9 The set of integers with square less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 100$], but not the latter set.
The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major.

When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.
For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and the must exist an element x of B that is not an element of A. That is, A is a proper subset of B iff $\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$ is true.
A useful way to show that two sets have the same elements is to show that each set is a subset of the other.
Showing two sets are equal: To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.
$\equiv \forall x (x \in A \rightarrow x \in B) \land \forall x (x \in B \rightarrow x \in A)$
$\equiv \forall x (x \in A \leftrightarrow x \in B)$
$\equiv A = B$

Sets may have other sets as members. For instance, we have the sets
$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$
Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

Theorem 1. For every set S, (I.) $\emptyset \subseteq S$ (II.) $S \subseteq S$
Th.1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

Proof (I.)
Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x (x \in \emptyset \rightarrow x \in S)$ is true.
Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. (Vacuous proof)
It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x (x \in \emptyset \rightarrow x \in S)$ is true ∎

The size of a set

Set are used extensively in counting problems, and for such applications we need to discuss the size of sets.

Definition 4. Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Definition 5. A set is said to be infinite if it is not finite.

Example 12 Because the null set has no elements, it follows that $|\emptyset| = 0$.

Power Sets

Many problems involve testing all combinations of elements of a set if they satisfy some property.
To consider all such combinations of elements of a set S, we build a new set that has as its members all the subsets of S.

Definition 6. Given a set S, the **power set** S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Example 14 What is the power set of the set $\{0, 1, 2\}$?
The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$.
Hence, $\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
Note that the empty set and the set itself are members of this set of subsets.

Example 15 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?
The empty set has exactly one subset, namely, itself. Consequently $\mathcal{P}(\emptyset) = \{\emptyset\}$.
The set $\{\emptyset\}$ has exactly two subsets, namely $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore, $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
If a set has n elements, then its power set has 2^n elements.

Cartesian Products

The order of elements in a collection is often important. Because set are unordered, a different structure is needed to represent ordered collections.

Definition 7. The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_i as its ith element.
Note that (a, b) and (b, a) are not equal unless $a = b$.

Definition 8. Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence $A \times B = \{(a, b) \mid a \in A \land b \in B\}$

Example 17 What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.
Note that the Cartesian products $A \times B$ and $B \times A$ are not equal unless $\underbrace{A = \emptyset \text{ or } B = \emptyset}_{\text{So that } A \times B = B \times A = \emptyset}$ or $A = B$.

Example 18 Show that the Cartesian product $B \times A$ is not equal to $A \times B$, where A and B are as in Ex. 17.
The Cartesian product $B \times A$ is $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$. This is not equal to $A \times B$, which was found in Ex. 17.

Definition 9. The Cartesian product of the sets $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.
In other words, $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$.

Example 19 What is the Cartesian product $A \times B \times C$, where $A = \{0,1\}$, $B = \{1,2\}$, and $C = \{0,1,2\}$?
The Cartesian product $A \times B \times C$ consist of all ordered triples (a,b,c), where $a \in A, b \in B$, and $c \in C$.
Hence, $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Remark: $(A \times B) \times C$ is not the same as $A \times B \times C$

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. More generally, $A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$.

Example 20 Suppose that $A = \{1,2\}$. It follows that $A^2 = \{(1,1), (1,2), (2,1), (2,2)\}$ and $A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$

Using Set Notation with Quantifiers.

$\forall x \in S (P(x))$ is shorthand for $\forall x (x \in S \to P(x))$

$\exists x \in S (P(x))$ is shorthand for $\exists x (x \in S \land P(x))$.

Example 22 What do the statements $\forall x \in \mathbb{R}(x^2 \geq 0)$ and $\exists x \in \mathbb{Z}(x^2 = 1)$ mean?
$\forall x \in \mathbb{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$: "The square of every real number is nonnegative."
$\exists x \in \mathbb{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$: "There is an integer whose square is 1."

Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$

Note that $\forall x P(x)$ is true over the domain U iff the truth set of P is the set U.

Likewise, $\exists x P(x)$ is true over the domain U iff the truth set of P is nonempty $(P \neq \emptyset)$

Example 23 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$", $Q(x)$ is "$x^2 = 2$" and $R(x)$ is "$|x| = x$".

(I.) The truth set of P, $\{x \in \mathbb{Z} \mid |x| = x\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

(II.) The truth set of Q, $\{x \in \mathbb{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This the empty set because there are no integers x for which $x^2 = 2$.

(III.) The truth set of R, $\{x \in \mathbb{Z} \mid |x| = x\}$, Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbb{N}$, the set of nonnegative integers.

2.2 Set Operations

Definition 1. Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or both.
An element x belongs to the union of the sets A and B iff x belongs to A or B. This tells us that $A \cup B = \{x \mid x \in A \lor x \in B\}$

Principle of inclusion-exclusion
$|A \cup B| = |A| + |B| - |A \cap B|$
(Chap. 9)

Definition 2. Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
An element x belongs to the intersection of A and B iff x belongs to A and B. This tells us that $A \cap B = \{x \mid x \in A \land x \in B\}$

Definition 3. Two sets are called disjoint if their intersection is the empty set. $(A \cap B = \emptyset = \{\})$

Definition 4. Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not B.
The difference of A and B is also called the complement of B with respect to A. It is sometimes denoted by $A \setminus B$
An element x belongs to the difference of A and B iff $x \in A$ and $x \notin B$. This tells us that $A - B = \{x \mid x \in A \land x \notin B\}$

Definition 5. Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.
Therefore, the complement of the set A is $U - A$. An element belongs to $\bar{A}$ iff $x \notin A$. This tells us that $\bar{A} = \{x \in U \mid x \notin A\}$
We can express the difference of A and B as the intersection of A and complement of B. That is, $A - B = A \cap \bar{B}$.

$A - B = \{x \mid x \in A \land x \notin B\}$
$= \{x \mid x \in A \land x \in \bar{B}\}$
$= \{x \mid x \in (A \cap \bar{B})\}$
$= A \cap \bar{B}$ ∎

Set Identities

The identities given can be proved directly from the logical equivalences.

(I.) Identity laws: $A \cap U = A$, $A \cup \emptyset = A$

(II.) Idempotent laws: $A \cup A = A$, $A \cap A = A$

(III.) Domination laws: $A \cup U = U$, $A \cap \emptyset = \emptyset$

(IV.) Complement laws: $A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$

(V.) Complement law: $\bar{\bar{A}} = A$

(VI.) Commutive laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$

(VII.) Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

(VIII.) Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(IX.) Absorption laws: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

(X.) De Morgan's laws: $\overline{A \cap B} = \bar{A} \cup \bar{B}$, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

This identity says that the complement of the intersection of two sets is the union of their complements.

Once we have proved set identities, we can use them to prove new identities. In particular, we can apply a string of identities, one in each step, to take us from one side of a desired identity to the other.

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. $(\forall x ((x \in A \to x \in B) \land (x \in B \to x \in A))$
$\equiv \forall x (x \in A \leftrightarrow x \in B)$
$\equiv A = B$

We generally use a direct proof to do this.

Example 11 Use set builder notation and logical equivalences to establish the first De Morgan's law $\overline{A \cap B} = \bar{A} \cup \bar{B}$

We can prove this identity with the following steps

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$, def. of complement.
$= \{x \mid \neg(x \in (A \cap B))\}$, def. of does not belong symb.
$= \{x \mid \neg(x \in A \land x \in B)\}$, def. of intersection
$= \{x \mid \neg(x \in A) \lor \neg(x \in B)\}$, De Morgan's law for logical equivalences
$= \{x \mid x \notin A \lor x \notin B\}$, def. of does not belong symb.
$= \{x \mid x \in \bar{A} \lor x \in \bar{B}\}$, def. of complement.
$= \{x \mid x \in \bar{A} \cup \bar{B}\}$, def. of union
$= \bar{A} \cup \bar{B}$, meaning of set builder notation ∎

Example 14 Let A, B, and C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$.

We have $\overline{A \cup (B \cap C)} = \bar{A} \cap \overline{(B \cap C)}$, De Morgan's law
$= \bar{A} \cap (\bar{B} \cup \bar{C})$, De Morgan's law
$= (\bar{B} \cup \bar{C}) \cap \bar{A}$, commutative law for intersection
$= (\bar{C} \cup \bar{B}) \cap \bar{A}$, commutative law for union.

We summarize the three different ways for proving set identities in Table 3.

TABLE 3 Methods of Proving Set Identities.

Description	Method
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

Membership table

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid. ◀

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

Definition 6. The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.
To denote the union of the sets $A_1, A_2, \ldots, A_n$, we use the notation $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$.

Definition 7. The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.
We use the notation $A_1 \cap A_2 \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

2.3 Functions

Definition 1. Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
If f is a function from A to B, we write $f: A \to B$.
Remark: Functions are sometimes also called mappings or transformations.

Functions are specified in many different ways: (I.) Often we give a formula, such as $f(x) = x+1$, to define a function.
(II.) Other times we use a computer program to specify a function.
(III.) A function $f: A \to B$ can also defined in terms of relation from A to B, that contains one and one only, ordered pair (a,b) for every element $a \in A$, $f(a) = b$
Recall that a relation from A to B is just a subset of $A \times B$.

Definition 2. If f is a function from A to B, we say that A is the domain of f and B is the codomain.
If $f(a) = b$, we say that b is the image and a is a preimage of b
The range, or image, of f is the set of all images of elements of A.
Also, if f is a function from A to B, we say that f maps A to B.

Note that if we change either the domain or codomain of a function, we obtain a different function. If we change the mapping of elements, then we also obtain a different function.

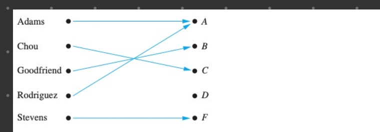


FIGURE 1 Assignment of grades in a discrete mathematics class.

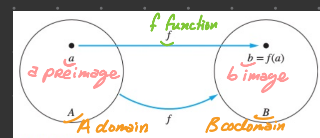
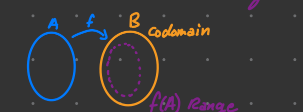


FIGURE 2 The function f maps A to B.

Remark: The codomain is the set of possible values of the function and the range is the set of all elements of the codomain that are achieved as the value of f for at least one element of the domain. range $\subseteq$ codomain



Definition 3. Let f_1 and f_2 be functions from A to $\mathbb{R}$. The $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
$(f_1 f_2)(x) = f_1(x) f_2(x)$.

"Range"
When f is a function from A to B, the image of a subset of A can also be defined.

Definition 4. Let f be a function from A to B and let S be a subset of A.
The image of S under the function f is the subset of B that consist of the images of the elements of S.
We denote the image of S by $f(S)$, so $\underbrace{f(S)}_{\text{A set}} = \{t \mid \exists s \in S (t = f(s))\}$

We also use the shorthand $\{f(s) \mid s \in S\} \equiv f(S)$ This notation $f(S)$ is potentially ambiguous.

Example 1 What are the domain, codomain, and range of the function that assigns grades to students?
Let G be the function that assigns a grade to a student in our discrete mathematics class.
Note that $G(\text{Adams}) = A$, for instance. The domain of G is the set $\{$Adams, Chou, Goodfriend, Rodriguez, Stevens$\}$ and the codomain is the set $\{A, B, C, D, F\}$. The range of G is the set $\{A, B, C, F\}$, because each grade except D is assigned to some student.
G(Domain of G) — A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.

Example 2 Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), ...
Each pair consist of a graduate student and this student's age.

Example 7 Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.
The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$

One-to-One and Onto Functions

Definition 5. A function f is said to be **one-to-one**, or an **injection** iff $f(a)=f(b)$ implies that $a=b$ for all a and b in the domain of f.

Note that a function f is one-to-one iff $f(a)\neq f(b)$ whenever $a\neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Remark: We can express using quantifier as $\forall a\forall b(f(a)=f(b)\to a=b)$ or equivalently $\forall a\forall b(a\neq b\to f(a)\neq f(b))$ where the universe of discourse is the domain of the function.

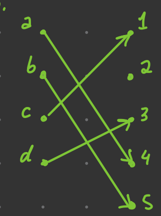
Definition 6. A function f is **increasing** if $\forall x\forall y(x<y\to f(x)\le f(y))$, **strictly increasing** if $\forall x\forall y(x<y\to f(x)<f(y))$

decreasing if $\forall x\forall y(x<y\to f(x)\ge f(y))$, and **strictly decreasing** if $\forall x\forall y(x<y\to f(x)>f(y))$, where the universe of discourse is the domain of f.

Conditions that guarantee that a function is one-to-one.

Example 8 Determine whether the function f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ with $f(a)=4$, $f(b)=5$, $f(c)=1$ and $f(d)=3$ is one-to-one.

The function f is one-to-one because f takes on different values at the four elements of its domain.



Example 9 Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-to-one.

The function $f(x)=x^2$ is not one-to-one because, for instance, $f(1)=f(-1)=1$, but $1\neq-1$.

Example 10 Determine whether the function $f(x)=x+1$ from the set of real numbers to itself is one-to-one.

Suppose that x and y are real numbers with $f(x)=f(y)$, so that $x+1=y+1$. This means that $x=y$.

Hence, $f(x)=x+1$ is a one-to-one function from $\mathbb{R}$ to $\mathbb{R}$.

Example 11 Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker.

In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then $f(x)\neq f(y)$ because the two workers x and y must be assigned different jobs.

Example 12 The function $f(x)=x^2$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ is strictly increasing. Suppose that x and y are positive real numbers with $x<y$.

Multiplying both sides of this inequality by x gives, $x^2<xy$.

Similarly, multiplying both sides by y gives, $xy<y^2$.

Hence, $f(x)=x^2<xy<y^2=f(y)$.

$x\neq y\to f(x)\neq f(y)$

However, the function $f(x)=x^2$ from $\mathbb{R}$ to the set of the set of nonnegative real numbers is not strictly increasing because $-1<0$, but $f(-1)=(-1)^2=1$ is not less than $f(0)=(0)^2=0$.

For some functions, the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called onto functions.

Definition 7. A function f from A to B is called **onto**, or a **surjection**, iff for every element $b\in B$ there is an element $a\in A$ with $f(a)=f(b)$.

Remark: A function f is onto if $\forall y\exists x(f(x)=y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

Definition 8. The function f is an **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto. $\forall a\forall b[f(a)=f(b)\to(a=b)]\wedge\forall y\exists x[f(x)=y]$

$[f(X)=Y]\equiv\forall a[a\in X\leftrightarrow a\in Y]$

Example 17 Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3,4\}$ with $f(a)=4$, $f(b)=2$, $f(c)=1$, and $f(d)=3$. Is f a bijection?

The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value.

It is onto because all four elements of the codomain are images of elements in the domain.

Hence, f is a bijection.

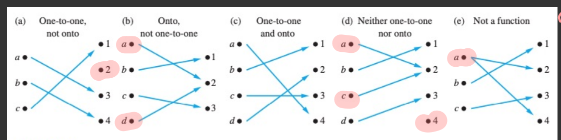
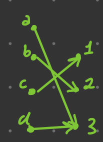


FIGURE 5 Examples of different types of correspondences.

(e) Not a function, because it sends an element to two different elements.

Example 13 Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a)=3$, $f(b)=2$, $f(c)=1$, and $f(d)=3$. Is f an onto function?



Because all three elements of the codomain are images of elements in the domain, we see that f is onto. Note that if the codomain were $\{1,2,3,4\}$, then f would not be onto.

Example 14 Is the function $f(x)=x^2$ from the set of integers to the set of integers onto?

The function f is not onto because there is no integer x with $x^2=-1$, for instance.

Example 15 Is the function $f(x)=x+1$ from the set of integers to integers onto?

This function is onto, because for every integer y there is an integer x such that $f(x)=y$.

To see this, note that $f(x)=y$ iff $x+1=y$, which iff $x=y-1$.

Note that $y-1$ is also an integer, and so, is in the domain of f.

Example 16 Consider the function f in Ex.11 that assigns jobs to workers. The function f is onto if for every job the is a worker assigned this jobs.

The function f is not onto when there is at least one job that has no worker assigned it.

* Suppose that f is a function from a set A to itself.

If A is finite, then f is one-to-one iff it is onto.

Example 18 Let A be a set. The identity function on A is the function $\imath_A: A\to A$, where $\imath_A(x)=x$ for all $x\in A$. In other words, the identity function $\imath_A$ is function that assigns each element to itself. The function $\imath_A$ is one-to-one and onto, so it is a bijection

To summarize what needs to be shown to establish whether a function is one-to-one and whether it is onto.

Suppose that $f: A\to B$.

To show that f is injective Show that if $f(x)=f(y)$ for arbitrary $x, y\in A$, then $x=y$.

To show that f is not injective Find particular elements $x, y\in A$ such that $x\neq y$ and $f(x)=f(y)$.

To show that f is surjective Consider an arbitrary element $y\in B$ and find an element $x\in A$ such that $f(x)=y$.

To show that f is not surjective Find a particular $y\in B$ such that $f(x)\neq y$ for all $x\in A$.

$f: A\to B$

$f(A)=B$

Equal sets

Inverse Functions and Compositions of Functions

Definition 9. Let f be a one-to-one correspondence (Bijective) from the set A to B. The **inverse function** of f is the function that assigns to an element b belonging to B to the unique element a in A such that $f(a)=b$.

The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b)=a$ when $f(a)=b$

If a function f is not a one-to-one correspondence, we cannot define an inverse function of f. When f is not a one-to-one correspondence, either it is not one-to-one or it is not onto.

If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain.

If f is not onto, for some element b in the codomain, no element a in the domain exist for which $f(a)=b$

Consequently, if f is not a one-to-one correspondence, we cannot assign to each element b in the codomain a unique element a in the domain such that $f(a)=b$ (because for some b there is either more than one such a or no such a).

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

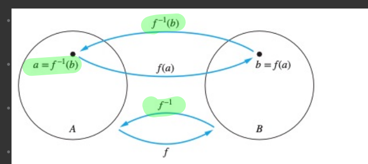


FIGURE 6 The function f^{-1} is the inverse of function f.

Example 19 Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a)=2$, $f(b)=3$, and $f(c)=1$. Is f invertible, and if it is, what is its inverse?

The function is invertible because it is a one-to-one correspondence.

The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1)=c$, $f^{-1}(2)=a$, and $f^{-1}(3)=b$.

Example 20 Let $f:\mathbb{Z}\to\mathbb{Z}$ be such that $f(x)=x+1$. Is f invertible, and if it is, what is its inverse?

The function f has an inverse because it is a one-to-one correspondence, as follows from Ex 10 and 15.

To reverse the correspondence, suppose that y is the image of x, so that $y=x+1$. Then, $x=y-1$.

This means that $y-1$ is the unique element of $\mathbb{Z}$ that is sent to y by f. Consequently, $f^{-1}(y)=y-1$.

Example 21 Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x)=x^2$. Is f invertible?

Because $f(-2)=f(2)=4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible

Note we can also show that f is not invertible because it is not onto.

Sometimes we can restrict the domain or the codomain of a function, or both, to obtain an invertible function, as Ex.22 illustrate

Example 22 Show that if we restrict the function $f(x)=x^2$ in Ex.21 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

The function $f(x)=x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one.

To see this, note that if $f(x)=f(y)$, then $x^2=y^2$, so $x^2-y^2=(x+y)(x-y)=0$. This means that $x+y=0$ or $x-y=0$, so $x=-y$ or $x=y$.

Because both x and y are nonnegative, we must have $x=y$. So this function is one-to-one.

Furthermore, $f(x)=x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root.

That is, if y is a nonnegative real number, there exist a nonnegative real number x such that $x=\sqrt{y}$, which means that $x^2=y$.

Because the function $f(x)=x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible.

Its inverse is given by the rule $f^{-1}(y)=\sqrt{y}$.

Definition 10. Let g be a function from the set A to B and let f be a function from the set B to C.

The **composition** of the functions f and g, denoted for all $a\in A$ by $f\circ g$, is the function from A to C defined by $(f\circ g)(a)=f(g(a))$

The domain of $f\circ g$ is the domain of g.

The range of $f\circ g$ is the image of the range of g with respect to the function f.

Note that the composition $f\circ g$ cannot be define unless the range of g is a subset of the domain of f $(g(A)\subseteq B)$

Example 23 Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a)=b$, $g(b)=c$ and $g(c)=a$. Let f be the function from the set $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a)=3$, $f(b)=2$, and $f(c)=1$. What is the composition of f and g, and what is the composition of g and f?

The composition $f\circ g$ is defined by $(f\circ g)(a)=f(g(a))=f(b)=2$, $(f\circ g)(b)=f(g(b))=f(c)=1$, and $(f\circ g)(c)=f(g(c))=f(a)=3$

Note that $g\circ f$ is not defined, because the range of f is not a subset is not a subset of the domain of g.

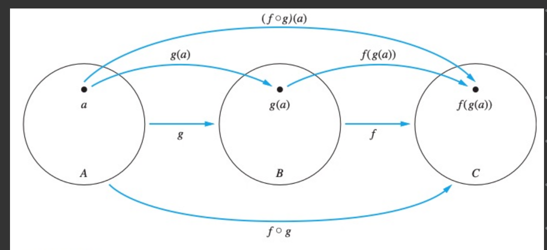


FIGURE 7 The composition of the functions f and g.

Example 24 Let f and g be the functions from the set of integers to the set of integers defined by $f(x)=2x+3$ and $g(x)=3x+2$. What is the composition of f and g? What is the composition of g and f?

Both the composition $f\circ g$ and $g\circ f$ are defined.

Moreover, $(f\circ g)(x)=f(g(x))=f(3x+2)=2(3x+2)+3=6x+7$

and $(g\circ f)(x)=g(f(x))=f(2x+3)=3(2x+3)+2=6x+11$

Remark: Note that even though $f\circ g$ and $g\circ f$ are defined for the function f and g, $f\circ g$ and $g\circ f$ are not equal.

In other words, the commutative law does not hold for the composition of function.

Example 25 Let f and g be functions defined by $f:\mathbb{R}\to\mathbb{R}^+\cup\{0\}$ with $f(x)=x^2$ and $g:\mathbb{R}^+\cup\{0\}\to\mathbb{R}$ with $g(x)=\sqrt{x}$. What is the function $(f\circ g)(x)$?

The domain of $(f\circ g)(x)=f(g(x))$ is the domain of g, which is $\mathbb{R}^+\cup\{0\}$, the set of nonnegative real numbers.

If x is a nonnegative real number, we have $(f\circ g)(x)=f(g(x))=f(\sqrt{x})=(\sqrt{x})^2=x$. The range of $f\circ g$ is the image of the range of g with respect to f. This is the set $\mathbb{R}^+\cup\{0\}$, the set of nonnegative real numbers.

Summarizing, $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$ and $f(g(x))=x$ for all x.

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. $f:A\to B$, $(f^{-1}\circ f)(a)=f^{-1}(f(a))=f^{-1}(b)=a$

$(f\circ f^{-1})(b)=f(f^{-1}(a))=f(a)=b$

Consequently $f^{-1}\circ f=\imath_A$ and $f\circ f^{-1}=\imath_B$, where $\imath_A$ and $\imath_B$ are identity functions on the sets A and B, respectively.

That is, $f^{-1}\circ f^{-1}=f^{-1}(f^{-1})=f$.

The Graphs of Functions

The graph of the function is often displayed pictorially to aid in understanding the behavior of the function.

Definition 11. Let f be a function from the set A to B. The graph of the function f is the set of ordered pairs $\{(a,b) \mid a\in A \text{ and } f(a)=b\}$. From the definition, the graph of a function f from A to B is the subset of $A\times B$ containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry.

Some Important Functions

Definition 12. The floor function assigns to the real number x the largest integer that is less than or equal to x, denoted by $\lfloor x\rfloor$.
The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x, denoted by $\lceil x\rceil$.

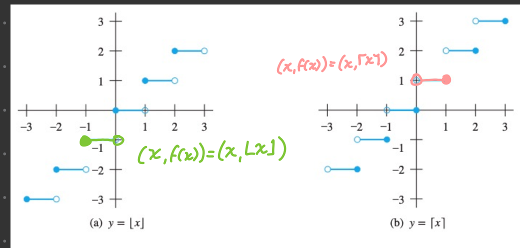


FIGURE 10 Graphs of the (a) floor and (b) ceiling functions.

Because these functions appear so frequently in discrete mathematics, it is useful to look over these identities.

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (n is an integer, x is a real number)

(1a)	$\lfloor x\rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x\rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x\rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x\rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x\rfloor \le x \le \lceil x\rceil < x+1$
(3a)	$\lfloor -x\rfloor = -\lceil x\rceil$
(3b)	$\lceil -x\rceil = -\lfloor x\rfloor$
(4a)	$\lfloor x+n\rfloor = \lfloor x\rfloor + n$
(4b)	$\lceil x+n\rceil = \lceil x\rceil + n$

There are also many statements about these functions that may appear to be correct, but actually are not.

A useful approach for considering statements about the floor function is to let $x=n+\varepsilon$, where $n=\lfloor x\rfloor$ is an integer, and ε, the fractional part of x, satisfies the inequality $0\le\varepsilon<1$.
Similarly, when considering statements about the ceiling functions, it is useful to write $x=n-\varepsilon$, where $n=\lceil x\rceil$ and $0\le\varepsilon<1$.

Direct proofs.
Proof (4a) $\lfloor x+n\rfloor=\lfloor x\rfloor+n$.
Suppose that $\lfloor x\rfloor=m$, where m is a positive integer.
By property (1a), it follows that $m\le x<m+1$.
Adding n to all three quantities in this chain of two inequalities shows that $m+n\le x+n<m+n+1$.
Using (1a) again, we see that $\lfloor x+n\rfloor=\underbrace{m}_{=\lfloor x\rfloor}+n=\lfloor x\rfloor+n$. ∎

There are certain types of functions that will be used throughout the text. These include polynomial, logarithmic, and exponential functions. Another function we will use throughout this text is the factorial function $f:\mathbb{N}\to\mathbb{Z}^+$ denoted by $f(n)=1\cdot2\cdots(n-1)\cdot n=n!$

Partial Functions

Definition 13. A partial function $f:A\to B$ is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B.
We say that f is undefined for elements in A that are not in the domain of definition of f.
When the domain of definition of f equals A, we say that f is a total function.
Remark: We write $f:A\to B$ to denote that f is a partial function from A to B.

2.4 Sequences and Summations

A sequence is a discrete structure used to represent an ordered list. They are also an important data structure in computer science.

Definition 1. A sequence is a function from a subset of the set of integers (usually either $\mathbb{N}$ or $\mathbb{Z}^+$) to a set S.
We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
We use the notation $\underbrace{\{a_n\}}_{(a_n)_{n\in\mathbb{N}}}$ to describe the sequence.

Definition 2. A geometric progression is a sequence of the form $a, ar, ar^2, \dots, ar^n, \dots$ where the initial term a and the common ratio r are real numbers.
Remark: A geometric progression is a discrete analogue of the exponential function $f(x)=ar^x$.

Definition 3. An arithmetic progression is a sequence of the form $a, a+d, a+2d, \dots, a+nd, \dots$ where the initial term a and the common difference d are real numbers.
Remark: An arithmetic progression is a discrete analogue of the linear function $f(x)=dx+a$

Finite sequence of the form $a_1, a_2, \dots, a_n$ are also called strings, also denoted by $a_1a_2\cdots a_n$. The empty string, denoted by λ, is the string that has no terms.

Recurrence Relations (Chap. 6: Induction et Récursion)

There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

Definition 4. A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in term one or more of the previous terms of the sequence, namely $a_0, a_1, \dots, a_{n-1}$ for integers n with $n\ge n_0$, where n_0 is a nonnegative integer.
A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation (A recurrence relation is said to recursively define a sequence ... explained later in Chapter 5)

We will study Fibonacci sequence in depth in Chapters 5 and 8, where we will see why it is important for many applications, including modeling the population growth of rabbits.
Fibonacci numbers occur naturally in the structures of plants and animals, such as in the arrangement of sunflower seeds in a seed head and in the shell of the chambered nautilus.

Definition 5. The Fibonacci sequence, $f_0, f_1, f_2, \dots$, is defined by the initial conditions $f_0=0, f_1=1$ and the recurrence relation $f_n=f_{n-1}+f_{n-2}$ for $n=2,3,4,\dots$
We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called closed formula, for the terms of the sequence.

Example 10 | Solve the RR and initial condition in Ex 5. Many methods have been developed for solving RR. Here, we will introduce a straightforward know as iteration via several examples.
We can successively apply the RR in Ex 5, starting with the initial condition $a_1=2$, and working upward until we reach a_n to deduce a closed formula for the sequence. We see that

$$a_2=2+3$$
$$a_3=(2+3)+3=2+3\cdot2$$
$$a_4=(2+3\cdot2)+3=2+3\cdot3$$
$$\vdots$$
$$a_n=a_{n-1}+3=(2+3\cdot(n-2))+3=2+3(n-1)$$

Forward substitution

or
$$a_n=a_{n-1}3$$
$$=(a_{n-2}+3)+3=a_{n-2}+3\cdot2$$
$$=(a_{n-3}+3)+3\cdot2=a_{n-3}+3\cdot3$$
$$\vdots$$
$$=a_2+3(n-2)=(\underbrace{a_1}_{2}+3)+3(n-2)=2+3(n-1)$$

Backward substitution

We can also successively apply the RR in Ex 5, starting with the term a_n, and working downward until we reach the initial condition $a_1=2$ to deduce this same formula.

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …

Example 27 | Display the graph of the function $f(x)=x^2$ from the set of integers to the set of integers.
The graph of f is the set of ordered pairs of the form $(x,f(x))=(x,x^2)$, where x is an integer.

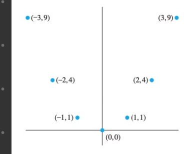


FIGURE 9 The graph of $f(x)=x^2$ from Z to Z.

Example 31 | Prove that if x is a real number, then $\lfloor 2x\rfloor=\lfloor x\rfloor+\lfloor x+\frac{1}{2}\rfloor$
To prove this statement we let $x=n+\varepsilon$, where n is an integer and $0\le\varepsilon<1$. There are two cases to consider, depending on whether ε is less than, or greater than or equal to $\frac{1}{2}$.

We first consider the case when $0\le\varepsilon<\frac{1}{2}$. In this case, $2x=2n+2\varepsilon$ and $\lfloor 2x\rfloor=2n$ because $0\le2\varepsilon<1$.
Similarly, $x+\frac{1}{2}=n+(\frac{1}{2}+\varepsilon)$, so $\lfloor x+\frac{1}{2}\rfloor=n$, because $0\le\frac{1}{2}+\varepsilon<1$.
Consequently, $\lfloor 2x\rfloor=2n$ and $\lfloor x\rfloor+\lfloor x+\frac{1}{2}\rfloor=n+n=2n$

Next, we consider the case when $\frac{1}{2}\le\varepsilon<1$. In this case, $2x=2n+2\varepsilon=(2n+1)+(2\varepsilon-1)$. Because $0\le2\varepsilon-1<1$, it follows that $\lfloor 2x\rfloor=2n+1$.
Because $\lfloor x+\frac{1}{2}\rfloor=\lfloor n+(\frac{1}{2}+\varepsilon)\rfloor=\lfloor n+1+(\varepsilon-\frac{1}{2})\rfloor$ and $0\le\varepsilon-\frac{1}{2}<1$, it follows that $\lfloor x+\frac{1}{2}\rfloor=n+\frac{1}{2}$.
Consequently, $\lfloor 2n\rfloor=2n+1$ and $\lfloor x\rfloor+\lfloor x+\frac{1}{2}\rfloor=n+(n+1)=2n+1$.
This concludes the proof. ∎

Example 32 | Prove or disprove that $\lceil x+y\rceil=\lceil x\rceil+\lceil y\rceil$ for all real numbers x and y.
Although this statement may appear reasonable, it is false.
A counterexample is supplied by $x=\frac{1}{2}$ and $y=\frac{1}{2}$. With these values we find that $\lceil x+y\rceil=\lceil\frac{1}{2}+\frac{1}{2}\rceil=\lceil 1\rceil=1$, but $\lceil x\rceil+\lceil y\rceil=\lceil\frac{1}{2}\rceil+\lceil\frac{1}{2}\rceil=1+1=2$ ∎

Example 1 | Consider the sequence $\{a_n\}$, where $a_n=\frac{1}{n}$. The list of the terms of this sequence, beginning with a_1, namely, $a_1, a_2, a_3, a_4, \dots$ starts with $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Example 2 | The sequence $\{b_n\}$ with $b_n=(-1)^n$, $\{c_n\}$ with $c_n=2\cdot5^n$ and $\{d_n\}$ with $d_n=6\cdot(\frac{1}{3})^n$ are geometric progressions with initial term and common ratio equal to 1 and −1, 2 and 5, and 6 and $\frac{1}{3}$, respectively, if we start at $n=0$.
The list of terms $b_0, b_1, b_2, b_3, b_4, \dots$ begins with $1, -1, 1, -1, 1, \dots$;
the list of terms $c_0, c_1, c_2, c_3, c_4, \dots$ begins with $2, 10, 50, 250, 1250, \dots$;
and the list of terms $d_0, d_1, d_2, d_3, d_4, \dots$ begins with $6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$;

Example 3 | The sequences $\{s_n\}$ with $s_n=-1+4n$ and $\{t_n\}$ with $t_n=7-3n$ are both arithmetic progressions with initial terms and common differences equal to −1 and 4, and 7 and −3, respectively, if we start at $n=0$. The list of terms $s_0, s_1, s_2, s_3, \dots$ begins with $-1, 3, 7, 11, \dots$ and the list of terms $t_0, t_1, t_2, t_3, \dots$ begins with $7, 4, 1, -2, \dots$

Example 5 | Let $\{a_n\}$ be a sequence that satisfies the recurrence $a_n=a_{n-1}-a_{n-2}$ for $n=2,3,4,\dots$, and suppose that $a_0=3$ and $a_1=5$.
What are a_2 and a_3?
We see from the recurrence relation that $a_2=a_1-a_0=5-3=2$ and $a_3=a_2-a_1=2-5=-3$. We can find a_4, a_5, and each successive term in a similar way.

Example 9 | Determine whether the sequence $\{a_n\}$, where $\underbrace{a_n=3n}_{(\mathrm{I}_0)}$ for every nonnegative integer n, is a solution of the RR $a_n=2a_{n-1}-a_{n-2}$ for $n=2,3,4,\dots$
Answer the same question where $\underbrace{a_n=2^n}_{(\mathrm{II}_0)}$ and where $\underbrace{a_n=5}_{(\mathrm{III}_0)}$.

(I_0) Suppose that $a_n=3n$ $\forall n\in\mathbb{Z}^+\cup\{0\}$. Then, for $n\ge2$, we see that $2a_{n-1}-a_{n-2}=2(3(n-1))-3(n-2)=3(2(n-1)-(n-1)+1)=3((n-1)+1)=3n=a_n$. Therefore, $\{a_n\}$, where $a_n=3n$, is a solution of the RR.

(II_0) Suppose that $a_n=2^n$, $\forall n\in\mathbb{N}$. Note that $a_0=1$, $a_1=2$, and $a_2=4$. Because $2a_1-a_0=2\cdot2-1=3\ne\underset{\substack{\|\\2}}{}a_2$, we see that $\{a_n\}$, where $a_n=2^n$, is not a solution of the RR.

(III_0) Suppose that $a_n=5$ $\forall n\in\mathbb{N}$. Then for $n\ge2$, we see that $a_n=2a_{n-1}-a_{n-2}=2\cdot5-5=5=a_n$. Therefore, $\{a_n\}$, where $a_n=5$, is a solution of the RR.

Lucas sequence. $L_n=L_{n-1}+L_{n-2}$ with initial conditions $L_1=1$ and $L_2=2$.
The same RR as the Fib. sequence, but with different initial conditions)

Next, we consider the addition of the terms of sequence. For this we introduce summation notation.

We use the notation to express the sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from sequence $\{a_n\}$, $\underbrace{\sum_{j=m}^{n} a_j, \ \sum_{j=m}^{n} a_j, \text{ or } \sum_{m \le j \le n} a_j}_{= a_m + a_{m+1} + \ldots + a_n}$

The usual laws for arithmetic apply to summations, we have $\sum_{j=1}^{n}(ax_j + by_j) = a\sum_j x_j + b\sum_j y_j$

Sometimes it is useful to shift the index of a summation in a sum. This is often done when two sums need to be added but their indices of summation do not match.
When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand.

Theorem 1. Geometric series

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Proof. Let $S_n = \sum_{j=0}^{n} ar^j$. $r \cdot S_n = r \cdot \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} ar^{j+1}$

$= \sum_{k=1}^{n+1} ar^k$, with $k = j+1$

$= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a)$

$= S_n + (ar^{n+1} - a)$

From these equalities, we see that

$r \cdot S_n = S_n + (ar^{n+1} - a)$

If $r \neq 1$, then $S_n = \dfrac{ar^{n+1}-a}{r-1}$

If $r = 1$, then $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$ ∎

We can also use summation notation to add all values of a function, or terms of an indexed set. That is, we write $\sum_{s \in S} f(s)$ to represent the sum.

TABLE 2 Some Useful Summation Formulae.

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Example 20 | $\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2 = \sum_{k=2}^{6} (j-1)^2$

Example 21 | Double summation arise in many contexts (as in the analysis of nested loops in computer programs).
An example of a double summation is $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$.

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i = 6 + 12 + 18 + 24 = 60$$

Example 23 | Find $\sum_{k=50}^{100} k^2$

First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$, with $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$

2.5 Cardinality of Sets

Definition 1. The sets A and B have the same cardinality iff there is a one-to-one correspondence from A to B.
When A and B have the same cardinality, we write $|A| = |B|$

Definition 2. If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B.
And we write $|A| \le |B|$

Definition 3. A set that is either finite or has the same cardinality as the set of positive integers is called countable.
When an infinite set S is countable, we denote $|S| = \underbrace{\aleph_0}_{\text{aleph null.}}$

Theorem 1. If A and B are countable sets, then $A \cup B$ is also countable.

Theorem 2. Schröder-Bernstein Theorem
If A and B are sets with $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$
In otherwords, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

2.6 Matrices

Matrices are used throughout discrete mathematics to express relationships between elements in sets.

Definition 1. A matrix is an rectangular array of numbers. A matrix with m row and n columns is called an $m \times n$ matrix. A matrix with the same number of rows as columns is called square.

Definition 2. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ A convenient shorthand notation for expressing the matrix A is to write $A = [a_{ij}]$

Matrix Arithmetic

Definition 3. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, $A + B = [a_{ij} + b_{ij}]$

Definition 4. Let A be an $m \times k$ matrix and B, $k \times n$ matrix, $AB = [c_{ij}]$ with $\underbrace{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}}_{c_{ij} = \sum_k a_{ik}b_{kj} = a^T b}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

FIGURE 1 The product of $A = [a_{ij}]$ and $B = [b_{ij}]$.

Transposes and Powers of Matrices

Definition 5. The identity matrix of order n is the $n \times n$ matrix $I_n = [\delta_{ij}]$ where $\delta_{ij} = \underbrace{\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}}_{= \mathbb{1}_{(i=j)}}$ Kronecker delta

Hence, $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when A is $m \times n$, we have $A \cdot I_n = I_m A = A$

And $A^0 = I_n$, when $A_{n \times n}$

Definition 6. Let $A = [a_{ij}]$ be an $m \times n$ matrix. $A^T = [b_{ij}]$, $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$
$j = 1, 2, \ldots, m$.

Definition 7. A square matrix A is called symmetric if $A = A^T$

Zero-One Matrices

Definition 8. The join of A and B: $a_{ij} \vee b_{ij}$, $b_1 \vee b = \begin{cases} 1 & \text{if } b_1 = 0 \text{ or } b_2 = 1 \\ 0 & \text{Otherwise} \end{cases}$

The meet of A and B: $a_{ij} \wedge b_{ij}$, $b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{Otherwise} \end{cases}$

Definition 9. Boolean product

$A \odot B = [c_{ij}]$, where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$

Analogue to the ordinary product of matrices, but with $\begin{cases} + \equiv \vee \\ \cdot \equiv \wedge \end{cases}$

Definition 10. Boolean power

$A^{[r]} = \underbrace{A \odot A \odot A \odot \cdots A}_{r \text{ times}}$ and $A^{[0]} = I_n$) RR. $A^{[n]} = A^{[n-1]} \odot A$