Lundi 5 juin 2023

6 Counting

6.1 The Basics of Counting

(I.) The product rule

A procedure that can be broken down into a sequence of tasks.

E.g. n_1 ways of doing the first task and n_2 ways for the second task. Then, there are $n_1 \cdot n_2$ ways to do the procedure.

Generalization

Suppose that a procedure is carried out by performing the tasks $T_1, T_2, T_3, \ldots, T_n$. If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous task were done then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_n$ way to carry out the procedure.

$\prod_{i=1}^{n} n_i$ ways to carry out the procedure ← Proved by mathematical induction

Example 6 | Counting functions

How many functions are there from a set with m elements to a set with n elements?

n elements in the codomain
m elements in the domain

Hence, by the product rule there are $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m \text{ times}} = n^m$ functions.

Example 7 | Counting One-to-One functions

How many one-to-one functions are there from a set with m elements to one with n elements?

When $m > n$, there are none.

Now let $m \le n$, suppose the elements in the domain are $a_1, a_2, \ldots, a_m$

There are n ways to choose the value of the function at a_1.

Because the function is One-to-One, the value of the function at a_2 can be picked in $n-1$ ways (Value used for a_1 cannot be used again)

By the product rule, there are $n \cdot (n-1)(n-2) \cdot \ldots \cdot (n-m+1)$

Or simply, $P(n,m) = \frac{n!}{(n-m)!} = n(n-1)(n-2) \cdot \ldots \cdot (n-m+1)$

Example 8 | The North American numbering plan (NANP)

Old plan: $\underline{NYX}$ $\underline{NNX}$ $\underline{XXXX}$

$N \in [2,9]$
$|N| = (9-2+1) = 8$
$X \in [0,9]$
$|X| = 10$
$Y \in \{0,1\}$
$|Y| = 2$

$8 \cdot 2 \cdot 10 \cdot 8^2 \cdot 10 \cdot 10^4$
$= 160 \cdot 640 \cdot 10\,000$
$= 1\,024\,000\,000$

New plan: $\underline{NXX}$ $\underline{NXX}$ $\underline{XXXX}$

$[8 \cdot 10^2]^2 \cdot 10^4$
$= 800^2 \cdot 10\,000$
$= 6\,400\,000\,000$

(II.) The Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways then there are $n_1 + n_2$ ways to do the task.

Generalization

If $n_i \neq n_j$, $\forall i,j$ $1 \le i < j \le m$ then $n_1 + n_2 + \ldots + n_m$ ways

$\sum_{i=1}^{m} n_i$ ways

Example 16 | Password, which is 6-8 characters long where each character is an uppercase letter or digit.

How many passwords are there that contains at least 1 digit?

$P_6 = \underbrace{36^6}_{\text{All length 6 passwords}} - \underbrace{26^6}_{\text{length 6 passwords that do not have any digit}}$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

$P_6 + P_7 + P_8$

(III.) The Subtraction Rule (Inclusion-Exclusion of two sets)

$n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways

Principle of Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example 18 | How many bit strings of length 8 either start with 1 bit or end with 2 bits 00?

Start with 1: $\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{2^7 \text{ ways}}$

End with 00: $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{2^6 \text{ ways}}\ \underline{0}\ \underline{0}$

Both: $\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 \text{ ways}}\ \underline{0}\ \underline{0}$

Thus, by the principle of Incl-Excl.

$2^7 + 2^6 - 2^5$
$= 128 + 64 - 32$
$= 160$

(IV.) The Division Rule

It can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w. Hence, $\frac{n}{d}$ ways

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6.2 The Pigeonhole Principle

Theorem 1 | The Pigeonhole Principle (Dirichlet drawer princ.)

If k is a positive integer and $k+1$ or more objects a placed into k boxes, then there is at least one box containing two or more of the objects.

Proof by contradiction

Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k+1$ objects.

← Proof by contraposition

Theorem 2 | The Generalized Pigeonhole Princ.

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Example 10 | During a month with 30 days a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some consecutive days during which the team must play exactly 14 games

Let a_j be the number of games played on or before the jth day of the month

Then $a_1, a_2, \ldots a_{30}$ is an increasing sequence of distinct positive int with $1 \le a_j \le 45$.

Moreover, $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also an inc. sequence with $15 \le a_j + 14 \le 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are less than or equal to 59

Hence, by the pigeonhole principle two of these int are equal.

a_j and $(a_j + 14)$ $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$

This means that exactly 14 games were played from day $j+1$ to day i.

Theorem 3 | Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

The Ramsey number $R(m,n)$

6.3 Permutations and Combinations

(I.) Permutations (Ordered arrangement)

Theorem 1 | R-permutation

$P(n,r) = n(n-1)(n-2) \cdot \ldots \cdot (\underbrace{n-r+1}_{n-(r-1)})$

where $1 \le r \le n$

And $P(n,0) = 1$ } There is exactly 1 way to order zero elements

Corollary 1 | $P(n,r) = \frac{n!}{(n-r)!}$ where $0 \le r \le n$

(II.) Combinations (Order doesn't matter)

Theorem 2 | R-combinations

$C(n,r) = \frac{n!}{r!(n-r)!}$ where $0 \le r \le n$

product rule

$P(n,r) = C(n,r) \cdot \underbrace{P(r,r)}_{\text{Number of ways it can be done}} \Leftrightarrow C(n,r) = \frac{P(n,r)}{P(r,r)}$) Division rule

Corollary 2 |

$C(n,r) = C(n, n-r)$

$0 \le r \le n$

{ Double counting proof / Bijective proof

6.4 Binomial Coefficient and Identities

Theorem 1 | The Binomial Theorem

$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \underbrace{\binom{n}{0}}_{=1} x^n + \binom{n}{1} x^{n-1} y + \ldots + \underbrace{\binom{n}{n}}_{=1} y^n$

$= \binom{n}{n-j}$

Corollary 1 | $\sum_{k=0}^{n} \binom{n}{k} = 2^n$) Proof using Th 1 with $x=1$, $y=1$

Corollary 2 | $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$) $x=-1$, $y=1$

$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \ldots$

Corollary 3 | $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$) $x=1$, $y=2$

Theorem 2 | Pascal's Identity

$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, $k \le n$

Pascal Triangle

$\ldots \binom{6}{4} \binom{6}{5} \ldots$
$\ldots \binom{7}{5} \ldots$

$\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$

Theorem 3 | Vandermonde's Identity

$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$

Corollary 4 | $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Theorem 4 | $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$

6.5 Generalized Permutations and Combinations

(III.) Permutations with Repetition

Theorem 1| R-permutations with repetition

n^R

(IV.) Combinations with Repetition

Theorem 2| R-combinations with reps.

$$C(n+R-1, R) = C(n+R-1, n-1)$$

No more $0 \le R \le n$, $R \ge n$ possible

Example 5| How many solutions does the equation $x_1 + x_2 + x_3 = 11$ where $x_1, x_2, x_3 \in \mathbb{N}$

$$\binom{3+11-1}{11} = \binom{13}{11} = \binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = 78$$

if $x_1 \ge 1$, $x_2 \ge 2$, $x_3 \ge 3$

$R = 11-1-2-3 = 5$

$$\binom{3+5-1}{5} = \binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{2 \cdot 1} = 21$$

Permutation with Indistinguishable Objects

⚠ Care must be taken to avoid counting things more than once

Example 7| How many different strings can be made by reordering the letters of the word SUCCESS

$$\underbrace{\binom{7}{3}}_{S}\underbrace{\binom{4}{2}}_{C}\underbrace{\binom{2}{1}}_{U}\underbrace{\binom{1}{1}}_{E}$$

Theorem 3| Permutations of n objects where there are n_1 indistinguishable object of type 1, n_2 indistinguishable obj. of type 2, ⋮ n_k indistinguishable obj. of type k.

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Proof.

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} * \dots * \binom{n-n_1-\dots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)}!} \cdot \frac{\cancel{(n-n_1)}!}{n_2!\,\cancel{(n-n_1-n_2)}!} * \dots * \frac{\cancel{n-n_1-\dots-n_{k-1}}}{n_k!\,(\underbrace{n-n_1-\dots-n_{k-1}-n_k}_{=0!\,=1})!}$$

$$= \frac{n!}{n_1!\, n_2! \times \dots * n_k!}$$

QED

(I.) Distinguish. Obj. and Distinguish. boxes

$$\frac{n!}{n_1!\, n_2! \times \dots * n_k!}$$

Example 8| How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

$n = 52$
$R = 5$

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{\underbrace{5!5!5!5!}_{\text{5 indistinguish. obj of four different type.}}\underbrace{32!}_{\text{32 obj of fifth type}}}$$

Theorem 4| Dist. obj / Dist. boxes

$$\frac{n!}{n_1!\, n_2! \times \dots * n_k!}$$

Same as permutation with indistinguish. obj.

Distributing Objects into Boxes

Closed formulae

(I.) Distinguishable. obj.
(II.) Indistinguish. obj
into distinguish. boxes

No closed formulae

(III.) Distinguish. obj.
(IV.) Indistinguish. obj
into indistinguish. boxes

(II.) Indistinguish. Obj. and Distinguish. boxes

$$\binom{n+R-1}{R}$$

Example 9| How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

$n = 8$
$R = 10$

$$\binom{8+10-1}{10} = \binom{17}{10} = \frac{17!}{10!\,(17-10)!}$$

(III.) Distinguish. obj. and Indistinguish. boxes

Stirling numbers of the second kind

$$\sum_{j=1}^{k} S(n,j), \text{ where } S(n,j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

$$= \sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

, n distinguish obj. k indisting. boxes

Example 10| How many ways are there to put 4 different employees into 3 indistinguishable offices, when each office can contain any number of employees?

4 emp in 1 office $\binom{4}{4}\binom{0}{0}\binom{0}{0} = 1 \cdot 1 \cdot 1 = 1$

3 emp in 1 office, 1 emp in another $\binom{4}{3}\binom{1}{1}\binom{0}{0} = 4 \cdot 1 \cdot 1 = 4$

2 emp in 1 office, 2 emp in another $\frac{\binom{4}{2}\binom{2}{2}\binom{0}{0}}{2!} = \frac{6 \cdot 1 \cdot 1}{2!} = \frac{6}{2!} = 3$

2 emp in 1 office, 1 emp in another, 1 emp in the last $\frac{\binom{4}{2}\binom{2}{1} \cdot \binom{1}{1}}{2!} = \frac{6 \cdot 2 \cdot 1}{2!} = \frac{12}{2} = 6$

Division Rule

(IV.) Indist. obj and Indist. boxes

$$a_1 + a_2 + a_3 + \dots + a_j = n$$

, where $a_1, a_2, \dots a_j \in \mathbb{N}$ with $a_1 > a_2 > \dots > a_j$