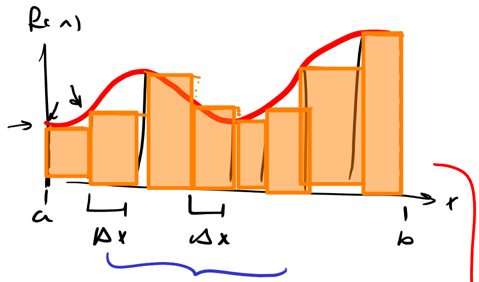
Integración numérica

↳ La principal idea es calcular el área bajo la curva de una función conocida de forma aproximada

$$I = \int_a^b f(x)\,dx$$



Una definición usual (matemáticamente) es la de las sumas de Riemann

$$I = \lim_{n\to\infty} \sum_{i=0}^{n} f(a+\Delta x\, i)\left(\frac{b-a}{n}\right)$$

donde $\frac{b-a}{n} = \Delta x$

En el límite las áreas coinciden

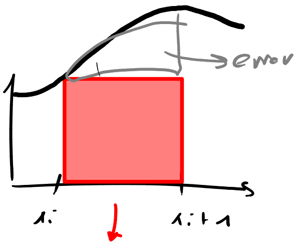
Lo que podemos hacer es interpolar entre los puntos x_i y x_{i+1} y calcular el área de un elemento. La interpolación la podemos realizar de varias formas. ps

$$f(x) \approx \underbrace{f(x_i)}_{\text{orden cero}} + f'(x_i)(x-x_i) + \frac{f''(x_i)(x-x_i)^2}{2} + O[(x-x_i)^3]$$

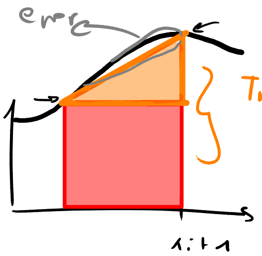
orden cero → Sumas de Riemann

orden lineal → Regla del trapecio

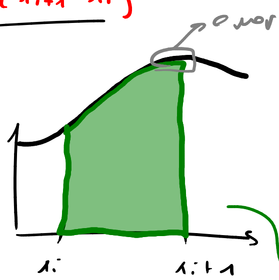
Orden cuadrático → Método de Simpson



Rectángulos de área $f(x_i)(x_{i+1}-x_i)$



Trapecios de área $\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$



Para una parábola, su área es $\int_0^{\Delta x}(ax^2+bx+c) = \frac{1}{3}a\Delta x^3 + \frac{b}{2}\Delta x^2 + c\Delta x$

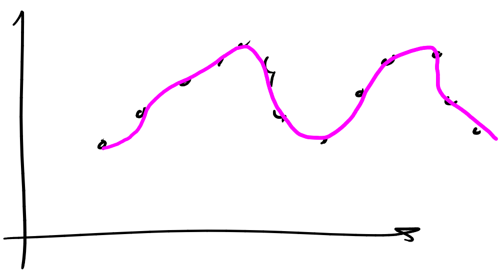
¿Cómo escribimos a, b, c en términos de $f(x)$?

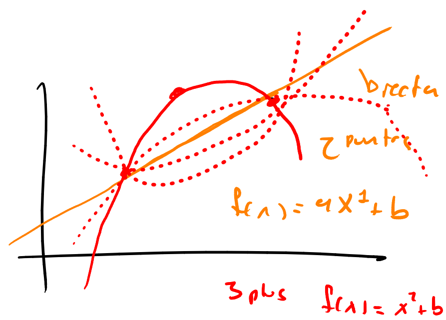
↳ Interpolando la curva.

x	y
x_0	y_0
x_3	y_2
⋮	
x_i	y_i
⋮	
x_{N-1}	y_{N-1}

N datos

$$f(x) = \sum_{i=0}^{N-1} a_i x^i$$





Interpolación de Lagrange (sup N-datos)

2 puntos → recta

3 puntos → parábola

$$f(x) = \sum_{i=0}^{N-1} y_i L_i(x) \,; \qquad L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{N-1} \frac{(x-x_j)}{(x_i-x_j)}$$

para $N=2 \rightarrow f(x) = \sum_{i=0}^{1} y_i L_i(x) = y_0 L_0(x) + y_1 L_1(x)$

$$y = mx+b \,, \quad m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y - y_0}{x - x_0}$$

$$y = \left(\frac{y_1-y_0}{x_1-x_0}\right)(x-x_0) + y_0 = \frac{1}{x_1-x_0}\left[(y_1-y_0)(x-x_0) + y_0(x_1-x_0)\right]$$

$$\Rightarrow y = \frac{1}{x_1-x_0}\left[\underline{y_1 x - y_1 x_0} - y_0 x + \cancel{y_0 x_0} + y_0 x_1 - \cancel{y_0 x_0}\right]$$

$$= \frac{1}{x_1-x_0}\left[y_1(x-x_0) - y_0(x-x_1)\right]$$

$$= y_1 \frac{(x-x_0)}{x_1-x_0} - y_0 \frac{(x-x_1)}{x_1-x_0}$$

$-(x_1-x_0) = x_0 - x_1$

$$y = y_0 \underbrace{\frac{(x-x_1)}{x_0-x_1}}_{L_0(x)} + y_1 \underbrace{\frac{(x-x_0)}{x_1-x_0}}_{L_1(x)}$$

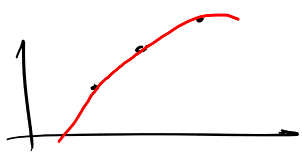
$L_0(x) = \frac{x-x_1}{x_0-x_1}$ $\qquad L_1(x) = \frac{x-x_0}{x_1-x_0}$

$\sum_i a_i := a_0 + a_1 + a_2 + \dots + a_n$

$\prod_{i=0}^{r} a_i := a_0 a_1 a_2 a_3 a_4 a_r \dots a_n$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{1} \frac{(x-x_j)}{x_i-x_j}$$

→ Interpolar 3 puntos (x_i, y_i), $i = (0,1,2)$



$$f(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{(x-x_j)}{(x_i-x_j)}$$

$$= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-x_1)}{(x_0-x_1)} \frac{(x-x_2)}{(x_0-x_2)}$$

$$\int_{x_0}^{x_0+h} f(x)\,dx$$

$x_i = x_0 + hj$ $\quad j = 0,1,2$

Reescribiendo

$x_0 = x_0$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$x_0 - x_1 = -h$

$x_0 - x_2 = -2h$

$x_1 - x_2 = h - 2h = -h$

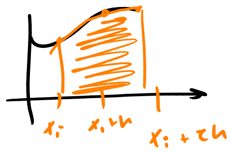
$$f(x) = y_0 \frac{(x-x_0-h)(x-x_0-2h)}{(-h)(-2h)} + y_1 \frac{(x-x_0)(x-x_0-2h)}{-h^2} + y_2 \frac{(x-x_0)(x-x_0-h)}{(2h)h}$$

$u = x - x_0$

$$\left(\frac{u}{h}-1\right) \qquad \left(\frac{u}{h}-2\right) \qquad \frac{u}{h}\left(\frac{u}{h}-2\right) \qquad \frac{u}{h}\left(\frac{u}{h}-1\right)$$

$$f(u) = y_0 \frac{(u-h)(u-2h)}{2h^2} - \frac{u(u-2h)}{h^2} y_1 + \frac{u(u-h)}{2h^2} y_2$$

$$t = u/h \Rightarrow f(t) = \underbrace{\frac{y_0}{2}(t-1)(t-2) - y_1 t(t-2) + y_2 \frac{t(t-1)}{2}}$$



Lo que queremos era:

$$I_i = \int_{x_0}^{x_0+2h} f(x)\,dx = \int_0^{2h} f(u)\,du = \int_0^2 f(t)\,h\,dt = h\int_0^2 f(t)\,dt$$

$u = x - x_0$, $du = dx$

$t = u/h$, $h\,dt = du$

$$I_i = h\int_0^2 dt\left[\frac{t^2-3t+2}{2}\right]y_0 - hy_1\int_0^2 dt\,(t^2-2t) + hy_2\int_0^2 \frac{t^2-t}{2}\,dt$$

$$= \frac{hy_0}{2}\left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_0^2 - hy_1\left(\frac{t^3}{3} - t^2\right)\Big|_0^2 + \frac{hy_2}{2}\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_0^2$$

$$= \frac{hy_0}{\cancel{2}}\left(4\frac{\cancel{8}}{3} - \frac{\cancel{4}\cdot 3}{\cancel{2}} + \cancel{4}2\right) - hy_1\left(\frac{8}{3} - 4\right) + \frac{hy_2}{\cancel{2}}\left(4\frac{\cancel{8}}{3} - \frac{\cancel{4}}{\cancel{2}}\right)$$

$$= hy_0\left(\frac{4-9+6}{3}\right) - hy_1\left(\frac{8-12}{3}\right) + hy_2\left(\frac{4-3}{3}\right)$$

$$= hy_0\frac{1}{3} + hy_1\frac{4}{3} + hy_2\frac{1}{3} = h\left(\frac{y_0}{3} + \frac{4}{3}y_1 + \frac{y_2}{3}\right) = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$\Rightarrow \quad I_i = \frac{h}{3}\left(f(x_i) + 4f(x_i+h) + f(x_i+2h)\right)$$

$h \to 2h'$

$h' = \frac{h}{2} \to \frac{h}{3} \to \frac{h}{6}$

$x_i \to x_i$

$x_i+h \to x_{i+1}$

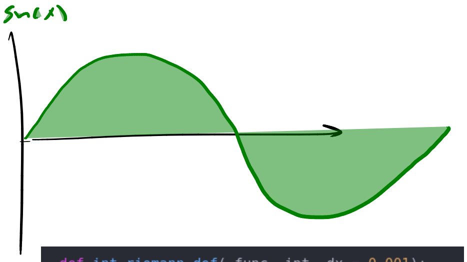
$x_i+2h \to x_{i+2}$

Pruebas

Integral definida

$$\int_0^{2\pi} \sin(x)\,dx = -\cos x \Big|_0^{2\pi} = \cos x \Big|_{2\pi}^{0} = \cos(0) - \cos(2\pi) = 0$$

sin(x)



```python
def int_riemann_def( func, int, dx = 0.001):
    res = 0
    x = np.arange(int[0], int[1], dx)
    i = 0
    while i < len(x):
        res += func(x[i]) * dx
        i += 1
    return res
```

```
❯ python3 1-Riemman.py
-7.54842135276e-08
```

≈ 0 ✓

Integral indefinida

$$f(x) = \sin(x) \Rightarrow F(x) = \int \sin(x)\,dx = -\cos(x) + C$$

y $\underbrace{F(x) = \Delta f(x)}_{}= \int_{x_0}^{x} \sin(t)\,dt$

Hay que guardar el valor de la integral en cada paso

Notemos que calculamos $\Delta f(x)$, no $f(x)$.

es decir

$$f(x) = \int_{x_0}^{x} \sin(t)\,dt + f(x_0)$$

tenemos que tener este valor, o bien, graficar $\Delta f(x)$

```python
def int_riemann( func, int, y_0 = 0, dx = 0.001):
    x = np.arange(int[0], int[1], dx)
    res = 0
    sol = np.zeros(len(x))  # sol = []
    i = 0
    while i < len(x):
        res += func(x[i]) * dx
        sol[i] = res  # sol.append(res)
        i += 1
    return [np.array(x), np.array(sol)+ y_0]

int = [0,np.pi*2]
print(int_riemann_def(f, int))
```

con $y_0 = -1$