Definición de grupos $\rightarrow$ 1) Multiplicación por escalar
2) Asociatividad
3) Identidad
4) Inversa

operador diferencial

Paréntesis de Poisson $\vec{y} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$, $\{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} = \underbrace{\left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\right)}_{D_f(g)} g = \underbrace{\left[\left(\frac{\partial f}{\partial \vec{y}}\right)^T \mathbb{J}\frac{\partial}{\partial \vec{y}}\right]}_{D_f(g)} g$

Einstein

o bien $\{f,g\} = \alpha_i(f)\frac{\partial}{\partial y_i} g$

P.D Los paréntesis de Poisson cumplen con la identidad de Jacobi

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

Calculemos $\{f,\{g,h\}\} + \{g,\{h,f\}\} = \{f,\{g,h\}\} - \{g,\{f,h\}\}$

$$= [D_f \circ D_g - D_g \circ D_f]h$$

$$= [D_f(D_g(\cdot)) - D_g(D_f(\cdot))]h$$

Propongamos $= \left[\underbrace{A_i \frac{\partial}{\partial q_i} + B_i \frac{\partial}{\partial p_i}}_{\text{Combinación lineal de las derivadas}}\right]h$

$\hookrightarrow$ No hay derivadas cruzadas por la independencia lineal

Como el operador no depende de h, usemos funciones de prueba

- $h = q_i \implies \{g,h\} = \{g,q_i\} = \cancel{\frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial p_i}} - \frac{\partial q_i}{\partial q_i}\frac{\partial g}{\partial p_i} = -\frac{\partial g}{\partial p_i}$

$$\Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} = \{f, -\frac{\partial g}{\partial p_i}\} + \{g, +\frac{\partial f}{\partial p_i}\}$$

$$= -\{f, \frac{\partial g}{\partial p_i}\} + \{g, \partial f/\partial p_i\}$$

$$= -\frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial p_i^2} + \frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q_i \partial p_i} + \frac{\partial g}{\partial q_i}\frac{\partial^2 f}{\partial p_i^2} - \frac{\partial g}{\partial p_i}\frac{\partial^2 f}{\partial q_i \partial p_i}$$

$$= -\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

$$= \frac{\partial}{\partial p_i}\{g,f\} =$$

Funciones

- Análogamente si $h = p_i$, ent $\{f,\{g,h\}\} + \{g,\{h,f\}\} = -\frac{\partial}{\partial q_i}\{g,f\}$

$\hookrightarrow$ operadores diferenciales que dependen de h

$$\Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} = D_h(\{f,g\}) = \{h,\{g,f\}\}$$

$$\Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \checkmark$$

Integrales de movimiento (Cantidades conservadas) $\longrightarrow$ $\frac{d}{dt}f = 0 = \{f, \underbrace{\mathcal{H}}_{\text{Hamiltoniano}}\} + \frac{\partial f}{\partial t} \Rightarrow \frac{\partial f}{\partial t} = -\{f, \mathcal{H}\} = \{\mathcal{H}, f\}$

- Podemos construir 1 integral de movimiento si tenemos dos

Sean f, g, tales que $\frac{\partial f}{\partial t} = \{\mathcal{H}, f\}$, $\frac{\partial g}{\partial t} = \{\mathcal{H}, g\}$

Empleamos Jacobi $\{\mathcal{H}, \{f,g\}\} + \{f, \underbrace{\{g,\mathcal{H}\}}_{-\partial_t g}\} + \{g, \underbrace{\{\mathcal{H}, f\}}_{\partial_t f}\} = 0$

$$\Rightarrow \{\mathcal{H}, \{f,g\}\} = \{f, \partial_t g\} - \{g, \partial_t f\} = \{f, \partial_t g\} + \{\partial_t f, g\} \quad \dots (1)$$

Pero vemos que $\frac{\partial}{\partial t}\{f,g\} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$

$$= \partial^2_{tq_i} f\, \partial_{p_i} g + \partial_{q_i} f\, \partial_{tp_i} g - \partial^2_{tp_i} f\, \partial_{q_i} g - \partial_{p_i} f\, \partial^2_{tq_i} g$$

$$= \partial_{q_i}(\partial_t f)\partial_{p_i} g - \partial_{p_i}(\partial_t f)\partial_{q_i} + \partial_{q_i} f\, \partial_{p_i}(\partial_t g) - \partial_{p_i} f\, \partial_{q_i}(\partial_t g)$$

$$= \{\partial_t f, g\} + \{f, \partial_t g\}$$

$$= \underbrace{\{\mathcal{H}, \{f,g\}\}}_{\times(-1)} \qquad \text{por } (1)$$

$$\Rightarrow \frac{d}{dt}\{f,g\} = \{\{f,g\}, \mathcal{H}\} + \frac{\partial}{\partial t}\{f,g\} = 0 \Rightarrow \{f,g\} \longrightarrow \text{Integral de movimiento}$$

Teorema de Noether

Supongamos una transformación infinitesimal tal que $F_2 = \mathbb{1} + \varepsilon \underbrace{G(q,P)}_{\text{F. generadora}}$

Habíamos definido $\delta\vec{y} = J\varepsilon \frac{\partial G}{\partial \vec{y}}$, y por las condiciones diferenciables de F_2

$\delta\vec{y}$: $(\vec{P}-\vec{p})_y(\vec{Q},\vec{q})$

$$\mathcal{H} + \frac{\partial F_2}{\partial t} = \underbrace{K}_{\text{Nueva hamiltoniana}} \Rightarrow K - \mathcal{H} = \delta\mathcal{H} = \frac{\partial F_2}{\partial t} = \varepsilon\frac{\partial G}{\partial t} \quad \dots (2)$$

Pero $\delta\mathcal{H} = \left(\frac{\partial \mathcal{H}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{H}}{\partial p_i}\delta p_i\right) = \left(\frac{\partial \mathcal{H}}{\partial \vec{y}}\right)^T \delta\vec{y} = \varepsilon\left(\frac{\partial \mathcal{H}}{\partial \vec{y}}\right)^T J \frac{\partial G}{\partial \vec{y}} = \varepsilon\{\mathcal{H}, G\} \quad \dots (3)$

Si G deja invariante al hamiltoniano $\delta\mathcal{H} = 0 \Rightarrow \boxed{\{\mathcal{H}, G\} = \{G, \mathcal{H}\} = 0}$ Por (3)

$$\Rightarrow \frac{\partial G}{\partial t} = 0 \quad \text{Por } (2)$$

$$\boxed{\Rightarrow \frac{dG}{dt} = \{G, \mathcal{H}\} + \frac{\partial G}{\partial t} = 0}$$ Teorema de Noether

¿Por qué funciona el principio de correspondencia?

La cuantización canónica de un sistema consiste en agarrar el hamiltoniano de un sistema dinámico y elevar a las variables dinámicas en operadores. Es decir

Oscilador armónico

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{1}{2} m \omega^2 \vec{r} \cdot \vec{r}$$

Cuantización canónica

$$\Rightarrow \quad \mathcal{H} = \frac{\vec{p} \cdot \vec{p}}{2m} + \frac{1}{2} m \omega^2 \vec{r} \cdot \vec{r} \longrightarrow \quad \mathcal{H} \longrightarrow \hat{\mathcal{H}} = \frac{\hat{\mathbb{P}}^2}{2m} + \frac{1}{2} m \omega^2 \hat{\mathbb{r}}^2$$

con $\{\vec{r}, \vec{p}\} = \mathbb{1}$

↳ corchetes de Poisson

$\vec{r} \longrightarrow \hat{\mathbb{r}}$; $\hat{\mathbb{r}} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$

$\vec{p} \longrightarrow \hat{\mathbb{P}}$; $\hat{\mathbb{P}} |\vec{r}\rangle = -i\hbar \nabla_{\vec{r}} |\vec{r}\rangle$

con $[\hat{\mathbb{r}}, \hat{\mathbb{P}}] = i\hbar \, \mathbb{1}$

↳ Conmutador

Primero, demostremos lo siguiente ($f_i \in T'Q$)

P.D $\{f_1, f_2 f_3\} = f_2 \{f_1, f_3\} + \{f_1, f_2\} f_3$

$$\{f_1, f_2 f_3\} = \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)^T \mathbb{J} \frac{\partial}{\partial \vec{\eta}} (f_2 f_3) = \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)^T \mathbb{J} \left[\left(\frac{\partial}{\partial \vec{\eta}} f_2\right) f_3 + f_2 \left(\frac{\partial}{\partial \vec{\eta}} f_3\right)\right]$$

$$\Rightarrow \{f_1, f_2 f_3\} = \underline{\left(\frac{\partial}{\partial \vec{\eta}} f_1\right)^T \mathbb{J} \left(\frac{\partial}{\partial \vec{\eta}} f_2\right) f_3} + \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)^T \mathbb{J} f_2 \left(\frac{\partial}{\partial \vec{\eta}} f_3\right)$$

$$= \{f_1, f_2\} f_3 + \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)^T \mathbb{J} \left(f_2 \frac{\partial}{\partial \vec{\eta}} f_3\right)$$

$$= \{f_1, f_2\} f_3 - \left(f_2 \frac{\partial}{\partial \vec{\eta}} f_3\right)^T \mathbb{J} \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)$$

$$= \{f_1, f_2\} f_3 - f_2 \left(\frac{\partial}{\partial \vec{\eta}} f_3\right)^T \mathbb{J} \left(\frac{\partial}{\partial \vec{\eta}} f_1\right)$$

$$= \{f_1, f_2\} f_3 - f_2 \{f_3, f_1\}$$

$$\therefore \quad \underline{\{f_1, f_2 f_3\} = f_2 \{f_1, f_3\} + \{f_1, f_2\} f_3} \; //$$

→ Notemos que es un producto ordenado y no conmutamos ninguna función

Ahora, supongamos cuatro funciones y calculemos

$\{f_1 f_2, f_3 f_4\} = ¿?$ ⟶ 2 Hay dos formas de hacerlo con el resultado anterior

→ $\{f_{12}, f_3 f_4\}$, con $f_{12} = f_1 f_2$

→ $\{f_1 f_2, f_{34}\}$, con $f_{34} = f_3 f_4$

Hagámoslo de las dos maneras

$$\{f_1 f_2, f_3 f_4\} = \{f_{12}, f_3 f_4\} = f_3 \{f_{12}, f_4\} + \{f_{12}, f_3\} f_4$$

$$= - f_3 \{f_4, \underline{f_{12}}\} - \{f_3, \underline{f_{12}}\} f_4$$

$$= - f_3 \{f_4, \underline{f_1 f_2}\} - \{f_3, \underline{f_1 f_2}\} f_4$$

$$= - f_3 \left(f_1 \{f_4, f_2\} + \{f_4, f_1\} f_2 \right) - \left(f_1 \{f_3, f_2\} + \{f_3, f_1\} f_2 \right) f_4$$

$$\Rightarrow \{f_1f_2, f_3f_4\} = \{f_{12}, f_3f_4\} = -f_3f_1\{f_4,f_2\} - f_3\{f_4,f_1\}f_2 - f_1\{f_3,f_2\}f_4 - \{f_3,f_1\}f_2f_4 \quad \dots (1)$$

Ahora, calculamos

$$\{f_1f_2, f_3f_4\} = \{f_1f_2, f_{34}\} = -\{f_{34}, f_1f_2\} \qquad 1 \longleftrightarrow 3, \quad 2 \longleftrightarrow 4$$

y de forma análoga $= f_1f_3\{f_2,f_4\} + f_1\{f_2,f_3\}f_4 + f_3\{f_1,f_4\}f_2 + \{f_1,f_3\}f_4f_2 \quad \dots (2)$

Igualando $(1)=(2) \longrightarrow \{f_1f_2, f_3f_4\} = \{f_1f_2, f_3f_4\}$

$$\Rightarrow -f_3f_1\{f_4,f_2\} - \cancel{f_3\{f_4,f_1\}f_2} - \cancel{f_1\{f_3,f_2\}f_4} - \{f_3,f_1\}f_2f_4 = f_1f_3\{f_2,f_4\} + f_1\{f_2,f_3\}f_4 + f_3\{f_1,f_4\}f_2 + \{f_1,f_3\}f_4f_2$$

$$\Rightarrow \qquad = -f_1f_3\{f_4,f_2\} - \cancel{f_1\{f_3,f_2\}f_4} - \cancel{f_3\{f_4,f_1\}f_2} - \{f_3,f_1\}f_4f_2$$

$$\Rightarrow f_3f_1\{f_4,f_2\} + \{f_3,f_1\}f_2f_4 = f_1f_3\{f_4,f_2\} + \{f_3,f_1\}f_4f_2$$

$$\Rightarrow (f_3f_1 - f_1f_3)\{f_4,f_2\} = \{f_3,f_1\}(f_4f_2 - f_2f_4) \longrightarrow$$ No hemos conmutado ninguna función

$$\Rightarrow [f_3,f_1]\{f_4,f_2\} = \{f_3,f_1\}[f_4,f_2]$$

$$\Rightarrow [f_1,f_3]\underbrace{\{f_2,f_4\} = \{f_1,f_3\}}[f_2,f_4]$$

La única forma de satisfacer esto es si $\{f_k,f_l\} \propto [f_k,f_l]$

Si $\{f_k,f_l\} = \frac{1}{i\hbar}[f_k,f_l] \longrightarrow$ Obtenemos la mecánica cuántica!

Ejemplo:

i) Evolución temporal

$$\frac{d}{dt}f = \{f,H\} + \frac{\partial f}{\partial t} \longrightarrow \frac{d}{dt}\hat{f} = \frac{1}{i\hbar}[\hat{f},\hat{H}] + \partial_t \hat{f}$$

↳ Ec. de movimiento en el esquema de Heisenberg

ii) Teorema de Noether

$$\{G,H\} = \{H,G\} = 0 \longrightarrow [\hat{G},\hat{H}] = [\hat{H},\hat{G}] = 0$$

H: Hamiltoniano

G: Función generadora de una transformación que deja invariante a H

$\Rightarrow G$ es el resultado de una simetría del sistema

$H \longrightarrow \hat{H}$

$\hat{G}$: Operador que, aplicado al sistema, lo deja invariante

$\Rightarrow$ ¡$\hat{G}$ es una simetría del sistema!