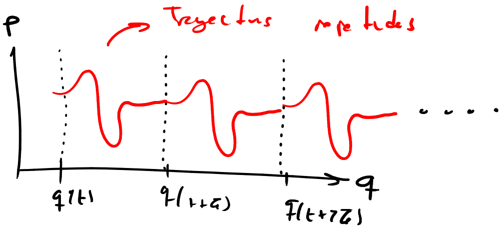
Variables de ángulo - acción

↓

Sistemas periódicos → Rotaciones

Sistemas periódicos → Oscilaciones

$\vec{q}(t) = \vec{q}(t+\tau)$ → periodo



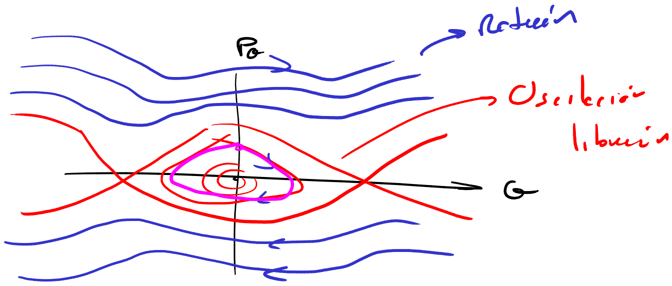
Ejemplo es el péndulo

$\mathcal{H} = \frac{1}{2m} p_\theta^2 + mgl\sin^2\theta$

$l = cte$

si $\theta \ll 1$ $\quad \mathcal{H} = E \approx \frac{1}{2m} p_\theta^2 + mgl\,\theta^2$

Dibujo del espacio fase



$\mathcal{H}_J \longrightarrow \mathcal{H}\left(\vec{q}, \frac{dW}{dq}\right) = E(\vec{\alpha}) = \alpha_1 = E(\vec{\gamma}(\vec{\alpha}))$

$\vec{\gamma} = \vec{\gamma}(\vec{\alpha}) \equiv$ ctes $\neq$ ctes de integración

↳ $\vec{P}$

↳ $J_i(\vec{\alpha}) \longrightarrow \{J_i\}$ conjunto de funciones independientes

↳ Variables de acción

Para 1D $\longrightarrow$ $\boxed{J = \oint p\,dq} = \int_t^{t+\tau} p \frac{dq}{dt}\,dt$ $\quad [p \cdot q]$ (=) acción → variable momento [=] [s]

$E \cdot t$ $\quad$ [E·s]

$\mathcal{H} = \alpha_1$ ↓

$W = W(q, \alpha)$ $\longrightarrow$ $\mathcal{H} = \mathcal{H}(J)$, $W = W(q, J)$ $\longrightarrow$ Cond. def.

$w = \frac{\partial W}{\partial J}$ = variable de ángulo ... (1)

$\Rightarrow \dot{w} = \frac{\partial \mathcal{H}(J)}{\partial J} = \nu(J)$ → cte

$\Rightarrow \underbrace{w = \nu t + \beta}_{(1)}$ → cte

¿Qué significado tiene ν?

$W = W(q) \longrightarrow dW = \frac{\partial w}{\partial q} dq \longrightarrow \Delta w = \oint \frac{\partial}{\partial q}(w)\,dq = \oint \frac{\partial}{\partial q}\left(\frac{\partial W}{\partial J}\right) dq$

$= \frac{d}{dJ} \oint \frac{\partial}{\partial q} W\,dq = \frac{d}{dJ} \oint p\,dq = \frac{dJ}{dJ} = 1$

1 periodo

$\Delta w = 1 = \nu \tau$

$\longrightarrow \boxed{\nu(J) = \frac{1}{\tau} = \text{frecuencia}}$

↳ Sin resolver el sistema dinámico

↳ frecuencia del sistema

↳ T. Fourier

$w(t) = \nu t + \beta$

$w(t+\tau) = \nu t + \nu\tau + \beta$

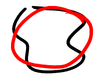
$\Delta w = \nu \tau$

Generalización para sistemas totalmente separables

$$W = \sum_j W_j(q_j; \vec{\alpha}) \Rightarrow p_j = \frac{\partial W}{\partial q_j} = \frac{dW_j}{dq_j} = p_j(q_j, \vec{\alpha})$$

$$J_j = \oint p_j \, dq_j = \oint \frac{dW_j}{dq_j} dq_j \rightarrow \text{Variables de acción}$$

→ ¿qué pasa si tengo variables cíclicas?

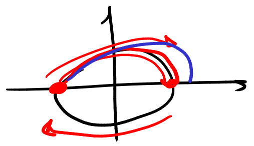


$$\dot{p}_j = 0 \Rightarrow p_j = \alpha_j \equiv ctes \Rightarrow J_j = \oint p_j \, dq_j = p_j \oint dq_j = p_j \, 2\pi$$

1D $\mathcal{H} = \alpha_1 = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2$ → ω

↳ despejamos p → $p^2 = 2m\alpha_1 - m^2\omega^2 x^2 \Rightarrow p_\pm = \pm m\omega \sqrt{\frac{2\alpha_1}{m\omega^2} - x^2} = \frac{dW}{dq}$

$$p = 0 \text{ si } x = \pm\sqrt{\frac{2\alpha_1}{m\omega^2}} = x_\pm$$



$$J = \oint p \, dx = \int_{x_-}^{x_+} p_+ dq - \int_{x_+}^{x_-} p_- dq = 2 \int_{x_-}^{x_+} p_+ dq = 2m\omega \int_{x_-}^{x_+} \sqrt{\frac{2\alpha_1}{m\omega^2} - x^2} \, dx$$

$a^2 - x^2$

$\frac{x^2}{a^2} = \sin\beta$

$$\Rightarrow J = 2m\omega \left[\cancel{\frac{1}{2} x \sqrt{\frac{2\alpha_1}{m\omega^2} - x^2}} + \frac{\alpha_1}{m\omega^2} \arcsin\left(\frac{x}{\sqrt{\frac{2\alpha_1}{m\omega^2}}} \right) \right] \Bigg|_{x_-}^{x_+}$$

$\cancel{\ }\to 0$

$$= \frac{2\alpha_1 \omega}{\omega^2} \left(\arcsin(+1) - \arcsin(-1) \right)$$

$$= 2\alpha_1 \frac{1}{\omega} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right)$$

$$J = \frac{2\pi}{\omega} \alpha_1$$

$$\mathcal{H} = \alpha_1 = \frac{J\omega}{2\pi} = \mathcal{H}(J) \Rightarrow \frac{d\mathcal{H}(J)}{dJ} = \frac{d}{dJ}\left(\frac{\omega}{2\pi} J \right) = \frac{\omega}{2\pi} = \nu$$

frec. angular

$$\nu = \frac{\omega}{2\pi} \checkmark \rightarrow \underline{\text{frecuencia}} = \frac{1}{\tau}$$

$$\rightarrow \nu \rightarrow \cancel{\left[\frac{\omega}{2\pi}\right]}$$