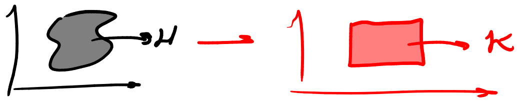
Tangent Bundle   P. de Hamilton

Transformaciones canónicas

↳ Automorfismos : $\mathcal{T} : T\dot{Q} \longrightarrow T^*Q$

$(q_i, p_i) \longrightarrow (Q_i, P_i)$

↳ (a) E.H.
$$\begin{cases} -\dot{p}_i = \dfrac{\partial \mathcal{H}}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \dfrac{\partial \mathcal{H}}{\partial t} = \dfrac{d\mathcal{H}}{dt} = -\dfrac{\partial L}{\partial t} \end{cases}$$



$$-\dot{P}_i = \frac{\partial \mathcal{H}}{\partial Q_i}$$
$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i}$$
$$\frac{\partial K}{\partial t} = \frac{dK}{dt}$$

→ Construir una función generadora $F \longrightarrow F_1 = \sum_{ij} q_i Q_j - F_1(q_i, Q_j)$

P. de Hamilton →

$$S = \int_{t_0}^{t} L(\{q_i\}, \{p_i\}, t)\, dt = \int_{t_0}^{t} \left(\sum p_i \dot{q}_i - \mathcal{H}(\{q_i\}\{p_i\}, t) \right) dt$$

(1) $\delta q_i = 0 = \delta p_i$

(2) $\delta P_i = 0 = \delta Q_i$

$\delta S = 0 \Rightarrow$ E.H.

$$\mathcal{T}(S) = S' = \int_{t_0} \left(\sum_i P_i \dot{Q}_i - K(\{Q_i\}, \{P_i\}, t) \right) \Rightarrow \delta S' = 0 \longrightarrow \text{E.H.}$$

Lo que quiero $S = \mathcal{T}(S) = S' \longrightarrow \delta S = \delta S'$ ✓

$F_1 = F_1(\{Q_i\}, \{q_i\}) = \sum_{ij} Q_i q_j$

$\longrightarrow F = F(\{Q_i\}, \{q_i\}, \{P_i\}, \{p_i\}, t)$

$$\longrightarrow \sum p_i \dot{q}_i - \mathcal{H} = \sum P_i \dot{Q}_i - K + \frac{d}{dt} F$$

Función generadora de las T.C.

→ Por ser lineal, puedo "cuadrar" más E.H para que mezcle su forma

$Q_j = Q_j(\{q_i\}, \{p_i\}, t)$

$P_j = P_j(\{q_i\}, \{p_i\}, t)$

(3) $\mathcal{T} : (q, p_i) \longrightarrow (Q, P) \quad \exists \quad \mathcal{T}^{-1} : (Q, P) \longrightarrow (q, p)$

Integración por partes

$$\delta S = \delta \int_{t_0}^{t} L\, dt = \int_{t_0}^{t} \delta \left(\sum \dot{q}_i p_i - \mathcal{H} + \frac{dF}{dt} \right) dt$$

$$\int \delta L = \int \frac{dL}{dt} \delta t\, dt = \cancel{L \delta t} \Big|_{t_1}^{t} - \int \frac{dL}{\partial t}\, dt$$

$\cancel{\dfrac{dF}{dt} \delta t}$

δq

Si $F_1 = F_1(\{Q_i\}, \{q_i\})$

$q_i, p_i \rightarrow Q, P_i$

$$\frac{d}{dt} F_1 = \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial t}$$

(a)

$\dot{q}_i \longrightarrow p_i = \dfrac{\partial F_1}{\partial q_i}$ (b)

$\dot{Q}_i \longrightarrow 0 = P_i + \dfrac{\partial F_1}{\partial Q_i}$ (c)

lo demás $\longrightarrow$ $\boxed{-\mathcal{H} = -K + \dfrac{\partial F_1}{\partial t}}$

$$\sum p_i \dot{q}_i - \mathcal{H} = \sum P_i \dot{Q}_i - K + \frac{d}{dt} F$$

$F_1 = \sum_j Q_j q_j$

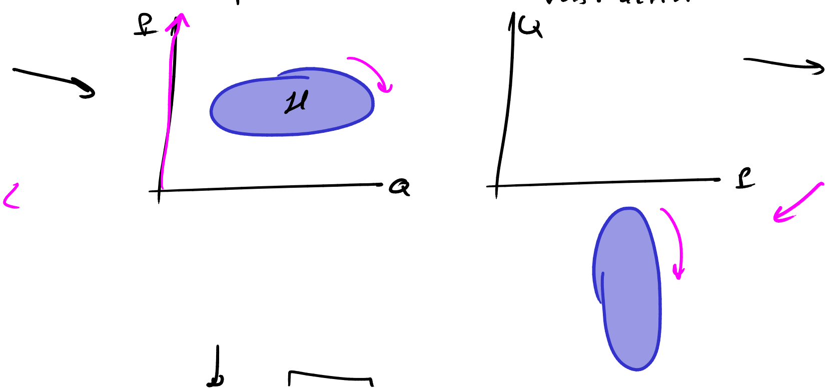
$p_i = \frac{\partial F_1}{\partial q_i} = Q_i$

$q_i = \frac{\partial F_1}{\partial Q_i} = -P_i$

$\mathcal{H} = \mathcal{K}$

$\Longrightarrow \Upsilon \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} -P_i \\ Q_i \end{pmatrix}$

trasf pasiva — Trans. activa



$F_1(\{q_i\},\{Q_i\}) = F_2(\{q_i\},\{P_i\}) - \sum_i Q_i P_i$

$$\boxed{\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_i \checkmark \\ Q_i &= \frac{\partial F_2}{\partial P_i} = \frac{\partial F_2}{\partial P_i} = q_i \\ \mathcal{H} &= \mathcal{K} - \cancel{\frac{\partial F_2}{\partial t}}^{\,0} \end{aligned}} \checkmark$$

si $F_1 = \sum_j q_i Q_i$

$\Rightarrow F_2 = \sum q_i P_i$ ✓

$F_2 \longrightarrow \Upsilon = \mathbb{1}$ trans. Identidad

→ Grupo

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Goldstein — Canonicas segun de Einstein

Generating Function	Generating Function Derivatives		Trivial Special Case		
$F = F_1(q, Q, t)$	$p_i = \dfrac{\partial F_1}{\partial q_i}$	$P_i = -\dfrac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$,	$Q_i = p_i$,	$P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \dfrac{\partial F_2}{\partial q_i}$	$Q_i = \dfrac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$,	$Q_i = q_i$,	$P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\dfrac{\partial F_3}{\partial p_i}$	$P_i = -\dfrac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$,	$Q_i = -q_i$,	$P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\dfrac{\partial F_4}{\partial p_i}$	$Q_i = \dfrac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$,	$Q_i = p_i$,	$P_i = -q_i$

Siempre $\longrightarrow \boxed{\mathcal{H} = \mathcal{K} - \frac{\partial F}{\partial t}}$ $\longrightarrow F_i \to$ No son una base de las T.C. *

$F_2 = F_2(\{q_i\}, \{P_i\}, t) \longrightarrow F_2 = \sum_i f_i(\{q_i\}, t)\, P_i$

Se debe cumplir

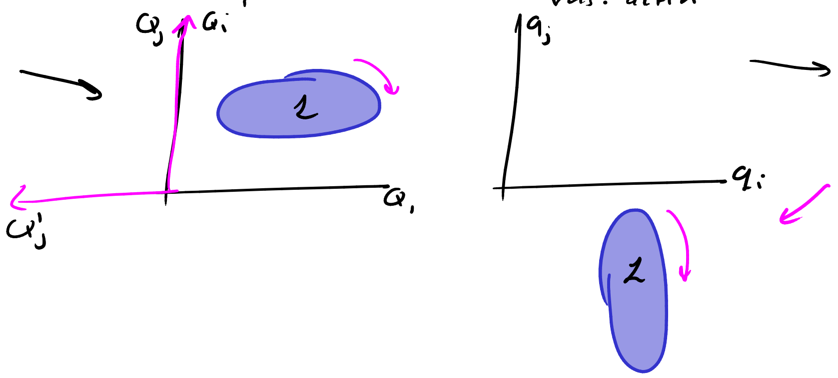
$Q_i = Q_i(\{q_i\})$

$Q_i := \frac{\partial F_2}{\partial P_i} = f_i(\{q_i\}, t) = Q_i(\{q_i\}, t)$

$p_i = \frac{\partial F_2}{\partial q_i} = P_i \frac{\partial f_i}{\partial q_i} \longrightarrow P_i = \frac{p_i}{\partial f / \partial q_i} = P(\{p, q_i\}, t)$

trasf pasiva

Tras. activa



$\longrightarrow$ Transformaciones de punto son canónicas.

Dejan invariante $\mathcal{L}$

$\Rightarrow$ Ecs. E-L

$\Rightarrow$ Ecs. mov

$\Rightarrow$ E.H

$\to F_2 = \sum_i f_i(\{q_i\}, t)\, P_i + g(\{q_i\}, t)$

$Q_i := \frac{\partial F_2}{\partial P_i} = f_i(\{q_i\}, t) = Q_i(\{q_i\}, t)$

$\longrightarrow p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial f_j}{\partial q_i} P_j + \frac{\partial g}{\partial q_i}$

$\longrightarrow \vec{p} = \nabla_{\vec{q}} \vec{f} \cdot \vec{P} + \nabla_{\vec{q}}\, g$

$\vec{P} = (\nabla_{\vec{q}} \vec{f})^{-1} (\vec{p} - \nabla_{\vec{q}}\, g)$

$\hookrightarrow \vec{P} = \vec{P}(\vec{q}, \vec{p}, t) \longleftarrow \mathcal{H} \to \vec{q}, \vec{p}$ $\to$ Necesarias

$\longrightarrow \vec{Q} = \vec{Q}(\vec{q}, t) \longleftarrow$ quitar $\vec{p}$

Notación rusa

$q_i \longrightarrow \vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_{3N-k} \end{pmatrix}$

$p_i \longrightarrow \vec{p}$

$Q_i \longrightarrow \vec{Q}$

$P_i \longrightarrow \vec{P}$

$f_i \to \vec{f}$

$\frac{\partial}{\partial q_i} \longrightarrow \frac{\partial}{\partial \vec{q}} = \nabla_{\vec{q}} = \begin{pmatrix} \partial/\partial q_1 \\ \partial/\partial q_2 \\ \vdots \\ \partial/\partial q_{3N-k} \end{pmatrix}$

$\to \frac{\partial f_j}{\partial q_i} = \frac{\partial \vec{f}}{\partial \vec{q}} = \nabla_{\vec{q}} \vec{f}$

→ ¿Por qué T.C? → Escribir el $\mathcal{H} \to K$ de tal forma que sea más fácil resolver $K \to \mathcal{H}$ ✓

Oscilador armónico

$\omega = \sqrt{k/m}$

$$\mathcal{H} = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \frac{1}{2m}\left(f^2(P)\cos^2 Q + \cancel{m^2\omega^2} \frac{f^2(P)\sin^2 Q}{\cancel{m^2\omega^2}}\right)$$

$$= \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right)$$

$$= \frac{f^2(P)}{2m} \to Q \to \text{variable cíclica}$$

$\to \mathcal{H} = K = \text{cte} \to$

$$p = f(P)\cos Q$$

$$q = \frac{f(P)}{m\omega}\sin Q$$

$$F_1 = \frac{m\omega}{2} q^2 \cot Q$$

$$p = \frac{\partial F_1}{\partial q} = m\omega \cot Q \, q$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q} \to q^2 = \frac{P\,2}{m\omega}\sin^2 Q$$

→ No tengo una raíz cuadrada

$$\to q = +\sqrt{\frac{2P}{m\omega}}\sin Q$$

$$\Rightarrow p = m\omega q \cot Q = m\omega\sqrt{\frac{2P}{m\omega}}\sin Q \frac{\cos Q}{\sin Q}$$

$$p = \sqrt{2m\omega P}\cos Q$$

$$f(P) = \sqrt{2m\omega P}$$

$$\mathcal{H} = \frac{f^2(P)}{2m} = \frac{2m\omega P}{2m} = \omega P \longrightarrow \boxed{\mathcal{H} = \omega P = E} ✓$$

$$\to P = E/\omega$$

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega \Rightarrow Q = \omega t + \varphi$$

1 integral

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \varphi)$$

$$p = \sqrt{2mE}\cos(\omega t + \varphi)$$