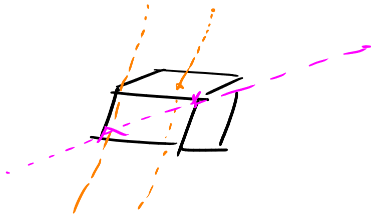
Tensor de inercia

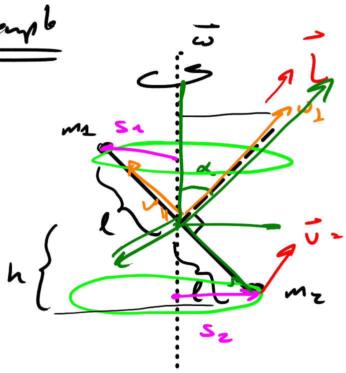
$$\overleftrightarrow{I} \in \mathcal{M}_{3\times 3}(\mathbb{R}) \qquad I_{ij} = \int dm \left(\|\vec{r}\|^2 \delta_{ij} - r_i r_j \right)$$

$$T = \tfrac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} = \underbrace{T_{CM}} + \underbrace{T_{rot}} = \underbrace{\tfrac{1}{2} m \dot{\vec{R}} \cdot \dot{\vec{R}}} + \underbrace{\tfrac{1}{2} \vec{\omega}^T \overleftrightarrow{I} \vec{\omega}}$$

$$\vec{p} = m\vec{v} \quad , \quad \vec{L} = I\vec{\omega} \longrightarrow \overleftrightarrow{I}\vec{\omega}$$



Exemplo

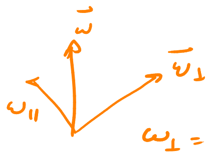


$m = m_1 = m_2$

$$\vec{L} = \sum_i (\vec{r}_i \wedge \vec{p}) \Rightarrow \vec{L} = 2\ell |\vec{p}| = \overbrace{2m\omega\ell^2\cos(\omega t)}^{L}\, \hat{\omega}_\perp$$

$$|\vec{p}| = m|\vec{v}| = m\omega r_\perp = m\omega\ell\cos(\omega t)$$

$$\vec{v} = \omega r_\perp = \omega \ell \cos(\omega t)$$



$\omega_\perp = \omega\cos\alpha$

$\omega_\parallel = \omega\sin\alpha$

$$\vec{L} = \underbrace{L\cos\alpha\, \hat{e}_z}_{L_z} + \underbrace{L \sin\alpha}_{L_x, L_y} \left(\cos\omega t\, \hat{e}_x + \sin\omega t\, \hat{e}_y \right)$$

$$\vec{\tau} = \frac{d}{dt}\vec{L} = L\sin\alpha\,\omega\left(-\sin\omega t\, \hat{e}_x + \cos\omega t\, \hat{e}_y\right) = \tau_x \hat{e}_x + \tau_y \hat{e}_y$$

$$\|\vec{\tau}\| = L\omega\sin\alpha$$

$$\vec{L} = \overleftrightarrow{I}\vec{\omega} \qquad \vec{\omega} = (0,0,\omega) = \omega\hat{e}_z \qquad I_{ij} = \sum_{\beta=1}^{2} m_\beta \left(\|\vec{r}_\beta\|^2 \delta_{ij} - r_{\beta i} r_{\beta j} \right)$$

$\vec{r}_2$: $x_2 = s\cos\omega t$, $y_2 = s\sin\omega t$, $z_2 = -h$

$\vec{r}_1$: $x_1 = -s\cos\omega t$, $y_1 = -s\sin\omega t$, $z_1 = h$

$$\boxed{\begin{matrix} -zx \\ -yz \\ -xy \end{matrix}}$$

$s^2 = x^2 + y^2$

$h = \ell\sin\alpha$

$s = \ell\cos\alpha$

$$I_{xx} = m_1\left(s^2\sin^2\omega t + h^2\right) + m_2\left(s^2\sin^2\omega t + h^2\right) = 2m\left(s^2\sin^2\omega t + h^2\right)$$

$$I_{yy} = 2m\left(s^2\cos^2\omega t + h^2\right)$$

$$I_{zz} = 2m s^2$$

$$I_{zy} = m_1(-z_1 y_1) + m_2(-z_2 y_2) = +m s \sin\omega t\, h + m s \sin\omega t\, h$$

$$= 2m s h \sin\omega t$$

$$I_{xy} = -2m s^2 \sin\omega t \cos\omega t$$

$$I_{xz} = 2m s h \cos\omega t$$

$$\overleftrightarrow{I} = 2m \begin{pmatrix} s^2\sin^2\omega t + h^2 & s^2 \sin\omega t\cos\omega t & sh\cos\omega t \\ s^2\sin\omega t\cos\omega t & s^2\cos^2\omega t + h^2 & sh\sin\omega t \\ sh\cos\omega t & sh\sin\omega t & s^2 \end{pmatrix}$$

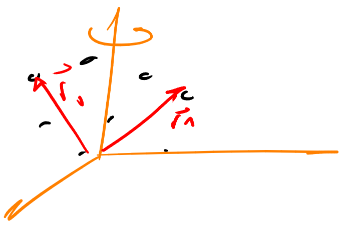
$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{pmatrix} = \begin{pmatrix} I_{xz} \\ I_{yz} \\ I_{zz} \end{pmatrix}\omega$$

$$\vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}\begin{pmatrix} I_{xz} \\ I_{yz} \\ I_{zz} \end{pmatrix}\omega = \begin{pmatrix} \frac{d}{dt} I_{xz} \\ \frac{d}{dt} I_{yz} \\ 0 \end{pmatrix}\omega = \underline{2m\omega^2} sh \begin{pmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{pmatrix}$$

$$\vec{\tau} = \tau_x \hat{e}_x + \tau_y \hat{e}_y = 2m\omega^2 sh\left(-\sin\omega t\, \hat{e}_x + \cos\omega t\, \hat{e}_y\right)$$

$\longrightarrow$ Calcular $\overleftrightarrow{I}$

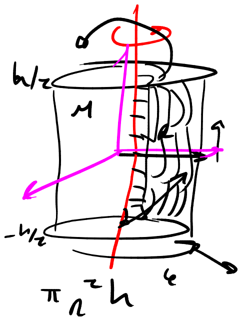
1) Identificar los elementos de masa / masas



2) Los escribo en sus coordenadas cartesianas (x_i, y_i, z_i)

3) Calculo los elementos de $\overleftrightarrow{I}$

$$I_{ij} = \sum_\beta m_\beta\left[\|\vec{r}_\beta\|^2\delta_{ij} - r_{\beta i} r_{\beta j}\right] = \int dm\left(\|\vec{r}\|^2\delta_{ij} - r_i r_j\right)$$



dist. de masa uniforme

$$\boxed{\frac{\Delta m}{\Delta V} = \frac{M}{\pi s^2 h}}$$

$$dm = \frac{M}{\pi r^2 h}\,dV$$

$$\underline{dv = s\,d\varphi\,ds\,dz}$$

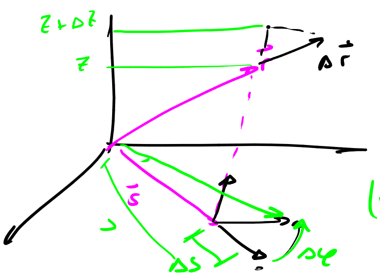
$x^2 + y^2 + z^2 - z^2$

$$I_{zz} = \int dm\,(x^2+y^2) = \int dm\, s^2$$

$$= \int_{vol} \frac{M}{\pi r^2 h} s^2\,dV = \frac{M}{\pi r^2 h}\int_{-h/2}^{h/2}\int_0^r\int_0^{2\pi} s^2\, s\,d\varphi\,ds\,dz$$

$$= \frac{M}{\pi r^2 h}\int_{-h/2}^{h/2}\int_0^r\int_0^{2\pi} s^3\,d\varphi\,ds\,dz$$

ΔV



$$\Delta V = s\Delta\varphi z s - (s+\Delta s)\Delta\varphi(z+\Delta z)(s+\Delta s)$$
$$\approx \underline{s\Delta s\Delta z\Delta s} + O(\Delta^2) \to 0$$

$(s+\Delta s)\Delta\varphi$

$z+\Delta z$

$s+\Delta s$

$$I_{zz} = \frac{M}{\pi r^2 h}\int_0^r\left[\int_{-h/2}^{h/2}\left[\int_0^{2\pi} s^3\,d\varphi\right]dz\right]ds$$

$$= s^3\int_0^{2\pi} d\varphi = s^3\varphi\Big|_0^{2\pi} = s^3 2\pi$$

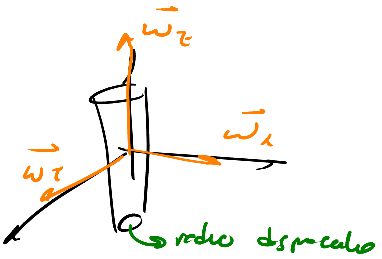
$$= \int_{-h/2}^{h/2} s^3 2\pi\,dz = 2\pi s^3\int_{-h/2}^{h/2} dz = 2\pi s^3 z\Big|_{-h/2}^{h/2}$$

$$= 2\pi s^3 h$$

$$\int_0^r 2\pi s^3 h\,ds = 2\pi h\int_0^r s^3\,ds = \frac{2\pi h}{4} r^4$$

$$I_{zz} = \frac{\cancel{2}\,\cancel{\pi}\,\cancel{h}\,R^{\cancel{4}}}{\cancel{4}\;2}\,\frac{M}{\cancel{\pi}\,\cancel{R^2}\,\cancel{h}} = \frac{1}{2} M R^2$$

I_{ij} qué tan difícil es que gire en la dirección i el objeto, debido a una rotación en la dirección j



$$\boxed{\frac{R}{h} \ll 1}$$

$$\overleftrightarrow{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

$$I_z = \frac{1}{2} M R^2$$

$$I = \frac{1}{12} M h^2$$