Trabajo y energía

03/11/21

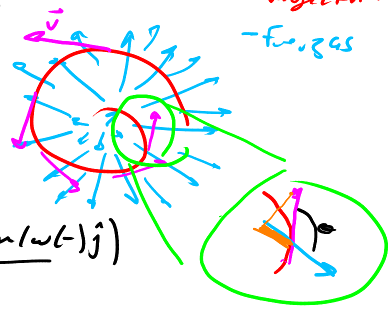
Definición integral — Definición diferencial — Integral de línea: - Trayectoria - Fuerzas

$$W_{ab} = -\int_a^b \vec{F}\cdot\vec{v}\,dt = -\int_a^b \vec{F}\cdot d\vec{r} \longrightarrow \boxed{dW = -\vec{F}\cdot d\vec{r}}$$

W_{ab} → $W(t_a) - W(t_b)$ → Trabajo

dt → Parametrización de la trayectoria

$\int_a^b$ → Integral de línea



$$\vec{r}(t) = e^{-\gamma t}\underbrace{(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath})}_{\text{espiral}}$$

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}(e^{-\gamma t}\hat{r}) = e^{-\gamma t}(-\gamma\hat{r} + \dot{\theta}\hat{\theta})$$

$$\vec{F}\cdot d\vec{r} = F\cos\theta\, dr$$

$$\hookrightarrow \int_{t_a}^{t_b} \underbrace{\vec{F}\cdot(e^{-\gamma t}(-\gamma)\hat{r} + e^{-\gamma t}\dot{\theta}\hat{\theta})}_{\in \mathbb{R}}\,dt$$

→ Energía cinética → Surge de considerar la 2da ley de Newton

$$\int dW = -\int \vec{F}\cdot d\vec{r} = \int -\vec{F}\cdot\vec{v}\,dt \qquad \vec{F} = \dot{\vec{p}} = m\frac{d\vec{v}}{dt} \leftarrow \text{2da ley}, \; m = cte$$

$$= -\int m\vec{v}\cdot\frac{d\vec{v}}{dt}\,dt \qquad \frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}(v^2) = 2\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$= -\int \underbrace{\frac{d}{dt}\left(\frac{m}{2}v^2\right)dt}_{\text{T. F. cálculo}}$$

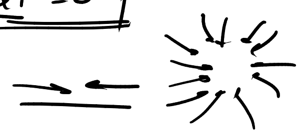
$$= -\Delta\Big(\underbrace{\frac{m}{2}v^2}_{E_c}\Big)$$

→ Fuerzas conservativas

①, ②, ③ → Definiciones de campos conservativos

$$-\Delta(E_c) = -\int \vec{F}\cdot d\vec{r} \qquad \vec{F}\cdot d\vec{r} = -dU \quad \text{(conservación; Potencial)}$$

$$\boxed{\nabla\times\vec{F} = 0} \; ③$$



$$= \int dU = \Delta U \; ② \longrightarrow \Delta U = -\Delta E_c$$

$$U_2 - U_1 = E_{c1} - E_{c2}$$

$$= \int \frac{dU}{dt}\,dt \qquad \Rightarrow E_{c1} + U_1 = E_{c2} + U_2 = E_T$$

→ Que la integral no depende de la trayectoria

$$=$$

$$-\vec{F}\cdot\vec{v} = \frac{dU(x,y,z)}{dt} = \frac{dU}{dx}\frac{dx}{dt} + \frac{dU}{dy}\frac{dy}{dt} + \frac{dU}{dz}\frac{dz}{dt}$$

$$-\vec{F}\cdot\vec{v} = \underbrace{\left(\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}\right)}_{\nabla U}\cdot\vec{v}$$

$$\nabla U \longrightarrow \boxed{\vec{F} = -\nabla U} \; ① \quad \text{F. conservativa}$$

→ $\vec{F} = -\frac{q_1 k}{r^2} \to U = -\frac{k}{r}$

$F = -kx \to U = \frac{1}{2}kx^2$

$F = -mg \to U = -mgz$

Resumen

· Def. de trabajo → $dW = -\vec{F}\cdot d\vec{r}$

· T. E-T → $\Delta W_{ab} = -\int_b^a \vec{F}\cdot d\vec{r}$

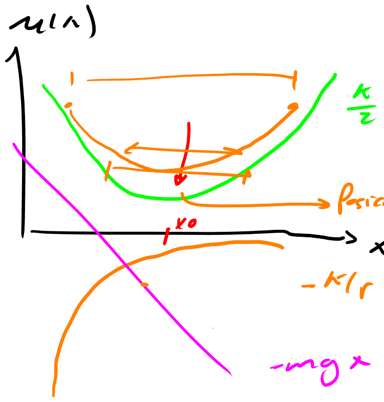
· F. Conservativas → $\underbrace{\Delta E_c = -\Delta U}$ $\Longrightarrow$ $E_T \equiv E_c + U$ (cte)

$$\frac{d}{dt}(E_c + U) = 0$$

Cómo interactúa $\vec{F}$

cómo se mueve la partícula $\vec{p}$

¿Qué nos dice el potencial de la dinámica?



$\frac{k}{2}(x-x_0)^2 \to F = -k(x-x_0)$

↳ $x - x_0 = A\cos\left(\sqrt{\frac{k}{m}}\,t + \varphi_0\right)$

↳ ¿Son puntos estables o inestables?

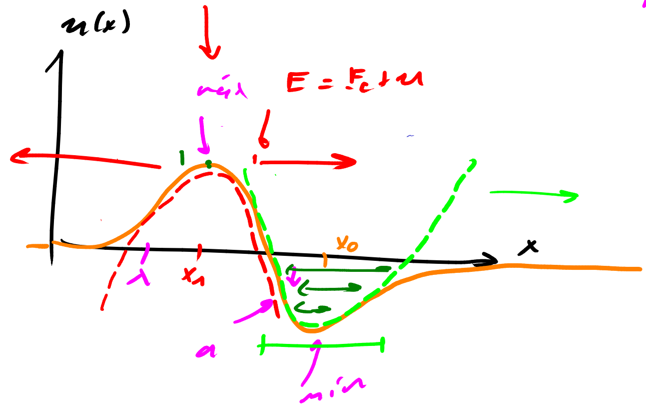
$\vec{F} = -\nabla U \to F = -\frac{dU}{dx}$

máx y mín → Puntos de equilibrio y no se mueve

$-\frac{dU}{dx} = 0 = F \to \vec{a} = 0$

$U(x) = \cancel{U_0} + \cancel{\frac{dU}{dx}\Big|_{x_0}(x-x_0)} + \frac{1}{2}\frac{d^2U}{dx^2}\Big|_{x_0}(x-x_0)^2 + \dots$

$U(x) \simeq \frac{1}{2}\underbrace{\frac{d^2U}{dx^2}\Big|_{x=x_0}}_{k}(x-x_0)^2 = \frac{k}{2}(x-x_0)^2$



$\frac{d^2U}{dx^2} > 0$

$-\frac{dU}{dt} = F = \underbrace{-k}_{<0}(x-x_0) \to x - x_0 = A\cos(\omega t + \varphi_0)$

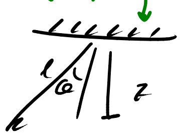
↳ atractiva

↓

Posición de equilibrio estable

$E_c + U(\theta) = E$

Ejemplo

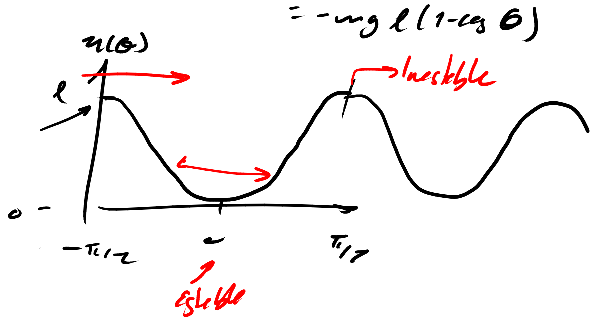


$F = -\frac{d^2U}{dx^2}\Big|_{x=x_1}(x-x_0) > 0$ (<0)

↳ Repulsiva

Posición de equilibrio inestable

$U = -mgz$
$= -mgl(1-\cos\theta)$



Otro ejemplo: Péndulo invertido

$$U = mgz + \cancel{U_0}^{\,0} = mgl\cos\varphi$$

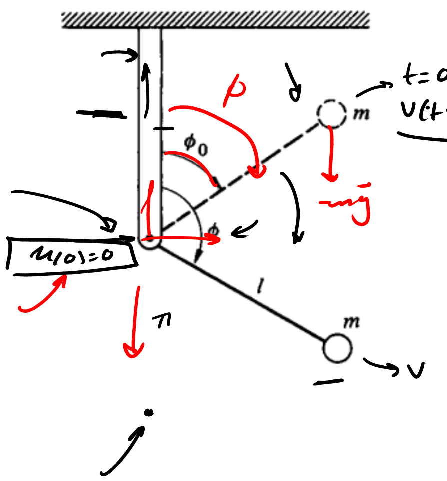
$\varphi=0 \to mgl$

$\varphi=\pi$

$-mgl$

$\varphi=\pi/2 \quad U=0$

$$\vec F = m\frac{d^2\vec x}{dt^2}$$



$t=0$

$V(t=0)=0$

$t=0 : V(0) \longrightarrow z(t_0) = l\cos\varphi_0$

$t \;: V(t) \longrightarrow z(t) = l\cos[\varphi(t)]$

$$\Delta E_c = \frac{1}{2}m\left[\cancel{V_0^2}^{\,0} - V^2(t)\right]$$

$$\Delta U = mg\,z(0) - mg\,z(t)$$
$$= mgl\left[\cos\varphi_0 - \cos\varphi\right]$$

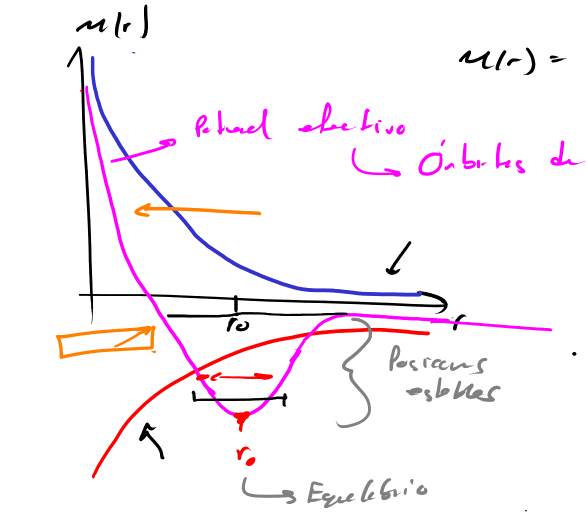
$$\underline{\Delta E_c = -\Delta U} \Rightarrow \cancel{m}\frac{1}{2}\cancel{m}V^2 = \cancel{m}gl\left[\cos\varphi_0 - \cos\varphi\right]$$

$$\Rightarrow V = \sqrt{2gl(\cos\varphi_0 - \cos\varphi)}$$

$$\frac{dV}{d\varphi} = \frac{1}{2}\frac{2gl\sin\varphi}{\sqrt{\;}} = 0$$

$$\sin\varphi = 0 \Rightarrow \underbrace{\varphi = 0,\pi}_{\text{coincide con nuestras posiciones}}$$

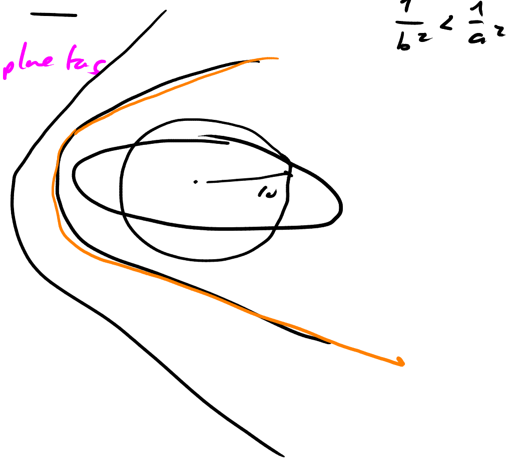
Ejemplo: Órbitas celestes

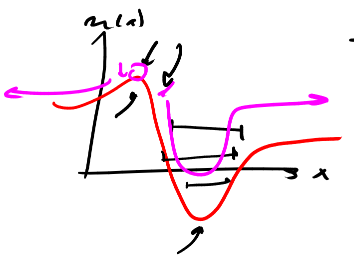


$$U(r) = -\frac{k_0}{r} + \frac{k_1}{r^2}$$

$a < b < 1 \qquad a^2 < b^2$

$$\frac{1}{b^2} < \frac{1}{a^2}$$





$$-\frac{dU}{dx} = 0 = F = ma \Rightarrow \underline{a=0}$$

$$k = \frac{d^2U}{dx^2} > 0 \longrightarrow \text{posición de equilibrio estable}$$

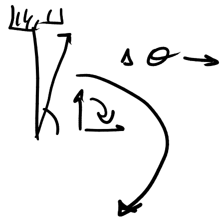
$$U = \frac{1}{2}\left(\frac{d^2U}{dx^2}\right)_{x=x_0}(x-x_0)^2 \longrightarrow \vec F = -\frac{dU}{dx} < 0$$
$$= -\left(\frac{d^2U}{dx^2}\right)_{x_0}(x-x_0)$$

$\underbrace{\qquad\qquad}_{k}$

> 0

$$\ddot x = -\frac{|k|}{m}x$$

$\hookrightarrow \sin\left(\sqrt{\frac{k}{m}}\,x\right)$



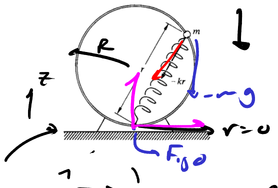
$$\frac{d^2U}{dx^2} < 0 \longrightarrow \text{inestables}$$

$$\vec F = -\left(\frac{d^2U}{dx^2}\right)_{x=x_0}(x-x_0) > 0$$

$\hookrightarrow$ repulsivas

$$e^{\pm\sqrt{\frac{|k|}{m}}\,t}$$

$$\ddot x = -\frac{-|k|}{m} \quad \boxed{\ddot x = \frac{k}{m}x}$$

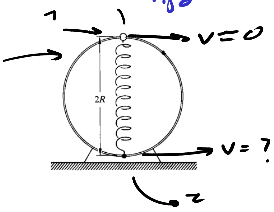


$$\mathcal{U} = +mgz + \frac{k}{2}r^2 + \mathcal{U}_0$$

$$\Delta \mathcal{U}_g = -\int \vec{F}\cdot d\vec{l}$$

$$d\vec{l} = \hat{e}_z dz \quad = -\int -mg(dz) = mgz$$

$$-\frac{d(mgz)}{dz} = F_g = -mg$$



$$\frac{1}{2}mV_0^2 = 0 = E_c^{(1)}$$

$$mg2R \quad , \quad \frac{k}{2}(2R)^2$$

$$\frac{1}{2}mv^2 = E_c^{(2)}$$

$$0 \quad , \quad 0$$

$$\Delta \mathcal{U}_r = -\int (-kr)\hat{e_r}\cdot(\hat{e}_r dr)$$

$$= \int kr\,dr = k\int r\,dr = \frac{1}{2}kr^2$$

$$\Delta E_c = -\Delta(\mathcal{U}_g + \mathcal{U}_r)$$

$$\cancel{E_c^{(1)}} + E_c^{(2)} = \cancel{\mathcal{U}^{(2)}} + \mathcal{U}^{(1)}$$

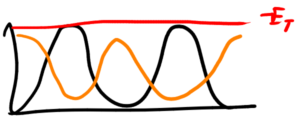
$$\frac{1}{2}\cancel{m}v^2(t) = \cancel{m}g2R + \frac{k}{2}\frac{(2R)^2}{m}$$

$$v = \sqrt{g4R + \frac{k}{m}4R^2}$$

- Identificar que hay fc. conservativas
- Calcular los potenciales
- T. ET

Fuerzas disipativas

si $\vec{F} = -\nabla\mathcal{U} \longrightarrow E_T = E_c + \mathcal{U} \longrightarrow \frac{d}{dt}(E_T) = 0$



$$\vec{F} = \underbrace{-\nabla\mathcal{U}}_{\text{conservativa}} + \underbrace{\vec{F}_{dis}}_{\text{no-conservativa}}$$

$$dw = -\vec{F}\cdot d\vec{r}$$

$$\Delta E_c = \int dw = -\int \overbrace{\nabla\mathcal{U}\cdot dr}^{\vec{F}} - \int \vec{F}_{dis}\cdot d\vec{r}$$

$$= -\Delta\mathcal{U} - \int \vec{F}_{dis}\cdot d\vec{r}$$

$$E_T = E_c^{(1)} + \mathcal{U}^{(1)} + \underset{=0}{Q^{(1)}} = E_c^{(2)} + \mathcal{U}^{(2)} - \underbrace{\int \vec{F}_{dis}\cdot d\vec{r}}_{Q^{(2)}}$$

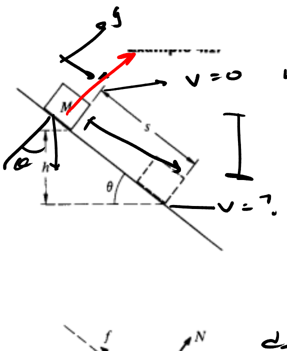
$$\frac{\Delta E_T}{\Delta_T} = \underbrace{\frac{\Delta(E + \mathcal{U})}{\Delta t} + \frac{\Delta Q}{\Delta t}}_{\Delta t \to 0} = 0$$

Energía perdida por unidad de tiempo

$$\boxed{\frac{d}{dt}(E_c + \mathcal{U}) = -\frac{dQ}{dt}}$$

$$\frac{J}{s} = P$$

$$\boxed{\Delta(E_c + \mathcal{U}) = -\Delta Q}$$

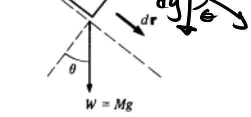


$v = 0 \quad mgh = \mathcal{U}$

$E_c + \mathcal{U} = mgh$

$v = ? \quad \mathcal{U} = 0$

$E_c + \mathcal{U} = \frac{1}{2}mv^2 \longrightarrow$ si no hay fricción $\boxed{v = \sqrt{2gh}}$



$$Q = \int \vec{F}_H\cdot d\vec{l} = \int_0^s dx(-mg\cos\theta)\mu = \int_h^0 \frac{dy}{\sin\theta}(-mg\cos\theta)\mu$$

$$Q = \int_h^0 dy\, mg\cot\theta\,\mu = hmg\cot\theta\,\mu$$

$$mgh = \frac{1}{2}mv^2 + hmg\cot\theta\,\mu \Rightarrow \underline{v = \sqrt{2gh(1-\mu\cot\theta)}}$$