Transformaciones canónicas

- $\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \longleftrightarrow \vec{\xi} = \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}$
- $\dot{\vec{\eta}} = \mathbb{J} \frac{\partial}{\partial \vec{\eta}} \mathcal{H} \longrightarrow \dot{\vec{\xi}} = \mathbb{J} \frac{\partial}{\partial \vec{\xi}} \mathcal{K}$
- $\{\vec{\eta}, \vec{\eta}\} = \{\vec{\xi}, \vec{\xi}\} = \mathbb{J}$

¿Cuándo son útiles?

1) $\mathcal{H} \longrightarrow \mathcal{K}$ de un problema conocido o ya resuelto

Casos particulares

2) $\vec{q} \longrightarrow \vec{Q}$ donde $\forall i$ Q_i es cíclica

Sistemas estacionarios

Si $\frac{\partial \mathcal{H}}{\partial t} = 0$, ent. $\vec{P} = \vec{\alpha} = cte$

$\Rightarrow \mathcal{K} = \mathcal{K}(\vec{\alpha})$ y $\vec{\omega} = \frac{\partial}{\partial \vec{\alpha}} \mathcal{K}$

$\Rightarrow \vec{Q} = \vec{\omega} t + \vec{\beta}$; $\vec{\alpha}, \vec{\beta}$ valores a determinar

3) $\vec{Q} = \vec{\beta} = ctes$, $\vec{P} = \vec{\alpha} = ctes$

$\Rightarrow \vec{q} = \vec{q}(\vec{\beta}, \vec{\alpha}, t)$, $\vec{p} = (\vec{\beta}, \vec{\alpha}, t)$

Escenario más general

¿Cómo encontrar a las transformaciones que cumplen 1), 2) o 3)?

— Hamilton-Jacobi

Para que se cumpla 3), podemos pedir que $\mathcal{K} = \mathcal{H} + \frac{\partial F}{\partial t} = 0$

$$\Rightarrow \dot{\vec{\xi}} = \mathbb{J} \frac{\partial}{\partial \vec{\xi}} \mathcal{K} \Rightarrow \dot{\vec{\xi}} = \begin{pmatrix} \dot{\vec{Q}} \\ \dot{\vec{P}} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} \vec{\beta} \\ \vec{\alpha} \end{pmatrix}$$

Supongamos que $F = F_2(\vec{q}, \vec{P}, t) \Longrightarrow \vec{p} = \frac{\partial}{\partial \vec{q}} F_2$ y $\vec{Q} = \frac{\partial F_2}{\partial \vec{P}}$

También válido para $F_1(\vec{q}, \vec{Q})$ → para $F_1 \rightarrow \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}}$

Con $\mathcal{K} = \mathcal{H}(\vec{q}, \vec{p}, t) + \frac{\partial F}{\partial t} = 0 \Longrightarrow \mathcal{H}\left(\vec{q}, \frac{\partial}{\partial \vec{q}} F_2, t\right) + \frac{\partial F_2}{\partial t} = 0$

Resolver para F_2

→ $3N - \ell + 1$ variables: $\vec{q}$'s y el tiempo

Ecuación diferencial de Hamilton-Jacobi

Ec. Diferencial Parcial no-lineal

$F_2 \equiv$ Función principal de Hamilton

↳ No lineal por $\mathcal{H} \sim \vec{p} \cdot \vec{p} = \left(\frac{\partial F_2}{\partial \vec{q}}\right) \cdot \left(\frac{\partial}{\partial \vec{q}} F_2\right)$

Notemos que debe haber $(s+1)$ integrales,

↳ Pero de la ec. diferencial, resolvemos para $\frac{\partial}{\partial \vec{q}} F_2$ y $\frac{\partial}{\partial t} F_2 \Rightarrow$ Solución es $F = F_2(\vec{q}, t; \vec{\alpha}) + \alpha_{3N-\ell+1}$

$\vec{P}$

↳ Libertad que nos damos

No importa en la formulación

Notemos que $\vec{P} = \vec{\alpha} = ctes$, entonces

condición de diferenciabilidad

Einstein

Acción

$$\frac{dF_2}{dt} = \frac{\partial F_2}{\partial q_i} \frac{\partial q_i}{\partial t} + \underbrace{\frac{\partial F_2}{\partial t}}_{\text{Ec. de Hamilton}} = \frac{\partial F_2}{\partial q_i} \dot{q}_i - \mathcal{H} = p_i \dot{q}_i - \mathcal{H} \Longrightarrow F_2 = \int (p_i \dot{q}_i - \mathcal{H})\, dt = S(t) - S(t_0)$$

Integrando

La acción $S=F_2$ es nuestra función principal de Hamilton

$$\underbrace{\mathcal{H}\left(\vec{q}, \frac{\partial}{\partial \vec{q}} S, t\right) + \frac{\partial S}{\partial t} = 0}_{HJ} \qquad S = S(\vec{q}, t; \vec{\alpha}) \quad \vec{P} = \vec{\alpha}$$

Dicho esto: ¿Cómo encontramos la transformación canónica?

1) Escribimos $\mathcal{H} = \mathcal{H}\left(\vec{q}, \vec{p} = \frac{\partial S}{\partial \vec{q}}, t\right)$

2) Resolvemos la ec. de HJ para $S = S(\vec{q}, t; \vec{\alpha})$, con $\vec{\alpha} = \vec{P}$

3) Calculamos $\vec{Q} = \frac{\partial S(\vec{q}, t; \vec{\alpha})}{\partial \vec{\alpha}} = \vec{Q}(\vec{q}, t; \vec{\alpha}) = \vec{\beta}$ * $\longrightarrow$ si F_1, entonces $\vec{P} = \vec{\alpha} = \frac{\partial S}{\partial \vec{Q}} = \frac{\partial S}{\partial \vec{\beta}}$

y resolvemos $\vec{q} = \vec{q}(\vec{\beta}, \vec{\alpha}, t) \longrightarrow$ 3N-ℓ ecuaciones .... (3.1)

4) Calculamos $\vec{p} = \frac{\partial}{\partial \vec{q}} S(\vec{q}, t; \vec{\alpha}) = \vec{p}(\vec{q}, t; \vec{\alpha})$

y sustituyamos en (3.1) $\vec{p} = \vec{p}(\vec{\beta}, \vec{\alpha}, t)$ ... (3.2)

4) Consideremos las condiciones iniciales $\vec{q}_0 = \vec{q}(t=t_0)$, $\vec{p} = \vec{p}(t=t_0)$

$\Rightarrow \underbrace{\vec{\alpha} = \vec{\alpha}(\vec{q}_0, \vec{p}_0, t_0)}_{(3.2)}, \quad \underbrace{\vec{\beta} = \vec{\beta}(\vec{q}_0, \vec{p}_0, t_0)}_{(3.1)}$

5) Sustituyendo todo, tenemos el problema resuelto

Ejemplo: Oscilador armónico (1D) $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$, $p = \frac{\partial S}{\partial x}$

1) $\mathcal{H} = \frac{1}{2m}\Big(\underbrace{\frac{\partial S}{\partial x}}_{p}\Big)^2 + \frac{1}{2} m \omega^2 x^2$

2) $\mathcal{H}\left(x, \frac{\partial S}{\partial x}, t\right) + \frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2} m \omega^2 x^2 + \frac{\partial S}{\partial t} = 0$ .... (2.1)

Como queremos $p = \alpha \equiv cte$, queremos que

$S = S(x, t; p=\alpha) = S(x, t; \alpha)$

No depende de t — No depende de x

Proponemos que $S(x, t; \alpha) = \underbrace{W(x; \alpha)}_{\text{Función característica de Jacobi}} + V(t; \alpha)$

$\Rightarrow$ (2.1) se reescribe como

$$\underbrace{\underbrace{\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2} m \omega^2 x^2}_{=\alpha}}_{\text{Depende de } x} + \underbrace{\underbrace{\frac{\partial V}{\partial t}}_{-\alpha}}_{\text{Depende de } t} = 0$$

$$\begin{cases} \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2} m \omega^2 x^2 = \overset{cte}{\alpha} \\ \frac{\partial V}{\partial t} = -\underbrace{\alpha}_{cte} \end{cases}$$

$$\frac{\partial V}{\partial t} = -\alpha \longrightarrow V = -\alpha \Delta t + V(t_0) = -\alpha \Delta t + V_0$$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x^2 = \alpha \longrightarrow \left(\frac{\partial W}{\partial x}\right)^2 = 2m\alpha - m^2\omega^2 x^2 = m^2\omega^2\left(\frac{2\alpha}{m\omega^2} - x^2\right)$$

$$\Rightarrow W = \int dx\, m\omega \left(\frac{2\alpha}{m\omega^2} - x^2\right)^{1/2} = W(x;\alpha)$$

$$\Rightarrow S(x,t;\alpha) = W(x;\alpha) + V(t;\alpha)$$

$$= \int dx\, m\omega \sqrt{\underbrace{\frac{2\alpha}{m\omega^2}}_{a^2} - x^2} - \alpha \Delta t + V_0$$

podemos hacer $V_0 = 0$ sin problemas

$$= \int dx\, m\omega \sqrt{\frac{2\alpha}{m\omega^2}} \sqrt{1 - \left(x\sqrt{\frac{m\omega}{2\alpha}}\right)^2} - E\Delta t + V_0$$

$$= m\omega\left(\frac{2\alpha}{m\omega^2}\right) \cdot \int d\underbrace{\left(x\sqrt{\frac{m\omega}{2\alpha}}\right)}_{\sin\theta} \overbrace{\sqrt{1-\underbrace{\left(x\sqrt{\frac{m\omega}{2\alpha}}\right)}_{\sin\theta}^2}}^{\cos\theta} - E\Delta t + V_0$$

$$= \frac{m\omega}{2}\left(\frac{2\alpha}{m\omega^2}\right)\left[x\sqrt{\frac{m\omega^2}{2\alpha}}\sqrt{1-\left(x\sqrt{\frac{m\omega}{2\alpha}}\right)^2} + \arcsin\left(x\sqrt{\frac{m\omega}{2\alpha}}\right)\right] - \Delta t\alpha + V_0$$

$$= \frac{m\omega}{2}\left(x\sqrt{\frac{2\alpha}{\omega^2} - x^2} + \frac{2\alpha}{m\omega^2}\arcsin\left(x\sqrt{\frac{m\omega^2}{2\alpha}}\right)\right) - \Delta t\alpha + V_0$$

Recordatorio

$$\int dy \sqrt{1-y^2} \qquad y = \sin\theta, \quad dy = \cos\theta\, d\theta$$

$$\int d\theta \cos\theta \sqrt{1-\sin^2\theta} = \int d\theta \cos^2\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\Rightarrow \int d\theta \cos^2\theta = \frac{1}{2}\int d\theta(\cos(2\theta) + 1) = \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) = \frac{1}{2}(\sin\theta\cos\theta + \theta)$$

e identificamos $\boxed{P = \alpha}$

$$\vec{Q} = \frac{\partial S(\vec{q}, t; \vec{\alpha})}{\partial \vec{\alpha}} = \vec{Q}(\vec{q}, t; \vec{\alpha}) = \vec{\beta}\ \star \longrightarrow \text{si } F_1, \text{ entonces } \vec{P} = \vec{\alpha} = \frac{\partial S}{\partial \vec{Q}} = \frac{\partial S}{\partial \vec{\beta}}$$

3) $$Q = \frac{\partial S}{\partial \alpha} = \beta = cte$$

Tenemos $$S = \int dx\, m\omega \sqrt{\frac{2\alpha}{m\omega^2} - x^2} - \alpha\Delta t + V_0 \Rightarrow \frac{\partial S}{\partial \alpha} = \int dx\, \cancel{m\omega} \frac{1}{\cancel{2}} \frac{\frac{\cancel{2}}{m\omega^2}}{\sqrt{\frac{2\alpha}{m\omega^2} - x^2}} - \Delta t$$

$$= \int \frac{dx}{\omega} \frac{1}{\sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{1-\left(x\sqrt{\frac{m\omega^2}{2\alpha}}\right)^2}} - \Delta t$$

$$\Rightarrow \frac{\partial S}{\partial \alpha} = \frac{1}{\omega}\int \frac{d\overbrace{\left(x\sqrt{\frac{m\omega^2}{2\alpha}}\right)}^{\sin\theta}}{\sqrt{1-\left(x\sqrt{\frac{m\omega^2}{2\alpha}}\right)^2}} - \Delta t$$

$y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{d\theta\cos\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta \frac{\cos\theta}{\cos\theta} = \int d\theta = \theta$$

$$= \frac{1}{\omega}\arcsin\left(x\sqrt{\frac{m\omega^2}{2\alpha}}\right) - \Delta t = \beta \Rightarrow \beta + \Delta t = \frac{\arcsin\left(x\omega\sqrt{m/2\alpha}\right)}{\omega}$$

$$\Rightarrow x\omega\sqrt{\frac{m}{2\alpha}} = \sin[\omega(\beta + \Delta t)]$$

$$\Rightarrow \underline{x = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}}\sin[\omega(\Delta t + \beta)]} = x(t;\beta,\alpha)$$

4) $p = \frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - x^2} = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - \underbrace{\frac{1}{\omega^2}\frac{2\alpha}{m}\sin^2[\omega(\Delta t+\beta)]}_{x^2}}$

$\Rightarrow p = m\cancel{\omega}\sqrt{\frac{2\alpha}{m\cancel{\omega^2}}}\sqrt{1-\sin^2[\omega(\Delta t+\beta)]} = \sqrt{2\alpha m}\cos[\omega(\Delta t+\beta)] = p(t;\beta,\alpha)$

5) Condiciones iniciales

$t_0 = 0\ ,\ P(t=t_0)=0\ ,\ x(t_0)=x_0 \Rightarrow P\Big|_{t=t_0} = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - x^2} = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - x_0^2} = 0$

Entonces $\alpha = \frac{1}{2}m\omega^2 x_0^2$

Notemos que β [=] tiempo
α [=] Energía

además por las condiciones iniciales y por tener un sistema conservativo

$\alpha = E \equiv$ Energía total del sistema

como $\beta + \Delta t = \frac{1}{\omega}\arcsin\left(x\omega\sqrt{\frac{m}{2\alpha}}\right) \longrightarrow \beta + t - t_0 = \frac{1}{\omega}\arcsin\left(x\omega\sqrt{\frac{m}{2\frac{1}{2}m\omega^2}}\right)$

$= \frac{1}{\omega}\arcsin\left(\frac{x}{x_0}\right)$

aplicando condiciones iniciales $t=t_0=0$ $\Rightarrow \beta = \frac{1}{\omega}\arcsin(1) = \frac{\pi}{2\omega}$

$\Rightarrow \begin{pmatrix} x \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} \beta = t_a \\ \alpha = E \end{pmatrix}$ → ¡Nuestra transformación canónica, generada por S, es ésta!

Recordemos que

$x = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}}\sin[\omega(t+\beta)] = \sqrt{\frac{2E}{m\omega^2}}\sin\left(\omega t + \frac{\pi}{2}\right) = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t)$ ($P = E$, $\beta = Q$)

$P = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - x^2} = \sqrt{2Em - m^2\omega^2\frac{2E}{m\omega^2}\cos^2\omega t} = \sqrt{2mE}\,|\sin\omega t| = -\sqrt{2mE}\sin\omega t$

¡que sí son las soluciones del oscilador armónico!

Del punto 3) obtuvimos que $x = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}}\sin[\omega(\Delta t+\beta)]$

4) $p = \sqrt{2\alpha m}\cos[\omega(\Delta t+\beta)]$

Todo esto viendo que si la trasformación era $F_2 = S(\vec{q},\vec{P})$. Pero vimos que podíamos tomar también los mismos resultados con $F_1 = F_1(\vec{q},\vec{Q})$. Recordando que en 3) $\longrightarrow$ $\vec{P} = \vec{\alpha} = -\frac{\partial F_1}{\partial \vec{Q}} = -\frac{\partial F_1}{\partial \vec{\beta}}$

Entonces:

$$3) \Rightarrow x = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}}\sin[\omega(\Delta t+\beta)] \Rightarrow \sqrt{\alpha} = \frac{x\omega}{\sin[\omega(\Delta t+\beta)]}\sqrt{\frac{m}{2}}$$

Sustituyendo $\sqrt{\alpha}$ en

$$4) \quad p = \sqrt{2\alpha m}\cos[\omega(\Delta t+\beta)] = \omega m x\frac{\cos[\omega(\Delta t+\beta)]}{\sin[\omega(\Delta t+\beta)]} = m\omega x\cot[\omega(\Delta t+\beta)]$$

y como $p = \frac{\partial F_1}{\partial q} \Rightarrow \frac{\partial F_1}{\partial x} = m\omega x\cot[\omega(\Delta t+\beta)]$

$$\Rightarrow \int\frac{\partial F_1}{\partial x}dx = F_1(\underset{q}{x},\underset{Q}{\beta}) + f_1(\beta,t) = \frac{1}{2}x^2 m\omega\cot[\omega(\Delta t+\beta)] + f_1(\beta,t)$$

$\cot' x = -\csc^2 x = \frac{-1}{\sin^2 x}$

$$\text{Con } \alpha = -\frac{\partial F_1}{\partial \beta} = -\frac{\partial f_1}{\partial \beta} - \left[\frac{1}{2}x^2 m\omega\frac{-1}{\sin^2[\omega(\Delta t+\beta)]}(\omega)\right]$$

$$= -\frac{\partial f_1}{\partial \beta} + \left[x\omega\sqrt{\frac{m}{2}}\frac{1}{\sin(\,)}\right]^2 = -\frac{\partial f_1}{\partial \beta} + (\sqrt{\alpha})^2$$

$$\Rightarrow \frac{\partial f_1}{\partial \beta} = 0 \Rightarrow f_1 = f_1(t)$$

¿Cómo la cumplimos?

$$\Rightarrow F_1(\underset{q}{x},\underset{Q}{\beta}) = \frac{m\omega x^2}{2}\cot[\omega(\Delta t+\beta)] + f_1(t)$$

$F_1 \rightarrow$ solución de la ec. de HJ

Sustituyendo en la ec. de Hamilton-Jacobi

$\mathcal{H}(\vec{q}, \frac{\partial F_1}{\partial q}, t) + \frac{\partial F_1}{\partial t} = 0$

$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega x^2$ ↳ y escribirlo en términos de $q = x$, $Q = \beta$ y t.

$$\mathcal{H} = \frac{1}{2m}\Big(\underbrace{\frac{\partial F_1}{\partial x}}_{p}\Big)^2 + \frac{1}{2}m\omega x^2 = \frac{1}{2m}\left[m\omega x\cot[\omega(\Delta t+\beta)]\right]^2 + \frac{1}{2}m\omega^2 x^2$$

$$= \frac{1}{2}m\omega^2 x^2\left[\cot^2[\omega(\Delta t+\beta)] + 1\right]$$

$$= \frac{m\omega^2 x^2}{2}\frac{1}{\sin^2[\omega(\Delta t+\beta)]}$$

$\cot^2 x + 1 = \csc^2 x = \frac{1}{\sin^2 x}$

$$\frac{\partial F_1}{\partial t} = \frac{m\omega x^2}{2}\frac{-1\,(\omega)}{\sin^2[\omega(\Delta t+\beta)]^2} + \frac{\partial f_1}{\partial t} = -\mathcal{H} + \frac{f_1}{\partial t}$$

¿Por qué no consideramos $\frac{\partial x}{\partial t}$?

$$\Rightarrow 0 = \mathcal{H} - \frac{\partial F_1}{\partial t} = \mathcal{H} - \mathcal{H} + \frac{f_1}{\partial t} \Rightarrow \frac{\partial f_1}{\partial t} = \frac{\partial f_1}{\partial \beta} = 0 \Rightarrow f_1 = cte$$

$$\text{Sup } f_1 = cte = 0 \qquad F_1 = \frac{m\omega^2 x^2}{2}\cot[\omega(\Delta t+\beta)] = \frac{m\omega^2}{2}q^2\cot[\omega Q]$$

si $\Delta t = 0$
$q = x$
$Q = \beta$.