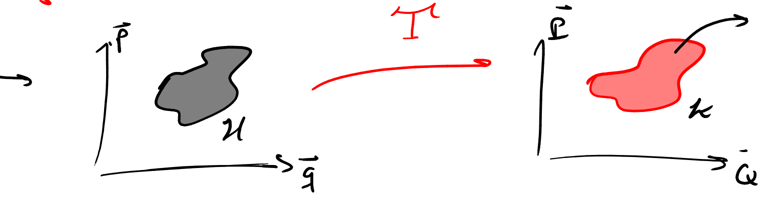
Transformaciones canónicas



$Z(3N-\ell)$

Conservaba las Ecs. de Hamilton invariantes

$\rightarrow \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$

$\rightarrow \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{K}}{\partial t} = \frac{d\mathcal{H}}{dt}$

Función generadora

$$\sum p_i \dot{q}_i - \mathcal{H} = \sum P_i \dot{Q}_i - \mathcal{K} + \frac{dF}{dt}$$

$F = F(\vec{q}, \vec{p}, \vec{Q}, \vec{P}, t)$

$\hookrightarrow$ 4 T. básicas

$$\boxed{-\mathcal{H} = -\mathcal{K} + \frac{\partial F}{\partial t}}$$

$F_1 = F_1(q, \vec{Q}, t)$

$\hookrightarrow p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$

$F_2 = F_2(\vec{q}, \vec{P}, t)$

$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$

Caso "trivial"

$F_2 = \sum_i q_i P_i \rightarrow F_2 = \mathbb{1}$

Transformaciones directas $\rightarrow$ No depende del tiempo

Conversión de Energía

$\vec{Q} = \vec{Q}(\vec{q}, \vec{p})$
$\vec{P} = \vec{P}(\vec{q}, \vec{p})$

$$\rightarrow \dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j}$$

$$\mathcal{H}(\vec{q}, \vec{p}) \rightarrow \mathcal{K} = \mathcal{H}(\vec{Q}, \vec{P}) \rightarrow \frac{\partial \mathcal{H}}{\partial P_i} = \frac{\partial \mathcal{H}}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial P_i} = \dot{Q}_i$$

$\rightarrow$ Imponemos

$$\boxed{\frac{\partial p_j}{\partial P_i} = \frac{\partial Q_i}{\partial q_j}} \quad \boxed{-\frac{\partial Q_i}{\partial p_j} = \frac{\partial q_j}{\partial P_i}}$$

Transformaciones canónicas directas (No hay dependencia expl. en t)

$$\dot{P} = \frac{\partial P_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} = -\frac{\partial \mathcal{H}}{\partial Q_i} = -\frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial Q_i} - \frac{\partial \mathcal{H}}{\partial p_j}\frac{\partial p_j}{\partial Q_i}$$

$$\rightarrow \boxed{\frac{\partial P_i}{\partial q_j} = -\frac{\partial q_j}{\partial Q_i}} \quad \boxed{\frac{\partial P_i}{\partial p_j} = \frac{\partial p_j}{\partial Q_i}}$$

Igualando bases

$\dot{q}, \dot{p}$ son l.i.

Notación simpléctica

$\rightarrow$ Goldstein

$\rightarrow \vec{\eta} \in \mathbb{R}^{2n}, \quad n = 3N-\ell$

$$\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \begin{matrix}]\, n \text{ elementos} \\ {}_{n+1} \\ {}_{2n} \end{matrix} \qquad \eta_i = q_i, \quad \eta_{n+i} = p_i$$

$$\left(\frac{\partial \mathcal{H}}{\partial \vec{\eta}}\right)_i = \frac{\partial \mathcal{H}}{\partial q_i} = \nabla_{\vec{\eta}} \mathcal{H}, \quad \left(\frac{\partial \mathcal{H}}{\partial \vec{\eta}}\right)_{i+n} = -\frac{\partial \mathcal{H}}{\partial p_i}$$

$$\mathbb{J} \in \mathcal{M}^{(\mathbb{R})}_{2n \times 2n} \rightarrow \mathbb{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in \mathcal{M}^{(\mathbb{R})}_{n \times n}$$

$$\boxed{\dot{\vec{\eta}} = \mathbb{J}\frac{\partial \mathcal{H}}{\partial \vec{\eta}} = \mathbb{J}\,\nabla_{\vec{\eta}} \mathcal{H}}$$

Simpléctica

Ecuaciones Hamilton

$$\mathbb{J}^{-1} = \mathbb{J}^{T} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{p}_1 \\ -\dot{p}_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix},$$

$$\mathbb{J}^2 = \mathbb{J}\mathbb{J} = -\mathbb{1} \rightarrow \det \mathbb{J} = +1$$

¿Cómo se ven las T. C en esta notación?

$$\underline{\vec{\xi} = \vec{\xi}(\vec{\eta})} \rightarrow \dot{\xi}_i = \underbrace{\frac{\partial \xi_i}{\partial \eta_j}}_{M_{ij}} \dot{\eta}_j \rightarrow \dot{\vec{\xi}} = \mathbb{M}\,\dot{\vec{\eta}}$$

Trasformación $\mathcal{T} \rightarrow \underline{\mathbb{M}}$

$$\dot{\vec{\xi}} = \mathbb{M}\dot{\vec{\eta}} = \mathbb{M}\,\mathbb{J}\,\frac{\partial \mathcal{H}}{\partial \vec{\eta}} = \mathbb{M}\,\mathbb{J}\,\mathbb{M}^T \frac{\partial \mathcal{H}}{\partial \vec{\xi}} \longrightarrow \dot{\vec{\xi}} = \underbrace{\mathbb{M}\,\mathbb{J}\,\mathbb{M}^T}_{\mathbb{J}} \frac{\partial \mathcal{H}}{\partial \vec{\xi}}$$

$$\frac{\partial \mathcal{H}}{\partial \eta_i} = \frac{\partial \mathcal{H}}{\partial \xi_j}\frac{\partial \xi_j}{\partial \eta_i} \quad \mathbb{M}^T$$

T. canónica

$$\boxed{\dot{\vec{\xi}} = \mathbb{J}\,\nabla_{\vec{\xi}}\mathcal{H}}$$

Condición simpléctica

$$\underline{\mathbb{M}\mathbb{J}\mathbb{M}^T = \mathbb{J}} \longrightarrow \mathbb{M}\mathbb{J} = \mathbb{J}\,(\mathbb{M}^T)^{-1}$$
$$\longrightarrow \mathbb{J}\mathbb{M} = (\mathbb{M}^T)^{-1}\mathbb{J}$$
$$\longrightarrow \mathbb{M}^T\mathbb{J}\mathbb{M} = \mathbb{J}$$

$\mathbb{M} \rightarrow$ Matriz simpléctica

$\hookrightarrow \mathcal{T}$ (trasformación canónica)

$\mathbb{M} = \mathbb{M}(F) \longrightarrow$ No es método general !

Trasformaciones canónicas infinitesimales

conv. einstein

$$F_2(\vec{q},\vec{P}) = \underbrace{q_k P_k}_{\mathbb{1}} + \underbrace{\epsilon}_{\epsilon \ll 1}\, G(\vec{q},\vec{P})$$

F_2: Función generadora de las T.C.

G: F. generadora de las T.C.I

Sea despreciable ϵ^2

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon\frac{\partial G}{\partial q_j} \Rightarrow \delta p_j = P_j - p_j = -\epsilon\,\partial G/\partial q_j$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon\frac{\partial G}{\partial P_j} \Rightarrow \delta q_j = Q_j - q_j = \epsilon\,\partial G/\partial p_j$$

$$\rightarrow \left.\begin{array}{l} \vec{Q} = \vec{q} + \delta\vec{q} \\ \vec{P} = \vec{p} + \delta\vec{p} \end{array}\right\} \quad \vec{\xi} = \vec{\eta} + \underbrace{\delta\vec{\eta}}\, \rightarrow \delta\vec{\eta} = \epsilon\,\mathbb{J}\,\frac{\partial G}{\partial \vec{\eta}}$$

Propuest.

$$\vec{\xi}(t_0) \rightarrow \vec{\xi}(t_0 + dt) \Rightarrow \mathcal{T} \rightarrow \mathbb{M}$$ ¿cómo se ve?

$$\underline{\mathbb{M} = \frac{\partial \vec{\xi}}{\partial \vec{\eta}}} = \frac{\partial}{\partial \vec{\eta}}\left(\vec{\eta} + \delta\vec{\eta}\right) = \mathbb{1} + \frac{\partial}{\partial \vec{\eta}}(\delta\vec{\eta}) = \mathbb{1} + \frac{\partial}{\partial \eta}\left(\epsilon\,\mathbb{J}\frac{\partial G}{\partial \vec{\eta}}\right)$$

$$\Rightarrow \mathbb{M} = \mathbb{1} + \epsilon\,\mathbb{J}\,\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}$$

$$\boxed{\underbrace{\nabla_{\vec{\eta}}\nabla_{\vec{\eta}}}\, G = \frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \eta_j} G}$$

$$\Rightarrow \mathbb{M}^T = \mathbb{1} - \epsilon\,\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}\,\mathbb{J}$$

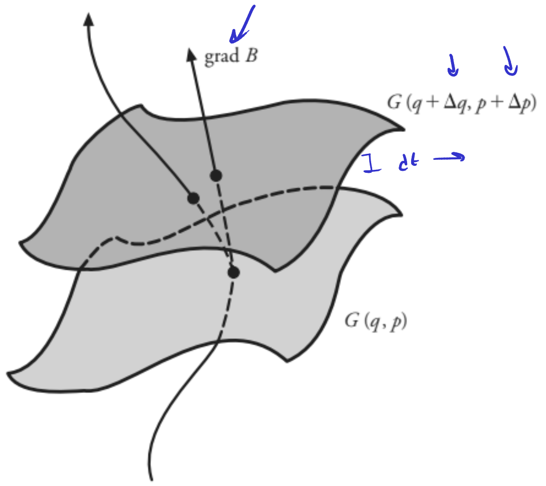
$$\begin{pmatrix} 0 & -\mathbb{1} \\ +\mathbb{1} & 0 \end{pmatrix} = -\mathbb{J} = \mathbb{J}^T$$

$$\mathbb{M}\mathbb{J}\mathbb{M}^T = \underbrace{\left(\mathbb{1} + \epsilon\,\mathbb{J}\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}\right)\mathbb{J}}\left(\mathbb{1} - \epsilon\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}\,\mathbb{J}\right)$$

$$= \left(\mathbb{J} + \epsilon\,\mathbb{J}\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}\,\mathbb{J}\right)\left(\mathbb{1} - \epsilon\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}\,\mathbb{J}\right)$$

$$= \mathbb{J} + \epsilon\,\mathbb{J}\cancel{\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}}\,\mathbb{J} - \epsilon\,\mathbb{J}\cancel{\frac{\partial^2 G}{\partial \vec{\eta}\,\partial \vec{\eta}}}\,\mathbb{J} + O(\epsilon^2)$$

$$\boxed{\Rightarrow \mathbb{M}\,\mathbb{J}\,\mathbb{M}^T = \mathbb{J}}$$


grad B
$G\,(q+\Delta q, p+\Delta p)$
I dt
$G\,(q, p)$