Ecuaciones de Hamilton y la Acción

Cambiemos la definición de la acción

$$S=\int_{t_1}^{t_2} L\, dt \longrightarrow S=\int_{t_1}^{t_2}\Big[\underbrace{T(\{q_i\},\{\dot q_i\},t)}_{\text{Energía cinética}}+\underbrace{W(\{q_i\},\{\dot q_i\},t)}_{\text{Trabajo realizado por las fuerzas aplicadas}}\Big]dt$$

Separemos W en fuerzas conservativas y disipativas $\Rightarrow$ $W=\underbrace{-(V-V_0)}_{\text{Diferencia de potencial}}+W^d$

$$\Rightarrow S=\int_{t_1}^{t_2}(\underbrace{T-V}_{L}+V_0+W^d)\,dt=\int_{t_1}^{t_2}(L+W^d)\,dt+V_0(t_2-t_1)$$

Del principio de Hamilton, hacemos una variación $\delta S=0$, donde $\delta f=\left(\frac{\partial f}{\partial \varepsilon}\right)_{\varepsilon=0} df$ sin una variación temporal

$$\Rightarrow \delta S=\int_{t_1}^{t_2}(\delta L+\delta W^d)\,dt+V_0\,\cancelto{0}{\delta(t_2-t_1)}$$

definiendo $\delta W^d=\sum_{n=1}^{3N-\ell} Q_n\,\delta q_n$

$$\delta L=\sum_{i=1}^{3N-\ell}\left(\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot q_i}\underbrace{\delta\left(\frac{dq_i}{dt}\right)}_{=\frac{d}{dt}(\delta q_i)}\right)+\cancelto{0}{\frac{\partial L}{\partial t}\delta t}$$

$$\Rightarrow \delta S=\int_{t_1}^{t_2}dt\sum_{n=1}^{3N-\ell}\left(\frac{\partial L}{\partial q_n}+Q_n\right)\delta q_n+\sum_{n=1}^{3N-\ell}\int_{t_1}^{t_2}dt\,\frac{\partial L}{\partial \dot q_n}\frac{d}{dt}(\delta q_n)$$

$$=\int_{t_1}^{t_2}\frac{d}{dt}\left(\delta q_n\frac{\partial L}{\partial \dot q_n}\right)-\int_{t_1}^{t_2}dt\,\delta q_n\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_n}\right)=\cancelto{0}{\delta q_n\frac{\partial L}{\partial \dot q_n}\Big|_{t_1}^{t_2}}-\int_{t_1}^{t_2}dt\,\delta q_n\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_n}\right)$$

$$\Rightarrow \delta S=\int_{t_1}^{t_2}dt\sum_{n=1}^{3N-\ell}\Big(\underbrace{\frac{\partial L}{\partial q_n}+Q_n-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_n}\right)}_{=0}\Big)\delta q_n=0$$

Sup. un conjunto l.i.

$$\therefore\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_n}\right)-\frac{\partial L}{\partial q_n}=Q_n$$

$\rightarrow$ Ec. de Euler Lagrange.

¿Y las ecuaciones de Hamilton?

Recordemos que $H=\sum \dot q_i p_i - L \Rightarrow L=\sum_i \dot q_i p_i - H(\{q_i\},\{p_i\},t)$

$$\Rightarrow S=\int_{t_1}^{t_2}\left(\sum_i p_i\dot q_i - H + W^d\right)dt$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left[\sum_i \left(p_i\, \delta\left(\frac{dq_i}{dt}\right) + \dot q_i\, \delta p_i \right) - \delta \mathcal{H} + \delta W^d \right] dt$$

$$= \int_{t_1}^{t_2} dt \left[\sum_i \underbrace{p_i \frac{d}{dt}(\delta q_i)} + \dot q_i\, \delta p_i - \sum_i \left(\frac{\partial \mathcal H}{\partial q_i}\delta q_i + \frac{\partial \mathcal H}{\partial p_i}\delta p_i + \frac{\partial \mathcal H}{\partial t}\cancel{\delta t}^{\,0} \right) + \sum_i Q_i\, \delta q_i \right]$$

$$\int_{t_1}^{t_2} p_i \frac{d}{dt}(\delta q_i) = \int_{t_1}^{t_2} \frac{d}{dt}(p_i\, \delta q_i) - \int_{t_1}^{t_2} \delta q_i \frac{dp_i}{dt}\, dt$$

$$= \cancel{(p_i\, \delta q_i)\Big|_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} \delta q_i\, \dot p_i\, dt$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} dt \sum_i \left[\left(-\dot p_i - \frac{\partial\mathcal H}{\partial q_i} + Q_i\right)\delta q_i + \left(\dot q_i - \frac{\partial \mathcal H}{\partial p_i}\right)\delta p_i \right] = 0$$

Notamos que pedimos que $\delta q_i = \delta p_i = 0$ en $t_{1/2}$.

$$\Rightarrow \quad \dot q_i = \frac{\partial \mathcal H}{\partial p_i}, \qquad -\dot p_i = \frac{\partial \mathcal H}{\partial q_i} - Q_i \qquad \text{Ecuaciones de Hamilton} \quad //$$

¿Cómo podemos hacer más débil esta constricción en los tiempos?

definamos

$$q_i(t;\epsilon) = q_i(t;0) + \underset{\text{parámetro de error}}{\epsilon}\, \delta q_i$$

$$\Rightarrow \delta q_i = \frac{\partial q_i}{\partial \epsilon}\Big|_{\epsilon=0} dq. \longrightarrow \text{Variación } \delta$$

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L\, dt \longrightarrow \Delta S = \Delta \int_{t_1}^{t_2} L\, dt = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \underbrace{L(t;\epsilon)}_{\text{Variaciones del camino}} dt - \int_{t_1}^{t_2} \underbrace{L(t;0)}_{\text{Camino correcto}} dt$$

$$\Rightarrow \Delta S = \int_{t_1+\Delta t_1}^{t_1} L(t;\epsilon)\, dt + \int_{t_2}^{t_2+\Delta t_2} L(t;\epsilon)\, dt + \int_{t_1}^{t_2} \left[L(t;\epsilon) - L(t;0)\right] dt$$

$\Delta t \ll 1$

$$\Delta S = \Delta \int_{t_1}^{t_2} L\, dt = L(t_2)\,\Delta t_2 - L(t_1)\,\Delta t_1 + \int_{t_1}^{t_2} \delta L\, dt$$

$$\Rightarrow \Delta S = L(t_2)\,\Delta t_2 - L(t_1)\,\Delta t_1 + \sum_n \int_{t_1}^{t_2} \cancel{\left[\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_n}\right]}^{\,0\ (\text{E-L.})} dt + \overbrace{\frac{\partial L}{\partial \dot q_n}}^{p_n} \delta q_n \Big|_{t_1}^{t_2}$$

ahora no son necesariamente cero

$$\Rightarrow \Delta S = \left(L(t)\,\Delta t - \sum_n p_n\, \delta q_n\right)\Big|_{t_1}^{t_2}$$

¿Cómo podemos escribir Δq_i en lugar de δq_i?
↳ consideremos la dependencia temporal

$$\delta q_i = \left(\frac{\partial q_i}{\partial \epsilon}\right)\bigg|_{\epsilon=0} d\epsilon \longrightarrow \Delta q_i = \left(\frac{d q_i}{d\epsilon}\right)\bigg|_{\epsilon=0} d\epsilon = \left(\frac{\partial q_i}{\partial \epsilon} + \dot{q}_e \frac{dt_\epsilon}{d\epsilon}\right)\bigg|_{\epsilon=0} d\epsilon$$

$$\Rightarrow \Delta q_i = \delta q_i + \dot{q}_i \Delta t \,, \quad \Delta t = d\epsilon \frac{dt}{d\epsilon}\bigg|_{\epsilon=0}$$

* Notemos que $\delta\left(\frac{df}{dt}\right) = \frac{d}{dt}(\delta f)$ pero $\Delta\left(\frac{df}{dt}\right) \neq \frac{d}{dt}(\Delta f)$

$$\Rightarrow \Delta S = \left(L\Delta t - \sum_i p_i \delta q_i\right)\bigg|_{t_1}^{t_2} = \left[L\Delta t + \sum_i p_i\left(\Delta q_i - \dot{q}_i \Delta t\right)\right]\bigg|_{t_1}^{t_2}$$

$$= \left[\Delta t\left(\underbrace{L - \sum_i p_i \dot{q}_i}_{-\mathcal{H}}\right) + \sum_i p_i \Delta q_i\right]\bigg|_{t_1}^{t_2}$$

$$= \left[\sum_i p_i \Delta q_i - \mathcal{H}\Delta t\right]\bigg|_{t_1}^{t_2}$$

Teorema de mínima acción

Sup. a) Sistema conservativo $\Longrightarrow \frac{\partial L}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} = \frac{d\mathcal{H}}{dt} = 0 \Rightarrow \mathcal{H} = T + V = E$

b) $\mathcal{H}$ se conserva en su valor real y sus variaciones $\Rightarrow \Delta\mathcal{H} = 0 \Rightarrow -\Delta V = \Delta T$

c) $\Delta q_i = 0$ en los extremos (pero no Δp_i ni Δt)

Como a) $\frac{d\mathcal{H}}{dt} = 0 \Rightarrow \mathcal{H} = \text{cte}$ y con c) $\Delta q\big|_{t_1} = \Delta q\big|_{t_2} = 0 \Rightarrow \Delta S = -\mathcal{H}(\Delta t_1 - \Delta t_2)$

e igualmente $\Delta S = \Delta\int_{t_1}^{t_2} L\,dt = \Delta\left(\int_{t_1}^{t_2}\sum_i p_i \dot{q}_i\,dt - \int_{t_1}^{t_2}\mathcal{H}\,dt\right) = \Delta\int_{t_1}^{t_2}\sum_i \dot{q}_i p_i\,dt - \Delta\int_{t_1}^{t_2}\mathcal{H}\,dt =$

$$= \Delta\int_{t_1}^{t_2}\sum \dot{q}_i p_i\,dt - \Delta\left(\mathcal{H}(t_2 - t_1)\right)$$

$$= \Delta\int_{t_1}^{t_2}\sum \dot{q}_i p_i\,dt - \mathcal{H}(\Delta t_2 - \Delta t_1)$$

$$\Rightarrow \cancel{\Delta S} = \overbrace{\Delta\int_{t_1}^{t_2}\sum_i \dot{q}_i p_i\,dt}^{A} + \cancel{\Delta S} \Rightarrow \Delta A = 0$$

En realidad S es la primera función principal de Hamilton, no la acción.
↳ Fregmann.

↓ Nolting lo llama el funcional de la acción.