# Formalismo de Hamilton

Describimos la dinámica del sistema a partir de el lagrangiano $\mathcal{L}$, una función de estado y las ecuaciones de Euler-Lagrange

$$\mathcal{L} = \mathcal{L}(\underbrace{\{q_i\}, \{\dot{q}_i\}}_{\text{Coordenadas y velocidad generalizada}}, t) \quad \text{t.q.} \quad \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\mathcal{L}\right) - \frac{\partial}{\partial q_i}\mathcal{L} = \underbrace{Q_i}_{\text{Fuerzas generalizadas}}$$

Lo anterior lo definimos con dos principios:

d'Alembert — Hamilton $\delta S = 0$

Principio diferencial

$$0 = \sum_\alpha (\vec{F}_\alpha - m\ddot{\vec{r}}_\alpha) \cdot \delta\vec{r}_\alpha$$

Principio Integral → Extremal

$$S = \int_t^{t_f} \underbrace{\mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t)}_{\text{Lagrangiano}}\, dt \quad \text{Acción}$$

En algunos casos

$$\underbrace{\mathcal{L}}_{\text{Lagrangiano}} = T - V$$

Caso "fácil"

F. Conservativas

$$\xrightarrow{\text{Definimos}} \quad \underbrace{\mathcal{H}}_{\text{Hamiltoniano}} = \sum_i \dot{q}_i P_i - \mathcal{L}$$

$$\mathcal{H} = E = T + V$$

F. Conservativas
sólo si esto pasa

Suponiendo que la Energía cinética es una función homogénea de la velocidad

$$T = \sum \tfrac{1}{2} m_i \dot{r}_i = \sum a_i \dot{q}_i^2$$

Nuestra motivación era cuando el tiempo era una coordenada cíclica

Coordenadas cíclicas $\longrightarrow \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t) \longrightarrow \frac{\partial \mathcal{L}}{\partial q_k} = 0$ El Lagrangiano no depende de $q_k$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \cancel{\frac{\partial \mathcal{L}}{\partial q_k}}^{0} = \cancel{Q_k}^{\text{Suponiendo } 0} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \underbrace{P_k}_{\text{Momento generalizado}} = cte \quad \text{Cos de conservación}$$

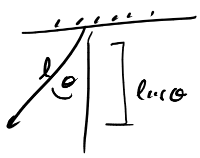
Si el tiempo era cíclico

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \frac{d}{dt}\mathcal{L}(\{q_i\}, \{\dot{q}_i\}) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right)$$

Notamos que

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right) = \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}_{= \frac{\partial \mathcal{L}}{\partial q_i}}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \Rightarrow \frac{d}{dt}\Big(\underbrace{\overbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}^{P_i}\dot{q}_i - \mathcal{L}}_{\mathcal{H}}\Big) = 0$$

— Calculamos el Hamiltoniano para el péndulo




$\mathcal{L} = \frac{1}{2} m l^2 \dot\theta^2 - mgl(1-\cos\theta) \Rightarrow$ No hay variables cíclicas ni cantidades conservadas

$\{q_i\} = \{\theta\}$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = m l^2 \dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{m l^2} \Rightarrow \dot\theta^2 = \frac{p_\theta^2}{m^2 l^4}$$

sabemos que $H = p_\theta \dot\theta - \mathcal{L} = p_\theta \dot\theta - \left[\frac{1}{2} m l^2 \dot\theta^2 - mgl(1-\cos\theta)\right]$

$$= p_\theta\left(\frac{p_\theta}{ml^2}\right) - \frac{1}{2}\cancel{ml^2}\left(\frac{p_\theta^2}{m^{\cancel{2}} l^{\cancel{4}}}\right) + mgl(1-\cos\theta)$$

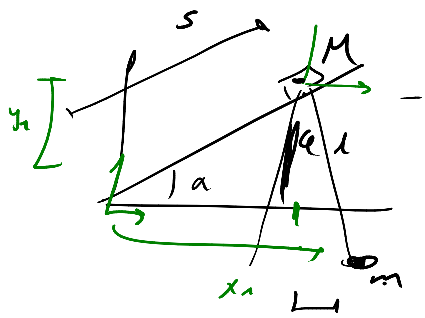
$$= \frac{p_\theta^2}{ml^2}\left(1-\frac{1}{2}\right) + mgl(1-\cos\theta)$$

$$= \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta) = T + V$$

Pues

$H = \sum_i p_i \dot q_i - \mathcal{L}$

Vamos a complicar más el Hamiltoniano




$$\begin{cases} x_1 = s\cos\alpha \rightarrow \dot x_1 = \dot s \cos\alpha \\ y_1 = s\sin\alpha \rightarrow \dot y_1 = \dot s \sin\alpha \\ x_2 = x_1 + l\sin\theta \rightarrow \dot x_2 = \dot x_1 + \dot\theta l \cos\theta \\ y_2 = y_1 - l\cos\theta \rightarrow \dot y_2 = \dot y_1 + l\dot\theta \sin\theta \end{cases} \quad \dot x_1^2 + \dot y_1^2 = \dot s^2$$

$$\dot x_2^2 = \dot x_1^2 + 2\dot x_1 \dot\theta l \cos\theta + \dot\theta^2 l^2 \cos^2\theta$$

$$\dot y_2^2 = \dot y_1^2 + 2\dot x_2 \dot\theta l \sin\theta + \dot\theta^2 l^2 \sin^2\theta$$

$$\Rightarrow T = \frac{M}{2}(\underbrace{\dot x_1^2 + \dot y_1^2}_{\dot s^2}) + \frac{m}{2}(\dot x_2^2 + \dot y_2^2)$$

$$= \frac{M}{2}\dot s^2 + \frac{m}{2}\left(\overbrace{\dot x_1^2 + \dot y_1^2}^{\dot s^2} + 2\dot\theta l [\underbrace{\overset{\dot s\cos\alpha}{\dot x_1}\cos\theta + \overset{\dot s \sin\alpha}{\dot y_1}\sin\theta}] + \dot\theta^2 l^2\right)$$

$$= \frac{M}{2}\dot s^2 + \frac{m}{2}\left(\dot s^2 + 2\dot\theta \dot s l \overbrace{\cos(\alpha-\theta)} + \dot\theta^2 l^2\right)$$

$$\Rightarrow V = -Mgy_1 - mgy_2 = -Mgs\sin\alpha - mg(\overbrace{s\sin\alpha}^{y_1} - l\cos\theta) = -(M+m)sg\sin\alpha + mgl\cos\theta$$

$$\mathcal{L} = T - V \longrightarrow p_s = \frac{\partial \mathcal{L}}{\partial \dot s} = \frac{\partial T}{\partial \dot s} = M\dot s + m\dot s + \dot\theta l\cos(\alpha-\theta) m = (M+m)\dot s + l\cos(\alpha-\theta) m\dot\theta$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{\partial T}{\partial \dot\theta} = m\dot s l\cos(\alpha-\theta) + ml^2\dot\theta = ml\cos(\alpha-\theta)\dot s + ml^2\dot\theta$$

Notamos que

$$\begin{pmatrix} P_s \\ P_\theta \end{pmatrix} = \begin{pmatrix} M+m & l\cos(\alpha-\theta)m \\ l\cos(\theta-\theta)m & m \end{pmatrix}\begin{pmatrix} \dot s \\ \dot\theta \end{pmatrix} \Rightarrow \begin{pmatrix} \dot s \\ \dot\theta \end{pmatrix} = \frac{\begin{pmatrix} m & l\cos(\alpha-\theta)m \\ l\cos(\alpha-\theta)m & M+m \end{pmatrix}}{m(M+m)-l^2m^2\cos^2(\alpha-\theta)}\begin{pmatrix} P_s \\ P_\theta \end{pmatrix}$$

Entonces, escribimos el Hamiltoniano

$$\mathcal{H} = P_s\dot s + P_\theta\dot\theta - \mathcal{L} = P_s\Bigg(\overbrace{\frac{mP_s + l\cos(\alpha-\theta)mP_\theta}{m(M+m)-l^2m^2\cos^2(\alpha-\theta)}}^{\dot s}\Bigg) + P_\theta\Bigg(\overbrace{\frac{l\cos(\alpha-\theta)mP_s + (M+m)P_\theta}{m(M+m)-l^2m^2\cos^2(\alpha-\theta)}}^{\dot\theta}\Bigg) - \mathcal{L}$$

$$\Rightarrow \mathcal{H} = \frac{mP_s^2+(M+m)P_\theta^2}{m[(M+m)-l^2m\cos^2(\alpha-\theta)]} + \frac{2P_sP_\theta l\cancel{m}\cos(\alpha-\theta)}{\cancel{m}[(M+m)-l^2m\cos^2(\alpha-\theta)]} - T + V$$

$$= \frac{mP_s^2+(M+m)P_\theta^2}{m[(M+m)-l^2m\cos^2(\alpha-\theta)]} + \frac{2P_sP_\theta l\cos(\alpha-\theta)}{[(M+m)-l^2m\cos^2(\alpha-\theta)]} + (M+m)sg\sin\alpha + mgl\cos\theta +$$

$$-\frac{M}{2}\dot s^2 - \frac{m}{2}\left(\dot s^2 + 2\dot\theta\dot s l\cos(\alpha-\theta) + \dot\theta^2 l^2\right)$$

$$\Rightarrow \mathcal{H} = \frac{mP_s^2+(M+m)P_\theta^2}{m[(M+m)-l^2m\cos^2(\alpha-\theta)]} + \frac{2P_sP_\theta l\cos(\alpha-\theta)}{[(M+m)-l^2m\cos^2(\alpha-\theta)]} + (M+m)sg\sin\alpha + mgl\cos\theta +$$

$$-\frac{\dot s^2}{2}(M+m) + 2\dot\theta\dot s l\cos(\alpha-\theta)\frac{m}{2} + \frac{m\dot\theta^2 l^2}{2}$$

calculos

$$\dot s^2 = \frac{(P_s m + l\cos(\alpha-\theta)mP_\theta)^2}{m^2\underbrace{[(M+m)-l^2m\cos^2(\alpha-\theta)]}_{\Delta}{}^2} = \frac{P_s^2m^2 + 2l\cos(\alpha-\theta)mP_\theta P_s + l^2\cos^2(\alpha-\theta)m^2P_\theta^2}{m^2[(M+m)-l^2m\cos^2(\alpha-\theta)]^2}$$

$$\Rightarrow \frac{1}{2}(M+m)\dot s^2 = \frac{1}{2}\frac{(M+m)P_s^2}{\Delta^2} + \frac{2l\cos(\alpha-\theta)P_\theta P_s(M+m)}{2\Delta^2 m} + \frac{l^2\cos^2(\alpha-\theta)(M+m)}{2\Delta^2}$$

$$\dot\theta^2 = \frac{(l\cos(\alpha-\theta)mP_s + (M+m)P_\theta)^2}{m^2[(M+m)-l^2m\cos^2(\alpha-\theta)]^2} = \frac{l^2\cos^2(\alpha-\theta)m^2P_s^2 + 2l\cos(\alpha-\theta)m(M+m)P_sP_\theta + (M+m)^2P_\theta^2}{m^2[(M+m)-l^2m\cos^2(\alpha-\theta)]^2}$$

$$\frac{ml^2\dot\theta^2}{2} = \frac{ml^4\cos^2(\alpha-\theta)P_s^2}{2\Delta^2} + \frac{2l^3\cos(\alpha-\theta)(M+m)P_sP_\theta}{2\Delta^2} + \frac{(M+m)^2l^2}{m\Delta^2\,2}$$

$$\dot s\dot\theta = \frac{(P_s m + l\cos(\alpha-\theta)mP_\theta)(l\cos(\alpha-\theta)mP_s + (M+m)P_\theta)}{m^2[(M+m)-l^2m\cos^2(\alpha-\theta)]}$$

$$= \frac{P_s^2(m + l\cos(\alpha-\theta)m) + P_\theta^2(M+m+l\cos(\alpha-\theta)m) + P_sP_\theta(m^2l\cos(\alpha-\theta) + (Mm+m^2)l\cos(\alpha-\theta))}{m^2[(M+m)-l^2m\cos^2(\alpha-\theta)]}$$

cc

$$\dot{\ell}\,\gamma(\alpha-\theta)\,m = \frac{P_s^2\,\ell\cos(\alpha-\theta)}{\Delta^2} + \frac{P_\theta^2\,M\,\ell\cos(\alpha-\theta)}{\Delta^2 m} + \frac{P_\theta^2\,[1+\ell\ell\cos(\alpha-\theta)]}{\Delta^2} - \frac{P_s P_\theta\, m\,\ell\cos^2(\alpha-\theta)}{\Delta^2} - \frac{M\,\ell\cos(\alpha-\theta) P_s P_\theta}{}$$

y como ya me harté... vamos a usar Mathemática

$s' = \dot{s}$ $\quad a = \alpha \quad$ $\theta' = \dot{\theta}$

```
T = M/2 (s')^2 + m/2 ((s')^2 + 2 (θ') (s') l Cos[a - θ] + l^2 (θ')^2)
V = - (m + M) s g Sin[a] + M g l Cos[θ]
```

$\frac{1}{2} M (s')^2 + \frac{1}{2} m \left((s')^2 + 2\, l \,\mathrm{Cos}[a-\theta]\, s' \theta' + l^2 (\theta')^2\right)$

$g\, l\, M\, \mathrm{Cos}[\theta] + g\,(-m - M)\, s\, \mathrm{Sin}[a]$

```
L = T - V // FullSimplify
```

$\frac{1}{2} \times \left(-2\, g\, l\, M \,\mathrm{Cos}[\theta] + 2\, g\,(m+M)\, s\,\mathrm{Sin}[a] + (m+M)(s')^2 + 2\, l\, m\,\mathrm{Cos}[a-\theta]\, s'\theta' + l^2 m (\theta')^2\right) \longrightarrow T - V = L$

```
{{m + M, l Cos[a - θ] m}, {l Cos[a - θ] m, m}};
{ds, dth} = Inverse[%].{ps, pθ}
```

$\longrightarrow$ nuestra matriz de $\begin{pmatrix}\dot{s}\\ \dot{\theta}\end{pmatrix} = M \begin{pmatrix}P_s\\ P_\theta\end{pmatrix}$

$\left\{\frac{m\,ps}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, p\theta\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}, \frac{(m+M)\, p\theta}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, ps\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right\} \longrightarrow \begin{pmatrix}P_s\\ P_\theta\end{pmatrix} = \mathcal{H}^{-1}\begin{pmatrix}\dot{s}\\ \dot{\theta}\end{pmatrix}$

$\mathcal{H} =$ `ps*ds + pθ*dth - L /. {s' → ds, θ' → dth}` $\longleftarrow$ Cambio $\dot{\theta}$ y $\dot{s}$ por $P_s$ y $\theta_s$

$p\theta\left(\frac{(m+M)\,p\theta}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, ps\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right) + ps\left(\frac{m\,ps}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, p\theta\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right) +$
$\frac{1}{2}\left(-l^2 m \left(\frac{(m+M)\,p\theta}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, ps\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right)^2 - 2\, l\, m\, \mathrm{Cos}[a-\theta]\left(\frac{(m+M)\,p\theta}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, ps\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right)\right.$
$\left.\left(\frac{m\,ps}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, p\theta\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right) - (m+M)\left(\frac{m\,ps}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2} - \frac{l\, m\, p\theta\, \mathrm{Cos}[a-\theta]}{m^2 + m M - l^2 m^2 \mathrm{Cos}[a-\theta]^2}\right)^2 + 2\, g\, l\, M\, \mathrm{Cos}[\theta] - 2\, g\,(m+M)\, s\, \mathrm{Sin}[a]\right)$

simplificamos $\longrightarrow$ `FullSimplify[%]`

$\frac{1}{2 m \left(m + M - l^2 m \,\mathrm{Cos}[a-\theta]^2\right)^2}\left((m+M)\left(m\, ps^2 - (-2 + l^2)(m+M)\, p\theta^2\right) + m\left(l\,\mathrm{Cos}[a-\theta]\left(2 \times (-2 + l^2)(m+M)\, ps\, p\theta - l\,\mathrm{Cos}[a-\theta]\left(l^2 m\, ps^2 + (m+M)\, p\theta^2 - 2\, l\, m\, ps\, p\theta\, \mathrm{Cos}[a-\theta]\right)\right) + 2\, g\, l\, M \left(m + M - l^2 m\,\mathrm{Cos}[a-\theta]^2\right)^2 \mathrm{Cos}[\theta] - 2\, g\,(m+M)\, s \left(m + M - l^2 m\,\mathrm{Cos}[a-\theta]^2\right)^2 \mathrm{Sin}[a]\right)\right)$

↳ y nunca nos iba a salir !!

¿Por qué escribimos a $\mathcal{H}$ en términos de $p$'s y no de $\dot{q}$'s?

| Función de estado | Ecuación a resolver | # de ecuaciones |
|---|---|---|
| Lagrangiano $\mathcal{L}$ | $\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}_{\text{Ecuación de 2do grado}} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$ | $3N - \ell$ de $2^{do}$ grado |
| ↓ $\mathcal{H} = \sum_i P_i \dot{q}_i - \mathcal{L}$ | $\underbrace{\frac{dP_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial P_i}}_{\text{Ecuaciones de primer grado}}$ Dependencia explícita de $P$ → Ecs. de Hamilton (Siguiente semana) | $2\cdot(3N - \ell)$ de $1^{er}$ grado |

Al usar $\mathcal{H}$ en lugar de $\mathcal{L}$ obtuvimos el doble de ecuaciones... pero son de un orden menor y más fáciles por lo tanto.

Para pasar de $\mathcal{L}$ a $\mathcal{H}$ empleamos una Transformada de Legendre → Siguiente semana

# ¿Qué es una transformada?

↳ Operación que permite escribir a una función de otra forma pero con la misma información.

P. Ejemplo

Transformada de Fourier

↓

Útil para escribir una función como una suma de exponenciales imaginarias

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t) = F(\omega)$$

$$\mathcal{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\omega t}}{2\pi} F(\omega) = f(t)$$

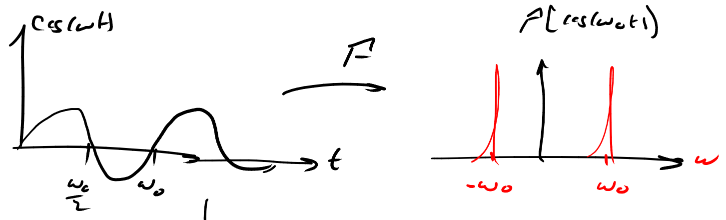
$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) \Rightarrow \mathcal{F}[\cos\omega t] = \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\frac{\left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)}{2}$$

$$= \underbrace{\int_{-\infty}^{\infty} dt\, \frac{e^{-i(\omega+\omega_0)t}}{2}}_{\frac{1}{2}\delta(\omega+\omega_0)} + \underbrace{\int_{-\infty}^{\infty} dt\, \frac{e^{i(\omega-\omega_0)t}}{2}}_{\frac{1}{2}\delta(\omega-\omega_0)}$$

$$\Rightarrow \mathcal{F}[\cos(\omega_0 t)] = \frac{1}{2}\delta(\omega+\omega_0) + \frac{1}{2}\delta(\omega-\omega_0)$$

↓

Visto en el tiempo




↳ Pasa de una función periódica o infinita a dos puntos nada más. Pero con la misma información.