Integrales de camino y de línea

Análisis de sistemas eléctricos en sistemas ingenieriles

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DEFINITION: Path Integrals The ***path integral***, or the ***integral of*** $f(x, y, z)$ ***along the path*** $\mathbf{c}$, is defined when $\mathbf{c}: I = [a, b] \to \mathbb{R}^3$ is of class C^1 and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on I. We define this integral by the equation

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt.$$

Sometimes $\int_{\mathbf{c}} f\, ds$ is denoted

$$\int_{\mathbf{c}} f(x, y, z)\, ds$$

or

$$\int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt.$$

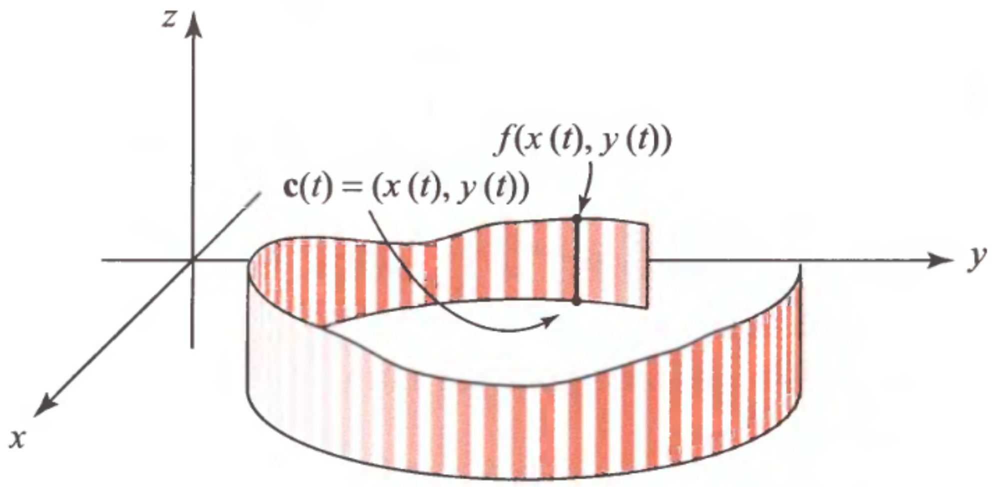
If $\mathbf{c}(t)$ is only piecewise C^1 or $f(\mathbf{c}(t))$ is piecewise continuous, we define $\int_{\mathbf{c}} f\, ds$ by breaking $[a, b]$ into pieces over which $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuous, and summing the integrals over the pieces.

The Path Integral for Planar Curves

An important special case of the path integral occurs when the path $\mathbf{c}$ describes a plane curve. Suppose that all points $\mathbf{c}(t)$ lie in the xy plane and f is a real-valued function of two variables. The path integral of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

When $f(x, y) \geq 0$, this integral has a geometric interpretation as the "area of a fence."



DEFINITION: Line Integrals Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt;$$

that is, we integrate the dot product of $\mathbf{F}$ with $\mathbf{c}'$ over the interval $[a, b]$.

As is the case with scalar functions, we can also define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ if $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$ is only piecewise continuous.

$$\int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

We next consider a useful technique for evaluating certain types of line integrals. Recall that a vector field $\mathbf{F}$ is a *gradient vector field* if $\mathbf{F} = \nabla f$ for some real-valued function f. Thus,

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

THEOREM 3: Line Integrals of Gradient Vector Fields Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and that $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

THEOREM 7: Conservative Fields Let $\mathbf{F}$ be a C^1 vector field defined on $\mathbb{R}^3$ except possibly for a finite number of points. The following conditions on $\mathbf{F}$ are all equivalent:

(i) For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(ii) For any two oriented simple curves C_1 and C_2 that have the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

(iii) $\mathbf{F}$ is the gradient of some function f; that is, $\mathbf{F} = \nabla f$ (and if $\mathbf{F}$ has one or more exceptional points where it fails to be defined, f is also undefined there).

(iv) $\nabla \times \mathbf{F} = \mathbf{0}$.

A vector field satisfying one (and, hence, all) of the conditions (i)–(iv) is called a ***conservative vector field***.[6]

$\mathbf{F} = \operatorname{grad} f$. Indeed, choose $\mathbf{c}$ to be the path shown in Figure 8.3.2, so that

$$f(x, y, z) = \int_0^x F_1(t, 0, 0)\, dt + \int_0^y F_2(x, t, 0)\, dt + \int_0^z F_3(x, y, t)\, dt,$$

where $\mathbf{F} = (F_1, F_2, F_3)$.

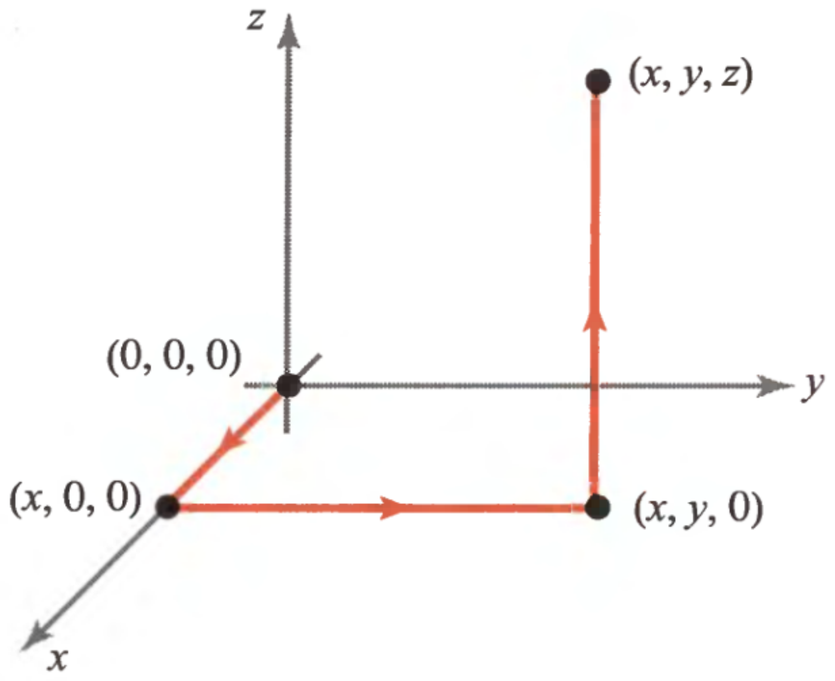


Figure 8.3.2 A path joining $(0, 0, 0)$ to (x, y, z).