Ejemplos de H-J

Oscilador armónico

1D

$$H=\frac{1}{2m}\left[p^2+m^2\omega^2x^2\right] \longrightarrow H=\frac{1}{2m}\left[p_x^2+p_y^2+m^2\omega_x^2x^2+m^2\omega_y^2y^2\right]$$

2D — si $\omega_x\neq\omega_y \Rightarrow$ Oscilador 2D anisotrópico

$=E\,t$

$V=P_iq_i$

$$S(x,t;\alpha=P)=W(x;\alpha)+V(t;\alpha) \longrightarrow S(x,y,t;\alpha_x,\alpha_y)=W_x(x;\alpha_x,\alpha_y)+W_y(y;\alpha_x,\alpha_y)-\alpha t$$

$$=W(x;\alpha)-\alpha t$$

$$\underbrace{\frac{1}{2m}\left[\left(\frac{dW}{dx}\right)^2+m^2\omega^2x^2\right]=\alpha}$$

$$\longrightarrow \frac{1}{2m}\Big[\overbrace{\underbrace{\left(\frac{dW_x}{dx}\right)^2+m^2\omega_x^2x^2}_{2m\alpha_x}}^{f(x,\,dW_x/dx)}+\overbrace{\underbrace{\left(\frac{dW_y}{dy}\right)^2+m^2\omega_y^2y^2}_{2m\alpha_y}}^{g(y,\,dW_y/dy)}\Big]=\alpha=H\left(f\left(x,\frac{\partial W_x}{\partial x}\right),g\left(y,\frac{\partial W_y}{\partial y}\right)\right)$$

$$x=\sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t+\beta)$$

$$p=\sqrt{2m\alpha}\cos(\omega t+\beta)$$

$$\alpha=\alpha_x+\alpha_y=E$$

$$x_i=\sqrt{\frac{2\alpha_{x_i}}{m\omega_{x_i}^2}}\sin(\omega_{x_i}t+\beta_{x_i}) \qquad x_i=\{x,y\}$$

$$p_{x_i}=\sqrt{2m\alpha_{x_i}}\cos(\omega_{x_i}t+\beta_{x_i})$$

2D $\longrightarrow$ Oscilador isótropo $\quad \omega_x=\omega_y\equiv\omega$

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+m^2\omega^2r^2\right),\quad S=W_r(r,\alpha_r,\alpha_\theta)+W_\theta(\theta;\alpha_r,\alpha_\theta)-\alpha t$$

$\hookrightarrow$ θ cíclica $\Rightarrow$ $W_\theta=\theta P_\theta$ $\qquad P_\theta=\frac{\partial S}{\partial\theta}=\alpha_\theta \Rightarrow P_\theta=\alpha_\theta\equiv\text{cte}$

$$=\theta\alpha_\theta$$

$$H\text{-}J\longrightarrow \frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2+\frac{\alpha_\theta^2}{r^2}+\omega^2m^2r^2\right]=\alpha$$

$\beta_x=0,\ \beta_y=\beta$

$$x^2+y^2=r^2 \Rightarrow r=\sqrt{\frac{2\alpha}{m\omega}}\sqrt{\sin^2\omega t+\sin^2(\omega t+\beta)}$$

$p_r=m\dot r$

$$\theta=\arctan\left(\frac{\sin\omega t}{\sin(\omega t+\beta)}\right)$$

$p_\theta=mr^2\dot\theta$

$\longrightarrow$ Campo central **2D**

$$HJ\to \frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2+\frac{\alpha_\theta^2}{r^2}\right]+V(r)=\alpha_1 \quad \ldots\ldots (1)$$

$$W=W_r(r;\alpha_1,\alpha_\theta)+\underbrace{W_\theta(\theta,\alpha_1,\alpha_\theta)}_{\theta\alpha_\theta}$$

De (1) despejamos

$$\frac{dW_r}{dr}=\left[2m(\alpha_1-V(r))-\frac{\alpha_\theta^2}{r^2}\right]^{1/2}=\frac{\partial W}{\partial r}$$

$$\Rightarrow W=\int dr\sqrt{2m(\alpha_1-V(r))-\frac{\alpha_\theta^2}{r^2}}+\theta\alpha_\theta$$

Escojemos

$E(\vec\alpha)\longrightarrow \alpha_1=E \Rightarrow \omega_i=\delta_{i1} \Rightarrow Q_1=t+\beta_1$

$$Q_i=\beta_i \quad i\geq 2$$

$$Q_1=\frac{\partial W}{\partial\alpha_1}=t+\beta_1=\int dr\frac{m}{\sqrt{2m(\alpha_1-V(r))-\frac{\alpha_\theta^2}{r^2}}}$$

$$Q_2 = \beta_2 = \frac{\partial W}{\partial \alpha_\theta} = \theta - \int dr \frac{1}{r^2} \frac{\alpha_\theta}{\sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}}} = \theta_0$$

si identificamos:

$$\beta_2 = \theta_0 \,, \quad u = \frac{1}{r} \quad \Rightarrow \quad \theta - \theta_0 = \Delta\theta = \int \frac{du}{\sqrt{\frac{2m}{l^2}}\sqrt{E - V(\frac{1}{u}) - \frac{u^2}{2m}}}$$

$$\alpha_\theta = l \,, \quad \alpha_1 = E$$

si $V \sim 1/r$

Solución de las Ec. de Newton

$\frac{1}{r}(\theta)$

$\downarrow$

$r(\theta)$

$$V = -\frac{k}{r} \Rightarrow V(\tfrac{1}{u}) = -ku$$

= Campo central en 3D

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V(r) \,, \quad W = W_r(r;\alpha_1,\alpha_\theta,\alpha_\varphi) + W_\theta(\theta;\alpha_1,\alpha_\theta,\alpha_\varphi) + W_\varphi(\varphi;\alpha_1,\alpha_\theta,\alpha_\varphi)$$

$\hookrightarrow$ φ es cíclica $\Rightarrow$ $\boxed{W_\varphi = \varphi\,\alpha_\varphi = \varphi\, p_\varphi}$ $\quad \vec{L}\cdot\hat{e}_z = l_z$

$$\mathcal{H}_J: \frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{r^2}\left[\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\theta}\right]\right] + V(r) = E = \alpha_1$$

l_z

$$\mathcal{H}\Big(r, \frac{dW_r}{dr}, \underbrace{f\Big(\theta, \frac{dW_\theta}{d\theta}\Big)}_{cte}\Big) = E = \alpha_1$$

$$\bullet \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\theta} = \alpha_\theta^2 \quad \cdots (2)$$

$$\alpha_\theta^2 = \|\vec{L}\|^2$$

Sustituyendo en (2)

$$\boxed{\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m(\alpha_1 - V(r))} \longrightarrow \text{igual al caso } L$$

Goldstein + Nolting

10 $\quad$ 3.5