Integrales de superfice

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DEFINITION: Parametrized Surfaces A ***parametrization of a surface*** is a function $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$. The ***surface*** S corresponding to the function $\mathbf{\Phi}$ is its image: $S = \mathbf{\Phi}(D)$. We can write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

If $\mathbf{\Phi}$ is differentiable or is of class C^1 [which is the same as saying that $x(u, v)$, $y(u, v)$, and $z(u, v)$ are differentiable or C^1 functions of (u, v)], we call S a ***differentiable or a*** C^1 ***surface***.

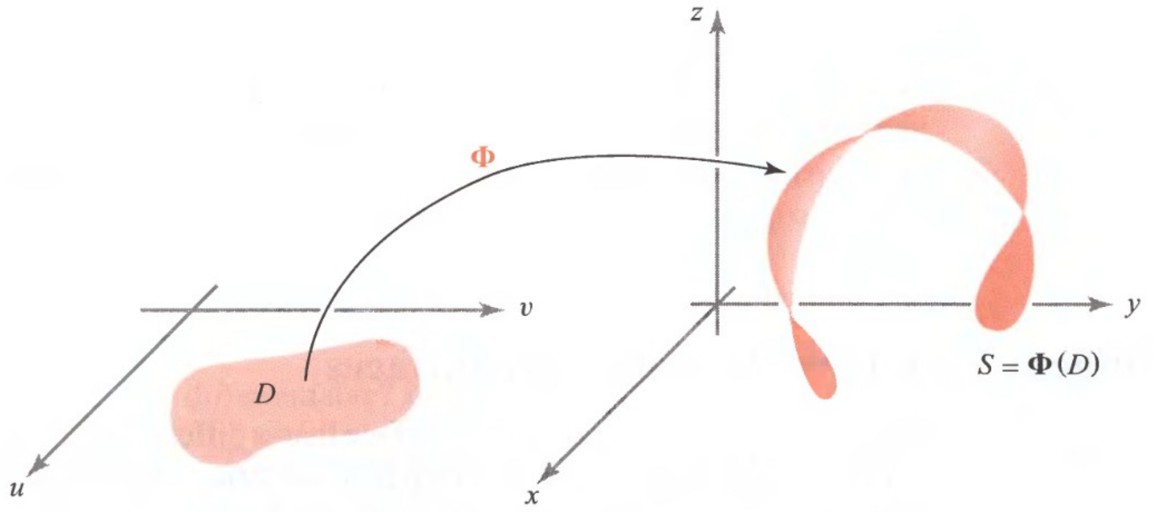

Figure 7.3.5 $\mathbf{\Phi}$ "twists" and "bends" D onto the surface $S = \mathbf{\Phi}(D)$.

Tangent Vectors to Parametrized Surfaces

$$\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

$$\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

Regular Surfaces

Because the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent to two curves on the surface at a given point, the vector $\mathbf{T}_u \times \mathbf{T}_v$ ought to be normal to the surface at the same point.

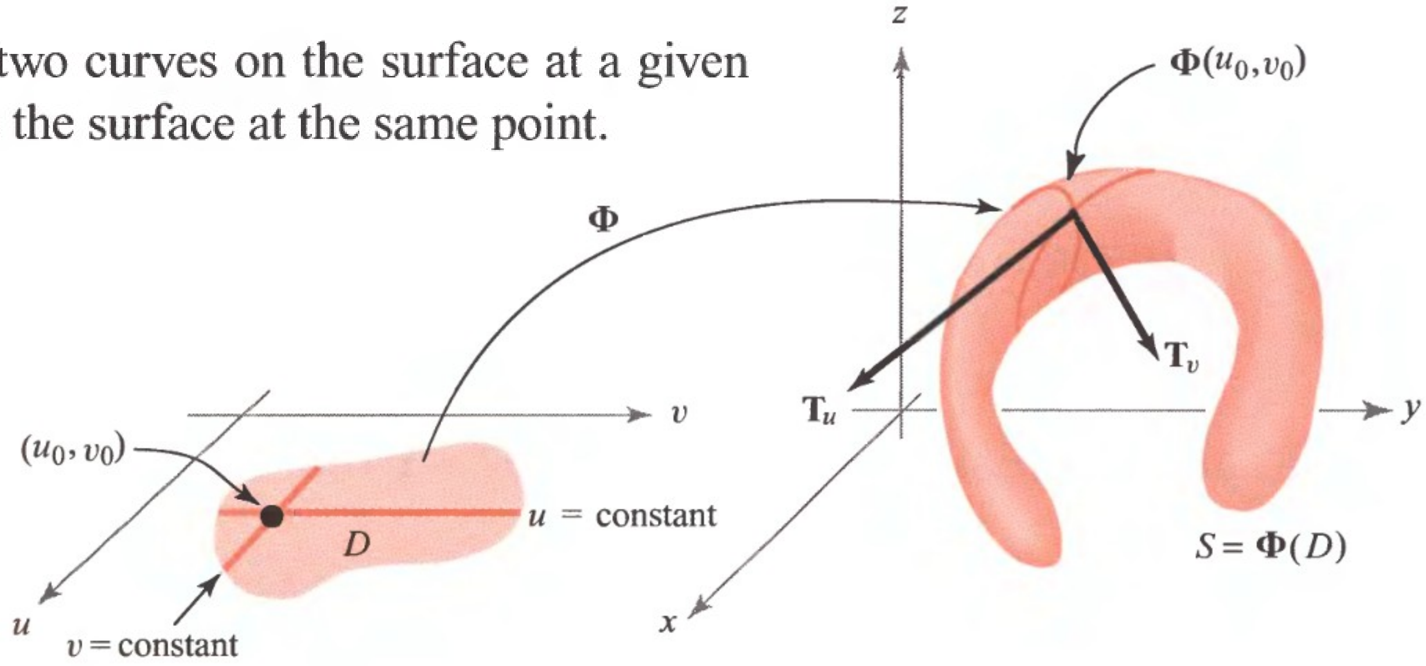

Figure 7.3.7 The tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ that are tangent to the curve on a surface S, and hence tangent to S.

DEFINITION: Area of a Parametrized Surface We define the ***surface area***[10] $A(S)$ of a parametrized surface by

$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv \tag{1}$$

where $\|\mathbf{T}_u \times \mathbf{T}_v\|$ is the norm of $\mathbf{T}_u \times \mathbf{T}_v$. If S is a union of surfaces S_i, its area is the sum of the areas of the S_i.

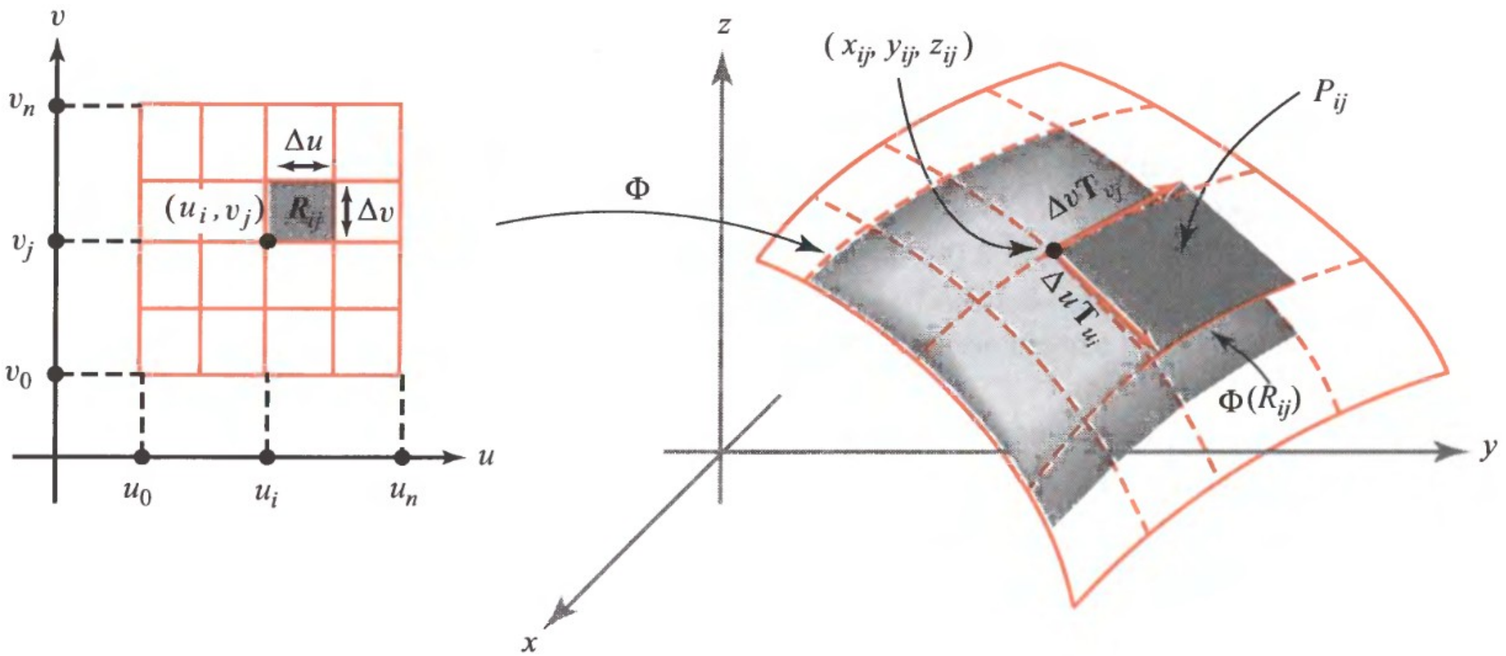

Figure 7.4.1 $\|\mathbf{T}_{u_i} \times \mathbf{T}_{v_j}\| \, \Delta u \, \Delta v$ is equal to the area of a parallelogram that approximates the area of a patch on a surface $S = \mathbf{\Phi}(D)$.

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$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2},$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du \, dv.$$

DEFINITION: The Integral of a Scalar Function Over a Surface If $f(x, y, z)$ is a real-valued continuous function defined on a parametrized surface S, we define the ***integral of f over S*** to be

$$\iint_S f(x, y, z)\, dS = \iint_S f\, dS = \iint_D f(\Phi(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv. \quad (1)$$

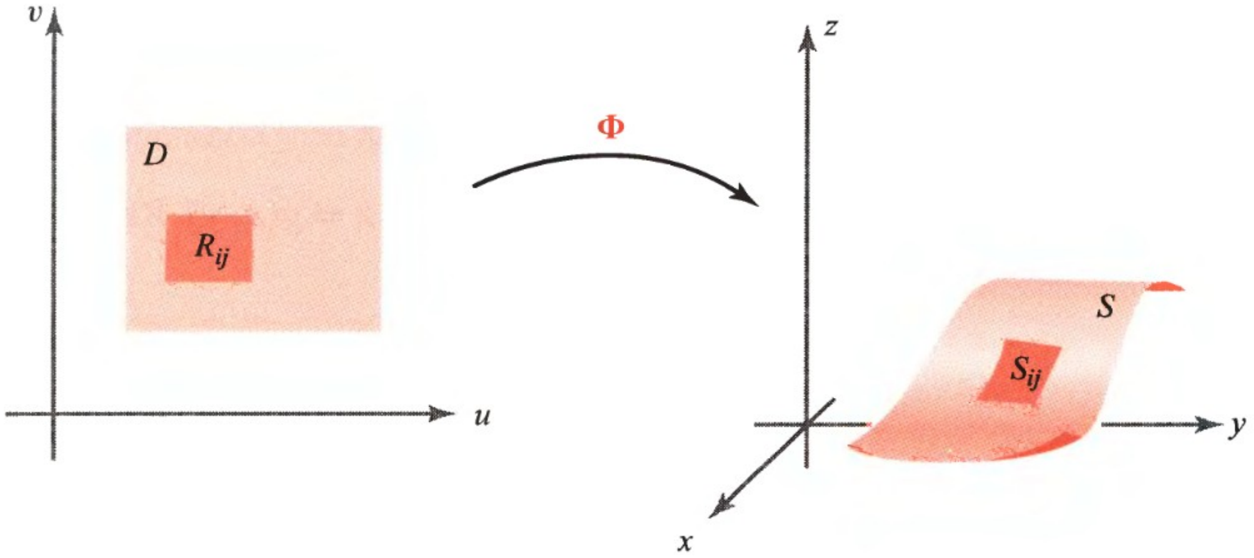

Figure 7.5.1 Φ takes a portion R_{ij} of D to a portion of S.

DEFINITION: The Surface Integral of Vector Fields Let $\mathbf{F}$ be a vector field defined on S, the image of a parametrized surface $\mathbf{\Phi}$. The ***surface integral*** of $\mathbf{F}$ over $\mathbf{\Phi}$, denoted by

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S},$$

is defined by (see Figure 7.6.1))

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv.$$

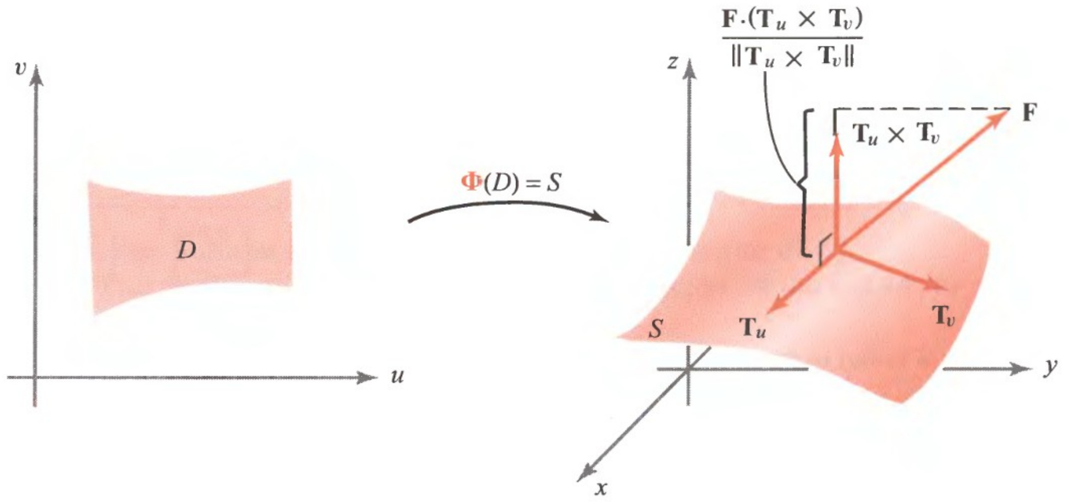

Figure 7.6.1 The geometric significance of $\mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)$.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_\Phi \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\,du\,dv$$
$$= \iint_D \mathbf{F}\cdot\left(\frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}\right)\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$$
$$= \iint_D (\mathbf{F}\cdot\mathbf{n})\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv = \iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S f\,dS,$$

where $f = \mathbf{F}\cdot\mathbf{n}$. We have thus proved the following theorem.

THEOREM 5 $\iint_S \mathbf{F}\cdot d\mathbf{S}$, the surface integral of $\mathbf{F}$ over S, is equal to the integral of the normal component of $\mathbf{F}$ over the surface. In short,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

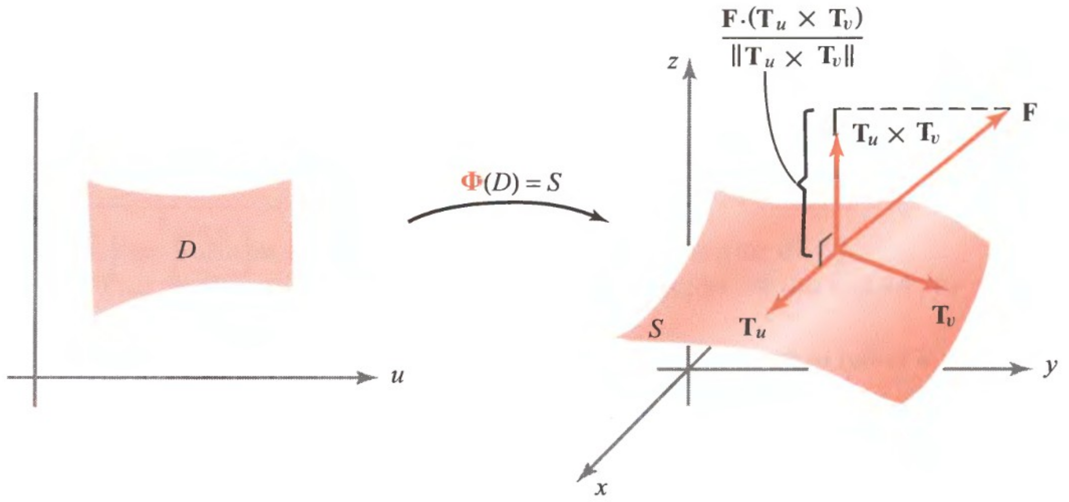

Figure 7.6.1 The geometric significance of $\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)$.

THEOREM 1: Green's Theorem Let D be a simple region and let C be its boundary. Suppose $P\colon D \to \mathbb{R}$ and $Q\colon D \to \mathbb{R}$ are of class C^1. Then

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

THEOREM 2: Area of a Region If C is a simple closed curve that bounds a region to which Green's theorem applies, then the area of the region D bounded by $C = \partial D$ is

$$A = \frac{1}{2} \int_{\partial D} x\,dy - y\,dx.$$