

# Essentials of Multiple Scattering Theory

Graduate Course Notes

February 18, 2025

The study of wave propagation in complex media has long been a central topic in optics, acoustics, and condensed matter physics. In particular, understanding the optical response of ensembles of nanoparticles has garnered significant attention due to its relevance in fields such as nanophotonics, plasmonics, and metamaterials. Two main theoretical frameworks are typically employed to model these systems: *multiple scattering theory* and *effective medium theories* (EMTs).

Multiple scattering theory explicitly accounts for the interactions between individual scatterers and provides an accurate description of wave propagation in structured and disordered media. The Foldy-Lax equations, formulated in the mid-20th century, offer a rigorous way to model the self-consistent field scattered by a collection of particles. In contrast, effective medium theories aim to replace a heterogeneous system with a homogeneous medium having equivalent macroscopic optical properties, often utilizing approximations such as Maxwell-Garnett and Bruggeman models. While EMTs offer computational simplicity and analytical tractability, they are generally less accurate in scenarios involving strong scattering, near-field interactions, and structural correlations.

The pioneering works of Lord Rayleigh on wave scattering in periodic structures laid the groundwork for modern multiple scattering approaches. Later developments by Foldy, Lax, and others extended these concepts to disordered ensembles of scatterers, leading to applications in optical random media, nanophotonic devices, and metamaterial design. Recent advances in nanophotonics have driven renewed interest in these theoretical approaches, particularly in the context of metasurfaces, plasmonic biosensors, and photonic crystals.

In this paper, we discuss the fundamental aspects of multiple scattering theory and its practical applications in describing the optical response of nanoparticle ensembles. We derive and analyze the Foldy-Lax equations, emphasizing their relevance in the study of strongly scattering media. Furthermore, we compare these approaches with effective medium theories, highlighting their respective advantages and limitations in different optical regimes. Our goal is to provide a comprehensive overview of these frameworks and illustrate their role in understanding the behavior of complex optical systems.

# 1 The Foldy-Lax Equations

The Foldy-Lax framework provides a way to describe wave propagation in a medium containing multiple scatterers. Consider a wave field  $\psi(\mathbf{r})$  scattered by  $N$  discrete scatterers located at positions  $\mathbf{r}_j$ . The total field at a point  $\mathbf{r}$  is given by the sum of the incident field  $\psi_0(\mathbf{r})$  and the scattered contributions:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \sum_{j=1}^N G(\mathbf{r}, \mathbf{r}_j) f_j \psi(\mathbf{r}_j), \quad (1)$$

where  $G(\mathbf{r}, \mathbf{r}_j)$  is the Green's function describing propagation from scatterer  $j$  to point  $\mathbf{r}$ , and  $f_j$  is the scattering amplitude of the  $j$ th scatterer.

The self-consistent equation for the field at each scatterer position leads to the Foldy-Lax hierarchy:

$$\psi(\mathbf{r}_i) = \psi_0(\mathbf{r}_i) + \sum_{j \neq i} G(\mathbf{r}_i, \mathbf{r}_j) f_j \psi(\mathbf{r}_j), \quad (2)$$

which must be solved for the fields  $\psi(\mathbf{r}_j)$  at each scatterer.

## 2 Applications

The Foldy-Lax equations have been extensively used to model:

- Light propagation in disordered media (e.g., photonic glasses, random lasers).
- Acoustics in bubbly liquids and granular materials.
- Electron transport in mesoscopic systems.

## 3 Comparison with Effective Medium Theories

While multiple scattering theory explicitly accounts for individual scatterers, effective medium theories (EMTs) seek to replace the complex heterogeneous medium with a homogeneous one having some *effective* properties. Effective medium theories (EMTs) offer an alternative approach by replacing the inhomogeneous system with a homogeneous medium possessing equivalent optical properties. Common EMTs include:

- **Maxwell-Garnett Theory:** Describes dilute mixtures and assumes that inclusions are embedded in a host medium without significant interparticle interactions.
- **Bruggeman Approximation:** Treats the medium as a symmetric mixture of constituents and accounts for higher particle concentrations.

- **Near-Zero Refractive Index Models:** Used in the context of metamaterials to describe unconventional wave propagation.

While EMTs provide valuable insights into the macroscopic optical response, they inherently neglect multiple scattering effects, phase coherence, and near-field coupling between nanoparticles. As such, their accuracy deteriorates in dense or resonant scattering systems.

### 3.1 Differences

- Multiple scattering theory retains information about individual scatterers and their spatial distribution, whereas EMTs describe only averaged macroscopic behavior.
- EMTs work well in the dilute regime but fail when strong correlations or resonance effects occur.
- The Foldy-Lax equations are explicit in treating interactions, while EMTs approximate them.

### 3.2 Similarities

- Both approaches seek to describe wave propagation in complex media.
- In the weak scattering limit, EMT results can sometimes be derived from multiple scattering formalisms.
- Both methods rely on Green's function techniques.

The interplay between multiple scattering theories and effective medium approximations remains a key area of study in nanophotonics and optical physics. While EMTs offer practical tools for designing bulk optical materials, multiple scattering approaches provide deeper insights into mesoscopic and strongly scattering systems. Understanding the limits and applicability of each method is crucial for advancing the design of optical nanostructures, metasurfaces, and disordered photonic materials.