Approximations to the Foldy-Lax Hierarchy: From Single Scattering to Beyond QCA

Graduate Course Notes

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1 Introduction

The Foldy-Lax hierarchy provides a fundamental framework for describing the multiple scattering of waves in a medium with discrete scatterers. Given its complexity, solving it exactly is impractical for large systems, necessitating various approximations. In this document, we review key approximations, from the simplest Single Scattering Approximation (SSA) to the more sophisticated Quasicrystalline Approximation (QCA) and beyond. These approximations have been extensively discussed in multiple works, including [1, 2, 3, 4].

2 Single Scattering Approximation (SSA)

The simplest approximation assumes each scatterer interacts only with the incident field, neglecting multiple scattering.

$$\mathbf{E}(\mathbf{r}_i) \approx \mathbf{E}_0(\mathbf{r}_i). \tag{1}$$

The scattered field is then:

$$\mathbf{E}_i^{\mathrm{sc}}(\mathbf{r}) = \mathbf{G}(\mathbf{r}, \mathbf{r}_i)\mathbf{T}_i\mathbf{E}_0(\mathbf{r}_i), \tag{2}$$

where $\mathbf{T}_i$ is the individual scatterer's T-matrix and $\mathbf{G}$ is the Green's function.

3 Independent Scattering Approximation (ISA)

The ISA improves upon SSA by averaging over all possible configurations of scatterers. It assumes no positional correlations, making it suitable for low-density random media [1].

4 Coherent Potential Approximation (CPA)

The CPA replaces the disordered medium with an effective homogeneous medium. The effective permittivity ε_{eff} is determined self-consistently:

$$\Sigma(\mathbf{k}) = \rho \int T(\mathbf{k}, \mathbf{k}') G_{\text{eff}}(\mathbf{k}') d^3k'. \tag{3}$$

While CPA accounts for multiple scattering, it ignores spatial correlations [3].

5 Quasicrystalline Approximation (QCA)

QCA introduces short-range positional correlations via the pair distribution function $g(\mathbf{r}, \mathbf{r}')$:

$$\Sigma(\mathbf{k}) = \rho \int g(\mathbf{r}) T(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r. \tag{4}$$

This improves accuracy for moderate densities but still neglects long-range interactions [4].

6 Beyond QCA: Advanced Models

For high-density media, more advanced methods are required:

- **Dense Media Model (DMM)**: Extends QCA by including three-body and higher-order correlations.
- **Self-Consistent Coherent Scattering Approximation (SCCSA)**: Similar to CPA but accounts for phase coherence effects.
- **Monte Carlo Methods**: Direct numerical solutions of the Foldy-Lax equations [2].

7 Summary of Approximations

Approximation	**Key Assumption**	**Strengths**	**Weaknesses**
SSA	No interactions	Simple, valid for dilute systems	Fails at high densities
ISA	No correlations	Useful for random media	Ignores multiple scatterin
CPA	Effective medium	Self-consistent	Ignores spatial correlation
QCA	Short-range correlations	Improved accuracy	Ignores long-range effects
DMM	Higher-order correlations	Accurate at high densities	Computationally intensive
SCCSA	Phase coherence effects	More precise than CPA	Requires numerical metho

Table 1: Comparison of different approximations to the Foldy-Lax hierarchy.

8 Conclusion

The Foldy-Lax hierarchy provides a rigorous framework for multiple scattering theory. Various approximations improve its tractability, each with different levels of complexity and accuracy. Understanding these approximations is crucial for applications in disordered photonic materials and wave propagation in complex media.

References

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