Scientific Inquiry, vol. 7, No. 2, December, 2006, pp. 111 – 124 *IIGSS Academic Publisher*

A NEW DEFINITION FOR THE DEGREE OF GREY INCIDENCE*

SIFENG LIU[a)†], ZHIGENG FANG[a)] and YI LIN[b)‡]

[a)] *College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China;* [b)] *Mathematics Department, Slippery Rock University, Slippery Rock, PA 16057, USA*

(received May 02, 2005; In final form October 14, 2005)

Based on the definition of degree of grey incidence introduced by Julong Deng, a new definition of absolute degree of grey incidence is given in this paper. A simplified method for calculating the new absolute degree of grey incidence is established. Some properties of the new concept are studied. Compared to the original definition, the new concept has many advantages, such as (1) it satisfies the property of symmetry, (2) the order of grey incidences remains stable, and (3) the amount of numerical computation is less.

Keywords: absolute degree of grey incidence

1. INTRODUCTION

In 1982, the grey systems theory was introduced by Deng Ju-long in China. It was a new theory and method applicable to the study of problems with unascertained and very few data and/or poor information. This theory studies unascertained systems with partially known and partially unknown information by drawing out valuable information using the idea of generating and mining the partially known information. It can describe correctly and monitor effectively the systemic operational behavior of systems (Liu and Forrest, 1998). Grey incidence analysis is one of the major components of the grey systems theory (Chen, 2001; Liu et al., 2004).

The fundamental idea of the grey incidence analysis is that the closeness of a relationship is judged based on the level of similarity of the geometrical patterns of sequence curves (Deng, 1985; 1992). The more similar the curves are, the higher degree of incidence between the sequences; and, vice versa. Grey incidence analysis can be applied to studies of various sample sizes and distributions with small amounts of computation. In general, applications of the grey incidence analysis do not result in disagreements between quantitative analysis and qualitative analysis (Lin and Liu, 2000).

Based on the following definition for the degree of grey incidence (Deng, 1985; 1992)

$$\gamma(X_0, X_i) = \frac{1}{n}\sum_{k=1}^{n}\gamma_{0i}(k),$$

* This work was supported by the National Natural Science Foundation of China(Grant No.70473037), the Science Foundation for Doctoral Program of China (20020287001), the Natural Science Foundation of Jiangsu Province (Key project, BK2003211), and the Science Foundation for Innovation Group of NUAA (1009-260812).

† Corresponding author, e-mail: sfliu@nuaa.edu.cn.

‡ e-mail: jeffrey.forrest@sru.edu.

where

$$\gamma_{0i}(k) = \gamma(x_0(k), x_i(k)) = \frac{\min_i \min_k |x_0(k) - x_i(k)| + \zeta \max_i \max_k |x_0(k) - x_i(k)|}{|x_0(k) - x_i(k)| + \zeta \max_i \max_k |x_0(k) - x_i(k)|},$$

a so-called absolute degree of grey incidence is given as follows

$$\varepsilon_{0i} = \varepsilon(X_0, X_i) = \frac{1 + |s_0| + |s_i|}{1 + |s_0| + |s_i| + |s_0 - s_i|},$$

where

$$s_0 = \int_1^n (X_0 - x_0(1))dt,\ s_i = \int_1^n (X_i - x_i(1))dt,$$

and

$$s_0 - s_i = \int_1^n [(X_0 - x_0(1)) - (X_i - x_i(1))]dt.$$

With this definition in place, we will establish a simplified method to carry out the calculation of absolute degree of grey incidences when sequences X_k's are given. After that several important properties of this degree of grey incidences are studied. Some of the properties are expected to be practically useful in applications.

2. THE DEFINITION OF ABSOLUTE DEGREE OF GREY INCIDENCE

Assume that

$$X_i = (x_i(1), x_i(2), \cdots, x_i(n))$$

stands for a behavioral sequence of data, and that the zigzagged line

$$(x_i(1) - x_i(1), x_i(2) - x_i(1), \cdots, x_i(n) - x_i(1))$$

is denoted as $X_i - x_i(1)$. Let

$$s_i = \int_1^n (X_i - x_i(1))dt. \tag{1}$$

PROPOSITION 2.1. 1. When X_i is an increasing sequence, then $s_i \geq 0$,
2. when X_i is a decreasing sequence, then $s_i \leq 0$, and
3. when X_i is a vibrating sequence, then the sign of s_i is not fixed.

The proof of this proposition is based on definitions of increasing, decreasing and vibrating sequences and properties of integrations. Here, all the details are omitted. ☐

DEFINITION 2.1. Let $X_i = (x_i(1), x_i(2), \cdots, x_i(n))$ be the same as above and D a sequence operator

$$X_i D = (x_i(1)d, x_i(2)d, \cdots, x_i(n)d),$$

where

$$x_i(k)d = x_i(k) - x_i(1)$$ □

k = 1, 2, ..., n.. Then D is called a zero starting point operator with X_iD as the image of zero starting point of X_i, denoted

$$X_i D = X_i^0 = (x_i^0(1), x_i^0(2), \cdots, x_i^0(n)).$$

PROPOSITION 2.2. Assume that the images of zero starting point of two behavioral sequences

$$X_i = (x_i(1), x_i(2), \cdots, x_i(n)) \text{ and } X_j = (x_j(1), x_j(2), \cdots, x_j(n))$$

are

$$X_i^0 = (x_i^0(1), x_i^0(2), \cdots, x_i^0(n)) \text{ and } X_j^0 = (x_j^0(1), x_j^0(2), \cdots, x_j^0(n)),$$

respectively. Let

$$s_i - s_j = \int_1^n (X_i^0 - X_j^0)dt.$$

Then the following hold true:

1. If X_i^0 is always above X_j^0, then $s_i - s_j \geq 0$.
2. If X_i^0 is always underneath X_j^0, then $s_i - s_j \leq 0$, and
3. If X_i^0 and X_j^0 alternate their positions, then the sign of $s_i - s_j$ is not fixed. □

DEFINITION 2.2. The sum of time intervals between consecutive observations of a sequence X_i is called the length of X_i.

It should be noted that two sequences of the same length may not have the same number of observations, for example,

$$X_1 = (x_1(1), x_1(3), x_1(6)),$$
$$X_2 = (x_2(1), x_2(3), x_2(5), x_2(6)),$$

and

$$X_3 = (x_3(1), x_3(2), x_3(3), x_3(4), x_3(5), x_3(6)).$$

Even though these sequences all have 5 as their length, they have different numbers of observation values.

DEFINITION 2.3. Assume that two sequences X_0 and X_i are of the same length, and s_0 and s_i are defined as in equ. (1). Then

$$\varepsilon_{0i} = \frac{1+|s_0|+|s_i|}{1+|s_0|+|s_i|+|s_i-s_0|}$$

is called the absolute degree of (grey) incidence of X_0 and X_i.

Upto this point, we only introduced the concept of absolute degree of grey incidence for same length sequences. As for sequences of different lengths, several methods can be used to define this concept. For example, one can either delete the extra values of the longer sequence, or employ the grey modeling method GM(1, 1) developed for predictions to prolong the shorter sequence to the length of the longer sequence so that the concept of absolute degree of grey incidence can be defined for the modified sequences as that of the original sequences.

3. THE LEMMAS AND THEOREMS

PROPOSITION 3.1. Assume that the length of two sequences X_0 and X_i are the same. Let

$$X_0^{'} = X_0 - a, \text{ and } X_i^{'} = X_i - b,$$

where a and b are constants. If the absolute degree of grey incidences of $X_0^{'}$ and $X_i^{'}$ is $\varepsilon_{0i}^{'}$, then $\varepsilon_{0i}^{'} = \varepsilon_{0i}$.

In fact, when X_0 and X_i are moved horizontally, the values of s_0, s_i and $s_0 - s_i$ are not changed. So ε_{0i} is not changed. □

DEFINITION 3.1. If the time intervals of any two consecutive observation values of a sequence X have the same length, then X is called an equal time interval sequence.

LEMMA 3.1. Assume that X is an equal time interval sequence. If the time interval length $l \neq 1$, then the time axis transformation

$$t: T \to T, \; t \mapsto t/l$$

can transform X into an 1-time-interval sequence.

Proof. Under the transformation defined here, the sequence

$$X_i = (x_i(l), x_i(2l), \cdots, x_i(nl))$$

is transformed to the following with t changed to t/l and kl changed to $k/l/l = k$,

$$(x_i(1), x_i(2), \cdots, x_i(n)),$$

which is an 1-time-interval sequence.

LEMMA 3.2. Assume that X_0 and X_i are 1-time-interval sequences of the same length, and

$$X_0^0 = (x_0^0(1), x_0^0(2), \cdots, x_0^0(n)) \text{ and } X_i^0 = (x_i^0(1), x_i^0(2), \cdots, x_i^0(n))$$

are the zero images of X_0 and X_i. Then, we have

$$|s_0| = \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2} x_0^0(n)\right|,\ |s_i| = \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2} x_i^0(n)\right|,$$

and

$$|s_i - s_0| = \left|\sum_{k=2}^{n-1} (x_i^0(k) - x_0^0(k)) + \frac{1}{2}(x_i^0(n) - x_0^0(n))\right|.$$

Proof. We will complete our argument with three situations.

Case 1. X_0 and X_i are either both increasing or both decreasing sequences, and the zigzagged lines X_i^0 and X_0^0 do not intersect.

In this case, $|s_0|$, $|s_i|$ and $|s_i - s_0|$ are determined by areas of the following triangles with curvilinear sides, respectively:

$$X = 0,\ X = X_0^0,\ t = n,\ X = 0,\ X = X_i^0,\ t = n,$$

and

$$X = X_0^0,\ X = X_i^0\ t = n.$$

They are sums of little areas of $n - 1$ small trapezoids of height 1.The little trapezoids for $|s_0|$ have their base lengths as follows:

$$0, \left|x_0^0(2)\right|, \left|x_0^0(3)\right|, \cdots, \left|x_0^0(n)\right|.$$

The little trapezoids for $|s_i|$ have their base lengths as follows:

$$0, \left|x_i^0(2)\right|, \left|x_i^0(3)\right|, \cdots, \left|x_i^0(n)\right|.$$

And the little trapezoids for $|s_i - s_0|$ have their base lengths as follows:

$$0, \left|x_i^0(2) - x_0^0(2)\right|, \left|x_i^0(3) - x_0^0(3)\right|, \cdots, \left|x_i^0(n) - x_0^0(n)\right|.$$

So,

$$|s_0| = \frac{1}{2}\left|x_0^0(2)\right| + \frac{1}{2}\left(\left|x_0^0(2)\right| + \left|x_0^0(3)\right|\right) + \cdots + \frac{1}{2}\left(\left|x_0^0(n-1)\right| + \left|x_0^0(n)\right|\right) = \sum_{k=1}^{n-1}\left|x_0^0(k)\right| + \frac{1}{2}\left|x_0^0(n)\right|,$$

$$\begin{aligned} |s_i| &= \frac{1}{2}\left|x_i^0(2)\right| + \frac{1}{2}\left(\left|x_i^0(2)\right| + \left|x_i^0(3)\right|\right) + \cdots + \frac{1}{2}\left(\left|x_i^0(n-1)\right| + \left|x_i^0(n)\right|\right) \\ &= \sum_{k=2}^{n-1}\left|x_i^0(k)\right| + \frac{1}{2}\left|x_i^0(n)\right|. \end{aligned}$$

and

$$
\begin{aligned}
|s_i - s_0| &= \frac{1}{2}\left|x_i^0(2) - x_0^0(2)\right| + \frac{1}{2}\left(\left|x_i^0(2) - x_0^0(2)\right| + \left|x_i^0(3) - x_0^0(3)\right|\right) \\
&\quad + \cdots + \frac{1}{2}\left(\left|x_i^0(n-1) - x_0^0(n)\right| + \left|x_i^0(n) - x_0^0(n)\right|\right) \\
&= \sum_{k=2}^{n-1}\left|x_i^0(k) - x_0^0(k)\right| + \frac{1}{2}\left|x_i^0(n) - x_0^0(n)\right|.
\end{aligned}
$$

From the assumption that X_0 and X_1 are either both increasing or both decreasing and that the zigzagged lines X_i^0 and X_0^0 do not intersect, it follows that for k = 2, 3, …, n, all $x_0^0(k)$'s have the same sign, all $x_i^0(k)$'s have the same sign, and all $x_i^0(k) - x_0^0(k)$'s have the same sign. Therefore, we have

$$
\begin{aligned}
|s_0| &= \sum_{k=2}^{n-1}\left|x_0^0(k)\right| + \frac{1}{2}\left|x_0^0(n)\right| = \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right|, \\
|s_i| &= \sum_{k=2}^{n-1}\left|x_i^0(k)\right| + \frac{1}{2}\left|x_i^0(n)\right| = \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
|s_i - s_0| &= \sum_{k=2}^{n-1}\left|x_i^0(k) - x_0^0(k)\right| + \frac{1}{2}\left|x_i^0(n) - x_0^0(n)\right| \\
&= \left|\sum_{k=2}^{n-1}(x_i^0(k) - x_0^0(k)) + \frac{1}{2}(x_i^0(n) - x_0^0(n))\right|.
\end{aligned}
$$

Case 2. Both X_0 and X_i are vibrating sequences and the zigzagged lines X_i^0 and X_0^0 do not intersect. Since X_i^0 and X_0^0 do not intersect, from the discussion in 1, it follows that

$$
|s_i - s_0| = \left|\sum_{k=2}^{n-1}(x_i^0(k) - x_0^0(k)) + \frac{1}{2}(x_i^0(n) - x_0^0(n))\right|.
$$

In the following, we will look at $|s_0|$ and $|s_i|$.

Since X_0 is a vibrating sequence, s_0 equals the algebraic sum of various parts, bounded by $X = 0$, $X = X_i^0$, and $t = n$, while taking the parts above $X = 0$ positive and the parts beneath $X = 0$ negative.

Assume that $x_0^0(k)$ (k = 2, 3, …, n) changes sign only once, and

$$
x_0^0(m), x_0^0(m+1)
$$

is the only pair of points where the change of signs occurs. Assume that $x_0^0(m) > 0$ and $x_0^0(m+1) < 0$. Then

$$
x_0^0(k) > 0,\ k = 2, 3, \ldots, m-1, m, \text{ and } x_0^0(k) < 0,\ k = M+1, m+2, \ldots, n.
$$

Denote

$$s_{0m} = \int_m^{m+1} X_0^0 dt .$$

Then, we have

$$s_0 = \frac{1}{2}\left|x_0^0(2)\right| + \frac{1}{2}(\left|x_0^0(2)\right| + \left|x_0^0(3)\right|) + \cdots + \frac{1}{2}(\left|x_0^0(m-1)\right| + \left|x_0^0(m)\right|) + s_{0m}$$
$$- \frac{1}{2}(\left|x_0^0(m+1)\right| + \left|x_0^0(m+2)\right|) - \frac{1}{2}(\left|x_0^0(n-1)\right| + \left|x_0^0(n)\right|) \cdot$$

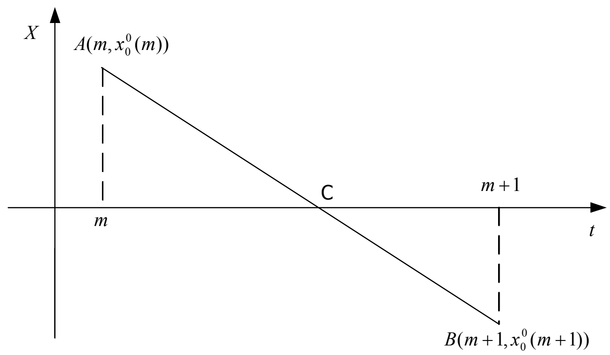

Figure 1

We now compute s_{0m}. As shown in Figure 1, the equation of the straight-line *AB* is

$$X = x_0^0(m) + (t-m)(x_0^0(m+1) - x_0^0(m)) \cdot$$

So, the intersection of *AB* and $X = 0$ is given at point *C*, where

$$C(m + \frac{x_0^0(m)}{x_0^0(m+1) - x_0^0(m)}, 0) \cdot$$

Therefore

$$s_{0m} = \frac{1}{2}\left|x_0^0(m)\right| \cdot \left|\frac{x_0^0(m)}{x_0^0(m+1) - x_0^0(m)}\right| - \frac{1}{2}\left|x_0^0(m+1)\right| \cdot \left|1 - \left|\frac{x_0^0(m)}{x_0^0(m+1) - x_0^0(m)}\right|\right|$$
$$= \frac{1}{2}(\left|x_0^0(m)\right| + \left|x_0^0(m+1)\right|) \cdot \left|\frac{x_0^0(m)}{x_0^0(m+1) - x_0^0(m)}\right| - \frac{1}{2}\left|x_0^0(m+1)\right| \cdot$$

Since $x_0^0(m) > 0$ and $x_0^0(m+1) < 0$, we have that

$$\left|\frac{x_0^0(m)}{x_0^0(m+1) - x_0^0(m)}\right| = \frac{\left|x_0^0(m)\right|}{\left|x_0^0(m+1) - x_0^0(m)\right|} = \frac{\left|x_0^0(m)\right|}{\left|x_0^0(m+1)\right| + \left|x_0^0(m)\right|} \cdot$$

Hence,

$$s_{0m} = \frac{1}{2}\left|x_0^0(m)\right| - \frac{1}{2}\left|x_0^0(m+1)\right| \cdot$$

Now, by considering

$\left|x_0^0(k)\right| = x_0^0(k)$, k = 2, 3, …, m, , and $\left|x_0^0(k)\right| = -x_0^0(k)$, $k = m+1, m+2, \ldots, n$,

it follows that

$$\left|s_0\right| = \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right|.$$

As for the case with $x_0^0(m) < 0$ and $x_0^0(m+1) > 0$, similar argument can be given.

As for the case with several pairs of points of sign changes, we can consider each pair individually, and obtain the following

$$\left|s_0\right| = \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right|.$$

Similarly, it can be proven that when X_i is a vibrating sequence, we also have

$$\left|s_i\right| = \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|.$$

Case 3. X_0 and X_i are vibrating sequences with intersecting X_i^0 and X_0^0. From 2, we have already had

$$\left|s_0\right| = \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right|,$$

and

$$\left|s_i\right| = \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|.$$

Now, $s_i - s_0$ equals the algebraic sum of various parts bounded by $X = X_0^0$, $X = X_i^0$ and $t = n$, taking the parts with X_i^0 on the top of X_0^0 positive and the other parts negative. Similar to 2, it can be proven that

$$\left|s_i - s_0\right| = \left|\sum_{k=2}^{n-1}(x_i^0(k) - x_0^0(k)) + \frac{1}{2}(x_i^0(n) - x_0^0(n))\right| \cdot \square$$

THEOREM 3.1. Assume that X_0 and X_i are two sequences of the same length, same time distances, and equal time interval. Then

$$\begin{aligned}\varepsilon_{0i} = &\left[1 + \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right| + \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|\right] \\ &\times\left[1 + \left|\sum_{k=2}^{n-1} x_0^0(k) + \frac{1}{2}x_0^0(n)\right| + \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|\right. \\ &\left.+ \sum_{k=2}^{n-1}\left(x_i^0(k) - x_0^0(k)\right) + \frac{1}{2}\left(x_i^0(n) - x_0^0(n)\right)\right]^{-1}.\end{aligned}$$

Proof. From Lemma 3.1, it can be assumed that X_0 and X_i are all 1-time interval sequences. So, from Lemma 3.2 and Definition 2.3, the result follows. □

THEOREM 3.2. Assume that two sequences X_0 and X_i have the same length, and that they have different lengths of time intervals or at least one of them is a non-equal-time-interval sequence. If the method of mean generations is used to fill in relevant blanks so that the sequences become sequences with the same relevant time steps and equal-time-intervals, then the absolute degree ε_{0i} of grey incidence is unchanged.

Proof . This argument will be completed with discussions of several cases.

Case 1. X_0 and X_i are sequences with the same corresponding time steps and non-equal time intervals. Without loss of generality, we can assume that there is only one pair of points with 2 as their (time) distance and that all other intervals of consecutive entries have length 1. Assume that the zero starting point images of X_0 and X_i are

$$X_0^0 = (x_0^0(1), x_0^0(2), \cdots, x_0^0(m), x_0^0(m+2), \cdots, x_0^0(n))$$

and

$$X_i^0 = (x_i^0(1), x_i^0(2), \cdots, x_i^0(m), x_i^0(m+2), \cdots, x_i^0(n)),$$

respectively. Now, we only need to fill in the gaps $x_0^0(m+1)$ and $x_i^0(m+1)$. Define

$$x_0^0(m+1) = \frac{1}{2}(x_0^0(m) + x_0^0(m+2))$$

and

$$x_i^0(m+1) = \frac{1}{2}(x_i^0(m) + x_i^0(m+2)) \cdot$$

Now X_0 and X_i are all equal-time-interval sequences. Let

$$s_{0m} = \int_m^{m+2} X_0^0 dt,\ s_{im} = \int_m^{m+2} X_i^0 dt,$$

and

$$s_{im} - s_{0m} = \int_m^{m+2} (X_i^0 - X_0^0) dt.$$

Before $x_0^0(m+1)$ and $x_i^0(m+1)$ are placed in the sequences, we have

$$s_{0m} = x_0^0(m) + x_0^0(m+2),\ s_{im} = x_i^0(m) + x_i^0(m+2),$$

and

$$s_{im} - s_{0m} = (x_i^0(m) - x_0^0(m)) + (x_i^0(m+2) - x_0^0(m+2)).$$

After $x_0^0(m+1)$ and $x_i^0(m+1)$ are placed in the sequences, we have

$$\begin{aligned} s_{0m} &= \frac{1}{2}(x_0^0(m) + x_0^0(m+1)) + \frac{1}{2}(x_0^0(m+1) + x_0^0(m+2)) \\ &= x_0^0(m+1) + \frac{1}{2}(x_0^0(m) + x_0^0(m+2)) \\ &= \frac{1}{2}(x_0^0(m) + x_0^0(m+2)) + \frac{1}{2}(x_0^0(m) + x_0^0(m+2)) \\ &= x_0^0(m) + x_0^0(m+2). \end{aligned}$$

Similarly, we can prove that the values for s_{im} and $s_{im} - s_{0m}$ do not change either. Therefore, $|s_0|$, $|s_i|$, and $|s_i - s_0|$ do not change, and nor does s_{0i}.

Similar to Lemma 2.2, it is not difficult to reason that no matter whether or not the pairs of points

$$(x_0^0(m), x_0^0(m+2)),\ (x_i^0(m), x_i^0(m+2))$$

and

$$(x_i^0(m) - x_0^0(m)),\ (x_i^0(m+2) - x_0^0(m+2))$$

represent sign changes, the conclusion above holds true.

Case 2. X_0 and X_i are equal-time-interval sequences of different lengths with unequal corresponding time intervals. Without loss of generality, we may assume that X_0 is a 1-time-interval sequence and X_i a 2-time-interval sequence. Let the zero starting point images of X_0 and X_i be respectively given as

$$X_0^0 = (x_0^0(1), x_0^0(2), x_0^0(3), \cdots, x_0^0(2n+1))$$

and

$$X_i^0 = (x_i^0(1), x_i^0(3), x_i^0(5), \cdots, x_i^0(2n+1)).$$

We now only need to fill in blanks in X_i^0 with

$$x_i^0(2k) = \frac{1}{2}(x_i^0(2k-1) + x_i^0(2k+1)),\ k = 1, 2, \ldots, n$$

to transform X_0^0 and X_i^0 into equal-time-interval sequences with corresponding time intervals the same. The rest of the proof is the same as in Case 1 and is omitted here.

Case 3. X_0 and X_i have different time intervals and at least one of them is non-equal-time-interval sequence. In this case, the method of mean generations can be used as mentioned in Case 2 to fill in the blanks in the sequences so that X_0 and X_i are transformed into sequences with the corresponding intervals the same. Now, as in Case 1, the method of mean generations can be applied to transform X_0 and X_i into equal-time-interval sequences. As for the argument of an unchanging s_{0i}, it can be given in a way similar to that in Case 1.

As for the situation of multiple blanks between two neighboring entries in the sequences, we can fill in the blanks one after another applying mean generations. And also the graphic method can be used. By joining the first left data point and the first right data point of the blanks with a straight line, we could get points from straight line according to time sequence to fill in the blanks. Similarly, it can be proven that the absolute degree of grey incidences is unchanged. All details are omitted here. □

4. PRPERTIES OF ABSOLUTE DEGREE OF GREY INCIDENCES

Applying our discussion above, we can prove the following results. All details are omitted.

THEOREM 4.1. The absolute degree of grey incidences ε_{0i} satisfies the following properties:

(1) $0 < \varepsilon_{0i} \le 1$;

(2) ε_{0i} is only related to the geometric shapes of X_0 and X_i, and has nothing to do with the special positions of X_0 and X_i. In other words, moving horizontally does not change the value of the absolute degree of grey incidences;

(3) Any two sequences are not absolutely unrelated. That is, ε_{0i} never equals zero;

(4) The more X_0 and X_i are geometrically similar, the greater ε_{0i};

(5) When X_0 and X_i are parallel, or X_i^0 is vibrating around X_0^0 with the area of the parts with X_i^0 on top of X_0^0 being equal to that of the parts with X_i^0 beneath X_0^0, $\varepsilon_{0i} = 1$;

(6) When any one of the data values in X_0 and X_i changes, ε_{0i} also changes accordingly;

(7) When the lengths of X_0 and X_i change, ε_{0i} also changes accordingly;

(8) $\varepsilon_{00} = \varepsilon_{ii} = 1$; and

(9) $\varepsilon_{0i} = \varepsilon_{i0}$. □

5. AN EXAMPLE

Assume that sequences

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5), x_0(7)) = (10,9,15,14,14,16)$$

and

$$X_1 = (x_1(1), x_1(3), x_1(7)) = (46,70,98)$$

are given. Find the absolute degree ε_{01} of incidence of the sequences X_0 and X_1 .

Solution. (1) Transform X_1 into a sequence with the same corresponding time intervals as X_0. Let

$$x_1(5) = \frac{1}{2}(x_1(3) + x_1(7)) = \frac{1}{2}(70 + 98) = 84$$
$$x_1(2) = \frac{1}{2}(x_1(1) + x_1(1)) = \frac{1}{2}(46 + 70) = 58$$

and

$$x_1(4) = \frac{1}{2}(x_1(3) + x_1(5)) = \frac{1}{2}(70 + 84) = 77 \, .$$

So, we have a new sequence X_1 in the place of the original X_1:

$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(7)) = (46,58,70,77,84,98) \, .$$

(2) Transform X_0 and X_1 into equal-time-interval sequences. Let

$$x_0(6) = \frac{1}{2}(x_0(5) + x_0(7)) = \frac{1}{2}(14 + 16) = 15$$

and

$$x_1(6) = \frac{1}{2}(x_1(5) + x_1(7)) = \frac{1}{2}(84 + 98) = 91 \, .$$

So the new sequence X_0 and X_1 look as follows:

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5), x_0(6), x_0(7)) = (10,9,15,14,14,15,16)$$

and

$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(6), x_1(7)) = (46,58,70,77,84,91,98) \, ,$$

and, they are all 1-time-interval sequences.

(3) Computing zero starting point images of X_0 and X_1 gives that

$$X_0^0 = (x_0^0(1), x_0^0(2), x_0^0(3), x_0^0(4), x_0^0(5), x_0^0(6),, x_0^0(7)) = (0,-1,5,4,4,5,6)$$

and

$$X_1^0 = (x_1^0(1), x_1^0(2), x_1^0(3), x_1^0(4), x_1^0(5), x_1^0(6),, x_1^0(7)) = (0,12,24,31,38,45,52).$$

(4) Find $|s_0|$, $|s_i|$, and $|s_i - s_0|$.

$$|s_0| = \left|\sum_{k=2}^{6} x_0^0(k) + \frac{1}{2} x_0^0(7)\right| = 20,$$

$$|s_1| = \left|\sum_{k=2}^{6} x_i^0(k) + \frac{1}{2} x_1^0(7)\right| = 176,$$

and

$$|s_1 - s_0| = \left|\sum_{k=2}^{6} (x_1^0(k) - x_0^0(k)) + \frac{1}{2}(x_1^0(7) - x_0^0(7))\right| = 156.$$

(5) Compute the absolute degree of grey incidences.

$$\varepsilon_{01} = \frac{1+|s_0|+|s_1|}{1+|s_0|+|s_1|+|s_1-s_0|} = \frac{197}{353} \approx 0.5581 \cdot \square$$

6. THE CONCLUDING REMARKS

Compared with the definition for grey incidences given by (Deng, 1985; 1992), the absolute degree of grey incidences, given in this paper, has many advantages such as (1) it satisfies the properties of symmetry, (2) the order of grey incidences remains stable, and (3) it requires smaller amount of computation, etc. The absolute degree of grey incidences is a numerical characteristic for the relationship of closeness between two sequences. In the course of systems analysis or relation analysis between systems' characteristic behaviors and relevant factors' behaviors, we are mainly interested in the ordering of the absolute degree of grey incidences between the systems' characteristic behaviors and each relevant factor's behavioral sequence. So, the importance of magnitudes of the absolute degree of grey incidences is relative.

References

Chen, S. L. (2001) "The grey element and the space of grey element (in Chinese)." *The Journal of Kunming University of Science and Technology*, January, 131 - 136.

Deng, J. L. (1985) *Grey Control Systems*. Huazhong University of Science and Technology Publisher, Wuhan.

Deng, J. L. (1992) *An Introduction to Grey Mathematics--Grey Hazy Set*. Huazhong University of Science and Technology Publisher, Wuhan.

Lin, Y., and Liu, S. F., (2000) "A systemic analysis with data (I)." *International Journal of General Systems*, vol.29, no.6, 989 - 999.

Lin, Y., and Liu, S. F., (2000) "A systemic analysis with data (II)." *International Journal of General Systems*, vol. 29, no. 6, 1001 - 13.

Liu, S. F. (1996) "Axioms on grey degree." *The Journal of Grey System*, vol. 8, no. 4, 396 - 400.

Liu, S. F., and Dang, Y. G., (2004) "Technical change and funds for science and technology." *Kybernetes: The International Journal of Systems & Cybernetics,* vol. 33, no. 2, 295 – 302.

Liu, S. F., Dang, Y. G., and Fang, Z.G., (2004) *Grey Systems Theory and Its Application.* (3rd edition). Science Press, Beijing.

Liu, S. F., and Deng, J. L., (2000) "The range suitable to GM(1,1)." *Theory and Practice on Systems Engineering*, vol. 20, no. 5, 121 - 124.

Liu, S. F., and Lin, Y., (1998) *An Introduction to Grey Systems: Foundations, Methodology and Applications.* IIGSS Academic Publisher, Grove City, PA.

Liu, S. F., and Lin, Y. (2005) *Grey Information: Theory and Practical Applications*. Springer-Verlag, London.

Yang, J. H., (1998) "On grey curved surface." *The Journal of Wuhan Chemical College*, January, 32 - 35.