Learning Objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Perform transformation of a vector from one Cartesian basis to another.
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention

Vector transformation

- Vector magnitude and direction do not depend on basis
- When defined on an orthonormal basis, like rectangular Cartesian, the transformation to other orthonormal bases is simple, with real coefficients.

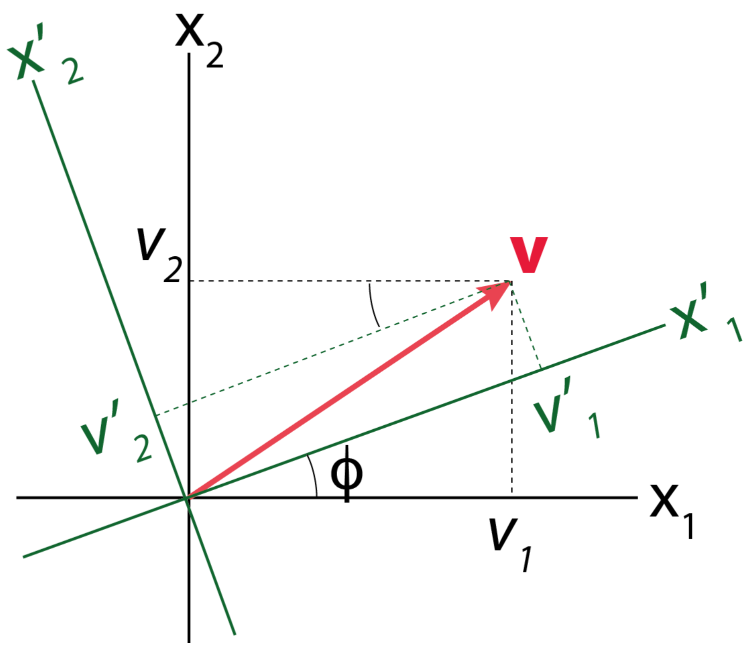
$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \qquad \text{if } i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \left|\hat{\mathbf{e}}_i\right|^2 = 1$$

check out Khan Academy lectures on orthonormal bases

Vector transformation

physical parameters should not depend on coordinate frame



for vectors on an **orthonormal** basis, the transformed vector **v**' depends on **v**:

$$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$$
$$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

Vector transformation

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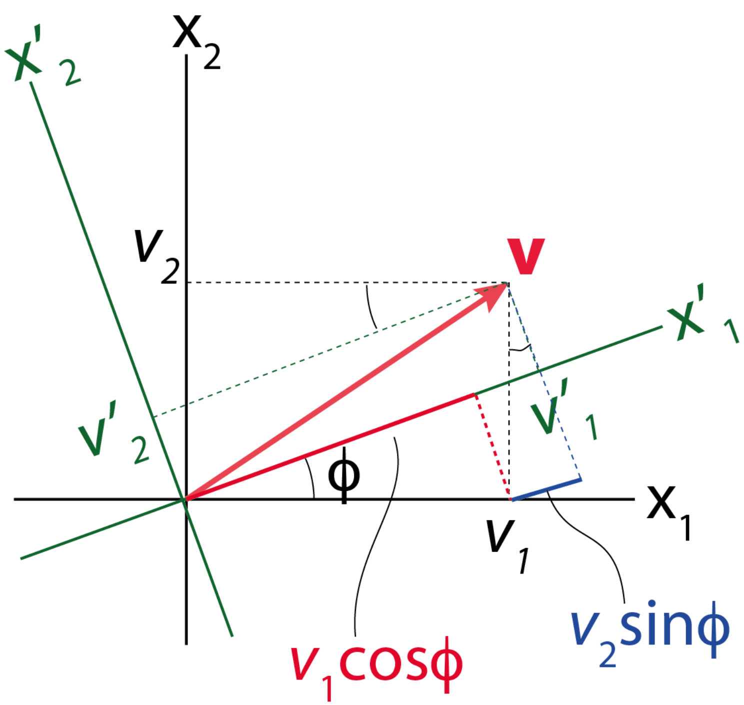

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$$v'_1 = \cos\phi \; v_1 + \sin\phi \; v_2$$

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Vector transformation

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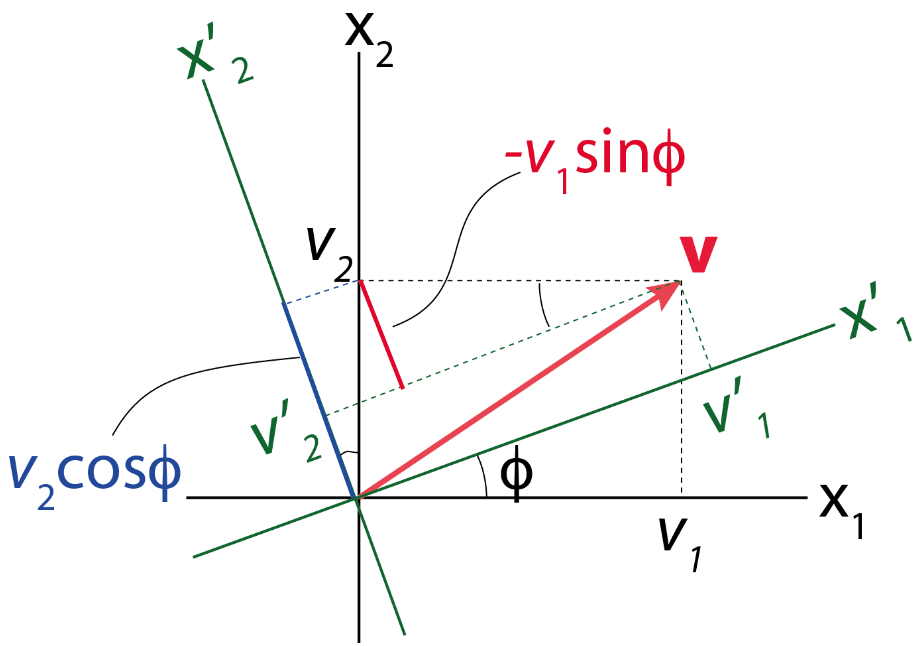

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Vector transformation

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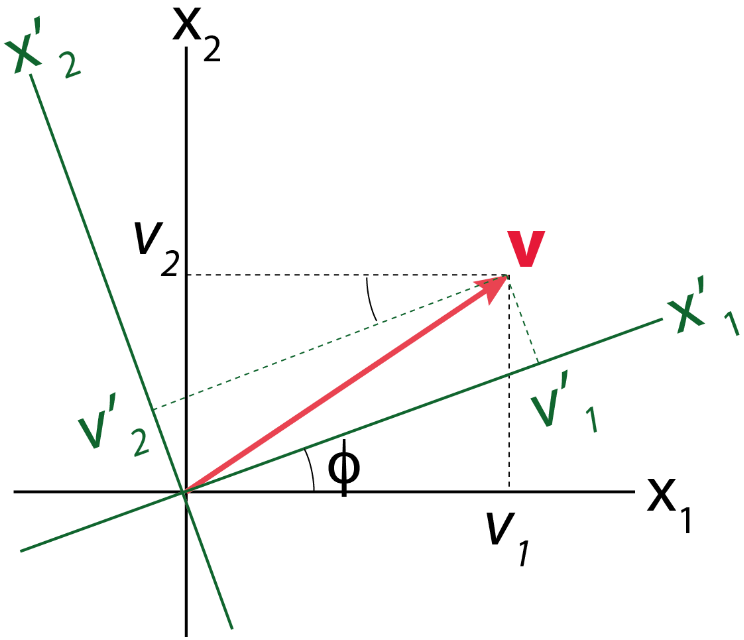


for vectors on **orthonormal** basis, transformed vector **v**' depends on **v**:

$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$

$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$

$$\Rightarrow \quad \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\alpha_{11} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{12} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2$$

$$\alpha_{21} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{22} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2$$

Transformation orthonormal bases

New basis vector in terms of old basis

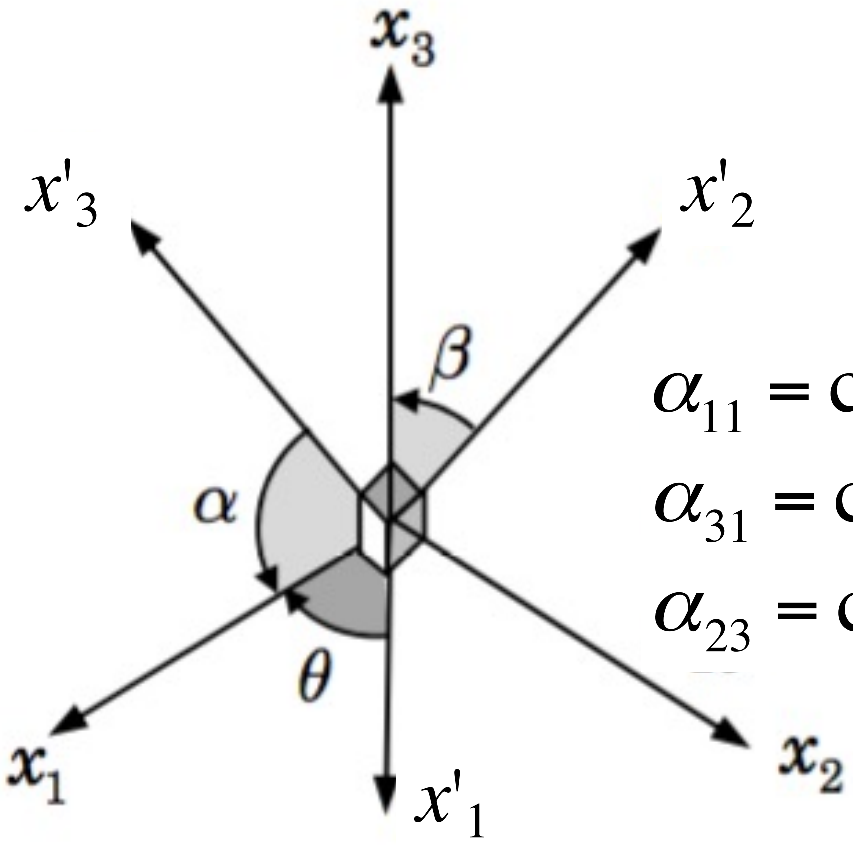
$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

$\cdot\hat{\mathbf{e}}_1$ on both sides yields:

$\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_1 = \alpha_{i1}$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



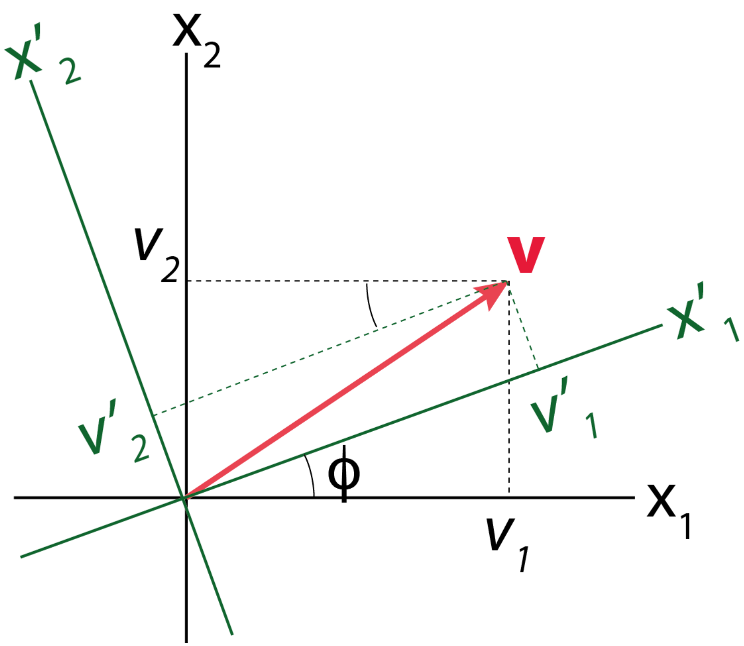
$$\alpha_{11} = \cos\theta$$
$$\alpha_{31} = \cos\alpha$$
$$\alpha_{23} = \cos\beta$$

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} \qquad \mathbf{v}' = \begin{bmatrix} \cos\phi & \overbrace{\cos(90-\phi)}^{\text{angle } x'_1 \text{ and } x_2} \\ \underbrace{\cos(90+\phi)}_{\text{angle } x'_2 \text{ and } x_1} & \cos\phi \end{bmatrix} \mathbf{v}$$

A word of caution!

vector represented in new basis

$$\mathbf{v}' = \mathbf{A}\mathbf{v}$$

vector represented in old basis

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v}$$

Clockwise rotation by ϕ

Matrix describing change of basis

New basis vectors are **rows**

new basis vector

$$\hat{\mathbf{e}}' = \mathbf{A}^T \hat{\mathbf{e}}$$

old basis vector

$$\hat{\mathbf{e}}' = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \hat{\mathbf{e}}$$

Anticlockwise rotation by ϕ

Matrix describing basis transformation

New basis vectors are **columns**

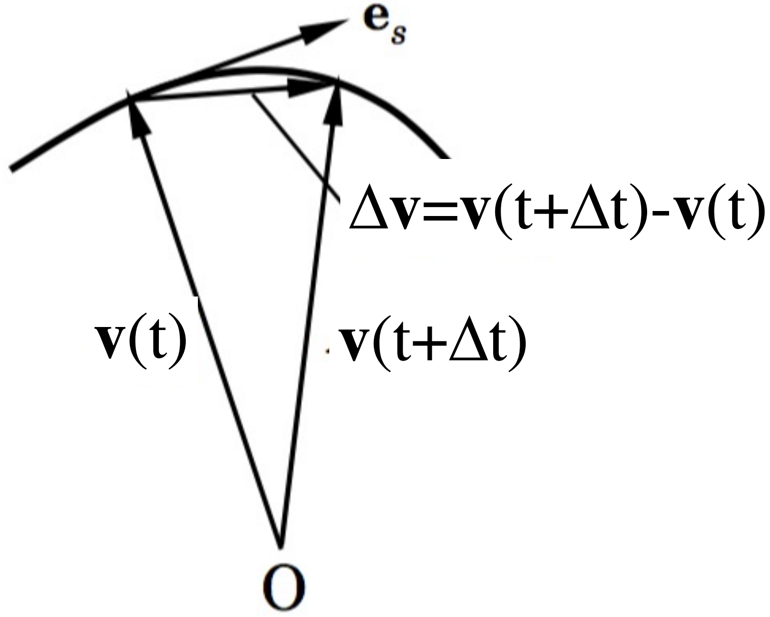
Vector Calculus

- mathematics that handles differentiation and integration of vector fields
- allows us to describe spatial variation in scalar and vector fields.

Vector derivatives

Scalar: e.g. time

$$\boxed{\frac{d\mathbf{v}}{dt} = \begin{pmatrix} \dfrac{dv_1}{dt} & \dfrac{dv_2}{dt} & \dfrac{dv_3}{dt} \end{pmatrix}}$$



$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$

usually has a different direction than **v**

remember: $\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$

$$\frac{d\mathbf{v}}{dt} = \frac{dv_1}{dt}\hat{\mathbf{e}}_1 + \frac{dv_2}{dt}\hat{\mathbf{e}}_2 + \frac{dv_3}{dt}\hat{\mathbf{e}}_3 + \underbrace{v_1\frac{d\hat{\mathbf{e}}_1}{dt} + v_2\frac{d\hat{\mathbf{e}}_2}{dt} + v_3\frac{d\hat{\mathbf{e}}_3}{dt}}_{\text{for Cartesian systems} = 0}$$

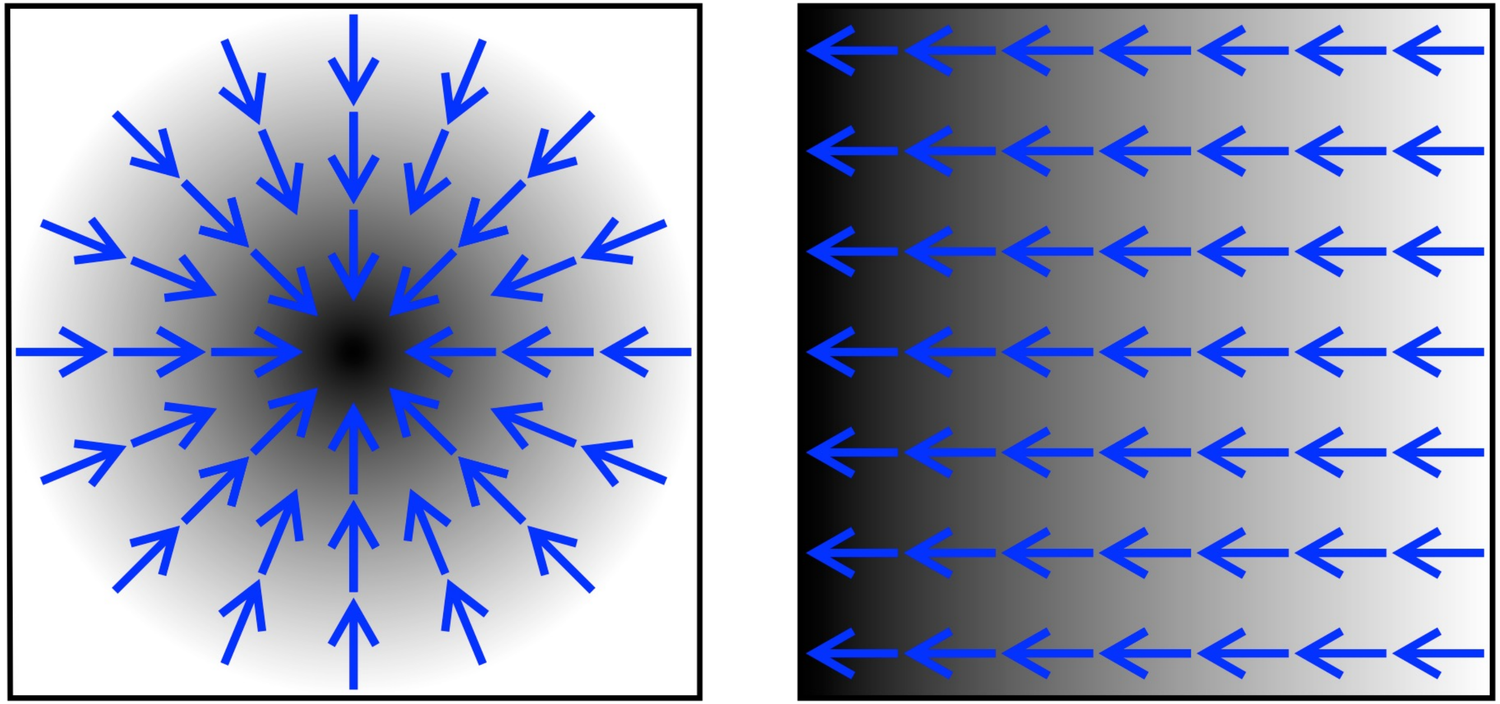
Del operator

$$\nabla = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix}$$

Has some properties of a vector, but not all

$$\mathbf{v} \cdot \nabla \phi \neq \left(\nabla \cdot \mathbf{v} \right) \phi$$

Vector derivatives: Gradient



ϕ - high ϕ - low

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right)$$

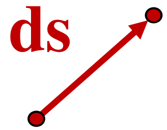
∂ - partial derivative

The gradient is a vector measure of change in scalar field with distance

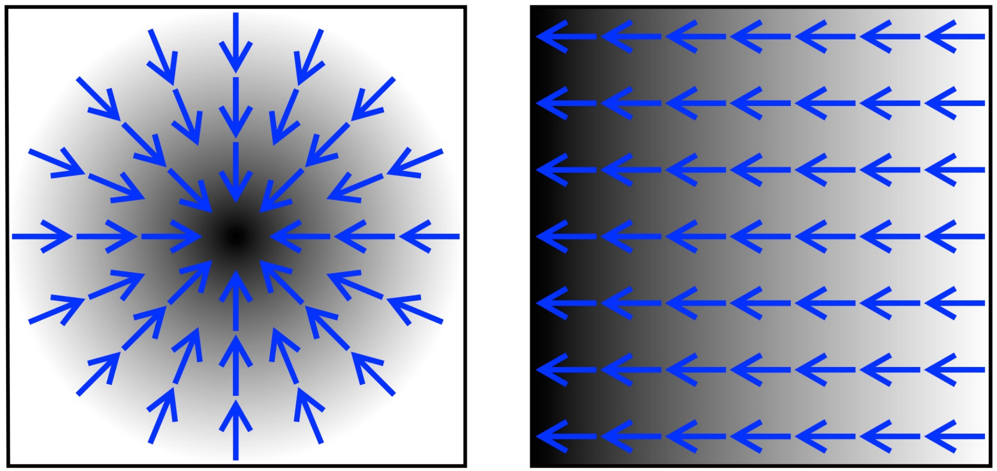
Vector derivatives
directional derivative: space

Change in ϕ over small **ds**

$$d\phi = \frac{\partial \phi}{\partial x_1} ds_1 + \frac{\partial \phi}{\partial x_2} ds_2 + \frac{\partial \phi}{\partial x_3} ds_3$$



$$d\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right) \cdot \begin{pmatrix} ds_1 \\ ds_2 \\ ds_3 \end{pmatrix} = \nabla \phi \cdot \mathbf{ds}$$



$\nabla \phi$

Exercise 5

Vector products with derivatives **divergence, curl**

- Divergence of a vector:
  scalar

$$\nabla \cdot \mathbf{v} = \sum_{i=1,3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

- Curl of a vector:
  vector

$$\nabla \times \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

Useful calculus theorems

Gauss or divergence theorem: $\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$

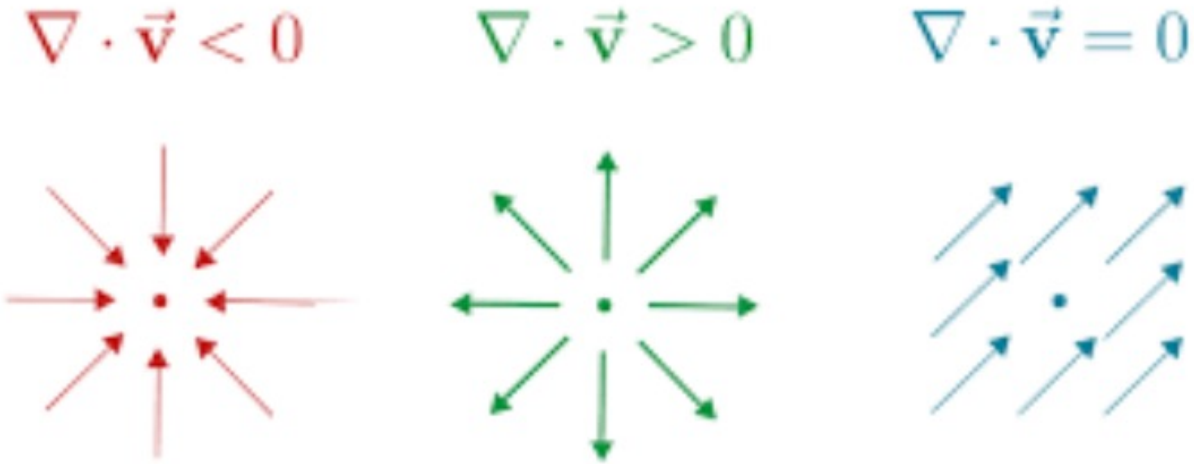
Stokes or curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Take volume V enclosed by a closed surface S
within a vector field $\mathbf{v}$ with continuous partial derivatives

Flow perpendicular to the boundary: $\mathbf{v} \cdot \hat{\mathbf{n}}$
Flow parallel to the boundary: $\mathbf{v} \cdot \hat{\mathbf{t}}$

These can be used to simplify integration over volumes or closed surfaces as well as to gain understanding of the meaning of div and curl

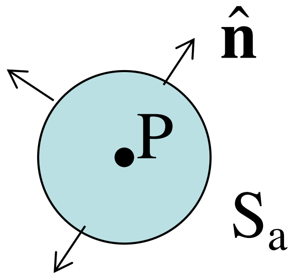
Divergence of a vector field



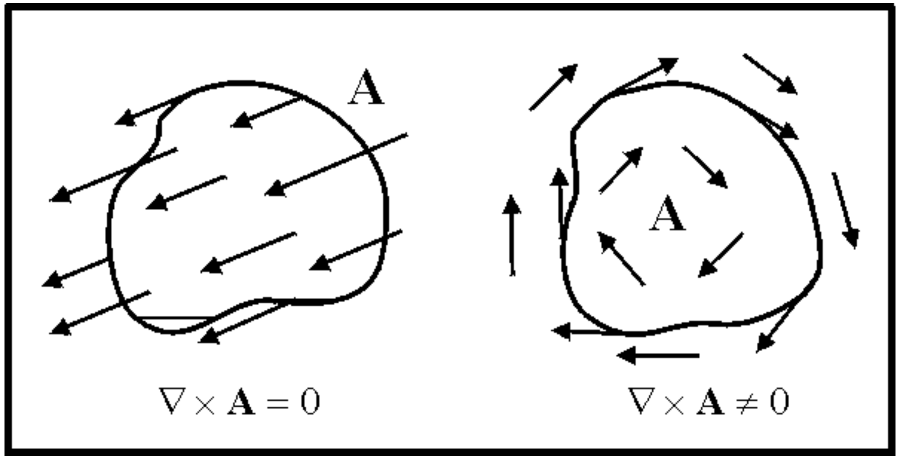
$$\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$$

Imagine a very small sphere, radius a, with boundary S_a around a point P

$$(\nabla \cdot \mathbf{v})_P \frac{4}{3}\pi a^3 = \oint_{S_a} \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$$



Divergence of a vector field represents the net outward flux per unit volume, i.e. measure of source/sink of flow



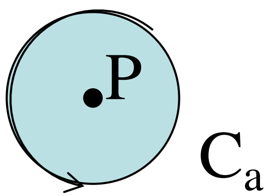
Curl of a vector field

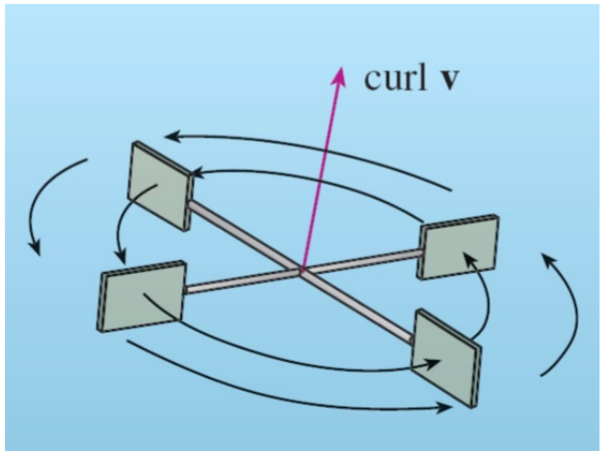
Amount of turn or spin, vorticity, in a vector field

Curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Right-hand side larger if velocities more parallel to the boundary, spinning in consistent direction, circulation around the boundary

Imagine a very small disk, radius a, with boundary C_a around a point P





$$\left(\nabla \times \mathbf{v}\right)_P \pi a^2 = \oint_{C_a} \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$$

Learning objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention

Try yourself

- Please try **Exercise 5** in the notebook, and if time, start **Exercise 6**

- Exercises can be continued in afternoon workshop