Outline Lecture 3

Part 1: Kinematics

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Strain Tensor

chapter3.ipynb

Part 2: Conservation Equations

- Conservation of Mass
- Conservation of Momentum

chapter4.ipynb

Learning Objectives
Conservation Equations

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.

Today:

- Use the conservation of mass equation
- Use the conservation of linear momentum equation, i.e. balance body forces and stresses

Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
2. Dynamics – equations that describe force balance, conservation of linear and angular momentum
3. Thermodynamics – relations temperature, heatflux, stress, entropy

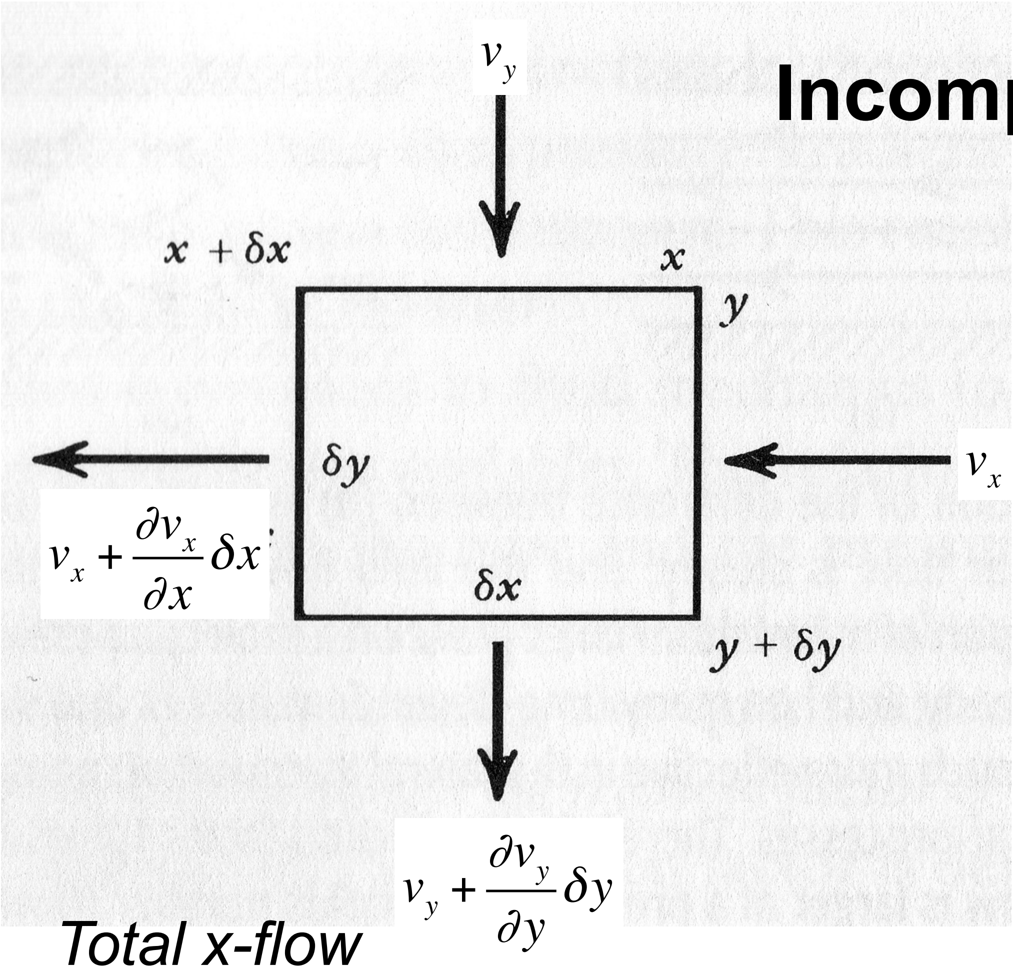
Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Kinematics: Conservation of Mass

- Describes that no material lost during flow or deformation
- Material-in balances material-out
- Take into account any potential changes in density (e.g. due to changes in temperature, pressure, phase)

2-D Conservation of Mass



Incompressible (ρ constant)

Continuity Equation

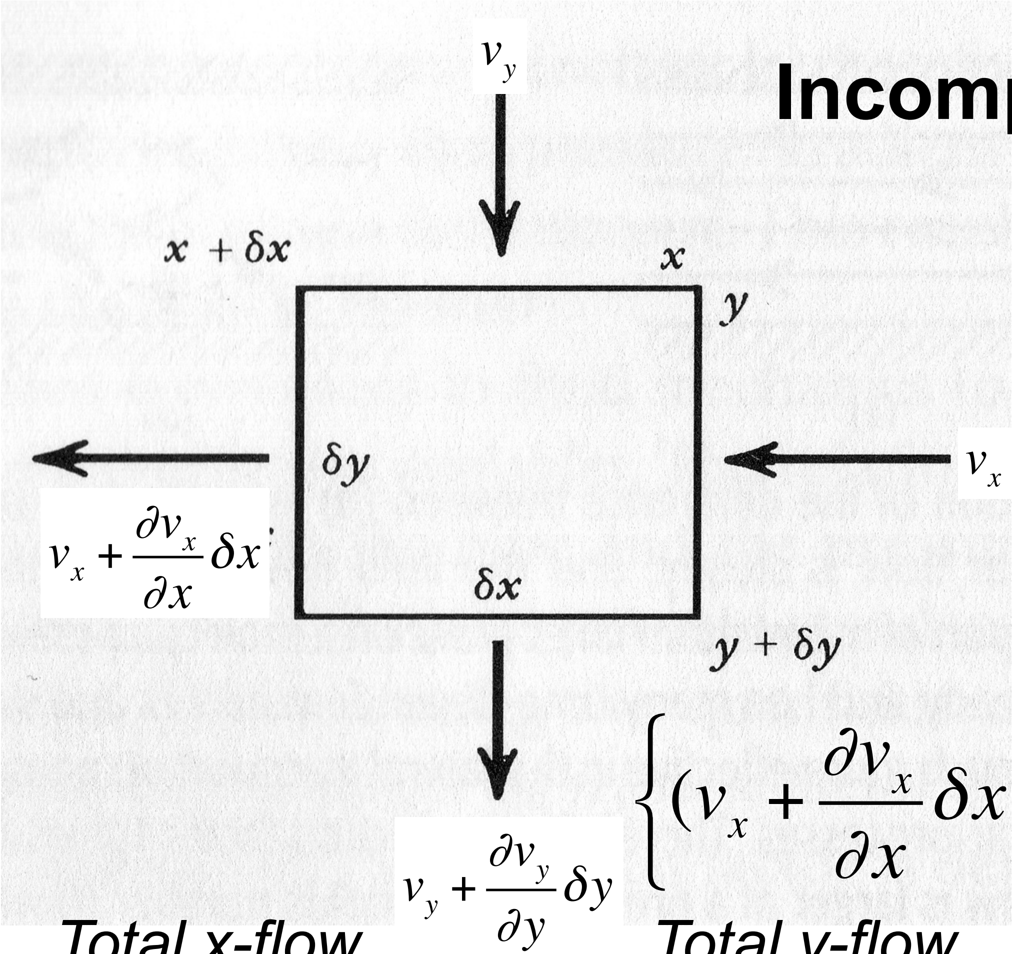
x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

Total x-flow

$$\left\{(v_x + \frac{\partial v_x}{\partial x}\delta x) - v_x\right\}\delta y$$

2-D Conservation of Mass



Incompressible (ρ constant)

Continuity Equation

x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

$$\left\{(v_x + \frac{\partial v_x}{\partial x}\delta x) - v_x\right\}\delta y$$

<u>*Total x-flow*</u> <u>*Total y-flow*</u> *Per unit area*

$$\left\{(v_x + \frac{\partial v_x}{\partial x}\delta x) - v_x\right\}\delta y + \left\{(v_y + \frac{\partial v_y}{\partial y}\delta y) - v_y\right\}\delta x = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

i.e., $\boxed{\nabla \cdot \mathbf{v} = 0}$ which also applies in 3-D

No volume changes!

Conservation of Mass

Full expression: compressible

$$\frac{D\rho dV}{Dt} = 0$$

ρ – density
dV – infinitesimal volume

density changes

$$\frac{D\rho}{Dt} dV + \rho \frac{DdV}{Dt} = 0$$

volume changes

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

In spatial description: $\dfrac{D\rho}{Dt} = \dfrac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$, so $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

ρ(time) advected

Dynamics: Conservation of Momentum

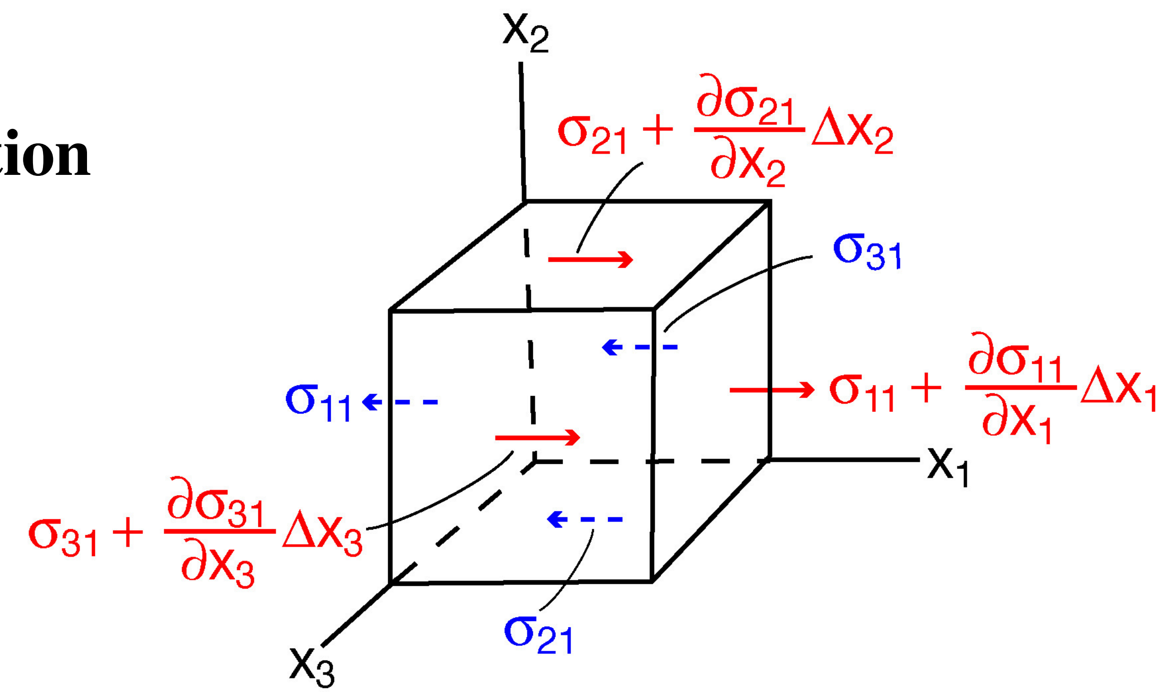
- Linear force balance = Newton's second law, **F**=m**a**
- Relates force **F** to motion, acceleration **a** – hence also called "equation of motion"
- Conservation angular momentum assumed in symmetry of stress tensor

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



$f_1\Delta x_1\Delta x_2\Delta x_3+$
$(\sigma_{11}+\Delta x_1\partial\sigma_{11}/\partial x_1 - \sigma_{11})\Delta x_2\Delta x_3 +$
$(\sigma_{21}+\Delta x_2\partial\sigma_{21}/\partial x_2 - \sigma_{21})\Delta x_1\Delta x_3 +$
$(\sigma_{31}+\Delta x_3\partial\sigma_{31}/\partial x_3 - \sigma_{31})\Delta x_1\Delta x_2$
$= \rho\Delta x_1\Delta x_2\Delta x_3\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

$\Rightarrow\ \mathbf{f} + \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2\mathbf{u}/\partial t^2$

Try yourself

chapter4.ipynb

Exercises 1 and **2**
now/tomorrow

Learning Objectives
Kinematics & Conservation

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Use the conservation of mass equation
- Use the conservation of linear momentum equation, i.e. balance body forces and stresses