Dimensions and Dimensional Analysis

Stephen Neethling

Outline

- The Advection-Diffusion Equation
  - Non-dimensionalising the equation
- Dimensional Analysis
  - Buckingham-Pi Theory
- Instabilities
  - Rayleigh-Taylor
  - Kelvin-Helmholtz
  - Plateau-Rayleigh

Learning Objectives

- Learn how to non-dimensionalise variables and equations
- Understand how to solve the advection-diffusion equation
  - Numerical and analytical solutions
- Learn how to carry out a dimensional analysis using the Buckingham-Pi theory
- Learn how to apply dimensional analysis to stability problems

Advection-Diffusion Equation

- We need an equation to demonstrate dimensional analysis on and the advection-diffusion equation is a good one to start with:
  - Consider a fluid with a chemical dissolved in it. The chemical will move with the fluid, but will also spread out due to diffusion
    - This is the classical example of advection-diffusion, but the equation appears in many other contexts as well
    - Even the Navier-Stokes equation has an advection-diffusion component at its heart, though for momentum rather than a substance (viscosity results in the diffusion of momentum):

$$\mathbf{f} = \mathbf{v}\,C - D\,\nabla C$$

$$\frac{\partial C}{\partial t} = -\nabla \cdot \mathbf{f}$$

- Where the lhs is the equation for the flux and the rhs is the continuity equation

Advection-Diffusion Equation

- If we substitute the flux equation into the continuity equation we get the following equation (assuming that the diffusion coefficient is constant):

$$\frac{\partial C}{\partial t} = -C\,\nabla \cdot \mathbf{v} \; - \mathbf{v} \; \cdot \nabla \mathrm{C} + D\nabla^2 \mathrm{C}$$

- If the fluid volume is conserved (i.e. $\nabla \cdot \mathbf{v} = 0$) then we arrive at the more familiar form of the advection-diffusion equation:

$$\frac{\partial C}{\partial t} = -\mathbf{v} \; \cdot \nabla \mathrm{C} + D\nabla^2 \mathrm{C}$$

Advection-Diffusion in 1D

$$\frac{\partial C}{\partial t} = -v \ \frac{\partial C}{\partial x} + D \frac{\partial^2 C}{\partial x^2}$$

- 1^{st} derivative is the term associated with the motion of the species due to the movement of the liquid (advection)
  - Because of this 1^{st} order terms are often referred to as "advective" even if the underlying physical mechanism isn't advection
- 2^{nd} derivative is the term associated with the diffusion of the species
  - Similarly to above, 2^{nd} order terms are often referred to as "diffusive" even if the underlying mechanism isn't diffusion

- This equation is sometimes called Burger's equation

Example:
Transport and Diffusion across a membrane

- Lets consider a membrane through which a fluid is being passed:
  - The fluid has a superficial velocity v and the chemical has a diffusivity, D
  - Within the fluid is a chemical that has a concentration C_0 on the upstream side
  - On the downstream side of the membrane it rapidly reacts and can be assumed to have a concentration of zero
  - The membrane has a thickness of h

- At first glance it appears that there are 4 different things that you can adjust in this problem:
  - C_0, h, D and v
  - How many of these things are actually independent?
    - Are there combinations of things that we can change without changing the form of the solution?

Non-dimensionalising the equation

- This is a useful way of finding the number of actual parameters in a problem
  - Also allows the underlying dependencies to be determined in a way that is independent of, for instance, the scale of the system

- Start by introducing dimensionless versions of the dependent and independent variables
- Concentration and distance have obvious ways of being made dimensionless:

$$C^* = \frac{C}{C_0} \qquad\qquad x^* = \frac{x}{h}$$

- With time there are two options:
  - This is because there is no variable that only has units of time and it is found in two different variables, v and D
  - We can choose either an advection time scale or a diffusion time scale

$$t^* = \frac{v\,t}{h} \quad \text{or} \quad t^* = \frac{D\,t}{h^2}$$

(note that there isn't a "correct" choice, with the appropriate one depending on the system and personal preference)

Non-dimensionalising the equation

- Substituting for the dimensional variables and rearranging the equation leads to the following expression (using the diffusive time scale):

$$\frac{\partial C^*}{\partial t^*} = -\frac{vh}{D}\frac{\partial C^*}{\partial x^*} + \frac{\partial^2 C^*}{\partial {x^*}^2}$$

$$C^* = \frac{C}{C_0} \qquad x^* = \frac{x}{h} \qquad t^* = \frac{D\,t}{h^2}$$

- Note that the dimensionless solution only depends on one dimensionless parameter – the Peclet number

$$Pe = \frac{vh}{D}$$

What does the Peclet number represent?

$$Pe = \frac{vh}{D}$$

- As with most dimensionless groups, the Peclet number is a ratio of two effects
- The Peclet number represents the relative importance of advection compared to diffusion
  - At low Peclet numbers ($Pe \ll 1$) diffusion dominates
  - At high Peclet numbers ($Pe \gg 1$) advection dominates

Numerical Solution

- Explicit numerical solution for this equation based on finite differencing is relatively straight forward to implement
  - This is known as a Forward Time Centred Space (FTCS) scheme

$$\frac{\partial C^*}{\partial t^*} \approx \frac{C(x^*, t^* + \Delta t^*) - C(x^*, t^*)}{\Delta t^*}$$

$$\frac{\partial C^*}{\partial x^*} \approx \frac{C(x^* + \Delta x^*, t^*) - C(x^* - \Delta x^*, t^*)}{2\Delta x^*}$$

$$\frac{\partial^2 C^*}{\partial {x^*}^2} \approx \frac{C(x^* + \Delta x^*, t^*) + C(x^* - \Delta x^*, t^*) - 2C(x^*, t^*)}{{\Delta x^*}^2}$$

Upwind Schemes

- Advection (1^{st} order) terms can cause numerical instability, especially in explicit solvers and when diffusive terms are weak
  - I.e. using the central difference is fine if Pe is small, but can be problematic for large Pe
- One way to improve the stability is to use an upwind scheme
  - Use a forward or backward difference depending on the sign of advective velocity

$$v\frac{\partial C^*}{\partial x^*} \approx v\begin{cases} \dfrac{C(x^*,t^*) - C(x^* - \Delta x^*, t^*)}{\Delta x^*} & if\ v > 0 \\ \dfrac{C(x^* + \Delta x^*, t^*) - C(x^*,t^*)}{\Delta x^*} & if\ v < 0 \end{cases}$$

Numerical Solution

- The governing equation

$$\frac{\partial C^*}{\partial t^*} = -Pe\ \frac{\partial C^*}{\partial x^*} + \frac{\partial^2 C^*}{\partial {x^*}^2}$$

- …can be approximated as

$${C^*}_{i,j} = C({x^*}_0 + i\Delta x^*, {t^*}_0 + j\Delta t^*)$$

$${C^*}_{i,j+1} \approx {C^*}_{i,j} + \Delta t^* \left(\frac{-Pe\left({C^*}_{i,j} - {C^*}_{i-1,j}\right)}{\Delta x^*} + \frac{\left({C^*}_{i+1,j} + {C^*}_{i-1\ j} - 2{C^*}_{i\ j}\right)}{{\Delta x^*}^2} \right)$$

- For stability in this explicit scheme:
  - The dimensionless rate of advection is Pe, while the dimensionless diffusion coefficient is 1

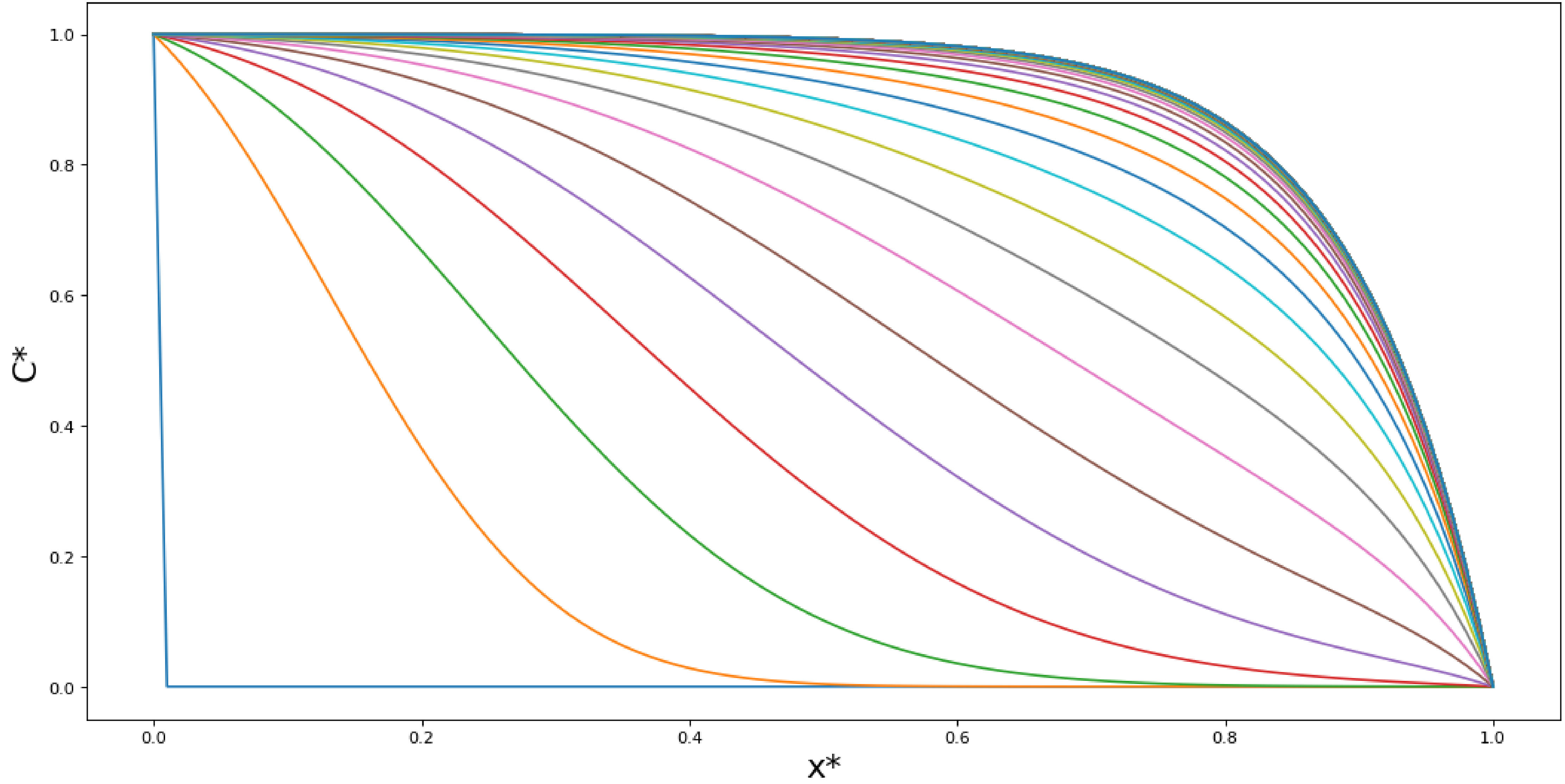
$$\Delta t^* \ll min\left(\frac{\Delta x^*}{Pe}, \frac{{\Delta x^*}^2}{2}\right) \qquad \Delta t^* \ll min\left(\frac{\Delta x^*}{"v"}, \frac{{\Delta x^*}^2}{2"D"}\right)$$

  - More on why this is the case later in the course

Python code for the implementation

```
%pylab inline

imax=101
L=1.0
#set initial condition as zero concentration
Cold=zeros(imax)
Cnew=zeros(imax)

x=linspace(0,L,imax)

#set Peclet number
Pe = 10.0

dx=L/(imax-1)

#Time step set based on stability criterion
#advective stability criterion
dt=dx/(Pe)
#include diffusive stability criterion
dt = 0.2*min(dt,0.5*(dx**2))

#set inlet concentration
Cnew[0]=1.0
Cold[0]=1.0

t=0.0
tout=0.0
dtout=0.01

if (dt>dtout):
    dt=dtout

#run for 1.0 non-dimensional time steps
t_max = 1.0

figure(num=None, figsize=(16, 8), dpi=80, facecolor='w', edgecolor='k')
xlabel('x*', fontsize=20)
ylabel('C*', fontsize=20)

plot(x,Cold)
tout+=dtout

#continue until the maximum time is exceeded
while (t<t_max):
    #explicitely set new concentration based on old concentration
    Cnew[1:-1]=dt*((1.0/(dx**2))*(Cold[2:]+Cold[:-2]-2.0*Cold[1:-1])-(Pe/dx)*(Cold[1:-1]-Cold[:-2]))+Cold[1:-1]

    t+=dt

    #plot a line at the specified time interval
    if (t>=tout):
        plot(x,Cnew)
        tout+=dtout

    #swap the old and the new arrays of concentration values
    Ctemp=Cold
    Cold=Cnew
    Cnew=Ctemp

show()
```

Results for Pe = 10 (Each line is 0.01 dimensionless time units apart)



Exercise

Download and run the code

- How does changing the Peclet number change the shape of the solution?
- Note that it also changes the time taken to run the simulation
  - Explicit solvers have strict time step constraints that change with both the resolution used and the physical parameters

Calculations

Analytical Solutions

- It is straight forward to get an analytical solution to this equation if we assume steady-state
  - The behaviour that is approached as time tends to infinity
  - The PDE becomes an ODE
    - Note that we could find an analytical solution for the full PDE using separation of variables, though it is quite a bit more complex to do than the steady state solution and involves infinite series

$$\frac{d^2C^*}{d{x^*}^2} = Pe\,\frac{dC^*}{dx^*}$$

- Since C^* is 1 at $x^* = 0$ and C^* is 0 at $x^* = 1$ the following expression can be obtained (simple calculus, but quite a bit of algebra!):

$$C^* = \frac{e^{Pe} - e^{Pe\,x^*}}{e^{Pe} - 1}$$

Exercise

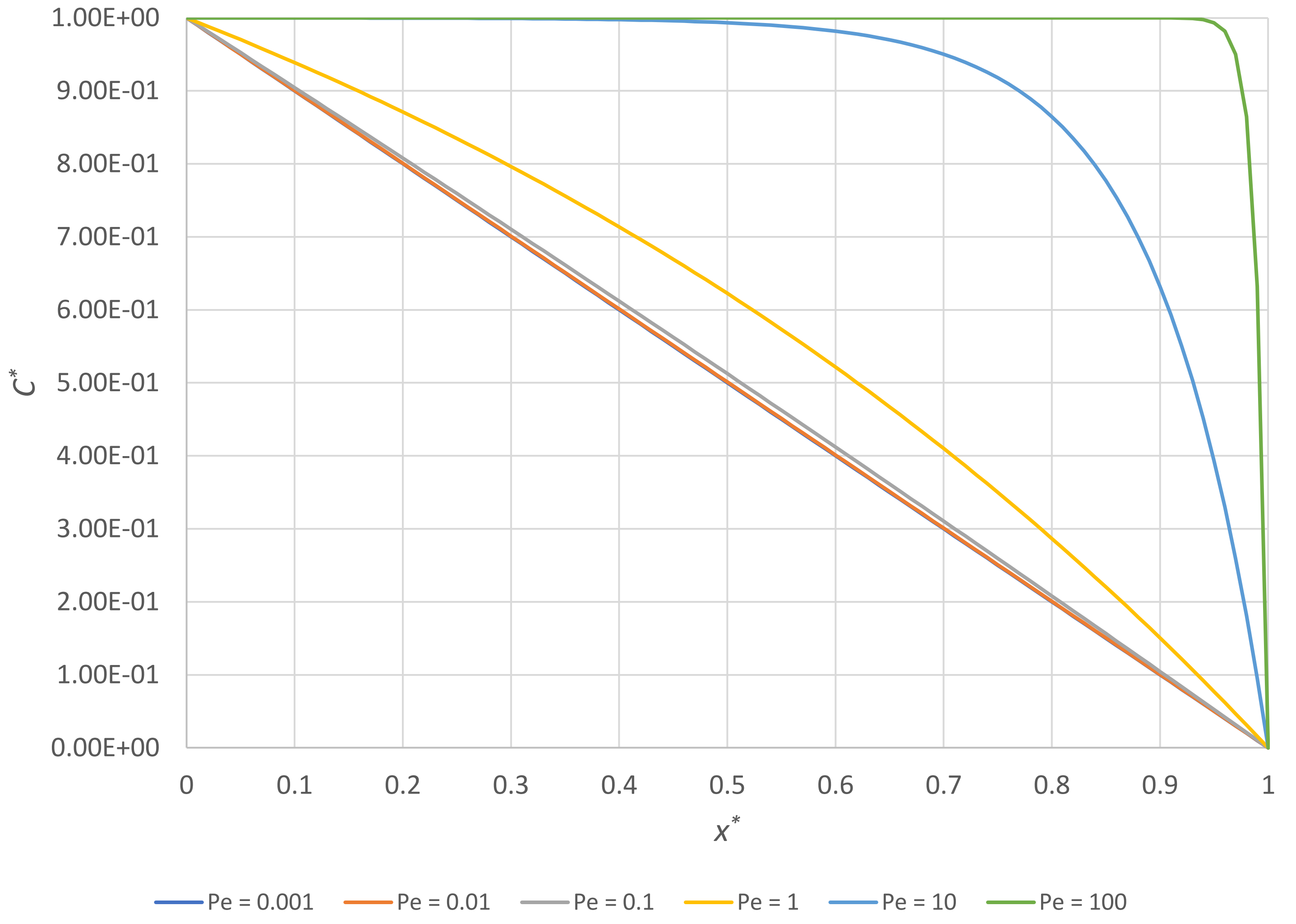
- Show for yourself that that the solution to $\dfrac{d^2C^*}{d{x^*}^2} = Pe\,\dfrac{dC^*}{dx^*}$

is $C^* = \dfrac{e^{Pe} - e^{Pe\,x^*}}{e^{Pe} - 1}$

(assuming that C^* is 1 at $x^* = 0$ and C^* is 0 at $x^* = 1$)

Calculations

Does the analytical solution make sense?



$$C^* = \frac{e^{Pe} - e^{Pe\,x^*}}{e^{Pe} - 1}$$

Dimensional Analysis

- As you have seen, we can obtain the relevant dimensionless groups and dependencies by non-dimensionalising the governing equations
  - This requires that we have a theory/model to describe the system

- We can still work out the relevant dimensionless groups even if we don’t have a model for the system
  - How many things do we need to vary to study the system?
  - What are the dimensionless quantities that should be changed?

Number of Parameters

- The first question we can ask is how many parameters are required to describe the system?

- To do this we need to know how many variables are required to describe the system
  - E.g. length scale, viscosity, density, velocity, diffusivity etc.
- …as well as how many base dimensions these variables involve
  - E.g. distance, time, mass etc.

- **Because any resultant model must be dimensionally consistent, the number of dimensionless groups required to specify a system is equal to the number of variables minus the number of base dimensions**

Note on Base Dimensions

To determine the base dimensions to use, find the minimum number required for the variables in the system

- E.g. Energy, force and pressure can all be broken down into dimensions of mass, length and time in different proportions.
- You must not break down a dimension if breaking it down would introduce more dimensions than are eliminated
  - Might be the case if the dimension (e.g. force) appears in more than one variable and one or more of its internal dimensions do not appear in any other variables in the system (e.g. no other variables involve mass or time)

How do we determine the dimensionless groups? Buckingham Pi Theory

- The first thing to do is to identify the variables involved in a system and their associated dimensions
- As an example I will use a fluid flowing down a pipe under the influence of gravity
  - Relevant variables: Density (ρ), viscosity (μ), pipe diameter (d), velocity (v) and gravity (g)
  - Dimensions: Length (L), mass (M) and time (T)

$$\rho = \frac{M}{L^3} \qquad \mu = \frac{M}{L\,T} \qquad d = L \qquad v = \frac{L}{T} \qquad g = \frac{L}{T^2}$$

- We can already determine the number of dimensionless groups required to describe this system
  - There are 5 variables and 3 base dimensions
  - The number of dimensionless groups is the number of variables minus the number of base dimensions
  - This system requires 2 dimensionless groups to describe it
- How do we now find these dimensionless groups?

Buckingham Pi Theory - Continued

- We can write this problem as a matrix of variables and their associated dimensions:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \rho & \mu & d & v & g \end{array} \\
\begin{array}{c} M \\ L \\ T \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -3 & -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 \end{pmatrix} = A
\end{array}
$$

Buckingham Pi Theory - Continued

- We now need to find combinations of the variables that result in the total of each of the dimensions in the combination being zero (i.e. the combination is dimensionless)
  - We already know that for this problem there can be two such independent combinations
- Mathematically this can be expressed as follows, where $\mathbf{x}$ is a vector containing the number of each of the variables that need to be combined and $\mathbf{0}$ is a zero vector:

$$\mathrm{A}\,\mathbf{x} = \mathbf{0}$$

Buckingham Pi Theory - Continued

$$A\,\mathbf{x} = \mathbf{0}$$

- Because A is not square we can't simply invert (and if it was square and invertible, the only solution would be the trivial one of a dimensionless group containing no variables!)

- For a small A we could quite easily find valid $\mathbf{x}$s by inspection
  - The system is underspecified and so there are actually infinitely many solutions

- I will show a more rigorous way to achieve it

Buckingham Pi Theory - Continued

- Let us split A into a square portion and a remaining portion:

$$\mathrm{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -3 & -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 \end{pmatrix} \qquad \mathrm{A}_1 = \begin{pmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathrm{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & -2 \end{pmatrix}$$

- We must similarly split $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} x_\rho \\ x_\mu \\ x_d \\ x_v \\ x_g \end{pmatrix} \qquad \mathbf{x_1} = \begin{pmatrix} x_\rho \\ x_\mu \\ x_d \end{pmatrix} \qquad \mathbf{x_2} = \begin{pmatrix} x_v \\ x_g \end{pmatrix}$$

Buckingham Pi Theory - Continued

- We can now express the problem as follows:

$$\mathrm{A}_1\mathbf{x_1} + \mathrm{A}_2\mathbf{x_2} = \mathbf{0} \longrightarrow \mathrm{A}_1\mathbf{x_1} = -\mathrm{A}_2\mathbf{x_2}$$

- Or in a solvable form:

$$\mathbf{x_1} = -\mathrm{A}_1{}^{-1}\mathrm{A}_2\mathbf{x_2}$$

Note that if A_1 is singular (i.e. there is no valid $A_1{}^{-1}$) then you need to split the original A into a different a A_1 and A_2

- Our choice of which variables are specified and which are calculated is essentially arbitrary

Buckingham Pi Theory - Continued

- While 2 independent dimensionless groups can be produced for this problem, there is more than one way to achieve this
  - Each set of values for $\mathbf{x_2}$ produces a different set of values for $\mathbf{x_1}$
  - We can choose any two independent $\mathbf{x_2}$ vectors to form our dimensionless groups
- The easiest thing to do ensure that the dimensionless groups are independent is to have a single variable specified for each vector
  - Two independent dimensionless groups would result from having (1, 0) and (0, 1) as two vectors for $\mathbf{x_2}$
    - We know that they are independent because they each contain a variable that the other one does not contain

Buckingham Pi Theory - Continued

- We can obtain the following inverse

$$\mathrm{A}_1{}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

- Using the equations from the previous slides this results in

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 2 \\ -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_\rho \\ x_\mu \\ x_d \\ x_v \\ x_g \end{pmatrix}$$

- Which represent the following two dimensionless groups

$$N_1 = \frac{\rho d v}{\mu} \quad \text{and} \quad N_2 = \frac{\rho^2 g d^3}{\mu^2}$$

Reynolds Number

$$Re = \frac{\rho d v}{\mu}$$

- The Reynolds number is a ubiquitous dimensionless group in fluid dynamics
  - Much more on it in a later lecture
  - Important use in predicting the onset of turbulence
- It represents a balance between inertial and viscous forces

$$Re \propto \frac{Inertial\ Force}{Viscous\ Force} \propto \frac{\rho v \frac{dv}{dx}}{\mu \frac{d^2 v}{dx^2}} \propto \frac{\rho v \frac{v}{d}}{\mu \frac{v}{d^2}} = \frac{\rho d v}{\mu}$$

You should be able to recognise the inertial and viscous terms from the Navier-Stokes equation

Galileo Number

$$Ga = \frac{\rho^2 g d^3}{\mu^2}$$

- Represents the balance between the gravitational, inertial and viscous forces
  - Will appear in problems with gravitationally driven flows
- It is proportional to the inertial force times the gravitational force divided by the viscous force squared

$$Ga \propto \frac{Inertial\ Force\ \times Gravity\ Force}{Viscous\ Force^2} \propto \frac{\rho v \frac{dv}{dx} \rho g}{\left(\mu \frac{d^2 v}{dx^2}\right)^2} \propto \frac{\rho v \frac{v}{d} \rho g}{\left(\mu \frac{v}{d^2}\right)^2} = \frac{\rho^2 g d^3}{\mu^2}$$

Other Dimensionless Combinations

- Note that Buckingham Pi analysis allows you to identify how many dimensionless groups are required to specify the system and allows you to identify a set of suitable numbers
- …, but these numbers are not unique and other valid combinations are possible
- If the dimensionless groups found are N_1, N_2 … N_i then a new valid dimensionless can be created:

$$N_{new} = {N_1}^{n1}{N_2}^{n2} \dots {N_i}^{ni}$$

- The main restriction on doing this is that the new set of dimensionless numbers must contain all the same variables as the original set
  - I.e. you must not use the above relationship to eliminate a variable from the set of dimensionless groups

Other Dimensionless Combinations
Our Problem

- In our case, for instance we could divide the Galileo number by the Reynolds
  - This results in a new dimensionless group that represents the ratio of the gravitational to the viscous force (if this dimensionless group has a name, I don't know it!)

$$Re = \frac{\rho d v}{\mu} \qquad N_2 = \frac{\rho g d^2}{\mu\, v}$$

- Alternatively we can get another set that both have names: Divide the Reynolds number squared by the Galileo number
  - This results in the Froude number – Ratio of inertial to gravity forces

$$Re = \frac{\rho d v}{\mu} \qquad Fr = \frac{v^2}{g d}$$

Choosing the Appropriate Numbers

- As all of these are valid combinations the appropriate ones to use are subjective, but should be based on an understanding of the system

- Use dimensionless groups that represent the main interactions in the system

- Often useful to try and keep the dimensionless groups associated with the main input and output variables in dimensionless groups of their own
  - This will allow us to study the dimensionless outputs as functions of the dimensionless inputs

Example

- We wish to analyse a droplet of water emerging from a needle and, in particular, its average diameter, d_d, when it detaches from a needle of diameter d_n
  - Initially we can assume that the droplets are being formed slowly:

    We therefore only need to consider the variables associated with the static shape:
    - Surface tension, γ
    - Gravity, g
    - Water density, ρ
- How many dimensionless groups are required to satisfy this system?
  - Derive suitable ones
- What additional variables may be required in a dynamic system?
  - What additional dimensionless groups may this require?

Calculations

Other Non-Dimensional Quantities

- Dimensional analysis gives a method for determining appropriate combinations of dimensional quantities
- Remember that any quantities that are already dimensionless need to also be included in any analysis
  - Exponents
  - Shape
    - For simple shapes there may be dimensionless groups made of dimensional variables that can specify the problem (e.g. aspect ratios)
    - More complex shapes may not be as simply quantified

Instabilities

- Instabilities can be tricky to model
  - Inherently time dependent
  - Can be triggered by small fluctuations
    - Numerical errors may trigger instabilities more easily than they really occur
    - … or models can be more stable than the real world and instabilities are not triggered

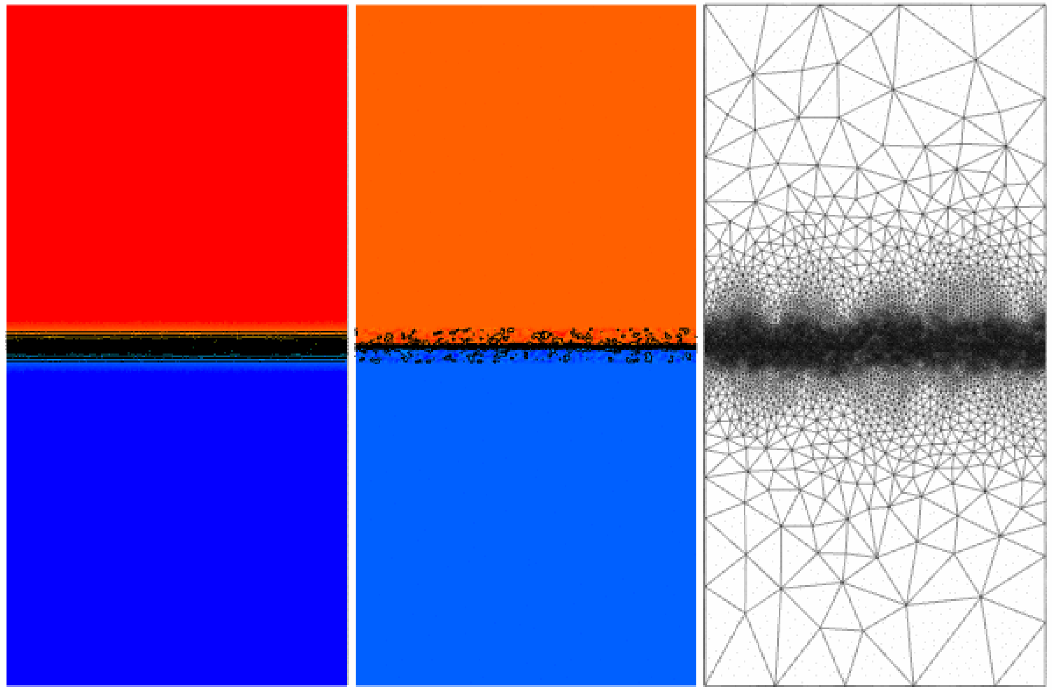
- Dimensional analysis useful for investigating and quantifying instabilities
  - Both the conditions for the onset and the subsequent behaviour

- An important instability is that associated with the transition from laminar to turbulent flow
  - Characterised by the Reynolds number
  - We will not be looking at this instability here as we will be doing more on this transition in the Fluid Mechanics lectures

Rayleigh-Taylor Instability

- A dense fluid over a less dense one
- Inherently unstable unless there are interfacial effects
  - Surface tension or diffusion can potentially stop the instability
- Density, gravity and viscosity can all impact the size of the instabilities and how fast they grows

Simulation of Rayleigh-Taylor instability with miscible fluids using Fluidity



Dimensional Analysis - Rayleigh-Taylor

- To reduce the size of our analysis lets assume that both fluids have the same kinematic viscosity, υ
  - $\upsilon = \frac{\mu}{\rho}$(units: m^2/s)
  - If the viscosities were different then the ratio of the viscosities could appear as one of the dimensionless groups
- There are two densities ρ_h and ρ_l
- Gravity, g, is the other physical property of the system

- We want to investigate two parameters:
  - λ is the wavelength of the instability
    - We could equivalently have used the wavenumber of the instability, which is proportional to the inverse of the wavelength
  - t is the timescale of the instability
    - E.g. how long the instability takes to grow to a given size

Dimensional Analysis - Rayleigh-Taylor

- We have 6 variables (v, g, ρ_h, ρ_l, λ and t) and 3 dimensions (M, L and T)

- This implies that 3 dimensionless groups are required to define the system
  - We want two of them to involve λ and t – keep what we want to predict in separate groups
  - There will then be a 3rd independent group

$$
\begin{array}{cc}
 & \begin{array}{cccccc} v & g & \rho_h & \rho_l & \lambda & t \end{array} \\
\begin{array}{c} M \\ L \\ T \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & -3 & -3 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Dimensional Analysis - Rayleigh-Taylor

- We can solve using the x associated with either ρ_l, λ or t as one and the others zero for each of the variables

- This will result in the following 3 dimensionless groups:

$$\lambda^* = \frac{\lambda\, g^{1/3}}{\nu^{2/3}} \qquad t^* = \frac{t\, g^{2/3}}{\nu^{1/3}} \qquad N_\rho = \frac{\rho_h}{\rho_l}$$

- For the density contrast it is actually more usual to use the Atwood number:

$$A = \frac{\rho_h - \rho_l}{\rho_h + \rho_l}$$

…but this is just a function of the above dimensionless group anyway

$$A = \frac{N_\rho - 1}{N_\rho + 1}$$

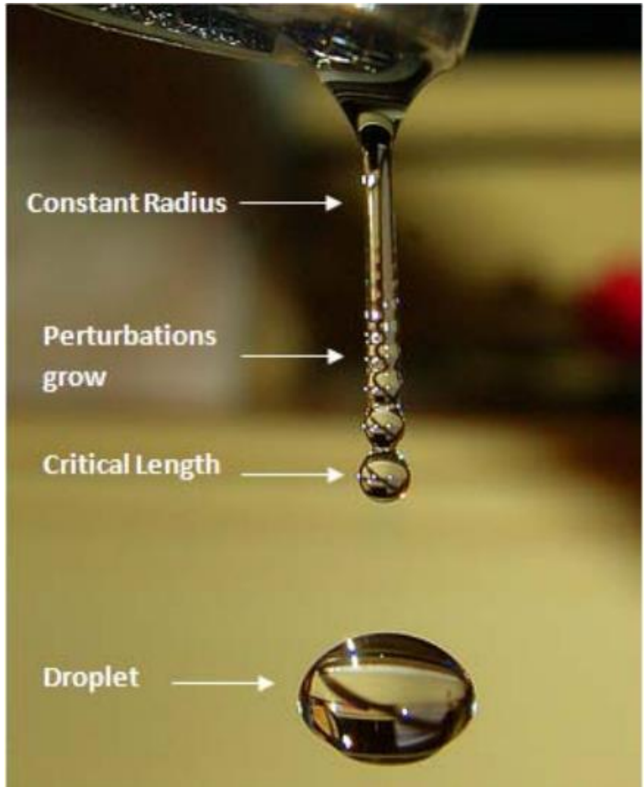
Dimensional Analysis - Rayleigh-Taylor

$$\lambda^* = \frac{\lambda\, g^{1/3}}{\nu^{2/3}} \qquad t^* = \frac{t\, g^{2/3}}{\nu^{1/3}} \qquad A = \frac{\rho_h - \rho_l}{\rho_h + \rho_l}$$

- Having done this analysis we can say, for instance, that the dimensionless wavelength of the instability at a given dimensionless time should be a function of the Atwood number only (assuming the same starting geometry)
- This, of course, assumes that we were correct in our identification of the important variables in the system!

Plateau-Rayleigh Instability

- Confusingly there are two common instabilities named after Lord Rayleigh
  - The other person associated with this instability is Joseph Plateau, who was a Belgian physicist who did a lot of work on foam, much of it after going blind
- The Plateau-Rayleigh instability is what causes a stream of liquid to break into droplets
- This phenomena is used in ink-jet printing



Plateau-Rayleigh Instability

- Surface Tension will try to break the stream into droplets
  - Surface area can be reduced by forming a neck and, ultimately, breaking up the stream into spheres
- Momentum of the fluid counteracts this tendency
  - This will depend on the velocity of the liquid in the jet and its density
- Note that viscosity plays a role, but it is a secondary one for fast flowing jets
- The perturbations will grow with time
- … with a size which depends on the parameters of the system
  - Can control droplet size in an ink-jet printer for instance

Dimensional Analysis
Plateau-Rayleigh

- There are a number of parameters in this system:
  - Density of the fluid, ρ
  - Velocity of the fluid, v
  - Surface tension, γ
  - Initial radius of the stream (or nozzle in the printing context), r

- There are two important variables that need to be investigated:
  - The size of the instabilities (related to the size of the droplets formed), l_{crit}
  - The time taken for the instability to grow, t_{crit}

Exercise

Dimensional Analysis Plateau-Rayleigh

- We have 6 variables ($\rho, v, \gamma, r, l_{crit}$ and t_{crit})
- 3 dimensions (M, L and T)

- There are therefore 3 dimensionless groups
- Show that a suitable set are:

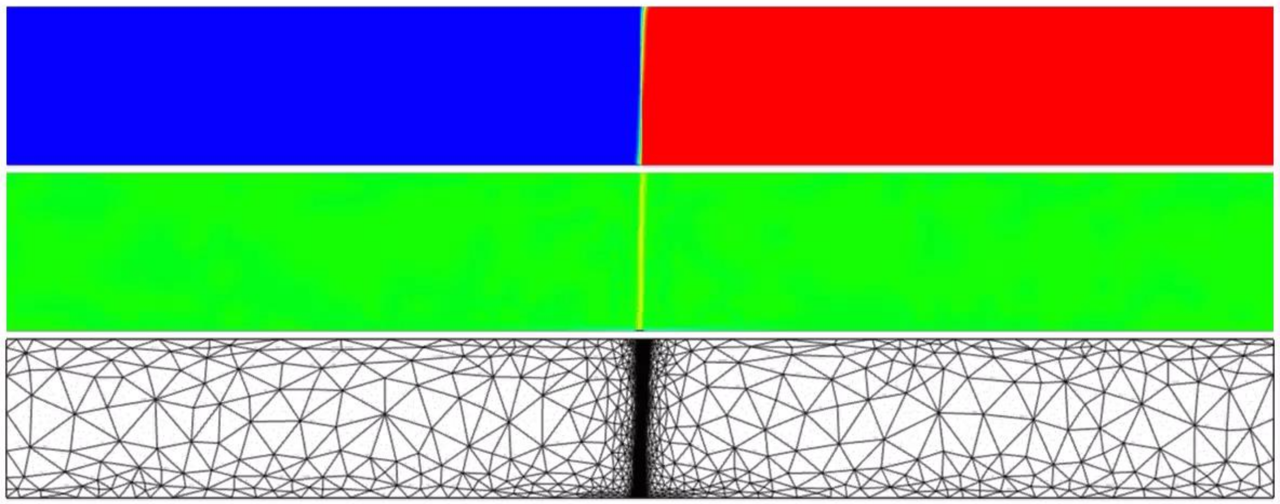
$$l^* = \frac{l_{crit}}{r} \qquad t^* = t_{crit}\sqrt{\frac{\gamma}{\rho r^3}} \qquad N_2 = \frac{v^2 \rho r}{\gamma}$$

- If you wish to include viscosity, μ, what would be an appropriate additional dimensionless group?
  - Hint: It is mainly the inertia that the viscosity will be acting against

Kelvin-Helmholtz Instabilities

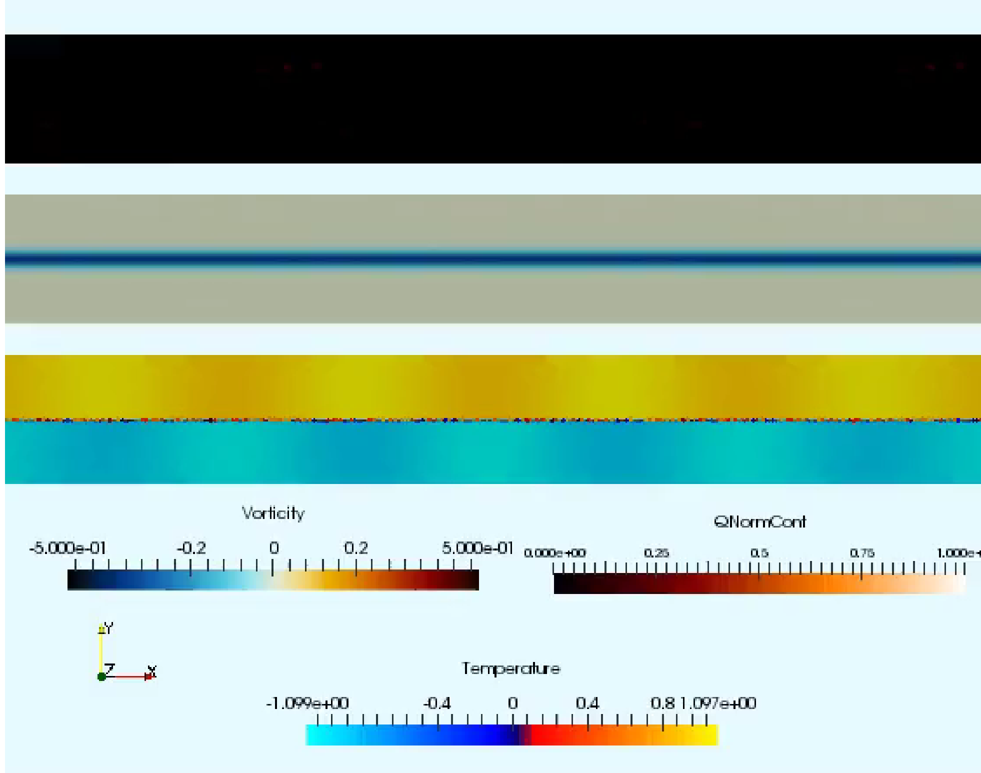
- Occur when two different fluids of different densities flow over one another with an interface between them or when different regions of a single density stratified fluid are experiencing shearing flow
  - This is different to the Rayleigh-Taylor instability in that the fluids are initially stably stratified – dense fluid is below the light fluid

Simulation with Kelvin-Helmholtz instabilities
(lock exchange problem)



Kelvin-Helmholtz instability seen in clouds

Dimensional Analysis
Kelvin-Helmholtz



- We have the following physical variables:
  - Density of the two fluids - ρ_h and ρ_l
  - Gravity -g
  - Difference in velocity between the fluids - Δv
  - Viscosity - μ
    - Assume both fluids have the same viscosity
- We wish to investigate the size of the instability generated - h
- Derive a set of dimensionless groups to characterise this problem
  - Ensure that one of them represents a dimensionless height - h^*

Calculations

Some Commonly Used Dimensionless Groups

Name	Formula	Ratio
Reynolds Number	$Re = \frac{\rho d v}{\mu}$	Ratio of fluid inertial and viscous forces
Bond Number	$Bo = \frac{\rho g d^2}{\gamma}$	Ratio of gravity to capillary forces
Froude Number	$Fr = \frac{v}{\sqrt{gd}}$	Ratio of inertia to gravity (actually the sqrt of the ratio of the forces). Important for fluid waves
Nusselt Number	$Nu = \frac{hd}{k}$	Ratio of convective to conductive heat transfer
Prandtl Number	$Pr = \frac{C_{PB}\mu}{k}$	Ratio of viscous to thermal diffusion
Peclet Number	$Pe = \frac{dv}{D}$	Ratio of advection to diffusion
Schmidt Number	$Sc = \frac{\mu}{\rho D}$	Ratio of viscous to molecular diffusion
Laplace Number	$La = \frac{\gamma \rho d}{\mu^2}$	Ratio of surface tension to momentum dissipation

This is just a small selection of dimensionless numbers that mainly reflects my research interests – All fields of Physics and Engineering will have their own collection of important dimensionless numbers