Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
2. Dynamics – equations that describe force balance, conservation of linear and angular momentum
3. Thermodynamics – relations temperature, heatflux, stress, entropy

Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Thermal parameters

Can you name 4 material parameters that affect temperatures or how material responds to changes in temperature

Each of these may depend on T, P, phase, composition,…

Thermal parameters

k - thermal conductivity (W/m/K)

A - heat production (W/m^3)

$\mathbf{C_P}$ - heat capacity (specific heat) at constant pressure (J/kg/K)

$\boldsymbol{\alpha}$ - thermal expansion coefficient (1/K)

$$\alpha = (1/V)[\partial V/\partial T]_P = (1/\rho)[\partial \rho/\partial T]_P$$

$\boldsymbol{\kappa}$ - thermal diffusivity $k/\rho/C_P$ (m^2/s)

Each of these may depend on T, P, phase, composition,…

Learning Objectives

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.
- Understand basic properties of elastic and viscous rheology and understand how the choice of rheology leads to different forms of the momentum conservation equation
- Using tensor analysis to obtain relations between the main isotropic elastic parameters

Rheology

$$\boxed{\text{deformation } (\boldsymbol{\varepsilon}) = \textit{rheology} \cdot \text{stress } (\boldsymbol{\sigma})}$$

material response to stress, depends on material, P,T, time, deformation history, environment (volatiles, water)

- *elastic*
- *viscous*
- *brittle*
- *plastic*

• experiments under simple stress conditions

$\Rightarrow$ strain evolution under constant stress, stress-strain rate diagrams

• thermodynamics + experimental parameters

• ab-initio calculations

Recap Fluid - Solid

- What is a solid?

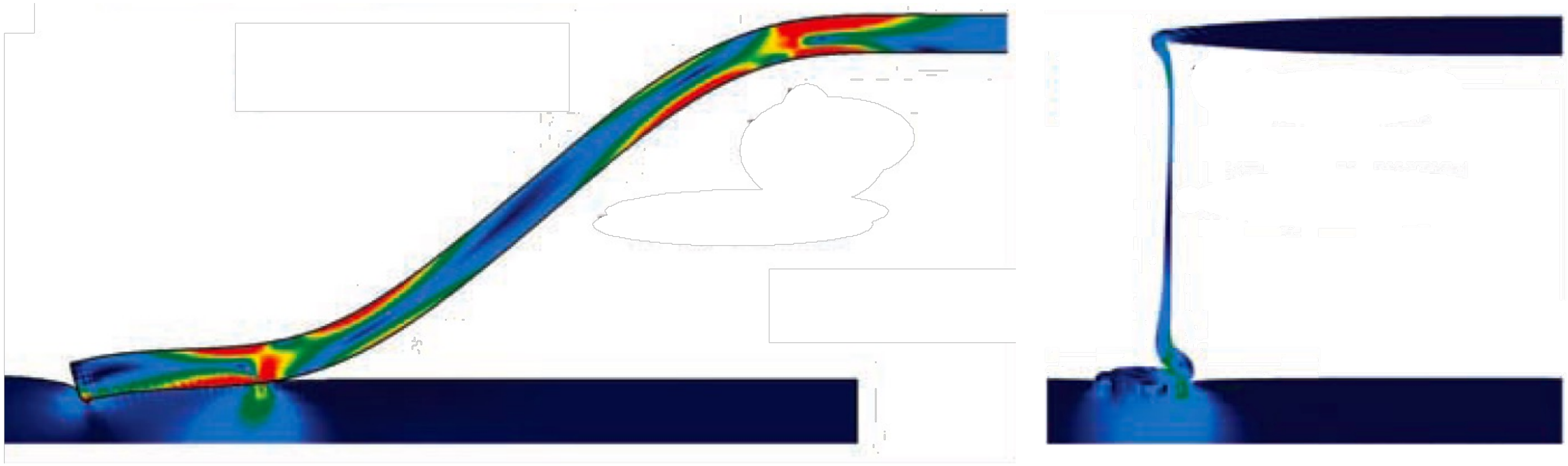
  A solid acquires finite deformation under stress

  stress σ ~ *strain* ε

- What is a fluid?

  A material that flows in response to applied stress

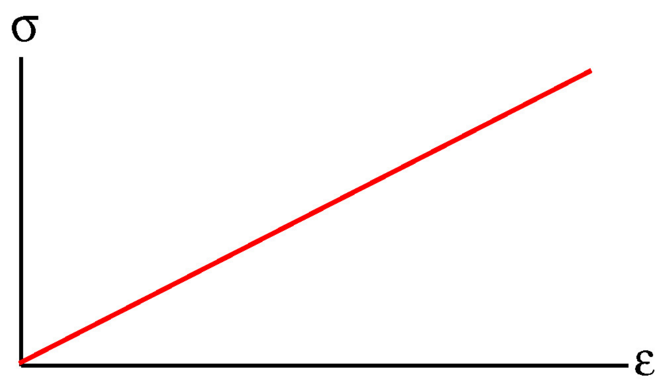
  stress σ ~ *strain rate* D

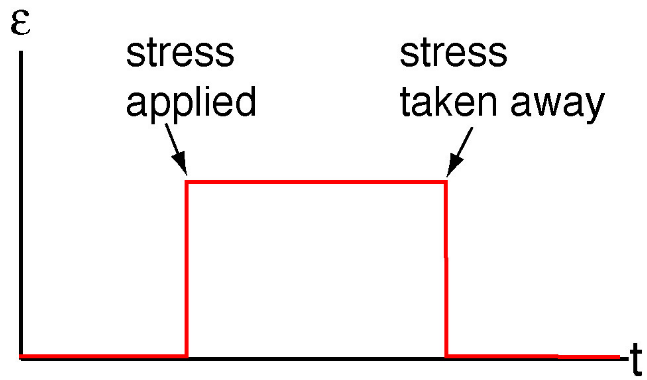


Figures from Funiciello et al. (2003a)

Elasticity

- linear response to load applied
- instantaneous
- completely recoverable
- below threshold (yielding) stress
- *dominates behaviour of coldest part of tectonic plates on time scales of up to 100 m.y. => fault loading*
- *on time scale of seismic waves, the whole Earth is elastic*





- $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ - Hooke's law
  C_{ijkl} - rank 4 elasticity tensor
  3^4 elements, up to 21 independent

Elasticity tensor

C_{ijkl} $3^4=81$ elements (for n=3)

- symmetry of σ_{ij} and ε_{kl}
  $\Rightarrow$ only 36 independent elements

$P=\boldsymbol{\sigma}:\mathbf{D}\approx \boldsymbol{\sigma}:D\varepsilon/Dt=DU/Dt$

- conservation of elastic energy $U=\boldsymbol{\sigma}:\varepsilon=\mathbf{C}:\varepsilon:\varepsilon \geq 0$

$\Rightarrow C_{ijkl}=C_{klij}$

$\Rightarrow$ only 21 independent elements - most general form of **C**

- other symmetries further reduce the number of independent elements

Elasticity tensor

For example, for *isotropic* media
only 2 independent elements (λ, μ):

What is isotropic?

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl}+\alpha\delta_{ik}\delta_{jl}\varepsilon_{kl}+\beta\delta_{il}\delta_{jk}\varepsilon_{kl}$$
$$= \lambda\delta_{ij}\varepsilon_{kk}+\alpha\varepsilon_{ij}+\beta\varepsilon_{ji}$$
$$= \lambda\delta_{ij}\theta+(\alpha+\beta)\varepsilon_{ij}$$

3 isotropic rank 4 tensors:
$\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$

$$\Rightarrow \sigma_{ij}= \lambda\theta\delta_{ij}+2\mu\varepsilon_{ij}$$

Hooke's law for isotropic material: 2 independent coefficients

Lamé constants

λ and **μ** : $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Bulk and shear modulus

K and **G** : $-p = K\,\theta$ *isotropic*

$\sigma'_{ij} = G\,\varepsilon'_{ij}$ *deviatoric*

$-p = \sigma_{kk}/3$

$\theta = \varepsilon_{kk}$

Determine relation to Lamé constants in Exercise 5

Young's modulus and Poisson's ratio

E and **ν** : $E = \sigma_{11}/\varepsilon_{11}$, $\nu = -\varepsilon_{33}/\varepsilon_{11}$ (uniaxial stress)

Determine in *optional Exercise 6*

Wave equation

For infinitesimal deformation:
spatial coordinates ≈ material coordinates

v_i (spatial) $\approx \partial u_i/\partial t$
a_i (spatial) $\approx \partial v_i/\partial t = \partial^2 u_i/\partial t^2$

Equation of motion: $\quad f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2 \quad (1)$

Elastic rheology: $\quad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad (2)$

Substitute (2) in (1) if (infinitesimal) deformation is consequence of force balance

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

$\nabla \cdot \sigma$ = Write vector equation (*see notebook)*

$$\frac{\partial u_k}{\partial x_k} = \frac{\partial u_j}{\partial x_j} = \nabla \cdot \mathbf{u}$$

$$\frac{\partial^2}{\partial x_j^2} = \nabla^2$$

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

$$\nabla\cdot\sigma = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$$

Using: $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u}$

=> $\boxed{\rho\partial^2\mathbf{u}/\partial t^2 = \mathbf{f} + (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times\nabla\times\mathbf{u}}$

what type of deformation do the two terms represent?

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

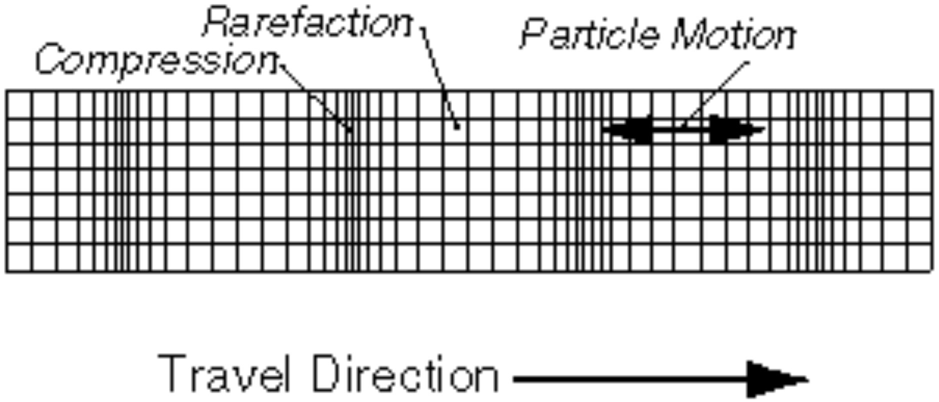
$$\nabla \cdot \sigma = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u}$$

Using: $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$

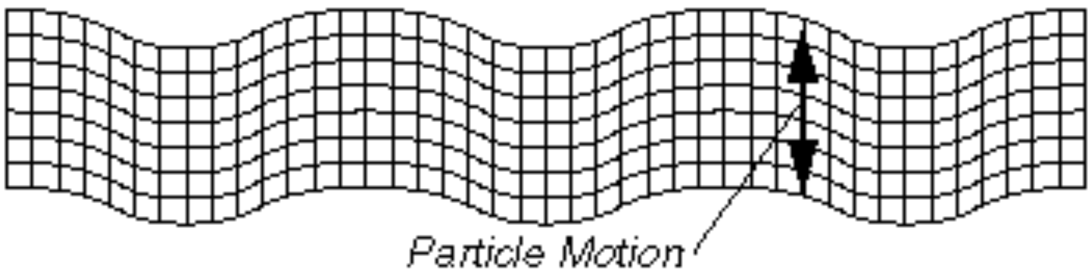
=> $$\boxed{\rho\partial^2\mathbf{u}/\partial t^2 = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}}$$

compressional *shear*

P wave


Rarefaction
Particle Motion
Compression
Travel Direction

S wave


Particle Motion

Recap Fluid - Solid

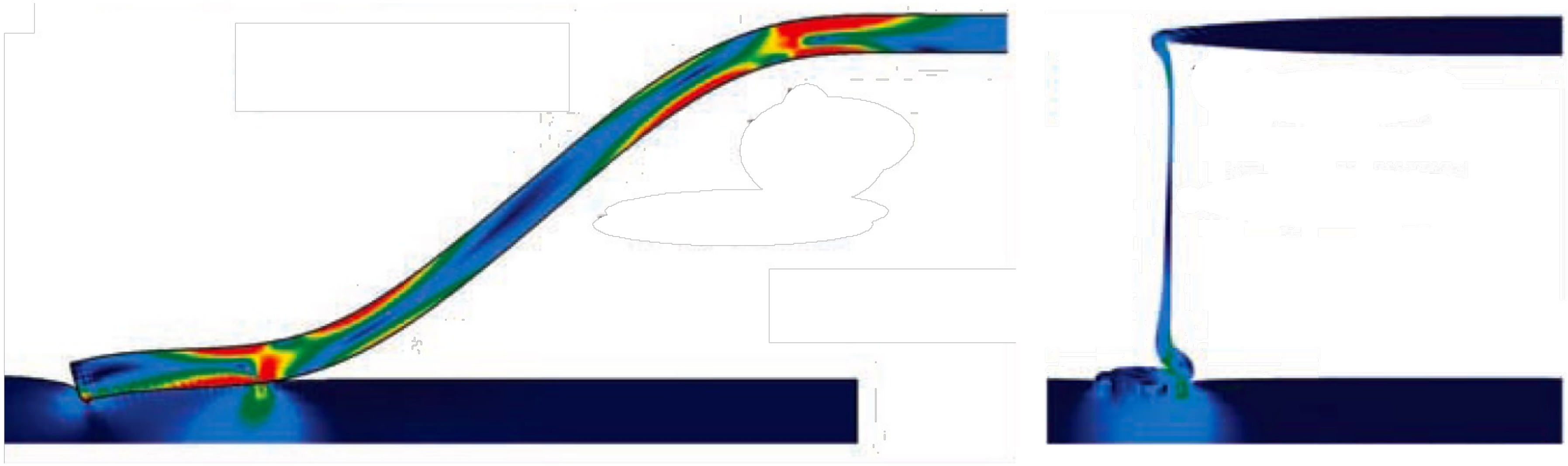
- What is a solid?

  A solid acquires finite deformation under stress

  stress σ ~ *strain* ε

- What is a fluid?

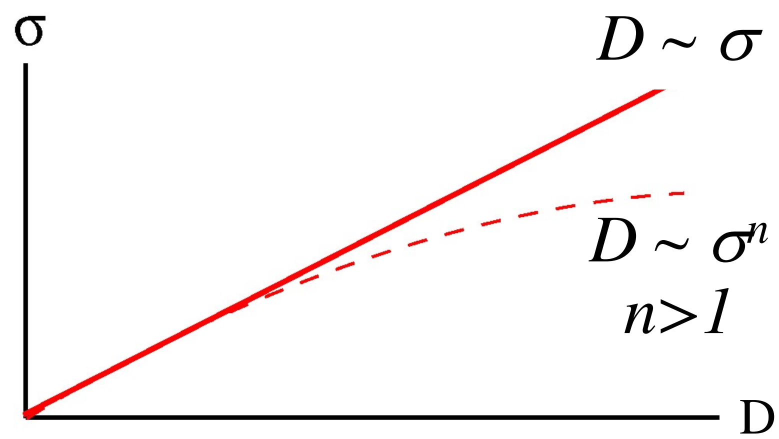
  A material that flows in response to applied stress *stress* σ ~ *strain rate* D

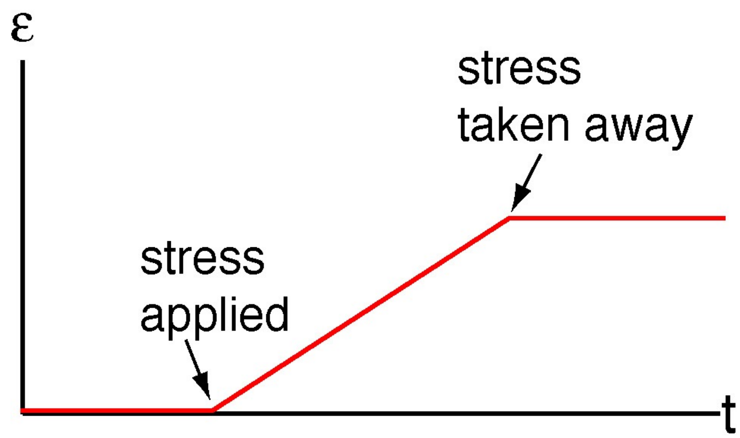


Figures from Funiciello et al. (2003a)

Viscous Flow

- steady state flow at constant stress
- permanent deformation
- linear (Newtonian) or non-linear (e.g., Powerlaw) relation between strain rate and stress
- isotropic stress does not cause flow
- *on timescales > years base tectonic plates and mantle deform predominantly viscously -> plate motions, postseismic deformation, but also glaciers, magmas*





Hydrostatics

Fluids can not support shear stresses

i.e. if in rest/rigid body motion: $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$

and this normal stress is the same on any plane: $\boldsymbol{\sigma} = -p\mathbf{I}$

p is *hydrostatic pressure*

In force balance: $\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0$

$$-\nabla p = -\mathbf{f}$$

In gravity field $\dfrac{\partial p}{\partial z} = \rho g \quad \Rightarrow \quad p_2 - p_1 = \rho g h$

where $h = z_2 - z_1$

Newtonian Fluids

In general motion: $\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}'$

In Newtonian fluids,
deviatoric stress varies ***linearly*** with ***strain rate***, $\mathbf{D}$
$D_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$

For *isotropic*, Newtonian fluids, 2 *material parameters*:

Viscous stress tensor $\sigma'_{ij} = \zeta D_{kk}\delta_{ij} + 2\eta D_{ij}$

where ζ is ***bulk viscosity*** and η **(*shear*) *viscosity***, $\Delta = D_{kk} = \nabla \cdot \mathbf{v}$

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

p not always mean normal stress: $\sigma_{kk} = -3p + (3\zeta + 2\eta)D_{kk}$

Consider a Newtonian shear flow with velocity field $v_1(x_2)$, $v_2=v_3=0$

What is $\mathbf{D}$? What is $\boldsymbol{\sigma}$?

Exercise 7

Illustrates that η represents resistance to shearing

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

Show that: $$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Assuming constant η

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

Show that: $$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Assuming constant η

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \cancel{\frac{\partial^2 v_j}{\partial x_i \partial x_j}}\right)$$

Because $$\frac{\partial v_j}{\partial x_j} = \Delta = 0$$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

Navier Stokes equation of motion:

$$-\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + \mathbf{v} \cdot \nabla\, \mathbf{v} \right]$$

Assuming constant η

Together with continuity, 4 equations, 4 unknowns (p, v_x, v_y, v_z)

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes for compressible Newtonian Flow

For compressible fluids: $\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

$$\sigma_{ij} = -p\delta_{ij} + \zeta \frac{\partial v_k}{\partial x_k}\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Assuming constant η, ζ

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \zeta \frac{\partial^2 v_j}{\partial x_i \partial x_j} + \eta\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}\right)$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \left(\zeta + \eta\right)\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \left(\zeta + \eta\right)\nabla\left(\nabla \cdot \mathbf{v}\right) + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for compressible Newtonian Flow

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

$$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

Navier Stokes equation of motion:

$$-\nabla p + \left(\zeta + \eta\right)\nabla\Delta + \eta\nabla^2\mathbf{v} + \mathbf{f} = \rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\,\mathbf{v}\ \cdot\ \mathbf{v}\right]$$

Assuming constant ζ, η

\+ Conservation of mass:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0$$

6 equations
6 unknowns
$(p, v_x, v_y, v_z, \rho, T)$

\+ Energy equation

\+ Equation of state for ρ(T,p)

Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
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Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Learning Objectives

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.
- Understand basic properties of elastic and viscous rheology and understand how the choice of rheology leads to different forms of the momentum conservation equation
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Outline

- Conservation equations
- Energy equation
- Rheology
- Elasticity and Wave Equation
- Newtonian Viscosity and Navier Stokes

More reading on the topics covered in this lecture can be found in, for example: Lai et al. Ch 4.14-4-16, 6.18, Ch 5.1-5.6, Ch 6.1-6.7; Reddy parts of Ch 5 & Ch 6

Try yourself

For this part of the lecture, first try **Exercise 5** and **7** in *chapter4.ipynb*

Then complete any remaining exercises in *chapter4.ipynb:*

Exercise 1, 2, 3, 4, 5, 7, 8

- Additional practise: ***in the text***
- Advanced practise: **Exercise 6**

Coursework 1

- In-class test, pen-paper, not open book.
- Wednesday 25 January 10:00-11:30
- Will be based on analytical content of all lectures, in particular lectures 1-5
- Understand material covered in lectures and slides, practise class exercises (with answers)
- Study guide released next week

Outline of course

- *Part 1: Analytical background*
  1. Intro vector/tensor calculus (*SG*)
  2. Stress tensor (*SG*)
  3. Kinematics and strain (*SG*)
  4. Conservation equations (*SG*)
  5. Dimensional Analysis (*SN*)
- *Part 2: Numerical techniques (advanced)*
  6. Interpolation and quadrature (*MP*)
  7. Ordinary differential equations (*MP*)
  8. Partial differential equations and finite difference (*MP*)
- *Part 3: Numerical solutions*
  9. Potential flow (*SN*)
  10. Navier-Stokes (*SN*)
  11. Nonlinear rheology and turbulence (*SN*)
  12. Finite Element Method (*MP*)