Exercise 4 – Friendship paradox and Barabási-Albert model

A. Your friends have more friends than you do.

(1) Show that the average number of friends of friends in a (social) network (e.g. the number of next-nearest connections) can be written as:

$$\langle k_{\text{nn}} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} \quad \text{where} \quad \langle k^n \rangle = \frac{\sum_i k^n}{N}.$$

Hint: Show first that the total number of friends of friends reads $\sum_{i=1}^{N} k_i^2$, $N =$ number of nodes.

(2pts)

(2) Calculate an explicit expression for scale-free networks in terms of γ, k_{min} and k_{max}. For the evaluation of the normalization constant integrate from k_{min} to k_{max}.

(2pts)

(3) Demonstrate the existence of the friendship paradox within the Barabsi-Albert model by considering the limit case $\gamma \to 3$ (use the rule of l'Hôpital).

Assess the outcomes in scenarios where super-hubs are present: $k_{\text{max}} \to \infty$.

(3pts)

(4) Examine the differences between your findings from above and the outcomes from a random network with a Poisson degree distribution:

$$p(k) = \frac{e^{-\lambda} \lambda^k}{k!}.$$

Begin with the calculation of the first two moments and their ratio to find the average number of friends of friends.

(2pts)

(5) Take the result from (2) and substitute k_{max} with the natural cutoff* to get an approximate expression for the scaling of $\langle k_{\text{nn}} \rangle$ with the number of nodes N. Consider again the limiting case $\gamma \to 3$.

*See largest expected hub.

(3pts)

B. Barabási-Albert model

Consider the Barabási-Albert model without preferential attachment. The degree dynamics is given by:

$$\frac{d}{dt}k_i(t) = m\Pi(k) \quad \text{with an uniform distribution} \quad \Pi(k) = \frac{1}{N(t)}$$

such that a every time t a new node attaches randomly to any node in the network. For simplicity you can assume that $N = t$.

(1) Calculate the degree dynamics and find an expression for $k_i(t)$. Discuss briefly how it compares to the degree dynamics with preferential attachment.

(3pts)

(2) Calculate the degree distribution p_k following the steps from the continuum theory (first approach from the lecture, not the master equation). (4 pts)