Exercise 4 – Friendship paradox and Barabási-Albert model

A. Your friends have more friends than you do.

(1) Show that the average number of friends of friends in a (social) network (e.g. the number of next-nearest connections) is given by:

$$\langle k_{\mathrm{nn}} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} \quad \text{where} \quad \langle k^n \rangle = \frac{\sum_i k^n}{N}$$

Hint: Show first that the total number of friends of friends reads $\sum_{i=1}^{N} k_i^2$, $N =$ number of nodes.

(2pts)

(2) Calculate an explicit expression for scale-free networks in terms of $\gamma, k_{\min}$ and $k_{\max}$.

(2pts)

(3) Show the presence of the friendship paradox by considering the limit case $\gamma \to 3$ (use the rule of l'Hôpital).

(3pts)

(4) Compare your finding with a random network having a Poissonian degree distribution. Calculate the first two moments and then their ratio in order to obtain the average number of friends of friends.

(2pts)

(5) Take the result from (2) and substitute $k_{\max}$ with the natural cutoff* to get an approximate expression for how $\langle k_{\mathrm{nn}} \rangle$ depends on the number of nodes N. Consider again the limiting case $\gamma \to 3$. Discuss, supported by a diagram, how $\langle k_{\mathrm{nn}} \rangle$ behaves for increasing network size.

*See largest expected hub.

(3pts)

B. Barabási-Albert model

Consider the Barabási-Albert model without preferential attachment. The degree dynamics is given by:

$$\frac{d}{dt}k_i(t) = m\Pi(k) \quad \text{with an uniform distribution} \quad \Pi(k) = \frac{1}{N(t)}$$

such that a every time t a new node attaches randomly to any node in the network. For simplicity you can assume that $N = t$.

(1) Calculate the degree dynamics and find an expression for $k_i(t)$. Discuss briefly how it compares to the degree dynamics with preferential attachment.

(3pts)

(2) Calculate the degree distribution p_k following the steps from the continuum theory (first approach from the lecture, not the master equation). (4 pts)