Computable categoricity relative to a c.e. degree

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Outline

1. Overview on computable categoricity

    Historical overview

2. Computable categoricity relative to a degree

    Focusing on the c.e. degrees

    Main result

    Outline of strategies

Overview on computable categoricity

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Example

Let $L = (A, <_L)$ be a computable linear ordering. Two elements $a, b \in A$ are said to be **adjacent** if $a <_L b$ and there is no $c \in A$ such that $a <_L c <_L b$.

L is c.c. if and only if it has only finitely many pairs of adjacent elements (Remmel [7]).

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Historically, there have been two approaches in exploring the connection between c.c.-ness and relatively c.c.-ness: an **algebraic** perspective and a **model theoretic** perspective.

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- Goncharov, Lempp, and Solomon [5] showed that an ordered abelian group is c.c. if and only if it has finite rank.

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To do the back-and-forth construction, we only need to be able to compute $\leq_L$ and $\leq_{L'}$, and so the isomorphism will be computable in L'.

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Model theory can help answer that question.

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A **formally Σ_1 Scott family** for $\mathcal{A}$ is a c.e. set W of $\exists$-formulas with a fixed finite set of parameters such that

(1) for every $\overline{a} \in \mathcal{A}$, there is a $\varphi(\overline{x}) \in W$ where $\mathcal{A} \models \varphi(\overline{a})$, and
(2) for every $\overline{a}, \overline{b} \in \mathcal{A}$, if $\mathcal{A} \models \varphi(\overline{a})$ and $\mathcal{A} \models \varphi(\overline{b})$, then $(\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})$.

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Corollary

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Let $\mathcal{A}$ be a computable structure. $\mathcal{A}$ is **computably categorical relative to a degree d** if for every **d**-computable copy $\mathcal{B}$ of $\mathcal{A}$, there exists a **d**-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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Question: Are there structures where they are only c.c. relative to certain degrees **d**?

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Fact (Downey, Harrison-Trainor, Melnikov [2])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ^0_1 Scott family.

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This implies that $\mathcal{A}$, as in the statement above, must be c.c. relative to all degrees above $\mathbf{0}''$.

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The contrapositive also gives us that if $\mathcal{A}$ does not have a $\mathbf{0}''$-computable Σ^0_1 Scott family, then it is not c.c. relative to *any* $\mathbf{d} \geq \mathbf{0}''$.

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Question: What happens between $\mathbf{0}$ and $\mathbf{0}''$?

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Theorem (V.)

Let $P = (P, \leq)$ be a countably infinite partially ordered set and suppose we partition P as $P = P_0 \sqcup P_1$. There exists a computable c.c. directed graph $\mathcal{G}$ and an embedding h of P into the c.e. degrees where $\mathcal{G}$ is c.c. relative to each degree in $h(P_0)$ and is not c.c. relative to each degree in $h(P_1)$.

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We also have the following notation for $p \in P$.

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$$\overline{D_p} := \bigoplus_{q \neq p} A_q.$$

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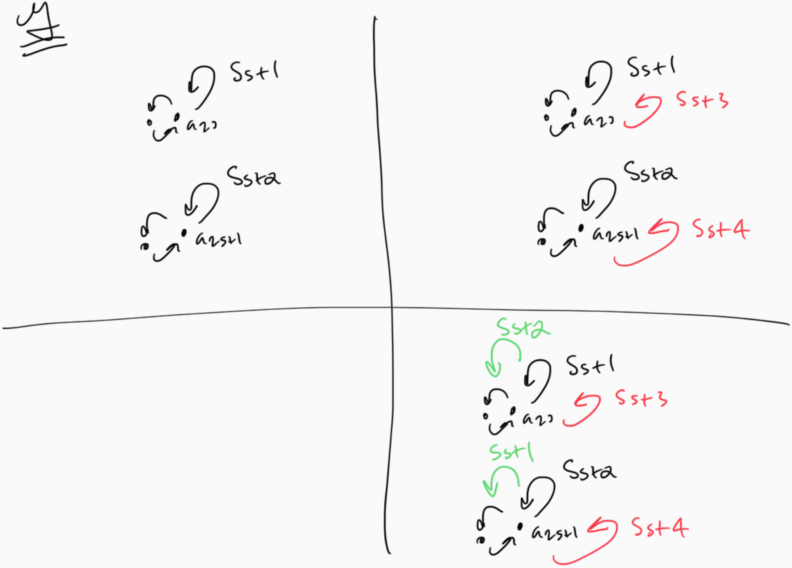
Definition

The root node a_{2s} in our graph $\mathcal{G}$ with its loops is the $2s$**th connected component** or just the $2s$th component of $\mathcal{G}$.

Configuration of loops in $\mathcal{G}$



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Let s be the current stage of the construction and let α be an S_e-strategy.

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2. If n_α is defined and $f_\alpha[s-1]$ is defined for all $m < n_\alpha$, α looks for copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components of $\mathcal{G}[s]$.

3. If no copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components are found, α takes no additional action at stage s, retains the value of n_α, and sets $f_\alpha[s] = f_\alpha[s-1]$.

Basic strategies: S_e

3. If no copies of the $2n_\alpha$th and $(2n_\alpha + 1)$st components are found, α takes no additional action at stage s, retains the value of n_α, and sets $f_\alpha[s] = f_\alpha[s-1]$. If copies are found, α extends $f_\alpha[s-1]$ to $f_\alpha[s]$ by matching the components in $\mathcal{G}[s]$ to the copies found in $\mathcal{M}_e[s]$ and increments n_α by 1.

Basic strategies: T_i^p

Let $p \in P_0$. Our basic strategy to satisfy all T_i^p requirements to make $\mathcal{G}$ c.c. relative to D_p is similar to our S_e-strategy. Let α be a T_i^p-strategy.

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For each n, we try to find copies of the $2n$th and $(2n+1)$st components of $\mathcal{G}$ in $\mathcal{M}_i^{D_p}$.

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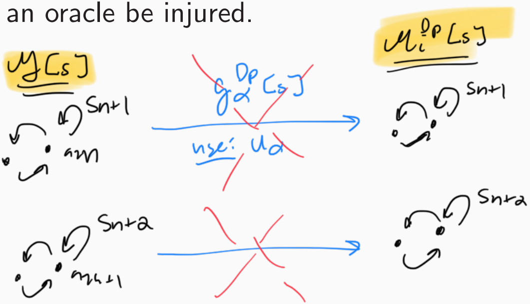
For each n, we try to find copies of the $2n$th and $(2n+1)$st components of $\mathcal{G}$ in $\mathcal{M}_i^{D_p}$. But now because D_p is a c.e. set, loops in $\mathcal{M}_i^{D_p}$ can be injured or embeddings using a finite part of D_p as an oracle be injured.



if at a later stage, a # $x < u_\alpha$ enters D_p, then $g_\alpha^{D_p}[s]$ will disappear on the $2n$th and $(2n+1)$st components of $\mathcal{G}$.

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If $D_p[t] \neq D_p[s]$, then α will update its parameter n_α accordingly depending on what type of injury occurred. Otherwise, it will proceed to try and match the $2n_\alpha$th and $(2n_\alpha + 1)$st components of $\mathcal{G}$ for the n_α parameter it had at the beginning of stage s.

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Finally, for $q \in P_1$, we do the following to satisfy all R_e^q requirements to make $\mathcal{G}$ not c.c. relative to D_q.

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$\underline{\mathcal{B}_q}$

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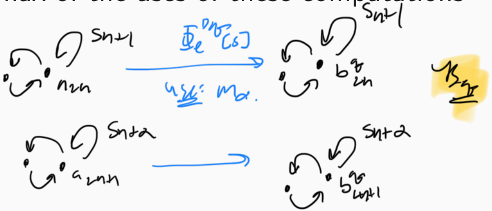
$\mathcal{M}$



Basic strategies: R_e^q

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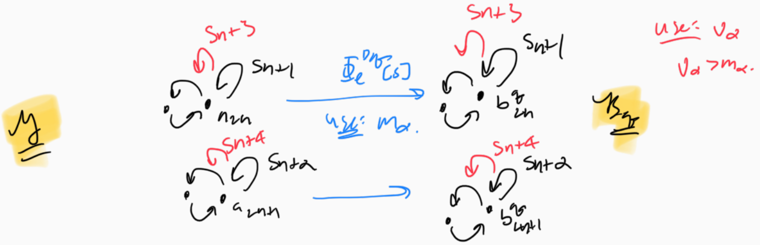
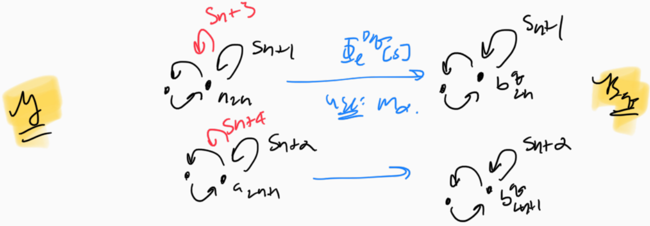
4. α now issues a challenge to all higher priority requirements which are S_e and T^p_i: they must now extend their embeddings, if possible, to include these new loops.



Basic strategies: R_e^q

5. If all higher S_e and T_i^p requirements can meet this challenge and α becomes eligible to act again at a later stage, it enumerates v_α into A_q. This makes the $(5n+3)$- and $(5n+4)$-loops in $\mathcal{B}_q$ disappear.



6. α reattaches a $(5n+3)$-loop to b^q_{2n+1} and a $(5n+4)$-loop to b^q_{2n}. It also attaches a $(5n+1)$-loop to a_{2n+1} and to b^q_{2n+1}, and a $(5n+2)$-loop to a_{2n} and to b^q_{2n}.

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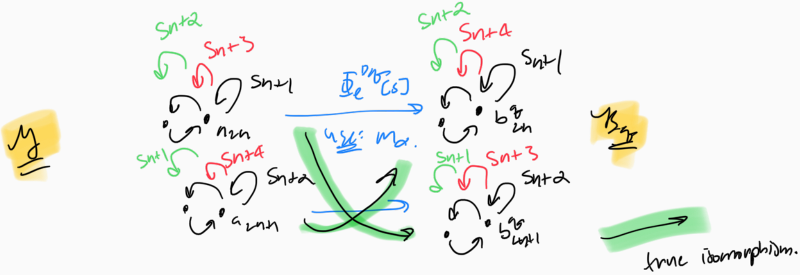
Our final configuration of loops in $\mathcal{B}_q$ is now:



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Interaction 2

The N_e^p-strategy enumerating numbers into A_p to achieve independence of degrees: this is resolved on a tree of strategies and by letting T_i^p check for any changes in D_p up to a finite part each stage.

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An R_e^q-strategy β and a T_i^p-strategy α when $q < p$ in P and T_i^p is of higher priority than R_e^q:

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Interaction 3

An R^q_e-strategy β and a T^p_i-strategy α when $q < p$ in P and T^p_i is of higher priority than R^q_e: the T^p_i-strategy needs an additional step for when it is challenged to enumerate any uses associated to the $2n_\beta$th and $(2n_\beta + 1)$st components of $\mathcal{G}$ into A_p. This lets us lift uses for T^p_i.

Thank You

Thanks for attending my talk! I'd be happy to answer any questions.

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