Computable categoricity relative to a c.e. degree

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Outline

1. Preliminaries

2. Relativizing categoricity

   Relative computable categoricity

   Δ^0_α-computable categoricity

3. Categoricity relative to a degree

   Current work and future directions

Preliminaries

Question

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A computable structure $\mathcal{A}$ is **computably categorical** if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, there exists a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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- Computable ordered groups of finite rank (Gončarov, Lempp, Solomon [GLS03]).

The given conditions in each example are both necessary and sufficient for computable categoricity.

Relativizing categoricity

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Remark

If a computable structure is relatively computably categorical, then it is computably categorical.

The converse is not true in general.

Algebraic characterization of computable categoricity

For a class of structures, if there is a purely algebraic characterization of computable categoricity, then a computably categorical structure $\mathcal{A}$ will often also be relatively computably categorical.

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The connection between an algebraic characterization of computable categoricity and the equivalence of plain and relativized computable categoricity was clarified by the following result.

Theorem (Ash, Knight, Manasse, and Slaman [Ash+89]; Chisholm [Chi90])

A structure is relatively computably categorical if and only if it has a formally Σ_1 Scott family.

Plain but not relative

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Kudinov [Kud96] showed that the assumption of 2-decidability could not be lowered to 1-decidability.

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Definition

A computable structure $\mathcal{A}$ is **relatively** Δ^0_α-**categorical** if for any copy $\mathcal{B}$ of $\mathcal{A}$, there is a $\Delta^0_\alpha(\mathcal{B})$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Plain and relative Δ^0_α-categoricity

There are some examples of classes of structures where plain and relative Δ^0_α-categoricity coincide.

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Similar to the case for computable categoricity and relative computable categoricity, plain and relative Δ^0_α-categoricity need not coincide. There are several examples in a paper by Fokina, Harizanov, and Turetsky [FHT19] (trees of finite and infinite heights, etc.).

Categoricity relative to a degree

A newer relativization

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Definition

For $X \in 2^{\mathbb{N}}$, a computable structure $\mathcal{A}$ is **computably categorical relative to a degree X** if for every X-computable copy $\mathcal{B}$ of $\mathcal{A}$, there is an X-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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Fact

A computable structure $\mathcal{A}$ is **relatively computably categorical** *if for all $X \in 2^{\mathbb{N}}$, $\mathcal{A}$ is computably categorical relative to X.*

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Fact (Downey, Harrison-Trainor, Melnikov [DHTM21])

If $\mathcal{A}$ is a computable structure and it is computably categorical relative to some degree $\mathbf{d} \geq \mathbf{0}''$, then $\mathcal{A}$ has a $\mathbf{0}''$-computable Σ^0_1 Scott family. So, $\mathcal{A}$ is computably categorical relative to all $\mathbf{d} \geq \mathbf{0}''$.

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So at $\mathbf{0}''$ and above, any computable structure $\mathcal{A}$ will settle on whether it is computably categorical relative to all degrees or to none of them.

Question

What happens between $\mathbf{0}$ and $\mathbf{0}''$?

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Theorem (Downey, Harrison-Trainor, Melnikov [DHTM21])

There is a computable structure $\mathcal{A}$ and c.e. degrees $\mathbf{0} = Y_0 <_T X_0 <_T Y_1 <_T X_1 <_T \dots$ such that

(1) *$\mathcal{A}$ is computably categorical relative to Y_i for each i,*

(2) *$\mathcal{A}$ is not computably categorical relative to X_i for each i,*

(3) *$\mathcal{A}$ is computably categorical relative to $\mathbf{0}'$.*

Partial orders of c.e. degrees

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Theorem (V. [Vil24])

Let $P = (P, \leq)$ be a computable partially ordered set and let $P = P_0 \sqcup P_1$ be a computable partition. Then, there exists a computable directed graph $\mathcal{G}$ and an embedding h of P into the c.e. degrees where

(1) *$\mathcal{G}$ is computably categorical;*

(2) *$\mathcal{G}$ is computably categorical relative to each degree in $h(P_0)$; and*

(3) *$\mathcal{G}$ is not computably categorical relative to each degree in $h(P_1)$.*

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(3) making $\mathcal{G}$ computably categorical relative to all degrees in $h(P_0)$; and

(4) making $\mathcal{G}$ not computably categorical relative to any degree in $h(P_1)$.

We use a tree of strategies to organize restraints and parameters.

Future directions: in the generic degrees

Definition

A degree $\mathbf{d}$ is **low for isomorphism** if for every pair of computable structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{d}} \mathcal{B}$ if and only if $\mathcal{A} \cong_{\Delta^0_1} \mathcal{B}$.

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This means that there *cannot* be a computable structure $\mathcal{A}$ which is not computably categorical but is computably categorical relative to $\mathbf{d}$ for a 2-generic degree $\mathbf{d}$.

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Conjecture

There exists a 1-generic G such that there is a computable directed graph $\mathcal{A}$ where $\mathcal{A}$ is not computably categorical but is computably categorical relative to G.

Future directions: identifying pathological behavior in classes of structures

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For structures other than directed graphs, can you produce an example which witnesses the pathological behavior in the poset result?

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There are some results in the literature that give a negative result for certain classes of structures already.

Theorem (Bazhenov [Baz14])

For every degree $\mathbf{d} < \mathbf{0}'$, a computable Boolean algebra is $\mathbf{d}$-computably categorical if and only if it is computably categorical.

Future directions: identifying pathological behavior in classes of structures

Conjecture

For the following classes of structures, there exists a computable example in each class which witnesses the pathological behavior in the poset result: symmetric, irreflexive graphs; partial orderings; lattices; rings with zero-divisors; integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups.

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This is based on the codings given in a paper by Hirschfeldt, Khoussainov, Shore, and Slinko in [Hir+02]. In this paper, they specified codings which satisfied certain conditions and thus preserved several computability theoretic properties of structures, such as the degree spectrum or computable dimension.

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Thanks

Thank you for your attention!

I'd be happy to answer any questions.