Exercise 1

Q.1 →

1) Calculation of $f_x(x)$ (pdf):

Since we know that

$$f_x(x) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\,dy$$

breaking down the above limits we get,

$$f_x(x) = \int_{-\infty}^{0} 0\cdot dy + \int_{0}^{1-x} 2\cdot dy + \int_{+1-x}^{+\infty} 0\cdot dy \quad \text{for } 0 \le x \le 1$$

$$f_x(x) = 2(1-x)$$

$$f_x(x) = \begin{cases} 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

2) Calculation of $f_y(y)$ (pdf):

Again we know that,

$$f_y(y) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\,dx$$

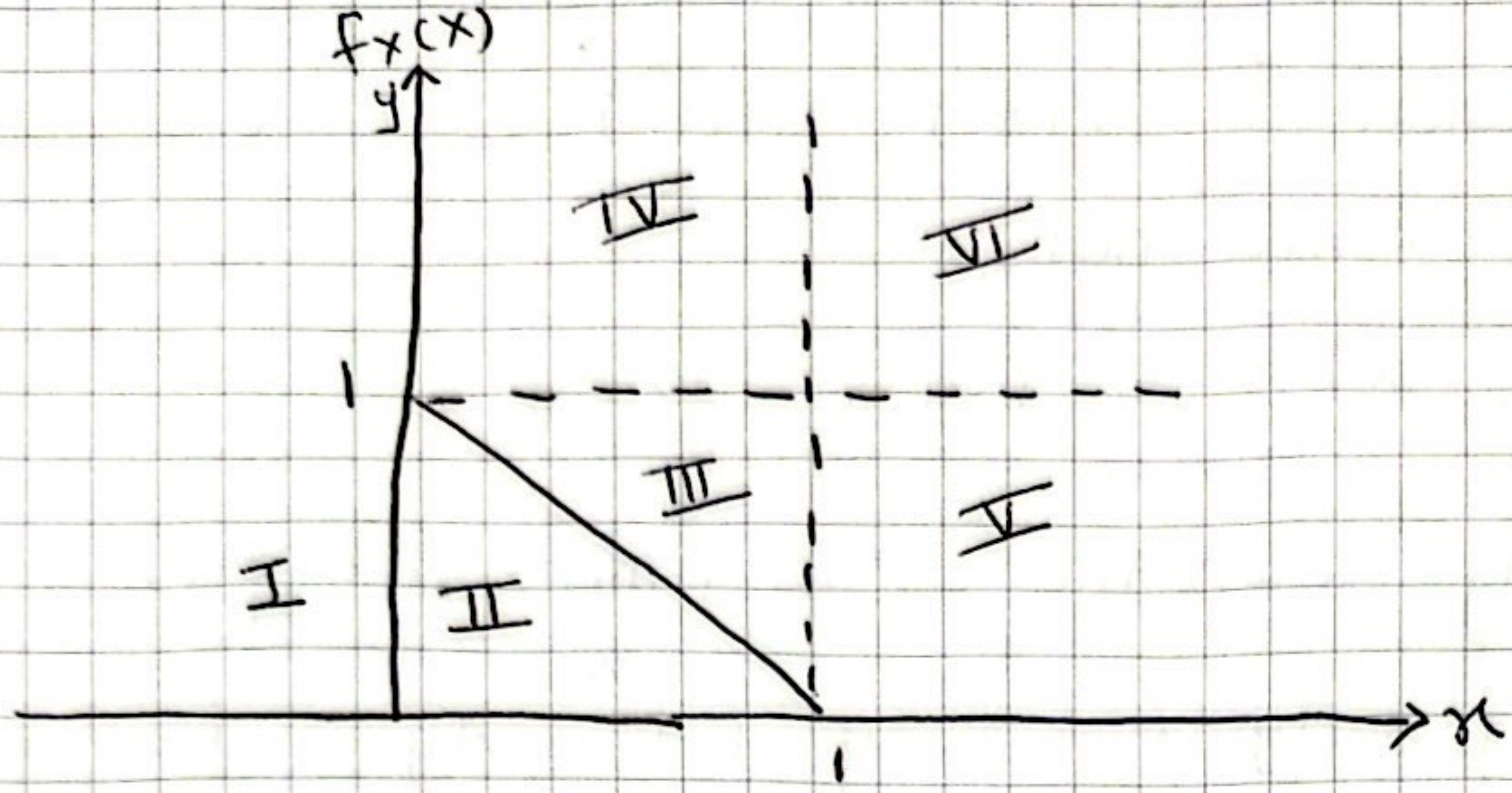
breaking down the above limits we get,

$$f_y(y) = \int_{-\infty}^{0} 0\cdot dx + \int_{0}^{1-y} 2\cdot dx + \int_{1-y}^{+\infty} 0\cdot dx \quad \text{for } 0 \le y \le 1$$

$$f_y(y) = 2(1-y)$$

$$f_y(y) = \begin{cases} 2(1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

3> Calculation of $f_{xy}(x,y)$ (cdf):-



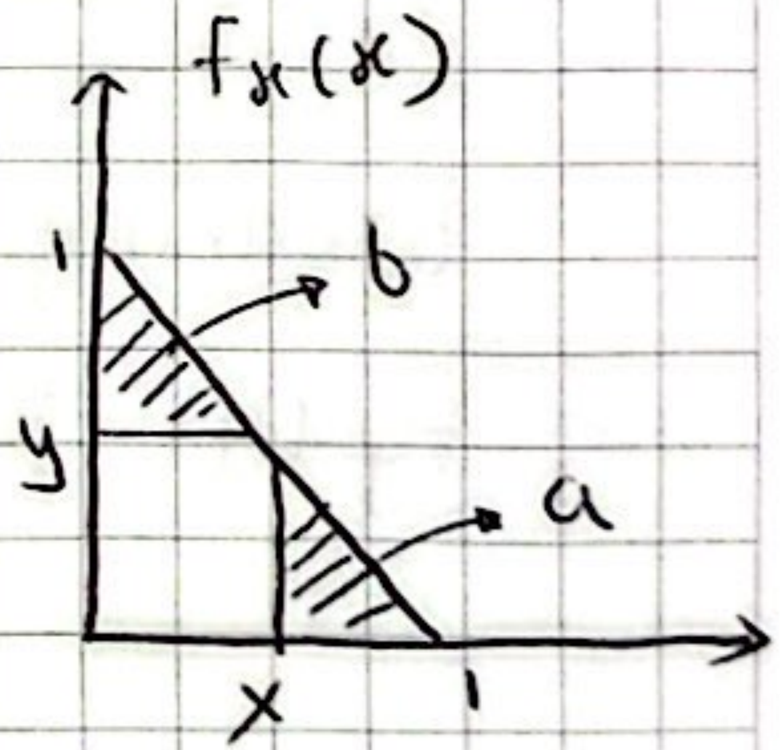
a) $x \le 0$ and $y \le 0$

$$f_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} 0 \cdot dy\,dx = 0$$

b) $x \geq 0$; $y \geq 0$; $x+y \le 1$

$$f_{xy}(x,y) = \int_{0}^{x}\int_{0}^{y} 2 \cdot dy\,dx = \int_{0}^{x} 2y \cdot dx = 2xy$$

c) $x \le 1$, $y \le 1$, $x+y \geq 1$



$$f_{xy}(x,y) = 1 - \iint_{a} 2\,dx\,dy - \iint_{b} 2\,dy\,dx$$

$$F_{xy}(x,y) = 1 - \int_{y}^{1}\int_{0}^{1-y} 2\,dx\,dy - \int_{x}^{1}\int_{0}^{1-x} 2\,dy\,dx$$

$$F_{xy}(x,y) = 1 - \int_{y}^{1} [2x]_{0}^{1-y}\,dy - \int_{x}^{1} [2y]_{0}^{1-x}\,dx$$

$$F_{xy}(x,y) = 1 - \left[2y - \frac{2y^2}{2}\right]_y^1 - \left[2x - \frac{2x^2}{2}\right]_x^1$$

$$F_{xy}(x,y) = 1 - (2-1-2y+y^2) - (2-1-2x+x^2)$$

$$F_{xy}(x,y) = 1 - (1-y)^2 - (1-x^2)$$

d) $0 \le x \le 1 \quad , \quad y \geqslant 1$

$$F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(x,y)\,dy\,dx$$

$$= \int_0^x\int_0^{1-x} 2\,dy\,dx$$

$$F_{xy}(x,y) = \int_0^x [2y]_0^{1-x}\,dx = \int_0^x (2-2x)\,dx$$

$$= \left[2x - \frac{2x^2}{2}\right]_0^x$$

$$F_{xy}(x,y) = \left[2x - x^2\right] = 1 - (1-x)^2$$

e) $0 \le y \le 1 \quad , \quad x \geqslant 1$

$$F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(x,y)\,dy\,dx$$

$$F_{xy}(x,y) = \int_0^{1-y}\int_0^{y} 2\,dy\,dx = \int_0^{1-y} [2y]_0^y\,dx$$

$$F_{xy}(x,y) = \int_0^{1-y} [2y-0]\,dx = [2yx]_0^{1-y}$$

$$= 2y - 2y^2 \Rightarrow 1 - (1-y)^2$$

f) $y \geq 1,\ x \geq 1$

$$F_{xy}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{xy}(x,y)\, dy\, dx = 1$$

Because cdf is a probability for a random variable $\leq$ a veriable.

$$F_{xy}(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \text{ and } y \leq 0 \\ & \text{for } x \leq 0 \text{ or } y \leq 0 \\ 2xy & \text{for } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 1-(1-y)^2-(1-x)^2 & \text{for } x \leq 1,\ y \leq 1,\ x+y \geq 1 \\ 1-(1-x)^2 & \text{for } 0 \leq x \leq 1,\ y \geq 1 \\ 1-(1-y)^2 & \text{for } 0 \leq y \leq 1,\ x \geq 1 \\ 1 & \text{for } y \geq 1,\ x \geq 1 \end{cases}$$

4> Calculation of $F_x(x)$

$$F_x(x) = F_{xy}(x, +\infty)$$

$$F_x(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - (1-x)^2 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

5> Calculation of $F_y(y)$

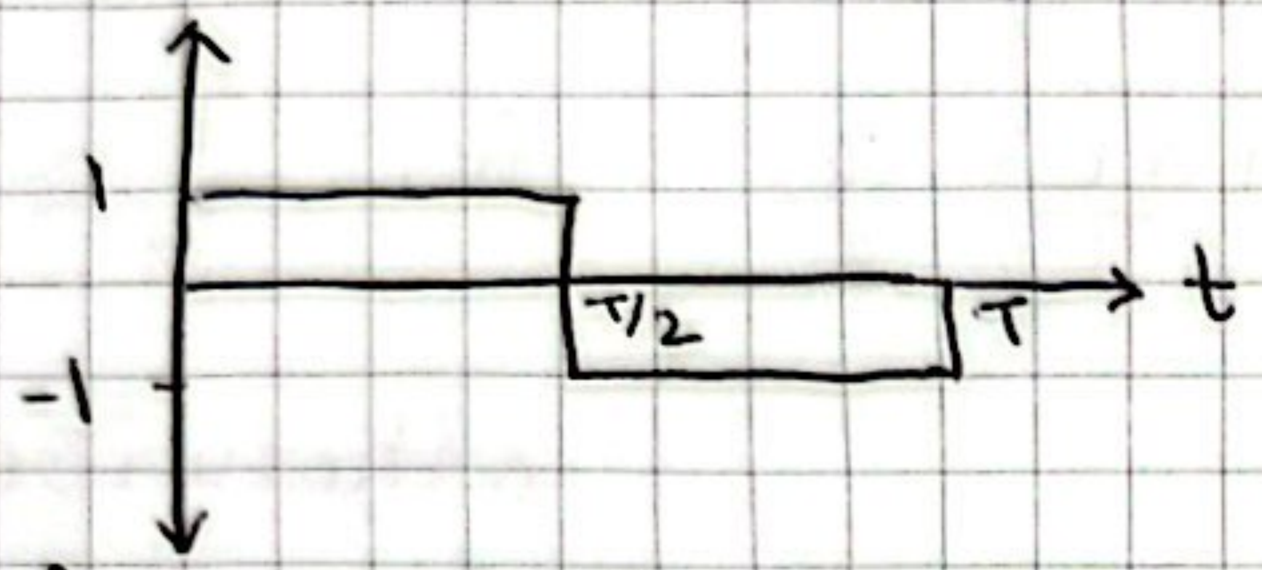
$$F_y(y) = F_{xy}(+\infty, y)$$

$$F_y(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ 1 - (1-y)^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y \geq 1 \end{cases}$$

Q.2 →

for $a(\xi_1) = 0$, $P_0 = \frac{1}{4}$

$$x(\xi_1, t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{T}{2} \\ -1 & \text{for } T/2 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

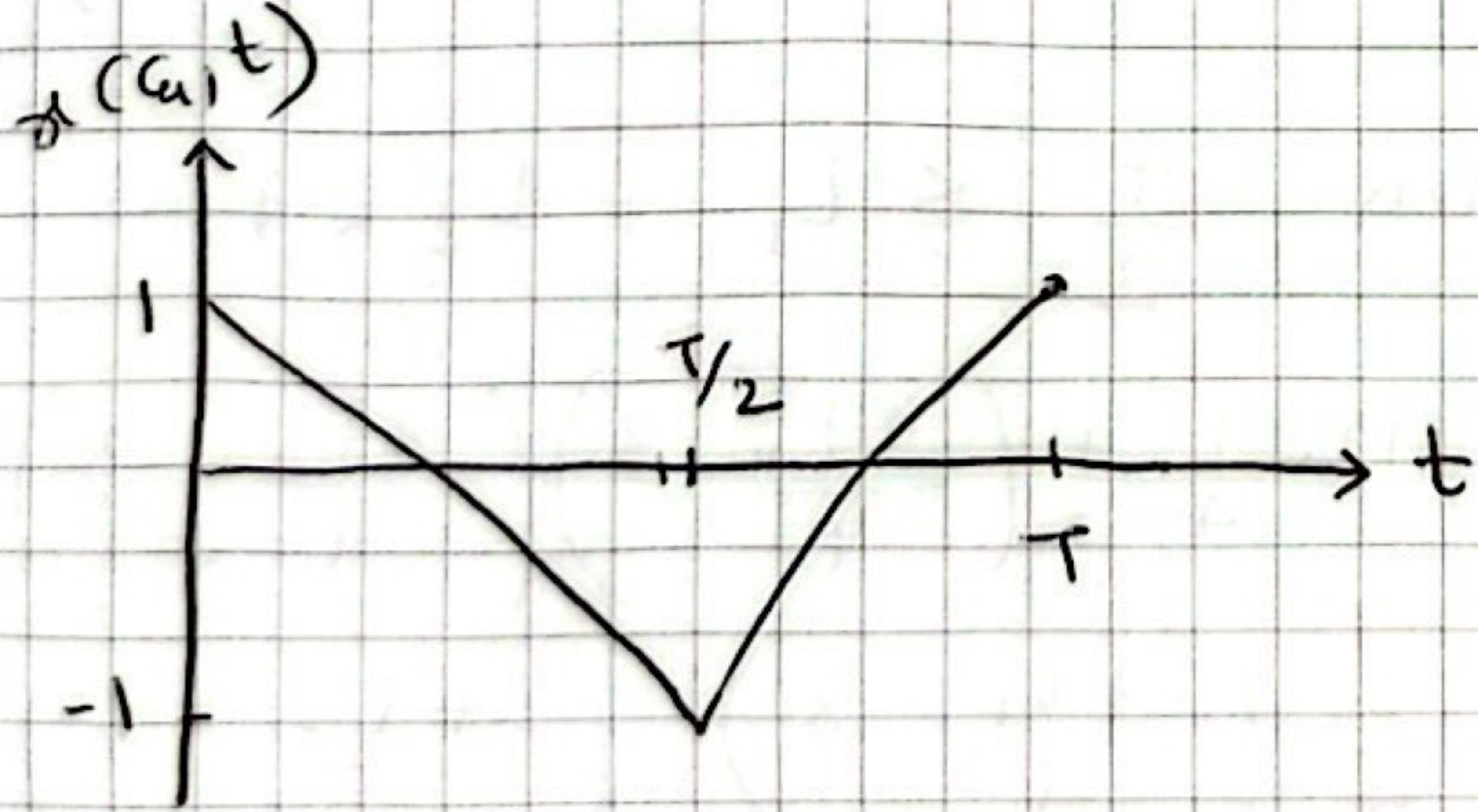


for $a(\xi_1) = 1$, $P_1 = 3/4$

$$x(\xi_1, t) = \begin{cases} 1 - \frac{4t}{T} & \text{for } 0 \le t \le T/2 \\ -3 + \frac{4t}{T} & \text{for } T/2 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

so taking the values of t at different time interval in above equation.

$$x(\xi_1, t) = \begin{cases} 1 & \text{when } t = 0 \\ -1 & \text{when } t = T/2 \\ 1 & \text{when } t = T \end{cases}$$



b) Calculate the mean $m_x^{(1)}(t)$

Sol^n Since we know that,

$$m_x^{(1)} = x \cdot pdf(x) \qquad (1)$$

Mean

$$m_x^{(1)} = \begin{cases} 1 \cdot \frac{1}{4} + \left(1 - \frac{4t}{T}\right)\frac{3}{4} & \text{for } 0 \le t < \frac{T}{2} \\ -1 \cdot \frac{1}{4} + \left(-3 + \frac{4t}{T}\right)\frac{3}{4} & \text{for } \frac{T}{2} \le t < T \\ 0 & \text{otherwise} \end{cases}$$

$$m_x^{(1)} = \begin{cases} 1 - \frac{3t}{T} & \text{for } 0 \le t < \frac{T}{2} \\ -\frac{5}{2} + \frac{3t}{T} & \text{for } \frac{T}{2} \le t < T \\ 0 & \text{otherwise} \end{cases}$$

c) Autocorrelation $S_{xx}(t_1, t_2)$

Case 1 :- when $t_1, t_2 < \frac{T}{2}$

$$S_{xx}(\tau) = (1)(1) \cdot \frac{1}{4} + \left(1 - \frac{4t_1}{T}\right)\left(1 - \frac{4t_2}{T}\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = \frac{1}{4} + \frac{3}{4}\left[\left(1 - \frac{4t_1}{T}\right)\left(1 - \frac{4t_2}{T}\right)\right]$$

Case 2 :- when $\frac{T}{2} \le t_1, t_2 < T$

$$S_{xx}(\tau) = (-1) \cdot (-1) \cdot \frac{1}{4} + \left(\frac{4t_1}{T} - 3\right)\left(\frac{4t_2}{T} - 3\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = \frac{1}{4} + \frac{3}{4}\left(\frac{4t_1}{T} - 3\right)\left(\frac{4t_2}{T} - 3\right)$$

(t

Case 3 when $0 \le t_1 < \frac{T}{2}$ & $\frac{T}{2} \le t_2 < T$

$$S_{xx}(\tau) = (1)\cdot(-1)\left(\frac{1}{4}\right) + \left(1 - \frac{4t_1}{T}\right)\left(\frac{4t_2}{T} - 3\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = -\frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t_1}{T}\right)\left(\frac{4t_2}{T} - 3\right)$$

Case 4

when $\frac{T}{2} \le t_1 < T$ and $0 \le t_2 < \frac{T}{2}$

$$S_{xx}(\tau) = (-1)\cdot(1)\cdot\frac{1}{4} + \left(\frac{4t_1}{T} - 3\right)\left(1 - \frac{4t_2}{T}\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = -\frac{1}{4} + \frac{3}{4}\left(\frac{4t_1}{T} - 3\right)\left(1 - \frac{4t_2}{T}\right)$$

d) Calculate the variance $\sigma_x^2(t)$?

→ since we know that

$$\sigma_x^2 = m_x^{(2)} - (m_x^{(1)})^2$$

$$m_x^{(2)} = \left\{\begin{array}{ll} (1)\cdot\frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 & 0 \le t < \frac{T}{2} \\ (1)\cdot\frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 & \frac{T}{2} \le t < T \\ \frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 & 0 \le t < \frac{T}{2} \\ \frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 & \frac{T}{2} \le t < T \end{array}\right\}$$

So now, for $0 \le t < T/2$

$$\sigma_x^2 = m_x^{(2)} - (m_x^{(1)})^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 - \left(1 - \frac{3t}{T}\right)^2$$

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$$= \frac{1}{4} + \frac{3}{4}\left(1 + \frac{16t^2}{T^2} - \frac{8t}{T}\right) - \left(1 + \frac{9t^2}{T^2} - \frac{6t}{T}\right)$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{12t^2}{T^2} - \frac{6t}{T} - 1 - \frac{9t^2}{T^2} + \frac{6t}{T}$$

$$= \frac{3t^2}{T^2}$$

for $\frac{T}{2} \leq t < T$

$$\sigma_x^2 = m_x^{(2)} - (m_x^{(1)})^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 - \left(\frac{-5}{2} + \frac{3t}{T}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{16t^2}{T^2} + 9 - \frac{24t}{T}\right) - \left(\frac{9t^2}{T^2} + \frac{25}{4} - \frac{15t}{T}\right)$$

$$= \frac{1}{4} + \frac{12t^2}{T^2} + \frac{27}{4} - \frac{18t}{T} - \frac{9t^2}{T^2} - \frac{25}{4} + \frac{15t}{T}$$

$$= \frac{3}{4} + \frac{3t^2}{T} - \frac{3t}{T}$$

$$\therefore \quad \sigma_x^2 = \begin{cases} \dfrac{3t^2}{T^2} & \text{for } 0 \leq t < \dfrac{T}{2} \\[2ex] \dfrac{3}{4} + \dfrac{3t^2}{T^2} - \dfrac{3t}{T} & \text{for } \dfrac{T}{2} \leq t < T \\[2ex] 0 & \text{otherwise} \end{cases}$$