Task 5.1

(a) → $y(S,t) = e^{-a(S)t} \cos(x(S))$ for $t \geq 0$

$$m_y(t1)(t) = E[y(S,t)] = E[e^{-a(S)t} \cos(x(S))]$$

$$m_y(1)(t) = E[e^{-a(S)t}] \cdot E[\cos(x(S))]$$

$$f_a(a) = 1 \quad \text{for } a \in [0,1]$$

$$E[e^{-a(S)t}] = \int_0^1 e^{-at} f_a(a)\,da = \int_0^1 e^{-at}\,da$$

$$E[e^{-a(S)t}] = \left[\frac{1-e^{-t}}{t}\right] \quad \text{for } t > 0$$

$$f_x(x) = \frac{1}{\pi} \quad \text{for } x \in [0,\pi]$$

$$E[\cos(x(S))] = \int_0^\pi \cos(x) f_x(x)\,dx = \int_0^\pi \cos(x) \frac{1}{\pi}\,dx$$

$$E[\cos(x(S))] = \frac{1}{\pi}\int_0^\pi \cos(x)\,dx = \frac{1}{\pi}[\sin(x)]_0^\pi$$

$$E[\cos(x(S))] = \frac{1}{\pi}(0-0) = 0$$

$$m_y(1)(t) = E[e^{-a(S)t}] \cdot E[\cos(x(S))] = \left(\frac{1-e^{t}}{t}\right) \cdot 0 = 0$$

(b) → Covariance function $C_{yy}(t_1, t_2)$

$$C_{yy}(t_1,t_2) = S_{yy}(t_1,t_2) - M_y\cancel{(t_1)}^{\,0} \cdot M_y(t_2)$$

$$C_{yy}(t_1,t_2) = S_{yy}(t_1,t_2)$$

$$S_{yy}(t_1,t_2) = E\{y(s,t_1)\, y(s,t_2)\}$$

$$C_{yy}(t_1,t_2) = E[e^{-a(s)t_1} \cos(x(s)) \cdot e^{-a(s)t_2} \cdot \cos(x(s))]$$

$$C_{yy}(t_1,t_2) = E[e^{-a(s)(t_1+t_2)} \cos^2(x(s))]$$

$$C_{yy}(t_1,t_2) = E[e^{-a(s)(t_1+t_2)}] \cdot E[\cos^2(x(s))]$$

$$f_a(a) = 1 \quad \text{for } a \in [0,1]$$

$$E[e^{-a(s)(t_1+t_2)}] = \int_0^1 e^{-a(t_1+t_2)} f_a(a)\,da = \int_0^1 e^{-a(t_1+t_2)}\,da$$

$$E[e^{-a(s)(t_1+t_2)}] = \frac{1 - e^{-(t_1+t_2)}}{t_1+t_2}$$

$$f_x(x) = \frac{1}{\pi} \quad \text{for } x \in [0,\pi]$$

$$E[\cos^2(x(s))] = \int_0^{\pi} \cos^2(x) f_x(x)\,dx = \int_0^{\pi} \cos^2(x) \frac{1}{\pi}\,dx$$

$$E[\cos^2(x(s))] = \frac{1}{\pi}\int_0^{\pi} \frac{1+\cos(2x)}{2}\,dx$$

$$E[\cos^2(x(s))] = \frac{1}{2\pi}\int_0^{\pi} 1\,dx + \frac{1}{2\pi}\int_0^{\pi} \cos(2x)\,dx$$

$$\frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

$$E\left[\cos^2(x(s))\right] = \frac{1}{2}$$

$$C_{yy}(t_1,t_2) = E\left[e^{-a(s)(t_1+t_2)}\right] \cdot E\left[\cos^2(x(s))\right]$$

$$C_{yy}(t_1,t_2) = \left(\frac{1 - e^{(t_1+t_2)}}{t_1+t_2}\right) \cdot \frac{1}{2}$$

$$C_{yy}(t_1,t_2) = \frac{1 - e^{-(t_1+t_2)}}{2(t_1+t_2)}$$

Task 5.2 → Task 2.2.2

Random Process:

$Z(\zeta,t) = X(\zeta,t) + Y(\zeta,t)$

$X(\zeta,t)$ & $Y(\zeta,t)$ → statistically independent.

$E\{X(\zeta,t)\} = m_x$

$E\{Y(\zeta,t)\} = m_y$

$C_{XX}(\tau) = e^{-\frac{|\tau|}{T_1}} \quad T_1 > 0$

$C_{YY}(\tau) = e^{-\frac{|\tau|}{T_2}} \quad T_2 > 0$

$S_{ZZ}(\tau) = ?$

$S_{ZZ}(\tau) = E\{Z(\zeta,t_1)\, Z(\zeta,t_2)\}$

$= E\{[X(\zeta,t_1) + Y(\zeta,t_1)][X(\zeta,t_2) + Y(\zeta,t_2)]\}$

$= E\{X(\zeta,t_1)X(\zeta,t_2) + X(\zeta,t_1)Y(\zeta,t_2) + Y(\zeta,t_1)X(\zeta,t_2)$
$+ Y(\zeta,t_1)Y(\zeta,t_2)\}$

Since $X(\zeta,t)$ and $Y(\zeta,t)$ are statistically independent;
we have;

$S_{ZZ}(\tau) = E\{X(\zeta,t_1)X(\zeta,t_2)\} + E\{X(\zeta,t_1)\}E\{Y(\zeta,t_2)\} +$
$E\{Y(\zeta,t_1)\}E\{X(\zeta,t_2)\} + E\{Y(\zeta,t_1)Y(\zeta,t_2)\}$

$S_{ZZ}(\tau) = S_{XX} + M_xM_y + M_yM_x + S_{YY}$

$$S_{zz}(\tau) = S_{xx}(\tau) + S_{yy}(\tau) + 2M_xM_y$$

from $C_{xx}(\tau) = S_{xx}(\tau) - M_x^{(1)}(t_1)M_x^{(1)}(t_2)$ Eqn 4.16 Pg 70

but for a stationary process;

$$M_x^{(1)}(t_1) = M_x^{(1)}(t_2)$$

therefore;

$$C_{xx}(\tau) = S_{xx}(\tau) - (M_x^{(1)})^2 \text{ and}$$

$$C_{yy}(\tau) = S_{yy}(\tau) - (M_y^{(1)})^2.$$

Substituting for S_{xx} and S_{yy};

$$S_{xx}(\tau) = C_{xx}(\tau) + (M_x^{(1)})^2$$

$$S_{yy}(\tau) = C_{yy}(\tau) + (M_y^{(1)})^2$$

$$S_{zz}(\tau) = C_{xx}(\tau) + M_x^2 + C_{yy}(\tau) + M_y^2 + 2M_xM_y$$

Substituting $C_{xx}(\tau)$ and $C_{yy}(\tau)$

$$S_{zz}(\tau) = e^{-\frac{|\tau|}{T_1}} + e^{-\frac{|\tau|}{T_2}} + M_x^2 + 2M_xM_y + M_y^2$$

$$\boxed{S_{zz}(\tau) = e^{-\frac{|\tau|}{T_1}} + e^{-\frac{|\tau|}{T_2}} + (M_x+M_y)^2}$$

$T_1 > 0$

$T_2 > 0$

AnyScanner

PSD $\rightarrow$ $S_{zz}(\omega)$

$$S_{zz}(\omega) = FFT\{S_{zz}(\tau)\}$$

$$S_{zz}(\omega) = FFT\{e^{-\frac{|\tau|}{T_1}} + e^{-\frac{|\tau|}{T_2}} + (M_x + M_y)^2\}$$

Using the linearity property of Fourier Transform:

$$S_{zz}(\omega) = FFT\{e^{-\frac{|\tau|}{T_1}}\} + FFT\{e^{-\frac{|\tau|}{T_2}}\} + FFT\{(M_x+M_y)^2\}$$

From the Fourier Transform tables:

$$FFT\{e^{-\alpha|\tau|}\} = \frac{2\alpha}{\alpha^2+w^2} \quad \text{and} \quad FT\{A\} = 2\pi A\delta(\omega)$$

for our case;

$$\alpha = \frac{1}{T_1} \quad \text{and} \quad \alpha = \frac{1}{T_2};$$

We have;

$$S_{zz}(\omega) = \frac{2\cdot 1/T_1}{\left(\frac{1}{T_1}\right)^2 + w^2} + \frac{2\cdot \frac{1}{T_2}}{\left(\frac{1}{T_2}\right)^2 + w^2} + 2\pi(M_x+M_y)^2\,\delta(\omega)$$

$$= \frac{2\cdot\frac{1}{T_1}\times T_1^2}{\left(\frac{1}{T_1^2} + w^2\right)\times T_1^2} + \frac{\left(2\cdot\frac{1}{T_2}\right)\times T_2^2}{\left(\frac{1}{T_2^2} + w^2\right)\times T_2^2} + 2\pi(M_x+M_y)^2\delta(\omega)$$

$$\boxed{S_{zz}(\omega) = \frac{2T_1}{1+w^2T_1^2} + \frac{2T_2}{1+w^2T_2^2} + 2\pi(M_x+M_y)^2\,\delta(\omega)}$$