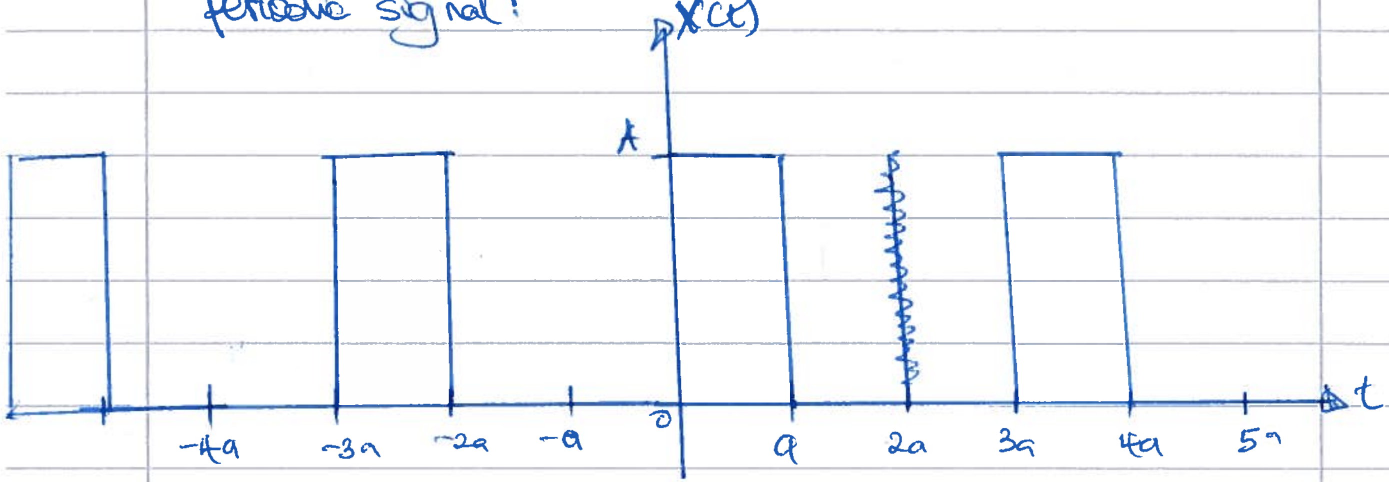
# Exercise 1 – 2025 – Solutions

a) Determine the autocorrelation function of the following periodic signal:




Solution:

→ Since its a periodic signal, the autocorrelation is given by:

$$S_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, x(t+\tau)\, dt$$ Pg 85 eqn 4.42,

We shall use:

$$S_{xx}(\tau) = \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, dt$$

→ Amplitude $= A$.

→ width $= a$

→ Period $= 3a$.

at $\tau = 0$

$$S_{xx}(0) = \frac{1}{3a} \int_0^{3a} A^2\, dt = \frac{1}{3a} \int_0^{a} A^2 dt$$

$$= \frac{1}{3a} \left[ A^2 t \right]_0^a = \frac{1}{3\cancel{a}} \cdot A^2 \cancel{a}$$

$$= \boxed{\frac{A^2}{3}}$$

at $\tau = a$

$S_{xx}(a) = 0$

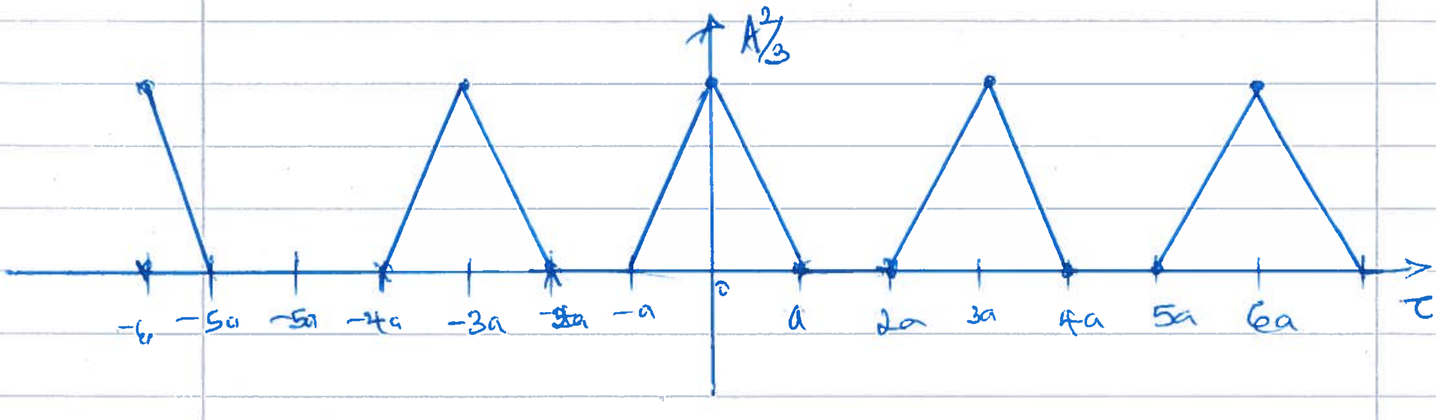
at $\tau = 2a$

$S_{xx}(2a) = 0$

at $\tau = 3a$

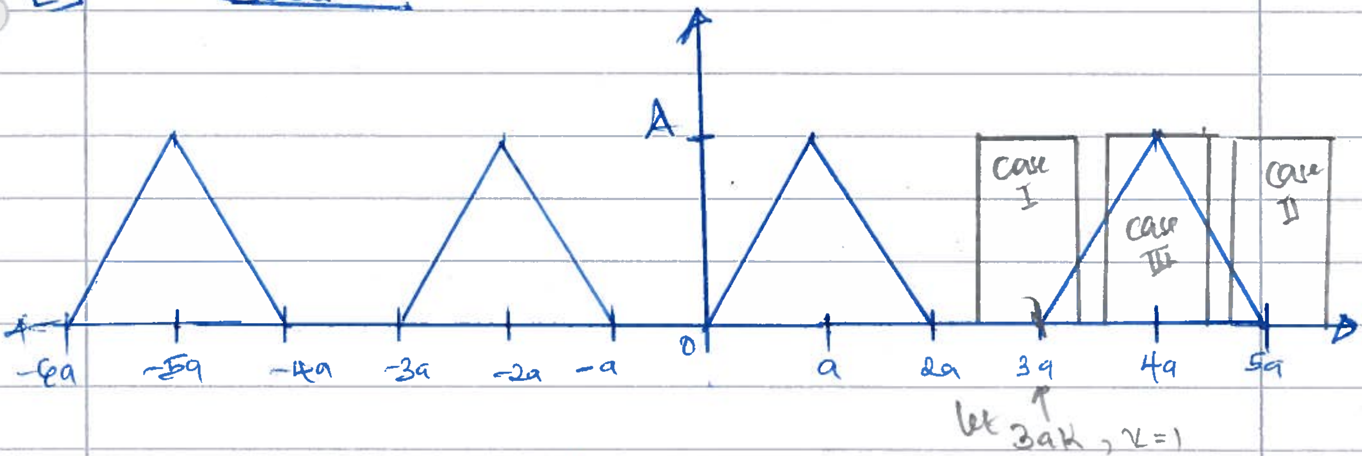
$S_{xx}(3a) = \frac{1}{3a} \cdot \cancel{A^2 a} = \frac{A^2}{3}$

→ The transition or the overlap increases & decreases linearly.



b

# Solution




$K \in \mathbb{Z}$

→ Overlap → how much they overlap.

We know when $\tau = 0$; No overlap

$S_{xy}(0) = 0$, Period $3a$ – Both signal

Autocorrelation → Period $= 3a$.

Lets define $x(t)$ using the three case

Case I: → Increasing overlap. – Increasing Area.

Case II: → decreasing overlap – decreasing area

Case III: → Complete overlap. – Maximum area.

from the Equation of line $y = mx + c$; $\overset{x(t)}{y} = mt + c$

two points: $(3ak, 0)$, $(3ak + a, A)$

$$m = \frac{A - 0}{(3ak + a) - 3ak} = A/a$$

$$m = \Delta y / \Delta x = \frac{y - 0}{t - 3ak} = A/a$$

$$\boxed{y = A/a\,(t - 3ak) \quad \text{for } 3ak \le t \le (3k+1)a}$$

the 2nd slope → Same but – negative slope

$m = -A/a$  $(4a, A) \Rightarrow (3ak + a, A)$ & $(3ak + 2a, 0)$

$$m = \frac{\Delta y}{\Delta x} = \frac{y - 0}{t - 3ak + 2a} = -A/a$$

$$y = -A/a\,(t - (3k+2)a) \quad \text{for } (3k+1)a \le t \le (3k+2)a$$

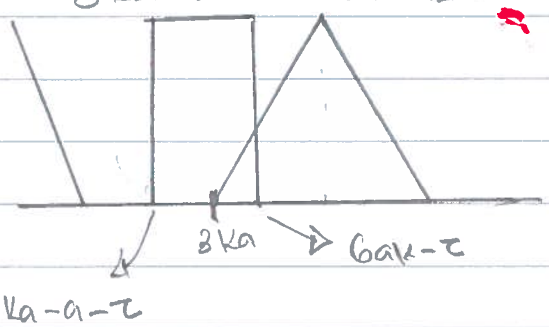
$$y = x(t) = -\frac{A}{a}\left(t-(3k+2)a\right)$$

$$= -\frac{A}{a}t + A(3k+2)$$

$$= \boxed{A(3k+2) - \frac{A}{a}t}$$

$$x(t) = \begin{cases} \frac{A}{a}(t-3ak) & 3ka \le t \le (3k+1)a \\ 3(k+2)A - \frac{A}{a}t & (3k+1)a \le t \le (3k+2)a \\ 0 & (3k+2)a \le 3(k+1)a = 3ka+3a \end{cases}$$

[Increasing overlap] Case I: $3ka \ge \tau \ge (3k-1)a$




$$S_{xy} = \frac{1}{3a}\int_{3ka}^{6ak-\tau} \frac{A}{a}(t-3ak)\,dt$$

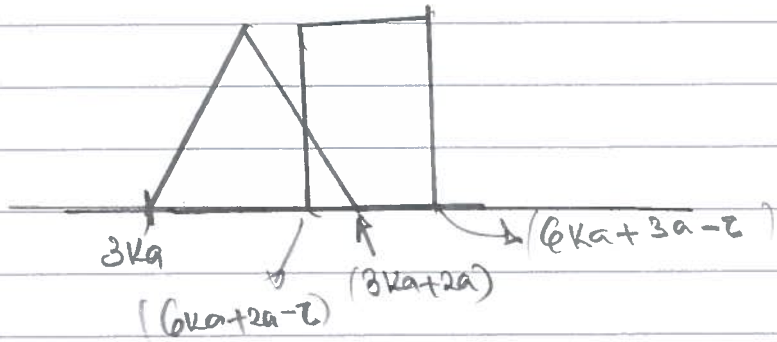
- Equivalent shifting

Eg $\int_3^5 t-3\,dt = \int_0^3 t\,dt$

$$= \frac{A}{3a^2}\int_0^{3ak-\tau} t\,dt$$

$$\boxed{= \frac{A}{6a^2}(3ak-\tau)^2 \quad \text{for } 3ak \ge \tau \ge (3k-1)a}$$

Case 2: $(3ka+a) \geq \tau \geq 3ak$




$$S_{xy} = \frac{1}{3a}\int_{6ka+2a-\tau}^{(3k+2a)} (3k+2)A - \frac{At}{a}\, dt$$

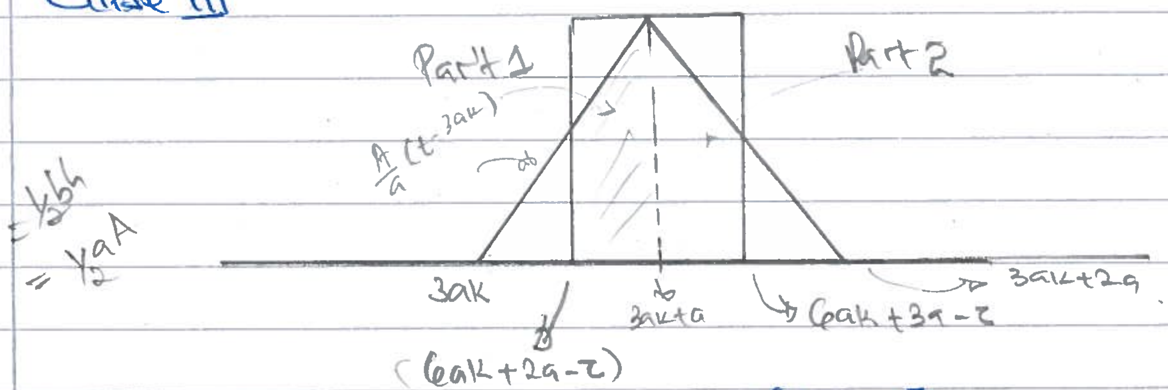
Shifting:

$$= \frac{A}{3a}\int_{3ak-\tau}^{0} \left(0 - \frac{At}{a}\right) dt = \frac{A}{3a^2}\left[-\frac{t^2}{2}\right]_{3ka-\tau}^{0}$$

$$= \frac{A}{3a^2}\left[0 + \frac{(3ak-\tau)^2}{2}\right]$$

$$\boxed{= \frac{A}{6a^2}(3ak-\tau)^2 \quad \text{for } 3ak-a \leq \tau \leq 3ak}$$

Case III




Part 1

$$F_1(\tau) = \frac{1}{3a}\left[\frac{a \cdot A}{2} - \frac{A}{a}\int_{3ak}^{6ak+2a-\tau} (t-3ak)\, dt\right]$$

$$= A/6 - A/3a^2 \int_0^{3ak+2a-\tau} t\,dt$$

$$= A/6 - A/3a^2 \left[t^2/2\right] = A/6 - A/6a^2 \left[3ak + 2a - \tau\right]^2$$

$$= \boxed{A/6 - A/6a^2 (3ak + 2a - \tau)^2}$$

Part 2:

$$F_2(\tau) = \frac{1}{3a}\left[\frac{a \cdot A}{2} - A \int_{6ak+3a-\tau}^{3ak+2a} ((3k+2) - 1/2\, t)\,dt\right]$$

$$= A/6 - A/3a \int_{3ak+\cancel{a}-\tau}^{0} (0 - t/2)\,dt$$

$$= A/6 - A/6a^2 (3ak + a - \tau)^2$$

$S_{xy} = F_1(\tau) + F_2(\tau)$ — skip to final solution)

$$= A/3 - A/6a^2 \left[(3ak + a - \tau)^2 + (3ak + 2a - \tau)^2\right]$$

$$= A/3 - \frac{A}{6a^2}\left[(3ak + a - \tau)(3ak + a - \tau) + (3ak + 2a - \tau)(3ak + 2a - \tau)\right]$$

$$\Rightarrow (3ak)^2 + 3a^2k - 3ka\tau + 3ka^2 + a^2 - a\tau - 3\tau ka - a\tau + \tau^2$$

$$\Rightarrow [(3ka)^2 + 6ka^2 + a^2 - 6ka\tau - 2a\tau + \tau^2]$$

$$(3ak + 2a - \tau)^2 = (3ka)^2 + 6ka^2 - 3ka\tau + 6ka^2 + 4a^2 - 2\tau a - 3k\tau a + \tau^2$$

$$= (3ka)^2 + 12ka^2 - 6k\tau a + 4a^2 - 4\tau a + \tau^2$$

$$= A/3 - A/6a^2 \left[2(3ka)^2 + 18ka^2 + 5a^2 - 12ka\tau - 6\tau a + 2\tau^2\right]$$

factor a out!

$$= \frac{A}{3} - \frac{A}{3a^2}\left[(3ka)^2 + 9ka^2 + \frac{5}{2}a^2 - 6ka\tau - 3a\tau + \tau^2\right]$$

$$= \frac{A}{3} - \frac{A}{3a^2}\left[\underbrace{(3ka)^2 + 9ka^2 + \frac{9}{4}a^2}_{} + \frac{1}{4}a^2 - 6ka\tau - 3a\tau + \tau^2\right]$$

$$= \frac{A}{3} - \frac{1}{4}a^2 \cdot \frac{A}{3a} - \frac{A}{3a^2}\left[(3ka + \frac{3}{2}a)^2 - 2\tau(3ka - \frac{3}{2}a) + \tau^2\right]$$

$$= \frac{A}{3} - \frac{A}{12} - \frac{A}{3a^2}\left[3ka + \frac{3}{2}a - \tau\right]^2$$

$$= \frac{A}{4} - \frac{A}{3a^2}\left(3ka + \frac{3}{2}a - \tau\right)^2$$

or

$$= \boxed{\frac{A}{4} - \frac{A}{3a^2}\left(-3ka - \frac{3}{2}a + \tau\right)^2}$$

for $3ak + a \leq \tau \leq 3ak + 2a$

$$\left.\begin{array}{ll} \text{I}: & \frac{A}{6} - \frac{\tau^2}{a^2} \quad \text{Parabola} \\ \text{II}: & \frac{A}{4} - \frac{A}{3a^2}(\tau - 1.5a)^2 \end{array}\right\} \text{ for } k=0$$

$\frac{A}{4} - \frac{A}{3a^2} \cdot \frac{9}{4}a^2$

$\frac{A}{4} - \frac{3A}{4} \cdot$

$\frac{A}{4}$