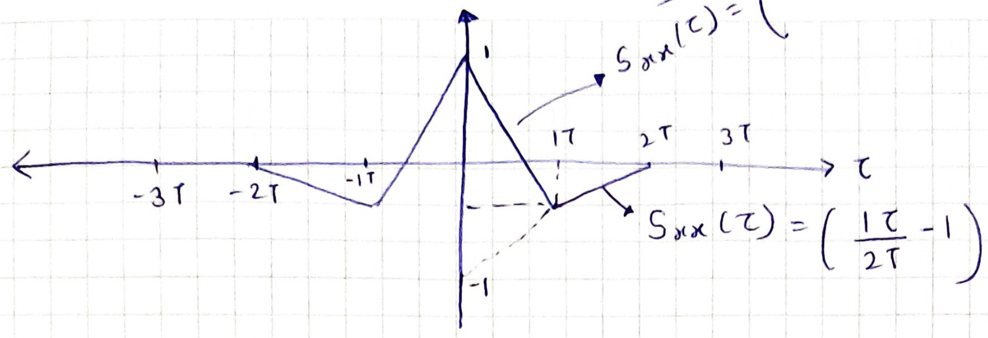
SSS (Exercise-3)



Power spectral Density $S_{xx}(\omega) = ?$

Solution →

$$\because \quad S_{xx}(\omega) = \int_{-\infty}^{+\infty} S_{xx}(\tau) \cdot e^{-j\omega\tau} \, d\tau$$

$$S_{xx}(\omega) = 2\int_{0}^{\infty} S_{xx}(\tau)\, e^{-j\omega\tau} \, d\tau$$

$$= 2\int_{0}^{2T} S_{xx}(\tau)\, e^{-j\omega\tau} \, d\tau$$

$$S_{xx}(\omega) = 2\left[\int_{0}^{T} \left(1-\frac{3\tau}{2T}\right) e^{-j\omega\tau} \, d\tau + \int_{T}^{2T} \left(-1+\frac{\tau}{2T}\right) \cdot e^{-j\omega\tau} \, d\tau\right]$$

$$\because \quad e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$$

$$S_{xx}(\omega) = 2\left[\int_{0}^{T} \left(1-\frac{3\tau}{2T}\right)(\cos\omega\tau - j\sin\omega\tau)\, d\tau + \int_{T}^{2T} \left(-1+\frac{\tau}{2T}\right)(\cos\omega\tau - j\sin\omega\tau)\, d\tau\right]$$

Now taking only the real part, because the function is real so,

$$S_{xx}(\omega) = 2\left[\int_{0}^{T} \left(1-\frac{3\tau}{2T}\right)\cos\omega\tau \, d\tau + \int_{T}^{2T} \left(\frac{\tau}{2T}-1\right)\cos\omega\tau \, d\tau\right]$$

$$S_{xx}(\omega) = 2\left[\int_{0}^{T} \cos\omega\tau \, d\tau - \frac{3}{2T}\int_{0}^{T} \tau\cos(\omega\tau)\, d\tau + \frac{1}{2T}\int_{T}^{2T} \tau\cos(\omega\tau)\, d\tau - \int_{T}^{2T} \cos(\omega\tau)\, d\tau\right]$$

using integration by parts

$$\because \int uv\,dx = u\int v\,dx - \int\left[\frac{d}{dx}(u)\int v\,dx\right]dx$$

$$S_{xx}(\omega) = 2\left[\frac{\sin(\omega\tau)}{\omega}\Bigg|_0^T - \frac{3}{2T}\left(\frac{\tau\sin(\omega\tau)}{\omega} + \frac{\cos(\omega\tau)}{\omega^2}\right)\Bigg|_0^T - \left(\frac{1}{\omega}\sin(\omega\tau)\right)\Bigg|_T^{2T} + \frac{1}{2T}\left(\frac{\tau\sin(\omega\tau)}{\omega} + \frac{\cos(\omega\tau)}{\omega^2}\right)\Bigg|_T^{2T}\right]$$

$S_{xx}(\omega) \Rightarrow$

$$2\left[\frac{\sin(\omega T)}{\omega} - \frac{3}{2T}\left(\frac{T\sin(\omega T)}{\omega} + \frac{\cos(\omega T)}{\omega^2} - \frac{1}{\omega^2}\right) - \left(\frac{\sin(2\omega T)}{\omega} - \frac{\sin(\omega T)}{\omega}\right) + \frac{1}{2T}\left(\frac{2T\sin(2\omega T)}{\omega} - \frac{T\sin(\omega T)}{\omega} + \frac{\cos(2\omega T)}{\omega^2}\right) - \frac{\cos(\omega T)}{\omega^2}\right]$$

$S_{xx}(\omega) \Rightarrow$

$$\frac{2}{\omega}\left[2\sin\omega T - 2\sin\omega T + \frac{3}{2\omega T} - \frac{3\cos\omega T}{2\omega T} + \frac{\cos 2\omega T}{2\omega T} - \frac{\cos\omega T}{2\omega T}\right]$$

$$S_{xx}(\omega) \Rightarrow \frac{2}{\omega^2 T}\left[-2\cos\omega T + \frac{3}{2} + \frac{\cos 2\omega T}{2}\right]$$

$$\left\{\begin{aligned} \because \cos 2A &= \cos^2 A - \sin^2 A \\ \sin^2 A &= 1 - \cos^2 A \\ \cos 2A &= 2\cos^2 A - 1 \end{aligned}\right\}$$

$$\therefore S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[-2\cos\omega T + \frac{3}{2} + \cos^2\omega T - \frac{1}{2}\right]$$

$$S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[-2\cos\omega T + 1 + \cos^2\omega T\right]$$

$$S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[1 - \cos\omega T\right]^2 \quad \text{--- (1)}$$

$\therefore$ $S_{xx}(\omega)$ is,

ω	$S_{xx}(\omega)$
0	0
π/T	$8T/\pi^2$
$2\pi/T$	0
$3\pi/T$	$8T/9\pi^2$
$4\pi/T$	0
$5\pi/T$	$8T/25\pi^2$