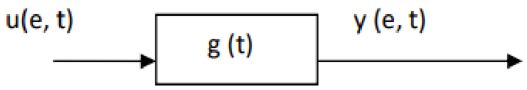
Task 4.3 Power spectral density and transfer function

A Gaussian noise signal u(e, t) is input to a linear system described by its impulse response g(t).



Some power spectral densities are given:

$$S_{uy} = \frac{S_1}{(1 - j\omega b)(1 + j\omega T_1)} \qquad\qquad S_{yy} = \frac{S_1}{1 + \omega^2 T_1^2}$$

a) Determine the transfer function $G(j\omega)$ of the linear system.
b) Is the system described by $G(j\omega)$ a causal system? Explain your statement.
c) Calculate the autocorrelation function $s_{uu}(\tau)$ of the input signal u(e, t).

Task 4.3

Solution:→

$$u(e,t) \longrightarrow \boxed{g(t)} \longrightarrow y(e,t)$$

Power Spectral densities are,

$$S_{uy} = \frac{S_1}{(1-j\omega b)(1+j\omega T_1)} \quad \text{and} \quad S_{yy} = \frac{S_1}{1+\omega^2 T_1^2}$$

- very important formulas for a stationary time depending continuous random process,

$$\boxed{\begin{aligned} S_{xy}(\omega) &= \underline{H}(j\omega)\, S_{xx}(\omega) \\ S_{yx}(\omega) &= \underline{H}^*(j\omega)\, S_{xx}(\omega) \\ S_{yy}(\omega) &= \underline{H}(j\omega)\, \underline{H}^*(j\omega)\, S_{xx}(\omega) \end{aligned}}$$

(a) $$G^*(j\omega) = \frac{S_{yy}}{S_{uy}} = \frac{\dfrac{S_1}{(1+\omega^2 T_1^2)}}{\dfrac{S_1}{(1-j\omega b)(1+j\omega T_1)}} = \frac{(1-j\omega b)(1+j\omega T_1)}{(1+\omega^2 T_1^2)}$$

$$= \frac{(1-j\omega b)(1+j\omega T_1)}{(1+j\omega T_1)(1-j\omega T_1)} = \frac{(1-j\omega b)}{(1-j\omega T_1)}$$

$$\boxed{\therefore \quad G^*(j\omega) = \frac{1+j\omega b}{1+j\omega T_1}}$$

(b) To identify the causal or non causal system, we must check the poles exist.

$$\therefore \quad 1 + j\omega T_1 = 0$$

$$j\omega T_1 = -1$$

$$\omega = \frac{-1}{jT_1}$$

(c) $\underline{\underline{S_{uy}(\tau)}} \rightarrow$

$$S_{uu} = \frac{S_{uy}}{G(j\omega)} = \frac{\dfrac{S_1}{(1-j\omega b)(1+j\omega T_1)}}{\dfrac{(1+j\omega b)}{(1+j\omega T_1)}} = \frac{S_1}{(1-j\omega b)(1+j\omega b)}$$

$$\therefore S_{uu} = \frac{S_1}{(1-j\omega b)(1+j\omega b)} = \frac{S_1}{1+\omega^2 b^2}$$

$$S_{uu}(\tau) = F^{-1}[S_{uu}] = F^{-1}\left[\frac{S_1}{1+\omega^2 b^2}\right]$$

$$= F^{-1}\left[\frac{S_1/b}{b(1/b^2+\omega^2)}\right]$$

$$\therefore S_{uy}(\tau) = \frac{S_1}{b} \cdot e^{-\frac{|\tau|}{b}}$$