Exercise #3

$a \to 10$
$b \to 15$

Task 2.2.6

$X(\xi, t) \to$ Random Process.

PSD

$$S_{xx}(\omega) = \begin{cases} S_0\left(1 - \frac{|\omega|}{2\omega_0}\right) & |\omega| \leq 2\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

$S(\xi, t) = 2A \sin(\omega_0 t + \phi(\xi))$; A = Constant

$\phi \to [0, 2\pi]$

$X(\xi, t)$ & $S(\xi, t) \to$ statistically independent.

$y(\xi, t) = S(\xi, t)\, X(\xi, t).$

$M_x^{(2)}$ of $X(\xi, t)$?

$M_x^{(2)} = E\{X^2(\xi, t)\}$

$S_{xx}(\tau) = E\{x(\xi, t) \cdot x(\xi, t+\tau)\}$

at $\tau = 0$

$S_{xx}(0) = E\{x(\xi, t)\, x(\xi, t)\} = M_x^{(2)}$

$\boxed{\text{Pg 65}}$ Eq 4.68, & Eq 4.69; $\to$ Pg 92

$$S_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)\, e^{j\omega}\, d\omega$$

for $\tau = 0$

$$S_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)\, d\omega$$

$$S_{xx}(0) = E\{x^2(\xi, t)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)\, d\omega.$$

$$M_x^{(2)} = \frac{1}{2\pi} \int_{-2\omega_0}^{2\omega_0} S_0\left(1 - \frac{|\omega|}{2\omega_0}\right) d\omega$$

because PSD is an even function (symmetric at zero)

$$M_x^{(2)} = \frac{1}{2\pi} \cdot 2 \int_0^{2\omega_0} S_0 \left(1 - \frac{\omega}{2\omega_0}\right) d\omega$$

$$= \frac{S_0}{\cancel{2}\pi} \int_0^{2\omega_0} \left(1 - \frac{\omega}{2\omega_0}\right) d\omega$$

$$= \frac{S_0}{\pi} \left[\omega - \frac{\omega^2}{4\omega_0}\right]_0^{2\omega_0}$$

$$= \frac{S_0}{\pi} \left[(2\omega_0 - 0) - \left(\frac{4\omega_0^2 - 0}{4\omega_0}\right)\right]$$

$$= \frac{S_0}{\pi} \left[2\omega_0 - \omega_0\right]$$

$$= \frac{S_0}{\pi} \cdot \omega_0$$

15

$$\boxed{M_x^{(2)} = \frac{S_0 \omega_0}{\pi}}$$ ✓

Alternative

$\boxed{Q2}$

$$M_x^{(2)} = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)\, d\omega = \frac{1}{2\pi} \int_{-2\omega_0}^{2\omega_0} S_0 \left(1 - \frac{|\omega|}{2\omega_0}\right) d\omega$$

$$= \frac{S_0}{2\pi} \left(\int_{-2\omega_0}^{2\omega_0} 1\, d\omega + \frac{1}{2\omega_0} \int_{-2\omega_0}^{0} \omega\, d\omega - \frac{1}{2\omega_0} \int_0^{2\omega_0} \omega\, d\omega \right)$$

$$= \frac{S_0}{2\pi} \left(4\omega_0 + \frac{1}{2\omega_0} \left[\frac{\omega^2}{2}\right]_{-2\omega_0}^{0} - \frac{1}{2\omega_0} \left[\frac{\omega^2}{2}\right]_0^{2\omega_0} \right)$$

$$= \frac{S_0}{2\pi} \left(4\omega_0 - \omega_0 - \omega_0 \right) = \frac{S_0}{\cancel{2}\pi} \cdot \cancel{2}\, \omega_0$$

$$= \frac{S_0 \omega_0}{\pi}$$

2

b) $S_{yy}(\omega) = ?$

$$S_{yy}(\omega) = FFT\{S_{yy}(\tau)\}$$

$$S_{yy}(\tau) = E\{y(\zeta,t_1)\, y(\zeta,t_2)\}$$

$$S_{yy}(\tau) = E\{s(\zeta,t_1)\, s(\zeta,t_2) \cdot x(\zeta,t_1)\, x(\zeta,t_2)\}$$

Since $s(\zeta,t)$ & $x(\zeta,t)$ are statistically independent;

$$S_{yy}(\tau) = E\{s(\zeta,t_1)\, s(\zeta,t_2)\} \cdot \underbrace{E\{x(\zeta,t_1)\, x(\zeta,t_2)\}}_{S_{xx}(\tau)}$$

$$S_{yy}(\tau) = E\{s(\zeta,t_1)\, s(\zeta,t_2)\} \cdot S_{xx}(\tau)$$

$$\Rightarrow \cancel{E\{s} \; S_{ss}(\tau) = E\{2A\sin(\omega_0 t_1 + \phi) \cdot 2A\sin(\omega_0 t_2 + \phi)\}$$

$$= 4A^2 E\{\sin(\omega_0 t_1 + \phi)\sin(\omega_0 t_2 + \phi)\}$$

from trig Identities:

$$\sin(A)\sin(B) = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$= 4A^2 E\left\{\frac{\cos(\omega_0 t_1 + \phi - \omega_0 t_2 - \phi) - \cos(\omega_0 t_1 + \phi + \omega_0 t_2 + \phi)}{2}\right\}$$

$$= 2A^2 E\{\cos\omega_0(t_1 - t_2) - \cos(\omega_0(t_1+t_2) + 2\phi)\}$$

$$= 2A^2 E\{\underbrace{\cos(\omega_0\tau)}_{\text{Constant}} - \cos(\omega_0(t_1+t_2) + 2\phi)\}$$

$$= 2A^2\cos(\omega_0\tau) - 2A^2 E\{\cos(\omega_0(t_1+t_2) + 2\phi)\}$$

Since $\phi(\zeta)$ is random phase shift uniformly spread over a full cycle; the Expectation is zero

$$S_{ss}(\tau) = 2A^2\cos(\omega_0\tau) - 2A^2(0) = 2A^2\cos(\omega_0\tau)$$

$$S_{yy}(\tau) = 2A^2 \cos(\omega_0 \tau) \cdot S_{xx}(\tau)$$

$$S_{yy}(\omega) = FFT\{S_{yy}(\tau)\}$$

$$= FFT\{2A^2 \cos(\omega_0 \tau) \cdot S_{xx}(\tau)\}$$

from fourier shifting property and Modulation Property:

$$FFT\{\cos(\omega_0 \tau) \cdot f(\tau)\} = \frac{1}{2}[F(\omega - \omega_0) + F(\omega + \omega_0)]$$

therefore:

$$S_{yy}(\omega) = 2A^2 \cdot \frac{1}{2}[S_{xx}(\omega - \omega_0) + S_{xx}(\omega + \omega_0)]$$

$$\boxed{S_{yy}(\omega) = A^2[S_{xx}(\omega - \omega_0) + S_{xx}(\omega + \omega_0)]}$$

Task 2.2.6 B

$M_x^{(2)}$: More hints

→ 1. The two processes $s(e,t)$ and $x(e,t)$ are statistically independent, therefore, ACFs of $y(e,t)$ can be calculated as the product of the ACF of $x(e,t)$ and the ACF of $s(e,t)$.

Both processes are stationary (for process $x(e,t)$ it's given in the task, for process $s(e,t)$ → [showed in lecture]

$$S_{yy}(\omega) = \int_{-\infty}^{+\infty} R_{ss}(\tau) R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

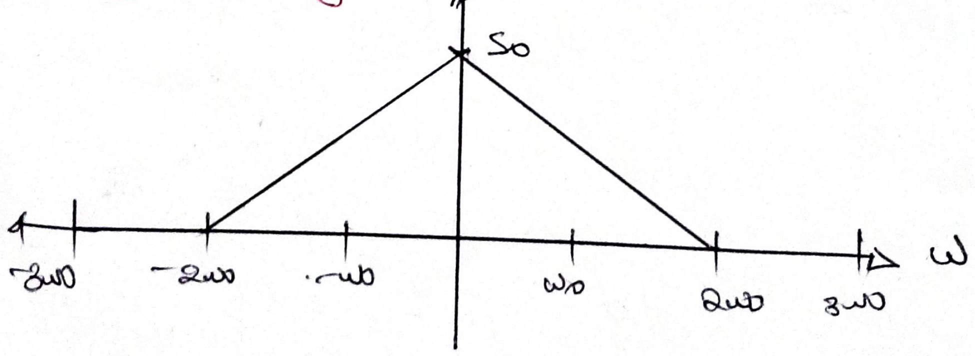
$$= \int_{-\infty}^{+\infty} 2A^2 \cos(\omega_0\tau) \cdot R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

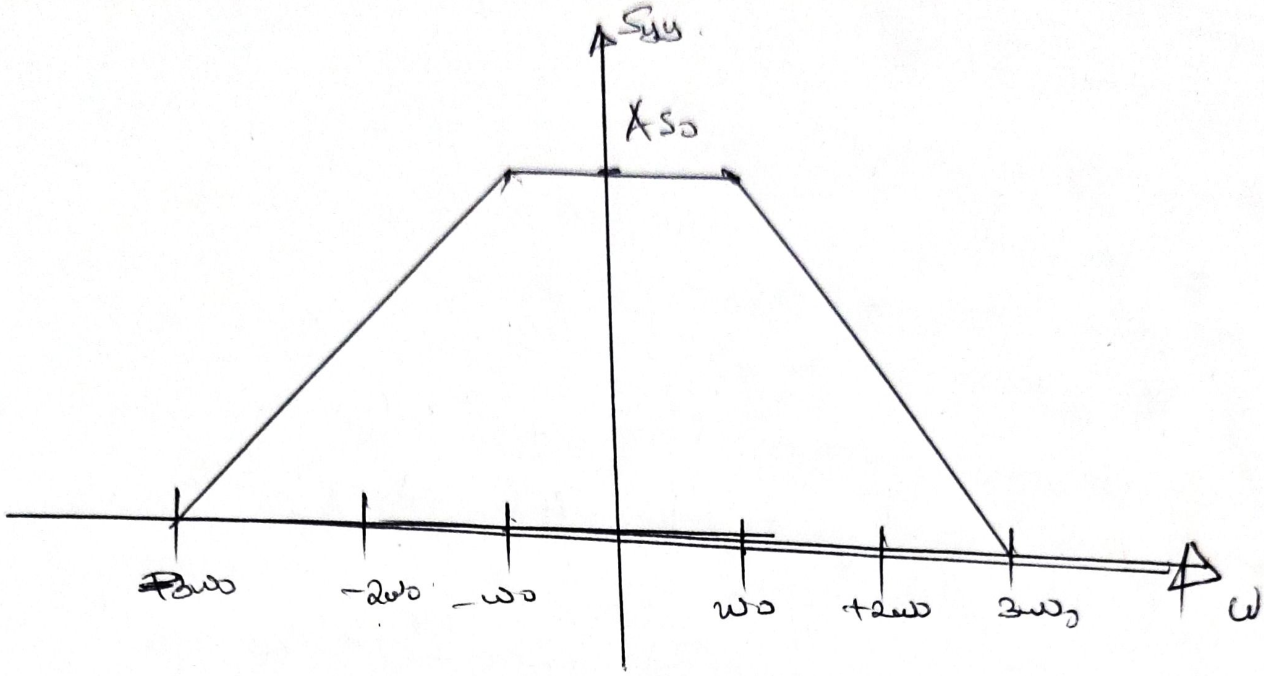
$$= A^2 \int_{-\infty}^{+\infty} R_{xx}(\tau) \left(e^{+j\omega_0\tau} + e^{-j\omega_0\tau} \right) e^{-j\omega\tau} d\tau$$

$$= A^2 \int_{-\infty}^{+\infty} R_{xx}(\tau) \left(e^{-j(\omega-\omega_0)\tau} + e^{-j(\omega+\omega_0)\tau} \right) d\tau$$

$$= A^2 \cdot S_{xx}(\omega-\omega_0) + A^2 S_{xx}(\omega+\omega_0)$$

sketch for S_{xx}





Syy
Xs₀
-3w₀
-2w₀
-w₀
w₀
+2w₀
3w₀
w