Task 5.3

a) →

The mean of the process is related to the long term behavior of the autocorrelation function as $\tau \to \infty$ and mean $(m_x^{(1)}) = 1$ (given)

$$m_x^{(1)} = \lim_{\tau \to \infty} S_{xx}(\tau) = 0$$

$$\Rightarrow \lim_{\tau \to \infty} a e^{-\alpha|\tau|} + b = 0$$

$$\Rightarrow 0 + b = 0 \quad \therefore \boxed{b = 0}$$

The standard deviation is related ~~th~~ to the autocorrelation function at $\tau \to 0$

Now, we know that,
standard deviation $(\sigma) = 1$ .. (given)

$$\sigma_x^2 = m_x^{(2)} - [m_x^{(1)}]^2$$

$$(1)^2 = m_x^{(2)} - [0]^2$$

$$\therefore m_x^{(2)} = 1$$

$$\therefore m_x^{(2)} = \lim_{\tau \to 0} S_{xx}(\tau) = 1$$

$$\Rightarrow \lim_{\tau \to 0} a \cdot e^{-\alpha|\tau|} + b = 1$$

put $b = 0$

$$\therefore \Rightarrow \lim_{\tau \to 0} a \cdot e^{-\alpha|\tau|} = 1$$

$$\Rightarrow a \cdot e^{0} = 1 \Rightarrow \boxed{a = 1}$$

b) →

$$S_{xy}(t_1, t_2) = E\{x(\zeta, t_1)\, y(\zeta, t_2)\}$$

---- (given)

$$\text{but } y(\zeta, t) = \int_{t_0}^{t} x(\zeta, \lambda)\, d\lambda \quad \text{(for } t > t_0\text{)}$$

$$\therefore S_{xy}(t_1, t_2) = E\left\{x(\zeta, t_1) \int_{t_0}^{t} x(\zeta, \lambda)\, d\lambda\right\}$$

$$= \int_{t_0}^{t} E\{x(\zeta, t_1) \cdot x(\zeta, \lambda)\}\, d\lambda$$

$$\text{Here } E\{x(\zeta, t_1) \cdot x(\zeta, \lambda)\} = S_{xx}(\tau)$$

$$\therefore S_{xy}(t_1, t_2) = \int_{t_0}^{t} S_{xx}(\tau)\, d\lambda$$

$$= \int_{t_0}^{t} a \cdot e^{-\alpha|\tau|} + b \, d\lambda$$

put the values $a=1$ and $b=0$

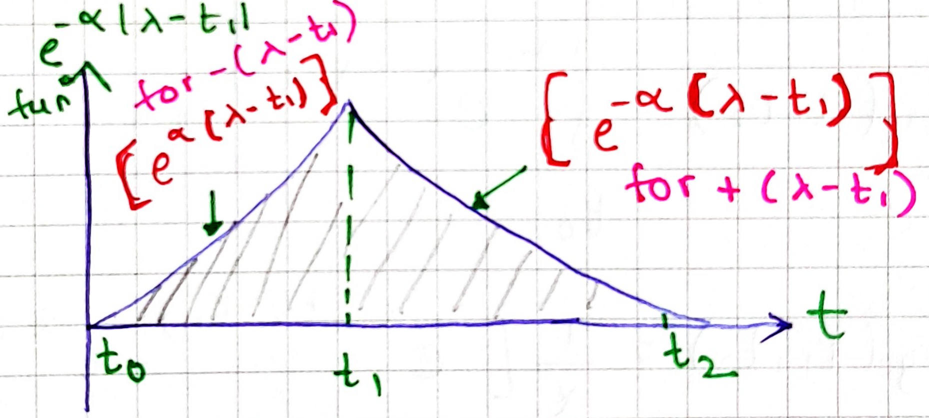
$$\therefore S_{xy}(t_1,t_2) = \int_{t_0}^{t} e^{-\alpha|\tau|}\, d\lambda$$

Here τ is nothing but the time difference which can be shown as $\tau = t_2 - t_1$

Here in this case, $\tau = \lambda - t_1$

$$\therefore S_{xy}(t_1,t_2) = \int_{t_0}^{t} e^{-\alpha|\lambda - t_1|}\, d\lambda$$

Case I :- for $t_0 < t_1 < t_2$ $\left[|\lambda - t_1| = \begin{matrix} -(\lambda - t_1)\alpha \\ +(\lambda - t_1) \end{matrix} \right]$



$$\therefore S_{xy}(t_1,t_2) = \int_{t_0}^{t_1} e^{\alpha(\lambda - t_1)}\, d\lambda + \int_{t_1}^{t_2} e^{-\alpha(\lambda - t_1)}\, d\lambda$$

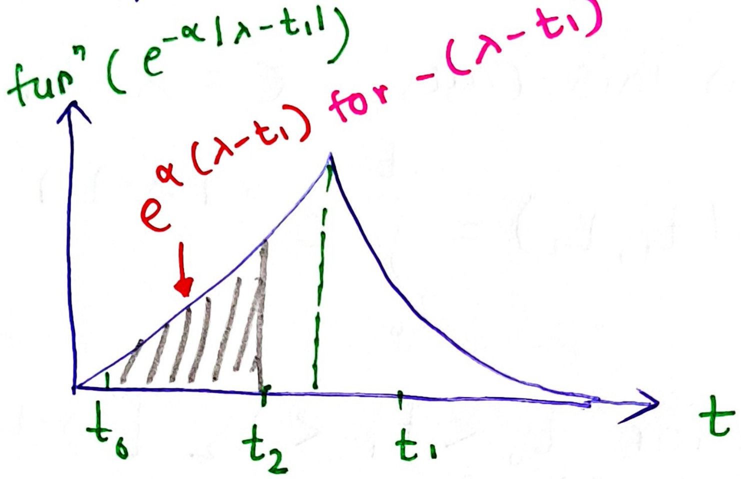
$$= \left.\frac{e^{\alpha(\lambda - t_1)}}{\alpha}\right|_{t_0}^{t_1} + \left.\frac{e^{-\alpha(\lambda - t_1)}}{-\alpha}\right|_{t_1}^{t_2}$$

$$= \frac{1}{\alpha}\left[1 - e^{\alpha(t_0 - t_1)} - \left[e^{-\alpha(t_2 - t_1)} - 1\right]\right]$$

$$\Rightarrow \frac{1}{\alpha}\left[2 - e^{\alpha(t_0-t_1)} - e^{-\alpha(t_2-t_1)}\right]$$

Case II :- for $t_0 < t_2 < t_1$

Here in this case integral will apply only on negative part.

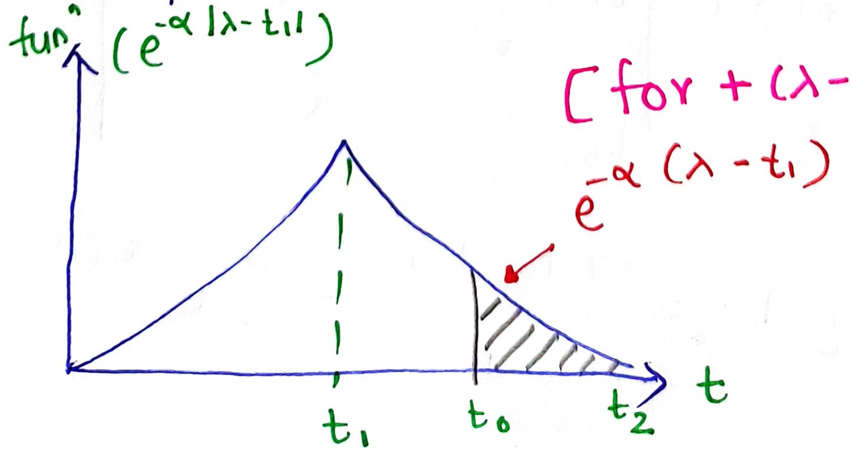


$$\therefore S_{xy}(t_1 t_2) = \int_{t_0}^{t_2} e^{\alpha(\lambda-t_1)}\, d\lambda$$

$$\therefore S_{xy}(t_1, t_2) = \frac{1}{\alpha}\left[e^{\alpha(t_2-t_1)} - e^{\alpha(t_0-t_1)}\right]$$

Case III :- for $t_1 < t_0 < t_2$

Now in this case integral will apply only on positive part.



$$\therefore S_{xy}(t_1, t_2) = \int_{t_0}^{t_2} e^{-\alpha(\lambda - t_1)} \, d\lambda$$

$$= -\frac{1}{\alpha}\left[e^{-\alpha(t - t_1)} - e^{-\alpha(t_0 - t_1)}\right]$$

Case IV :- 0 ; elsewhere.

c) → $y(\zeta, t)$ is non stationary process as the function ~~demp~~ depends on time t.
($S_{xy}(t_1, t_2)$)

Also remember →

- stationary random process → does not depend on τ or t
- weakly stationary random process → depends on τ
- Non-Stationary random process → depends on t.