Part 1

1.1 Probability

Task 1.1.1

In a random experiment three light bulbs are turned on simultaneously.

It must be assumed that every single light bulb may be faulty.

Define the simplest possible sample space for this random experiment contains the events

$$A_1 = \{exactly\ one\ light\ bulb\ burns\}$$

and

$$A_2 = \{maximum\ of\ two\ light\ bulbs\ burn\}$$

Suppose that the probabilities of the events A_1 *and* A_2 are given as

$$P(A_1) = \frac{1}{4} \quad und \quad P(A_2) = \frac{1}{2}$$

Task 1.1.2

The association of two disjoint events A and B is the certain event. From the event X and the events A and B are known, the conditional probabilities:

$$P(X|A) = \frac{1}{4}, P(X|B) = \frac{1}{3}, P(A|X) = \frac{1}{2}.$$

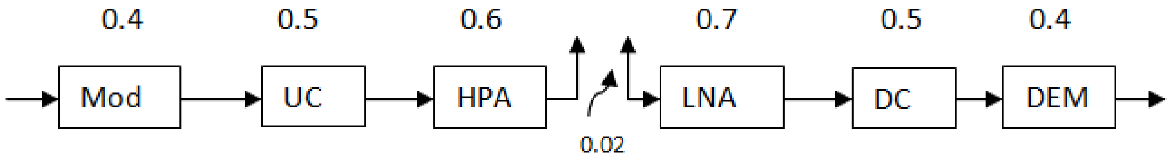
Determine P (A) and P (B).

Task 1.1.3

For the modules of an electronic device 10000 resistors of a certain resistance (Ω-value) are necessary. The resistors have been bought from three different manufacturers.
5000 pieces from manufacturer A_1 (1% of the resistors do not meet the specification),
3000 pieces from manufacturer A_2 (2% of the resistors do not meet the specification),
2000 pieces from manufacturer A_3 (5% of the resistors do not meet the specification).
How is the probability of the event B that an arbitrarily picked resistor is out of specification?

Task 1.1.4

For the radio transmission chain



the following probabilities for the components within a time interval T are given:

Modulator breakdown, event A_1: $P(A_1)=0.4$
Up-Converter break down, event A_2: $P(A_2)=0.5$
Power amplifier break down, event A_3: $P(A_3)=0.6$
Transmission media break down, event A_4: $P(A_4)=0.02$
Low noise amplifier break down, event A_5: $P(A_5)=0.7$
Down converter break down, event A_6: $P(A_6)=0.5$
Demodulator break down, event A_7: $P(A_7)=0.4$

How is the probability that no interrupt occurs?

1.2 Probability density function (PDF) and cumulative distribution function (CDF)

Task 1.2.1

The probability density function (pdf) of the random variable x (ζ) is given:

$$f_x(x) = \begin{cases} \frac{k}{8} e^{-\frac{x}{k}+2} & x \geq 0 \\ 0 & otherwise \end{cases}$$

a) Calculate the constant k.
b) Calculate the mean and the variance (m_x, σ_x^2) of the random variable x (ζ).
c) Calculate the probability $P(\{-1 \leq x(\,\zeta\,) < 2\})$.

Task 1.2.2

Given are two random variables x (ζ) and y (ζ). For the region $|x| \leq 4$ and $0 \leq y \leq 2 - \frac{|x|}{2}$ they have a constant joint density $f_{xy}(x, y)$.

a) Calculate the joint density $f_{xy}(x, y)$.
b) Are the random variables x (ζ) and y (ζ) statistically independent?
c) Calculate the correlation coefficient r_{xy}.
d) How is the probability that the sum of x (ζ) and y (ζ) exceeds the value (2 – y (ζ)) ?

Task 1.2.3

Given are two discrete statistically independent random variables x (ζ) and y (ζ). The variable x (ζ) shows the values +1 and -1 uniformly at random. The probability density $f_y(y)$ is given by:

$$f_y(y) = \frac{1}{3}\,\delta(y-1) + \frac{1}{3}\,\delta(y-2) + \frac{1}{3}\,\delta(y-3).$$

Another random variable z (ζ) is formed as follows:

$$z(\,\zeta\,) = y(\,\zeta\,) + x(\,\zeta\,).$$

a) Calculate the correlation coefficient r_{xy} and r_{xz}.
b) Sketch the probability density $f_z(z)$.

1.3 Random processes

Task 1.3.1

A random experiment is given by the toss of a coin. We distinguish two outcomes:

$$\zeta_1 = \{heraldic\ figures\} \quad \text{and}$$

$$\zeta_2 = \{number\}.$$

To each outcome we assign one of the following functions and thus obtain the random processes:

$$x(\zeta_1, t) = x_0 \cos(\pi \tfrac{t}{T})$$

and

$$x(\zeta_2, t) = x_0 \left| \sqrt{\frac{t}{T}} \right| \qquad t > 0, T > 0$$

Furthermore, we define the events

$$A_1 = \{\zeta_1\} \quad and \quad A_2 = \{\zeta_2\},$$

that occur with the following probabilities:

$$P(A_1) = P(A_2) = \tfrac{1}{2}.$$

a) Calculate the cumulative distribution function $F_x(x,t)$ for $t = \frac{T}{4}, \frac{T}{2}, T$.
b) Is the random process x (ζ, t) stationary, weakly stationary or non-stationary? (Give a reason)

Task 1.3.2

A random variable x (ζ) takes the value 1 with probability p and the value 2 with probability (1 – p). A random process y (ζ, t) is defined as follows:

$$y(\zeta, t) = \begin{cases} x_0 & kT \cdot x(\zeta) \le t < \left(k + \frac{1}{2}\right) T \cdot x(\zeta) \\ -x_0 & \left(k + \frac{1}{2}\right) T \cdot x(\zeta) \le t < (k+1) T\, x(\zeta) \\ & k = integer. \end{cases}$$

a) Sketch two sample functions of the process y (ζ, t).
b) Calculate the average $m_y(t)$ and the variance $\sigma_y^2(t)$ of the random process y (ζ, t).
c) Is y (ζ, t) stationary, weakly stationary or non-stationary? (rationale)

Task 1.3.3

Let a random process $x(\zeta, t) = a(\zeta)\sin\omega t + b(\zeta)\cos\omega t$. $a(\zeta)$ and $b(\zeta)$ are statistically independent random variables of zero-mean. The variances σ_a^2 and σ_b^2 are known.

a) Calculate the mean $m_x(t)$ and the variance $\sigma_x^2(t)$ of the random process x (ζ, t).
b) Make a statement about the wide-sense stationarity of the random process x (ζ, t).

Task 1.3.4

A pulse-width-modulated random process y (ζ, t) is defined as follows:

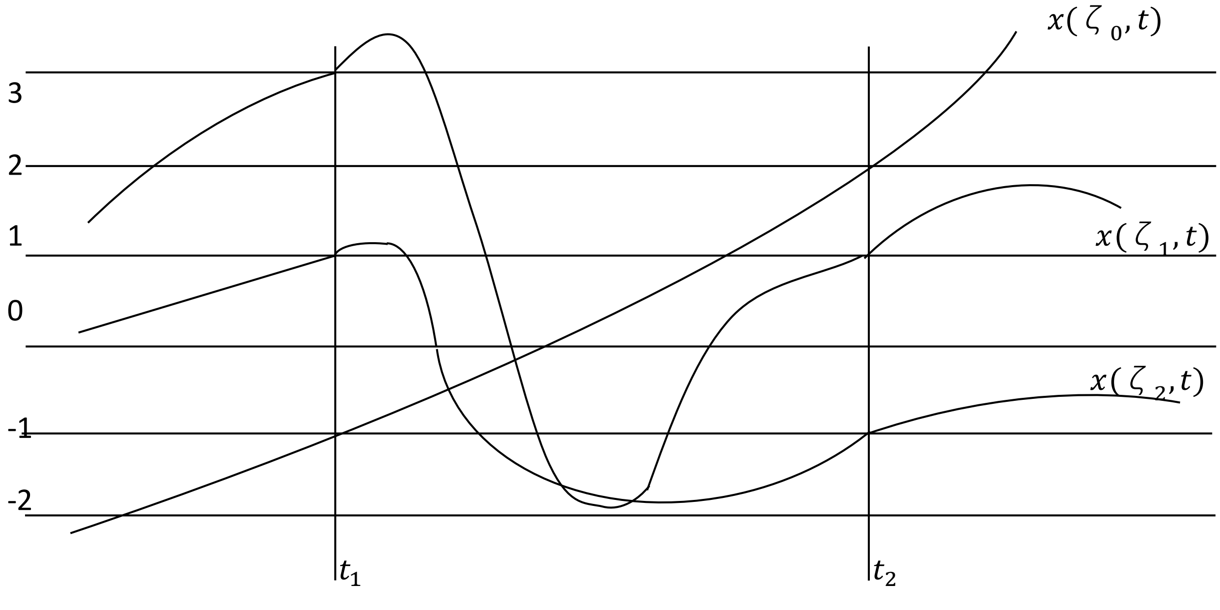
$$y(\zeta, t) = \begin{cases} +2 & iT \le t < iT + b(\zeta, iT) \\ -1 & iT + b(\zeta, iT) \le t < (i+1)T \end{cases} \qquad i \in Z$$

The (discrete-time) random process y (ζ, iT) is stationary. Its amplitudes are uniformly at random in the interval [0, T]. b (ζ, iT) and b (ζ, jT) are statistically independent for all i ≠ j.

a) Sketch a sample function of the random process.
b) Specify the average $m_y(t)$.

Task 1.3.5

Given a random process x (ζ, t) with an ensemble of three sample functions:



The sample functions occur with the following probabilities:

$$P(\{\zeta_0\}) = \tfrac{1}{2}, P(\{\zeta_1\}) = \tfrac{1}{3}, P(\{\zeta_2\}) = \tfrac{1}{6}.$$

Determine $s_{xx}(t_1, t_2)$, $i.e.$ the autocorrelation function at t_1 and t_2.

Task 1.3.6

A random variable x (ζ) takes the value x_1 with probability p and the value x_2 with probability (1 – p) . A random process y (ζ, t) is defined as follows:

$$y(\zeta,t) = \begin{cases} x(\zeta) & for\ t < T \\ -x(\zeta) & for\ t \geq T \end{cases}$$

a) Sketch a sample function of the random process y (ζ, t).
b) Calculate mean value, variance and autocorrelation function of the random process.
c) If the random process strictly stationary, weakly stationary or non-stationary?

Task 1.3.7

Given are two random variables $x_1(\zeta)$ and $x_2(\zeta)$. For any of the random variables one has

$$x_i(\zeta) = \begin{cases} +1 & with\ P = 1/2 \\ -1 & with\ P = 1/2 \end{cases} \qquad for\ i = 1,2.$$

The random variables are statistically independent. A random process is defined by

$$y(\zeta,t) = \begin{cases} x_1(\zeta) & for\ 0 \leq t < T \\ x_1(\zeta) + x_2(\zeta) & for\ T < -t < 2T \\ 0 & otherwise \end{cases}$$

a) Sketch each pattern of this random process function probability occurring with P>0.
b) Determine the probability density function $f_y(y,t)$, the average $m_y^{(1)}(t)$ and the scattering $\sigma_y(t)$.
c) Is the random process an ergodic process? (rationale)
d) Calculate the autocorrelation function $s_{yy}(t_1,t_2)$ and sketch it.

Task 1.3.8

Let $a(\zeta)$ be a random variable taking the value 0 with probability $P_0 = \frac{1}{4}$ and the value 1 with probability $P_1 = \frac{3}{4}$. A random process x (ζ, t) is defined as:

$$x(\zeta,t) = \begin{cases} 1 - \frac{4}{T}\, t\, a(\zeta) & for\ 0 \leq t < \frac{T}{2} \\ -1 + \left(\frac{4}{T}\, t - 2\right) a(\zeta) & for \frac{T}{2} \leq t < T \\ 0 & otherwise \end{cases}$$

a) Sketch all distinct pattern functions of the random process x (ζ, t).
b) Calculate the mean $m_x^{(1)}(t)$.
c) Calculate the autocorrelation function $s_{xx}(t_1,t_2)$.
d) Calculate the variance $\sigma_x^2(t)$.
e) Is the random process stationary? (Reason).

Task 1.3.9

A random process x (ζ, t) is given by

$$x(\zeta,t) = A(\zeta)\sin(\frac{t}{T}\pi + \varphi(\zeta)) \quad with\, T > 0.$$

The random variable $A(\zeta)$ takes the value +1 and 0 with equal probability. The probability density function of the random variable $\varphi(\zeta)$ is

$$f_\varphi(\varphi) = p\,\delta(\varphi) + (1-p)\delta\left(\varphi + \frac{\pi}{2}\right) \quad with\ 0 \le p \le 1$$

The random variable $A(\zeta)$ and $\varphi(\zeta)$ are statistically independent.

a) Sketch all distinct pattern functions of the random process x (ζ, t) and specify the probabilities with which the sample functions occur.
b) Sketch the cumulative distribution function $F_x(x,t)$ at the time t = 0 and t = T / 4
c) Calculate the mean $m_x^{(1)}(t)$ of the random process $x(\zeta,t)$.
d) Determine the covariance function $c_{xx}(t_1,t_2)$.
e) Why is the random process is not ergodic?

Task 1.3.10

Given is a random variable a (ζ) with the following probability density function:

$$f_a(a) = \frac{1}{3}\,\delta(a) + \frac{2}{3}\,\delta(a+1)$$

A random process is constructed as:

$$x(\zeta,t) = \begin{cases} (1 + a(\zeta))\sin\frac{2\pi}{T}\,t & t < 0 \\ a(\zeta)\frac{t}{T} + (1 + a(\zeta))\sin\frac{2\pi}{T}\,t & t \ge 0 \end{cases}$$

a) Sketch all pattern functions of the random process that occur with non-zero probability.
b) Calculate the mean $m_x(t)$ of the random process $x(\zeta,t)$.
c) Calculate the autocorrelation function $s_{xx}(t_1,t_2)$ of the random process x (ζ, t).
d) Is the random process x (ζ, t) ergodic? (rationale)

Task 1.3.11

Given is the joint density function

$$f_{xy}(x,y) = \begin{cases} 2 & for\ x \ge 0\ and\ x + y\ \le 1 \\ 0 & elsewhere \end{cases}$$

Calculate $f_x(x)$, $f_y(y)$, $F_{xy}(x,y)$, $F_x(x)$, $F_y(y)$

Task 1.3.12

The probability density function $f_a(a)$ of the random variable a (ζ) is given as:

$$f_a(a) = \begin{cases} 1 & for\ 1 \le a \le 2 \\ 0 & otherwise \end{cases}$$

Let a random process x (ζ, t) be

$$x(\zeta, t) = \begin{cases} t\, e^{-a(\zeta)t} & for\ t \ge 0 \\ 0 & otherwise \end{cases}$$

a) Calculate the mean $m_x^{(1)}(t)$ and
b) the autocorrelation function $s_{xx}(t_1, t_2)$.

Task 1.3.13

A non-stationary random process y (ζ, t) is defined as follows:

$$y(\zeta, t) = \begin{cases} e^{-a(\zeta)t} \cos x(\zeta) & for\ t \ge 0 \\ 0 & otherwise \end{cases}$$

The random variable $x\ (\zeta)$ is uniformly distributed in the interval $[0, \pi]$
and the random variable $a\ (\zeta)$ is uniformly distributed in the interval $[0,1]$.
The random variables $a\ (\zeta)$ and $x\ (\zeta)$ are statistically independent.
Calculate

a) The mean $m_y^{(1)}(t)$ and
b) The covariance function $c_{yy}(t_1, t_2)$.

Task 1.3.14

A stationary random process x (ζ, t) has the autocorrelation function

$$s_{xx}(\tau) = e^{-\alpha|\tau|} \quad for\ \alpha > 0$$

Another random process is defined by

$$y(\zeta, t) = \begin{cases} \int_0^t x(\zeta, \lambda)\, d\lambda & for\ t \ge 0 \\ 0 & otherwise \end{cases}$$

a) Calculate the mean $m_y^{(1)}(t)$
b) Determine the autocorrelation function of the random process y (ζ, t)
c) How is variance of the random process y (ζ, t)?
d) Is y (ζ, t) is a stationary random process?

Task 1.3.15

From a stationary random process x(ζ, t) the mean (=0), the standard deviation (=1) and the autocorrelation function

$$s_{xx}(\tau) = \; a\, e^{-\alpha|\tau|} + b$$

are known.

A random process y (ζ, t) is given by

$$y(\zeta\,, t) = \begin{cases} 0 & for\; t \le t_0 \\ \int_{t_0}^{t} x(\zeta\,, \lambda) d\lambda & for\; t > t_0 \end{cases}$$

a) Determine the constants a and b.
b) Determine the cross-correlation function

$$s_{xy}(t_1, t_2) = E\{x(\zeta\,, t_1)\; y\,(\zeta\,, t_2)\}$$

c) Is the process y (ζ, t) stationary?

Task 1.3.16

Given is a random process

$$x(\zeta\,, t) = \sin\frac{2\pi}{T}\;\left(t - \frac{\varphi(t)T}{3}\right)$$

$\varphi(\zeta)$ takes at the values 1, 2 or 3 with equal probability.

a) Calculate the mean $m_x(t)$
b) How is the autocorrelation function $s_{xx}(t_1, t_2)$?
c) Show that the process is wide-sense stationary.
d) The process x (ζ, t) is sampled at the instants kT. From the samples the discrete-time random process y (ζ, kT) is formed by the following method:

$$y(\zeta\,, kT) = \begin{cases} \sum_{i=1}^{k} x(\zeta\,, iT) & for\; k > 0 \\ 0 & otherwise \end{cases}$$

Calculate the auto-correlation function

$$s_{yy}(kT, lT) = E\{y(\zeta\,, kT)\; y(\zeta\,, lT)\}$$

Task 1.3.17

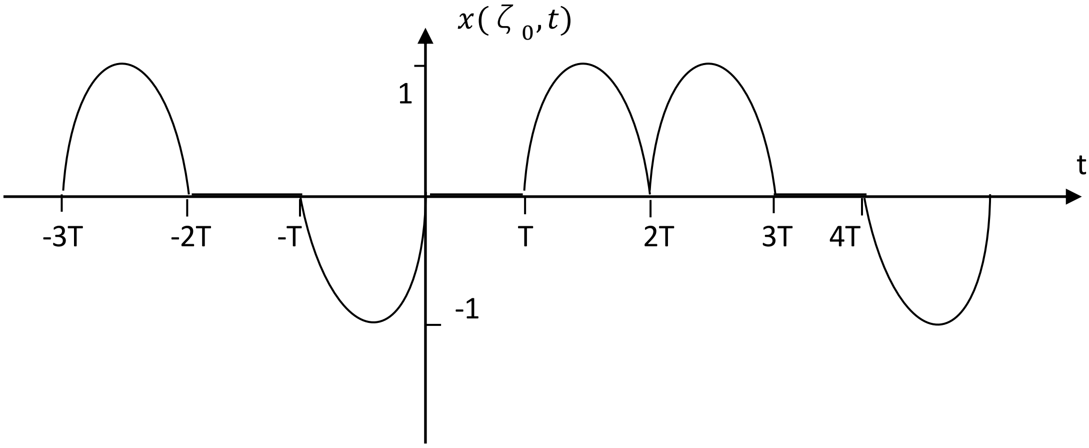
A random process is defined as:

$$x(\zeta,t) = \sum_{k=-\infty}^{\infty} a(\zeta,kT)\, s(t-kT)$$

$a(\zeta,kT)$ is a random variable with the values +1, -1, and 0. These values are equiprobale. The random variables a (ζ, iT) and a (ζ, jT), i ≠ j, are uncorrelated. Furthermore, it applies

$$s(t) = \begin{cases} \sin\frac{\omega_0}{2}\, t & for\ 0 \le t \le T \\ 0 & otherwise \end{cases}$$

with $\omega_0 = \frac{2\pi}{T}$. A sample function of the random process thus provides, for example, like this:



a) Calculate the mean $m_x^{(1)}(t)$ and the autocorrelation function $s_{xx}(t_1,t_2)$ of the random process x (ζ, t).
b) Is the random process x (ζ, t) stationary?

By sampling the random process x (ζ, t) at the instants of time t = iT + T / 2, i ∈ Z, a discrete random process y (ζ, iT) is produced.

c) Calculate mean, autocorrelation function and variance of this discrete random process y (ζ, iT).

Note: The autocorrelation function of a discrete random process y (ζ, iT) is defined by

$$s_{yy}(lT,kT) = E\{y(\zeta,lT)\, y(\zeta,kT)\}$$

Task 1.3.18

Let a discrete stationary random process x (ζ, t).
The outcomes of the process are the values $x_1 = -1, x_2 = 0 \ and \ x_3 = 1$
The probabilities of the occurrence of those outcomes are

$$P(\{x(\zeta, t+\tau) = x_i\} | \{x(\zeta, t) = x_j\}) = \begin{cases} \frac{1}{3}\left(1 + 2\, e^{-|\tau|}\right) & for \ i = j \\ \frac{1}{3}\left(1 - e^{-|\tau|}\right) & for \ i \neq j \end{cases} \qquad i, j = 1,2,3$$

a) Calculate the probabilities

$$P(\{x(\zeta, t) = x_i\}) \quad for \ i = 1,2,3.$$

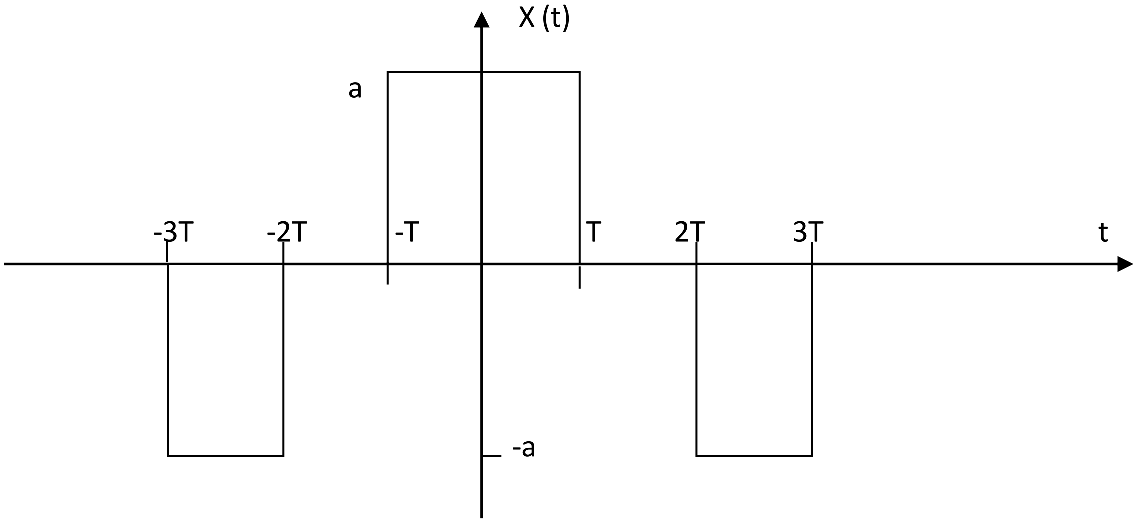
b) Calculate the ACF $s_{xx}(\tau)$

Part 2

2.1 Correlation functions of deterministic signals

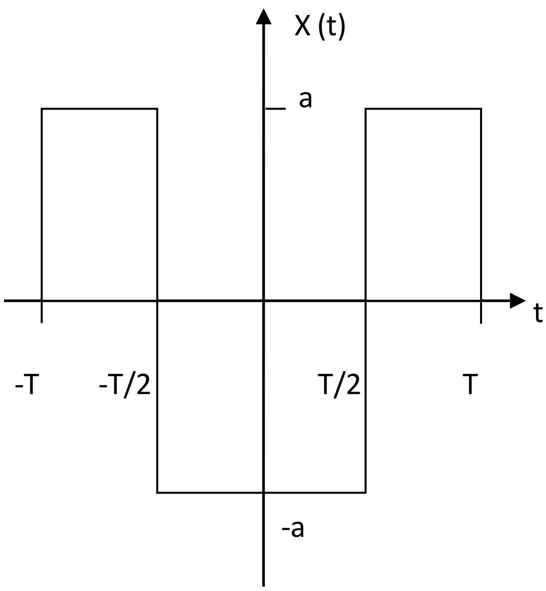
Task 2.1.1

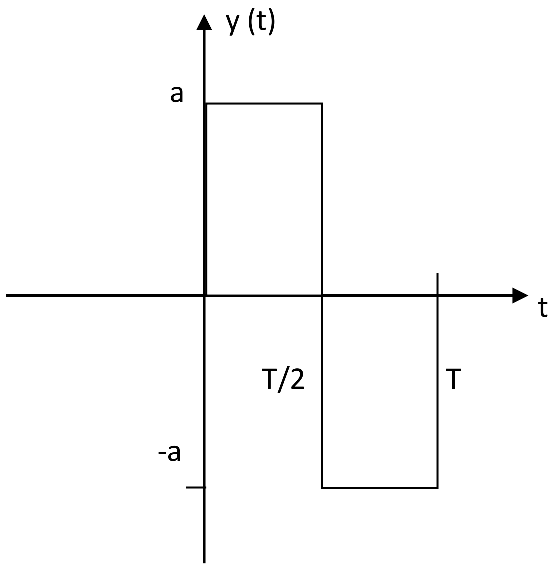
a) Given is the following signal x (t):



Sketch the autocorrelation function $\tilde{s}_{xx}(\tau)$.

b) Let two deterministic signals x (t) and y (t) of finite energy.

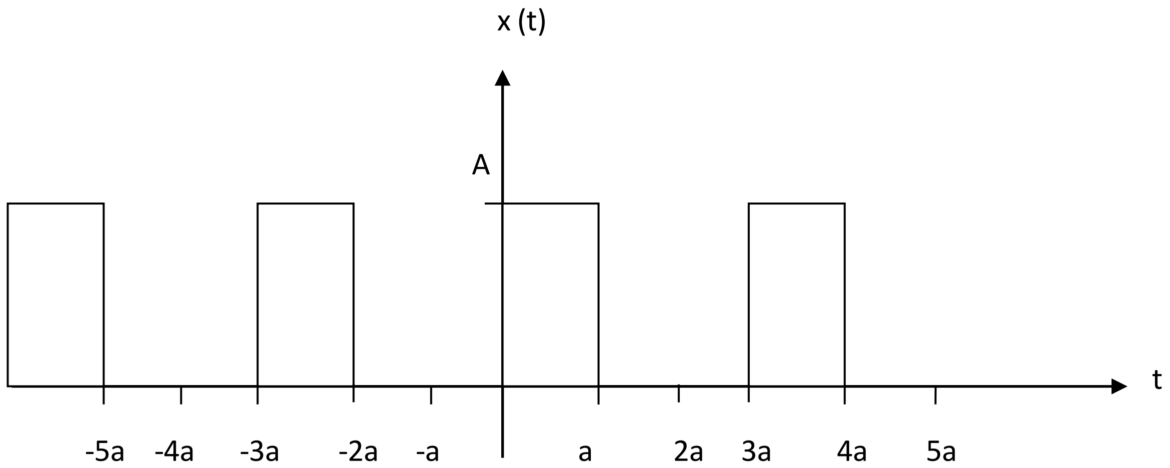




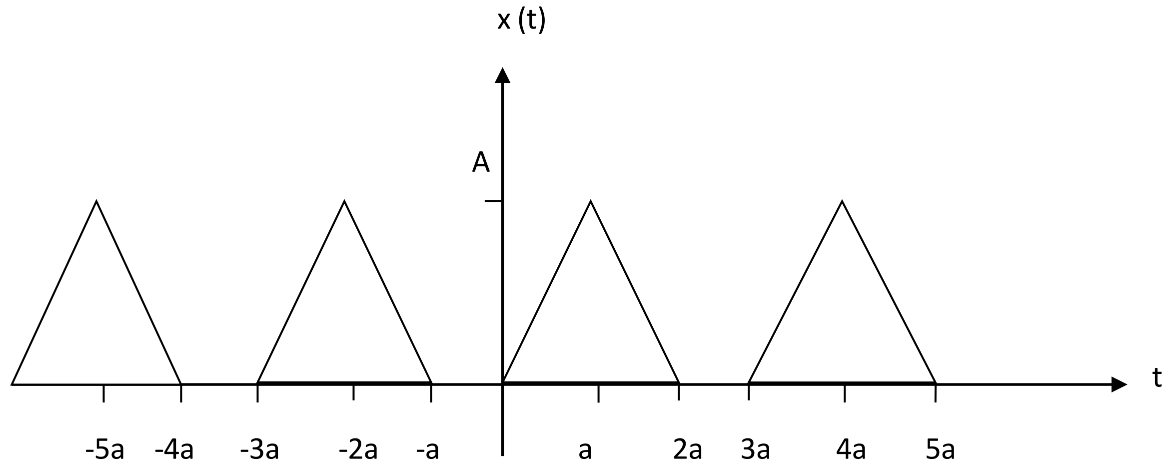
Sketch the cross-correlation function $\tilde{s}_{xy}(\tau)$.

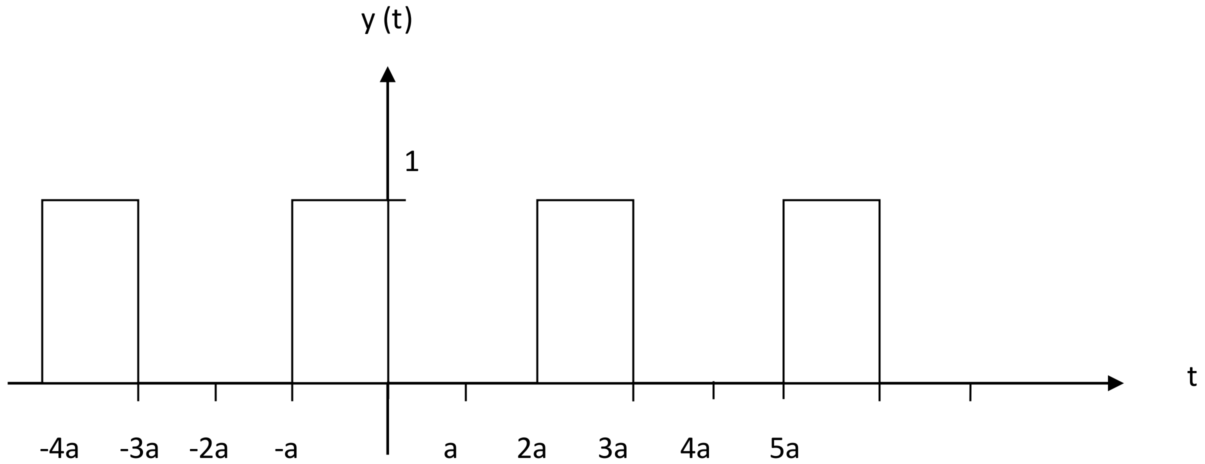
Task 2.1.2

a) Determine the autocorrelation function of the following periodic signal:



b) Two periodic signals x(t) and y(t):

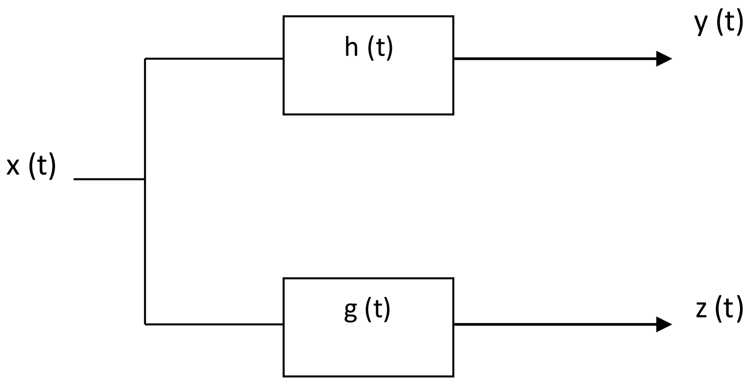




Calculate and sketch the cross-correlation function of these signals.

Task 2.1.3

Given is the following block diagram:



The two impulse responses h (t) and g (t) are:

$$h(t) = \begin{cases} A & 0 \le t < T \\ 0 & otherwise \end{cases} \qquad g(t) = \begin{cases} A & 0 \le t < \frac{T}{2} \\ 0 & otherwise \end{cases}$$

The input of the system the deterministic signal x (t).

a) Assuming that

$$x(t) = B \cdot (\delta(t) - \delta(t-2T) + \delta(t-3T))$$

sketch the functions,

$$s_{yy}(\tau) = \lim_{u\to\infty} \int_{-u}^{+u} y(t)y\,(t+T)dt$$

$$s_{zz}(\tau) = \lim_{u\to\infty} \int_{-u}^{+u} z(t)z(t+T)dt \qquad and$$

$$s_{yz}(\tau) = \lim_{u\to\infty} \int_{-u}^{+u} y(t)z(t+T)dt$$

b) Assume that

$$x(t) = B \sum_{k=-\infty}^{+\infty} (-1)^k\, \delta(t-2kT)$$

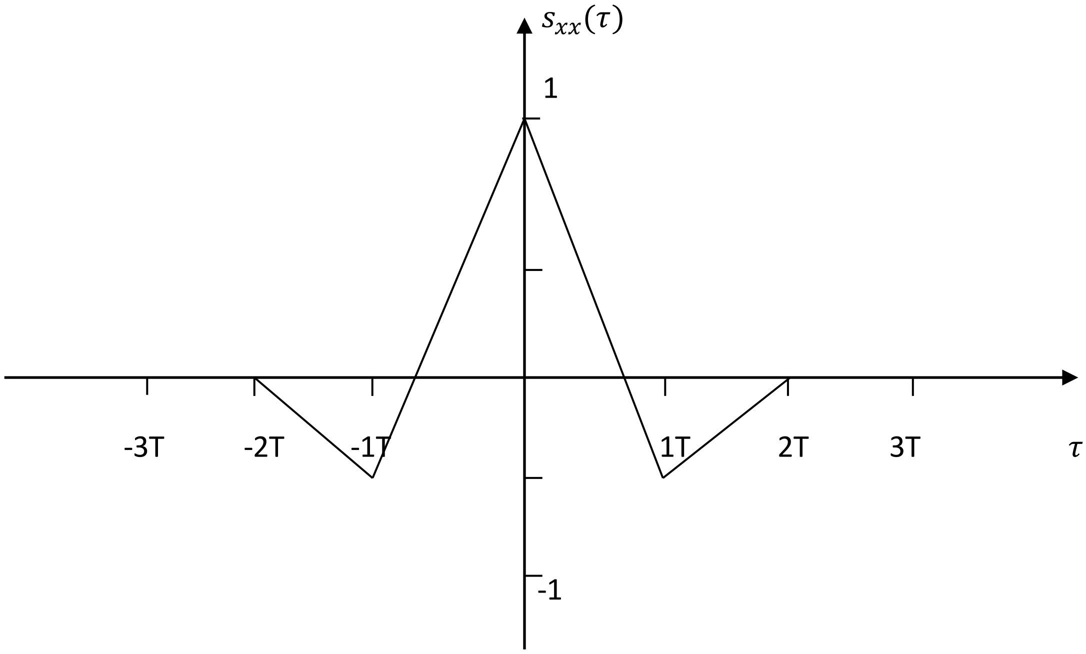
and sketch the function

$$s_{yy}(\tau) = \lim_{u\to\infty} \frac{1}{2u} \int_{-u}^{+u} y(t)y(t+T)dt$$

2.2 Power spectral density (PSD)

Task 2.2.1

This is the autocorrelation function $s_{xx}(\tau)$ of the stationary test random process x (ζ, t):



Calculate and sketch the power spectral density $S_{xx}(\omega)$.

Task 2.2.2

Given is the following random process:

$$z(\zeta, t) = x(\zeta, t) + y(\zeta, t)$$

The stationary random processes x (ζ, t) and y (ζ, t) are statistically independent.

The following moments are known:

$$E\{x(\zeta, t)\} = m_x$$

$$E\{y(\zeta, t)\} = m_y$$

$$c_{xx}(\tau) = e^{-\frac{|\tau|}{T_1}}, \quad T_1 > 0$$

$$c_{yy}(\tau) = e^{-\frac{|\tau|}{T_2}}, \quad T_2 > 0$$

Calculate the autocorrelation function $s_{zz}(\tau)$ and the power spectral density $S_{zz}(\omega)$ of the random process z (ζ, t).

Task 2.2.3

Given is the autocorrelation function $s_{xx}(\tau)$ of a stationary random process x (ζ, t) is:

$$s_{xx}(\tau) = 2 + \frac{1}{1 + \frac{\tau^2}{T^2}}$$

Another random process y (ζ, t) will be defined as follows:

$$y(\zeta, t) = x(\zeta, t) + x(\zeta, t - 8T)$$

a) Calculate $s_{yy}(\tau)$
b) Determine the mean values m_x, m_y and the mean square values $m_x^{(2)}, m_y^{(2)}$
c) Let $S_{xx}(\omega)$ be the power spectral density of x (ζ, t). Calculate the relationship between $S_{yy}(\omega)\ and\ S_{xx}(\omega)$.

Task 2.2.4

Given is the autocorrelation function $s_{xx}(\tau)$ of a stationary random process x (ζ, t) is:

$$s_{xx}(\tau) = \begin{cases} 1.25 - \frac{|\tau|}{T} & |\tau| < T \\ 0.25 & otherwise \end{cases}$$

A second random process y (ζ, t) is defined as:

$$y(\zeta, t) = x(\zeta, t) + a \quad ; \quad a = constant$$

Calculate and sketch the power spectral densities $S_{xx}(\omega)\ and\ S_{yy}(\omega)$.

Note: $s_{xx}(\tau)$ may be represented by the sum of two functions that can be separately transformed to the frequency domain.

Task 2.2.5

Given are two random processes

$x(\zeta, t) = 1 + \sin(\omega_0 t + \varphi_1(\zeta))$ and $y(\zeta, t) = 1 + \cos(\omega_o t + \varphi_2(\zeta))$

The random variables $\varphi_1(\zeta)\ and\ \varphi_2(\zeta)$ have the following probability densities:

$$f_{\varphi_i}(\varphi_i) = \begin{cases} \frac{1}{2\pi} & |\varphi_i| \le \pi \\ 0 & otherwise \end{cases} \qquad i = 1,2$$

a) Calculate $S_{xy}(\omega),\ if\ \varphi_1(\zeta)\ if\ \varphi_2(\zeta)$ are statistically independent.
b) Calculate $S_{xy}(\omega)\ for\ \varphi_1(\zeta) = \varphi_2(\zeta) = \varphi(\zeta)$.

Task 2.2.6

A stationary random process x (ζ, t) has the power spectral density

$$S_{xx}(\omega) = \begin{cases} S_0 \left(1 - \dfrac{|\omega|}{2\omega_0}\right) & |\omega| \leq 2\omega_0 \\ 0 & otherwise \end{cases}$$

Let another random process s (ζ, t):

$$s(\zeta, t) = 2A \sin(\omega_o t + \varphi(\zeta)) \quad ; \quad A = constant.$$

The phase angle $\varphi(\zeta)$ is uniformly at random in the interval [0, 2π]. The random processes x (ζ, t) and s (ζ, t) are statistically independent. A third random process will be constituted as amplitude modulation:

$$y(\zeta, t) = s(\zeta, t)x(\zeta, t)$$

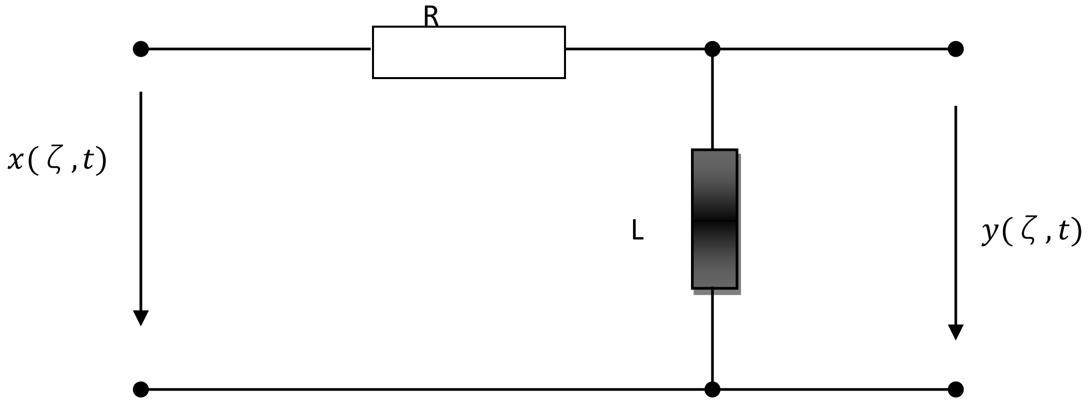
a) Determine the second moment $m_x^{(2)}$ of the random process x (ζ, t).
b) Calculate and sketch the power spectral density $S_{yy}(\omega)$ of the random process $y(, t)$.

Part 3: Linear Systems

3.1 Autocorrelation function, crosscorrelation function, power spectral density

Task 3.1.1

Given is the the linear system below:



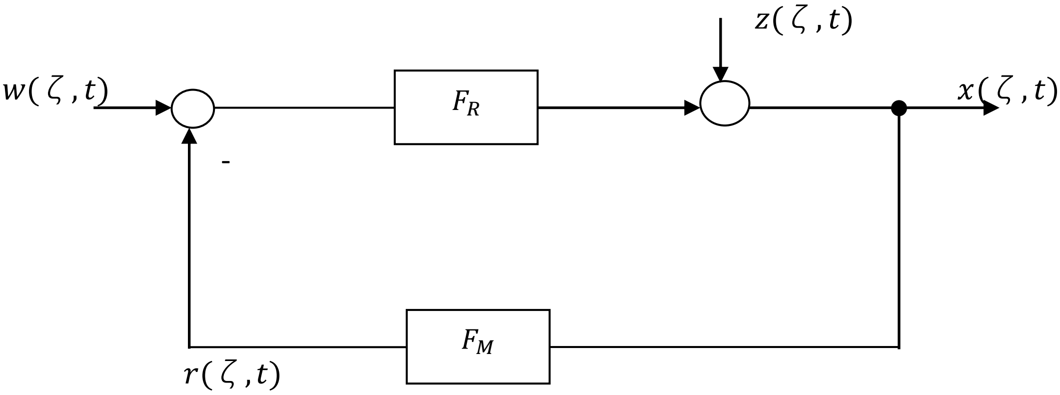
Let the stationary random process x (ζ, t):

$$x(\zeta, t) = A + B\sin(\omega_0 t + \varphi(\zeta))$$

a) Calculate the mean of the random process y (ζ, t).
b) Determine and sketch the power spectral densities $S_{xx}(\omega)$ *and* $S_{yy}(\omega)$.

Task 3.1.2

Given is a control loop:



The power spectral densities $S_{ww}(\omega)$ [of the reference signal w (ζ, t)] and $S_{zz}(\omega)$ [of the artifact z (ζ, t)] are known:

$$S_{ww}(\omega) = \frac{S_0}{1 + 4\,\omega^2 T^2}, \qquad S_{zz}(\omega) = \frac{S_0}{1 + 16\,\omega^2 T^2}$$

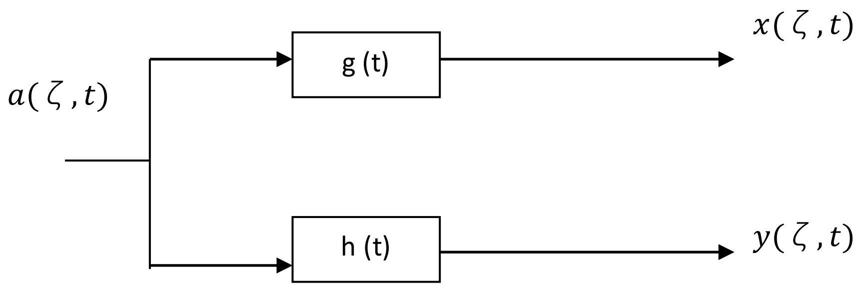
The two input signals w (ζ, t) and z (ζ, t) are orthogonal. It is known that

$$F_R(\omega) = \frac{3(1 + 2\,j\omega T)}{(1 + 4\,j\omega T)}, \qquad F_M(\omega) = \frac{1}{1 + 2\,j\omega T}$$

a) Determine the power spectral density $S_{xx}(\omega)$ of the output signal (ζ, t).
b) What is the cross power spectral density $S_{rx}(\omega)$?

Task 3.1.3

Given is the following block diagram:



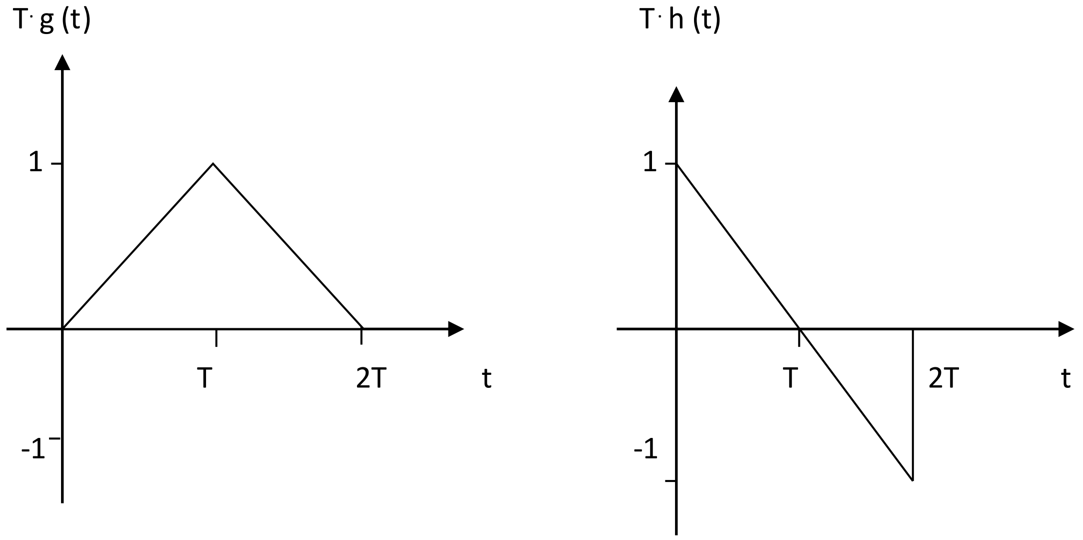
Let the random process $a(\zeta,t)$:

$$a(\zeta,t) = T \sum_{i=-\infty}^{+\infty} b(\zeta,iT)\,\delta(t-iT)$$

The random variable $b(\zeta,iT)$ takes the values +1 and -1 with equal probability. Furthermore it does apply that:

$$E\{b(\zeta,iT)\,b(\zeta,jT)\} = 0 \quad fuer\ i \neq j$$

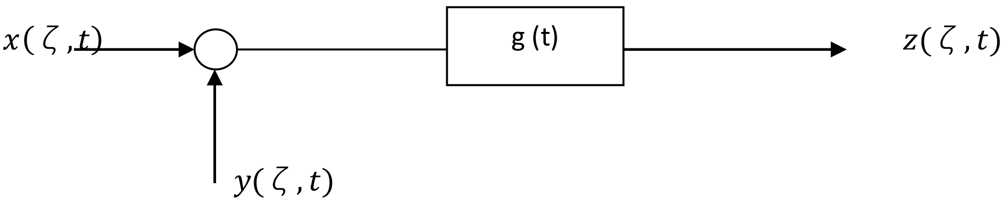
The impulse responses g (t) and h (t) are given by the following figures:



a) Sketch a pattern function of the random process $x(\zeta,t)$.
b) Show that the random process $x(\zeta,t)$ is not ergodic.
c) Calculate the expectation $E\{x(\zeta,t)\,y(\zeta,t)\}$ for all $t \in [-\infty,\infty]$.

Task 3.1.4

Given is the following block diagram:



The autocovariance function of the stationary random process $x(\zeta,t)$ is known:

$$c_{xx}(t,t+\tau) = \frac{1}{3}\, e^{-|\tau|}$$

The Fourier transform of g (t) is:

$$G(\omega) = \frac{1}{2 - 3\,j\omega - \,\omega^2}$$

The random process $y(\zeta,t)$ is defined by

$$y(\zeta,t) = 2\, sin^2\,(\omega_0 t + \,\varphi(\zeta))$$

Moreover the joint pdf of the random variable $\varphi(\zeta)$ and the random process $x(\zeta,t)$ is given for all t:

$$f_{x\varphi}(x,\varphi) = \begin{cases} \frac{1}{2\pi}\,\left(1 - \left|\frac{x}{2} - 1\right|\right) & for\ 0 \le \varphi < \pi\ \ and\ 0 \le x < 4 \\ 0 & otherwise \end{cases}$$

a) Calculate $S_{xx}(\omega)\ and\ S_{yy}(\omega)$.
b) Calculate $S_{zz}(\omega)$?

Part 4: Matlab tasks

Task 4.1 Autocorrelation function

Given the signal $x(t) = \sin(2\pi f t)$ with $f = 1Hz$. The signal is sampled for 5s at a sampling rate of 10 Hz

Use Matlab to

a) plot the signal in the timeframe $0 \leq t \leq 5$ (i.e. 5s, the s is omitted). Use correct axis labels and scaling.
b) Calculate the ACF of the sampled signal using a rectangular time window.
c) Plot the ACF calculated in part b) for $-3 \leq \tau \leq 5$.

Now please answer the next question without using Matlab

d) What would the plot from part c) look like if you could sample the signal in the time window -∞ $\leq t \leq +\infty$ and subsequently calculate the ACF for -∞ $\leq \tau \leq +\infty$?

And

e) Sketch the ACF that would result from part d) in the timeframe -5 $\leq \tau \leq 5$.

Task 4.2 Fast Fourier transform FFT and Hamming window

Let the signal

$$x(t) = \sin(2\pi f t)$$

with

$$f = 5\ Hz$$

Write Matlab programs.

a) Sample the signal x(t). The sampling frequency is $f_s = 50\ Hz$. Plot the sampled signal in the timeframe -10 $\leq t \leq 10$ (t in seconds). Use correct axis labels and scaling.
b) Cut out a time frame from x(t) in the range -5 $\leq t \leq 5$. Use a rectangular time window (=1 for -5 $\leq t \leq 5; = 0\ elsewhere$). Hint: multiply the rectangular time window pointwise with the sampled signal. Plot the result from -10 to +10. Use correct axis labels and scaling.
c) Calculate the FFT of the cut-out time frame (hint: 500 sampling points) and plot it. Use correct axis labels and scaling.
d) Once again cut out a time frame from x(t) in the range -5 $\leq t \leq 5$. This time use a Hamming window. Hint: Create a Hamming window using the corresponding Matlab function. Use the same sampling rate that you used for sampling x(t).
e) Calculate the FFT of the cut-out time frame of part d). Use correct axis labels and scaling.

Now please answer the next question without using Matlab

f) Compare the FFTs of c) and e) and discuss the differences.

Task 4.3 Power spectral density PSD

Let a random process

$$x(\zeta, t) = \sin(2\pi ft) + \sin(3\pi ft) + \alpha \cdot n(\zeta, t)$$

The frequency f is 300 Hz, α is 0.1, and n(ζ, t) is normally distributed random noise. An A/D converter takes samples of a pattern function of the process with a sampling frequency of 3 kHz. The length of the buffer of the A/D converter is 2048.

a) Write a Matlab program that calculates and plots the PSD (power spectral density) of the sampled pattern function random process x(ζ, t) using the Wiener-Khintchine theorem. Don't use the Matlab function for direct PSD calculation. Use the Matlab function "randn" for the noise. Plot the sampled time signal in the timeframe from +0.00s to +0.02s, and plot the PSD (positive frequencies only). Don't forget the axis labels.
b) Increase α to 0.3 and run your program again. What do you observe?
c) Increase α to 2.0 and run your program again. What do you observe?
d) Run your program with α set to 0.3 and the sampling frequency to 900 Hz.
e) Run your program with α set to 0.3 and the sampling frequency to 450 Hz. Explain the results.
f) Run your program with α set to 0.3 and a sampling frequency of 3 kHz. In contrast to b) the buffer of the A/D converter now should have a length of 8192.
g) Take the settings of f) but instead of normally distributed random noise add uniformly distributed noise.

Task 4.4 Sígnal analysis and digital signal synthesis

a) Analyze the sampled time signal given in the CSV file "testsignal". The only pre-knowledge is the sampling frequency which is 1 kHz. For reading the file, analyzing the data, and plotting the results use Matlab. Calculate and plot the PSD (Hint: use the Wiener-Khintchine theorem). Discuss your results.
b) Write a Matlab program. Synthesize the digital signal analyzed in task 4.1 based on the determined PSD using a sampling frequency of 2 kHz. The timeframe of the synthesized signal and the original signal should have the same length. Plot the first 100 sampling points of the signal. Plot the ACF of the signal from -0,01s to +0.01s.

Task 4.5 Kalman filter

Run the given Matlab program "robot_movement". Extend the program so that a Kalman filter calculates estimates of the robot's velocity. The program should plot the true velocity (ground truth), the robot's velocity readings, and the Kalman filter estimate of the velocity.

Task 4.6 Random process, Fast Fourier transform FFT and Hamming window

Let the random process $x(e,t) = \sin(2\pi f t + \varphi(e))$ with $f = 5\ Hz$.

Write Matlab programs.

First assume there's no random phase (i.e. $\varphi(e) = 0$).

a) Sample the signal x(t) for 10 seconds. The sampling frequency is $f_s = 100\ Hz$. Plot the sampled signal in the time frame $0 \leq t \leq 0.5$ (t in seconds). Use correct axis labels and scaling.
b) Cut out a time frame from x(t) in the range $2 \leq t \leq 5$. Use a rectangular time window. Plot the result from 0 to 10. Use correct axis labels and scaling.
c) Calculate the FFT of the cutou t time frame (hint: 300 sampling points) and plot it. Use correct axis labels and scaling.
d) Cut out a time frame as in part b but this time use a Hamming window instead of a rectangular window. Hint: Create a Hamming window using the corresponding Matlab function. Use the same sampling rate that you used for sampling x(t). The length of the Hamming window must be the same as the length of the time frame to be cut out.
e) Calculate the FFT of the signal part that has been cut out in part d). Use correct axis labels and scaling.
f) Compare the FFTs of c) and e) and discuss the differences.

Now $\varphi(e)$ is a uniformly distributed random variable in the interval $0 \leq \varphi(e) \leq 2\pi$. The srandom process $x(e,t)$ is sampled the same way as in part a.

g) Create 3 pattern functions of $x(e,t)$ and plot them in different colors in the same coordinate system.
h) Create another 100 pattern functions. Calculate the linear ensemble average at t=1s, t=2.3s and t=8.1s. Then calculate the time average of pattern function number 5, number 20 and number 75. Compare the results.
i) Apply the Wiener-Khintchine theorem to the pattern function number 20. Plot the ACR and the PSD.

Task 4.7 Power spectral density PSD

Let a random process

$$x(\zeta, t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \alpha \cdot n(\zeta, t)$$

The frequency f_1 = 30 Hz, f_2 = 50 Hz, α is 0.1, and $n(\zeta, t)$ is normally distributed random noise. An A/D converter (ADC) takes samples of a pattern function of the process with a sampling frequency of 500 Hz. The length of the ADC buffer is 2048.

a) Write a Matlab program that calculates the power spectral density (PSD) by applying the Wiener-Khintchine theorem to the sampled pattern function of the random process $x(\zeta, t)$. Don't use the Matlab function for direct PSD calculation. Use the Matlab function "randn" for the noise. Plot the sampled time signal in the timeframe from +0.00s to +0.06s, and plot the PSD (positive frequencies only). Don't forget the axis labels.
b) What maximum length in seconds can the signal have so that it can be stored completely in the ADC buffer?
c) Increase α to 0.8 and run your program again. What do you observe?
d) Increase α to 2.0 and run your program again. What do you observe? (ctd. on the next page)

e) Run your program with $\alpha = 0.3$ and a sampling frequency of 100 Hz. What do you observe?
f) Run your program with $\alpha = 0.3$ and a sampling frequency of 80 Hz. Explain the results.
g) Run your program with $\alpha = 0.3$ and a sampling frequency of 500 Hz. In contrast to b) the buffer of the ADC now should have a length of 8192. What's the difference compared to part a)?
h) Plot the PSD of part a), c) and d) in logarithmic scale (y-axis).

Task 4.8 Signal Analysis

Analyze the sampled time signal given in the CSV (comma separated values) file "testsignal". The only pre-knowledge is the sampling frequency which is 500 Hz. Use Matlab to read the file, determine the length of the signal in sampling points and seconds, analyze the signal and determine the PSD using the Wiener-Khintchine theorem. Use Matlab to find the highest peak of the PSD and read the corresponding frequency. Print it as the "main frequency". Plot the signal and the PSD. Discuss your results.

Task 4.9 Digital Signal Synthesis

Write a Matlab program. Your task is to synthesize a digital signal. Create a cosine signal. It's frequency should be the "main frequency" from task 4.9. The sampling frequency should be 2 kHz. The legth of the signal in seconds should be the same as the signal in task 4.1. Plot the first 100 sampling points of the signal. Plot the ACF of the signal from -1s to +1s.

Task 4.10 Wiener filter

Write a Matlab program.
Let a random process

$$x(\zeta, t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \alpha \cdot n(\zeta, t)$$

The frequencies are f_1=2.5Hz and f_2=3Hz. α is 0.2, and $n(\zeta, t)$ is normally distributed random noise. An A/D converter takes samples of a pattern function of the process with a sampling frequency of 1 kHz. The length of the buffer of the A/D converter is 4096.

a) Plot the signal from 0s to 1s.
b) Use a Wiener filter (hint: use a Matlab toolbox) to reduce the noise. Then plot it as in part a).
c) Repeat the whole procedure with α=0.5
d) Repeat c) with unitiformly distributed noise in the interval from 0 to 1.