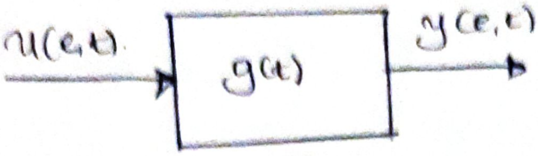
Exercise #4

Task 4.8 : PSD & Transfer function



$$S_{uy} = \frac{S_1}{(1-j\omega b)(1+j\omega T_1)} \quad ; \quad S_{yy} = \frac{S_1}{1+\omega^2 T_1^2}$$

Pg 124; Eqn 5.49, 5.50, 5.51

$\boxed{a}$

From the formulas :

$$S_{uy}(\omega) = G(j\omega)\, S_{uu}(\omega) \quad \dots \text{①}$$

$$S_{yu}(\omega) = G^*(j\omega)\, S_{uu}(\omega) \quad \dots \text{②}$$

$$S_{yy}(\omega) = G(j\omega)\, G^*(j\omega)\, S_{uu}(\omega) \quad \dots \text{③}$$

from Eqn ① and ③ 3

$$S_{uu}(\omega) = \frac{S_{uy}(\omega)}{G(j\omega)} \; ;$$

$$S_{yy} = \cancel{G(j\omega)}\, G^*(j\omega) \cdot \frac{S_{uy}(\omega)}{\cancel{G(j\omega)}}$$

$$G^*(j\omega) = \frac{S_{yy}(\omega)}{S_{uy}(\omega)}$$

$$= \frac{\cancel{S_1}}{1+\omega^2 T_1^2} \cdot \frac{(1-j\omega b)(1+j\omega T_1)}{\cancel{S_1}}$$

but $1+\omega^2 T_1^2 = (1+j\omega T_1)(1-j\omega T_1)$

$$\therefore G^*(j\omega) = \frac{(1-j\omega b)\cancel{(1+j\omega T_1)}}{\cancel{(1+j\omega T_1)}(1-j\omega T_1)}$$

$$G^*(j\omega) = \frac{1-j\omega b}{1-j\omega T_1}$$

the Conjugate is:

$$\boxed{G(j\omega) = \frac{1+j\omega b}{1+j\omega T_1}}$$

b Is the described system by $G(j\omega)$ a causal system?

Poles of the transfer function:

$s = 1+j\omega T_1 = 0$; $j\omega T_1 = -1$

$W = \frac{-1}{j\omega T_1}$; $W = j/T_1$

$s = \alpha + j\beta$

$\alpha = 1$; $\beta = \omega T_1$

The system is causal because the poles of the transfer function are lying in the upper halfplane of the complex plane.

Extra:

*[Further: if all poles are in the left halfplane of the Laplace s-plane, the system is stable]

c Calculate the ACF $S_{uu}(\tau)$ of the input signal $u(e,t)$

$$S_{uu}(\omega) = \frac{S_{uy}(\omega)}{G(j\omega)} = \frac{S_1}{(1-j\omega b)\cancel{(1+j\omega T_1)}} \cdot \frac{\cancel{(1+j\omega T_1)}}{(1+j\omega b)}$$

$$S_{uu}(\omega) = \frac{S_1}{(1-j\omega b)(1+j\omega b)} = \frac{S_1}{1+\omega^2 b^2}$$

$$S_{uu}(\tau) = FFT^{-1}\{S_{uu}(\omega)\}$$

$$S_{uu}(\tau) = FFT^{-1}\left\{\frac{S_1}{1+\omega^2 b^2}\right\}$$

from the fourier table;

$$FFT\{e^{-\alpha|t|}\} = \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\Rightarrow \quad \frac{(S_1)(1/b)^2 \times 2}{\left(\frac{1}{b}\right)^2 + \omega^2 \times 2} = \frac{S_1}{2b}\left[\frac{2(1/b)}{(1/b)^2+\omega^2}\right]$$

$$\therefore \alpha = 1/b$$

$$\boxed{S_{uu}(\tau) = \frac{S_1}{2b} e^{-\frac{1}{b}|\tau|}}$$