Probability Review - AMLI 2021

Introduction to Probability

Rolling a Die Creates a Random Variable

Random Variable

X	Probability(X)
1	⅙
2	⅙
3	⅙
4	⅙
5	⅙
6	⅙

Die Rolls Are Uniform Probabilities

- If we roll a 6-sided die, what is the probability of rolling a 1?
- What is the probability of rolling an even number?

The Expected Value is the "Average" Roll

- When we roll a 6-sided die, what is the "most likely" value?
- Imagine rolling the die 100 times, what would the "average" roll be?
- The expected value of a random variable can be thought of as the *mean* or *average*

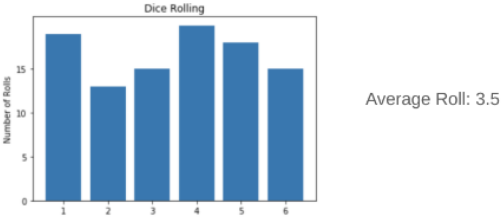

Figure 1: Probability Mass Function

Relationships Among Random Variables

- Independent variables: knowing one event has happened does not change the probability that the other happens
  - Probability of rolling a 1 and flipping a head
  - When X and Y are independent, $P(X \text{ and } Y) = P(X)P(Y)$
- Dependent variables: knowing one event has happened gives us new information, affecting the probabilty that the other happens
  - Probability that the sum of two-die rolls being a 5, if the first was a 3

Conditional Probability

- The probability of X given Y has occured is $P(X|Y)$
  - $P(\text{sum of two die} = 5 \mid \text{first die} = 3) = P(\text{sum is 5 if the first die is 3}) = \frac{1}{6}$
- Conditional probability expression

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

Conditional Probability: Worked Example

- What is the probability that the sum of die $= 5$, given the first die is 3?

Conditional Probabilities Are Not Joint Probabilities

- If we let X be the first die roll, Y be the second die roll, and Z be the sum of the two die rolls, then
  - Conditional probability:
    - $P(Z = 5|X = 3) \rightarrow P(\text{sum=5} \mid \text{first die} = 3) = \frac{1}{6}$
  - Joint probability:
    - $P(X = 3 \cap Y = 2) \cup (X = 2 \cap Y = 3)) \rightarrow$ $P(\text{roll 3 on die 1 and 2 on die 2 OR 2 on die 1 and 3 on die 2}) = \frac{2}{36}$
  - Probability of Z:
    - $P(Z = 5) = P(X + Y = 5) \rightarrow P(\text{sum of two die is 5}) = \frac{4}{36}$

Conditional, Joint, & Marginal Probabilities Are Related

- Let X and Y be random variables.
  - If X and Y are independent, then $P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = P(X)$
  - What is an example of this?
    - Let X be the outcome of rolling a 6-sided die and let Y be the outcome of flipping a coin. Suppose the outcome of Y is heads. What is the probability that the roll of the die is 3?

Conditional, Joint, & Marginal Probabilities Are Related

- Let X and Y be random variables.
- $P(X) = \sum_Y P(X|Y)P(Y)$
- What is an example of this?
  - Let X be the sum of rolling two dice and let Y be the outcome of the first die roll. What is the probability that X is 3?

Conditional, Joint, & Marginal Probabilities Are Related

- Let X and Y be random variables.
- $P(X) = \sum_Y P(X \cap Y)$
- What is an example of this?
  - Let X the be the outcome of the first die roll and Y be the outcome of a second die roll. What is the probability that X is 3?

Conditional, Joint, & Marginal Probabilities Are Related

Let X and Y be random variables.

1. If X and Y are independent then $P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$

   Conditional Probability

2. $P(X) = \sum_Y P(X|Y)P(Y)$

3. $P(X) = \sum_Y P(X \text{ and } Y)$

   Joint Probability

   Marginal Probability

Conditional Probabilities and Bayes' Theorem

- Sometimes we want to find $P(X|Y)$ when we already know $P(Y|X)$.
- In this case, we use Bayes' Theorem:
  - For random variables X and Y:

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

Bayes' Theorem: Example

- Earlier, we determined that $P(\text{that sum of two die is } 5|\text{first die is } 3) = \frac{1}{6}$.
- Given this value, use Bayes' Theorem to determine $P(\text{first die is } 3|\text{ sum of two die is } 5)$.

Sample Exercise: Peanut Chocolate Detector

- Problem statement: Suppose we have a new device that distinguishes whether or not a type of chocolate contains peanuts. If a chocolate contains peanuts, 99% of the time it correctly reports a positive result. Likewise, if a chocolate does not contain peanuts, 99% of of the time it correctly reports a negative result. Assume that 1% of all chocolates contain peanuts.
- Question: If the device reports that a chocolate contains peanuts, what is the probability that the chocolate actually does contain peanuts?

Sample Exercise: Peanut Chocolate Detector (Solution)

- Let N be a random variable indicating whether peanuts are in a chocolate bar.
- Let D be a random variable indicating whether peanuts are detected in a chocolate bar.