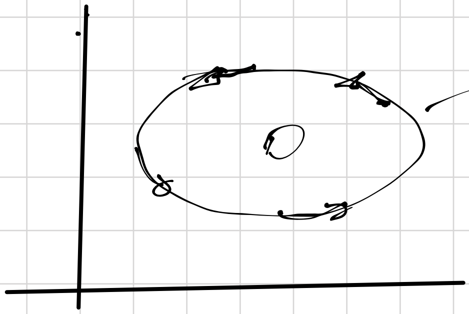
Teorema de Green



trayectoria cerrada.

orientación: contrario a las manecillas del reloj.

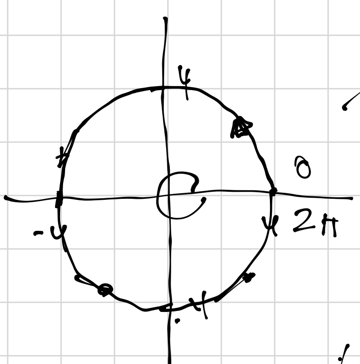
Sea C una curva cerrada simple con orientación positiva y suave por partes en el plano y sea D la región acotada por C. Si: P y Q tienen derivadas parciales continuas en una región abierta que contiene a D, entonces

$$\int F\cdot dr = \int P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Ej. Verifique el teorema de Green.

$$\oint y\,dx - x\,dy$$

donde C es el círculo con centro en el origen y radio 4



$x^2+y^2=16$ → $x = r\cos t = 4\cos t$

$y = r\,\text{sen}\,t = 4\,\text{sen}\,t$

$dx = -4\,\text{sen}\,t\,dt$

$dy = 4\cos t\,dt$

forma directa.

$$\int y\,dx - x\,dy$$

$$\int 4\,\text{sen}\,t(-4\,\text{sen}\,t) - 4\cos t(4\cos t)\,dt$$

$$\int -16\,\text{sen}^2 t - 16\cos^2 t\,dt = \int_0^{2\pi} -16(\text{sen}^2 t + \cos^2 t)\,dt$$

$$\int_0^{2\pi} -16\,dt = -16t\Big|_0^{2\pi} = -16(2\pi) = -32\pi$$

Teorema de Green

$$\int \underbrace{y}_{P}\,dx - \underbrace{x}_{Q}\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\iint(-1-1)\,dA = \int_0^{2\pi}\int_0^4 -2\,r\,dr\,d\theta$$

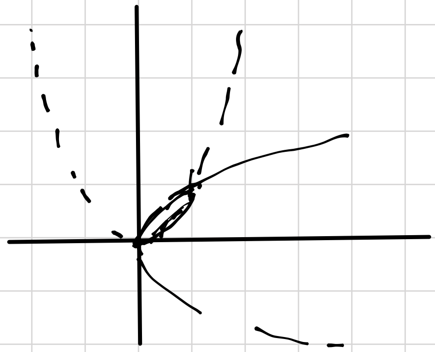
$$\int_0^{2\pi} -r^2\Big|_0^4 d\theta = \int_0^{2\pi} -(4)^2\,d\theta = \int_0^{2\pi} -16\,d\theta$$

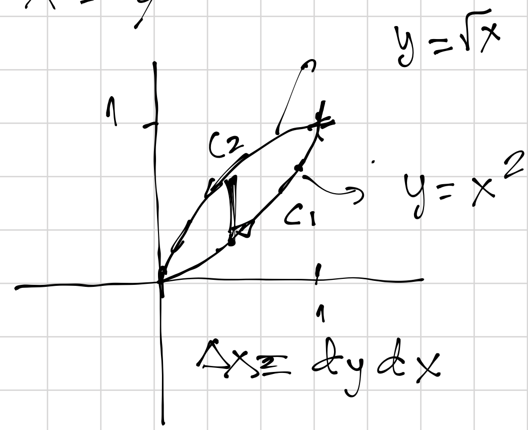
$$-16\,\theta\Big|_0^{2\pi} = -32\pi$$

Ej. Evaluar la integral de línea.

$$\oint (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

C: es la frontera de la región encerrada por las parabolas $y=x^2$ & $x=y^2$





$$\oint(\underbrace{y+e^{\sqrt{x}}}_{P})\,dx + (\underbrace{2x+\cos y^2}_{Q})\,dy = \iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

$$=\iint(2-1)\,dA = \iint dA = \int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx$$

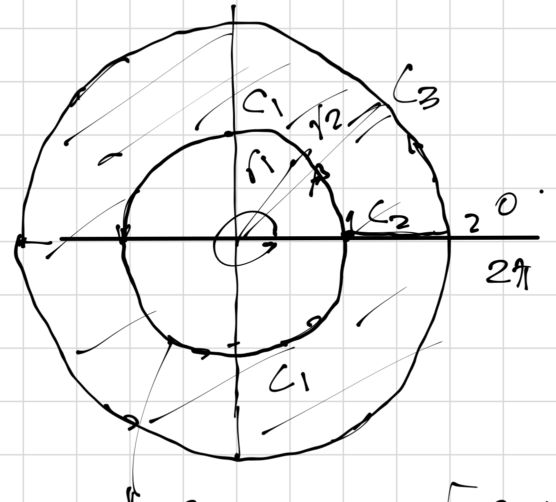
$$\int_0^1 y\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x^{1/2}-x^2)\,dx$$

$$\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\Big|_0^1 = \frac{2}{3}(1)^{3/2} - \frac{1}{3}(1)^3$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Ej. Evaluar $\oint x e^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$

donde C: es la frontera de la región entre los circulos $x^2+y^2=1$ & $x^2+y^2=4$



$$\int \underbrace{xe^{-2x}}_{P} dx + \underbrace{(x^4 + 2x^2y^2)}_{Q} dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint (4x^3 + 4xy^2) - (0)\, dA$$

$x^2+y^2=1 \to \sqrt{r^2}=\sqrt{1} \to r=1$ $\quad\to x = r\cos\theta$

$x^2+y^2=4 \to \sqrt{r^2}=\sqrt{4} \to r=2$ $\quad y = r\,\text{sen}\,\theta$

$$\iint (4r^3\cos^3\theta + 4r\cos\theta\, r^2\text{sen}^2\theta)\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_1^2 (4r^4\cos^3\theta + 4r^4\cos\theta\,\text{sen}^2\theta)\, dr\, d\theta$$

$$\int_0^{2\pi} \frac{4}{5}r^5\cos^3\theta + \frac{4}{5}r^5\cos\theta\,\text{sen}^2\theta \Big|_1^2 d\theta$$

$$\int_0^{2\pi} \frac{4}{5}(2^5-1^5)\cos^3\theta + \frac{4}{5}(2^5-1^5)\cos\theta\,\text{sen}^2\theta\, d\theta$$

$$\int_0^{2\pi} \left(\frac{124}{5}\cos^3\theta + \frac{124}{5}\cos\theta\,\text{sen}^2\theta\right) d\theta$$

$$\int \cos^3\theta = \int \cos\theta\cos^2\theta\, d\theta = \int \cos\theta(1-\text{sen}^2\theta)\, d\theta$$

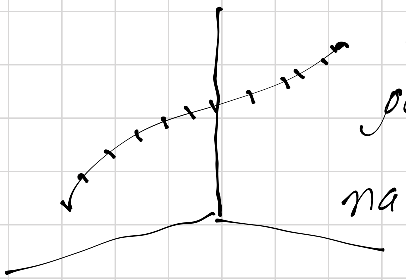
$$\int \cos\theta - \cos\theta\,\text{sen}^2\theta\, d\theta$$

$$\int_0^{2\pi} \frac{124}{5}\cos\theta - \cancel{\frac{124}{5}\cos\theta\,\text{sen}^2\theta} + \cancel{\frac{124}{5}\cos\theta\,\text{sen}^2\theta}\, d\theta$$

$$\frac{124}{5}\text{sen}\,\theta\Big|_0^{2\pi} = \frac{124}{5}(0) = 0$$

Integrales de Superficie

$\rho(x,y,z) \to$ densidad



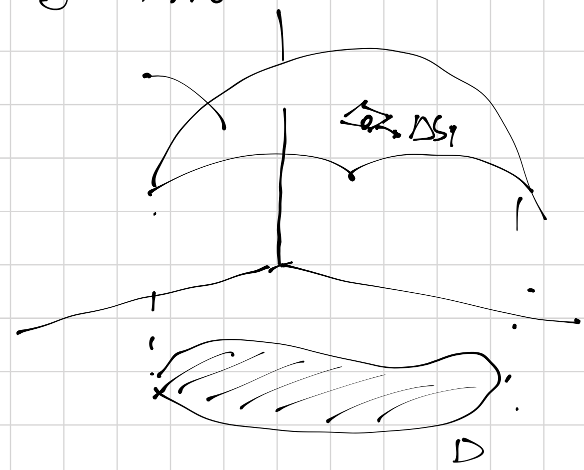
curva C

$\rho(x,y)$

masa = densidad . long.

$\Downarrow$

$\int \rho(x,y)\,ds$



$\rho(x,y,z) = \dfrac{\text{masa}}{\text{Area}}$

masa $= \rho(x,y,z)\cdot$ Area

area de la Superficie ΔS

$$\text{masa} = \rho(x_i,y_i,z_i)\,\Delta S_i$$

$$\text{masa} = \sum_{i=1}^{n} \rho(x_i,y_i,z_i)\,\Delta S_i$$

$$\text{masa} = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i,y_i,z_i)\,\Delta S_i = \iint \rho(x,y,z)\,ds.$$

Integral de Superficie

Sea f una función de tres variables x, y, z definido en una región del espacio que contiene a una superficie S. Entonces la integral de superficie de f sobre S es:

$$\iint f(x,y,z)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k,z_k)\,\Delta S_k$$

$\Downarrow$

$z = g(x,y)$

$$ds = \sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}\,dA$$

$$\iint f(x,y,z)\,ds = \iint f(x,y,g(x,y))\cdot\sqrt{1+g_x^2+g_y^2}\,dA$$

Ej. Evaluar $\iint xz^2\,ds$ donde S es la porción del cilindro $y = 2x^2+1$ en el primer octante, acotado por $x=0$, $x=2$, $z=4$ y $z=8$

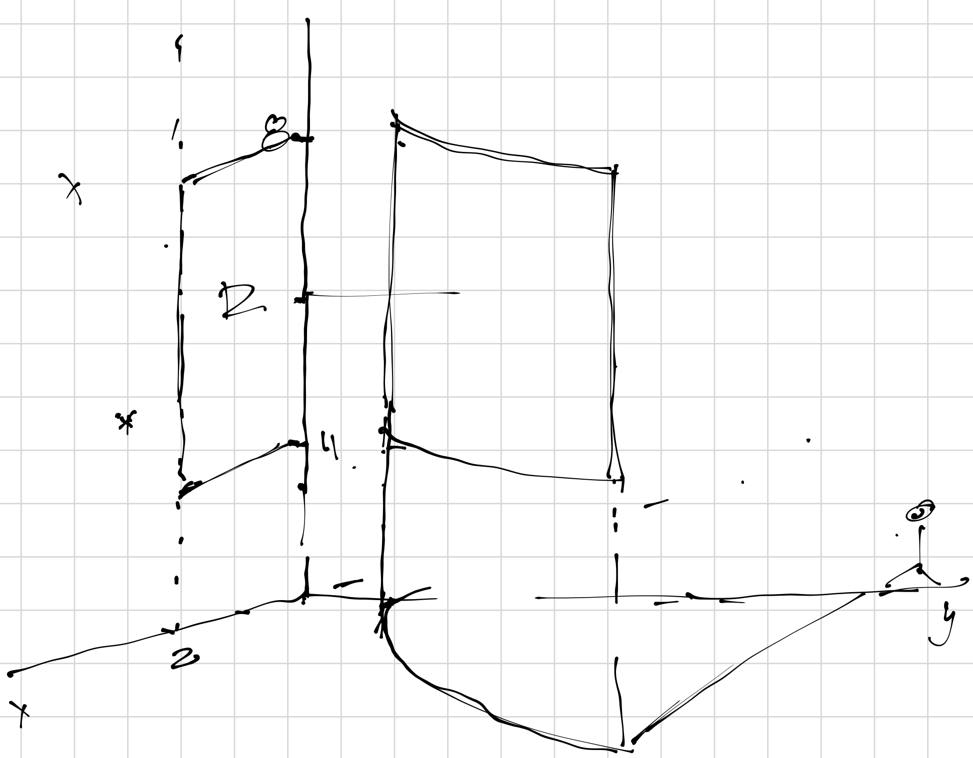
Proyecciones en otros planos

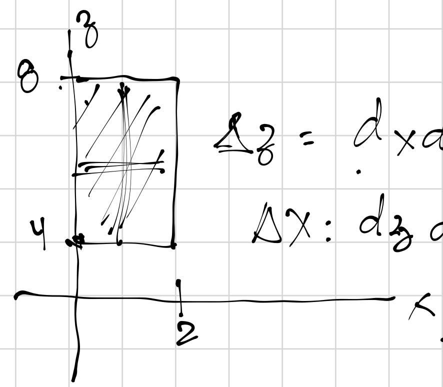
Si $y = g(x,z)$

$$\iint f(x,y,z)\,ds = \iint f(x,g(x,z),z)\sqrt{1+g_x^2+g_z^2}\,dA$$

Si $x = g(y,z)$

$$\iint f(x,y,z)\,ds = \iint f(g(y,z),y,z)\cdot\sqrt{1+g_y^2+g_z^2}\,dA$$





$\Delta z = dx\,dz$

$\Delta x: dz\,dx$

$y = 2x^2+1$

$y = g(x,z) = 2x^2+1$

$g_x = 4x$

$g_z = 0$

$$\iint xz^2\,ds = \iint xz^2\sqrt{1+(4x)^2+0^2}\,dA$$

$$\int_0^2\int_4^8 xz^2\sqrt{1+16x^2}\,dz\,dx$$

$$\int_0^2 \frac{1}{3} x z^3 \Big|_4^8 \sqrt{1+16x^2}\, dx$$

$$\frac{1}{3}[8^3-4^3] \int_0^2 x\sqrt{1+16x^2}\, dx$$

$$u = 1+16x^2$$

$$du = 32x\,dx \qquad \frac{du}{32} = x\,dx$$

$$\frac{1}{3}[8^3-4^3] \int_0^2 \frac{1}{32} u^{1/2}\, du$$

$$\frac{8^3-4^3}{96}\left[\frac{2}{3}u^{3/2}\right]_0^2$$

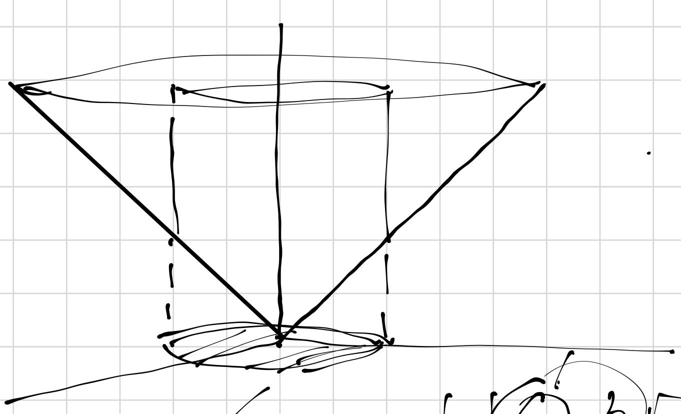
$$\frac{2}{3}\left[\frac{8^3-4^3}{96}\right](1+16x^2)^{3/2}\Big|_0^2$$

$$\frac{2}{3}\left[\frac{8^3-4^3}{96}\right]\left[(1+64)^{3/2}-(1)^{3/2}\right]$$

Ej. Evaluar $\iint f(x,y,z)\, ds$.

donde $f(x,y,z) = xz^3$

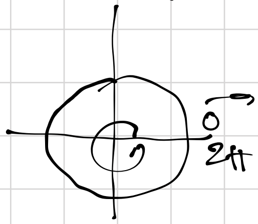
S es el cono de un solo manto $z=\sqrt{x^2+y^2}$ dentro del cilindro $x^2+y^2=1$



$$z = g(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$$

$$\iint x z^3\, ds$$

$$\iint x z^3 \sqrt{1+g_x^2+g_y^2}\, dA$$



$x^2+y^2=1 \to r^2=1 \to r=1$

$$g_x = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x) = \frac{x}{(x^2+y^2)^{1/2}}$$

$$g_x^2 = \frac{x^2}{x^2+y^2}$$

$$g_y = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}y) = \frac{y}{(x^2+y^2)^{1/2}}$$

$$g_y^2 = \frac{y^2}{(x^2+y^2)}$$

$$\iint xz^2\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dA$$

$$2x^2+2y^2$$

$$\iint xz^3\sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}\,dA$$

$$\iint xz^3\sqrt{\frac{2\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}}\,dA \qquad (z)^3=(\sqrt{x^2+y^2})^3$$

$$z^3=(x^2+y^2)^{3/2}$$

$$\sqrt{2}\iint xz^3\,dA$$

$$r \cdot r^3 \quad 3/\cancel{2} \cdot 0 = r^5$$

$$\sqrt{2}\iint x(x^2+y^2)^{3/2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^1 r\cos\theta\,(r^2)\,r\,dr\,d\theta$$

$$\sqrt{2}\int_0^{2\pi}\frac{1}{6}r^6\Big|_0^1\cos\theta\,d\theta$$

$$\frac{\sqrt{2}}{6}(1^6)\int_0^{2\pi}\cos\theta\,d\theta = \frac{\sqrt{2}}{6}(1)^6\,\operatorname{sen}\theta\Big|_0^{2\pi}$$

$$\frac{\sqrt{2}}{6}(1)^6(\operatorname{sen}2\pi - \operatorname{sen}0) = 0$$