Ecuación de Bernoulli

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^n \qquad n \neq 0,\; n \neq 1$$

$$\frac{d}{dx}\left[\, v = y^{1-n} \,\right]$$

$$\frac{dv}{dx} = (1-n)\frac{dy}{dx}$$

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x)$$

$$\Downarrow$$

Ej. Resolver

$$\left[\, \cancel{x}\frac{dy}{dx} - (1-x)\,y = x y^2 \,\right] * \frac{1}{\cancel{x}}$$

$$\frac{dy}{dx} - \frac{(1-x)}{x}\,y = \frac{\cancel{x}}{\cancel{x}}\,y^2 \qquad v = y^{1-2}$$

$$v = y^{-1}$$

$$\frac{dy}{dx} - \frac{(1-x)}{x}\,y = y^2 \overset{1}{\cancel{Q(x)}}$$

$$P(x) = (-1)\left(\frac{1-x}{x}\right) \qquad n = 2 \qquad Q(x) = 1$$

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x)$$

$$\frac{dv}{dx} + \overset{-1}{(1-2)}(-1)\left(\frac{1-x}{x}\right) v = (1-2)(1)$$

$$\rightarrow \frac{dv}{dx} + \frac{1-x}{x}\,v = -1$$

$$F.I. = e^{\int P(x)dx} = e^{\int \frac{1}{x} - 1\, dx} = e^{\ln x - x}$$

$$F.I. = e^{\ln x} e^{-x} = x e^{-x}$$

$$\frac{d}{dx}\left(x e^{-x} v\right) = -x e^{-x}$$

$$\int d(xe^{-x}v) = \int -xe^{-x}\,dx$$

$$xe^{-x}v = -\left[-xe^{-x} + \int -e^{-x}dx\right] \qquad \begin{matrix} u = x & du = dx \\ \int dv = \int e^{-x}dx & v = -e^{-x} \end{matrix}$$

$$xe^{-x}v = xe^{-x} + e^{-x} + c$$

$$v = \frac{xe^{-x} + e^{-x} + c}{xe^{-x}}$$

$$v = 1 + \frac{1}{x} + \frac{ce^{x}}{x} \qquad v = y^{-1}$$

$$y^{-1} = 1 + \frac{1}{x} + c\,\frac{e^{x}}{x}$$

Ej. Resolver $3(1+x^2)\dfrac{dy}{dx} = 2xy(y^3-1)$

$$3(1+x^2)\frac{dy}{dx} = 2xy^4 - 2xy$$

$$\left[3(1+x^2)\frac{dy}{dx} + 2xy = 2xy^4\right] * \frac{1}{3(1+x^2)}$$

$$\frac{dy}{dx} + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4 \qquad n = 4$$

$$P(x) = \frac{2x}{3(1+x^2)} \qquad Q(x) = \frac{2x}{3(1+x^2)}$$

$$v = y^{1-n} = y^{1-4} = y^{-3}$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

$$\frac{dv}{dx} + (1-4)\left(\frac{2x}{3(1+x^2)}\right)v = (1-4)\left(\frac{2x}{3(1+x^2)}\right)$$

$$\frac{dv}{dx} + (-3)\left(\frac{2x}{3(1+x^2)}\right)v = (-3)\left(\frac{2x}{3(1+x^2)}\right)$$

$$\frac{dv}{dx} - \frac{2x}{1+x^2}v = -\frac{2x}{1+x^2}$$

$u = 1+x^2$
$du = 2x\,dx$

$$F.I. = e^{\int p(x)dx} = e^{-\int \frac{2x}{1+x^2}dx}$$

$$F.I. = e^{-\int \frac{du}{u}} = e^{-\ln u} = e^{\ln u^{-1}} = u^{-1}$$

$$F.I. = (1+x^2)^{-1} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left((1+x^2)^{-1}v\right) = \left(\frac{-2x}{1+x^2}\right)\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2}$$

$$-\int \frac{du}{u^2} = -\int u^{-2}du$$

$$-(u^{-1})$$

$$\int d[(1+x^2)^{-1}v] = \int \frac{-2x}{(1+x^2)^2}dx$$

$$(1+x^2)^{-1}v = (1+x^2)^{-1} + C$$

$$\frac{(1+x^2)^{-1}}{(1+x^2)^{-1}} + \frac{C}{(1+x^2)^{-1}}$$

$$v = \frac{(1+x^2)^{-1} + C}{(1+x^2)^{-1}} = 1 + C(1+x^2)$$

$$\frac{C}{(1+x^2)^{-1}}$$

$v = y^{-3}$

$$\frac{C}{\frac{1}{(1+x^2)}}$$

Sol $\quad y^{-3} = 1 + C(1+x^2)$

Ej. Resolver $\quad y^{1/2}\frac{dy}{dx} + y^{3/2} = 1 \qquad y(0) = 4$

$$\left[y^{1/2}\frac{dy}{dx} + y^{3/2} = 1\right] * \frac{1}{y^{1/2}}$$

$$y^{3/2}y^{-1/2} = y^1$$

$$\frac{dy}{dx} + \frac{y^{3/2}}{y^{1/2}} = \frac{1}{y^{1/2}}$$

$$\frac{dy}{dx} + y = y^{-1/2} \rightarrow \text{Ec. de Bernoulli}$$

$$P(x) = 1 \qquad Q(x) = 1 \qquad n = -\frac{1}{2}$$

$$v = y^{1-n} = y^{1-(-\frac{1}{2})} = y^{3/2}$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

$$\frac{dv}{dx} + (1-(-\frac{1}{2}))(1)v = (1-(-\frac{1}{2}))(1)$$

$$\frac{dv}{dx} + \frac{3}{2}v = \frac{3}{2}$$

$$F.I = e^{\int \frac{3}{2}dx} = e^{\frac{3}{2}x}$$

$$\int \frac{d}{dx}[e^{3/2x}v] = \int \frac{3}{2}e^{\frac{3}{2}x}$$

$$e^{3/2x}v = e^{3/2x} + C$$

$$\boxed{v = 1 + Ce^{-3/2x}}$$

$$dv = \frac{3}{2}dx - \frac{3}{2}v\,dx = \frac{3}{2}dx(1-v)$$

$$\int \frac{dv}{1-v} = \int \frac{3}{2}dx$$

$$[-\ln(1-v) = \frac{3}{2}x + C] * -1$$

$$\rightarrow e^{\ln(1-v)} = e^{(-\frac{3}{2}x + C)} = e^{-3/2x}e^{C}$$

$$1 - v = Ce^{-\frac{3}{2}x} \rightarrow v = 1 - e^{-\frac{3}{2}x}$$

$$v = y^{3/2}$$

$$y(0) = 4$$

$$y^{3/2} = 1 + Ce^{-3/2x}$$

$$4^{3/2} = 1 + C\cancel{e^{-3/2(0)}}$$

$$\boxed{C = 7}$$

$$\boxed{y^{3/2} = 1 + 7e^{-3/2x}}$$

Ecuaciones Exactas

diferencial total de una función $z = f(x,y)$

$$z = f(x,y)$$

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$z = f(x,y) = c \quad \rightsquigarrow \quad dz = 0$$

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

a partir de un conjunto de familias de la curva $f(x,y) = c$ se puede generar una ecuación diferencial de primer orden si se calcula la diferencial total.

$$y\,dx + x\,dy = 0$$

$$d[xy] = 0$$

Ecuación diferencial exacta.

$$f(x,y) = xy = c$$

Ecuación exacta

una expresión diferencial $M(x,y)dx + N(x,y)dy$ es una diferencial exacta en una región R del plano xy si esta corresponde a la diferencial de alguna función $f(x,y)$ definida en R. una ecuación diferencial de primer orden de la forma

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

se conoce como una ecuación exacta si la expresión del lado izquierdo es una diferencial exacta.

Criterio para una diferencial exacta

Si $M(x,y)$ y $N(x,y)$ son continuas y tienen primeras derivadas parciales continuas en una región rectangular R definida por $a < x < b$

$c < y < d$ entonces una condición suficiente para que $M(x,y)$ y $N(x,y)$ sea una diferencial exacta es:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Procedimiento

1. $f(x,y) = C$

2. $$\frac{\partial f}{\partial x} = M(x,y)$$

$$\int \partial f = \int M(x,y)\,dx$$

$$\rightarrow f(x,y) = \int M(x,y) + h(y)$$

3. $$\frac{\partial f(x,y)}{\partial y} = \frac{\int M(x,y) + h(y)}{\partial y} = \frac{\int M(x,y)}{\partial y} + h'(y) = N(x,y)$$

$$\frac{d\,h(y)}{dy} = N(x,y) - \frac{\int M(x,y)}{\partial y}$$

$$\int d[h(y)] = \int \left[N(x,y) - \frac{\int M(x,y)}{\partial y} \right] dy$$

$$h(y) =$$

4. Sustituir la función $h(y)$ en la Ecuación planteada en el paso 2

$$f(x,y) = \int M(x,y) + h(y) = C$$

Ej. Resolver

$$\underbrace{(\operatorname{sen} y - y \operatorname{sen} x)}_{M}dx + (\underbrace{\cos x + x\cos y - y}_{N})dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\frac{\partial M}{\partial y} = \cos y - \operatorname{sen} x$

$\frac{\partial N}{\partial x} = -\operatorname{sen} x + \cos y$

si es exacta.

1. $f(x,y) = c$

2. $\frac{\partial f(x,y)}{\partial x} = M(x,y)$

$$\int \partial f(x,y) = \int (\operatorname{sen} y - y \operatorname{sen} x)\, dx$$

$$f(x,y) = x \operatorname{sen} y + y \cos x + h(y)$$

3. $\frac{\partial f(x,y)}{\partial y} = N(x,y)$

$$\cancel{x\cos y} + \cancel{\cos x} + h'(y) = \cancel{\cos x} + \cancel{x\cos y} - y$$

$$\frac{d}{dy} h(y) = -y$$

$$\int d\, h(y) = \int -y\, dy$$

$$h(y) = -\frac{1}{2} y^2$$

4. $f(x,y) = x \operatorname{sen} y + y \cos x - \frac{1}{2} y^2 = C$

$$\int (\operatorname{sen} y - y \operatorname{sen} x)\, dx + \int (\cos x + x\cos y - y)\, dy = \int 0$$

$x \operatorname{sen} y + y \cos x$

$y\cos x + x \operatorname{sen} y - \frac{1}{2} y^2 = C$

Sol $x \operatorname{sen} y + y \cos x - \frac{1}{2} y^2 = C$