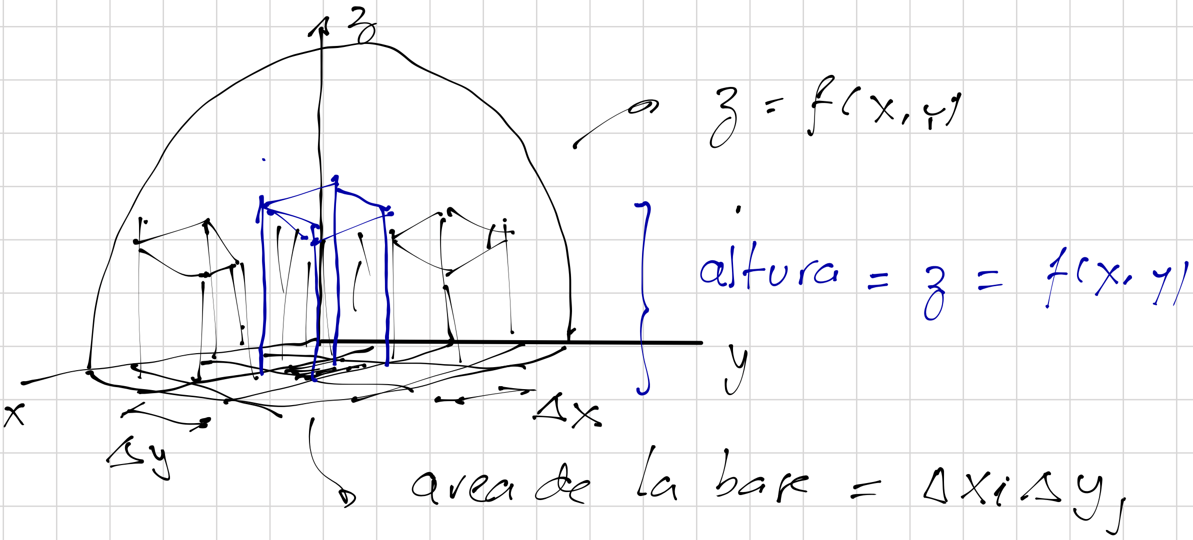
Volumen de superficies mediante integrales dobles



Volumen = (area de la base)(altura)

$$\Delta V = f(x_i, y_i)\Delta x_i \Delta y_j$$

$$\sum_{i=1}^{n} \Delta V = \sum_{i=n}^{\infty} f(x_i, y_i)\Delta x_i \Delta y_i$$

$$V = \lim_{n \to \infty} \sum_{i=n}^{\infty} f(x_i, y_i)\Delta x_i \Delta y_i$$

$$V = \int\int f(x, y)\, dx\, dy$$

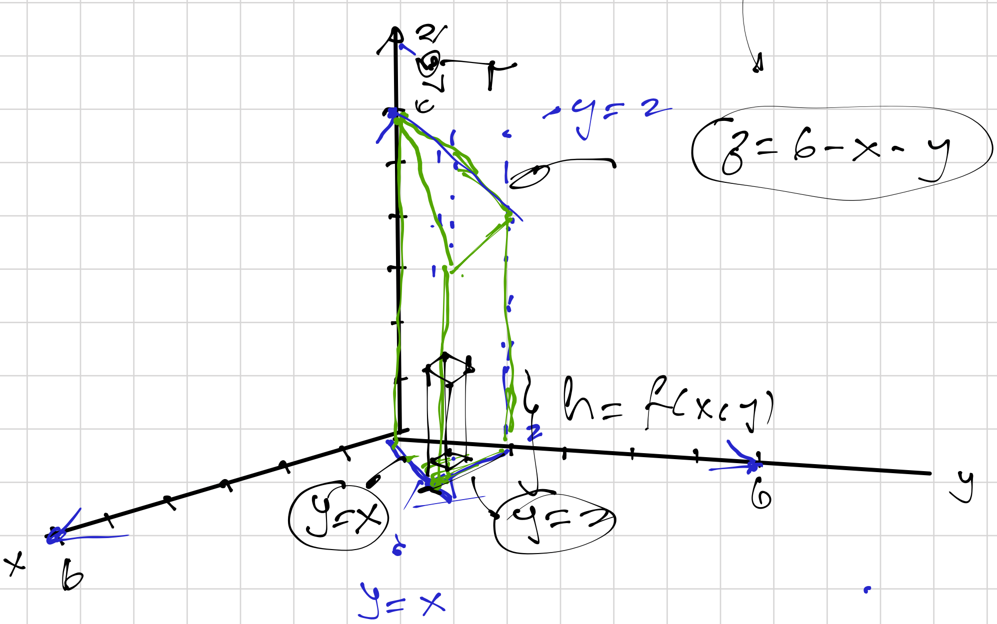
$$V = \int\int f(x, y)\, dy\, dx$$

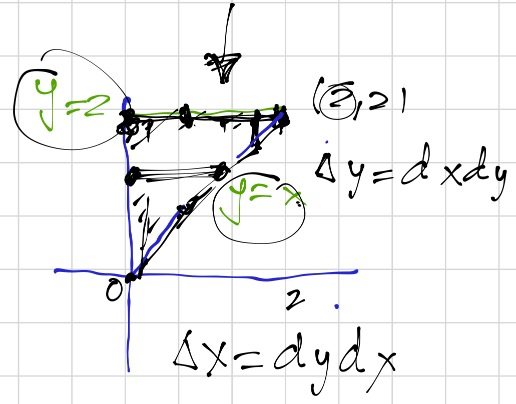
Volumen de una región sólida

Si f es integrable sobre una región plana y $f(x,y) \geq 0$ para toda (x,y) en R, el volumen de la región sólida acotada inferiormente por R y superiormente por la gráfica de f, se define como:

$$V = \iint_R f(x, y)\, dA$$

Ej. Determine el volumen del sólido limitado los planos $y = x$, $y = 2$ & $x + y + z = 6$





$$V = \iint f(x,y)\, dA = \int_0^2 \int_0^y (6-x-y)\, dx\, dy$$

$h = z = 6 - x - y$

$$V = \int_0^2 \int_x^2 (6-x-y)\, dy\, dx$$

$-\frac{3}{2}x^2$

$$V = \int_0^2 6y - xy - \frac{1}{2}y^2 \Big|_x^2 = \int_0^2 (12 - 2x - 2) - (6x - x^2 - \frac{1}{2}x^2)\, dA$$

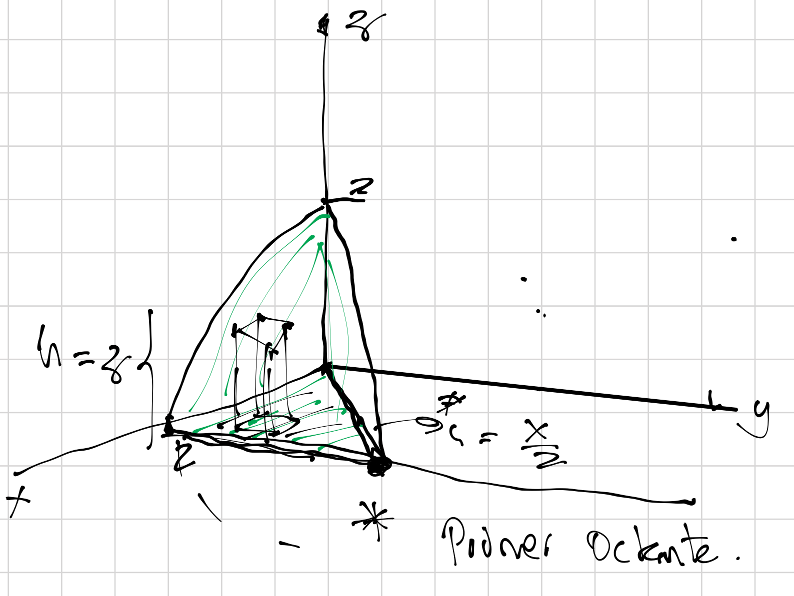
$$V = \int_0^2 (10 - 8x + \frac{3}{2}x^2)\, dx = 10x - 4x^2 + \frac{1}{2}x^3 \Big|_0^2$$

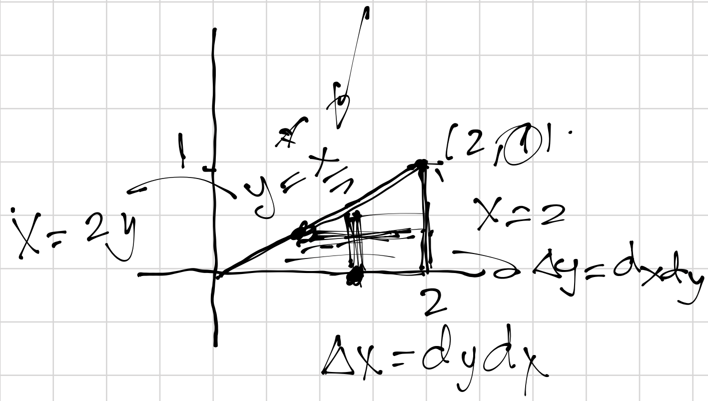
$$= 10(2) - 4(2)^2 + \frac{1}{2}(2)^3$$

$$V = 20 - 16 + 4 = 8u^3$$

Ej. Calcular el volumen del solido limitado por el cilindro $x^2 + z^2 = 4$ y los planos $x = 2y$, $y = 0$, $z = 0$ en el primer octante.

$y = \frac{x}{2}$





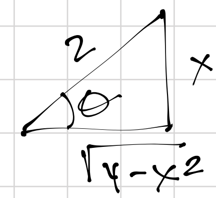
$z^2 = 4 - x^2 \qquad z = \sqrt{4-x^2}$

$V = \iint f(x,y)\,dA \quad \to \quad V = \int_0^1 \int_{2y}^{2} \sqrt{4-x^2}\,dx\,dy$

$V = \int_0^2 \int_0^{x/2} \sqrt{4-x^2}\,dy\,dx$

$V = \int_0^1$

sust. trig.



$\operatorname{sen}\theta = \frac{x}{2}$

$x = 2\operatorname{sen}\theta$

$V = \int_0^2 y\sqrt{4-x^2}\Big|_0^{x/2} = \int_0^2 \frac{x}{2}\sqrt{4-x^2}\,dx$

$u = 4 - x^2$

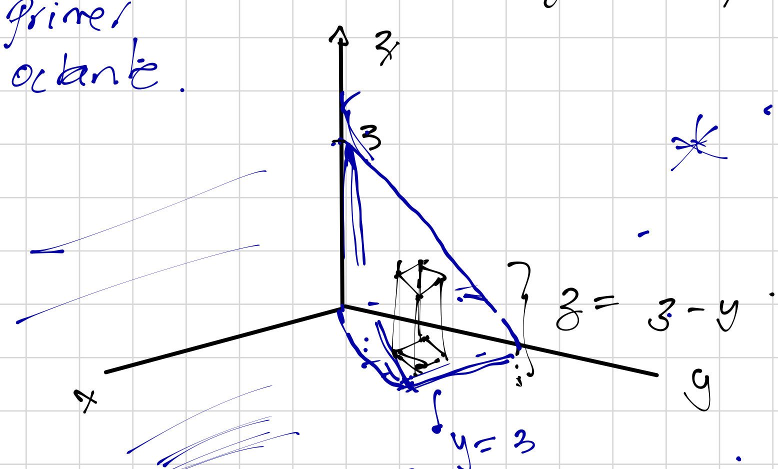
$du = -2x\,dx \qquad -\frac{du}{2} = x\,dx$

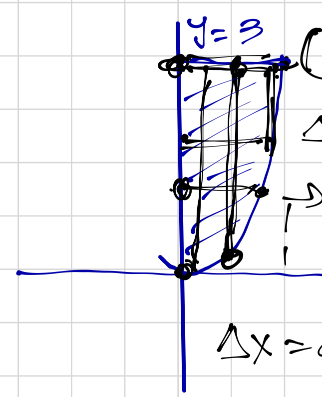
$-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\int_0^2 u^{1/2}\,du = -\frac{1}{4}\,\frac{u^{3/2}}{\frac{3}{2}} = -\frac{1}{3}(4-x^2)^{3/2}\Big|_0^2$

$V = -\frac{1}{6}\left[(4-4)^{3/2} - (4-0)^{3/2}\right] = \frac{8}{6}$

Ej. Determine el volumen del sólido acotado por las gráficas de $y = x^2$, $y + z = 3$ $\quad z = 0$

primer octante.





$\Delta y = dx\,dy$

$y = x^2$

$x = \sqrt{y}$

$\Delta x = dy\,dx$

$\sqrt{3} = \sqrt{x^2}$

$x = \sqrt{3}$

$V = \iint f(x,y)\,dA$

$V = \int_0^{\sqrt{3}} \int_{x^2}^{3} (3-y)\,dy\,dx \quad \to \quad V = \int_0^3 \int_0^{\sqrt{y}} (3-y)\,dx\,dy$

$V = \int_0^{\sqrt{3}} 3y - \frac{1}{2}y^2 \Big|_{x^2}^{3} dx$

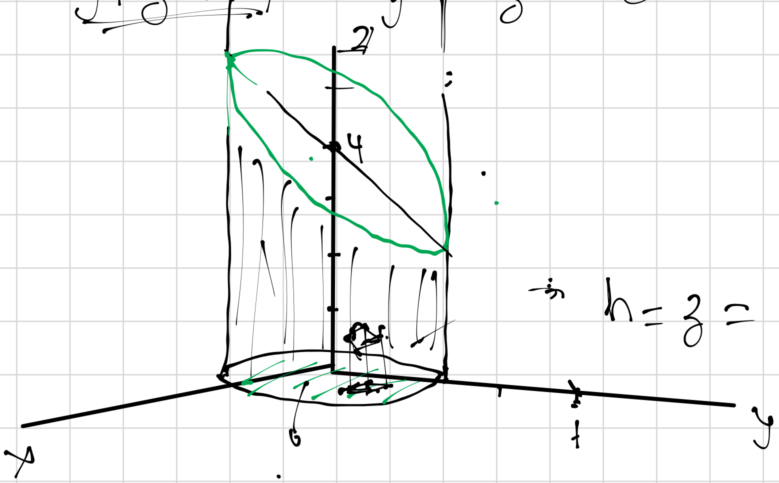
$$V = \int_0^{\sqrt{3}} \left(9 - \frac{9}{2}\right) - \left(3x^2 - \frac{1}{2}x^4\right) dx$$

$\frac{9}{2}$

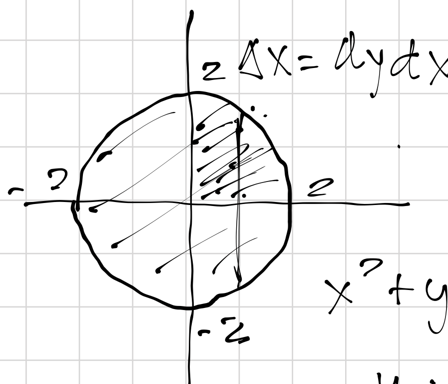
$$V = \frac{9}{2}x - x^3 + \frac{1}{10}x^5 \Big|_0^{\sqrt{3}}$$

$$V = \frac{9}{2}(\sqrt{3}) - (\sqrt{3})^3 + \frac{1}{10}(\sqrt{3})^5 =$$

Ej. Calcular el volumen del sólido limitado por la intersección del cilindro $x^2+y^2=4$, el plano $y+z=4$ y $z=0$



$\Rightarrow h = z = 4 - y$



$x^2+y^2=4$

$y = \pm\sqrt{4-x^2}$

$$V = \iint f(x,y)\, dA$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

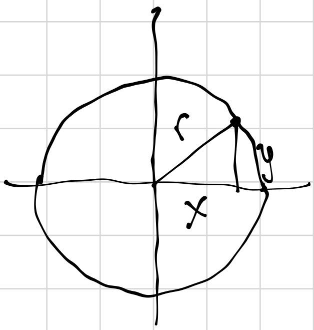
$$V = \int_{-2}^{2} 4y - \frac{1}{2}y^2 \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

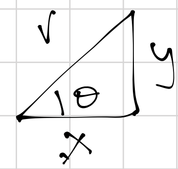
$$\int_{-2}^{2} \left[4\sqrt{4-x^2} - \frac{1}{2}(4-x^2)^{1/2\,\cdot 2}\right] - \left[4(-\sqrt{4-x^2}) - \frac{1}{2}(-\sqrt{4-x^2})^2\right] dx$$

$$\int_{-2}^{2} \underline{4\sqrt{4-x^2}} - \frac{1}{2}(4-x^2) + \underline{4\sqrt{4-x^2}} - \frac{1}{2}(4-x^2)\, dx$$

↑ sust. trigonométrica.

Coordenadas Rectangulares – coordenadas Polares





Por Pit.

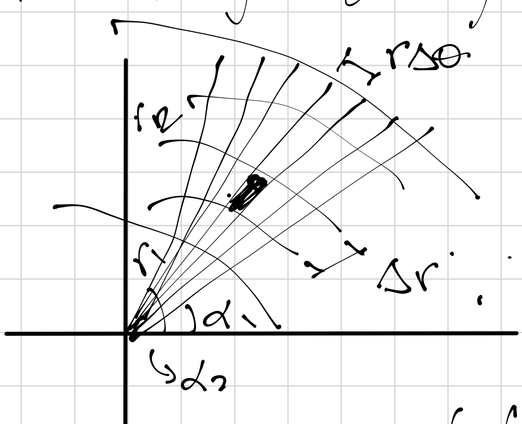
$$\boxed{x^2+y^2=r^2}$$

$$\operatorname{sen}\theta = \frac{y}{r} \qquad \boxed{y = r\operatorname{sen}\theta}$$

$$\cos\theta = \frac{x}{r} \rightarrow \boxed{x = r\cos\theta}$$

Diferencial del Área.

$dA = dx\,dy$ o $dy\,dx$



$$\Delta A \simeq r\Delta\theta\,\Delta r$$

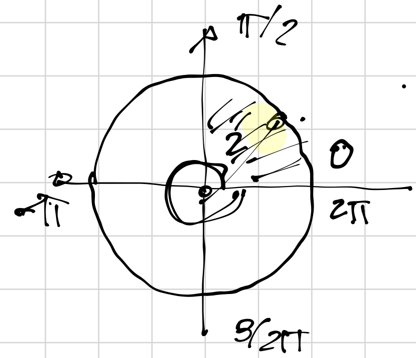
$$\Delta A \approx dA = r\,dr\,d\theta$$

$$V = \int\int f(x,y)\,dx\,dy$$

$$y = \sqrt{4-x^2}$$

$$V = \int\int f(r\cos\theta, r\operatorname{sen}\theta)\,r\,dr\,d\theta$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$



$$x^2+y^2=4$$

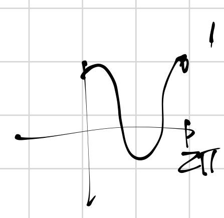
$$\sqrt{r^2}=\sqrt{4}$$

$$\boxed{r=2}$$

$4r - r^2 s - \theta$

$$V = \int_0^{2\pi}\int_0^{2} (4 - r\operatorname{sen}\theta)\,r\,dr\,d\theta$$

$$V = \int_0^{2\pi} 2r^2 - \frac{1}{3}r^3\operatorname{sen}\theta\Big|_0^2 \,d\theta$$



$$V = \int_0^{2\pi} \left(8 - \frac{8}{3}\operatorname{sen}\theta\right) d\theta$$

$$V = 8\theta + \frac{8}{3}\cos\theta \Big|_0^{2\pi}$$

$$V = \left(8(2\pi) + \frac{8}{3}\overset{1}{\cos 2\pi}\right) - \left(\overset{0}{8(0)} + \frac{8}{3}\overset{+1}{\cos(0)}\right)$$

$$V = 16\pi + \frac{8}{3} - \frac{8}{3} = \underline{16\pi}$$