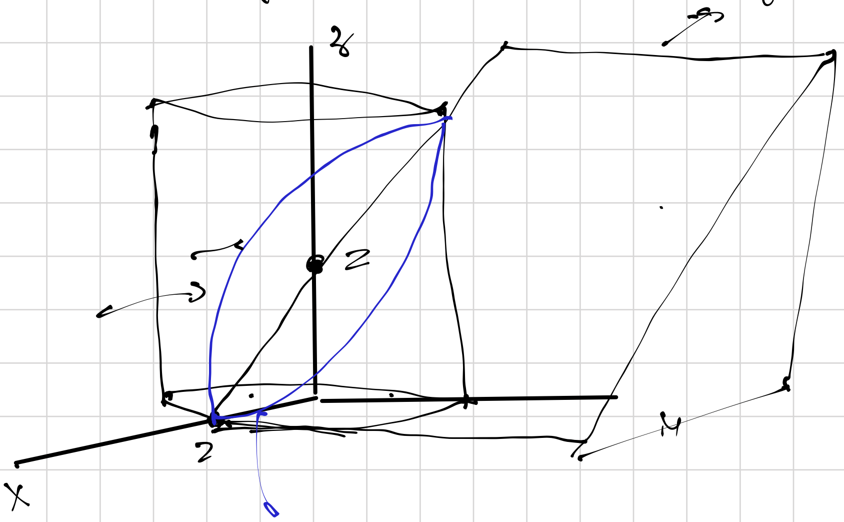
Funcion vectorial de la interseccion de superficies cuadraticas y planos.

Ej. Determine una funcion vectorial que represente la curva de interseccion de las superficies

$x^2+y^2=4$ $\qquad$ $x+z=2$



$x^2+y^2=r^2$

$x^2+y^2=4 \quad \rightarrow \quad x=r\cos t \qquad y=r\,\text{sen}\,t$

$r=2 \qquad x=2\cos t \qquad y=2\,\text{sen}\,t$

$x+z=2 \quad \rightarrow \quad z=2-x$

$z=2-2\cos t$

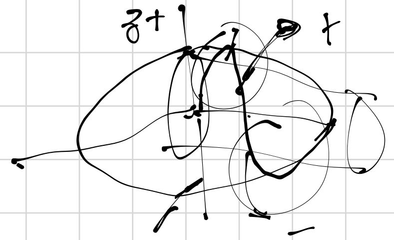
$r(t)=2\cos t\, i+2\,\text{sen}\,t\, j+(2-2\cos t)\, k$

Ej. Determine una funcion vectorial que represente la curva de interseccion del

semielipsoido $\quad x^2+y^2+4z^2=4$

cilindro $\quad x^2+z^2=1 \quad \rightarrow \quad z^2=1-x^2$

$x^2+y^2+4(1-x^2)=4 \qquad\qquad z=t$



$$x^2 + y^2 + x - 4x^2 = 4$$

$$x^2 - 4x^2 + y^2 = 0$$

$$-3x^2 + y^2 = 0 \qquad \sqrt{y^2} = \sqrt{3x^2}$$

$$y = \pm\sqrt{3}x \rightarrow y = +\sqrt{3}x$$

$x = t$ $\qquad y = +\sqrt{3}\,t$ $\qquad x = \frac{y}{\sqrt{3}}$

$$\sqrt{z^2} = \sqrt{1 - x^2} \qquad z = \pm\sqrt{1-x^2} \rightarrow z = +\sqrt{1 - t^2}$$

$$r(t) = t\,i + \sqrt{3}\,t\,j + \sqrt{1-t^2}\,k$$

$y = t$

$x = \frac{t}{\sqrt{3}}$

$z = \sqrt{1 - \frac{t^2}{3}}$

Ej. hiperboloide $z = x^2 - y^2$

cilindro $x^2 + y^2 = 1 \rightarrow r = 1$

$x = r\cos t \qquad y = r\,\text{sen}\, t$

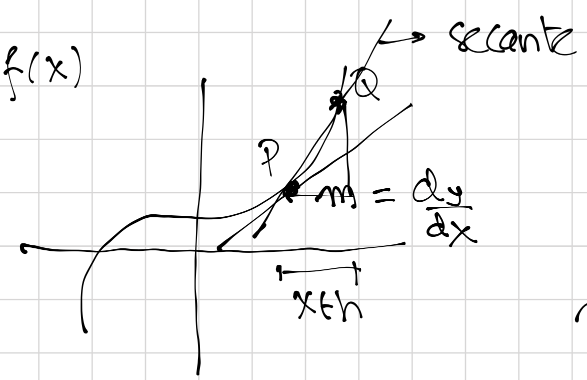
$x = \cos t \qquad y = \text{sen}\, t$

$z = \cos^2 t - \text{sen}^2 t$

$$r(t) = \cos t\, i + \text{sen}\, t\, j + (\cos^2 t - \text{sen}^2 t)\, k$$

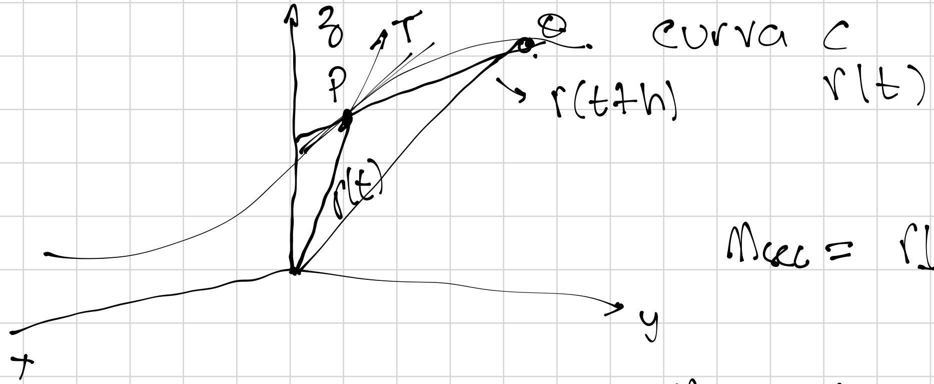
Derivadas de funciones vectoriales

$y = f(x)$



$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$M_{sec} = \frac{r(t+h) - r(t)}{h}$$

$$M_{tg} = \frac{dr}{dt} = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

Teorema

Si $r(t) = f(t)i + g(t)j + h(t)k$ donde f, g y h son funciones derivables, entonces

$$r'(t) = f'(t)i + g'(t)j + h'(t)k$$

Vector T → Vector tangente unitario

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$T(t) = \frac{f'(t)i + g'(t)j + h'(t)k}{\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}}$$

Ej. Determine $r'(t)$

$$r(t) = t^2 i + \cos(t^2) j + \operatorname{sen}^2 t\, k$$

$$r'(t) = 2t i - 2t \operatorname{sen}(t^2) j + 2 \operatorname{sen} t \cdot \cos t\, k$$

Ej. Determine $r'(t)$

$$r(t) = \frac{1}{1+t} i + \frac{t}{1+t} j + \frac{t^2}{1+t} k$$

$$i \to \frac{1}{1+t} \to (1+t)^{-1} \to -1(1+t)^{-2}(1) = \frac{-1}{(1+t)^2}$$

$j \to \dfrac{t}{1+t} \to \dfrac{(1+t)(1) - t(1)}{(1+t)^2} = \dfrac{1+\cancel{t}-\cancel{t}}{(1+t)^2} = \dfrac{1}{(1+t)^2}$

$k \to \dfrac{t^2}{1+t} \to \dfrac{(1+t)(2t) - t^2(1)}{(1+t)^2} = \dfrac{2t+2t^2-t^2}{(1+t)^2} = \dfrac{2t+t^2}{(1+t)^2}$

$r'(t) = -\dfrac{1}{(1+t)^2} i + \dfrac{1}{(1+t)^2} j + \dfrac{2t+t^2}{(1+t)^2} k$

Ej. Si $r(t) = e^{2t} i + e^{-2t} j + te^{2t} k$

Determine $T(0)$, $r''(0)$, $r'(t) \times r''(t)$

$r'(t) = 2e^{2t} i - 2e^{-2t} + (2te^{2t} + e^{2t}) k$

$r''(t) = 4e^{2t} i + 4e^{-2t} j + (4te^{2t} + 2e^{2t} + 2e^{2t}) k$

$r''(t) = 4e^{2t} i + 4e^{-2t} j + (4te^{2t} + 4e^{2t}) k$

$T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{2e^{2t} i - 2e^{-2t} j + (2te^{2t} + e^{2t}) k}{\sqrt{(2e^{2t})^2 + (-2e^{-2t})^2 + (2te^{2t} + e^{2t})^2}}$

$T(0) = \dfrac{2e^{2(0)} i - 2e^{-2(0)} j + 2(0)e^{2(0)} + e^{2(0)} k}{\sqrt{(2e^{2(0)})^2 + (-2e^{-2(0)})^2 + (2(0)e^{2(0)} + e^{2(0)})^2}}$

$T(0) = \dfrac{2i - 2j + 2k}{\sqrt{4+4+1}} = \dfrac{2i - 2j + 1k}{\sqrt{9}} = \dfrac{2}{3} i - \dfrac{2}{3} j + \dfrac{1}{3} k$

$r''(0) = 4e^{2(0)} i + 4e^{-2(0)} j + (4(0)e^{2(0)} + 4e^{2(0)}) k$

$r''(0) = 4i + 4j + 4k$

$r'(t) \times r'(t) = \begin{vmatrix} i & j & k \\ 2e^{2t} & -2e^{-2t} & 2te^{2t} + e^{2t} \\ 4e^{2t} & 4e^{-2t} & 4te^{2t} + 4e^{2t} \end{vmatrix}$

$(-1)^{i+j}$

$= [(-2e^{-2t})(4te^{2t}+4e^{2t}) - 4e^{-2t}(2te^{2t}+e^{2t})]i - 2e^{2t}(4te^{2t}+4e^{2t})$

$-4e^{2t}(2te^{2t}+e^{2t}) + [(2e^{2t})(4e^{-2t}) - (4e^{2t})(-2e^{-2t})$

$(-8t-8-8t-4)i - [(\cancel{8t}e^{4t}+8e^{4t}-\cancel{8t}e^{4t}-4e^{4t})]j$

$(8+8)K$

$r'(t) \times r''(t) = (-16t-12)i + 4e^{4t}j + 16K$

<u>Integrales de funciones vectoriales</u>

$r(t)$

$$\int_a^b r(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} r(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left[\sum_{i=1}^{n} f(t_i^*)\Delta t\right]i + \left[\sum_{i=1}^{n} g(t_i)\Delta t\right]j + \left[\sum_{i=1}^{n} h(t_i)^*\Delta t\right]K\right.$$

$$\int_a^b r(t)dt = \int_a^b f(t)dt\, i + \int_a^b g(t)dt\, j + \int_a^b h(t)dt\, K$$

$$\int r(t)dt = (F(t)+C_1)i + (G(t)+C_2)j + (H(t)+C_3)K$$

Ej. Evalúe la integral.

$$\int_1^4 (2t^{3/2}\,i + (t+1)\sqrt{t}\,K)\,dt$$

$i:\ \int_1^4 2t^{3/2}\,dt = 2\frac{t^{3/2+1}}{\frac{3}{2}+1} = \frac{2t^{5/2}}{\frac{5}{2}} = \frac{4}{5}t^{5/2}\Big|_1^4$

$\frac{4}{5}(4^{5/2}-1^{5/2}) = \frac{4}{5}(31) = \frac{124}{5}$

$K:\ \int_1^4 (t^{3/2}+t^{1/2})dt = \frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2}\Big|_1^4$

$$\left[\frac{2}{5}(4)^{5/2}+\frac{2}{3}(4)^{3/2}\right]-\left[\frac{2}{5}(1)^{5/2}+\frac{2}{3}(1)^{3/2}\right]$$

$$\left(\frac{64}{5}+\frac{16}{3}\right)-\left(\frac{2}{5}+\frac{2}{3}\right)=\frac{272}{15}-\frac{16}{15}=\frac{256}{15}$$

Sol $\quad \frac{124}{5} i+\frac{256}{15} K$

Ej. Determine

$$\int_0^1\left(\frac{1}{t+1} i+\frac{1}{t^2+1} j+\frac{t}{t^2+1} K\right) dt$$

$i \rightarrow \int \frac{1}{t+1} dt \quad \rightarrow u=t+1 \quad du=dt$

$$\int \frac{du}{u}=\ln u=\ln(t+1)\Big|_0^1$$

$$\ln(1+1)-\cancel{\ln(0+1)}^{\,0}=\ln(2)$$

$$j \rightarrow \int_0^1 \frac{1}{t^2+1} dt=tg^{-1} t\Big|_0^1=\underbrace{tg^{-1} 1}_{\frac{\pi}{4}}-\cancel{tg^{-1} 0}^{\,0}$$

$K \rightarrow \int_0^1 \frac{t}{t^2+1} dt \quad u=t^2+1 \quad du=2t\, dt$

$$dt=\frac{du}{2}$$

$$\frac{1}{2}\int \frac{du}{u}=\frac{1}{2}\ln u=\frac{1}{2}\ln(t^2+1)\Big|_0^1$$

$$\frac{1}{2}\ln(1^2+1)-\cancel{\frac{1}{2}\ln(0^2+1)}^{\,0}=\frac{1}{2}\ln 2$$

Sol $\quad \ln 2\, i+\frac{\pi}{4} j+\frac{1}{2}\ln 2\, K$