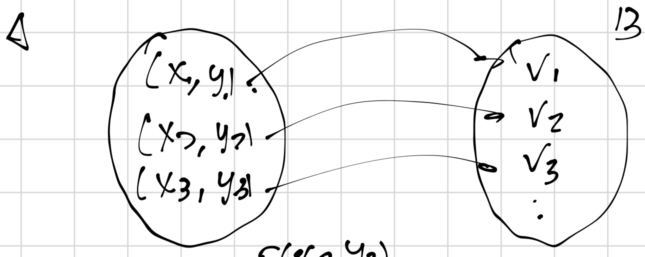
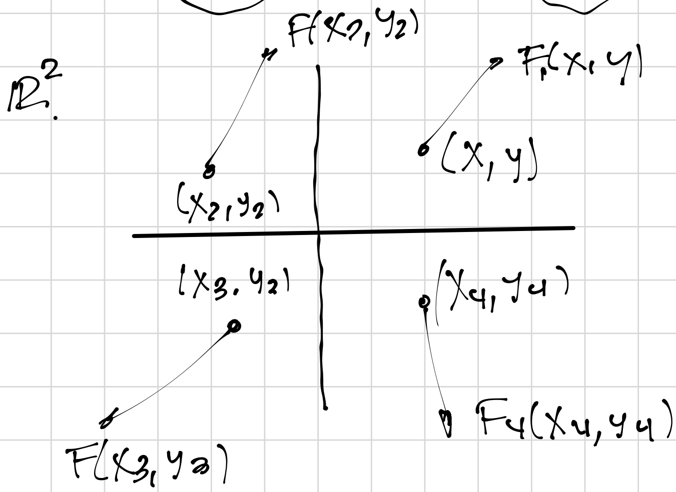
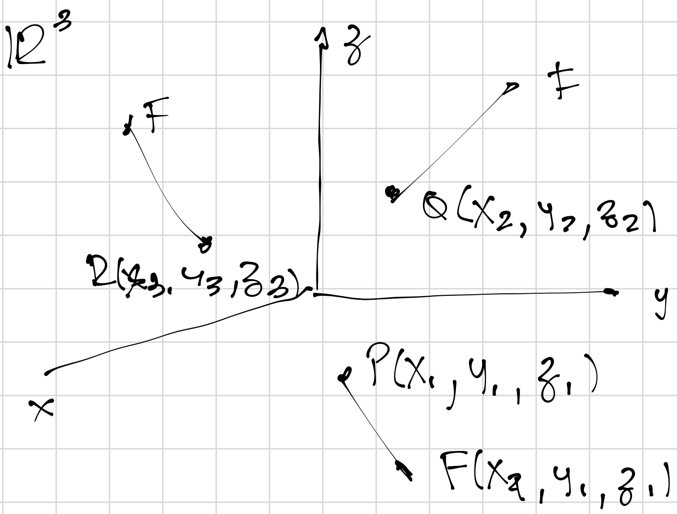
Calculo vectorial

Campo vectorial: Es una funcion cuyo dominio es un conjunto de puntos en $\mathbb{R}^2$ o $\mathbb{R}^3$ y cuyo rango es un conjunto de vectores V_2 o V_3.





$\longrightarrow F(x,y) = P(x,y)i + Q(x,y)j$

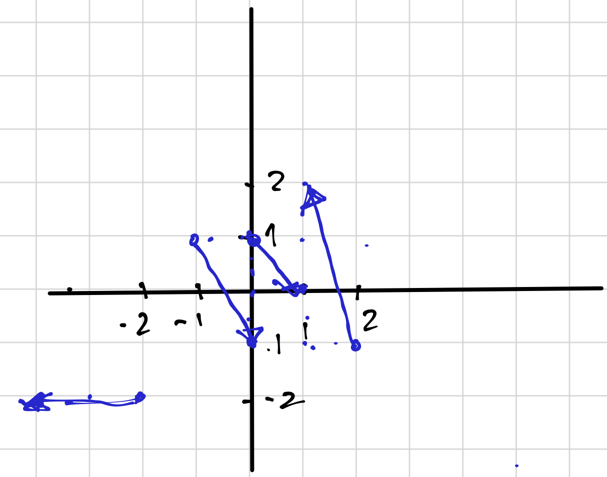


$\longrightarrow F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$

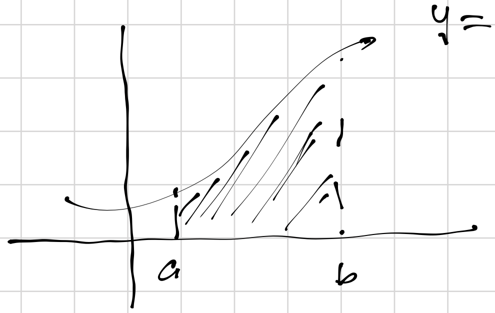
Ej. trace el campo vectorial.

$$F(x,y) = yi + (x-y)j + zk$$

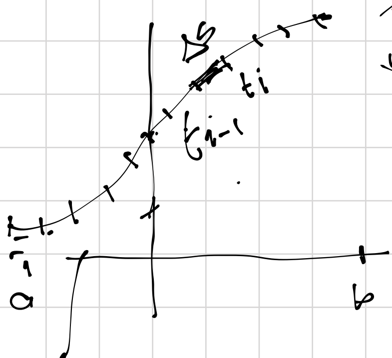
x	y	F
0	1	$F = i + (0-1)j = i - j$
-1	1	$F = i + (-1-1)j = i - 2j$
-2	-2	$F = -2i + (-2-(-2))j = -2i$
2	-1	$F = -i + (2-(-1))j = -i + 3j$



Integrales de línea



$$A = \int_a^b f(x)\,dx$$



$y = f(x)$. → integrales de línea.

alambre → densidad lineal → $\rho(x,y)$

⇓

Encontrar la masa del alambre.

$$\rho(x,y) = \frac{\text{masa}}{\text{longitud}}$$

masa $= \rho(x,y) \cdot$ longitud.

longitud $= \Delta s_i$

masa $= \rho(x_i, y_i)\,\Delta s_i$

masa $= \sum_{i=1}^{n} \rho(x_i, y_i)\,\Delta s_i$

masa $= \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i, y_i)\,\Delta s_i$

$$\boxed{\text{masa} = \int_C \rho(x,y)\,ds}$$

Definición: Integral de línea.

Si f se define en una curva suave C dada por las ecuaciones $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, la integral de línea de f a lo largo de C es:

$$\rightarrow \int_C f(x,y)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i$$

$ds \rightarrow$ longitud de la curva.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$- \int_C f(x,y)\,ds = \int^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \rightarrow \mathbb{R}^2$$

$$\int_C f(x,y,z)\,ds = \int f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Integral de línea en forma de diferencial.

con respecto de x.

$$\int_C f(x,y)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i$$

con respecto de y

$$\int f(x,y)\,dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta y_i$$

$$\int f(x,y)\,dx = \int f(x(t), y(t))\,x'(t)\,dt \quad +$$

$$\int f(x,y)\,dy = \int f(x(t), y(t))\,y'(t)\,dt$$

$$\int f(x,y)\,dx + \int f(x,y)\,dy = \int M(x,y)\,dx + N(x,y)\,dy$$

$\mathbb{R}^3$

$\int f(x,y)\,dz = \int f(x(t), y(t), z(t))\, z'(t)\,dt$

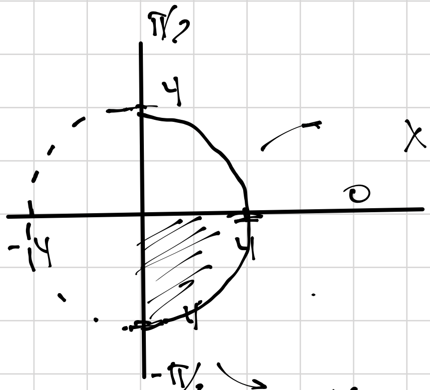
$\Downarrow$

$\int M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$

Ej. Evaluar la integral de linea

$\int x y^4\, ds$

donde C es la mitad derecho del circulo $x^2+y^2=16$



$x^2+y^2=16$

$\int x y^4\, ds$

$x = r\cos t \qquad y = r\,\text{sen}\, t \qquad x' = -4\,\text{sen}\, t$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad y' = 4\cos t$

$x = 4\cos t \qquad y = 4\,\text{sen}\, t$

$\int x y^4\, ds = \int (4\cos t)(4\,\text{sen}\, t)^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int 4\cos t \cdot 4^4\, \text{sen}^4 t \sqrt{(-4\,\text{sen}\, t)^2 + (4\cos t)^2}\, dt$

$4^5 \int \cos t\, \text{sen}^4 t \sqrt{16\,\text{sen}^2 t + 16\cos^2 t}\, dt$

$4^5 \int \cos t\, \text{sen}^4 t \sqrt{16(\text{sen}^2 t + \cos^2 t)}\, dt$

$(2)\, 4^6 \int_0^{\pi/2} \cos t\, \text{sen}^4 t\, dt \qquad u = \text{sen}\, t$

$\qquad\qquad\qquad\qquad\qquad\qquad du = \cos t\, dt$

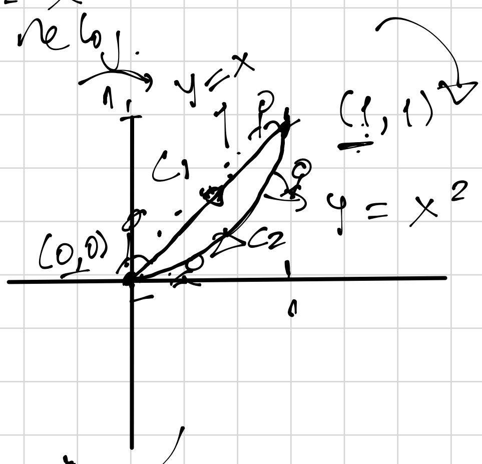
$2\cdot 4^6 \int_0^{\pi/2} u^4\, du = \frac{2\cdot 4^6}{5}\, u^5 \Big|_0^{\pi/2}$

$$\frac{2\cdot 4^6}{5}\,\operatorname{sen}^5 t\Big|_0^{\pi/2} = \frac{2\cdot 4^6}{5}\cdot(\operatorname{sen}\pi/2)^5$$

$$= \frac{2\cdot 4^6}{5}$$

Ej. Calcular $\int x\,dx$

donde C es la curva de intersección de $y=x$ y $y=x^2$ en sentido a favor de las manecillas del reloj.



$$\to \int x\,dx = \int_{C_1} x\,dx + \int_{C_2} x\,dx$$

Para C_1 $\quad y=x \quad x=t \quad y=t \quad x'(t)=1 \quad y'(t)=1$

$$\int x\,dx = \int x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

$$\int t\sqrt{(1)^2+(1)^2}\,dt = \sqrt{2}\int_0^1 t\,dt$$

$$\sqrt{2}\cdot\frac{1}{2}t^2\Big|_0^1 = \frac{\sqrt{2}}{2}\left[1^2-0^2\right] = \frac{\sqrt{2}}{2}$$

Para C_2 $\quad y=x^2 \quad \boxed{x=t} \quad y=t^2$

$$dx = 1\,dt \qquad dy = 2t\,dt$$

$$\int t\sqrt{(1)^2+(2t)^2}\,dt = \int_1^0 t\sqrt{1+4t^2}\,dt$$

$$u = 1+4t^2 \qquad du = 8t\,dt$$

$$\frac{du}{8} = t\,dt$$

$$= \frac{1}{8}\int_1^0 u^{1/2}\,du = \frac{1}{4\cancel{8}}\cdot\frac{u^{3/2}}{\frac{3}{2}} = \frac{1}{12}\left(1+4t^2\right)^{3/2}\Big|_1^0$$

$$= \frac{1}{12}\left[\left(1+4(0)^2\right)^{3/2} - \left(1+4(1)^2\right)^{3/2}\right] = \frac{1}{12}\left[1 - 5^{3/2}\right] = -0.8483$$

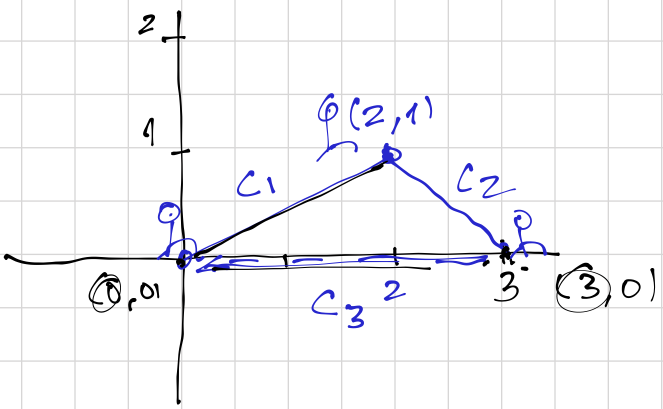
$$C = C_1 + C_2 = \frac{\sqrt{2}}{2} - 0.8493 = \underline{-0.1411}$$

Ej.

Evalue la integral de linea.

$$\int (x+2y)\,dx + x^2\,dy$$

donde C:
segmentos de recta de $(0,0)$ a $(2,1)$ y de $(2,1)$ a $(3,0)$



Para C_1 $\quad y - y_0 = m(x - x_0)$ $\quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{2-0} = \frac{1}{2}$

$$y - 0 = \frac{1}{2}(x - 0)$$

$$\boxed{y = \frac{1}{2}x}$$

$$dy = \frac{1}{2}\,dx$$

$$\int (x + 2(\tfrac{1}{2}x))\,dx + x^2(\tfrac{1}{2})\,dx$$

$$\int_0^2 (x + x)\,dx + \frac{1}{2}x^2\,dx$$

$$\int_0^2 (2x + \frac{1}{2}x^2)\,dx = x^2 + \frac{1}{6}x^3\Big|_0^2$$

$$2^2 + \frac{1}{6}(2)^3 = 4 + \frac{8}{6} = 4 + \frac{4}{3} = \frac{16}{3}$$

$+$

C_2 $(2,1) - (3,0)$

$$y = y_0 + m(x - x_0) \qquad m = \frac{0-1}{3-2} = \frac{-1}{1} = -1$$

$$y = 1 - 1(x-2)$$

$$y = 1 - x + 2 \rightarrow y = -x + 3 \qquad dy = -dx$$

$$\int_2^3 (x + 2(-x+3))dx + (x^2)(-dx)$$

$$\int_2^3 (x - 2x + 6 - x^2)\,dx$$

$$\int_2^3 (-x + 6 - x^2)\,dx$$

$$-\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3 \Big|_2^3$$

$$\left[-\frac{1}{2}(3)^2 + 6(3) - \frac{1}{3}(3^3)\right] - \left[-\frac{1}{2}(2)^2 + 6(2) - \frac{1}{3}(2)^3\right]$$

$$\left[-\frac{9}{2} + 18 - 9\right] - \left[-2 + 12 - \frac{8}{3}\right]$$

$$\left[-\frac{9}{2} + 9\right] - \left[10 - \frac{8}{3}\right]$$

$$-\frac{9}{2} - 1 + \frac{8}{3} = \frac{-27 - 6 + 16}{6} = -\frac{17}{6}$$

Para C_3 $\quad x = t \qquad y = 0 \qquad dx = dt$

$dy = 0$

$$\int (x + 2y)\,dx + x^2\,dy$$

$$\int (t + 2(0))\,dt + \cancel{t^2(0)}^{\,0}$$

$$= \int_3^0 t\,dt = \frac{1}{2} t^2 \Big|_3^0 = \frac{1}{2}(0^2 - 3^2) = \frac{-9}{2}$$

$$C = C_1 + C_2 + C_3$$

$$C = \frac{16}{3} - \frac{17}{6} - \frac{9}{2} =$$