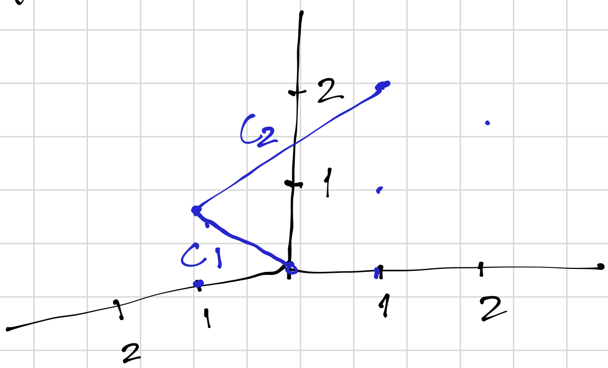
Ej. Evalue la integral de línea.

$$\int (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$

donde C consta de segmentos de recta de $(0,0,0)$ a $(1,0,1)$ y de $(1,0,1)$ a $(0,1,2)$.



$C = C_1 + C_2$

$r(t) = (1-t)\,r_0 + t\,r_1 \rightarrow [0,1]$

$r_0 \to P_0 \qquad r_1 \to P_1$

$C_1 \quad (0,0,0) \rightarrow (1,0,1)$

$r(t) = \cancel{(1-t)}\langle 0,0,0\rangle + t\langle 1,0,1\rangle$

$r(t) = t\,i + 0\,j + t\,k$

$x = t \rightarrow dx = dt$
$y = 0 \rightarrow dy = 0$
$z = t \qquad dz = dt$

$$\int_0^1 (0+t)\,dt + (t+t)\,0 + (t+0)\,dt$$

$$\int_0^1 2t\,dt = \frac{2}{2}t^2\Big|_0^1 = (1)^2 = 1$$

$C_2 \quad (1,0,1) \rightarrow (0,1,2)$

$r(t) = (1-t)\langle 1,0,1\rangle + t\langle 0,1,2\rangle \rightarrow [0,1]$

$r(t) = (1-t)\,i + 0j + (1-t)\,k + \cancel{0t}\,i + t\,j + 2t\,k$

$r(t) = (1-t)\,i + t\,j + (1+t)\,k$

$x = 1-t \qquad dx = -dt$
$y = t \qquad dy = dt$
$z = 1+t \qquad dz = dt$

$$\int_0^1 (t+1+t)(-dt) + (1-\cancel{t}+1+\cancel{t})\,dt + (1-\cancel{t}+t)\,dt$$

$$\int_0^1 (-2t-\cancel{1}+2+\cancel{1})\,dt = \int_0^1 -2t+2\,dt$$

$$-t^2+2t\Big|_0^1 = -(1)^2+2(1) = -1+2 = 1$$

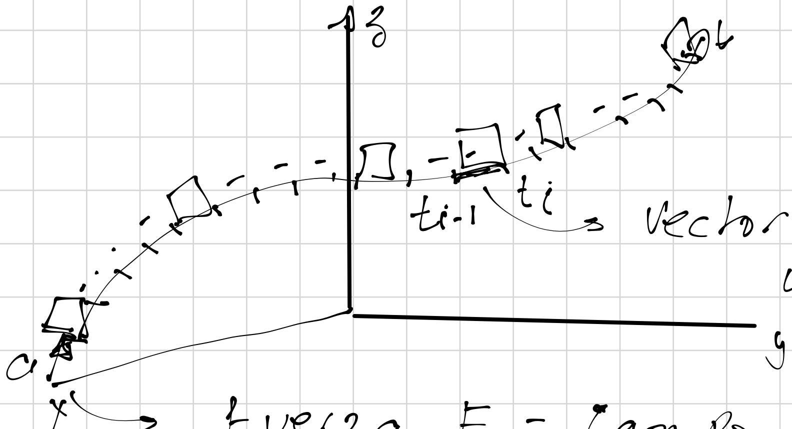
$$C = C_1 + C_2 = 1+1 = 2$$

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Integrales de línea de Campos vectoriales



curva

$r(t) = f(t)i + g(t)j + h(t)k$

$a \leq t \leq b$

→ vector tangente unitario

→ fuerza F = Campo vectorial.

$$F = Mi + Nj + Pk$$

trabajo = W = fuerza · distancia.

distancia = Δs_i

$$W = F(x_i, y_i, z_i) \cdot \Delta s_i$$

vector → escalar.

$$W = F(x_i, y_i, z_i) \cdot T(t_i)\,\Delta s_i$$

$$W = \sum_{i=1}^{n} F(x_i, y_i, z_i) \cdot T(t_i)\,\Delta s_i$$

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i, y_i, z_i) \cdot T(t_i)\,\Delta s_i$$

$$W = \int_a^b F(x,y,z) \circ T(x,y,z)\, ds = \int_a^b F \cdot T\, ds$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} \qquad ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$W = \int_a^b (Mi + Nj + PK) \cdot \left(\frac{\frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} K}{\|r'(t)\|} \right) \cdot \|r'(t)\|\, \cancel{dt}$$

$$W = \int_a^b M\,dx + N\,dy + P\,dz$$

$$W = \int_a^b F \cdot dr$$

Ej Evalue la integral $\int F \cdot dr$ donde c esta dada por

$$F(x,y) = xy^2 i - x^2 j$$

$$\boxed{r(t) = (t^3 i + t^2 j)} \quad \rightarrow \quad 0 \leq t \leq 1$$

$$\rightarrow \quad dr = (3t^2 i + 2t j)\, dt$$

K

$$\int_0^1 (xy^2 i - x^2 j) \cdot (3t^2 i + 2t j)\, dt \qquad \rightarrow r(t) = x(t) i + y(t) j$$

$$x = t^3, \quad y = t^2 \qquad x = t^3 \quad y = t^2$$

$$\int_0^1 (t^3 \cdot t^4 i - t^6 j) \cdot (3t^2 i + 2t j)\, dt$$

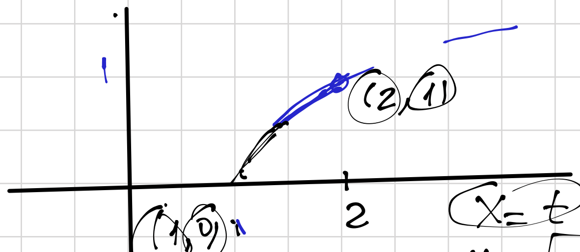
$$\int_0^1 (3t^9 - 2t^7)\, dt = \frac{3}{10} t^{10} - \frac{2}{8} t^8 \Big|_0^1$$

$$\frac{3}{10}(1)^{10} - \frac{1}{4}(1)^8 = \frac{12 - 10}{40} = \frac{2}{40} = \frac{1}{20}$$

Ej. Encontrar el trabajo realizado por la fuerza

$$F(x,y) = x^2 i + y e^x j$$

Sobre una particula que se mueve a lo largo de la parabola $x = y^2 + 1$ de $(1,0)$ a $(2,1)$



$x = t$

$y = \sqrt{t-1}$

$r(t) = ti + \sqrt{t-1}\,j$

$x = y^2 + 1 \to y = \sqrt{x-1}$

$x = t$

$\boxed{y = t}$

$x = t^2 + 1 \qquad x^2 = (t^2+1)^3$

$r(t) = (t^2+1)i + tj$

$dr = (2ti + j)dt$

$W = \int F \cdot dr$

$t^2 i + \sqrt{t-1}\, e^t$

$w = \int_0^1 (x^2 i + ye^x j) \cdot (2ti + j)dt$

$w = \int_0^1 ((t^2+1)^2 i + te^{t^2+1} j) \cdot (2ti + j)\,dt$

$w = \int_0^1 (2t(t^2+1)^2 + te^{t^2+1})\,dt$

$u = t^2+1$

$du = 2t\,dt$

$u = t^2 + 1 \qquad du = 2t\,dt$

$\frac{du}{2} = t\,dt$

$w = \int_0^1 u^2\,du + \int_0^1 \frac{1}{2} e^u\,du$

$\frac{1}{3}(t^2+1)^3 + \frac{1}{2}e^{t^2+1}\Big|_0^1$

$\left[\frac{1}{3}(1^2+1)^3 + \frac{1}{2}e^{1^2+1}\right] - \left[\frac{1}{3}(0^2+1)^3 + \frac{1}{2}e^{0^2+1}\right]$

$\left(\frac{8}{3} + \frac{1}{2}e^2\right) - \left(\frac{1}{3} + \frac{1}{2}e^1\right) =$

Ej. Calcular el trabajo realizado por la fuerza

$\vec{F}(x,y,z) = yz\,i + xz\,j + xy\,k$ que actua a lo largo de la curva dada por

$r(t) = t^3 i + t^2 j + t k$ de $t = 1$ a $t = 3$.

$P(1,1,1)$ a $Q(3,2,3)$

$W = \int F \cdot dr$ → distancia

$x = t^3 \qquad y = t^2 \qquad z = t$

$dx = 3t^2 dt \qquad dy = 2t\,dt \qquad dz = dt$

$dr = (3t^2 i + 2t j + k)\,dt$

$W = \int_1^3 (t^3 i + t^4 j + t^5) \cdot (3t^2 i + 2t j + k)\,dt$

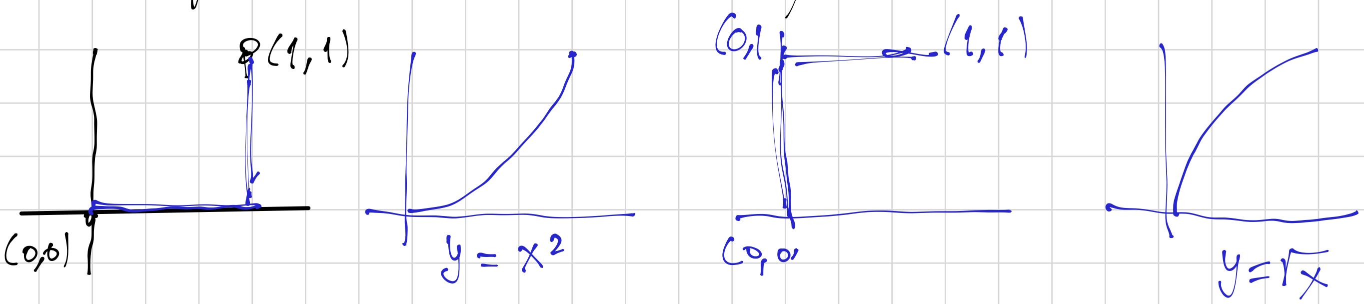
$W = \int_1^3 (3t^5 + 2t^5 + t^5)\,dt = \int_1^3 6t^5$

$\frac{6}{6} t^6 \Big|_1^3$

$W = (3^6 - 1^6)$

Teorema fundamental de las integrales de línea

→ Campo vectorial sea conservativo.
→ Independencia de la trayectoria.



Sea C una curva suave a trozos en una región abierta R dada por

$r(t) = x(t)\,i + y(t)\,j \qquad a \le t \le b$

Si $F(x,y) = Mi + Nj$ es un campo vectorial conservativo en R, y además M y N son continuos en R, entonces

$\int F \cdot dr = \int_a^b \nabla f \cdot dr = f(x(b), y(b)) - f(x(a), y(a))$

función potencial.

$F(x,y) = \nabla f(x,y) \Rightarrow$ diferencial total de una función f.

Criterio de Campos Conservativos

Si $F(x,y) = P(x,y)i + Q(x,y)j$ es un campo vectorial conservativo donde P y Q tienen derivadas parciales de primer orden en su dominio D, entonces a todo lo largo de D, se tiene

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \rightarrow$$

$$F(x,y) = P(x,y)\,i + Q(x,y)$$

$$\int P(x,y)\,dx + \int Q(x,y)\,dy$$

sol. de la EC dif. exacta.

$f(x,y)$

Campo vectorial es en R^3.

$$F(x,y,z) = Pi + Qj + RK$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial y}$$

$$= \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$$= \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

$\int P\,dx + \int Q\,dy + \int R\,dz$

$f(x,y,z)$

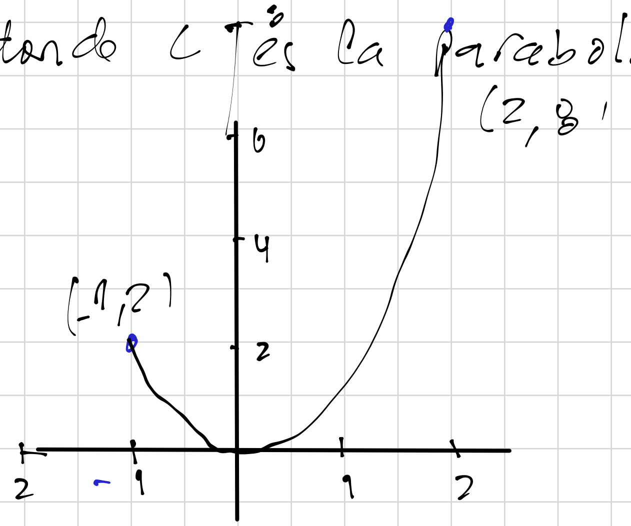
$y^3 + xy \qquad x^4 + xy \qquad z^5$

$f(x,y,z) = y^3 + xy + x^4 + z^5$

Ej Evaluar la integral $\int F \cdot dr$ donde

$F(x,y) = x^2 i + y^2 j$

donde C es la parabola $y = 2x^2$ de $(-1,2)$ a $(2,8)$



$F(x,y) = \underset{M}{x^2}\, i + \underset{N}{y^2}\, j$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$0 = 0 \rightarrow$ Si es un campo conservativo.

$\int x^2 dx + \int y^2 dy$

$\frac{1}{3}x^3 \qquad \frac{1}{3}y^3$

$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^2$

$$\int F \cdot dr = f(x,y)\Big|_{(-1,2)}^{(2,8)}$$

$$= \frac{1}{3}x^3 + \frac{1}{3}y^3\Big|_{(-1,2)}^{(2,8)}$$

$$\left[\frac{1}{3}(2)^3 + \frac{1}{3}(8)^3\right] - \left[\frac{1}{3}(-1)^3 + \frac{1}{3}(2)^3\right] =$$

$$\left(\cancel{\frac{8}{3}} + \frac{512}{3}\right) - \left(\frac{-1}{3} + \cancel{\frac{8}{3}}\right) = \frac{512}{3} + \frac{1}{3} = \frac{513}{3}$$

Ej Evaluar $\int F \cdot dr$

donde $F(x,y,z) = (y^2 z + 2xz^2)i + (2xyz)j + (xy^2 + 2x^2 z)k$

donde $C: \quad r(t) = \sqrt{t}\, i + (t+1)j + t^2 k \qquad 0 \le t \le 1$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z + 2xz^2 & 2xyz & xy^2 + 2x^2 z \end{vmatrix}$$

$$\frac{\partial}{\partial y}(xy^2 + 2x^2 z) = \frac{\partial}{\partial z}(2xyz)$$

$$2xy = 2xy \checkmark$$

$$\frac{\partial}{\partial x}(xy^2 + 2x^2 z) = \frac{\partial}{\partial z}(y^2 z + 2xz^2)$$

$$y^2 + 4xz = y^2 + 4xz \checkmark$$ → Es un campo conservativo.

$$\frac{\partial}{\partial x}(2xyz) = \frac{\partial}{\partial y}(y^2 z + 2xz^2)$$

$$2yz = 2yz \checkmark$$

$f(x,y,z)$

$$\int (y^2 z + 2xz^2)\,dx + \int (2xyz)\,dy + \int (xy^2 + 2x^2 z)\,dz$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$xy^2z + x^2z^2 \qquad xy^2z \qquad xy^2z + x^2z^2$$

$$f(x,y,z) = xy^2z + x^2z^2$$

$$0 \leq t \leq 1$$

$$x = \sqrt{t} \qquad y = t+1 \qquad z = t^2$$

$t=0$ $\quad x = \sqrt{0} \qquad y = 0+1 \qquad z = 0^2$

$$x = 0 \qquad y = 1 \qquad z = 0 \qquad P(0,1,0$$

$t=1$

$$x = \sqrt{1} \qquad y = 1+1 \qquad z = 1^2 \qquad Q(1,2,1)$$

$$x = 1 \qquad y = 2 \qquad z = 1$$

$$xy^2z + x^2z^2 \Big|_{(0,1,0)}^{(1,2,1)}$$

$$= (1)(2)^2(1) + (1^2)(1) \quad - \quad = \quad 4+1 = 5$$