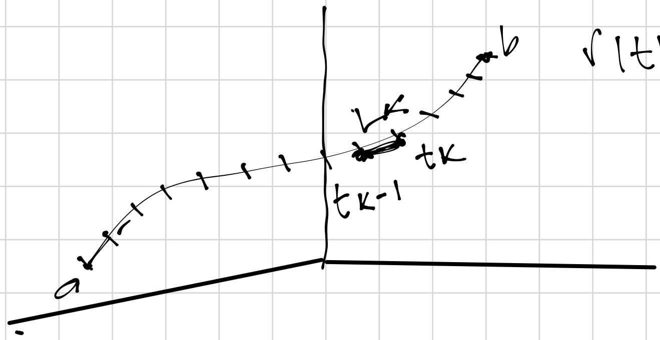
Longitud de una curva



$$r(t) = f(t)i + g(t)j + h(t)k \qquad a \le t \le b$$

$$a = t_0 < t_1 < t_2 < t_3 < \dots < t_n = b$$

$$L = \sum_{k=1}^{n} L_k$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$L_K = \sqrt{[f(t_K) - f(t_{K-1})]^2 + [g(t_K) - g(t_{K-1})]^2 + [h(t_K) - h(t_{K-1})]^2}$$

Teorema del valor medio

$$u_K^*, \; v_K^*, \; w_K^* \rightarrow (t_{K-1}, t_K)$$

$$f(t_K) - f(t_{K-1}) = f'(u_K^*)(t_K - t_{K-1}) = f'(u_K^*)\Delta t_K$$

$$g(t_K) - g(t_{K-1}) = g'(v_K^*)(t_K - t_{K-1}) = g'(v_K^*)\Delta t_K$$

$$h(t_K) - h(t_{K-1}) = h'(w_K^*)(t_K - t_{K-1}) = h'(w_K^*)\Delta t_K$$

$$L_K = \sqrt{[f'(u_K^*)\Delta t_K]^2 + [g'(v_K^*)\Delta t_K]^2 + [h'(w_K^*)\Delta t_K]^2}$$

$$L_K = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,\Delta t$$

$$L_K = \lim_{n \to \infty} \sum_{K=1}^{n} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,\Delta t$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

$$L = \int_a^b \| r'(t) \| \, dt$$

Ej. Calcular la longitud

$$r(t) = t^3 i + 6t j + 3t^2 k \qquad 0 \le t \le 3$$

$f(t) = t^3 \qquad g(t) = 6t \qquad h(t) = 3t^2$

$f'(t) = 3t^2 \qquad g'(t) = 6 \qquad h'(t) = 6t$

$$L = \int_0^3 \sqrt{(3t^2)^2 + (6)^2 + (6t)^2}\, dt$$

$$L = \int_0^3 \sqrt{9t^4 + 36 + 36t^2}\, dt$$

$9t^4 + 36t^2 + 36$

$3t^2 \times 6 \mid 18t^2$
$3t^2 \times 6 \mid 18t^2$
$36t^2$

$\rightarrow (3t^2+6)(3t^2+6) = (3t^2+6)^2$

$$L = \int_0^3 \sqrt{(3t^2+6)^2}\, dt = \int_0^3 3t^2 + 6\, dt$$

$$L = t^3 + 6t \Big|_0^3 = 3^3 + 6(3) = 27 + 18 = 45$$

Ej Calcular la longitud

$r(t) = \sqrt{2}\, t\, i + e^t j + e^{-t} k \qquad 0 \le t \le 1$

$f(t) = \sqrt{2}\, t \qquad g(t) = e^t \qquad h(t) = e^{-t}$

$f'(t) = \sqrt{2} \qquad g'(t) = e^t \qquad h'(t) = -e^{-t}$

$e^{-2t} = \frac{1}{e^{2t}}$

$$L = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\, dt$$

$$L = \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\, dt = \int_0^1 \sqrt{2 + e^{2t} + \frac{1}{e^{2t}}}\, dt$$

$$L = \int_0^1 \sqrt{\frac{2e^{2t} + e^{4t} + 1}{e^{2t}}}\, dt = \int_0^1 \frac{\sqrt{e^{4t} + 2e^{2t} + 1}}{\sqrt{e^{2t}}}\, dt$$

$$e^{4t} + 2e^{2t} + 1$$

$$\left.\begin{matrix} e^{2t} & 1 \\ e^{2t} & 1 \end{matrix}\right\} \quad \begin{matrix} e^{2t} \\ e^{2t} \\ \hline 2e^{2} \end{matrix} \quad \to (e^{2t}+1)(e^{2t}+1) = (e^{2t}+1)^2$$

$$e^{2t} \cdot e^{-t}$$

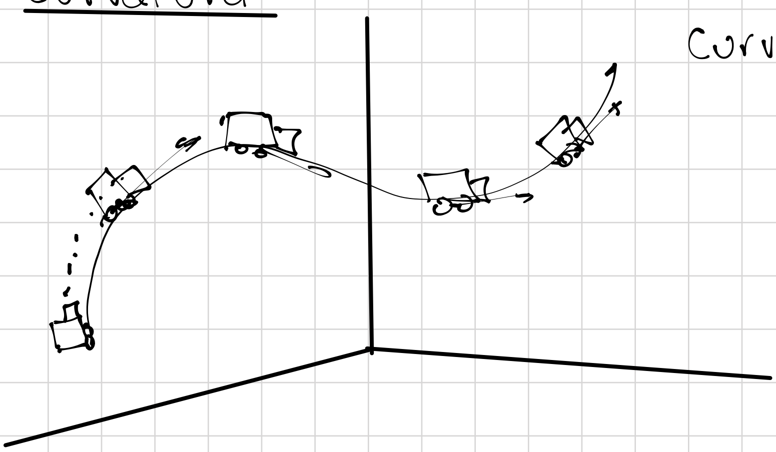
$$\frac{1}{e^t} = e^{-t}$$

$$L = \int_0^1 \frac{\sqrt{(e^{2t}+1)^2}}{\sqrt{e^{2t}}}\, dt = \int_0^1 \frac{e^{2t}+1}{e^t}\, dt$$

$$L = \int_0^1 e^t + e^{-t}\, dt = e^t - e^{-t}\Big|_0^1$$

$$L = (e^1 - e^{-1}) - (\cancel{e^0} - \cancel{e^{-0}}) = e^1 - e^{-1}$$

Curvatura



curva $C \to r(t)$ $\quad a \le t \le b$

La curvatura en un punto dado es una medida de lo rápido que la curva cambia de dirección en ese punto.

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Teorema 1

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

Teorema 2

$$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Ej. 1 Determine la curvatura de la curva

$$r(t) = t\,i + \frac{1}{2}t^2\,j + t^2\,k$$

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$$K(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

$$\frac{1}{\sqrt{1+5t^2}} = (1+5t^2)^{-1/2}$$

$$r'(t) = i + t\,j + 2t\,k$$

$$\|r'(t)\| = \sqrt{(1)^2+(t)^2+(2t)^2} = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i+t\,j+2t\,k}{\sqrt{1+5t^2}} = \frac{i}{\sqrt{1+5t^2}} + \frac{t}{\sqrt{1+5t^2}}j + \frac{2t}{\sqrt{1+5t^2}}k$$

$$i:\ (1+5t^2)^{-1/2} = -\frac{1}{2}(1+5t^2)^{-\frac{3}{2}}(10t) = \frac{-5t}{(1+5t^2)^{3/2}}$$

$$j:\ \frac{t}{\sqrt{1+5t^2}} = \frac{(1+5t^2)^{1/2}(1) - t\frac{1}{2}(1+5t^2)^{-1/2}(10t)}{\left((1+5t^2)^{1/2}\right)^2} =$$

$$\frac{(1+5t^2)^{1/2} - \frac{5t^2}{(1+5t^2)^{1/2}}}{(1+5t^2)^1} = \frac{1+\cancel{5t^2}-\cancel{5t^2}}{(1+5t^2)^{3/2}} = \frac{1}{(1+5t^2)^{3/2}}$$

$$k:\ \frac{2t}{\sqrt{1+5t^2}} = \frac{(1+5t^2)^{1/2}(2) - 2t(\frac{1}{2})(1+5t^2)^{-1/2}(10t)}{\left((1+5t^2)^{1/2}\right)^{2}}$$

$$= \frac{2(1+5t^2)^{1/2} - \frac{10t^2}{(1+5t^2)^{1/2}}}{(1+5t^2)^1} = \frac{2(1+5t^2)-10t^2}{(1+5t^2)^{3/2}}$$

$$= \frac{2+\cancel{10t^2}-\cancel{10t^2}}{(1+5t^2)^{3/2}} = \frac{2}{(1+5t^2)^{3/2}}$$

$$T'(t) = \frac{-5t}{(1+5t^2)^{3/2}}i + \frac{1}{(1+5t^2)^{3/2}}j + \frac{2}{(1+5t^2)^{3/2}}k$$

$$\|T'(t)\| = \sqrt{\left(\frac{-5t}{(1+5t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+5t^2)^{3/2}}\right)^2 + \left(\frac{2}{(1+5t^2)^{3/2}}\right)^2}$$

$$\|T'(t)\| = \sqrt{\frac{25t^2}{(1+5t^2)^3} + \left(\frac{1}{(1+5t^2)^3}\right) + \left(\frac{4}{(1+5t^2)^3}\right)}$$

$$\|T'(t)\| = \sqrt{\frac{25t^2+5}{(1+5t^2)^3}} = \sqrt{\frac{5(1+5t^2)}{(1+5t^2)^3}} = \sqrt{\frac{5}{(1+5t^2)^2}}$$

$$\|T'(t)\| = \frac{\sqrt{5}}{(1+5t^2)}$$

$$K(t) = \frac{\frac{\sqrt{5}}{(1+5t^2)^1}}{\frac{(1+5t^2)^{1/2}}{1}} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

Teorema-2

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$r'(t) = i + tj + 2tK$$

$$r''(t) = 0i + j + 2K$$

$$\|r'(t)\| = (1+5t^2)^{1/2} \rightarrow \|r'(t)\|^3 = \left[(1+5t^2)^{1/2}\right]^3 = (1+5t^2)^{3/2}$$

$$r'(t) \times r''(t) = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} i & j & K \\ 1 & t & 2t \\ 0 & 1 & 2 \end{vmatrix}$$

$$r'(t) \times r''(t) = (2t - 2t)i - (2-0)j + (1-0)K$$

$$r'(t) \times r''(t) = -2j + K$$

$$\|r'(t) \times r''(t)\| = \sqrt{(-2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$K(t) = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

Ej. Halle la curvatura de $r(t) = e^t\cos t\, i + e^t \operatorname{sen} t\, j + t k$ en el punto $\boxed{P(1,0,0)}$

$$r'(t) = (-e^t\operatorname{sen} t + e^t\cos t)\, i + (e^t\cos t + e^t\operatorname{sen} t)\, j + k$$

$$r''(t) = (\cancel{-e^t\cos t} - e^t\operatorname{sen} t - e^t\operatorname{sen} t + \cancel{e^t\cos t})\, i + (\cancel{-e^t\operatorname{sen} t} + e^t\cos t + e^t\cos t + \cancel{e^t\operatorname{sen} t})\, j + 0k$$

$x = e^t\cos t$ $\quad$ $y = e^t\operatorname{sen} t$ $\quad$ $\boxed{z = t}$

$P(1,0,0)$ $\quad$ $0 = e^t\operatorname{sen} t$ $\quad$ $z = t \rightarrow \boxed{t=0}$

$x = 1$ $\quad$ $t = 0$

$y = 0$ $\quad$ $0 = e^0\operatorname{sen} 0$ $\quad$ $0 = 0$

$z = 0$

$$r'(0) = (-e^0\cancel{\operatorname{sen} 0} + e^0\cancel{\cos 0})\, i + (e^0\cancel{\cos 0} + e^0\cancel{\operatorname{sen} 0})\, j + k$$

$$r'(0) = i + j + k$$

$$r''(t) = (-2e^t\operatorname{sen} t)\, i + (2e^t\cos t)\, j + 0k$$

$$r''(0) = (-2e^0\cancel{\operatorname{sen} 0})\, i + (2e^0\cancel{\cos 0})\, j + 0k$$

$$r''(0) = 2j$$

$$r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = (0-2)i - (0-0)j + (2-0)k$$

$$= -2i + 2k$$

$$\|r'(0) \times r''(0)\| = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$[\|r'(0)\|] = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$[\|r'(0)\|]^3 = (\sqrt{3})^3 = 3^{3/2}$$

$$K(0) = \frac{\sqrt{8}}{3^{3/2}}$$