Project 10 - Javier Salazar 1001144647.

1. Let $X_1(t) = 1 + X(t)$ be the Brownian motion starting at 1 and $X_2(t) = 2 + Y(t)$ be the Brownian motion starting at 2, where $X(t)$ and $Y(t)$ are independent standard Brownian motions started at $x = 0$ at time $t = 0$.

(a) Simulate n = 10,000 trials of the process $X_1(t)$ and $X_2(t)$ for $0 \leq t \leq 1$

(b) Find the frequency of $|X_2(1) - X_1(1)| \leq 1$ and compare with the exact P $(|X_2(1) - X_1(1)| \leq 1)$
Hint. $X_2(1) - X_1(1) \sim N(\mu, \sigma^2) = N(1, 2)$

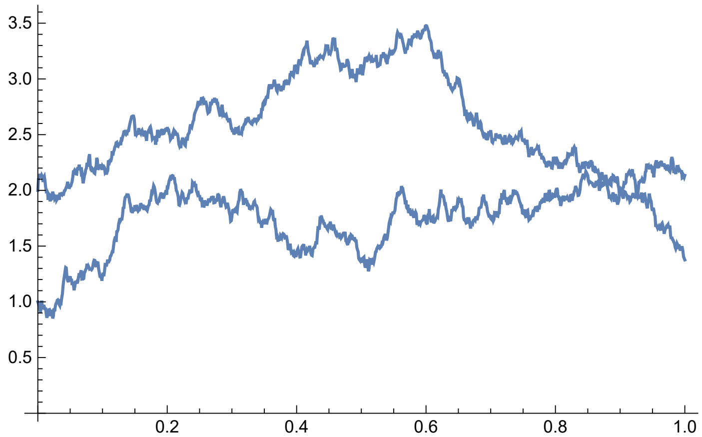
(c) Plot a sample of $X_1(t)$, $X_2(t)$, together on a single graph for $0 \leq t \leq 1$, for which $|X_2(1) - X_1(1)| \leq 1$

```
In[]:= n = 10000; x = 0; a = -1; b = 1; t = 1; h = .005; d = Sqrt[h] ; counts = 0; m = t/h;
Do[ s = 1;
   Do[s = s + Random[NormalDistribution[0, d]] - Random[NormalDistribution[0, d]], {m}];
   If [ a ≤ s ≤ b, counts = counts + 1],
  {n}];
Print["frequency(X2(t)- X1(t) belongs to [", a, "," , b, "]) = ", counts/n // N]
Print["exact probability = ", NIntegrate[1/(2 Sqrt[π]) E^(-(1-x)^2/4), {x, a, b}]];

frequency(X2(t)- X1(t) belongs to [-1,1]) = 0.419
exact probability = 0.42135
```

In[ ]:=
```
BrownianMotion[x_, t_, h_] := Module[{d = Sqrt[h], m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
   sums = FoldList[Plus, 0, g];
Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
brown = Table[{i * h, x + X[i * h]}, {i, 0, m}];
ListLinePlot[brown, Joined → True]];
g1 = BrownianMotion[1, 1, .001];
g2 = BrownianMotion[2, 1, .001];
Show[g1, g2, PlotRange → All]
```

Out[ ]=



2. Let $Y_x^y(s)$ be the Brownian Bridge for $0 \le s \le t = 2$, and $x = 0$, $y = 1$.

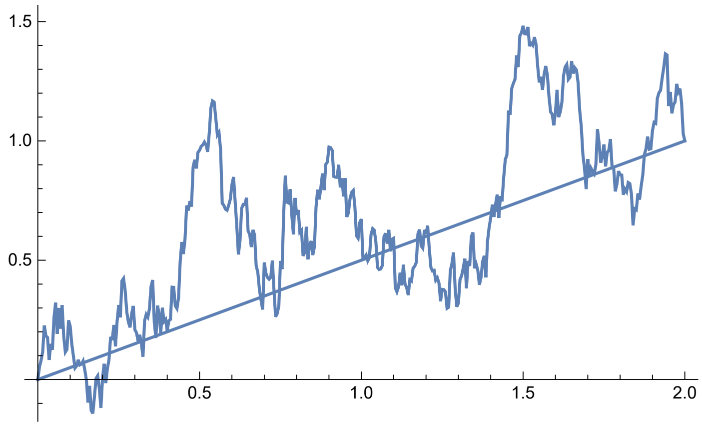
Simulate n = 10,000 trials of the process $Y_0^1(1)$ while counting the frequency of $Y_0^1(1)$ landing in [0,2], for (a) h = .005, and (b) h =.001 and Plot a single realization in (a) and (b) that lands in [0,2]

(c) compare the frequencies from (a) and (b) with the exact probability $P(0 \le Y_0^1(1) \le 2)$

(d) repeat (a)-(c) for n = 40,000

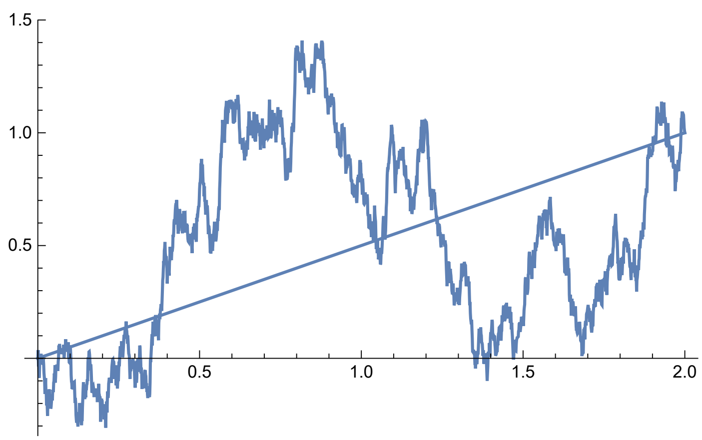
In[ ]:=
```
BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridgeMean = Table[{i * h, x - (i / m) * (x - y)}, {i, 0, m}];
bridge = Table[{i * h, x + X[i * h] - (i / m) * (X[m * h] + x - y)}, {i, 0, m}];
g1 = ListPlot[bridgeMean, Joined → True, DisplayFunction → Identity];
g2 = ListPlot[bridge, Joined → True, DisplayFunction → Identity];
Show[g1, g2, PlotRange → All, DisplayFunction → $DisplayFunction]];
BrownianBridge[0, 1, 2, .005]
```

Out[ ]=



In[ ]:=
```
BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridgeMean = Table[{i * h, x - (i / m) * (x - y)}, {i, 0, m}];
bridge = Table[{i * h, x + X[i * h] - (i / m) * (X[m * h] + x - y)}, {i, 0, m}];
g1 = ListPlot[bridgeMean, Joined → True, DisplayFunction → Identity];
g2 = ListPlot[bridge, Joined → True, DisplayFunction → Identity];
Show[g1, g2, PlotRange → All, DisplayFunction → $DisplayFunction]];
BrownianBridge[0, 1, 2, .001]
```

Out[ ]=



In[ ]:=
```
BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridge = Table[x + X[i * h] - (i / m) * (X[m * h] + x - y), {i, 0, m}];];
n = 10 000; x = 0; y = 1; a = 0; b = 2; t = 2; h = .005; d = √h ; counts = 0; m = t/h;
For[i = 1, i < n + 1, i++, BrownianBridge[x, y, t, h];
 If[0 ≤ bridge[[m / 2 + 1]] ≤ 2, counts = counts + 1, counts = counts]]
Print["frequency(Xx(t) belongs to [", a, "," , b, "]) = ", counts / n // N]
Print["exact probability = ", NIntegrate[1/√π e^-(u-1/2)^2, {u, a, b}]];
```

frequency(X_x(t) belongs to [0,2]) = 0.7429

exact probability = 0.743303

In[ ]:=
```
BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridge = Table[x + X[i * h] - (i / m) * (X[m * h] + x - y), {i, 0, m}];];
n = 10 000; x = 0; y = 1; a = 0; b = 2; t = 2; h = .001; d = √h ; counts = 0; m = t/h;
For[i = 1, i < n + 1, i++, BrownianBridge[x, y, t, h];
 If[0 ≤ bridge[[m / 2 + 1]] ≤ 2, counts = counts + 1, counts = counts]]
Print["frequency(Xx(t) belongs to [", a, "," , b, "]) = ", counts / n // N]
Print["exact probability = ", NIntegrate[1/√π e^-(u-1/2)^2, {u, a, b}]];
```

frequency(X_x(t) belongs to [0,2]) = 0.7437

exact probability = 0.743303

In[ ]:=
```
BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridge = Table[x + X[i * h] - (i / m) * (X[m * h] + x - y), {i, 0, m}];];
n = 40 000; x = 0; y = 1; a = 0; b = 2; t = 2; h = .005; d = √h ; counts = 0; m = t/h;
For[i = 1, i < n + 1, i++, BrownianBridge[x, y, t, h];
 If[0 ≤ bridge[[m / 2 + 1]] ≤ 2, counts = counts + 1, counts = counts]]
Print["frequency(Xx(t) belongs to [", a, "," , b, "]) = ", counts / n // N]
Print["exact probability = ", NIntegrate[1/√π e^-(u-1/2)^2, {u, a, b}]];
```

frequency(X_x(t) belongs to [0,2]) = 0.7402

exact probability = 0.743303

```
In[]:= BrownianBridge[x_, y_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
bridge = Table[x + X[i * h] - (i / m) * (X[m * h] + x - y), {i, 0, m}];];
n = 40000; x = 0; y = 1; a = 0; b = 2; t = 2; h = .001; d = √h; counts = 0; m = t/h;
For[i = 1, i < n + 1, i++, BrownianBridge[x, y, t, h];
 If[0 ≤ bridge[[m / 2 + 1]] ≤ 2, counts = counts + 1, counts = counts]]
Print["frequency(Xx(t) belongs to [", a, "," , b, "]) = ", counts / n // N]
Print["exact probability = ", NIntegrate[1/√π e^-(u-1/2)^2, {u, a, b}]];

frequency(Xx(t) belongs to [0,2]) = 0.7437

exact probability = 0.743303
```

3. Let $S(t) = e^{x+\mu t+\sigma X(t)} = s_0\, e^{\mu t+\sigma X(t)}$ with $s_0 = S(0) = e^x$, be the Geometric Brownian Motion with $x = 0$, $\mu = .10$, $\sigma = .3$, $0 \leq t$ (in years). This means the 1 share of Stock has price \$1 at time 0, with average annual return 10% and 30% volatility.

(a) Find the exact probability that the stock price after 1 year is at least \$1.15.

(b) Simulate n = 10,000 trials of the process S(1) while counting the frequency of $S(1) \geq 1.15$, for h =.001 and plot a single realization that is at least 1.15 after one year.

(c) compare the frequency (b) with the exact probability $P(1.15 \leq S(1))$ found in (a)

```
In[]:= BrownianGeometric[x0_, μ_, σ_, t_, h_] := Module[{d = √h, m = t/h},
g = Table[Random[NormalDistribution[0, d]], {m}];
sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
geometric = Table[x0 * e^(μ *i/m+σ*X[i*h]), {i, 0, m}];];
n = 10000; x = 1; μ = .1; σ = .3; t = 1; h = .001; d = √h; counts = 0; m = t/h;
For[i = 1, i < n + 1, i++, BrownianGeometric[x, μ, σ, t, h];
 If[ geometric[[m + 1]] ≥ 1.15, counts = counts + 1, counts = counts]]
Print["frequency S(1) >= 1.15 = ", counts / n // N]
(Log[1.15] - .1) / .3 // N;
Print["exact probability = ", NIntegrate[1/(.3 * √(2 π)) e^-((x-.1)^2/(2*.3^2)), {x, 0.13254, Infinity}]];

frequency S(1) >= 1.15 = 0.4412

exact probability = 0.456813
```

In[∘]:=
```
BrownianGeometric[x0_, μ_, σ_, t_, h_] := Module[{d = √h, m = t/h},
  g = Table[Random[NormalDistribution[0, d]], {m}];
  sums = FoldList[Plus, 0, g]; Table[X[i * h] = sums[[i + 1]], {i, 0, m}];
  geometric = Table[{i * h, x0 * ⅇ^(μ *i/m+σ*X[i*h])}, {i, 0, m}];
  drift = Table[{i * h, x0 * ⅇ^((i/m) * (μ + σ^2/2))}, {i, 0, m}];
  g1 = ListPlot[geometric, Joined → True, DisplayFunction → Identity];
  g2 = ListPlot[drift, Joined → True, DisplayFunction → Identity];
  Show[g1, g2, PlotRange → All, DisplayFunction → $DisplayFunction]];
BrownianGeometric[x, μ, σ, t, h]
```

Out[∘]=