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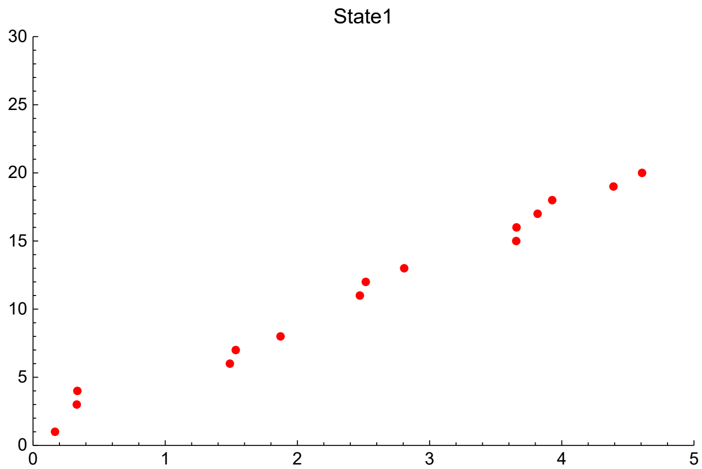
Project 3.

1. Assume that a department store at the border between State 1 and State 2 receives 4 calls per hour. Given that fact that
State 1 has twice as many people as State 2 analyze the marked Poisson Process N(t) during the hours of 4 PM - 9 PM.
(a) Simulate a sample of N(t) and mark each call as either type 1 or type 2 for the appropriate probability p = P(1)
and 1-p = P(2). Use ListPlot[.] to display the sample in three separate graphs as follows:
g1 ~ N1(t) = type 1 process marked with point size .04, g2 ~ N2 (t) = type 2 process marked with point size .02,
g3 ~ N(t) = N1(t) + N2(t) = combined process.
(b) Simulate N(t) = N1(t) + N2(t) n = 100, 1000, 10000 times, find the corresponding sample average and variance for
N(t), N1(t), N2(t) (use Mean[.] and Variance[.]) and compare with the corresponding exact values.

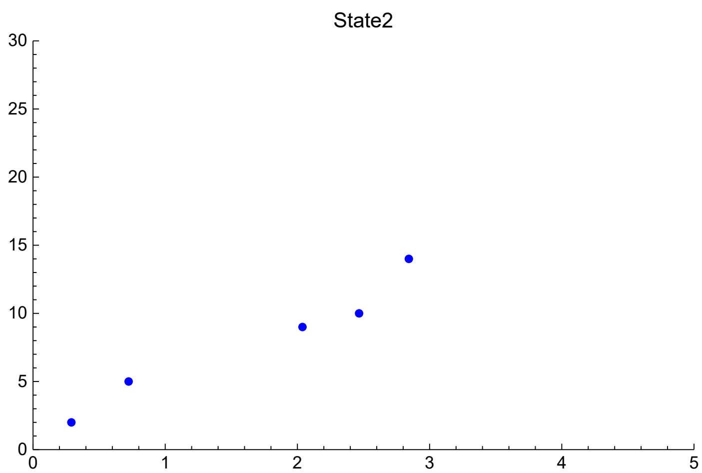
```
t = 5; λ = 4; s = 0; A1 = {}; A2 = {}; n = 0;
r = Random[ExponentialDistribution[λ]];
type = RandomInteger[{1, 3}]; s = s + r;
  While[s ≤ t, n = n + 1;
 If[type == 1 || type == 3, AppendTo[A1, {s, n}], AppendTo[A2, {s, n}]];
        r = Random[ExponentialDistribution[λ]];
 type = RandomInteger[{1, 3}];
        s = s + r]
```

In[∘]:=
```
g1 = ListPlot[A1, PlotStyle → Red, PlotStyle -> PointSize[.04],
   PlotRange → {{0, 5}, {0, 30}}, PlotLabel → State1]
g2 = ListPlot[A2, PlotStyle → Blue, PlotStyle -> PointSize[.02],
   PlotRange → {{0, 5}, {0, 30}}, PlotLabel → State2]
Show[{g1, g2}, PlotLabel → "Marked Poisson Process"]
```

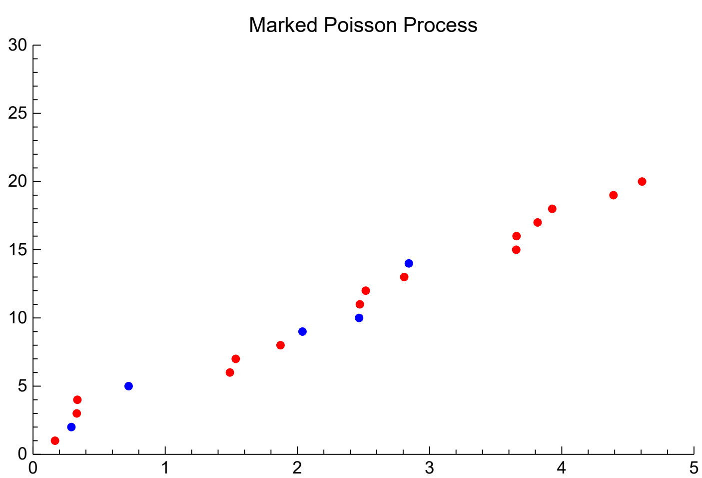
Out[∘]=



Out[∘]=



Out[∘]=



In[ ]:=
```
Clear[a, s, t, r, λ, m1, n];  n = 100; λ = 4 * 2/3; t = 5; m1 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m1, a], {n}]

Clear[a, s, t, r, λ, m2, n];  n = 100; λ = 4 * 1/3; t = 5; m2 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m2, a], {n}]
Mean[m1] // N
Variance[m1] // N
Mean[m2] // N
Variance[m2] // N
Mean[m1] + Mean[m2] // N
Variance[m1] + Variance[m2] // N
```

Out[ ]= 12.92

Out[ ]= 14.3572

Out[ ]= 6.3

Out[ ]= 5.48485

Out[ ]= 19.22

Out[ ]= 19.842

In[ ]:=

```
Clear[a, s, t, r, λ, m1, n];  n = 1000; λ = 4 * 2 / 3; t = 5; m1 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m1, a], {n}]

Clear[a, s, t, r, λ, m2, n];  n = 1000; λ = 4 * 1 / 3; t = 5; m2 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m2, a], {n}]
Mean[m1] // N
Variance[m1] // N
Mean[m2] // N
Variance[m2] // N
Mean[m1] + Mean[m2] // N
Variance[m1] + Variance[m2] // N
```

Out[ ]= 13.554

Out[ ]= 14.4415

Out[ ]= 6.771

Out[ ]= 6.65521

Out[ ]= 20.325

Out[ ]= 21.0967

In[ ]:=
```
Clear[a, s, t, r, λ, m1, n];  n = 10000; λ = 4 * 2/3; t = 5; m1 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m1, a], {n}]

Clear[a, s, t, r, λ, m2, n];  n = 10000; λ = 4 * 1/3; t = 5; m2 = {};
Do[a = 0;
 s = 0;
 Label[start];
 r = Random[ExponentialDistribution[λ]]; s = s + r;  If[s ≤ t, a = a + 1;
  Goto[start]];
 AppendTo[m2, a], {n}]
Mean[m1] // N
Variance[m1] // N
Mean[m2] // N
Variance[m2] // N
Mean[m1] + Mean[m2] // N
Variance[m1] + Variance[m2] // N
```

Out[ ]= 13.3325

Out[ ]= 13.5367

Out[ ]= 6.6435

Out[ ]= 6.61207

Out[ ]= 19.976

Out[ ]= 20.1488

```
(*The true m1 is 13.33, true m2 is 6.67,
 and the mean of m1 + m2 is 20. Increasing sample size results in a convergent mean)
```

2. Each cereal box contains a mascot from the set of 5 different kinds. Suppose a buyer comes to the store according
to Poisson process with the rate of one purchase per week. Let N = the minimum number the required trips to the
store in order to complete the five different mascots collection (Coupon Collecting Problem).
(a) Simulate 3 samples of $\{x_1, , \ldots, x_N\}$ where $x_i \in \{1, 2, 3, 4, 5\}$, x_N is the mascot completing the collection
and N = last purchase.
(b) Simulate n =1000, 10000, 100000 samples of N, find the sample average and compare with the exact value EN.

In[ ]:=
```
Clear[Num, Sample, outcome]
(Num = 0;
 Sample = {}; l = {};
 Label[start];
 m = l;
 outcome = RandomInteger[{1, 5}];
 AppendTo[Sample, outcome];
 Num = Num + 1; l = m ⋃ {outcome};
 If[l ≠ {1, 2, 3, 4, 5}, Goto[start]])
Sample
```

Out[ ]= {3, 2, 4, 5, 1}

In[ ]:=
```
Clear[Num, Sample, outcome]
(Num = 0;
 Sample = {}; l = {};
 Label[start];
 m = l;
 outcome = RandomInteger[{1, 5}];
 AppendTo[Sample, outcome];
 Num = Num + 1; l = m ⋃ {outcome};
 If[l ≠ {1, 2, 3, 4, 5}, Goto[start]])
Sample
```

Out[ ]= {5, 3, 1, 3, 4, 3, 1, 4, 5, 3, 2}

In[ ]:=
```
Clear[Num, Sample, outcome]
(Num = 0;
 Sample = {}; l = {};
 Label[start];
 m = l;
 outcome = RandomInteger[{1, 5}];
 AppendTo[Sample, outcome];
 Num = Num + 1; l = m ⋃ {outcome};
 If[l ≠ {1, 2, 3, 4, 5}, Goto[start]])
Sample
```

Out[ ]= {3, 3, 5, 3, 2, 1, 5, 3, 4}

In[ ]:=
```
average[n_] := (NumTotal = 0;
  Do[Num = 0;
   l = {};
   Label[start];
   m = l;
   outcome = RandomInteger[{1, 5}];
   Num = Num + 1;
   l = m ⋃ {outcome};
   If[l ≠ {1, 2, 3, 4, 5}, Goto[start]];
   NumTotal = NumTotal + Num, {n}];
  N[NumTotal/n])
```

In[ ]:=
```
Clear[t]
avg = 0;
exact := ∫_0^∞ (1 - (1 - e^(-t/5))^5) dt (*Alternatively, exact = 1 + 5/4 + 5/3 + 5/2 + 5*)
Print["The exact value is ", exact // N]
n = {1000, 10000, 100000};
For[i = 1, i < 4, i++, avg = average[n[[i]]]; Print[
  "For a sample size of n = ", n[[i]], ", the sample average is ", average[n[[i]]],
  " with an absolute relative error of ", Abs[(exact - avg) / exact] * 100, "%" ]]
```

```
The exact value is 11.4167

For a sample size of n = 1000, the sample average is
 11.405 with an absolute relative error of 1.69635%

For a sample size of n = 10000, the sample average is
 11.5131 with an absolute relative error of 0.253723%

For a sample size of n = 100000, the sample average is
 11.4329 with an absolute relative error of 0.127153%
```

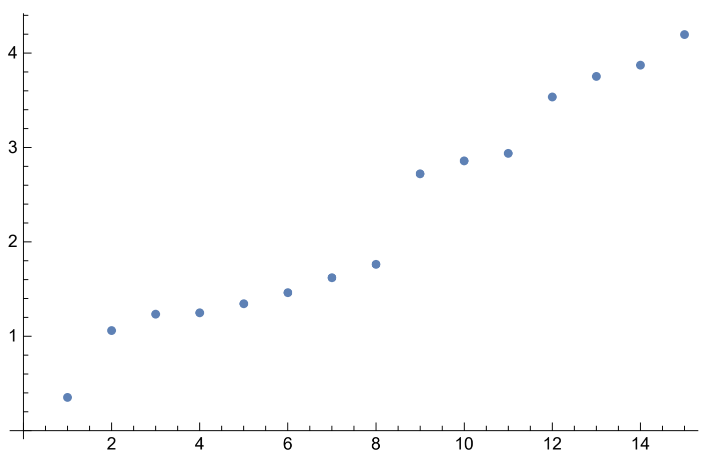
3. Let $\lambda = 2, t = 5$. First simulate N according to Poisson random variable with parameter λt. For this N simulate uniform random variables $U_1, U_2, \ldots, U_N$ from $[0, t]$ and re-order them from smallest to largest $\hat{U}_1 \leq \ldots \leq \hat{U}_N$.

The ordered sample $(\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_N) = (S_1, S_2, \ldots, S_N)$ are the arrival times of the Poisson Process N(t). This is an

alternative way of simulating Poisson Process without using exponentially distributed interarrival times.

Graph a sample of $N(u), 0 \leq u \leq t$.

In[ ]:=
```
n = RandomVariate[PoissonDistribution[2 * 5]];
A = Sort[RandomVariate[UniformDistribution[{0, 5}], {n}]];
ListPlot[A]
```

Out[ ]=


4. Consider the Shock Model $A(t) = \sum_{i=1}^{N(t)} A_i \, e^{-\alpha (t - S_i)}$ for $\alpha = \frac{1}{2}, \lambda = 3, t = 10$, where N(t) is Poisson $\sim \lambda$,

the amplitude of i.i.d. $A_i \sim A$ has a uniform distribution U[1, 5], and S_i is the arrival time of the i-th shock for N(t).

Simulate A(t) n = 1000 times. Find the sample average and compare with exact value EA(t) = $\frac{\lambda E[A]}{\alpha}(1 - e^{-\alpha t})$.

Graph a sample of A(u), $0 \le u \le t$.

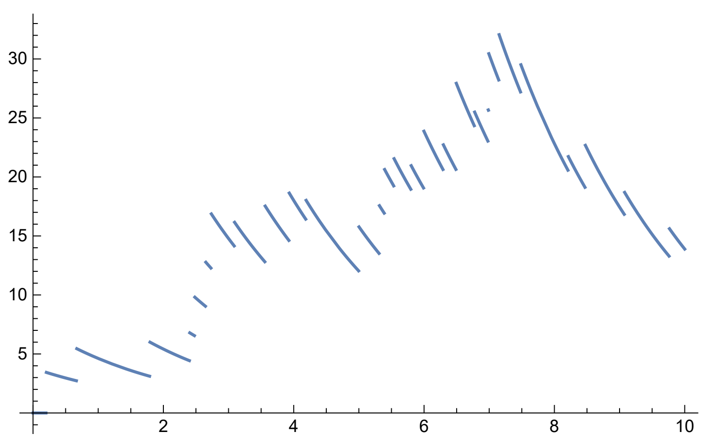
```
Clear[n, A, s, shockTotal, list];
exactvalue := 3 * 3 * 2 * (1 - Exp[-.5 * 10])
list = {};
For[j = 1, j ≤ 1000, j++,
 n = RandomVariate[PoissonDistribution[3 * 10]];
 A = RandomVariate[UniformDistribution[{1, 5}], {n}];
 s = Sort[Table[RandomReal[{0, 10}], {n}]];
 shockTotal := Sum[A[[i]] * Exp[-0.5 * (10 - s[[i]])], {i, 1, n}];
 AppendTo[list, shockTotal];]
Print["The exact value is ", exactvalue]
Print["For a sample size of n = ", 1000, ", the sample mean is ", Mean[list],
 " with an absolute relative error of ", Abs[(exact - Mean[list]) / exactvalue] * 100, "%"]
```

```
The exact value is 17.8787

For a sample size of n = 1000, the sample mean is
 17.8886 with an absolute relative error of 36.1993%
```

```
In[∘]:= Clear[n, A, s, shockTotal, list];
n = RandomVariate[PoissonDistribution[3 * 10]];
A = RandomVariate[UniformDistribution[{1, 5}], {n}];
s = Sort[Table[RandomReal[{0, 10}], {n}]];
shockTotal[u_] := Sum[A[[i]] * Exp[-0.5 * (u - s[[i]])] * UnitStep[u - s[[i]]], {i, 1, n}];
Plot[shockTotal[u], {u, 0, 10}]
```

Out[∘]=



5. Consider a compound Poisson Process $X(t) = \sum_{i=1}^{N(t)} Y_i$, where N(t) ~ Poisson with λ = 10/hour, t = 12 and

$Y_i \sim Y$ = Binomial B(m,p) with $p = \frac{1}{2}$, m = 100. By interpreting N(t) as the number of customers visiting a store between 9AM - 9PM and each customer spending \$Y, X(t) is the daily revenue for the store. Simulate X(t)

n = 1000 times, find the sample average and variance and compare with exact values EX(t) and VarX(t).

Graph a sample of X(u) , 0 ≤ u ≤ t.

In[ ]:=
```
Clear[n, A, s, compound, list];
list = {};
exactmean := 6000 (*Mean of binomial is mp = 100*.5 = 50. And 50*120 = 6000*)
exactvariance := 303 000
(*Mean = 100*.5*.5+50^2 = 25+2500 = 2525. And 2525*120 = 303000*)
For[j = 1, j ≤ 1000, j++,
 n = RandomVariate[PoissonDistribution[10 * 12]];
 Y = RandomVariate[BinomialDistribution[100, 0.5], {n}];
 s = Sort[Table[RandomReal[{0, 12}], {n}]];
 compound := Sum[Y[[i]], {i, 1, n}];
 AppendTo[list, compound];]
Print["The exact mean is ", exactmean, " and the exact variance is ", exactvariance]
Print["For a sample size of n = ", 1000, ", the sample mean is ",
 Mean[list] // N, " with an absolute relative error of ",
 Abs[(exactmean - Mean[list]) / exactmean] * 100 // N, "%" , " and sample variance is ",
 Variance[list] // N, " with an absolute relative error of ",
 Abs[(exactvariance - Variance[list]) / exactvariance] * 100 // N, "%"]
```

```
The exact mean is 6000 and the exact variance is 303 000

For a sample size of n = 1000, the sample mean is 5995.75 with an absolute relative error of
 0.0708667% and sample variance is 309 346. with an absolute relative error of 2.09426%
```

In[ ]:=
```
Clear[n, A, s, compound, list];
n = RandomVariate[PoissonDistribution[10 * 12]];
Y = RandomVariate[BinomialDistribution[100, 0.5], {n}];
s = Sort[Table[RandomReal[{0, 12}], {n}]];
compound[u_] := Sum[Y[[i]] * UnitStep[u - s[[i]]], {i, 1, n}];
Plot[compound[u], {u, 0, 12}]
```

Out[ ]=