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Project 5.

1. **Ehrenfest Model.** Consider the 2 urn model for N = 10.

(a) Find the transition matrix P

(b) Determine whether P is regular

(c) Find the period for each state

(d) Find the stationary distribution π

(e) Simulate n = 10000 samples starting at time 0 from the uniform distribution,
compute the frequencies of visiting each state and compare to π (code in notebook)

(f) Find the probabilities of the first return f_{ii}^{n} for states i = 0, 1, ... , 10 at time n = 5, 10, 20 and determine
the smallest N such that P(first return to i by time N) = $\sum_{n=1}^{N} f_{ii}^{n} \geq .99$ for i ∈ {3, 4, 5, 6, 7}.

(g) Animate a sample of the first 100 steps when starting from an initial state of choice

In[ ]:=

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/10 & 0 & 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/10 & 0 & 8/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/10 & 0 & 7/10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/10 & 0 & 6/10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/10 & 0 & 5/10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6/10 & 0 & 4/10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7/10 & 0 & 3/10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8/10 & 0 & 2/10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9/10 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

```
Print["The transition matrix P is ", P // MatrixForm]
Print["P is not regular because any
   power of P produces a matrix with at least one zero entry."]
Print["P has period 2."]
Solve[{π0, π1, π2, π3, π4, π5, π6, π7, π8, π9, π10} ==
    {π0, π1, π2, π3, π4, π5, π6, π7, π8, π9, π10}.P,
   {π0, π1, π2, π3, π4, π5, π6, π7, π8, π9, π10}];
Q[n_] := MatrixPower[P, n] // MatrixForm;
Q[6] // N;
```

The transition matrix P is $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & \frac{9}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & \frac{7}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10} & 0 & \frac{3}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{10} & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

P is not regular because any power of P produces a matrix with at least one zero entry.

P has period 2.

Solve: Equations may not give solutions for all "solve" variables.

```
In[ ]:= 1 = π0 + π1 + π2 + π3 + π4 + π5 + π6 + π7 + π8 + π9 + π10 /. {π1 → 10 π0, π2 → 45 π0, π3 → 120 π0,
      π4 → 210 π0, π5 → 252 π0, π6 → 210 π0, π7 → 120 π0, π8 → 45 π0, π9 → 10 π0, π10 → π0};
Print["The stationary distribution is ", {π1 → 10 π0, π2 → 45 π0, π3 → 120 π0, π4 → 210 π0,
    π5 → 252 π0, π6 → 210 π0, π7 → 120 π0, π8 → 45 π0, π9 → 10 π0, π10 → π0} /. π0 → 1/1024]

r := RandomInteger[{1, 10}]
n = 10000;
c0 = 0;
c1 = 0;
c2 = 0;
c3 = 0;
c4 = 0;
c5 = 0;
c6 = 0;
c7 = 0;
c8 = 0;
c9 = 0;
c10 = 0; x = 5;
Do[
  Which[x == 0, x = 1; c1 = c1 + 1,
   x == 1, If[r == 1, x = 0; c0 = c0 + 1, x = 2; c2 = c2 + 1],
   x == 2, If[r ≤ 2, x = 1; c1 = c1 + 1, x = 3; c3 = c3 + 1],
   x == 3, If[r ≤ 3, x = 2; c2 = c2 + 1, x = 4; c4 = c4 + 1],
   x == 4, If[r ≤ 4, x = 3; c3 = c3 + 1, x = 5; c5 = c5 + 1],
   x == 5, If[r ≤ 5, x = 4; c4 = c4 + 1, x = 6; c6 = c6 + 1],
   x == 6, If[r ≤ 6, x = 5; c5 = c5 + 1, x = 7; c7 = c7 + 1],
   x == 7, If[r ≤ 7, x = 6; c6 = c6 + 1, x = 8; c8 = c8 + 1],
   x == 8, If[r ≤ 8, x = 7; c7 = c7 + 1, x = 9; c9 = c9 + 1],
   x == 9, If[r ≤ 9, x = 8; c8 = c8 + 1, x = 10; c10 = c10 + 1],
   x == 10, x = 9; c9 = c9 + 1],
  {n}];
Print["frequencies(to visit i) = ", {c0, c1, c2, c3, c4, c5, c6, c7, c8, c9, c10}/n // N]
Print["exact π = ", {5/512, 45/1024, 15/128, 105/512, 63/256, 105/512, 15/128, 45/1024, 5/512, 1/1024} // N]
```

Set: Cannot assign to raw object 1.

The stationary distribution is $\{\pi1 \to \frac{5}{512}, \pi2 \to \frac{45}{1024}, \pi3 \to \frac{15}{128}, \pi4 \to \frac{105}{512}, \pi5 \to \frac{63}{256}, \pi6 \to \frac{105}{512}, \pi7 \to \frac{15}{128}, \pi8 \to \frac{45}{1024}, \pi9 \to \frac{5}{512}, \pi10 \to \frac{1}{1024}\}$

frequencies(to visit i) =
{0.0004, 0.0067, 0.0387, 0.1111, 0.2005, 0.2495, 0.2154, 0.1232, 0.0438, 0.0095, 0.0012}

exact π = {0.00976563, 0.0439453, 0.117188, 0.205078,
0.246094, 0.205078, 0.117188, 0.0439453, 0.00976563, 0.000976563}

In[∘]:=
```
M = P;
a[n_] := Do[M = P.(M * Io), {n - 1}];
a[5]; F[5] = M; F[5] // N // MatrixForm
a[10]; F[10] = M; F[10] // N // MatrixForm
a[20]; F[20] = M; F[20] // N // MatrixForm
Io = Table[1, {11}, {11}] - IdentityMatrix[11];
Do[M = P; Do[M = P.(M * Io), {k - 1}]; b[k] = M;, {k, 2, 200}]
```

$$B = \sum_{k=2}^{62} b[k] \mathbin{//} \text{MatrixForm} \mathbin{//} N$$

```
Print["N=62 is the smallest N where the p>0.99 for state 3,4,5,6"]
```

Out[∘]//MatrixForm=

0.	0.	0.	0.2016	0.	0.3024	0.	0.	0.	0.	0.
0.00756	0.	0.009	0.	0.26208	0.	0.1512	0.	0.	0.	0.
0.	0.02496	0.	0.04032	0.	0.2352	0.	0.0672	0.	0.	0.
0.0042	0.	0.04872	0.	0.07056	0.	0.1722	0.	0.0252	0.	0.
0.	0.01968	0.	0.072	0.	0.08736	0.	0.1056	0.	0.0072	0.
0.0012	0.	0.0528	0.	0.087	0.	0.087	0.	0.0528	0.	0.00
0.	0.0072	0.	0.1056	0.	0.08736	0.	0.072	0.	0.01968	0.
0.	0.	0.0252	0.	0.1722	0.	0.07056	0.	0.04872	0.	0.00
0.	0.	0.	0.0672	0.	0.2352	0.	0.04032	0.	0.02496	0.
0.	0.	0.	0.	0.1512	0.	0.26208	0.	0.009	0.	0.007
0.	0.	0.	0.	0.	0.3024	0.	0.2016	0.	0.	0.

Out[∘]//MatrixForm=

0.00202037	0.	$9. \times 10^{-7}$	0.	0.012812	0.	0.0920382	0.
0.	0.00927469	0.	0.000346961	0.	0.0432827	0.	0.0765427
0.00200657	0.	0.0189034	0.	0.00525016	0.	0.0700013	0.
0.	0.0120205	0.	0.0223983	0.	0.0216484	0.	0.0656795
0.0018196	0.	0.0313623	0.	0.0174976	0.	0.0419506	0.
0.	0.0124057	0.	0.0466316	0.	0.0135336	0.	0.0466316
0.00157026	0.	0.0376042	0.	0.0419506	0.	0.0174976	0.
0.	0.0116645	0.	0.0656795	0.	0.0216484	0.	0.0223983
0.00131901	0.	0.0391614	0.	0.0700013	0.	0.00525016	0.
0.	0.0105996	0.	0.0765427	0.	0.0432827	0.	0.000346961
0.00107601	0.	0.0388717	0.	0.0920382	0.	0.012812	0.

Out[∘]//MatrixForm=

0.	0.	0.	3.67031×10^{-9}	0.	0.000779902	0.	0
0.00145094	0.	$9. \times 10^{-17}$	0.	6.56976×10^{-6}	0.	0.00781877	
0.	0.0078324	0.	7.34062×10^{-10}	0.	0.000467941	0.	0
0.00164641	0.	0.0137823	0.	1.72279×10^{-6}	0.	0.0054396	
0.	0.0104417	0.	0.00929291	0.	0.00015598	0.	0
0.00166485	0.	0.0220753	0.	0.0022622	0.	0.0022622	
0.	0.0110239	0.	0.0181375	0.	0.00015598	0.	0
0.00166892	0.	0.0251533	0.	0.0054396	0.	1.72279×10^{-6}	
0.	0.0112856	0.	0.0229474	0.	0.000467941	0.	7.3
0.00166952	0.	0.026915	0.	0.00781877	0.	6.56976×10^{-6}	
0.	0.0114431	0.	0.0262909	0.	0.000779902	0.	3.6

Out[∘]//MatrixForm=

0.172858	0.	1.	1.	1.	0.999997	0.999126	0.962916	0.724121	0.2
0.0728584	0.558552	0.1	1.	1.	0.999998	0.999126	0.967161	0.724121	0.2
0.0825197	0.309503	0.887025	0.2	1.	0.999998	0.999339	0.967633	0.738598	0.2
0.059935	0.394906	0.558781	0.990825	0.3	0.999999	0.999392	0.972382	0.742217	0.3
0.052544	0.345793	0.807786	0.586893	0.999848	0.399999	0.999605	0.974417	0.761161	0.3
0.0476167	0.327326	0.77379	0.980655	0.499747	0.999999	0.499747	0.980655	0.77379	0.3
0.0459348	0.308859	0.761161	0.974417	0.999605	0.399999	0.999848	0.586893	0.807786	0.3
0.0434119	0.303781	0.742217	0.972382	0.999392	0.999999	0.3	0.990825	0.558781	0.3
0.0429581	0.291932	0.738598	0.967633	0.999339	0.999998	1.	0.2	0.887025	0.3
0.0411428	0.290886	0.724121	0.967161	0.999126	0.999998	1.	1.	0.1	0.5
0.0411428	0.281479	0.724121	0.962916	0.999126	0.999997	1.	1.	1.	

N=62 is the smallest N where the p>0.99 for state 3,4,5,6

In[∘]:=
```
Clear[c0, c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, x, n]
r := RandomInteger[{1, 10}]
n = 100;
c0 = 0;
c1 = 0;
c2 = 0;
c3 = 0;
c4 = 0;
c5 = 0;
c6 = 0;
c7 = 0;
c8 = 0;
c9 = 0;
c10 = 0; x = 5; state = {};
Do[
  Which[x == 0, x = 1; AppendTo[state, x]; c1 = c1 + 1,
   x == 1, If[r == 1, x = 0;
    AppendTo[state, x];
    c0 = c0 + 1, x = 2;
    AppendTo[state, x];
    c2 = c2 + 1],
   x == 2, If[r ≤ 2, x = 1;
    AppendTo[state, x];
    c1 = c1 + 1, x = 3;
    AppendTo[state, x];
    c3 = c3 + 1],
   x == 3, If[r ≤ 3, x = 2;
    AppendTo[state, x];
    c2 = c2 + 1, x = 4;
    AppendTo[state, x];
    c4 = c4 + 1],
   x == 4, If[r ≤ 4, x = 3;
    AppendTo[state, x];
    c3 = c3 + 1, x = 5;
    AppendTo[state, x];
    c5 = c5 + 1],
   x == 5, If[r ≤ 5, x = 4;
    AppendTo[state, x];
    c4 = c4 + 1, x = 6;
```

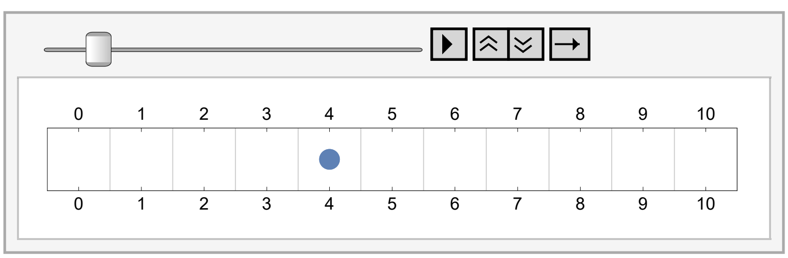
```
    AppendTo[state, x];
    c6 = c6 + 1],
   x == 6, If[r ≤ 6, x = 5;
    AppendTo[state, x];
    c5 = c5 + 1, x = 7;
    AppendTo[state, x];
    c7 = c7 + 1],
   x == 7, If[r ≤ 7, x = 6;
    AppendTo[state, x];
    c6 = c6 + 1, x = 8;
    AppendTo[state, x];
    c8 = c8 + 1],
   x == 8, If[r ≤ 8, x = 7;
    AppendTo[state, x];
    c7 = c7 + 1, x = 9;
    AppendTo[state, x];
    c9 = c9 + 1],
   x == 9, If[r ≤ 9, x = 8;
    AppendTo[state, x];
    c8 = c8 + 1, x = 10;
    AppendTo[state, x];
    c10 = c10 + 1],
   x == 10, x = 9; AppendTo[state, x]; c9 = c9 + 1],
  {n}];

Table[

  B[i] = ListPlot[{{state[[i]] - .5, 1 - .5}}, PlotStyle -> PointSize[.03],
     Frame → Automatic, AspectRatio → Automatic, Axes → None,
     FrameTicks → {{{-.5, 0}, {0.5, 1}, {1.5, 2}, {2.5, 3}, {3.5, 4}, {4.5, 5},
        {5.5, 6}, {6.5, 7}, {7.5, 8}, {8.5, 9}, {9.5, 10}}, None}, GridLines →
      {{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, {0, 1}}, PlotRange → {{-1, 10}, {0, 1}}];,

  {i, 1, 100}];
ListAnimate[Table [B[i], {i, 0, 100}], 2]
```

Out[ ]=



2. **Modulo Division.** Let Y_n be the sum of n independent rolls of a fair die and consider the problem of determining with what probability Y_n is divisible by 7 in a long run. Let X_n be the reminder when Y_n is divided by 7. Then X_n is a Markov chain with states {0, 1, 2, 3, 4, 5, 6}.

(a) Find the transition matrix P

(b) Determine whether P is regular and check it is doubly stochastic (i.e., sum in each row and column is 1)

(c) Find the stationary distribution π (i.e. the limiting probability of being divisible by 7) by taking large powers of P

(d) Simulate n = 100000 samples starting from 6 at time 0, find the frequencies of visiting each state and compare to π

In[ ]:=

```
Print ["Yes, P is regular because there exists an m such that
   P^m has all positive entries (namely, m=2). P is doubly stochastic."]
```

$$P = \begin{pmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \end{pmatrix};$$

```
Print["The transition matrix P is ", P // MatrixForm]

Q[n_] := MatrixPower[P, n] // MatrixForm;
Q[17] // N // MatrixForm // Rationalize;
Print["The stationary distribution for all the states is ", 1/7]
n = 100000;
vis0 = 0;
vis1 = 0;
vis2 = 0;
vis3 = 0;
vis4 = 0;
vis5 = 0;
vis6 = 0;
x = 6;
Do[
 Which[x == 0, (r = RandomReal[];
   Which[r ≤ 1/6, x = 1;
    vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
            x = 2 ;
    vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 3;
    vis3 = vis3 + 1, 3/6 < r ≤ 4/6, x = 4;
    vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 5;
```

```
vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 6;
vis6 = vis6 + 1]),
   x == 1, (r = RandomReal[]; Which[r ≤ 1/6, x = 0; vis0 = vis0 + 1, 1/6 < r ≤ 2/6,
          x = 2 ;
vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 3;
vis3 = vis3 + 1, 3/6 < r ≤ 4/6, x = 4;
vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 5;
vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 6;
vis6 = vis6 + 1]),
   x == 2, (r = RandomReal[]; Which[r ≤ 1/6, x = 1; vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
          x = 3;
vis3 = vis3 + 1, 2/6 < r ≤ 3/6, x = 0;
vis0 = vis0 + 1, 3/6 < r ≤ 4/6, x = 4;
vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 5;
vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 6;
vis6 = vis6 + 1]),
   x ==  3, (r = RandomReal[];
Which[r ≤ 1/6, x = 1;
vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
          x = 2 ;
vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 0;
vis0 = vis0 + 1, 3/6 < r ≤ 4/6, x = 4;
vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 5;
vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 6;
vis6 = vis6 + 1]),
```

```
    x == 4, (r = RandomReal[]; Which[r ≤ 1/6, x = 1; vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
          x = 2 ;
    vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 3;
    vis3 = vis3 + 1, 3/6 < r ≤ 4/6, x = 0;
    vis0 = vis0 + 1, 4/6 < r ≤ 5/6, x = 5;
    vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 6;
    vis6 = vis6 + 1]),
   x == 5, (r = RandomReal[]; Which[r ≤ 1/6, x = 1; vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
          x = 2 ;
    vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 3;
    vis3 = vis3 + 1, 3/6 < r ≤ 4/6, x = 4;
    vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 0;
    vis0 = vis0 + 1, 5/6 < r ≤ 1, x = 6;
    vis6 = vis6 + 1]),
   x == 6, (r = RandomReal[]; Which[r ≤ 1/6, x = 1; vis1 = vis1 + 1, 1/6 < r ≤ 2/6,
          x = 2 ;
    vis2 = vis2 + 1, 2/6 < r ≤ 3/6, x = 3;
    vis3 = vis3 + 1, 3/6 < r ≤ 4/6, x = 4;
    vis4 = vis4 + 1, 4/6 < r ≤ 5/6, x = 5;
    vis5 = vis5 + 1, 5/6 < r ≤ 1, x = 0;
    vis0 = vis0 + 1])
      ] ,
  {n}]
Print["The frequency of visiting states 0, 1, 2, 3, 4, 5, 6 is ", vis0/100000 // N,
 ", " , vis1/100000 // N, ", ", vis2/100000 // N, ", ", vis3/100000 // N,
 ", ",  vis4/100000 // N, ", ", vis5/100000 // N, ", ", vis6/100000 // N,
 " respectively. The associated error compared to the exact frequencies is within
   the error for a sample size of n = 1000 because 1/7 is approximately 0.1429"]
```

Yes, P is regular because there exists an m such that P^m has all positive entries (namely, m=2). P is doubly stochastic.

The transition matrix P is $\begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}$

The stationary distribution for all the states is $\frac{1}{7}$

The frequency of visiting states 0, 1, 2, 3, 4, 5, 6 is 0.14376, 0.14304, 0.14221, 0.14313, 0.14371, 0.14185, 0.1423 respectively. The associated error compared to the exact frequencies is within the error for a sample size of n = 1000 because 1/7 is approximately 0.1429

3. **Markov Game.** Consider a game based on random walk on {0, 1, 2, 3, 4} with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

which pays R(i) = i every time X_n visits state i = 0, 1, 2, 3, 4.

(a) Find the stationary distribution π

(b) Find the long run average game payoff

(c) Simulate n =1000 games and compare with the $E^{\pi}R$

In[ ]:=

```
P = ( 0    1/2  1/2  0    0
      1/2  0    1/2  0    0
      1/3  1/3  0    1/3  0
      0    0    1/2  0    1/2
      1/2  0    0    1/2  0 );

A = MatrixPower[P, 1000] // N // Rationalize;
Print["The stationary distributions are ", A[[1, 1]], ", ", A[[2, 2]], ", ", A[[3, 3]],
 ", ", A[[4, 4]], ", ", A[[5, 5]], " for states 0, 1, 2, 3, 4, respectively." ]
exactpayoff := (0 * A[[1, 1]] + 1 * A[[2, 2]] + 2 * A[[3, 3]] + 3 * A[[4, 4]] + 4 * A[[5, 5]]) // N
Print["The long run avergae game payoff is ", exactpayoff]
n = 1000; vis0 = 0; vis1 = 0; vis2 = 0; vis3 = 0; vis4 = 0; x = 0;
Do[
 Which[x == 0, (r = RandomInteger[];
   Which[r == 0, x = 1;
    vis1 = vis1 + 1, r == 1, x = 2;
    vis2 = vis2 + 1]),
      x == 1, (r = RandomInteger[];
   Which[r == 0, x = 0;
    vis0 = vis0 + 1, r == 1, x = 2;
    vis2 = vis2 + 1]),
      x == 2, (r = RandomReal[]; Which[r ≤ 1/3, x = 0; vis0 = vis0 + 1, 1/3 < r ≤ 2/3,
            x = 1; vis1 = vis1 + 1 , 2/3 < r ≤ 1, x = 3; vis3 = vis3 + 1]),
      x == 3, (r = RandomInteger[];
   Which[r == 0, x = 2;
    vis2 = vis2 + 1, r == 1, x = 4;
    vis4 = vis4 + 1]),
      x == 4, (r = RandomInteger[];
   Which[r == 0, x = 0;
    vis0 = vis0 + 1, r == 1, x = 3;
    vis3 = vis3 + 1])
      ] ,
 {n}]
payoff := (0 * vis0 + 1 * vis1 + 2 * vis2 + 3 * vis3 + 4 * vis4) / n // N;
Print["Simulating n = 1000 games the long run average payoff is ",
 payoff, ", and has an associated absolute relative error of ",
 Abs[(payoff - exactpayoff) / exactpayoff * 100], "%"]
```

The stationary distributions are $\frac{22}{87}$, $\frac{20}{87}$, $\frac{9}{29}$, $\frac{4}{29}$, $\frac{2}{29}$ for states 0, 1, 2, 3, 4, respectively.

The long run avergae game payoff is 1.54023

Simulating n = 1000 games the long run average payoff is
 1.608, and has an associated absolute relative error of 4.4%

4. **Traveler's Dilemma.** Four towns A, B, C, D are connected by roads as follows: A⟷ B, A⟷ D, B⟷ D, B⟷ C.

(a) draw 2 D graph with properly designated transition edges and corresponding probabilities (use Mathematica for drawing or cut and paste from another program).

Suppose each time, staying in whichever town, a traveler chooses at random the neighboring town to travel next day.

(b) What is the long run probability of spotting the traveler in any given town on any given day?

In[ ]:= $P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix};$

```
A = MatrixPower[P, 100] // N // Rationalize;
Print["The long run probabilities are ", A[[1, 1]], ", ", A[[2, 2]],
 ", ", A[[3, 3]], ", ", A[[4, 4]], " for states A, B, C, D, respectively." ]
```

The long run probabilities are $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{8}$, $\frac{1}{4}$ for states A, B, C, D, respectively.