Project 12 - Javier Salazar 1001144647

1. Consider a Weibull Distribution, which is widely used for modelling the lifetime of elements. Use *Mathematica* to evaluate

μ = Mean[WeibullDistribution[α,β]], σ^2 =Variance[WeibullDistribution[α,β]], density = PDF[WeibullDistribution[α,β],x]

and plot on a single graph densities for (α,β) = (2, 1), (2, 4), (3, 6).

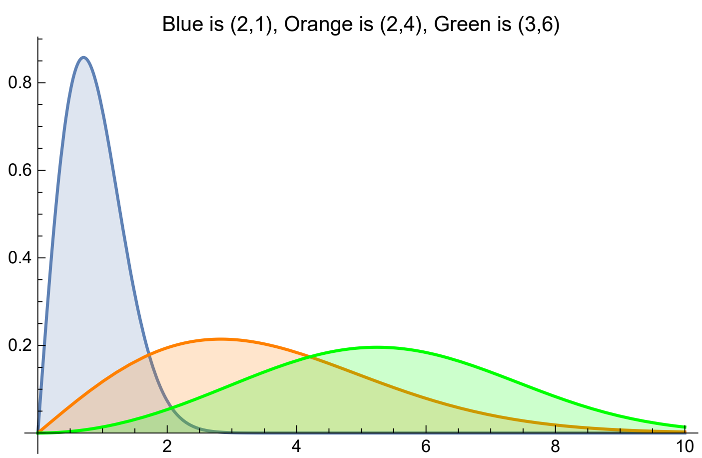
Remark. For $\beta = \frac{1}{\lambda}$ and α = 1 WeibullDistribution[α,β] turns into exponential distribution ~λ.

In[∘]:=
```
μ1 = Mean[WeibullDistribution[2, 1]];
μ2 = Mean[WeibullDistribution[2, 4]];
μ3 = Mean[WeibullDistribution[3, 6]];
σsquared1 = Variance[WeibullDistribution[2, 1]];
σsquared2 = Variance[WeibullDistribution[2, 4]];
σsquared3 = Variance[WeibullDistribution[3, 6]];
density1 = PDF[WeibullDistribution[2, 1], x];
density2 = PDF[WeibullDistribution[2, 4], x];
density3 = PDF[WeibullDistribution[3, 6], x];
data = {{".", "(2,1)", "(2,4)", "(3,6)"}, {"μ", μ1, μ2, μ3 // N}, {"σ²", σsquared1,
    σsquared2, σsquared3 // N}, {"density", density1, density2, density3}};
Grid[data, Alignment → Left, Spacings → {2, 1}, Frame → All, ItemStyle → "Text",
 Background → {{Gray, None}, {LightGray, None}}]
g1 = Plot[density1, {x, 0, 10}, Filling → Axis, PlotRange → Full,
   PlotLabel → "Blue is (2,1), Orange is (2,4), Green is (3,6)"];
g2 = Plot[density2, {x, 0, 10}, Filling → Axis, PlotStyle → Orange];
g3 = Plot[density3, {x, 0, 10}, Filling → Axis, PlotStyle → Green];
Show[g1, g2, g3]
```

Out[∘]=

.	(2,1)	(2,4)	(3,6)
μ	$\frac{\sqrt{\pi}}{2}$	$2\sqrt{\pi}$	5.35788
σ^2	$1-\frac{\pi}{4}$	$16\left(1-\frac{\pi}{4}\right)$	3.79198
density	$\begin{cases} 2e^{-x^2}x & x>0 \\ 0 & \text{True} \end{cases}$	$\begin{cases} \frac{1}{8}e^{-\frac{x^2}{16}}x & x>0 \\ 0 & \text{True} \end{cases}$	$\begin{cases} \frac{1}{72}e^{-\frac{x^3}{216}}x^2 & x>0 \\ 0 & \text{True} \end{cases}$

Out[∘]=



2. Consider a renewal process N(t) with X_i ~ WeibullDistribution[2,1] and m(t) = EN(t)

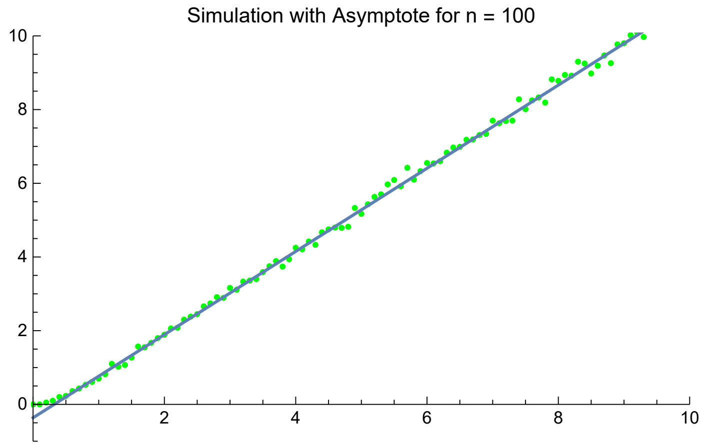
   (a) Simulate the sample average $\overline{m}(t) = \frac{N^1(t) + ... + N^n(t)}{n} \approx m(t)$, for $t \in \{0, 0.1, 0.2, ..., 10\}$, for n = 100, 1000

   (b) Graph $\overline{m}(t)$ together with the linear asymptote $\frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}$ (≈ m(t) for large t) , n =100, 1000

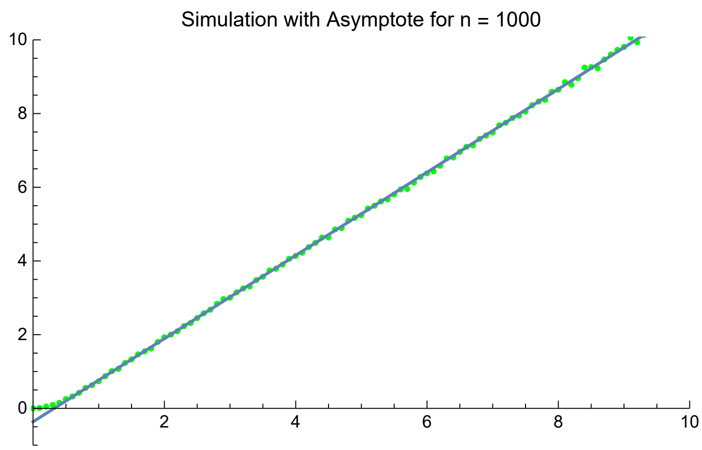
(c) Evaluate the error between $\overline{m}(s)$ and $\frac{t}{\mu}+\frac{\sigma^2-\mu^2}{2\mu^2}$ for $t\in\{0,1,2,...,10\}$, $n=1000$

```
In[ ]:= g1 = Plot[t / μ + (σsquared1 - μ1^2)/(2 μ1^2), {t, 0, 10}, AxesOrigin → {0, -1}];

n = 100; m̅ = {}; t = 10; dt = .1; p = t/dt;
Do[mt = {};
  Do[Numt = {}; s = 0; s = s + RandomVariate[WeibullDistribution[2, 1]];
    While[s < i * dt, AppendTo[Numt, s];
   s = s + RandomVariate[WeibullDistribution[2, 1]]];
          AppendTo[mt, Length[Numt]],
    {n}];
 AppendTo[m̅, Mean[mt]],
 {i, 0, p}]
sampleAverage = Table[{i * .1, m̅[[i + 1]]}, {i, 0, 100}] // N;
g2 = ListPlot[sampleAverage, PlotRange → {{0, 10}, {-1, 10}},
   PlotLabel → "Simulation with Asymptote for n = 100", PlotStyle → Green];
Show[g2, g1]

n = 1000; m̅ = {}; t = 10; dt = .1; p = t/dt;
Do[mt = {};
  Do[Numt = {}; s = 0; s = s + RandomVariate[WeibullDistribution[2, 1]];
    While[s < i * dt, AppendTo[Numt, s];
   s = s + RandomVariate[WeibullDistribution[2, 1]]];
          AppendTo[mt, Length[Numt]],
    {n}];
 AppendTo[m̅, Mean[mt]],
 {i, 0, p}]
sampleAverage = Table[{i * .1, m̅[[i + 1]]}, {i, 0, 100}] // N;
g3 = ListPlot[sampleAverage, PlotRange → {{0, 10}, {-1, 10}},
   PlotLabel → "Simulation with Asymptote for n = 1000", PlotStyle → Green];
Show[g3, g1]

Errors = Table[{i * .1,  m̅[[i + 1]] - (i / μ * .1 + (σsquared1 - μ1^2)/(2 μ1^2))}, {i, 0, 100, 10}] // N;
Grid[Errors, Alignment → Left, Spacings → {2, 1}, Frame → All, ItemStyle → "Text"]
```

Out[ ]=



Out[ ]=



Out[ ]=

0.	0.36338
1.	−0.0299989
2.	0.0366219
3.	−0.0167573
4.	−0.0151364
5.	−0.0365156
6.	−0.0298948
7.	−0.0512739
8.	−0.0156531
9.	0.0169677
10.	−0.112411

3. A lifetime of an element has WeibullDistribution[3,6], replacement time is Uniform[0,1] and replacement cost R_i equals

twice the replacement time. Consider a corresponding Renewal - Reward Process N(t) with $R(t) = \sum_{i=1}^{N(t)} R_i$.

(a) Simulate n =1000 samples of $\frac{R(t)}{t}$ for t = 100, 200, 500 and compare to $\lim_{t\to\infty} \frac{R(t)}{t}$

(b) Graph a typical sample of R(t) for t =100

In[ ]:=
```
t = 100; Rt = {}; n = 1000;
Do[ TotReward = 0; s = 0;
   r = RandomReal[] + RandomVariate[WeibullDistribution[3, 6]]; s = s + r;
   While[s < t, TotReward = TotReward + RandomReal[0, 2];
         r = RandomReal[] + RandomVariate[WeibullDistribution[3, 6]];
         s = s + r];
    AppendTo[Rt, TotReward / t],
  {n}];
Mean[Rt] // N
```

Out[ ]=
```
{0., 0.}
```

4. A specialty store marks down 50% discount on a $20 item, effective immediately whenever the item is not sold for 1 hour

while reverting back to the original price immediately after the first discounted item was sold. Suppose that buyers arrive

according to Poisson process (renewals are the moments of purchase). Let "off" corresponds to discounted

price and "on" be the regular price periods in the corresponding alternating renewal process, and the day begins with the

regular price

Consider 100 hours period starting with regular price and assume that buyers arrive with the rate of λ/per hour.

(a) Find the long-run proportion of time for the discounted price

From the hint, the long - run proportion of time for the discounted price, depdendent on λ, is $e^{-\lambda}$

(b) Find the long - run average price to the buyer. Calculate long - run average price for $\lambda = 3, 2, 1, \frac{1}{2}, \frac{1}{3}$

(c) Simulate n =1000 samples to find $\frac{\text{off- time in 100 hours}}{100}$ and compare to $\lim_{t\to\infty} \frac{\text{off- time in t hours}}{t}, \lambda = 1$

(d) Graph a typical sample of on - off cycles over 10 hours period , $\lambda = 1$ (*)

(*) Hint. Renewals $S_n = X_1 + ... + X_n$ are buyer arrivals with cycles $X_i = Z_i + Y_i$ = on + off = min(X_i,1) + [max(X_i,1) - 1].

Notice that if $X_i \le 1$ then off part of the cycle is 0. After simulating a sample $S_1, S_2, ... S_{N(t)}$ of renewals on [0, t]

decompose each X_i , as above, to find the on and off parts of the cycle. Graph on with the value of 20 and off with 10.

In[∘]:=
```
n = {3, 2, 1, 1/2, 1/3};
x = {0, 0, 0, 0, 0};
For[i = 0, i ≤ 4, λ = n[[i]]; x[[i]] = 20 (1 - ⅇ^-λ) + 10 ⅇ^-λ // N, i++]
data = {{"λ ", "long-term avg price"}, {n[[1]], x[[1]]},
   {n[[2]], x[[2]]}, {n[[3]], x[[3]]}, {n[[4]], x[[4]]}, {n[[5]], x[[5]]}};
Grid[data, Alignment → Left, Spacings → {2, 1}, Frame → All, ItemStyle → "Text",
 Background → {{Gray, None}, {LightGray, None}}]
```

Out[∘]=

λ	long–term avg price
3	19.5021
2	18.6466
1	16.3212
$\frac{1}{2}$	13.9347
$\frac{1}{3}$	12.8347

In[∘]:=
```
X := Random[PoissonDistribution[1]]
Z := Min[X, 1];
Y := Max[X, 1] - 1;
t = 100; n = 1000; Ont = {};
Do[TotOn = 0;
   u = 0; on = Z; off = Y;
  While[u + on + off < t, u = u + on + off; TotOn = TotOn + on;
      on = Z; off = Y
      ];
  If[on < t - u, TotOn = TotOn + on, TotOn = TotOn + t - u];
  AppendTo[Ont, TotOn / t],
  {n}];
Mean[Ont] // N;
Print["(off - time in 100 hours)/100 = ", 1 - Mean[Ont] // N,
 " and lim_{t→∞} (off - time in t hours)/t = ", 1/ⅇ // N , " with relative absolute error of ",
 Abs[1/ⅇ - (1 - Mean[Ont])] / (1/ⅇ) * 100 // N, " %"]
```

$\frac{\text{off - time in 100 hours}}{100}$ = 0.36584 and $\lim_{t\to\infty} \frac{\text{off - time in t hours}}{t}$ =
 0.367879 with relative absolute error of 0.554378 %

In[ ]:=

```
X := Random[PoissonDistribution[1]]
Z := Min[X, 1];
Y := Max[X, 1] - 1;
t = 10; OnIntervals = {};
Do[u = 0; on = Z;  off = Y;
  While[u + on + off < t,
       u = u + on + off; AppendTo[OnIntervals, {u - (on + off), u - off}];
       on = Z; off = Y
       ];
  If[on < t - u, AppendTo[OnIntervals, {u, u + on}]], AppendTo[OnIntervals, {u, t}]]
```

... Do: Non-list iterator AppendTo[OnIntervals, {u, t}] at position 2 does not evaluate to a real numeric value.

Out[ ]=

```
Do[u = 0;
 on = Z;
 off = Y;
 While[u + on + off < t, u = u + on + off;
  AppendTo[OnIntervals, {u - (on + off), u - off}];
  on = Z;
  off = Y];
 If[on < t - u, AppendTo[OnIntervals, {u, u + on}]], AppendTo[OnIntervals, {u, t}]]
```