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1. **Industrial Mobility and Peters's Principle.** Suppose a position at a large firm is filled by a person at one of the following levels : T = trainee, J = junior, S = senior. Let X(t) denote the level of the person holding a position at time t, and suppose X(t) evolves as a Markov process according to the transition rate matrix Q as follows:

$$Q = \begin{pmatrix} \backslash & T & J & S \\ T & -a_T & a_T & 0 \\ J & a_{JT} & -a_J & a_{JS} \\ S & a_S & 0 & -a_S \end{pmatrix}$$

In words, a trainee stays at the rank for an exponentially distributed time ~ a_T and then becomes a junior. A junior stays
at that rank for an exponentially distributed time ~ $a_J = a_{JT} + a_{JS}$ and subsequently will be promoted to a senior with probability $\frac{a_{JS}}{a_J}$ or the junior leaves the position with probability $\frac{a_{JT}}{a_J}$ and will be immediately replaced by a trainee. A senior stays
at that rank for an exponentially distributed time ~ a_S and then leaves (or gets fired according to the Peter's Principle of reaching the level of incompetence) and will be immediately replaced by a trainee. Suppose the mean time spent in positions T, J, S are as follows: 1, 2, 5 years ($-q_{ii} = \frac{1}{\text{mean time spent at } i}$ in the matrix Q). Assume in addition that the chance of being promoted from J to S is $\frac{2}{5}$.

(a) Find the transition probability matrix $P(t) = e^{Q\,t}$

In[∘]:=

```
<< BarCharts`
```

GeneralizedBarChart: Symbol GeneralizedBarChart appears in multiple contexts {BarCharts`, Global`}; definitions in context BarCharts` may shadow or be shadowed by other definitions.

General: BarCharts` is now obsolete. The legacy version being loaded may conflict with current functionality. See the Compatibility Guide for updating information.

In[∘]:=

```
Q = {{-1, 1, 0}, {3/10, -1/2, 1/5}, {1/5, 0, -1/5}};
P[t_] := MatrixExp[Q * t];
P[t] // MatrixForm
```

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 3/10 & -1/2 & 1/5 \\ 1/5 & 0 & -1/5 \end{pmatrix};$$

Out[∘]//MatrixForm=

$$\begin{pmatrix}
\frac{1}{5} + \frac{90\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} + \frac{10\sqrt{89}\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{90\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} + \frac{10\sqrt{89}\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} & \frac{2}{5} - \frac{130\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{10\sqrt{89}\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} + \frac{130\, e^{-\frac{17t}{20}+}}{-89+17\sqrt{8}} \\
\frac{1}{5} - \frac{31\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{3\sqrt{89}\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} + \frac{31\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} - \frac{3\sqrt{89}\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} & \frac{2}{5} + \frac{25\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} + \frac{5\sqrt{89}\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{25\, e^{-\frac{17t}{20}+\frac{\sqrt{89}}{20}}}{-89+17\sqrt{89}} \\
\frac{1}{5} - \frac{14\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{2\sqrt{89}\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} + \frac{14\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} - \frac{2\sqrt{89}\, e^{-\frac{17t}{20}+\frac{\sqrt{89}\,t}{20}}}{-89+17\sqrt{89}} & \frac{2}{5} + \frac{40\, e^{-\frac{17t}{20}-\frac{\sqrt{89}\,t}{20}}}{89+17\sqrt{89}} - \frac{40\, e^{-\frac{17t}{20}+\frac{\sqrt{89}}{20}}}{-89+17\sqrt{89}}
\end{pmatrix}$$

(b) Find the stationary distribution $\pi = (\pi_T, \pi_J, \pi_S)$ by taking $\lim_{t\to\infty} P(t)$ (each row is π)

In[ ]:=
```
M = Limit[P[t], t → Infinity];
M // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

```
(*πT=1/5, πJ=2/5, πS=2/5*)
```

(c) Find the stationary distribution π by solving $\pi Q = 0$

In[ ]:=
```
Solve[{πT, πJ, πS}.Q == {0, 0, 0}, {πT, πJ, πS}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[ ]= $\{\{\pi J \to 2\,\pi T,\ \pi S \to 2\,\pi T\}\}$

```
(*By linear algebra and inspection the solution is πT=1/5, πJ=2/5, πS=2/5*)
```

2. **Barber Shop.** There are 2 waiting chairs and 2 barbers. Customers arriving to fully occupied shop are turned away.

Assume that customers arrive according to Poisson process at rate of 5/hour (inter arrivals are exponential ~ $\lambda = 5$) and

the service rate is 2/hour (services are exponential ~ $\mu = 2$). If X(t) = the number of the customers in the shop at time t,

then it can be modelled by a birth-death process on states {0, 1, 2 ,3 , 4} with the following transition rate matrix

$$Q = \begin{pmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix}$$

Suppose that at the time of opening the shop at 8 AM there were 2 customers waiting.

(a) Using p(t) = p(0) P(t) find the probability that the shop will be full at: 9AM, 10AM, 12PM, 4PM

```
Q = ( -5   5   0   0   0
       2  -7   5   0   0
       0   4  -9   5   0
       0   0   4  -9   5
       0   0   0   4  -4 );

P[t_] := MatrixExp[Q * t];
P[t] // N // MatrixForm;
p[t_] := {0, 0, 1, 0, 0}.P[t];
For[i = 1, i ≤ 8, i = i * 2, Print[p[i] // N]] (*The last entry in the
 matrix is the probablity that the shop is full at the respective time*)
```

{0.0717358, 0.17338, 0.207755, 0.247386, 0.299743}

{0.0658251, 0.16359, 0.203281, 0.252612, 0.314692}

{0.0648914, 0.162212, 0.202746, 0.253408, 0.316742}

{0.0648758, 0.16219, 0.202737, 0.253421, 0.316776}

(b) Find the stationary distribution π

In[ ]:=
```
M = Limit[P[t] // N, t → Infinity];
M // MatrixForm
 (*Stationary distribution is along the diagonal for each respective state*)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.0648758 & 0.16219 & 0.202737 & 0.253421 & 0.316776 \\ 0.0648758 & 0.16219 & 0.202737 & 0.253421 & 0.316776 \\ 0.0648758 & 0.16219 & 0.202737 & 0.253421 & 0.316776 \\ 0.0648758 & 0.16219 & 0.202737 & 0.253421 & 0.316776 \\ 0.0648758 & 0.16219 & 0.202737 & 0.253421 & 0.316776 \end{pmatrix}$$

(c) Assume X(0) = 2. Simulate n = 365 days sample (assume the shop operates 8AM - 8PM). Compare the frequencies of states to the stationary distribution π.

(d) Repeat (c) for n = 10,000.

In[ ]:=
```
Q = {{-5, 5, 0, 0, 0}, {2, -7, 5, 0, 0}, {0, 4, -9, 5, 0}, {0, 0, 4, -9, 5}, {0, 0, 0, 4, -4}};
P[t_] := MatrixExp[Q * t];
P[t] // N // MatrixForm;
p[t_] := {0, 0, 1, 0, 0}.P[t];
p[365 * 12] // N
p[10000 * 12] // N
```

$$Q = \begin{pmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix};$$

General: Exp[−68542.8] is too small to represent as a normalized machine number; precision may be lost.

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General: Exp[−45537.1] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[ ]= {0.0648758, 0.16219, 0.202737, 0.253421, 0.316776}

General: Exp[-1.87789×10^{6}] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[-1.87789×10^{6}] is too small to represent as a normalized machine number; precision may be lost.

General: Exp[-1.24759×10^{6}] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[ ]= {0.0648758, 0.16219, 0.202737, 0.253421, 0.316776}

3. **Car Service.** A service is carried out sequentially as follows: T(tune-up), A(air-conditioning), B(brakes). The mean
service times are respectively 1,2,3 hours. Assume the services are independent and exponentially distributed. If X(t) denotes
the state of service at time t then it can be modelled by a birth-death process on states {1,2,3,4} where 4 corresponds to the
absorbing state (i.e., all three services 1-T, 2-A, 3-B have been completed and therefore the transition rate to leave 4 is 0).
Then the transition rate matrix has the form

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that the transition rate = $\frac{1}{\text{mean times spent at } a \text{ given state}}$.

(a) What is the probability that 5 hours after the service started the car is still in the brake repair stage?

```
Q = {{-1, 1, 0, 0}, {0, -1/2, 1/2, 0}, {0, 0, -1/3, 1/3}, {0, 0, 0, 0}};
P[t_] := MatrixExp[Q * t];
P[5] // N // MatrixForm (*0.189*)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.00673795 & 0.150694 & 0.367537 & 0.475031 \\ 0. & 0.082085 & 0.320372 & 0.597543 \\ 0. & 0. & 0.188876 & 0.811124 \\ 0. & 0. & 0. & 1. \end{pmatrix}$$

(b) What is the probability that the service has been completed at the end of the 6-th hour?

```
P[6] // N // MatrixForm (*0.864*)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.00247875 & 0.0946166 & 0.314004 & 0.5889 \\ 0. & 0.0497871 & 0.256645 & 0.693568 \\ 0. & 0. & 0.135335 & 0.864665 \\ 0. & 0. & 0. & 1. \end{pmatrix}$$

4. **Linear Growth Model with Immigration.** Let X(t) has the parameters $\lambda_0 = \lambda$, $\mu_0 = \mu$, $\mu_n = n\,\mu$, $\lambda_n = n\,\lambda + \theta$ for $n \geq 1$
and let $\lambda < \mu$. Then the expected population size $m(t) = x_0 e^{(\lambda-\mu)\,t} + \frac{\theta}{\lambda-\mu}[\,e^{(\lambda-\mu)\,t} - 1\,] \to \frac{\theta}{\mu-\lambda}$ = equilibrium size, as $t \to \infty$.
Let the population of 100 has the birth rate λ =1 /individual/year, death rate $\mu = 1.2$ individual/year, and the immigration factor
$\theta = 1$ /year.

(a) Simulate n = 1000 samples of the population size at time t = 30 years, calculate the sample average and compare to m(t).

```
Table[λp[n_] := n * 1 + 1, {n, 1, 1000}];
Table[μp[n_] := n * 1.2, {n, 1, 1000}];
t = 30; x0 = 100;
exactmean := 1 / (1.2 - 1)
TotCust = {};
Do[s = 0; NumCust = x0; n = NumCust;
 Label[start];
 r = Random[ExponentialDistribution[λp[n] + μp[n]]]; s = s + r;
 If[s ≤ t, If[RandomReal[{0, 1}] < λp[n] / (λp[n] + μp[n]), NumCust = NumCust + 1;
   n = NumCust;
   Goto[start], NumCust = NumCust - 1;
   n = NumCust;
   Goto[start]]];
 AppendTo[TotCust, NumCust]
 , {1000}]
Print["The average mean is ", Mean[TotCust] // N]
Print["The exact mean is ", exactmean]
```

```
The average mean is 5.236

The exact mean is 5.
```

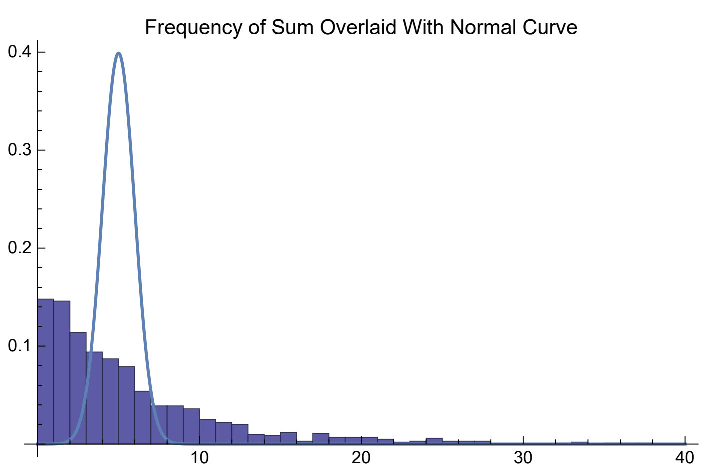
(b) Calculate probability frequencies of population size at t = 30 for n =1000 samples and graph the frequencies together with

the stationary distribution for comparison.

```
In[]:= bcount = BinCounts[TotCust, {0, 40, 1}];
Total[bcount]
Show[GeneralizedBarChart[Table[{.5 + (i - 1) * 1, bcount[[i]] / (1 * 1000), 1}, {i, 1, 40}],
  PlotLabel → "Frequency of Sum Overlaid With Normal Curve"],
 Plot[1 / Sqrt[2 * Pi] * Exp[- (x - 5)^2 / 2], {x, 0, 40}, PlotRange → All]]
```

```
Out[]= 1000
```

Out[]=



(c) Graph a single sample of population evolution in 30 years

In[ ]:=
```
Plot[100 * Exp[-1.2 * x] - 5 * (Exp[-1.2 * x] - 1), {x, 0, 30}, PlotRange → All]
```

Out[ ]=