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Project 8.

1. Area by Monte Carlo

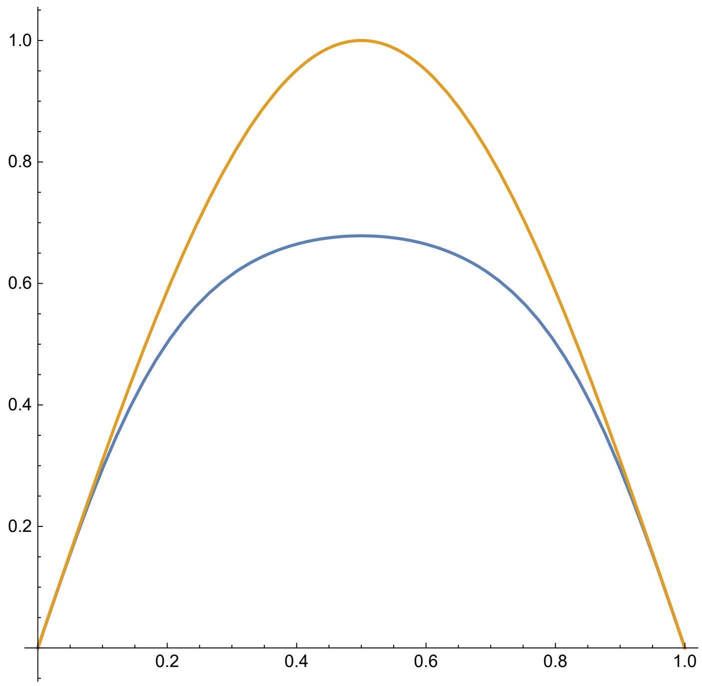
(a) Plot Sin[Sin[Sin[Sin[π x]]]] and Sin[π x] on the single graph on the interval [0,1]

(b) Compute the area between Sin[Sin[Sin[Sin[π x]]]] and Sin[π x] using Monte Carlo for n =1,000,000

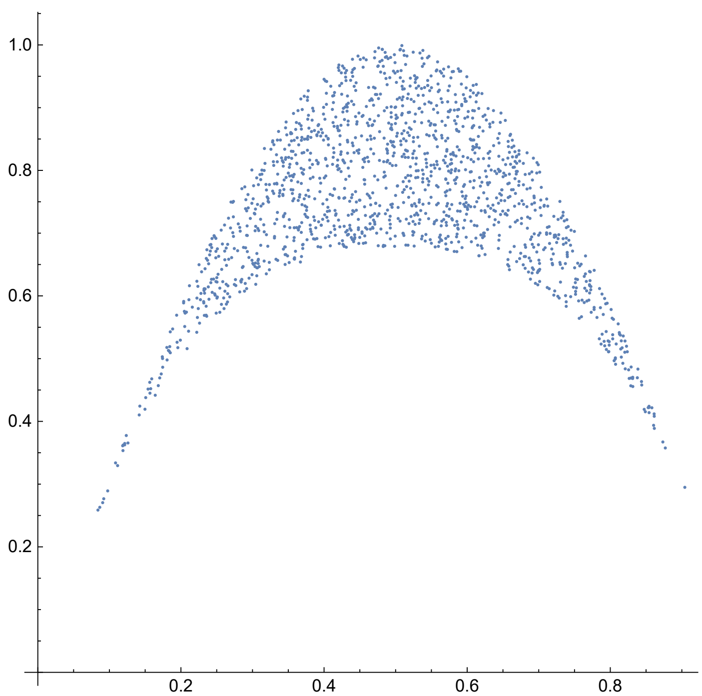
(c) Compare the simulated value with the value obtained from NIntegrate[.]

(d) Plot the simulated sample of n =10,000 points

In[ ]:=
```
Plot[{Sin[Sin[Sin[Sin[π x]]]], Sin[π x]}, {x, 0, 1}, AspectRatio → 1]
pie1[n_] := (hits = 0;
  Do[{x, y} = {RandomReal[{0, 1}], RandomReal[{0, 1}]};
   If[ Sin[Sin[Sin[Sin[π x]]]] ≤ y ≤ Sin[π x], hits = hits + 1], {i, 1, n}]; hits / n // N)
Print[ "Monte Carlo approximation = ", pie1[1000000]]
TrueValue := NIntegrate[Sin[π x] - Sin[Sin[Sin[Sin[π x]]]], {x, 0, 1}]
Print[ "True Value = ", TrueValue]
pie2[n_] := (hitsCount = 0;
  hitsPoints = {};
  Do[{x, y} = {RandomReal[{0, 1}], RandomReal[{0, 1}]};
   If[ Sin[Sin[Sin[Sin[π x]]]] ≤ y ≤ Sin[π x],
    hitsCount = hitsCount + 1; hits = AppendTo[hitsPoints, {x, y}]], {i, 1, n}];
  4 hitsCount / n // N)
pie2[10000];
ListPlot[hitsPoints, AspectRatio → 1]
```

Out[ ]=



```
Monte Carlo approximation = 0.14881
True Value = 0.148108
```

Out[ ]=


2. Integrals by standard Monte Carlo

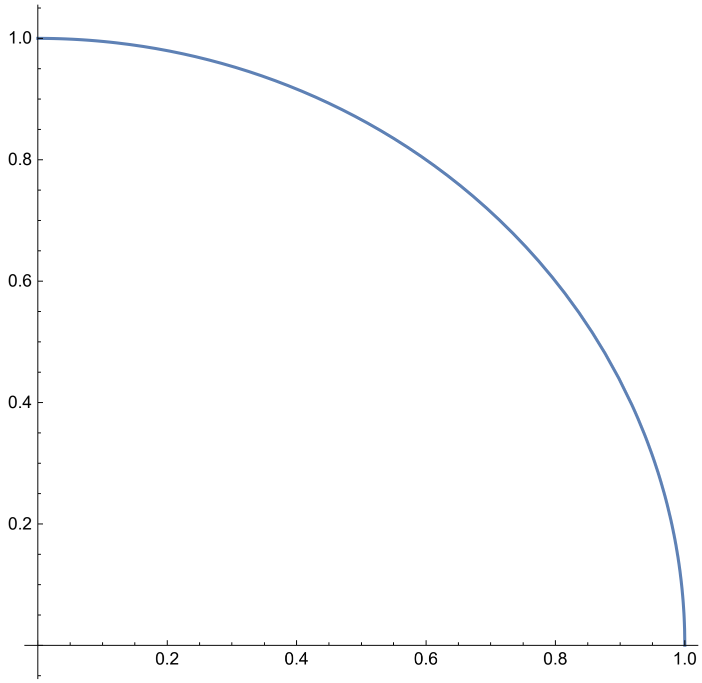
Pick your favorite definite integral on [0,1] whose exact values involve π or e. Plot the graph of the corresponding function. Simulate the approximate integral value by using Monte Carlo for n =

1,000,000 and compare the with exact value or the value obtained by numerical integration NIntegrate[.].

In[ ]:=

```
Plot[Sqrt[1 - x^2], {x, 0, 1}, AspectRatio → 1]
pie1[n_] := (hits = 0;
  Do[{x, y} = {RandomReal[{0, 1}], RandomReal[{0, 1}]};
   If[x^2 + y^2 ≤ 1, hits = hits + 1], {i, 1, n}]; hits/n // N)
Print["Monte Carlo approximation = ", 4 * pie1[1000000]]
Print["True Value = ", π // N]
```

Out[ ]=




```
Monte Carlo approximation = 3.14201
True Value = 3.14159
```

3. Integrals by Importance Sampling Monte Carlo

Let $f(x) = 4\sqrt{1 - x^2}$ and $I = \int_0^1 f(x)\,dx$. Find a linear probability density $\varphi(x) = a\,x + b$ that works well with *Importance Sampling* method of approximating the integral I. Namely, find a and b such that

$\mathrm{Var}(J_n(\varphi)) = \frac{1}{n}\left(\int_0^1 \frac{f^2(x)}{\varphi(x)}\,dx - I^2\right)$ is as small as possible. That is, find a and b which

minimize $\int_0^1 \frac{\left(4\sqrt{1-x^2}\right)^2}{a\,x+b}\,dx$ (or is close to minimum by numerical approximation)

(*) subject to: $a\,x + b > 0$ and $\int_0^1 (a\,x + b)\,dx = 1$

Hint. Use $a + 2\,(1 - a)\,x$ as parametrization for $a\,x + b$.

Compare the variance for the standard Monte Carlo $\mathrm{Var}(J_n(1)) = \frac{1}{n}(\int_0^1 f^2(x)\,dx - I^2)$ with the variance of the *Importance Sampling* $\mathrm{Var}(J_n(\varphi)) = \frac{1}{n}(\int_0^1 \frac{f^2(x)}{\varphi(x)}\,dx - I^2)$, by taking their ratio and interpret how many trials $m < n$ (by what factor) are sufficient for *Importance Sampling* to match the accuracy of the standard Monte Carlo.

Simulate $m < n = 1{,}000{,}000$ trials of *Importance Sampling* with your $\varphi(x)$ to approximate the integral I and compare with the standard Monte Carlo simulation corresponding to uniform distribution.

In[ ]:=
```
s = 0; n = 1000000;   (* standard sampling  ~ uniform distribution on [0,1] *)
Do[s = s + f[RandomReal[]], {n}];
a = MonteCarloIntegral = s / n // N;
exact = π // N;
Print["exact = ", exact]
Print["Monte Carlo = ", a]
Print["error = ", Abs[a - exact]]
```

```
exact = 3.14159

Monte Carlo = 3.14164

error = 0.0000506955
```

In[ ]:=
```
f[x_] := 4 Sqrt[1 - x^2]    (* 0 ≤ x ≤ 1 *)
φ[x_] := 1.7 - 1.4 x;   g[x_] := f[x] / φ[x];
Clear[y]
Solve[1.7 x - 0.7 x^2 == y, x] ;     (* 0 ≤ x ≤ 1 *)
H[u_] := 0.0714286 (17 - Sqrt[289 - 280 u]);
s = 0; n = 200000; (* importance sampling ~ density φ[x] on [0,1] *)
Do[s = s + g[H[RandomReal[]]], {n}];
a = MonteCarloIntegral = s / n // N;
exact = π // N;
Print["exact = ", exact]
Print["Monte Carlo = ", a]
Print["error = ", Abs[a - exact]]
Print["Importace sampling takes about 5x less iterations"]
```

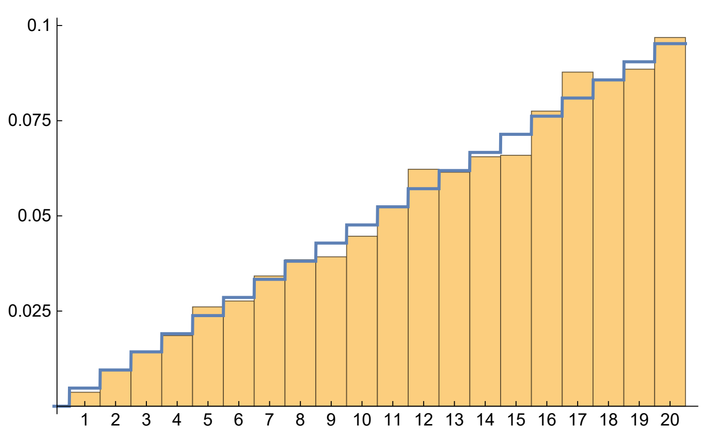
Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

```
exact = 3.14159

Monte Carlo = 3.14316

error = 0.00156335

Importace sampling takes about 5x less iterations
```

4. Generate a sample of 10,000 from a random variable X on {1, 2, . . . , 20} with probability distribution $P(X = k) = \frac{k}{210}$ and plot together the graph of obtained frequencies versus the exact distribution of X. Find the expected number of trials to generate 10,000 accepted outcomes.

In[∘]:=
```
Clear[u, i, p]
p = {1/210, 2/210, 3/210, 4/210, 5/210, 6/210, 7/210, 8/210, 9/210, 10/210, 11/210,
   12/210, 13/210, 14/210, 15/210, 16/210, 17/210, 18/210, 19/210, 20/210};
simu1 = {};
Do[{i = RandomInteger[{1, 20}], u = RandomReal[],
  If[u ≤ p[[i]] / (20/210), AppendTo[simu1, i]]}, {10000}]
g1 = Histogram[simu1, {0.5, 20.5, 1}, "Probability",
   Ticks → {{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20},
     {.025, .05, .075, .1, .125, 0.15}}];
g2 = Plot[Piecewise[{{1/210, 0.5 ≤ x < 1.5}, {2/210, 1.5 ≤ x < 2.5}, {3/210, 2.5 ≤ x < 3.5},
     {4/210, 3.5 ≤ x < 4.5}, {5/210, 4.5 ≤ x < 5.5}, {6/210, 5.5 ≤ x < 6.5},
     {7/210, 6.5 ≤ x < 7.5}, {8/210, 7.5 ≤ x < 8.5}, {9/210, 8.5 ≤ x < 9.5},
     {10/210, 9.5 ≤ x < 10.5}, {11/210, 10.5 ≤ x < 11.5}, {12/210, 11.5 ≤ x < 12.5},
     {13/210, 12.5 ≤ x < 13.5}, {14/210, 13.5 ≤ x < 14.5}, {15/210, 14.5 ≤ x < 15.5},
     {16/210, 15.5 ≤ x < 16.5}, {17/210, 16.5 ≤ x < 17.5}, {18/210, 17.5 ≤ x < 18.5},
     {19/210, 18.5 ≤ x < 19.5}, {20/210, 19.5 ≤ x < 20.5}}], {x, 0, 20.5}];
Show [g1, g2] (* blue is the exact distribution of p,
histogram of the simualted data is yellow *)
10000 / Length[simu1] // N
400 / 210 // N
```

Out[∘]=



Out[∘]= 1.93311

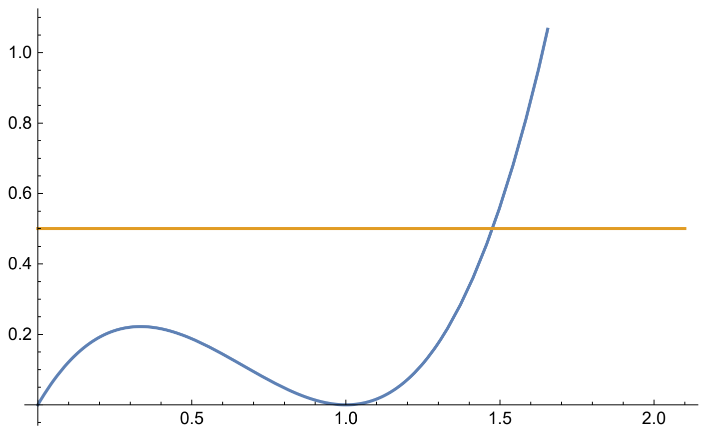
Out[∘]= 1.90476

5. Generate a sample of 10,000 from a random variable X on [0, 2] with density $f(x) = \frac{3}{2}x(x-1)^2$ for $g = \frac{1}{2}$ ~ U[0,2] and another g(x) of choice <u>that improves the acceptance rate</u>. Plot together the graph of obtained frequencies versus density f(x). Compare in both cases the average number of trials needed per acceptance to the theoretical average acceptance rate = c.

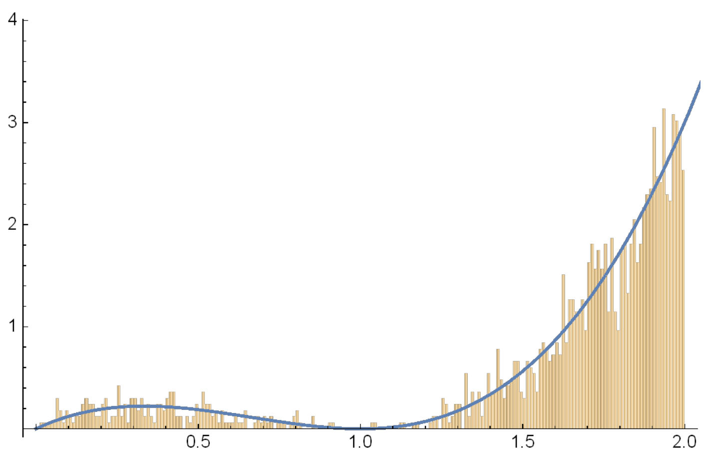
In[ ]:=

```
f[x_] := 3/2 x (x - 1)^2; g[x_] := 1/2; c = 6;
Plot[{f[x], g[x]}, {x, 0, 2.1}]
simu2 = {};
Do[y = RandomReal[{0, 2}];
 u = RandomReal[];
 If[ u ≤ 1/2 y (y - 1)^2, AppendTo[simu2, y]], {10000}]
Show[Histogram[simu2, {0, 2, 0.01}, "ProbabilityDensity",
  ChartElementFunction → "GlassRectangle"], Plot[f[x], {x, 0, 2.1}]]
10000 / Length[simu2] // N
```

Out[ ]=



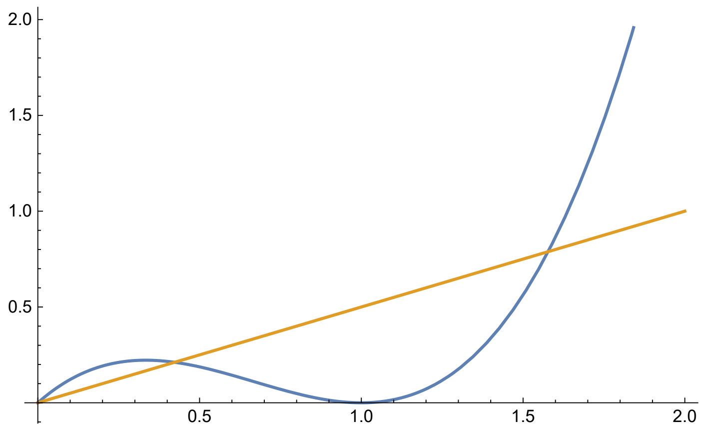
Out[ ]=



Out[ ]= 6.035

In[◦]:=
```
f[x_] := 3/2 x (x - 1)^2; g[x_] := 1/2 x; c = 3;
Plot[{f[x], g[x]}, {x, 0, 2}]
simu3 = {};
Do[y = Sqrt[RandomReal[]] + 1;
 u = RandomReal[];
 If[ u ≤ (y - 1)^2, AppendTo[simu3, y]], {10 000}]
10 000 / Length[simu3] // N
Show[Histogram[simu3, {0, 2, 0.01}, "ProbabilityDensity",
  ChartElementFunction → "GlassRectangle"], Plot[f[x], {x, 0, 2.1}]]
```

Out[◦]=



Out[◦]= 1.98926

Out[◦]=