

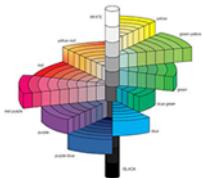
the HSL Metric Tensor

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Recently I was asked by Namaska¹ to provide them beautiful harmonies colours patterns to include in their production line. Harmony colour pattern is a vast and complex subject, one does not really know how to chose the correct colour which works with the set of colours in the palette chosen. So I send a request to the mathematician side of my brain, and he took his Pilot Falcon fountain pen and began to think.

Among the different colour's models there is one which attracted my attention: The HSL Colour Mode.



HSL is a cylindrical system coordinates which defines colour with 3 components:

h : is the hue of the colour, defined in radians

s : is the saturation of the colour and

l : is the lightness of the colour. Both saturation and lightness are bounded to the interval $[0,1]$

Then we have an \mathbb{R}^3 space which defines all the possible colours. I was wondering how to iterate from one colour to another keeping the harmony of the palette. In this cylindrical coordinates, we can now define a position vector:

$$\vec{u}(r) = s \cdot \cos(hue) \cdot \vec{i} + s \cdot \sinus(hue) \cdot \vec{j} + l \cdot \vec{k} \quad (1)$$

and then calculate it's partial derivatives, according to Einstein's notation, and obtain the vectors of local basis:

$$\frac{\partial u^\alpha}{\partial x^\beta} = e^\nu \quad (2)$$

Riemann's Metric Tensor is defined by the dot products among the vector basis, so calculation equation (1) yields:

$$g_{hsl} = \begin{pmatrix} s^2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad (\text{The HSL Metric Tensor})$$

¹Namaska is a french company that produces handmade hemp bags

Then we can have a quantification of dc which represents a differential amount of colour:

$$dc^2 = s^2 \cdot dh^2 + ds^2 + dl^2 \quad (3)$$

It is important to notice that the amount of hue in dc is multiplied by the value of saturation. This makes sense since we can consider saturation as a hue property. However, lightness seems to be an independent attribute of the hue. This makes sense.

With this mathematical object we can define the *norm* of a colour which is given by the equation:

$$\|\vec{c}\| = \sqrt{c_k \cdot g_{hsl} \cdot c^k} \quad (4)$$

And the angle of two colours can be evaluate as follows:

$$\text{angle}(c_1, c_2) = \arccos\left(\frac{g(c_1, c_2)}{\sqrt{g(c_1) \cdot g(c_2)}}\right) \quad (5)$$

Last, we can have the path from two colours, given by the line integral and considering a path λ connecting the two colours :

$$L(a, b) = \int_{\lambda} \sqrt{g_{hsl}(dc)} \quad (6)$$

We can assume that a valid path from colour a to colour b is somekind of a *Manhattan* distance, that is from a we will walk through the arc sdh , and then increment saturation and lightness in ds and dl respectively, so the equation (7) can be written as:

$$L(dh, ds, dl) = s \cdot |dh| + |ds| + |dl| \quad (\text{Charlotte colour distance})$$

Obviously Charlotte's distance is not a geodesic curve between the referenced colours, but this distance has very interesting properties regarding fast computing optimization, and my purpose is two find a distance that can separate two hsl colours, not the minimum distance between them.

Here comes the most sexy content, if maths didn't panic you, my dear reader, you will be rewarded. In order to test this mathematical tool I build a quick application, in casual coding style, (i.e Python) and I got very interesting results. I took several pictures references with good harmony and strong contrast, so my aesthetic part of my brain started with the following painting: Then I choose



Figure 1: Giovanna Tornabuoni from Ghirlandaio

a colour from the reference and I obtain the following colour palette:

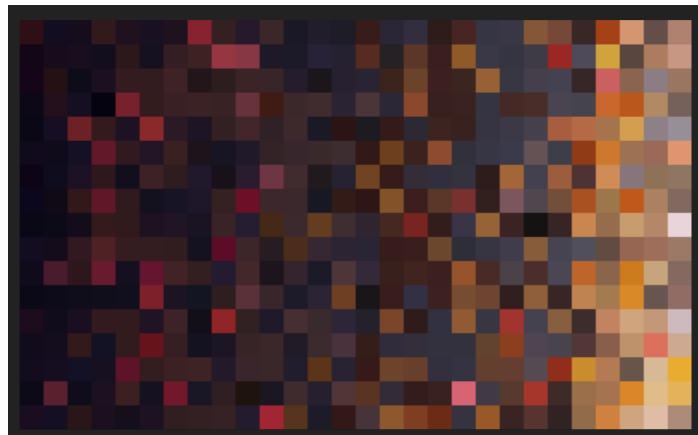


Figure 2: Giovanna Tornabuoni from Ghirlandaio

Then I choose another colour from the reference (the chosen colour in the reference is located in upper left corner in the palette) And at every run we

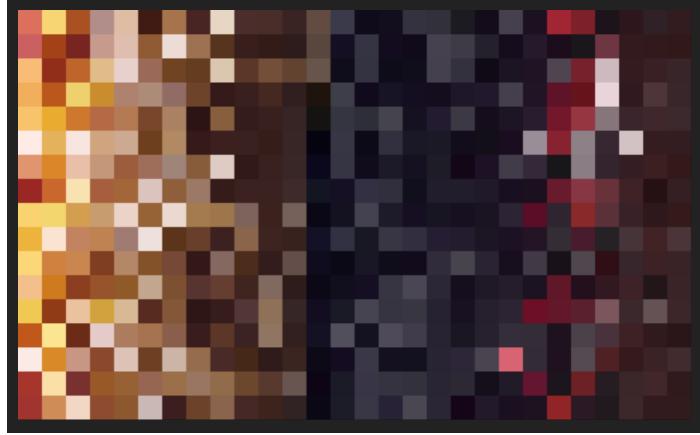


Figure 3: Giovanna Tornabuoni from Ghirlandaio

obtain a new arrangement of the colours, starting with a cold colour we would get a palette arranged from cold to warm, for example. Another use case of the HSL Metric Tensor is filtering colours according to a given colour reference, that is, the HSL Metric Tensor will give us which colours are close to the given reference according to Charlotte's distance.

Finally, some interesting results from this painting in order to automatically build color palettes given a starting colour:

Analytic	value
max distance (dark starting reference)	2,59
max distance (light starting reference)	3,59
max Neighbourhood distance	0,3770
average Neighbourhood distance	0,0050
variance	0,0003
mediane	0,0030

If you want to manually harmonize colour and you start with a (h,s,l) triple, I suggest to vary saturation and lightness in no more than 0,001 and for the hue angle you should use the following expression:

$$h = 0,0001/s \quad (7)$$

Have a nice day.