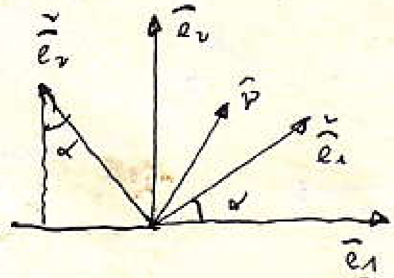
Exercise:

Find covector components for a given change of basis in $\bar{e}_i$

$$F = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$ This is a rotation matrix.

$$\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \alpha(\bar{v}) = v^1 + v^2$$



what are the components of $\bar{v}$ and $\alpha(\bar{v})$ in the new basis?

$$\tilde{\bar{v}}^i = B\bar{v}^i \quad \text{with} \quad F = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\tilde{\alpha}(\bar{v}) = F\alpha_i \quad \text{and} \quad B = F^{-1}$$ since F is an orthonormal transformation

then $F^{-1} = F^T$ so

$$B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Proof:

$$B \cdot F = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$B \cdot F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

So
$$\tilde{v}^i = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}}_{\{2,2\}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\{2,1\}} = \begin{bmatrix} \cos\alpha + \sin\alpha \\ -\sin\alpha + \cos\alpha \end{bmatrix}$$

$$\tilde{\alpha}_i = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha + \sin\alpha & -\sin\alpha + \cos\alpha \end{bmatrix}$$

so we have $\tilde{\alpha}_i \equiv \tilde{v}^i$ because the transformation was orthonormal. Let's try with another transformation