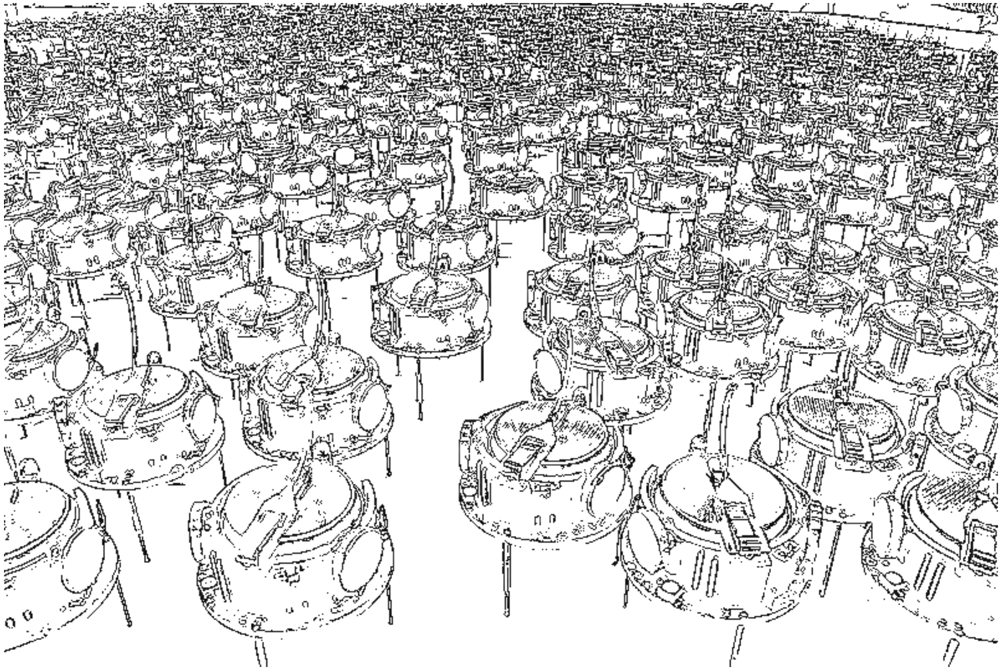
Intelligent Multi Agent Systems



Computing Solution Concepts

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Agenda

Now we know a lot about different solution concepts…

➡ But how to compute them?

➡ We will get to know a few algorithms for different solution concepts

Outline

Algorithms for:

- Zero Sum Games
- Computing Nash Equilibria
- Iterated Dominance

2 Player Zero-Sum Games

Recap: $u_1(a_1, a_2) = -u_2(a_1, a_2)$

➡ Optimal strategy: $U_1^*(\alpha) = -U_2^*(\alpha)$

➡ All Nash Equilibria have the same value $U_1^*(\alpha)$

➡ All Nash Equilibria are minmax and maxmin strategies

$$\arg\max_{\alpha_1} \min_{\alpha_2} u_1(\alpha_1, \alpha_2)$$

$$\arg\min_{\alpha_1} \max_{\alpha_2} u_2(\alpha_1, \alpha_2)$$

2 Player Zero-Sum Games

We can construct a linear program, that implements the minmax strategy for player 2:

$$\arg\min_{U_1^*, \alpha_2} \quad U_1^*$$

$$\text{s.t.} \quad \sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k \leq U_1^*, \quad \forall j \in A_1$$

$$\sum_{k \in A_2} \alpha_2^k = 1$$

$$\alpha_2^k \geq 0, \quad \forall k \in A_2$$

MinMax Strategies

$$\arg \min_{U_1^*, \alpha_2} \quad U_1^*$$

$$\text{s.t.} \quad \sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k \leq U_1^*, \quad \forall j \in A_1$$

$$\sum_{k \in A_2} \alpha_2^k = 1$$

$$\alpha_2^k \geq 0, \quad \forall k \in A_2$$

Variables: U_1^*, α_2^k

- First constraint says: U_1^* is the maximum value for player *1* (in combination with objective) if current strategy of player *2* is α_2^k
- We want to minimize the maximum value U_1^*
- Thats a MinMax Strategy for player 2!

… and vice versa: MaxMin Strategies

$$\arg\max_{U_1^*, \alpha_1} \quad U_1^*$$

$$\text{s.t.} \quad \sum_{j \in A_1} u_1(a_1^j, a_2^k)\alpha_1^j \geq U_1^*, \quad \forall k \in A_2$$

$$\sum_{j \in A_1} \alpha_1^j = 1$$

$$\alpha_1^j \geq 0, \quad \forall j \in A_1$$

Variables: U_1^*, α_1^j

- First constraint says: U_1^* is the worst case value for player *1* with current strategy α_1^j
- We want to maximize the worst-case value U_1^*
- Thats a MaxMin Strategy for player 1!

Useful alternative formulation (MinMax)

Formulation with Slack Variables:

$$\min \; U_1^*$$

$$\text{s.t.} \quad \sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k + r_1^j = U_1^*, \quad \forall j \in A_1$$

$$\sum_{k \in A_2} \alpha_2^k = 1$$

$$\alpha_2^k \geq 0, \;\; \forall k \in A_2, \qquad r_1^j \geq 0, \;\; \forall j \in A_1$$

Variables: U_1^*, α_2^k, r_1^j

- ➡ Slack variables must all be positive
- ➡ Therefore, the equality constraint is equal to the inequality constraint from the previous slides

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2 Player General Sum Games

Linear Program's are nice: Solving LP's is in P

Unfortunately, we can not formulate a linear program any more

➡ No opposing interests

➡ Can not minimize utility of other agent to maximize own utility

➡ Yet, General Sum Games can be formulated as Linear Complementarity Problem (LCP)

➡ Solving an LCP is PPAD complete: "Polynomial parity argument directed version" (almost as hard as NP-complete)

2 Player General Sum Games

General Sum Games can be formulated as Linear Complementarity Problem (LCP)

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k + r_1^j = U_1^*, \quad \forall j \in A_1$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k)\alpha_1^j + r_2^k = U_2^*, \quad \forall k \in A_2$$

$$\sum_{j \in A_1} \alpha_1^j = 1, \qquad \sum_{k \in A_2} \alpha_2^k = 1$$

$$\alpha_1^j \geq 0, \quad \forall j \in A_1, \qquad r_2^k \geq 0, \quad \forall k \in A_2$$

$$\alpha_2^k \geq 0, \quad \forall k \in A_2, \qquad r_1^j \geq 0, \quad \forall j \in A_1$$

$$r_1^j \alpha_1^j = 0, \qquad r_2^k \alpha_2^k = 0, \qquad \forall j \in A_1, \forall k \in A_2$$

2 Player General Sum Games

General Sum Games can be formulated as Linear Complementarity Problem (LCP)

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k + r_1^j = U_1^*, \quad \forall j \in A_1$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k)\alpha_1^j + r_2^k = U_2^*, \quad \forall k \in A_2$$

$$\sum_{j \in A_1} \alpha_1^j = 1, \qquad \sum_{k \in A_2} \alpha_2^k = 1$$

$$\alpha_1^j \geq 0, \ \forall j \in A_1, \qquad r_2^k \geq 0, \ \forall k \in A_2$$

$$\alpha_2^k \geq 0, \ \forall k \in A_2, \qquad r_1^j \geq 0, \ \forall j \in A_1$$

$$r_1^j \alpha_1^j = 0, \qquad r_2^k \alpha_2^k = 0, \qquad \forall j \in A_1, \forall k \in A_2$$

Differences to Zero-Sum Case:

- No objective, only constraints
- Constraints for both players in the formulation

Complementary condition:

- Slack variables always positive
- If action a_1^j is used (i.e. $\alpha_1^j > 0$) then $r_1^j = 0$

Linear Complementarity Problem

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k + r_1^j = U_1^*, \quad \forall j \in A_1, \qquad \sum_{j \in A_1} u_2(a_1^j, a_2^k)\alpha_1^j + r_2^k = U_2^*, \quad \forall k \in A_2$$

$$r_1^j \alpha_1^j = 0, \quad r_2^k \alpha_2^k = 0, \quad \forall j \in A_1, \forall k \in A_2$$

Why does it solve our problem?

- If action a_1^j is used (i.e. $\alpha_1^j > 0$) then $r_1^j = 0$
- *For each action* $\bar{a}_1^j$ *with* $\alpha_1^j > 0$ *it holds:*
  - The expected utility for agent 1 for all actions $\bar{a}_1^j$ is the same

    $$\sum_{k \in A_2} u_1(a_1^j, a_2^k)\alpha_2^k = U_1^*, \quad \forall j, \alpha_1^j > 0$$
  - The value U_1^* is the maximum expected utility agent 1 can get (assuming strategy from agent 2 is α_2)
  - Each action $\bar{a}_1^j$ is a best response to α_2

Linear Complementarity Problem

The same argument holds for player 2

- If the LCP is satisfied, we have found a (mixed strategy) Nash Equilibrium!

Ok, how do we solve a LCP?:

- There are many algorithms which do that…
- Most famous one: Lemke Howson Algorithm
- Only finds one Nash Equilibrium

Computationally very expensive:

- Solving an LCP is PPAD complete: "Polynomial parity argument directed version"
- People believe that PPAD is much harder then P (similar to NP)
- Worst case: Exponential in size of the game (?)
- But there is no proof… (also similar to NP)

Beyond finding an Equilibrium

We might want to find a equilibrium with one of the following properties:

- **Uniqueness:** Is there a unique Equilibrium for game G?
- **Pareto optimality:** Does there exist a strict Pareto Optimal equilibrium?
- **Guaranteed Payoff:** Does there exist a equilibrium where some player i gets an expected payoff of at least *v*?
- **Guaranteed social welfare:** Does there exist an equilibrium where the sum of all agent's utility is at least *k*?
- **Action inclusion:** Does there exist a equilibrium where player *i* plays action a_i with positive probability
- **Action exclusion:** Does there exist a equilibrium where player *i* plays action a_i with zero probability

Bad news: All these questions are NP-complete!

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Identifying dominated strategies

We can make the problem easier by deleting dominated strategies:

Definition:

In a strategic game player *i*'s action a_i'' strictly dominates another action a_i' if

- $u_i(a_i'',a_{-i}) > u_i(a_i',a_{-i})$ for every list a_{-i} of the other player's action

We say that a_i' is strictly dominated

Definition:

In a strategic game player *i*'s action a_i'' weakly dominates another action a_i' if

- $u_i(a_i'',a_{-i}) >= u_i(a_i',a_{-i})$ for every list a_{-i} of the other player's action
- $u_i(a_i'',a_{-i}) > u_i(a_i',a_{-i})$ for some list a_{-i} of the other player's action

We say that a_i' is weakly dominated

Identifying dominated strategies

However, actions can also be dominated by mixed strategies:

- M is not dominated by U or D
- But M is dominated by a mixed strategy that takes D and U with equal probability.

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

Detecting Dominated Strategies

Test for detecting strictly dominated strategies a_i^j :

$$\sum_{k \in A_i} \alpha_i^k u_i(a_i^k, a_{-i}) > u_i(a_i^j, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$\alpha_i^k \geq 0, \quad \forall k \in A_i \qquad \sum_{k \in A_i} \alpha_i^k = 1$$

There exists an α_i such that the expected utility of α_i is always larger than $u_i(a_i^j, a_{-i})$, no matter what the other agents do.

- However, this not a proper linear program
- No objective, we need weak inequalities
- We will also assume that all utilities are positive (> 0)

Detecting dominated strategies

Linear Program for detecting strictly dominated strategies:

$$\begin{aligned}
\text{minimize} \quad & \sum_{k \in A_i} \alpha_i^k \\
\text{subject to} \quad & \sum_{k \in A_i} \alpha_i^k u_i(a_i^k, a_{-i}) \geq u_i(a_i^j, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& \alpha_i^k \geq 0 && \forall k \in A_i
\end{aligned}$$

- No normalizing constraint
- Minimization of summed "probabilities"

Why is it a solution to our problem?

- If a solution exists with $\sum_k \alpha_i^k < 1$ then we can add $1 - \sum_k \alpha_i^k$ to some α_i^k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)

Iterated Dominance

Similar programs can be constructed for weakly dominated strategies.

This can be done by repeatedly solving our LPs:

- By deleting strategies, other strategies might become dominated.
- Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
- Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N}(|A_i| - 1)$ steps.
- Thus we need to solve $O((n \cdot \max_i |A_i|)^2)$ linear programs.

We might reduce the action sets considerably, simplifying the use of LCP

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