David Coeurjolly

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Distance Transform

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2006

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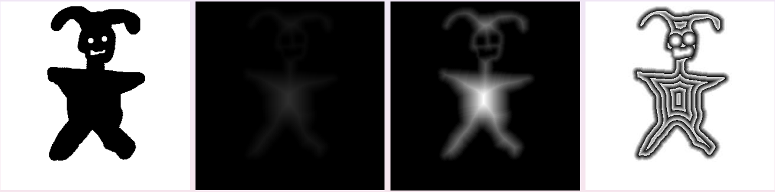
Statement

Context : region based shape analysis

Definition of the DT

Labeling of each pixel p of the object by the distance to the closest point q in the background



- Classical problem in discrete geometry
- From the DT we can compute:
  - direct measurements (*local width*,...)
  - differential estimators
  - medial axis or skeleton extraction

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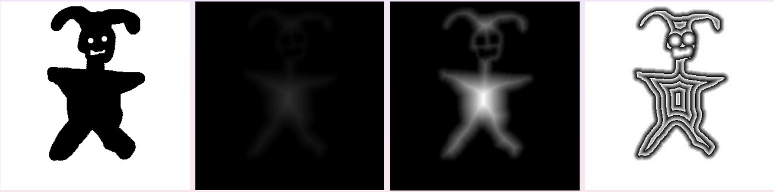

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Approximation of differential estimators

Idea

We consider the boundary of a shape as an implicit surface $f(x,y)=0$ where f is given by the DT

$$f(x,y)=0$$

$$\vec{g}(x,y)=(I_x,I_y)^T$$

$$\vec{t}(x,y)=\frac{(-I_x,I_y)^T}{\sqrt{I_x^2+I_y^2}}$$

$$k=\frac{-t^T Ht}{\|\vec{g}\|},\quad H=\begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$$

Approximation of differential estimators

Idea

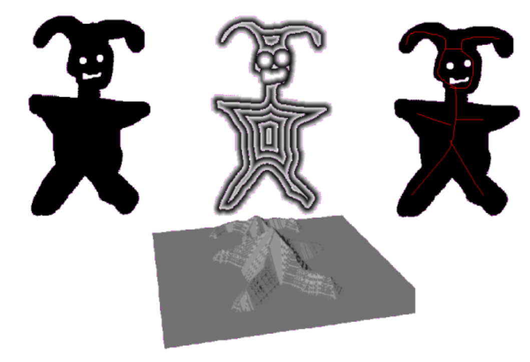
We consider the boundary of a shape as an implicit surface $f(x,y)=0$ where f is given by the DT


$$\begin{aligned} f(x,y) &= 0 \\ \vec{g}(x,y) &= (I_x, I_y)^T \\ \vec{t}(x,y) &= \frac{(-I_x, I_y)^T}{\sqrt{I_x^2 + I_y^2}} \end{aligned}$$

$$k = \frac{-t^T H t}{\|\vec{g}\|}, \quad H = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$$

Medial Axis and Skeleton extraction

Shape → DT → Medial Axis



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Discrete metrics

Constraint

Definition of a metric based on integer numbers

- Rounded metric $\lceil d \rceil$
- Approximate the Euclidean metric with integers: Chamfer masks
- DT based on sequences of chamfer masks
- Displacement based DT $(d_x, d_y)^T$
- DT based on the squared Euclidean distance SDT

Axioms of a metric

d is a metric on an non-empty set S iff:

$\forall p, q, r \in S$:

- $d(p,p) = 0$
- $d(p,q) = d(q,p)$
- $d(p,r) \leq d(p,q) + d(q,r)$

Ball

A ball of radius r with center p is the set of points q in S such that:

$$d(p,q) < r$$

Rounded Euclidean metric

If d is a metric then $\lceil d \rceil$ is also a metric

- $\lceil d(p,p) \rceil = 0$
- $\lceil d(p,q) \rceil = \lceil d(q,p) \rceil$
- $\lceil d(p,r) \rceil \leq \lceil d(p,q) \rceil + \lceil d(q,r) \rceil$ $(\forall a,b,c \in \mathbb{R}, \quad a+b \geq c \Rightarrow \lceil a \rceil + \lceil b \rceil \geq \lceil c \rceil)$

$\Rightarrow$ Yes !

And what about $\lfloor d \rfloor$ or $[d]$?

No!

Chamfer metrics

[Bor86, Ver91, Thi01, FM05]

Idea

First, we fix a set of elementary displacements and then we affect a weight to each step in order to approximate the Euclidean distance.

—	1	—
1	0	1
—	1	—

1	1	1
1	0	1
1	1	1

General masks

$(\rightarrow, \nearrow)$:

b	a	b
a	0	a
b	a	b

$(\rightarrow, \nearrow, \rightarrow + \nearrow)$:

$2b$	c	$2a$	c	$2b$
c	b	a	b	c
$2a$	a	0	a	$2a$
c	b	a	b	c
$2b$	c	$2a$	c	$2b$

$\ldots$

Chamfer metrics

[Bor86, Ver91, Thi01, FM05]

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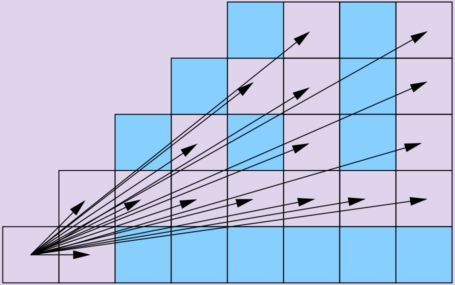
c	b	a	b	c
$2a$	a	0	a	$2a$
c	b	a	b	c
$2b$	c	$2a$	c	$2b$

...

Elementary displacements: *Farey series*

(u, v) is valid if $\frac{v}{u}$ is an irreducible fraction



Elementary displacements in a $m \times m$ mask $\Leftrightarrow$ fractions in the Farey series $\mathcal{F}_m$

How to compute the weights ?

Objectives

- The mask must form a metric:
  - $d(p,p) = 0$
  - $d(p,q) = d(q,p)$
  - $d(p,r) \leq d(p,q) + d(q,r)$
- Trade-off between the approximation of the Euclidean distance and the size of the mask

Cost function to minimize

$$d_{mask}(p,q) - d_{euc}(p,q)$$

Over a domain defined by linear constraints

- 3x3 mask : $b < 2a$ and $b > a$
- 5x5 mask : $2a < c$, $3b < 2c$ and $c < a + b$
- ...

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- ...

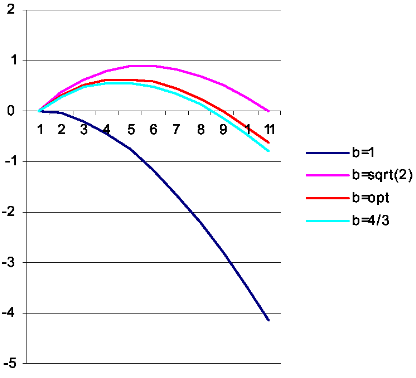
Weight analysis

3x3 masks [Bor86]

$$Diff = yb + (x-y)a - \sqrt{x^2+y^2}$$

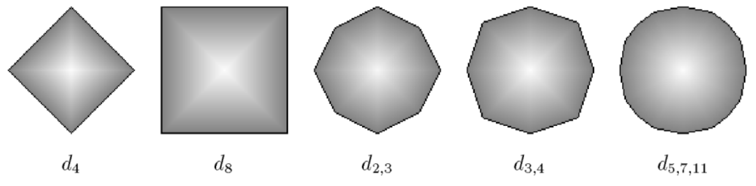
$\Rightarrow a_{opt} = 0.95509$ and $b_{opt} = 1.3693$

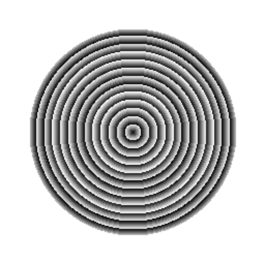
Example: $a = 1$,



- x-axis: the column $x = 10$ and $1 \leq y \leq 10$
- y-axis: the error function $yb + (x-y)a - \sqrt{x^2+y^2}$
- $b = 1$: d_8 or chessboard metric

Unit balls





(Chamfer ball images from [Thi01])

Sequence of chamfer masks

[RP66, MDKC00, Nag05]

Idea

We consider a set of chamfer masks (*e.g.* d_4 and d_8) and we switch the masks to find a better approximation of d_{euc} in a DT problem

Problems to solve

- Find the set of chamfer masks
- Find the best sequence of the masks to approximate d_{euc}

Chamfer metrics - summary

+ Simple computations
- Approximation of the Euclidean distance
- Not an isotropic representation of an object
+ DT is easy to implement

Main drawbacks

If you:

- change the shape of the pixels (*e.g.* elongated grids with factor λ),
- change size of the mask, or
- change the dimension of the image

you have to update the weights with a new optimization process

Exact Euclidean metrics

DT based on displacement vectors (d_x, d_y)

+ Error free representation
- Complex DT to obtain an error free computation
- We have to store two coordinates

DT based on the squared Euclidean distance (d^2_{euc})

+ Error free representation
+ Fast error free DT
- We have to store the square of integer numbers

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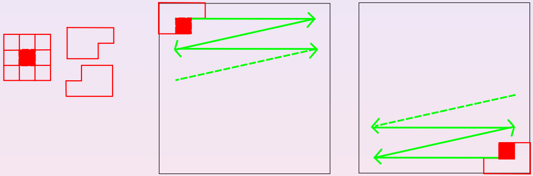
Computation of the Chamfer DT

[RP66, RP68, Bor86]

Sketch of the algorithm

Decomposition of the mask into two parts and double scan of the image to update the *min* distance



$$DT(i,j) = \min_{(k,l)\in Mask} (DT(i+k, j+l) + weight(k,l))$$

Initialization :

$$DT(i,j) = 0 \quad \text{if} \quad (i,j) \notin Object$$
$$DT(i,j) = +\infty \quad \text{if} \quad (i,j) \in Object$$

Rem: the displacement $(0,0)$ with weight 0 belongs to the mask...

Example

Using the d_{3-4} mask

3	3	3	3	3	3
3	6	6	6	6	4
3	6	9	9	8	4
3	6	9	12	8	
		3	6	4	

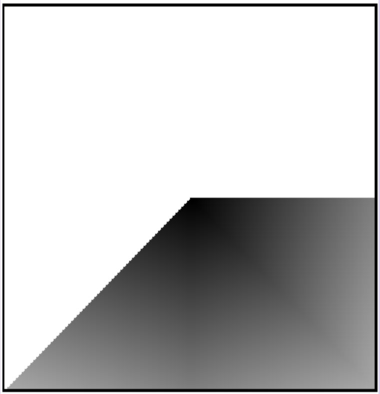
$\longrightarrow$

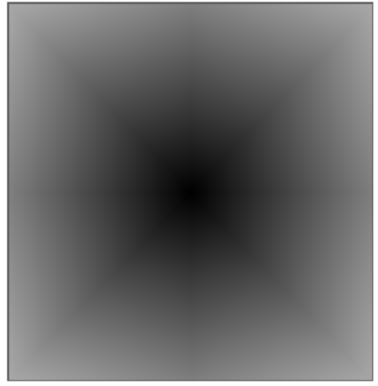
3	3	3	3	3	3
3	6	6	6	6	3
3	6	7	7	4	3
3	3	4	6	3	
		3	3	3	

(image from [Thi01])

Example

Single background pixel at the center of the image





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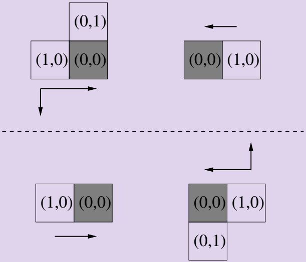
Vector DT

Idea - [Dan80, Mul92]

Store at each pixel the vector $\vec{v} = (x_p, y_p)$ such that $DT(p) = |v(p)|$

Danielson's algorithm [Dan80]

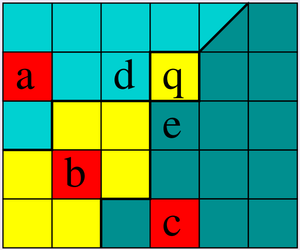
Multiple scan process with directional masks (4-connected). At each step we update $\vec{v} = (x_p, y_p)$ with the vector with the minimal distance.



- Top-down scan
- Bottom-up scan

Errors in Danielson's VDT

Local updates can lead to incorrect DT



- a, b and c are background pixels
- Danielson's algorithm: $a - d$, $c - e$ but $c - q$ or $a - q$
- Correct algorithm: $a - d$, $c - e$ and $b - q$

Algorithms exist to correct these pathological cases leading to error-free VDT [Cui99, CM99]

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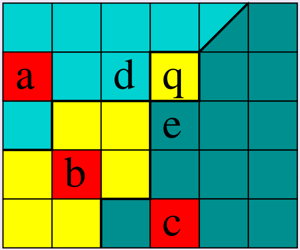

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Squared Euclidean Distance Transform

Idea

Store the square of the EDT

△ d_{euc}^2 is not a metric

Squared Euclidean Distance Transform

Idea

Store the square of the EDT

$\triangle$ d_{euc}^2 is not a metric

Squared Euclidean Distance Transform

[ST94, Hir96, MRH00]
Let P be the background of the object F, the SEDT at q in F is given by:

$$s(q) = \min_{p \in P}\{d^2_{euc}(p,q)\}$$

If we decompose the problem with $q(i,j)$, we have:

$$s(q) = \min_{p(x,y) \in P}\{(x-i)^2 + (y-j)^2\}$$

and:

$$g(i,j) = \min_x\{(x-i)^2\}$$

with

$$s(i,j) = \min_y\{(y-j)^2 + g(i,y)\}$$

⇒ dimensional decomposition of the DT computation

⇒ separable technique

Step 1: $g(i,j) = \min_x\{(x-i)^2\}$

Simple 2 scan algorithm

- Input row:

∞	■	∞	∞	∞	■	∞	∞

- $\rightarrow$:

∞	■	1	4	9	■	1	4

- $\leftarrow$:

1	■	1	4	1	■	1	4

Computational cost

- $O(n^2)$ for a $n \times n$ image
- $O(n^d)$ for a n^d image

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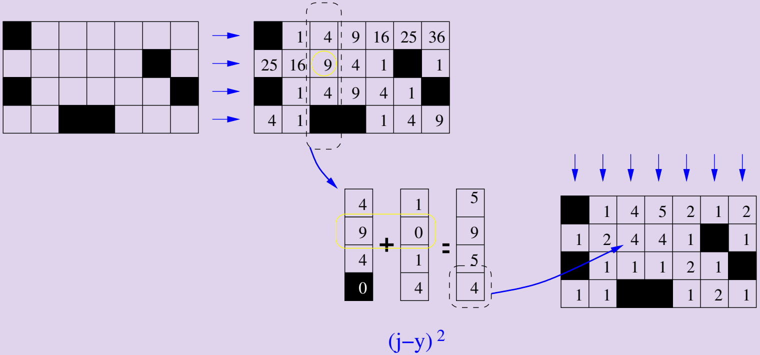
Computational cost

- $O(n^2)$ for a $n \times n$ image
- $O(n^d)$ for a n^d image

Step 2: $s(q) = \min_y \{(y-j)^2 + g(i,y)\}$

Straightforward algorithm

For each cell (i,j), scan the y to compute the *min*



$O(n^3)$ for a 2D image but we can design $O(n^2)$ algorithms [ST94]

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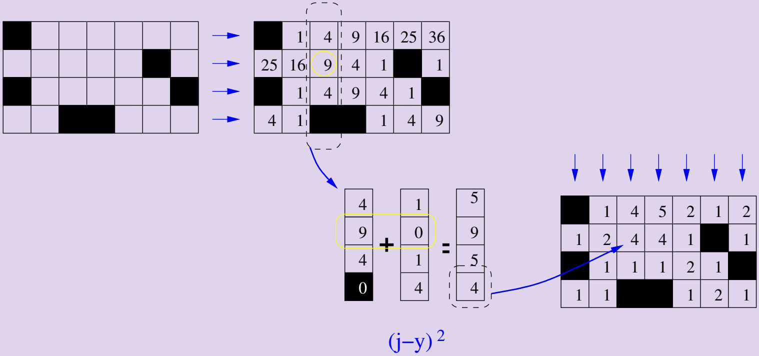
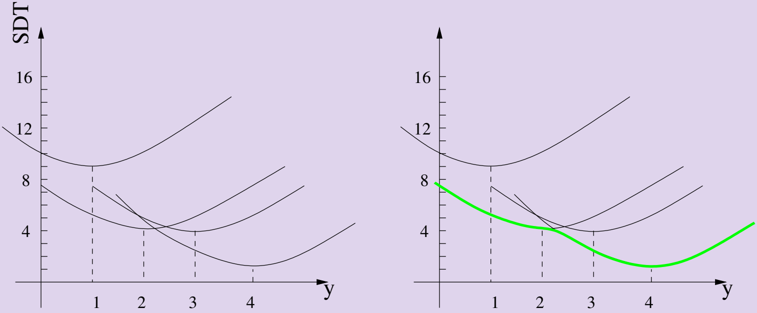


$O(n^3)$ for a 2D image but we can design $O(n^2)$ algorithms [ST94]

Optimal Step 2 algorithm: Parabola lower envelope computation

$\{(y-j)^2 + g(i,y)\}$: family of parabolas



column $[9,4,4,1]$ after step 1 and $[5,4,2,1]$ after step 2

[Hir96, MRH00]

Linear in time lower envelope computation

Sketch of the algorithm [Hir96, MRH00]

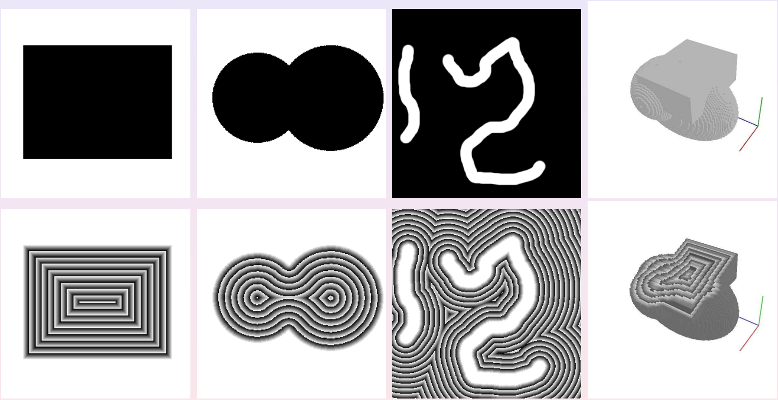
- We scan the parabolas and use a stack to store the parabolas that belong to the lower envelope
- When a new parabola is considered, this parabola may invalidate some parabolas in the stack $\rightarrow$ we pop the parabolas on the stack while the parabola on the top of the stack is invalidated by the new one
- When no more parabolas have to be considered, we compute the SDT map with the heights of the lower envelope parabolas

Computational analysis

Linear process in the number of parabolas

- $O(n^2)$ for a $n \times n$ image
- $O(n^d)$ for a n^d image

Some results of the SEDT



$\Rightarrow$ *optimal in time algorithms and error free DT whatever the dimension*

SDT - summary

- Optimal algorithms to compute error free SDT
- Trivial generalizations to d-dimensional objects
- Can handle elongated factors
- Based on a isotropic metric

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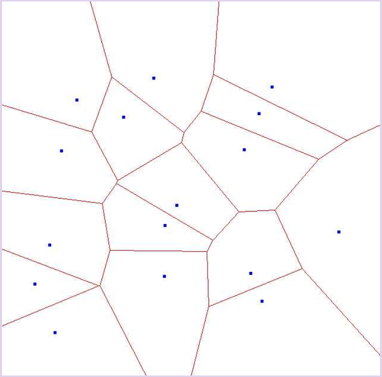
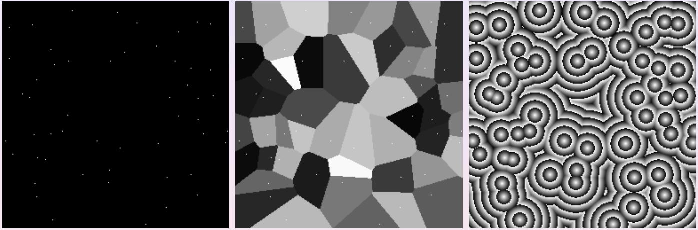
Voronoi diagram in Computational Geometry

Definition in $2-D$

Given a set of sites $\mathcal{S} = \{s_i\}$ in $\mathbb{R}^2$, the Voronoi diagram is a decomposition of the plane into cells $\mathcal{C} = \{c_i\}$ (one cell per site) such that for each point p in the open cell c_i, we have $d(p, s_i) < d(p, s_j)$ for $i \neq j$



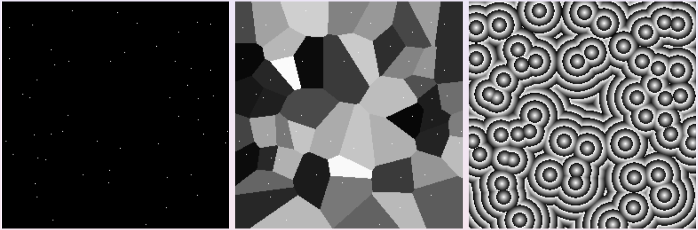
Voronoi diagrams and EDT



EDT $\Leftrightarrow$ rewriting the Voronoi diagram labeling of background points

DT algorithms based on the Voronoi diagram extraction exist to compute the EDT

Voronoi diagrams and EDT



EDT $\Leftrightarrow$ rewriting the Voronoi diagram labeling of background points

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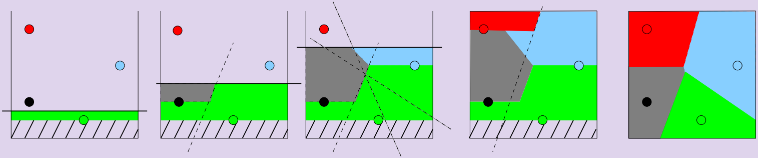
Sweep line technique to construct 2-D discrete Voronoi diagrams

[BGKW95, GM98]



- 2 scan process to construct the diagram
- When we move from a row to the next one, we use a process based on a stack of sites to update the Voronoi diagram on the current row

$O(n^2)$ for a 2-D image

Sweep line technique to construct 2-D discrete Voronoi diagrams

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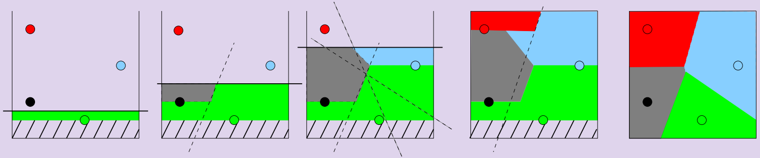

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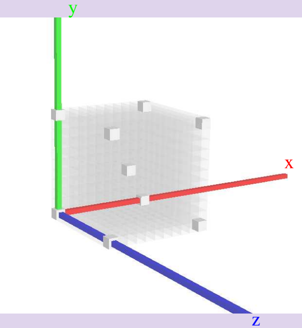
Generalizations to higher dimensions

Idea - [Coe02, CRMQR03]

Separable decomposition of the d-dimensional Voronoi diagram



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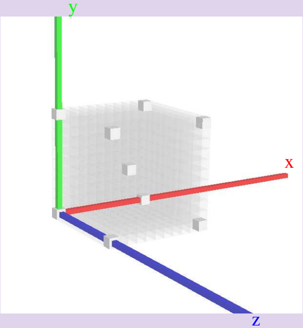

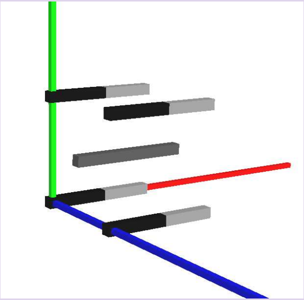

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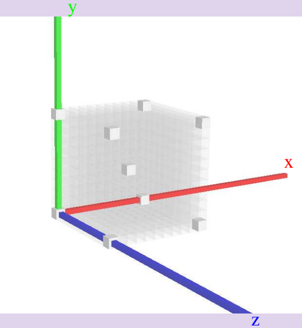

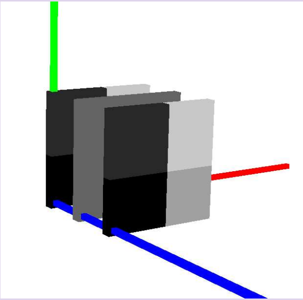

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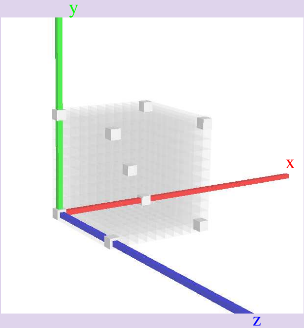
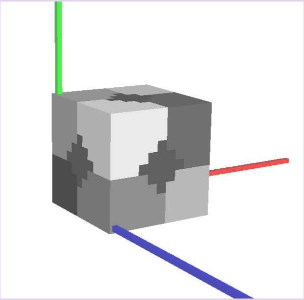




Computational cost

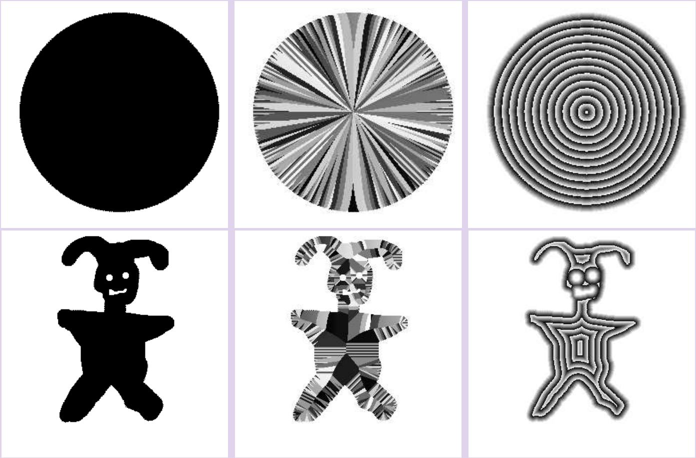
Analysis

- the problem is decomposed into several $1-D$ Voronoi diagram constructions
- each $1-D$ problem can be solved in linear time

$\Rightarrow O(n^2)$ for $2-D$ images and $O(n^d)$ for $d-D$ images

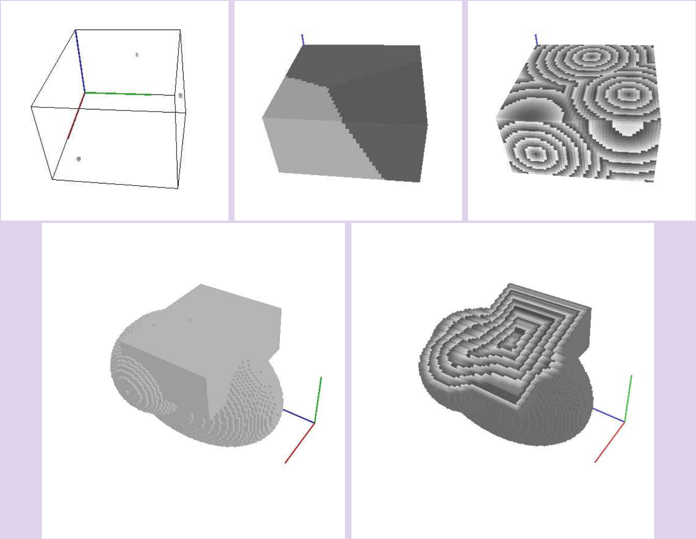
Results

2-D



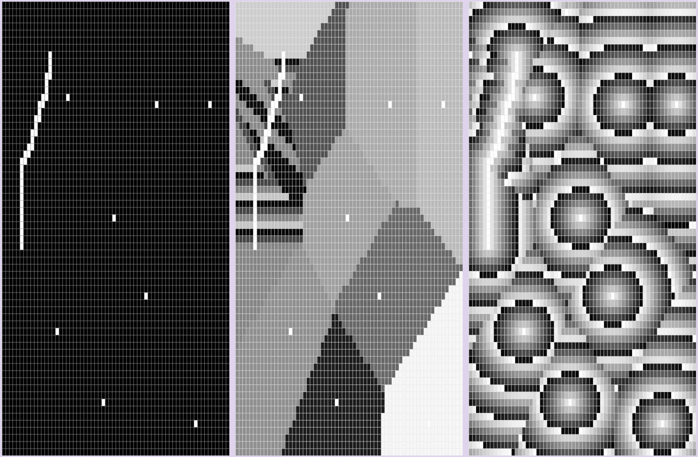
Results

3-D



Generalization for separable techniques

Anisotropic grids - [Coe02]

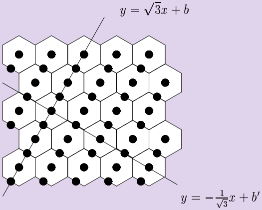


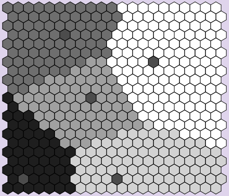
Hexagonal grids - [Coe02]

Generalization for separable techniques

Anisotropic grids - [Coe02]

Hexagonal grids - [Coe02]





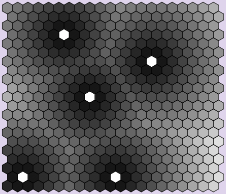


Table of contents

Conclusion

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- Links between DT and classical objects in the Computational Geometry
- We also have *Farey series* in DT problems !

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Technical Committee 18 “Discrete Geometry” of the International Association on Pattern Recognition (IAPR)

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Definitions

Applications

Discrete metrics

DT Algorithms

Chamfer based DT

Euclidean DT

Voronoi diagram based DT

Conclusion

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