



Recommendations on Statistical Randomness Test Batteries for Cryptographic Purposes

ELENA ALMARAZ LUENGO, Group of Analysis, Security and Systems (GASS), Department of Statistic and Operational Research, Faculty of Mathematical Science, Universidad Complutense de Madrid (UCM)
LUIS JAVIER GARCÍA VILLALBA, Group of Analysis, Security and Systems (GASS), Department of Software Engineering and Artificial Intelligence (DISIA), Faculty of Computer Science and Engineering, Office 431, Universidad Complutense de Madrid (UCM)

Security in different applications is closely related to the goodness of the sequences generated for such purposes. Not only in Cryptography but also in other areas, it is necessary to obtain long sequences of random numbers or that, at least, behave as such. To decide whether the generator used produces sequences that are random, unpredictable and independent, statistical checks are needed. Different batteries of hypothesis tests have been proposed for this purpose.

In this work, a survey of the main test batteries is presented, indicating their pros and cons, giving some guidelines for their use and presenting some practical examples.

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1 INTRODUCTION

In recent years, due to an increase in electronic transactions and the dissemination of sensitive information via networks, ensuring the security of the involved information has become quintessential. Therefore, there has been an increase in cryptographic studies with an aim to obtain

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Authors' addresses: E. Almaraz Luengo, Group of Analysis, Security and Systems (GASS), Department of Statistic and Operational Research, Faculty of Mathematical Science, Universidad Complutense de Madrid (UCM), Plaza de Ciencias 3, Ciudad Universitaria, Madrid, Spain, 28040; email: ealmaraz@ucm.es; L. J. García Villalba, Group of Analysis, Security and Systems (GASS), Department of Software Engineering and Artificial Intelligence (DISIA), Faculty of Computer Science and Engineering, Office 431, Universidad Complutense de Madrid (UCM), Calle Profesor José García Santesmases 9, Ciudad Universitaria, Madrid, Spain, 28040; email: javiergv@fdi.ucm.es.

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better encryption systems that, as well, require unpredictable values (random numbers), which are essential to the efficacy and success of any cryptographic tool. To precisely guarantee this security in data processing and the confidentiality of the information being transmitted, the numbers used in the generation of the different keys involved must exhibit certain properties: they must pass certain statistical tests that ensure (with a certain level of confidence) the randomness and uniformity of the sequences used, thus rendering the sequences irreproducible. Furthermore, any given series of generated values cannot be obtained by any function or mathematical model. The verification of the statistical tests is thus crucial to the development of efficacious sequences and to avoid predictable mathematical models that may potentially breach security systems.

Random numbers are the fundamental premise of simulation. It would be desirable to obtain pure random numbers that verify the conditions of independence and uniformity and that are not reproducible. In other words, pure random numbers allow the accomplishment of safe operations is the target. Commonly, generators are used to provide sequences that seem to be independent realizations of a Uniform random variable in the interval $(0,1)$ ($U(0,1)$), and we say “apparently,” because, in general, they tend to use mathematical algorithms that generate sequences that, although generated with a particular algorithm, verify the desired conditions abovementioned, namely, pseudo-random sequences. According to this, we can define two main classes of random numbers (this is a very common classification in literature. See Section 2.2. of [50] or Section 2.3. of [59]): “true” random numbers and pseudo-random numbers. There is one additional type, also defined in literature, known as quasi-random numbers.

Definition 1.1 (True Random Numbers). True random numbers are based on physical sources (entropy sources), which are sources of intrinsic natural randomness along with some processing function (see [52]). Therefore, they do not need an initial sequence (also known as “seed”) and are not expected to show periodic patterns or correlations between the obtained data.

Other advantages of true random numbers are the high level of security they provide in some applications (such as key generation) and that since there is no underlying algorithm that generates them, we should not worry about them being reproduced. On the contrary, some disadvantages are that their generation is time-consuming and computationally inefficient (hence, the speed of generators of this type of numbers cannot match the demand of bitrate that is often required), their installation and performance are troublesome and its generation is difficult. An interesting description of this type can be seen in [52].

Definition 1.2 (Pseudo-random Numbers). Pseudo-random numbers constitute a series of values that, although they are deterministically generated (as they are constructed through generation algorithms and an initial seed), have the appearance of being values that come from independent realizations of a uniform random variable [49].

The way pseudo-random numbers are generated makes them reproducible.

Definition 1.3 (Quasi-random Numbers). Quasi-random numbers are obtained through specific algorithms and the obtained sequences are distributed in a uniform way in the square or in the cube unit. For a (finite) quasi-random sequence the objective is that it fills a unit hypercube as uniformly as possible [23].

As the dimension increases, the main disadvantage of quasi-random numbers is that there are not specific hypothesis tests to evaluate the goodness of the obtained sequences [39].

In Cryptography, the most common generators used are **random number generators (RNGs)** and **pseudo-random number generators (PRNGs)**, both of which produce bit sequences (0-1). For the purposes of Cryptography, these obtained numbers must be unpredictable. This concept is

intimately related to the absence of correlations between the obtained numbers and their independence. The problem of randomness testing is becoming increasingly crucial, particularly because of the need of effective security of communications [51] due to today's widespread use of public communications, e-commerce (key management, electronic signatures), and so on.

In the case of RNGs, a non-deterministic seed (entropy source) and a generating function are necessary to produce randomness. The source of the entropy generally comes from some physical phenomenon like the noise of an electrical circuit, hardware sources, and so on. For more details you can see [52, 64].

This type of generator can potentially become problematic if the source of entropy is predictable. In order for this not to affect the sequence obtained in terms of its fundamental condition of unpredictability, it would be required to combine outputs obtained with different sources. However, this solution is not optimal, since the obtained sequences can be statistically inadequate. They could not pass the randomness or uniformity tests. Contrastingly, from a computational point of view, the generation of large sequences is ultimately time-consuming. This means that there exists a tendency to use PRNGs for their immanently better production of statistical and computational results.

As previously commented, in the case of the PRNGs, if the seed is known, given that the structure of the generator is fixed, the complete sequence could be determined, which is why special care must be taken to keep the seed secret. The seed can be generated by using a RNG.

There is a variety of studies on the generation of random numbers in the Computer Science area, Information Theory and Cryptography. Some interesting studies can be found in [18, 33] in whose works aspects related to the generation of random numbers and information theory appear. In [27] a theoretical framework is established for the generation of random sequences from the perspective of information theory.

In [60], the authors perform a detailed study on the generation of Gaussian random numbers. In [28], the authors address the uniform random number generation without information leakage and address the secure uniform random number generation with partial information leakage.

In [46], a new methodology to generate pseudo-random numbers is presented by using generative adversarial networks. To this effect, the authors validate their result by using **National Institute of Standards and Technology (NIST)** tests suite. Previously, in 1994, a method for generating pseudo-random numbers was presented (digital inversive method) [17].

However, works in which the properties of generators are studied in detail can also be found. For instance, in [21] properties of nonlinear filter of m-sequences (an important type of pseudo-random sequence generator) are studied. In [26], the authors study the power generator of pseudo-random numbers in the particular case of Blum-Blum-Shub generator. In Reference [22], the authors obtain a class of keystream generators with a large linear complexity.

There are also several works in which weaknesses of existing generators are revealed as in [3], where the author shows that the Marsaglia and Zaman's generator produces the successive digits of a rational b-adic number and, in this work, an efficient prediction algorithm for the sequences generated by this generator is obtained. In addition, there are other works that suggest improvements in the sequences obtained such as in [35], where a post-processing function to reduce or eliminate statistical weaknesses of physical random number generators is used; or in [1], where a compression method (using pseudo-chaotic systems) is applied to True Random Bit Generators.

As showcased in previous lines, the necessity of "good sequences" is a fact and a point of great interest. As a result, it is necessary to verify if the sequences meet the desired requirements. From

Table 1. Types of Errors in Hypothesis Testing

TRUE SITUATION	CONCLUSION	
	Not reject H_0	Reject H_0 (Accept H_1)
H_0 is true	No error	Type I error
H_1 is true	Type II error	No error

the statistical point of view, certain tests must be passed to ensure that the sequences obtained meet the desired conditions.

This article surveys the principal statistic batteries existing in the literature, analyzes them and gives some recommendations about their use. The rest of the article is organized as follows: in Section 2, the principal concepts about hypothesis testing are given. In Section 3, the principal batteries are explained and some examples are given. In Section 4, other hypothesis tests that are not included in the batteries described in the previous section are described. In Section 5, some recommendations about the use of the batteries are given. Finally, in Section 5, the conclusions are given.

2 HYPOTHESIS TESTING

To verify if the sequence obtained with a generator can be considered as a “true random sequence,” different hypothesis tests can be applied, given a specific level of significance to decide on this question. Randomness is a statistical property, that is, the sequence will be characterized in probabilistic terms and its properties will be compared with those expected in the case of truly random sequences.

In the following, the concept of hypothesis testing will be defined from a statistical perspective.

The probability function of a **random variable (r.v.)** X , $f_X(x, \theta)$ depends on a parameter θ that takes values in a parametric space Θ , in such a way that for each value $\theta \in \Theta$, the function $f_X(x, \theta)$ is different. A statistical hypothesis about the parameter is a guess about the specific values it may take.

The establishment of a hypothesis on the parameter implies dividing the space Θ into two separate parts, Θ_0 (formed by the set of values θ that satisfy the hypothesis) and Θ_1 (formed by the set of values θ that do not satisfy the hypothesis), so that $\Theta = \Theta_0 \cup \Theta_1$.

The hypothesis to be contrasted is called Null Hypothesis H_0 and the other, Alternative Hypothesis H_1 . The problem that arises in a hypothesis contrast is to determine which of the two hypotheses can be accepted.

A hypothesis contrast is a rule of decision by which one hypothesis or another is chosen in light of the information obtained through a sample taken from the population under study.

The solution given to the problem of contrasting the two hypotheses implies the possibility of succeeding or failing in the choice because of not knowing with certainty which is the true one. The situation is reflected in Table 1 (for more details, see [8]):

- If H_0 is true and it is not rejected, then the decision is correct.
- If H_0 is true and it is rejected, then the decision is incorrect and this error is called Type I Error.
- If H_1 is true and it is not rejected, then the decision is correct.
- If H_1 is true and it is rejected, then the decision is incorrect and this error is called Type II Error.

The above situations are unknown and uncontrolled to some extent. However, a certain degree of control can be established over them by knowing the probabilities of committing the above errors. The probability of Error Type I is defined as α and it is also known as the level of significance of the contrast, generally established in values of 0.01, 0.05, or 0.10. The probability of Error Type II is defined as β . In practice, it is usually used $1 - \beta$ or contrast power. The level of significance and contrast power are not independent.

In practice, the concept of *p-value* is often used. The *p-value* is defined as the probability of obtaining results of the test, at least, as extreme as the results actually observed. If the condition of $p\text{-value} < \alpha$ is satisfied, then H_0 will eventually be rejected.

It is interesting to note that we do not always work with an underlying distribution $f_X(x, \theta)$ of the population but this does not mean that we cannot carry out a hypothesis analysis, that is, non-parametric techniques have been developed. In other words, no hypotheses appear about a certain parameter θ of a population, but the process is based on a statistician who will not refer to the parameter. It is common to use certain sample information such as the frequency with which the data appear in the sample, its position, and so on. In this type, we can highlight goodness-of-fit contrasts or randomness contrasts.

For cryptographic purposes, hypothesis contrasting should be used to determine whether the sequence of numbers obtained by a generator satisfy the uniformity hypothesis (in the sequence of random or pseudo-random bits obtained, the expected number of zeros and ones is the same, i.e., if n is the length of the sequence, the expected number of zeros (ones) is $n/2$), scalability (if a sequence is random, any substring obtained from the first must also be random) and consistency (the behavior of the generator must be studied for different seeds, and the result of the sequence obtained in terms of its randomness must not depend on the chosen seed).

The most common tests used to decide on the uniformity of the sample are the tests χ^2 and **Kolmogorov-Smirnov (K-S)**.

The steps of the χ^2 test are as follows [49]:

- (1) Organize the n pseudo random numbers in k disjoint classes I_i , $1 \leq i \leq k$. Count the quantity of $u_i \in I_i$, denote this number by f_i . So, $f_1 + \dots + f_k = n$ is the size of the sample. Under H_0 , $P(I_i) = p_i$ where $p_1 + \dots + p_k = 1$, because the sample is composed by independent items, $f_i \sim B(n, p_i)$, where $B(n, p_i)$ represents a Binomial distribution with parameters n and p_i .
- (2) Consider the variable $V = \sum_{i=1}^k \frac{(f_i - np_i)^2}{np_i}$. The asymptotic distribution of V is chi-square with $k - 1$ degrees of freedom (χ_{k-1}^2) when $n \rightarrow \infty$.
- (3) Select $\alpha \in (0, 1)$. Get $\chi_{k-1, \alpha}^2$ from the quantiles table of the χ^2 distribution.
- (4) If $V > \chi_{k-1, \alpha}^2$, then reject H_0 . If $V \leq \chi_{k-1, \alpha}^2$, then there is no evidence for rejecting H_0 at level α .

K-S test consists of the following steps [49]:

- (1) Arrange the pseudo-random numbers u_i from order from low to high. Let $\{y_i\}$ be the resulting sequence.
- (2) Build the empirical distribution function $F_n(y_i) = i/n$, $i = 1, \dots, n$.
- (3) Compute de KS statistic: $D_n = \max_{1 \leq i \leq n} |F_n(y_i) - y_i|$, $0 \leq y_i \leq 1$.
- (4) Select $\alpha \in (0, 1)$ that represents the probability of rejecting H_0 . Get $D_{n, \alpha}$ from the table of KS quantiles.
- (5) If $D_n \leq D_{n, \alpha}$, then there is no evidence for rejecting H_0 at level α . If $D_n > D_{n, \alpha}$, then reject H_0 .

The principal differences between them are:

- K-S is easier to apply than χ^2 .
- K-S is not affected by the regroupings.
- K-S is used for small samples.
- The power of K-S test is higher than χ^2 test.
- Test χ^2 can be adapted in an easy way for the case of unknown population parameters.
- Test χ^2 can be used in case of discrete and continuous distributions.

These tests are widely used in various batteries that will be exposed in the next section. Its use is direct and indirect, i.e., in some of them, they appear as elements of the battery and, in other cases, the distributions and tests are used as tools for other tests of the batteries. To contrast the hypothesis of randomness there are also several approaches. One of which is the runs test, the best known, which will be explained later.

3 BATTERIES OF TESTS USED FOR HYPOTHESIS TESTING IN CRYPTOGRAPHY

The sequences of bits obtained by the generators are expected to pass the hypothesis tests (i.e., there is no evidence to reject the null hypothesis for the selected α level) of interest to contrast the desired cryptographical properties previously commented on.

In literature, there are several batteries that have been traditionally used in cryptography. Many authors have studied some particular examples or have given a general overview about them, for example, in [34], some of the most popular tests are described. In [51], the author explains some statistical tests depending on their focus, which is the aspect they analyze. In Section 6 of [58], or in Section 2 of [10] the most known batteries are mentioned.

In the following subsections, our objective is to explain in general terms the tests that compose the different batteries and to indicate the pros and cons of each one.

3.1 Knuth's Battery

The first test battery used in Cryptography was introduced by Knuth [34]. Its use is currently set apart and other options (other batteries) are preferred, for example, for its computational implementation, its facility in their uses, and so on. It includes 12 statistical tests that are described briefly in the following lines.

- Frequency test: it evaluates whether the elements of a sequence are uniformly distributed. K-S test is used for this purpose.
- Serial test: it is tested if each of the elements of the sequence happens with the same frequency and additionally if the pairs of possible results obtained from the sequence are independent and uniformly distributed. The test can be generalized for lists of three elements, four elements, and so on.
- Gap test: for each element in the sequence the amount of elements different from that element is counted before it happens again, the process is repeated for all its occurrences, and performs a χ^2 test on the lengths of the gaps.
- Poker test: this test considers n groups of five consecutive integers and classifies them according to the following seven patterns: the five integers that appear are all different ($abcde$), an integer is repeated twice and three different ones appear once each ($aabcd$), an integer is repeated twice, a different one is repeated twice and a third different integer appears only once ($aabbc$), an integer is repeated three times and two different ones appear once each ($aaabc$), an integer is repeated three times and a different one is repeated twice ($aaabb$),

the same integer is repeated four times and a different one appears once (*aaaab*), the same integer is repeated five times (*aaaaa*). Under the hypothesis of randomness and adjustment to a $U(0, 1)$, the probabilities of these modalities can be computed. Classes *aaaaa* and *aaaab* are often grouped together when applying the χ^2 test.

- Coupon collector's test: this test examines the amount of data needed to obtain at least one of each of the possible values. The lengths obtained are examined with the χ^2 test.
- Permutation test: this test considers blocks of k elements given a sequence of size n . Each of the blocks can have any of the $k!$ possible arrangements of these k values. If the sequence were random, then each possible arrangement would occur with equal probability: $1/k!$. The test consists of observing many blocks and compare the observed frequencies of each possible order with the expected ones by applying a χ^2 test.
- Runs test: the test is based on the length of the ascending or descending runs, understood as ascending run when each item in the series is greater than the previous one, and a descending run when each item in the series is less than the previous one. The number R of runs (+ if $u_i < u_{i+1}$, - if $u_i > u_{i+1}$) is calculated, the asymptotic distribution of R is $N(\mu = \frac{1}{3}(2n - 1), \sigma = \sqrt{\frac{1}{90}(16n - 29)})$, where $N(\mu, \sigma)$ represents the normal distribution with mean μ and variance σ^2 , consider $Z = \frac{R - \mu}{\sigma}$ whose asymptotic distribution is $N(0, 1)$, then a α level is selected, the related quantile $z_{\alpha/2}$ such that $P[Z \geq z_{\alpha/2}] = \alpha/2$ is taken and the randomness is accepted if and only if $|Z| < z_{\alpha/2}$.
- Maximum of t test: the sequence is divided into subsequences of equal length, the maximum value of each subsequence is selected and a K-S test is applied to the selected values.
- Collision test: this test is used in cases where the amount of data is significantly less than the number of categories to which they may belong, so that the probability that two or more data belong to the same category is low. Collision is defined as the case in which two data belong to the same category.
- Birthday Spacings test: m data from the sequence being tested are selected, which can be between 1 and n , they are treated as if they were m birthday in a year of n days, then the m days are ordered and the gaps between each pair are calculated and these gaps are grouped by size. The particular description will be given in the description of Diehard tests battery.
- Serial Correlation test: the serial correlation statistic is computed. This statistic measures how much one element of the sequence depends on the previous one. If the sequence were random, then the serial correlation values would have values very close to zero.
- Subsequences test: this test is applied in cases where more than one random number is required at a time.

Originally, Knuth's battery was developed for simulation applications the same as the next battery we will explain, Marsaglia's battery.

3.2 Diehard Tests Battery

In 1995, George Marsaglia [40] published a CD-ROM of random numbers along with a set of tests to determine if a sequence of numbers could be considered as a random sequence. He called this battery "Diehard tests." If a PRNG passes the tests on this battery, then it can be used in many quality scientific studies. In this battery, 16 tests are included, which are briefly described below (for more details, you can see the official site [40] or Section 3 of [24], in which a summary of the tests is given).

- Birthday spacing test: in this test m birthdays of a year of n days are selected, then a list of the time between two events (two consecutive birthdays) is made. Let j be the number of values

that appear more than once in that list, then the asymptotic distribution of j is Poisson of parameter $\lambda = m^3/(4n)$, $\mathcal{P}(m^3/(4n))$ (in practice that approximation is valid when $n \geq 2^{24}$ and $m = 2^9$). This test recommends $n > 2^{18}$ and in practice takes $n = 2^{24}$ and $m = 2^9$ so the distribution to work with in this case is $\mathcal{P}(\lambda = 2)$. Samples of size 500 are taken, and a χ^2 test of goodness of fit is done, which provides a p -value. The first test uses the 1–24 bits (counting from the left) of integers in the file. Then the file closes and reopens. The second one uses the bits 2–25, the third uses 3–26 bits, and so on up till bits 9–32. Each set of bits provides a p -value, and the nine p -values forms a sample to which a K-S test is applied. This is one of the most difficult test to pass and if a sequence passes this test, it is very common that that sequence passes most of the rest tests of this battery.

- Overlapping 5-permutation test (OPERM5): a sequence of one million 32-bit random integers is being studied. A set of 5 consecutive integers is selected and the 120 possible permutations between the 5 selected numbers must appear with equal probability along the rest of the analyzed series.
- Binary rank tests (for 31×31 , 32×32 , and 6×8 matrices): the aim of these tests is to check whether the rank of the obtained data matrices is consistent with the expected rank for matrices whose entries are random.
- Bitstream test: the data set is considered to be a bit stream $a = \{a_i\}$ and a two-letter alphabet $(0 - 1)$ is considered, so the a flow can be seen as a succession of words of 20 overlapping letters (the first word would be $a_1 a_2 \dots a_{20}$, the second would be $a_2 a_3 \dots a_{21}$ and so on). This test counts the number of words of 20 letters (20 bits) that are missing in a string of 2^{21} words of 20 overlapping letters and compares it with the expected number if the sequence were random. There are 2^{20} possible words of 20 letters. If the sequence were random with $2^{21} + 19$ bits, then the number of missing words f would be distributed according to a random variable $N(\mu = 141.909, \sigma = 428)$, then the number obtained in the sequence under study is compared with the quantile of the standard Normal distribution $Z = \frac{f - \mu}{\sigma}$. The test must be repeated 20 times.
- Tests **overlapping-pairs-sparse-occupancy (OPSO)**, **overlapping-quadruples-sparse-occupancy (OQSO)** and DNA: the OPSO test considers words of 2 letters chosen from an alphabet of 1024 letters. Each letter is determined by a dozen bits of a 32-bit integer in the sequence under analysis. The test generates 2^{21} (overlapping) words of 2 letters (from $2^{21} + 1$ “keystrokes”) and counts the number of words of 2 letters that does not appear in the sequence under analysis. If the sequence were random, then this amount should be distributed according to a random variable $N(\mu = 141,909, \sigma = 290)$; therefore, this test compares the number obtained in the concrete sample with what theoretically should be obtained through the normal random variable. This test takes 32 bits at a time from the test file and uses a designated set of ten consecutive bits. Then it restarts the file for the next 10 designated bits, and so on. The OQSO test is similar to the previous one but considering 4-letter words of a 32-letter alphabet where each letter is determined by a designated string of 5 consecutive bits of the test file, whose elements are assumed as random 32-bit integers. The DNA test considers a 4-letter alphabet (C, G, A, T) , determined by two designated bits in the sequence of random integers that is being tested. We then consider 10-letter words and test whether the number of missing words is compatible with the theoretical value that should be there, which is distributed according to a $N(\mu = 141,909, \sigma = 339)$.
- The count-the-1’s test on a stream of bytes: this test considers the file of numbers as a sequence of bytes (4 per 32-bit integer). Each byte can contain an amount of ones that varies in $\{0, \dots, 8\}$ with probabilities $1/256, 8/256, 28/256, 56/256, 70/256, 56/256, 28/256,$

8/256, and 1/256, respectively. The byte set provides an overlapping word of 5 letters, each letter can take values in the set $\{A, B, C, D, E\}$. The letters are determined by the number of ones in that byte (so, the letter to be taken depends on the number of ones in the sequence): numbers 0,1,2 correspond to A, 3 to B, 4 to C, 5 to D and finally, 6,7,8 to E. The number of overlapping words of 5 letters is 5^5 . From a string of 256,000 words of 5 letters (overlapped), counts are made in the frequencies of each word. The quadratic form in the weak inverse of the covariance matrix of the cell counts provides a χ^2 test. The test returns two p -values for both 5- and 4-letter cell counts.

- The count-the-1's test for specific bytes: this test studies the randomness of the sequence of 5-letter random words that are overlapping. It is similar to the previous one, in this case, the test is applied 25 times, first using the first byte, then the second byte and so on, and the corresponding p -values of χ^2 test are found.
- The parking lot test: in this test a car is parked (it is a circle with radius 1) in a square of size 100×100 . Then a second car is parked, then the third and so on. If a crash occurs, then the process for that particular car is replicated from the beginning choosing different random places for parking. Let n be the number of attempts and m the number of successfully parked, in practice it is chosen $n = 12,000$, simulation shows that m distributes (asymptotically) as a $N(\mu = 3,523, \sigma = 21.9)$. At the end a K-S test for 10 obtained p -values is performed to check if they come from a $U(0, 1)$ distribution.
- Minimum distance test: a square is taken again but in this case of side 10,000 and 8,000 points are chosen randomly in it. Let d be the minimum distance between the $\frac{n(n-1)}{2}$ pairs of points. If the points were evenly distributed, then d^2 would be distributed according to an exponential random variable with parameter $\lambda = 1/0.995$ ($\exp\{\lambda = 1/0.995\}$), then $1 - \exp\{-d^2/0.995\}$ should be distributed according to a $U(0, 1)$ random variable and a K-S test is applied to the resulting 100 values to check the uniformity for random points in the square.
- 3D spheres test: the aim of the test is to evaluate the randomness of triplets of sequential random numbers of uniform distribution. They are 4,000 random points chosen on a 1,000 side cube. At each point, we center a sphere sufficiently large to reach the next nearest point. The volume of the smallest sphere is distributed approximately according to an $\exp\{\lambda = 3/120\pi\}$ random variable, so the radius of the cube is distributed according to an $\exp\{\lambda = 1/30\}$ random variable, the average $\mu = 30$ is obtained through extensive simulation. This test generates 4000 spheres 20 times. Each min radius cubed results in a uniform variable through $1 - \exp\{-r^3/30\}$, then a K-S test is applied to the 20 p -values.
- The squeeze test: starting with $k = 2^{31}$, multiply by floating point values on $[0, 1)$ until $k = 1$. Let t be the number of iterations that is necessary to reduce k to 1. We repeat this test 100,000 times, and we count the number of times in which $t < 7$ and $t \geq 47$ and compare them to an expected exponential to get the p -value.
- The overlapping sums test: the sequence of integers is floated to get a sequence U_i of values from a $U(0, 1)$ distribution. Then the sums $S(j) = \sum_{i=j}^{100+j-1} U_j$ are defined. The $S(j)$ have an approximate Normal distribution, they are typified to obtain a set of independent values from a distribution of $N(0, 1)$, which are transformed into uniform variables for a K-S test. The p -values from ten K-S tests are given still another K-S test.
- The runs test: the number of runs (increasing and decreasing) is counted in the sequence of values to be tested. Runs are counted for sequences of length 10,000. The procedure is similar to the description previously explained, it is done 10 times, and then it is repeated.

- The craps test: it is considered a game consisting of 200,000 dice games, the number of wins n and the number of throws needed to finish each game l are counted. $n \sim$ (asymptotically) $N(\mu = 200,000p, \sigma = \sqrt{200,000p(1-p)})$, with $p = 244/495$. $l \in [1, \infty)$, but counts for all players greater than 21 are grouped with 21. A χ^2 test is made on the number of-throws. Each 32-bit integer taken from the file to be evaluated provides the value for the throw of a die by normalizing it to an interval $[0,1)$ and then multiplying it by 6 and adding 1 to the integer part of the result.

There are some disadvantages in this battery. The main one of them is that the parameters that appear in the tests are set by the software and cannot be modified like the allowed sample size. It is required that the sequences to be tested are in a binary file in the form of 32-bit integers (exactly). In addition, there are also difficulties regarding the technical handling of the same as, for example, in the data input [36].

3.3 Dieharder Battery

To solve the disadvantages of the previous battery, Brown et al. [6] introduced the Dieharder battery, which consists of 26 tests, including Diehard battery tests, K-S test, runs test, and the following are incorporated:

- Greatest Common Divisor Marsaglia-Tsang: this test is based on the calculus of the **Greatest Common Divisor (GCD)** of two numbers. Two random 32-bit positive integers u and v ($u, v \in [1, 2^s]$) are chosen and Euclid's algorithm is applied to them. As a result, three elements are obtained: (1) the number of iterations k that have been done to find GCD, (2) a variable length sequence of partial quotients, and (3) the GCD of u and v . If k and GCD are considered as random variables, then these are independent and identically distributed. For s big, the mass function of GCD is approximately $P[\text{GCD}(u, v) = j] = \frac{6}{\pi^2 j^2}$ and the distribution of k is approximately Binomial with parameters $n = 50$ and $p = 0.376$ (for a complete description of this test, see [41]). A standard goodness-of-fit test for χ^2 distributions is applied to both GCD and k . Deviations from randomness suggest nonconformity between empirical and theoretical distributions of k and GCD.
- Generalized minimum distance: Diehard Minimum distance test generalized to higher dimensions; see [19].
- Lagged sum test: this test calculates the mean of the sequence with n lags and compares it with the theoretical value.
- Permutations test: this test is similar to the Non-Overlapping Serial Test but with a different mapping of the input uniforms to the sample statistics. This test counts order permutations of random numbers, taken n at a time. There are $n!$ permutations equally probable and the samples are independent. In this way a simple χ^2 test on the count vector with $n! - 1$ degrees of freedom can be applied.
- Monobit: this test studies the number of zeros and ones in the sequence under study. If the sequence is random, then the amount is expected to be approximately the same. The study can be done in terms of proportions, the proportion of zeros and ones is expected to be approximately the same. This test is also known as a frequency test.
- Generalized Serial test: this test attempts to verify whether the sequences of successive numbers are equally distributed independently or not. This is a generalization of the serial test; see [25, 32, 34].
- Bit distribution test: this test performs the accumulation of the frequencies of all tuples of bits of size n (selected without overlap) in a list of random integers and compares the

empirical distribution of the sample with the theoretical (binomial) distribution by using a χ^2 test.

These tests are open source and can be used in R interface in Linux or Unix operating systems [6].

Further studies have been carried out to speed up the execution of the tests of this battery; see, for example, [62], where a solution is proposed to accelerate the statistical tests based on reconfigurable hardware, taking advantage of the parallelization of tasks and high frequencies.

To apply this battery, we should go to the *dieharder* website: <https://webhome.phy.duke.edu/rgb/General/dieharder.php> [6]. In this webpage the Dieharder battery can be downloaded. There are different versions and one can find the information about its installation and use. The files have extension *.rpm* and *.tgz*. Originally, the files are prepared to be used under Linux or Unix, but there is the possibility to work under Windows operating system through the R interface.

Next, we will show some examples of the application of this battery by using R interface. For this purpose, we need to work with the library *RDieharder* [16]. The available tests can be seen in Figure 1. Moreover, there are different and known generators that can be used in R; see Figure 2.

In Example 1, see Figure 3, we use 100 samples of numbers generated by the generator *mt19937* (Mersenne Twister 19937 generator) and seed 12345. We apply the test number 15. The *p-value* that is calculated is $p = 0.8347$, which is higher than 0.01, so the sequence clearly passes the test.

In Example 2, see Figure 4, we use 100,000 samples of numbers generated by the generator *randu* (particular linear congruential generator with parameters $m = 2^{31}$, $a = 65,539$, and $b = 0$) and seed 12345. We apply the test number 15. In this case $p\text{-value} < 2.2e^{-16}$, so the sequence clearly does not pass the test.

3.4 NIST Battery

The U.S. NIST in 2001 [52] developed a test suite called NIST battery, consisting of 15 randomness tests. The NIST battery is widely used mainly for quality certifications. As with the Dieharder battery, test parameters are fixed and this reduces its flexibility. NIST battery was developed to detect non randomness for cryptographic applications. Next, we will explain briefly the tests included in this battery, but for a complete guide it is recommended reading the official documentation [52] (also, there is a lot of bibliography in which some of these tests are explained, for example, in [57]):

- The Frequency (Monobit) test: the aim of this test is to study the proportion of the number of zeros and ones in the sequence. If this were random, then the ratio would be approximately the same, i.e., $1/2$. The other tests on the battery should be checked once this test has been passed.
- Frequency test within a block: the aim of this test is to study the proportion of ones in a block of size N . If the sequence were random, then this ratio would be expected to be $N/2$.
- The Runs test: the aim of this test is to study the runs of the sequence under study. A run of length k consists of k equal bits so that the pre-string bit and the $k + 1$ are different. If the sequence were random, then the runs of zeros and ones would be distributed uniformly.
- Tests for the Longest-Run-of-Ones in a block: this test studies the strings of ones in a block of size M . It studies whether the length of the largest streak present in the sequence is consistent with what should be expected for a random sequence.
- The Binary Matrix Rank test: this test studies the ranks of the (disjoints) submatrices formed from the sequence data. If the data were random, then it is expected that there would be no

```
> dieharderTests()
```

	names	id
1	diehard_birthdays	0
2	diehard_operm5	1
3	diehard_rank_32x32	2
4	diehard_rank_6x8	3
5	diehard_bitstream	4
6	diehard_opso	5
7	diehard_oqso	6
8	diehard_dna	7
9	diehard_count_1s_stream	8
10	diehard_count_1s_byte	9
11	diehard_parking_lot	10
12	diehard_2dsphere	11
13	diehard_3dsphere	12
14	diehard_squeeze	13
15	diehard_sums	14
16	diehard_runs	15
17	diehard_craps	16
18	marsaglia_tsang_gcd	17
19	sts_monobit	100
20	sts_runs	101
21	sts_serial	102
22	rgb_bitdist	200
23	rgb_minimum_distance	201
24	rgb_permutations	202
25	rgb_lagged_sum	203
26	rgb_kstest_test	204
27	dab_bytedistrib	205
28	dab_dct	206
29	dab_filltree	207
30	dab_filltree2	208
31	dab_monobit2	209
32	dab_birthdays1	210
33	dab_opso2	211

Fig. 1. Tests in Dieharder battery available in *RDieharder*.

dependence between the sub-strings of the original sequence. This test is included in the Diehard battery.

- The Discrete Fourier Transform (Spectral) test: this test studies the peak heights in the Discrete Fourier Transform of the sequence to detect possible periodic patterns in the sequence that would imply that the sequence under study would not be random. The objective is to detect when the number of peaks exceeding the 95% threshold is statistically different from 5%. An interesting paper about this test can be seen in [31].
- The Non-overlapping Template Matching test: this test studies the number of occurrences of pre-specified target strings to see if a significant number of a non-periodic pattern occurs. In this test, a window of m -bits is used to study a specific pattern of size m . If this pattern

```
> dieharderGenerators()
```

	names	id		names	id		names	id
1	borosh13	0	28	random256-glibc2	27	54	taus2	53
2	cmrg	1	29	random256-libc5	28	55	taus113	54
3	coveyou	2	30	random32-bsd	29	56	transputer	55
4	fishman18	3	31	random32-glibc2	30	57	tt800	56
5	fishman20	4	32	random32-libc5	31	58	uni	57
6	fishman2x	5	33	random64-bsd	32	59	uni32	58
7	gfsr4	6	34	random64-glibc2	33	60	vax	59
8	knuthran	7	35	random64-libc5	34	61	waterman14	60
9	knuthran2	8	36	random8-bsd	35	62	zuf	61
10	knuthran2002	9	37	random8-glibc2	36	63	stdin_input_raw	200
11	lecuyer21	10	38	random8-libc5	37	64	file_input_raw	201
12	minstd	11	39	random-bsd	38	65	file_input	202
13	mrg	12	40	random-glibc2	39	66	ca	203
14	mt19937	13	41	random-libc5	40	67	uvag	204
15	mt19937_1999	14	42	randu	41	68	AES_OFB	205
16	mt19937_1998	15	43	ranf	42	69	Threefish_OFB	206
17	r250	16	44	ranlux	43	70	XOR (supergenerator)	207
18	ran0	17	45	ranlux389	44	71	kiss	208
19	ran1	18	46	ranlxd1	45	72	superkiss	209
20	ran2	19	47	ranlxd2	46	73	R_wichmann_hill	400
21	ran3	20	48	ranlxs0	47	74	R_marsaglia_multic.	401
22	rand	21	49	ranlxs1	48	75	R_super_duper	402
23	rand48	22	50	ranlxs2	49	76	R_mersenne_twister	403
24	random128-bsd	23	51	ranmar	50	77	R_knuth_taocp	404
25	random128-glibc2	24	52	slatec	51	78	R_knuth_taocp2	405
26	random128-libc5	25	53	taus	52	79	empty	600
27	random256-bsd	26						

Fig. 2. Generators available in *RDieharder*.

```
> dhctest = dieharder(rng="mt19937", test=15, psamples=100, seed=12345)
> print(dhctest)
```

Diehard Runs Test

data: Created by RNG 'mt19937' with seed=12345, sample of size 100
p-value = 0.8347

Fig. 3. Example 1.

is not found, then the window is moved one position; on the contrary, if a pattern is found, then the window is reset to the bit after the found pattern, and the search starts again.

- The Overlapping Template Matching test: this test studies the number of occurrences of pre-specified target strings. As in the previous test, it uses a window of m -bits to search for a specific m -bit pattern. If the pattern is not found, then the window moves one bit position. Contrastingly, if a pattern is found the window moves only one bit before starting again.

```
> dhtest = dieharder(rng=41, test=15, psamples=100000, seed=123456)
> print(dhtest)
```

Diehard Runs Test

data: Created by RNG 'randu' with seed=123456, sample of size 100000
p-value <2.2e-16

Fig. 4. Example 2.

- Maurer's "Universal Statistical" test: this test studies the number of bits between matching patterns. The aim of this test is to detect if one sequence can be compressed substantially without losing information. A significantly compressible sequence is considered non-random. This test was introduced by Maurer in 1992 [45] and is called universal, because "*it can detect any significant deviation of a device's output statistics from the statistics of a truly random bit source when the device can be modeled as an ergodic stationary source with finite memory but arbitrary (unknown) state transition probabilities*" [45].
- The Linear Complexity test: this test studies a **linear feedback shift register (LFSR)**. If a sequence were random, then it would have long LFSRs; on the contrary if the LFSR is too short, the sequence is non-random.
- The Serial test: this test studies the frequency of all possible m -bit patterns overlapped throughout the sequence. The aim of the test is to determine if the number of occurrences of the 2^m m -bit overlapping patterns is approximately the same as would be expected for a random sequence (in that case each m -bit pattern has the same probability of appearing). In the case $m = 1$, the frequency test is obtained.
- The Approximate Entropy test: this test studies the frequency of all possible n -bit patterns superimposed along the whole sequence and compares the frequencies of two consecutive blocks (n and $n + 1$) with those that should appear in the case of a random sequence.
- The Cumulative Sums (Cusums) test: the random walk through the cumulative (adjusted) sum of -1 and $+1$ of the sequence digits, is defined. If the sequence were random, then the "excursions" of the random walk should be close to 0.
- The Random Excursions test: this test studies the number of cycles that have exactly k visits in a cumulative sum random walk defined as in the previous test. A cycle of a random walk consists of a sequence of steps of length one, taken at random that begin and return to the origin. The objective of the test is to determine if the number of visits to a certain state within a cycle deviates from what would be expected in the case of a random sequence.
- The Random Excursions Variant test: this test studies the total number of times a particular state is visited in a cumulative sum random walk. The aim of this test is to detect deviations from the expected number of visits to various states in the random walk.

The particular parameters that appear in the different tests of the NIST battery can be seen in Table 2.

This battery has been studied in detail in [56]. In particular, its authors established the minimum lengths that the bit sequences must have to be used by the battery and criticized the computational times required for the verification of each of the tests.

The battery has been studied from different perspectives. For example, in [48], the authors calculate the exact non-asymptotic distribution of the p -values generated by some tests included in

Table 2. NIST Tests and Parameters Values

NIST TESTS	Recommended sample size n (number of bites)																																				
The Frequency (Monobit) test	$n \geq 100$																																				
Frequency test within a block	$n \geq 100$ $n \geq MN$ $M \geq 20, M \geq 0.01n$ $N < 100$ Being M the block size and $N = \text{int}(n/M)$																																				
The Runs test	$n \geq 100$																																				
Tests for the Longest-Run-of-Ones in a block	If $n = 128$, then $M = 8$ If $n = 6,272$, then $M = 128$ If $n = 750,000$, then $M = 10^4$ Being M the block size pre-set in the test to accommodate to values 8, 128, and 10^4																																				
The Binary Matrix Rank test	$n \geq 38MQ$ Being M the number of rows of the matrix and Q the number of columns It is consider values $M = Q = 32$, so each sequence to be tested should consist of a minimum of 38912 bits																																				
The Discrete Fourier Transform (Spectral) test	$n \geq 1,000$																																				
The Non-overlapping Template Matching test	$m = 2 \dots 10$ preferably $m = 9$ or $m = 10$ Being m the length in bits of each template $M > 0.01n$ being M the length in bits of the substring to be tested $N = \text{int}(n/M)$ $N \leq 100$ Being N the number of independent blocks, initially fixed at $N = 8$																																				
The Overlapping Template Matching test	$n \geq 10^6$ $m = 9$ or $m = 10$ being the length in bits of the template $n \geq NM$, being M the length in bits of a substring of to be tested $M = 1,032$ in the test code Being N the number of independent blocks of n $N = 968$ in the test code																																				
Maurer's "Universal Statistical" test	The sequence of length n is divided into two segments of L -bit blocks The first segment consists of Q initialization blocks and the second one consists of K test blocks $n \geq (Q + K)L$ with $6 \leq L \leq 16, Q = 10 * 2^L, K \approx 1000 \cdot 2^L$ In particular the following recommendations are given: <table><thead><tr><th>n</th><th>L</th><th>Q</th></tr></thead><tbody><tr><td>$\geq 387,840$</td><td>6</td><td>640</td></tr><tr><td>$\geq 904,960$</td><td>7</td><td>1,280</td></tr><tr><td>$\geq 2,068,480$</td><td>8</td><td>2,560</td></tr><tr><td>$\geq 4,654,080$</td><td>9</td><td>5,120</td></tr><tr><td>$\geq 10,342,400$</td><td>10</td><td>10,240</td></tr><tr><td>$\geq 22,753,280$</td><td>11</td><td>20,480</td></tr><tr><td>$\geq 49,643,520$</td><td>12</td><td>40,960</td></tr><tr><td>$\geq 107,560,960$</td><td>13</td><td>81,920</td></tr><tr><td>$\geq 231,669,760$</td><td>14</td><td>163,840</td></tr><tr><td>$\geq 496,435,200$</td><td>15</td><td>327,680</td></tr><tr><td>$\geq 1,059,061,760$</td><td>16</td><td>655,360</td></tr></tbody></table>	n	L	Q	$\geq 387,840$	6	640	$\geq 904,960$	7	1,280	$\geq 2,068,480$	8	2,560	$\geq 4,654,080$	9	5,120	$\geq 10,342,400$	10	10,240	$\geq 22,753,280$	11	20,480	$\geq 49,643,520$	12	40,960	$\geq 107,560,960$	13	81,920	$\geq 231,669,760$	14	163,840	$\geq 496,435,200$	15	327,680	$\geq 1,059,061,760$	16	655,360
n	L	Q																																			
$\geq 387,840$	6	640																																			
$\geq 904,960$	7	1,280																																			
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$\geq 107,560,960$	13	81,920																																			
$\geq 231,669,760$	14	163,840																																			
$\geq 496,435,200$	15	327,680																																			
$\geq 1,059,061,760$	16	655,360																																			

(Continued)

Table 2. Continued

NIST TESTS	Recommended sample size n (number of bites)
The Linear Complexity test	$n \geq 106$ $500 \leq M \leq 5,000$ $N \geq 200$ Being M the length in bits of a block Being N the number of independent blocks of M bits, $n = MN$
The Serial test	$m < \text{int}(\log_2 n) - 2$ Being m the length in bits of each block
The Approximate Entropy test	$m < \text{int}(\log_2 n) - 5$ Being m the length of each block, in this case, the first block length used in the test. $m+1$ is the second block length used
The Cumulative Sums (Cusums) test	$n \geq 100$
The Random Excursions test	$n \geq 10^6$
The Random Excursions Variant test	$n \geq 10^6$

the battery, and propose some approximations. Following the same line, [55] works for the case of test items based on the binomial distribution and discover some inconsistencies in the second level tests of the tests. They therefore propose a methodology based on the Q -value as the metric of these second level tests to replace the original p -value without any further modification, and the first level tests remain unchanged. Additionally, they give the correction test for the proposed second level tests based on the Q -value.

To apply this battery, we should go to the NIST website: <https://csrc.nist.gov/Projects/Random-Bit-Generation/Documentation-and-Software> [9].

In this webpage all the information about the download and installation of this battery can be found. The source code of the files was written in ANSI C but it is possible to work under Windows operating system using an auxiliary program such as MinGW.

We will show some examples of the use of this battery. In Example 3, see Figure 5, we apply all tests of the battery of a sequence generated by generator 1 (linear congruential generator with parameters used by Fishman and Moore [20]) and 100 bitstreams.

The final result is given in a separate document .txt. In Figure 6, we show a fragment of this report. The results are represented via a table. The number of rows corresponds to the number of statistical tests applied. The number of columns, $q = 13$, are distributed as follows: columns 1-10 correspond to the frequency of p -values, column 11 is the p -value that arises via the application of a chi-square test, column 12 is the proportion of binary sequences that passed, and in column 13 the corresponding statistical test is shown. In this case, the sequence passes the tests.

Other example in which the sequence passes the test is Example 4, see Figure 7. In this case, we apply the NIST battery to an external file uploaded by the user. We use a sequence of numbers generated by *dev/urandom* and 100 bitstreams. To do this, we need to choose option 0 in the initial menu of the program.

A fragment of the final analysis report of Example 4 can be seen in Figure 8.

Now, we will use another sequence generated by a biased generator. We generate a sequence by using *urandom* in Python, and then we delete some occurrences of a certain number, so by its construction, this sequence is not random. We will apply NIST battery, and we will see the results in Example 5; see Figure 9.

A fragment of the final analysis report can be seen in Figure 10. Results that NIST interprets as non-randomness of the data are marked by an asterisk.

```

                                GENERATOR SELECTION
                                -----
                                [0] Input File
                                [2] Quadratic Congruential I
                                [4] Cubic Congruential
                                [6] Modular Exponentiation
                                [8] Micali-Schnorr
                                [1] Linear Congruential
                                [3] Quadratic Congruential II
                                [5] XOR
                                [7] Blum-Blum-Shub
                                [9] G Using SHA-1

Enter Choice: 1

                                STATISTICAL TESTS
                                -----
                                [01] Frequency
                                [03] Cumulative Sums
                                [05] Longest Runs of Ones
                                [07] Discrete Fourier Transform
                                [09] Overlapping Template Matchings
                                [11] Approximate Entropy
                                [13] Random Excursions Variant
                                [15] Linear Complexity
                                [02] Block Frequency
                                [04] Runs
                                [06] Rank
                                [08] Nonperiodic Template Matchings
                                [10] Universal Statistical
                                [12] Random Excursions
                                [14] Serial

                                INSTRUCTIONS

                                Enter 0 if you DO NOT want to apply all of the
                                statistical tests to each sequence and 1 if you DO.

Enter Choice: 1

                                Parameter Adjustements
                                -----
                                [1] Block Frequency Test - block length(M):      128
                                [2] NonOverlapping Template Test - block length(m): 9
                                [3] Overlapping Template Test - block length(m): 9
                                [4] Approximate Entropy Test - block length(m): 10
                                [5] Serial Test - block length(m):               16
                                [6] Linear Complexity Test - block length(M):    500

Select Test (0 to continue): 0
How many bitstreams? 100
    Statistical Testing In Progress.....

    Statistical Testing Complete!!!!!!!!!!!!
```

Fig. 5. Example 3.

3.5 TestU01 Battery

L’Ecuyer and Simard [36] proposed a battery called TestU01 implemented in C that is easier to apply from a computational point of view and that includes many of the main tests in the literature. It is a very extensive battery that includes many of the Diehard and NIST tests and other tests that discover problems in some generators that pass Diehard and NIST batteries successfully. It is a more flexible battery than the previous ones, the implementations are more efficient, and it can support larger sample sizes and a wider range of test parameters. Tests are available for bit strings and for real number sequences in the interval (0, 1).

It consists of six test batteries, three of which are oriented to sequences of values within the interval (0, 1) and are used to test the structure and output of the PRNGs (Small Crush, Crush, Big Crush) and the other three oriented for analyzing bit sequences (Rabbit, Alphabit, and BlockAlphabit—the

RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES												
generator is <Linear-Congruential>												
C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	P-VALUE	PROPORTION	STATISTICAL TEST
6	15	8	9	11	7	10	11	13	10	0.678686	99/100	Frequency
11	7	11	6	10	13	15	13	6	8	0.437274	99/100	BlockFrequency
6	12	17	10	3	11	10	15	9	7	0.080519	99/100	CumulativeSums
7	13	11	10	9	10	11	13	11	5	0.779188	99/100	CumulativeSums
12	10	11	8	9	8	14	11	9	8	0.935716	97/100	Runs
8	8	10	18	7	9	13	10	11	6	0.289667	100/100	LongestRun
12	9	9	8	8	7	18	10	10	9	0.455937	99/100	Rank
5	11	12	11	13	5	15	13	8	7	0.262249	100/100	FFT
13	13	10	9	12	7	11	9	8	8	0.897763	99/100	NonOverlappingTemplate
10	13	7	9	10	12	10	13	6	10	0.851383	100/100	NonOverlappingTemplate
6	15	11	6	10	8	8	13	14	9	0.419021	100/100	NonOverlappingTemplate
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	7	4	3	5	6	7	7	5	6	0.911413	56/56	RandomExcursionsVariant
7	3	5	5	6	7	3	6	10	4	0.455937	56/56	RandomExcursionsVariant
7	8	11	17	12	9	14	6	9	7	0.275709	100/100	Serial
13	8	7	9	17	8	14	8	9	7	0.304126	99/100	Serial
9	12	6	10	15	13	7	11	9	8	0.637119	99/100	LinearComplexity

The minimum pass rate for each statistical test with the exception of the random excursion (variant) test is approximately = 96 for a sample size = 100 binary sequences.

The minimum pass rate for the random excursion (variant) test is approximately = 53 for a sample size = 56 binary sequences.

For further guidelines construct a probability table using the MAPLE program provided in the addendum section of the documentation.

Fig. 6. Example 3. Fragment of the final analysis report for the sequence generated by the LCG.

latter applies the Alphabit test battery repeatedly to a generator or a binary file after rearranging the bits by blocks of different sizes (2, 4, 8, 16, 32 bits)). According to Gentle [23], the time needed to test a single generator with Big Crush can be more than ten hours on a high-end PC. However, with Small Crush it is two orders of magnitude faster, so it is advisable to start testing with Small Crush and to stop if the generator does not pass the tests. Otherwise, it would continue with Crush. If the generator does not pass the tests, then it will stop; otherwise, it will continue with Big Crush. On a 64-bit AMD Athlon processor running at 2.4 GHz, Rabbit takes about 2 seconds to test a 220-bit stream in a file and about 9 min for a 230-bit stream. Alphabit takes less than 1 second for 220-bit and about 1 minute for 230-bit [36]. The tests included in each suite in TestU01 battery can be seen in Table 4.

This battery can be downloaded in: <http://simul.iro.umontreal.ca/testu01/tu01.html> [37].

In this webpage, we can find the source files, the installation and user's guide. This battery is implemented in the ANSI C language but as in the previous subsection, it can be used in Windows. In case you want to work under Windows operating system you have to be careful to export the libraries correctly and this depends on your operating system version.

Now, we will show some examples of the use of this battery. First, we show Example 6. In this case, we use an internal example of the package TestU01, *birth1.c*. In this example, we use a

Table 3. Tests Included in Kunth, Diehard, Dieharder, NIST, ENT, and SPRNG Batteries

Battery Test	Knuth	Diehard	Dieharder	NIST	ENT	Crypt-X	SPRNG
Frequency	X		X	X		X	
Serial	X			X			X
Gap	X						X
Poker	X						X
Coupon collector's	X						X
Permutation	X						X
Runs	X	X	X	X		X	X
Maximum of t	X						X
Collision	X						X
Birthday Spacings	X	X	X				
Serial Correlation	X				X		
Subsequences	X						
OPERM5		X	X				
Binary rank		X	X	X			
Bitstream		X	X				
OPSO, OQSO and DNA		X	X				
The count-the-1's on a stream of bytes		X	X				
The count-the-1's for specific bytes		X	X				
The parking lot		X	X				
Minimum distance		X	X				
3D spheres		X	X				
The squeeze		X	X				
The overlapping sums		X	X				
The craps		X	X				
K-S			X				
Greatest Common Divisor Marsaglia-Tsang			X				
Generalized minimum distance			X				
Lagged sum			X				
Permutations			X				
Generalized Serial			X				
Bit distribution			X				
Frequency within a block				X			
Longest-Run-of-Ones in a block				X			
Spectral				X			
The Non-overlapping Template Matching				X			
The Overlapping Template Matching				X			
Maurer's "Universal Statistical"				X			
The Linear Complexity				X		X	
The (Approximate) Entropy				X	X		
Cusums				X			
The Random Excursions				X			
The Random Excursions Variant				X			
Chi-square					X		

(Continued)

Table 3. Continued

Test \ Battery	Knuth	Diehard	Dieharder	NIST	ENT	Crypt-X	SPRNG
Arithmetic Mean					X		
Monte Carlo Value for π					X		
Binary derivative						X	
Change point						X	
Sequence complexity						X	
Equidistribution							X
Sum of Distributions							X
Ising Model							X
Random Walk (2D)							X

LCG with parameters $m = 2,147,483,647$, $a = 397,204,094$, $b = 0$, and seed $s = 12,345$, the result we obtain of the analysis can be seen in Figure 11. It is important to see how the situation changes if the number of births does if. If $n = 1,000$, then the p -value is $0.04 > \alpha$, and when $n = 10,000$, the p -value $< \alpha$.

Now, we modify the file *birth1.c*, and we include the library *bbattery.h* to apply Small Crush. This is our Example 7. The results can be seen in Figure 12.

In Example 8, we apply Small Crush to a new file. In this case, we use a new generator by combining two LCGs (using the function *unif01_CreateCombAdd2*), the first one with parameters $m = 2^{31} - 1$, $a = 630,360,016$, $b = 0$ and seed $s = 12,345$ (used in some FORTRAN) and the other with parameters $m = 2,147,483,563$, $a = 16,807$, $b = 0$, and $s = 12,345$ (used, for example, in APL, IMSL, and SIMPL/I). The results are shown in Figure 13, the program prints a convenient summary: the names of all tests that produce a p -value outside the $[0.1, 0.99]$ interval.

3.6 ENT Battery

There are other less used batteries such as ENT battery, proposed by Walker [65], which is used by the random.org website. ENT claims to test for simulation and cryptographic purposes. ENT battery has two modes of working binary and byte, depending on the mode, different statistics will be calculated and displayed. This battery includes the following tests (Observation: from the statistical point of view, there is one test, χ^2 and the others are measures indicators):

- Entropy: this test calculates the entropy of the sequence under study. If the sequence were random, then it should be rich in entropy. The maximum theoretical entropy of a long sequence is 8 bits per byte, in byte mode, and one bit per bit in binary mode.
- Chi-square Test: this test computes the frequency of the symbols and compares it with the frequency expected in a uniform distribution this is done by a χ^2 statistic.
- Arithmetic Mean: the arithmetic mean of the sequence values is calculated. If the sequence were random, then the expected value would be in 0.5 in binary mode and 127.5 in byte mode.
- Monte Carlo Value for π : in this test, the sequence of numbers is interpreted as coordinates in a square and the number of points that fall within the circle inscribed in that square is counted. That number is taken as an estimation of π . If the sequence were not random, then the approximation to the true value would not be good.
- Serial Correlation Coefficient: the correlation between two consecutive symbols (bits or bytes) is calculated. This correlation is expected to be low in random sequences.


```
C: \Users\AL\Documents\ sts-2_1_2 \ sts-2.1.2 > assess.exe 1000000

          GENERATOR SELECTION
          -----

          [0] Input File           [1] Linear Congruential
          [2] Quadratic Congruential I [3] Quadratic Congruential II
          [4] Cubic Congruential     [5] XOR
          [6] Modular Exponentiation [7] Blum-Blum-Shub
          [8] Micali-Schnorr         [9] G Using SHA-1

Enter Choice: 0

User Prescribed Input File: C: \Users\AL\n0

          STATISTICAL TESTS
          -----

          [01] Frequency           [02] Block Frequency
          [03] Cumulative Sums      [04] Runs
          [05] Longest Runs of Ones [06] Rank
          [07] Discrete Fourier Transform [08] Nonperiodic Template Matchings
          [09] Overlapping Template Matchings [10] Universal Statistical
          [11] Approximate Entropy  [12] Random Excursions
          [13] Random Excursions Variant [14] Serial
          [15] Linear Complexity

INSTRUCTIONS

Enter 0 if you DO NOT want to apply all of the
statistical tests to each sequence and 1 if you DO.

Enter Choice: 1

          Parameter Adjustements
          -----

          [1] Block Frequency Test - block length(M):      128
          [2] NonOverlapping Template Test - block length(m): 9
          [3] Overlapping Template Test - block length(m): 9
          [4] Approximate Entropy Test - block length(m): 10
          [5] Serial Test - block length(m): 16
          [6] Linear Complexity Test - block length(M): 500

Select Test (0 to continue): 0
How many bitstreams? 100

Input File Format:
  [0] ASCII-A sequence of ASCII 0's and 1's
  [1] Binary-Each byte in data file contains 8 bits of data

Select input mode: 1

Statistical Testing In Progress.....

Statistical Testing Complete!!!!!!!!!!!!
```

Fig. 7. Example 4.

This battery can be downloaded in: <https://www.fourmilab.ch/random/> [65]. Installation instructions (to be done on Unix and Windows) can be found and the guidelines for its use and interpretation.

Now, we show Example 9. This is an application of the ENT battery to an internal file available in its designed software, the sequence *ent.c*. This file has many non-random patterns (for example, the character space will be extremely overrepresented, and reserved words like *int* will be repeated many times). The results are shown in Figure 14.

RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES												
generator is <Linear-Congruential>												
C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	P-VALUE	PROPORTION	STATISTICAL TEST
9	11	13	11	5	12	7	12	15	5	0.319084	97/100	Frequency
9	13	6	8	11	8	14	6	14	11	0.494392	98/100	BlockFrequency
11	11	9	14	13	8	13	3	6	12	0.275709	97/100	CumulativeSums
10	12	12	10	6	10	9	9	13	9	0.935716	98/100	CumulativeSums
6	8	16	7	11	13	6	9	12	12	0.350485	100/100	Runs
8	4	12	10	11	9	12	14	10	10	0.678686	99/100	LongestRun
12	12	12	9	10	11	12	9	5	8	0.851383	98/100	Rank
9	12	12	11	11	7	11	9	13	5	0.779188	99/100	FFT
13	11	10	12	14	9	9	6	10	6	0.699313	98/100	NonOverlappingTemplate
11	10	5	11	10	8	10	11	16	8	0.616305	99/100	NonOverlappingTemplate
10	11	3	15	9	9	9	8	13	13	0.350485	98/100	NonOverlappingTemplate
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	6	3	4	5	3	5	3	8	9	0.383827	53/54	RandomExcursionsVariant
7	7	1	2	9	2	8	5	6	7	0.108791	53/54	RandomExcursionsVariant
7	3	5	4	3	6	3	5	11	7	0.236810	52/54	RandomExcursionsVariant
5	5	6	3	1	9	4	7	8	6	0.319084	52/54	RandomExcursionsVariant
4	7	2	7	4	5	5	6	8	6	0.739918	52/54	RandomExcursionsVariant
10	9	15	8	7	6	15	12	8	10	0.455937	100/100	Serial
9	8	8	4	13	20	8	9	11	10	0.066882	99/100	Serial
13	10	9	4	11	10	12	9	9	13	0.719747	100/100	LinearComplexity

The minimum pass rate for each statistical test with the exception of the random excursion (variant) test is approximately = 96 for a sample size = 100 binary sequences.

The minimum pass rate for the random excursion (variant) test is approximately = 51 for a sample size = 54 binary sequences.

For further guidelines construct a probability table using the MAPLE program provided in the addendum section of the documentation.

Fig. 8. Example 4: Fragment of the final analysis report for the sequence n0 generated by *dev/urandom*.

Now, we apply the ENT battery for a sequence of 1,000,000 numbers generated by using a LCG with parameters $m = 2^{48}$, $a = 25,214,903,917$, and $b = 11$ with seed 123456789, Example 10. The results of the application of ENT battery can be seen in Figure 15 and as it can be seen the sequence is not good from the perspective of randomness.

In Example 11, we use the sequence n0 previously used in Example 4, and we apply ENT battery. The results can be seen in Figure 16. As it can be seen these results are compatible with the randomness of the sequence.

Despite the speed and simplicity of the ENT battery, it has certain weaknesses. For instance, it presents some problems of dependence between tests (for example, Entropy vs. Chi-square Test among others) and problems related to poorly designed generators with the presence of serial correlation in which the battery does not detect that the numbers obtained are “bad” (degenerated by design). Let us show this in Example 12. In this case, we use the generator defined in Algorithm 1 (which generates non-random sequences), we can pass with good results the ENT battery as we can see in Figure 17.

```

C: \Users\AL\Documents\ sts-2_1_2 \ sts-2.1.2 > 8000000

-----
GENERATOR SELECTION
-----

[0] Input File                [1] Linear Congruential
[2] Quadratic Congruential I  [3] Quadratic Congruential II
[4] Cubic Congruential        [5] XOR
[6] Modular Exponentiation    [7] Blum-Blum-Shub
[8] Micali-Schnorr            [9] G Using SHA-1

Enter Choice: 0

User Prescribed Input File: C: \Users\AL\epsilon0_0.bin

-----
STATISTICAL TESTS
-----

[01] Frequency                [02] Block Frequency
[03] Cumulative Sums          [04] Runs
[05] Longest Runs of Ones     [06] Rank
[07] Discrete Fourier Transform [08] Nonperiodic Template Matchings
[09] Overlapping Template Matchings [10] Universal Statistical
[11] Approximate Entropy      [12] Random Excursions
[13] Random Excursions Variant [14] Serial
[15] Linear Complexity

INSTRUCTIONS

Enter 0 if you DO NOT want to apply all of the
statistical tests to each sequence and 1 if you DO.

Enter Choice: 1

-----
Parameter Adjustements
-----

[1] Block Frequency Test - block length(M):      128
[2] NonOverlapping Template Test - block length(m): 9
[3] Overlapping Template Test - block length(m): 9
[4] Approximate Entropy Test - block length(m): 10
[5] Serial Test - block length(m): 16
[6] Linear Complexity Test - block length(M): 500

Select Test (0 to continue): 0
How many bitstreams? 100

Input File Format:
[0] ASCII-A sequence of ASCII 0's and 1's
[1] Binary-Each byte in data file contains 8 bits of data

Select input mode: 1

Statistical Testing In Progress.....

Statistical Testing Complete!!!!!!!!!!!!

```

Fig. 9. Example 5.

Now it is an issue of interest to show a comparison of the described batteries. In Table 5, we summarize some advantages and disadvantages of the previous test batteries.

3.7 Other Batteries of Statistical Tests

Other batteries are, for example, the one proposed by Vattulainen et al. [63], Cryp-X [7], or SPRNG [44]. Vattulainen et al. in 1995 proposed a battery based on the ISING model and random walks on

RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES													
generator is <Linear-Congruential>													
C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	P-VALUE	PROPORTION	STATISTICAL TEST	
1	92	1	0	0	0	2	1	1	2	0.000000 *	99/100	Frequency	
0	2	2	0	91	0	2	0	2	1	0.000000 *	100/100	BlockFrequency	
1	91	2	1	1	0	3	0	0	1	0.000000 *	99/100	CumulativeSums	
1	91	1	1	2	0	0	1	3	0	0.000000 *	99/100	CumulativeSums	
1	0	92	0	1	2	1	1	0	2	0.000000 *	99/100	Runs	
91	2	2	1	1	0	2	0	1	0	0.000000 *	100/100	LongestRun	
0	0	1	91	0	1	1	4	0	2	0.000000 *	100/100	Rank	
0	1	92	1	2	1	1	1	1	0	0.000000 *	100/100	FFT	
1	2	91	0	1	0	2	0	2	1	0.000000 *	99/100	NonOverlappingTemplate	
1	2	90	1	2	2	1	1	0	0	0.000000 *	100/100	NonOverlappingTemplate	
:	:	:	:	:	:	:	:	:	:	:	:	:	:
<p>The minimum pass rate for each statistical test with the exception of the random excursion (variant) test is approximately = 96 for a sample size = 100 binary sequences.</p> <p>The minimum pass rate for the random excursion (variant) test is approximately = 7 for a sample size = 8 binary sequences.</p> <p>For further guidelines construct a probability table using the MAPLE program provided in the addendum section of the documentation.</p>													

Fig. 10. Example 5: Fragment of the final analysis report for the biased sequence.

ALGORITHM 1: Degenerated generator

```

for  $1 \leq v \leq 16$  do
  for  $0 \leq i \leq 255$  do
    for  $0 \leq j \leq 255$  do
      if  $j = 250$  and  $i \bmod 4 = 0$  then
         $output \leftarrow i;$ 
         $output \leftarrow 251;$ 
      else  $output \leftarrow i;$ 
         $output \leftarrow j;$ 
      end
    end
  end
end

```

lattices, including the Autocorrelation, Cluster, n-block and random walk tests. The Cryp-X battery was introduced by the Information Security Research Center at Queensland University of Technology for commercial purposes. This battery includes 6 tests: Binary derivative, Change point, Frequency, Linear complexity, Runs and Sequence complexity. Crypt-X supports stream ciphers, block ciphers, and keystream generators. The **Scalable Parallel Random Number Generators Library (SPRNG)** battery implements a number of “parallel” versions of the tests proposed in the Knuth battery [34], as well as the “inherently parallel” sum-of-independent-distributions test. It

Table 4. Batteries and Test Included in TestU01

Battery Test	Small Crush (10 tests)	Crush (96 tests)	Big Crush (106 tests)	Rabbit (38 tests)	Alphabit (17 tests)	BlockAlphabit (17 tests)
Birthday Spacings	X	X	X			
Collision	X	X	X			
Coupon Collector	X	X	X			
Gap	X	X	X			
Hamming Independence	X	X	X	X	X	X
Maximum of t	X	X	X			
Random Walk One	X	X	X	X	X	X
Rank of a Binary Random Matrix	X	X	X	X		
Simplified Poker	X	X	X			
Weighted Distribution	X	X	X			
Appearance Spacings		X	X	X		
Autocorrelation		X	X	X		
Close Pairs Bit Match		X		X		
Closest Pairs		X	X			
Collision-Permutation		X	X			
Fourier 1				X		
Fourier 3		X	X	X		
GCD		X	X			
Hamming Correlation		X	X	X	X	X
Hamming Weights		X	X	X		
Lempel Ziv		X	X	X		
Linear Complexity		X	X	X		
Longest Head Run		X	X	X		
Periods in Strings		X	X	X		
Permutation		X	X			
Runs		X	X	X		
Runs of Bits		X	X			
Sample Correlation		X	X			
Sample Mean		X	X			
Sample Product		X	X			
Savir2		X	X			
Serial		X	X			
Sumcollector		X	X			
Multinomial Bits				X	X	X

also implements other tests, based on the ISING model and random walk tests [43]. The SPRNG battery is suitable for applications in parallel Monte Carlo simulation. This battery is widely used, some institutions that use it are (see <http://www.sprng.org/>): Aeronautical Systems Center Major Shared Resource Center, Department of Defense, Los Alamos National Lab, National Center for Computational Sciences, **Computer Services for Academic Research (CSAR)** of UK National **High Performance Computing (HPC)** Service, United Kingdom, Irish Centre for High-End Computing, Colorado School of Mines, University of California San Diego, Ateneo de Manila University, Philippines among others.

```

$ birth1.exe
*****

HOST =

ulcg_CreateLCG: m = 2,147,483,647, a = 397,204,094, c = 0, s = 12,345

smarsa_BirthdaySpacings test:
-----
N = 1, n = 1,000, r = 0, d = 10,000, t = 2, p = 1

Number of cells =  $d^t$  = 100,000,000
Lambda = Poisson mean = 2.5000
-----

Total expected number = N*Lambda: 2.50

Total observed number: 6

p-value of test: 0.04
-----

CPU time used : 00:00:00.00

Generator state:
s = 1,858,647,048

*****

HOST =

ulcg_CreateLCG: m = 2,147,483,647, a = 397,204,094, c = 0, s = 12,345

smarsa_BirthdaySpacings test:
-----
N = 1, n = 10,000, r = 0, d = 1,000,000, t = 2, p = 1

Number of cells =  $d^t$  = 1,000,000,000,000
Lambda = Poisson mean = 0.2500
-----

Total expected number = N*Lambda: 0.25

Total observed number: 44

p-value of test: 9.5e-82 *****
-----

CPU time used : 00:00:00.00

Generator state:
s = 731,506,484

```

Fig. 11. Example 6: Results of the analysis of *birth1.c* file.

A comparative between tests included in the previous described batteries can be seen in Table 3.

In addition to the previously mentioned tests, there are others that are not included in the explained batteries. For example, the **Strict Avalanche Criterion (SAC)** test proposed by Hernández et al. [29] the adaptive χ^2 test the Book Stack and Order tests propose by Ryabko et al. [53, 54], randomness test based on random walk proposed by Doganaksoy et al. [12] or based on Golomb's

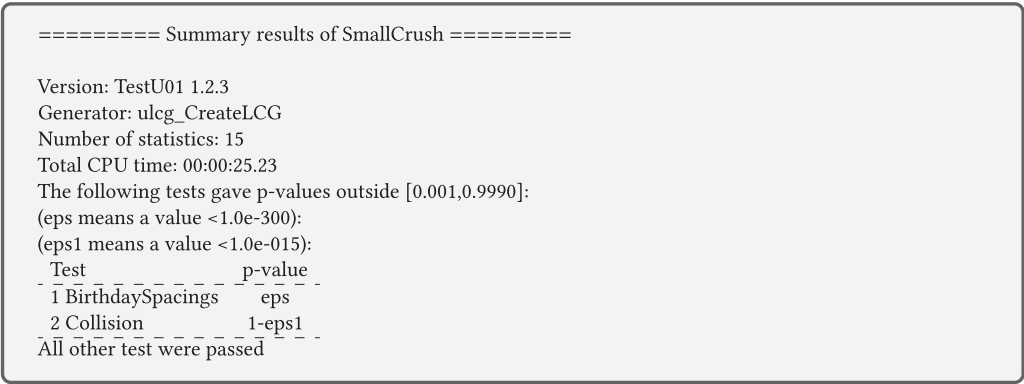


Fig. 12. Example 7: Results of the analysis of the modified *birth1.c* file.

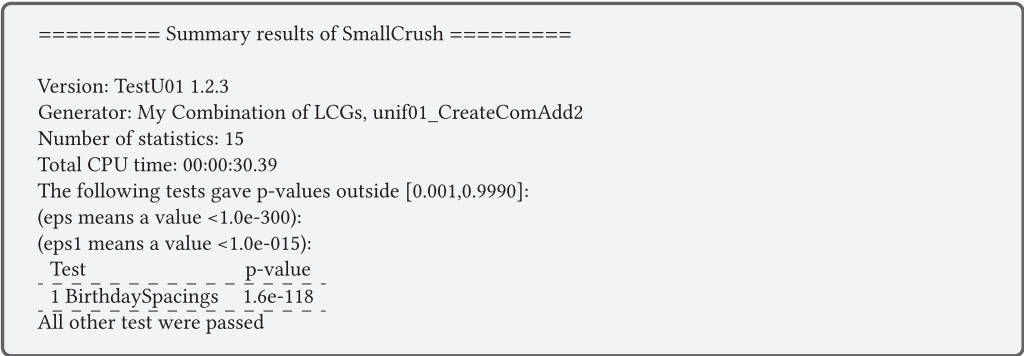


Fig. 13. Example 8: Results of the analysis of the combining generator.

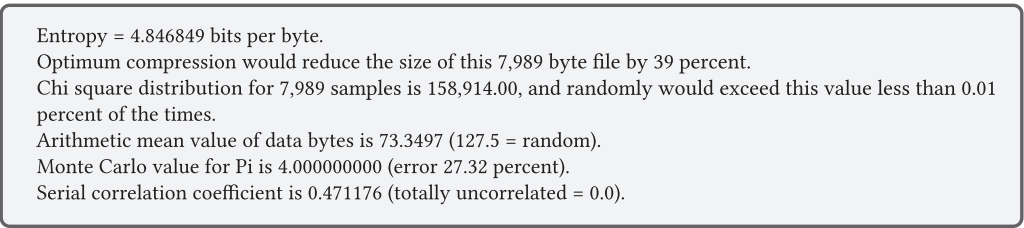


Fig. 14. Example 9: ENT results for the sequence *ent.c*.

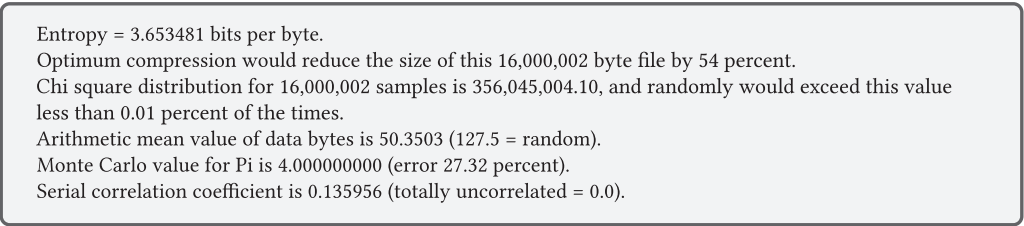


Fig. 15. Example 10: ENT results for the sequence *u* generated by a LCG.

Entropy = 7.999987 bits per byte.
 Optimum compression would reduce the size of this 15,000,000 byte file by 0 percent.
 Chi square distribution for 15,000,000 samples is 263.59, and randomly would exceed this value less than 34.26 percent of the times.
 Arithmetic mean value of data bytes is 127.5251 (127.5 = random).
 Monte Carlo value for Pi is 3.140208000 (error 0.04 percent).
 Serial correlation coefficient is -0.000196 (totally uncorrelated = 0.0).

Fig. 16. Example 11: ENT results for the sequence n0 generated by *dev/urandom*.

Entropy = 7.999912 bits per byte.
 Optimum compression would reduce the size of this 2097152 byte file by 0 percent.
 Chi square distribution for 2,097,152 samples is 256.00, and randomly would exceed this value less than 47.06 percent of the times.
 Arithmetic mean value of data bytes is 127.5005 (127.5 = random).
 Monte Carlo value for Pi is 3.140925540 (error 0.02 percent).
 Serial correlation coefficient is 0.000000 (totally uncorrelated = 0.0).

Fig. 17. Example 12: ENT results for a biased sequence generated by Algorithm 1.

postulates [14] and topological binary test proposed by [2]. An interesting paper that uses and describes some of these tests is [11].

4 SOME RECOMMENDATIONS

As described above, there are numerous test batteries to be applied, being the most recognized the Dieharder, the NIST and the TestU01. Many authors have studied different generators, properties, and so on, by using these principal batteries, see, for example, [4, 38, 42], in which the authors use these batteries to verify the goodness of the (random) generated sequences (in [42], the authors also use ENT and Diehard batteries). Sometimes, there are only a few tests of a battery, which are applied, for example, in [5] the authors apply the NIST tests: Frequency test, Cumulative Sums test and Runs test. In [30] the authors use TestU01 battery for testing randomness of the pseudo-random numbers generated by chaotic maps, and, to obtain more neutral test results, in [61] the authors use NIST battery to verify the goodness of the sequences they obtain by using a QRNG.

In other cases there is some particular test or measure of these batteries in which authors are interested. For example, in [66] the authors review and discuss the entropy estimators including the NIST SP 800-90B estimators and neural network-based estimators paying attention to their limitations on the time-varying data. In addition, they propose a scheme for the estimation of the entropy adopting change detection strategies to deal with this problem.

Which tests are the good ones? Which is the best battery? Do we have to use the same tests for true random numbers and for pseudo-random numbers? These questions have no general answer. The essential point is to be able to find a criterion that will tell us which battery is better, when to use one instead of the other, if there are substantial differences between them, and so on. Some of these questions have been dealt with in previous lines. We have described the tests that are included in each of them, and we have been able to verify that many of them are present in several of them, and we have indicated which test should be passed as a necessary (but not sufficient)

condition of randomness (runs test) or in what order, for example, one should work with TestU01. Unfortunately, no defined criteria exist to be able to answer the other questions raised, such as which battery is better. From a practical perspective and given that to obtain quality certification of products it is necessary the qualification of the US National Institute of Standards and Technology, we consider the use of the battery they propose of great importance (although, as we will see later, it also has some aspects that need to be improved). On the other hand, the Dieharder battery in its latest version has incorporated numerous tests included in the NIST battery and intends in future updates to include the entire NIST battery in addition to any tests considered relevant for testing randomness. From our point of view, the objective is good, but it does not go into detail about whether all the tests are actually necessary, whether there may be redundancies between them, and so on. In other words, it is not enough to build a compendium of all the tests that can be defined, but rather an analysis of them should be carried out to create a battery that contains the right and necessary number of tests that should be used. TestU01 also implements some tests included in other batteries such as Diehard or NIST batteries, and it can be considered a kind of platform on which almost all test could be implemented. Also, TestU01 battery is more flexible, the mode of implementation is more efficient, the battery can work with samples of big sizes and allows the user to work with more test parameters than other batteries.

That is why we will focus on the NIST and TestU01 batteries, but with some observations to take into account.

NIST provides guidance ([52], Section 5) on how to interpret the results obtained after the application of the tests and states that in order for the data to be considered random, all tests on the battery must be passed. This condition is too rigid, since, among other situations, truly random numbers have a high probability (80%) of failing at least one of the NIST battery tests. However, there are some statistical reasons that allow us to relax this restriction we will explain later.

NIST battery process the input data and gives the results of the analysis in several documents. We will pay attention to the Final Analysis Report file. This is a document .txt in which it appears the following information ([52], Section 5.7.):

- Ten columns whose entries correspond to the number of p -values that fall within intervals: $[0, 0.1)$, $[0.1, 0.2)$, ..., $[0.9, 1)$.
- One column named P-value in which the result for uniformity testing p -values computed for a given test are presented.
- One column named Proportion in which the proportion of sequences that pass a given test is presented.
- One column named Test in which the applied test is noticed.

The p -values computed by a single test should be distributed as a $U(0, 1)$ random variable. NIST uses one sample test χ^2 to verify the uniformity of the p -values. Note that χ^2 test works well when the number of tested sequences is at least 55. The values in the P-value column represent the p -value of this uniformity test. A computed small p -value indicates a problem of the generator, but it is very difficult to identify exactly which problem it is. The NIST battery documentation recommends taking a level $\alpha = 0.0001$ for this test. As it is known, the smaller the value of α , the higher the value of β (Type-II error). From a practical point of view, a small value of β is more important than a small value of α , that is, the probability of accepting a bad generator is wanted to be small (we need small values of β). Hence, from this perspective, we consider that NIST requirement should be relaxed and consider values $\alpha = 0.01$ or $\alpha = 0.001$.

In column Proportion, the proportion of sequences that pass a test is represented. The probability that a random sample passes a test is equal to $1 - \alpha$; for multiple sequences, this probability

can be approximated by $1 - \alpha$. According to the NIST battery documentation ([52], Section 4.2.1.), the proportion of passing sequences should fall into the interval

$$\left((1 - \alpha) - 3\sqrt{\frac{\alpha(1 - \alpha)}{k}}, (1 - \alpha) + 3\sqrt{\frac{\alpha(1 - \alpha)}{k}} \right), \quad (1)$$

k being the number of tested sequences. This interval is derived by using the Moivre-Laplace theorem, which assesses that the statistic \hat{p} (the approximation of the proportion) has an asymptotic $N(\mu, \hat{\sigma})$ distribution, where $\hat{\sigma} = \sqrt{\frac{\hat{p}(1 - \hat{p})}{k}}$ is the (estimated) standard deviation. In this case, if we set a confidence level $1 - \alpha$, then a constant l such that $P[-l \leq Z \leq l] = 1 - \alpha$ with $Z \sim N(0, 1)$, can be found to define the interval

$$\left(\hat{p} - l\sqrt{\frac{\hat{p}(1 - \hat{p})}{k}}, \hat{p} + l\sqrt{\frac{\hat{p}(1 - \hat{p})}{k}} \right). \quad (2)$$

Hence, in the NIST proposal $l = 3$ and its associate α level is 0.0028 (Type-I error), and note that $0.0028 < 0.01$ that is the interval that NIST gives does not reflect a confidence of 99%. If we want to work with a value $\alpha = 0.01$, then the correct selection of l should be 2.5758, and so the interval is

$$\left((1 - \alpha) - 2.5758\sqrt{\frac{\alpha(1 - \alpha)}{k}}, (1 - \alpha) + 2.5758\sqrt{\frac{\alpha(1 - \alpha)}{k}} \right). \quad (3)$$

The interpretation of the result is that in the 99% of the cases the proportion should fall into the confidence interval Equation (3) and, so, the critic region of the hypothesis test would be the complementary of that interval with probability $\alpha = 0.01$.

In addition, we consider that it is important to pay attention to some problems in the use of the discrete Fourier transform test. Following this point, in [47] the authors study in detail this test and point out that the most crucial problem is that its reference distribution from the statistical test is not derived mathematically, which is estimated numerically and it is concluded that this test should not be used unless the reference distribution is derived mathematically. Additionally, the authors propose a test, whose reference distribution is mathematically derived.

With these observations, we believe that the use of the NIST battery with this interpretation of the results would be more appropriate and statistically adjusted.

However, taking up the subject of possible dependence between tests, it is worth highlighting some studies whose main objective is precisely to detect them, for example, in [13, 15]. At present, this is a line of research that is being worked on.

TestU01, however, provides a very powerful platform where the user can select generators, specific statistical tests, complete pre-defined batteries and tools to apply tests to particular generators. This is an attempt to cover the bibliography and methodologies that have been designed over time. As mentioned before, the problem of execution and obtaining results is present and that is why it is recommended following a specific order when executing it.

The correspondence between tests in NIST battery and TestU01 battery can be seen in Table 6 where NIST battery tests and their corresponding functions on TestU01 battery are shown, in case there is no exact relation, similar functions are included, the latter is indicated with (*).

5 CONCLUSIONS

In this article, different hypothesis test batteries existing in the literature have been described in detail oriented to the verification of the desirable statistical properties for a data sequence either obtained with a RNG or a PRNG.

Table 5. Advantages and Disadvantages of Knuth's, Diehard, Dieharder, NIST, Test U01, and ENT Batteries

Battery	Advantages	Disadvantages
Knuth	First designed battery Establishes the basis for the design of the others batteries	Objectively bad generators can pass the tests ($p\text{-value} \geq \alpha$) Not currently used
Diehard	More rigorous tests than those presented by Knuth It is more difficult that $p\text{-value} > \alpha$ It is more complete than the previous one	Fixed parameters Limited sample size Data must be in a binary file in the form of 32-bit integers (exactly)
Dieharder	More complete battery than the previous ones Friendly interface Open source Can be used in R interface in Linux or Unix operating systems	Fixed parameters Recommended for random numbers rather than bit sequences
NIST	More complete battery than the previous ones Widely used Designed for cryptographic applications	Fixed parameters Dependence between tests
TESTU01	It includes most of the literature tests and implements the main RNGs Flexibility to choose subsets of tests Ability to work at a uniform number level as well as at a bit level	Depending on the selected subset it can take a considerable amount of computing time
ENT	Consists of five simple statistics, which are performed extremely quickly	Dependence between some of the obtained results

The batteries are composed by different hypothesis tests that can be applied separately or together with the rest of the battery. However, among the existing batteries it can be checked how in many cases the same tests are applied. In others, the number of tests to be used is extended or even some tests are changed for others generating new batteries. When applying a certain battery, the initial problems arise in selecting the level of significance of the tests that is essential for the rejection or not of the null hypothesis under study. Generally, this level is set at $\alpha = 0.01$ or $\alpha = 0.05$, and values lower than 0.01 make it very difficult to reject the null hypothesis and, therefore, the probability of deciding incorrectly on the randomness of the sequence increases. An interesting aspect would be to study how the response of the decision maker would vary, that is, how the rejection or not of the null hypothesis would vary depending on the variation of the α significance level and its statistical implications.

However, as seen throughout this article, in many of the batteries, the tests are designed under certain parameters that in some cases are preset, avoiding their modification by the analyst. This last one causes certain rigidity in its application and important statistical consequences as, for example, to increases in Type-I error. In this article, the different pros and cons of each of the existing batteries have been discussed. Moreover, we have given some recommendations about the use of the batteries.

Two main lines of research derived from our review can be extracted: the first is based on carrying out studies on the dependence or not of the different tests included in the batteries. The second and as a consequence of the above, is the design of efficient test batteries in which there is no overlap in the checks of the statistical properties.

Table 6. NIST vs TestU01 Batteries.

NIST TESTS	TESTING FUNCTION IN TESTU01 BATTERY
The Frequency (Monobit) test	sstring_HammingWeight2 with $L = n$
Frequency test within a block	sstring_HammingWeight2
The Runs test	sstring_Run
Tests for the Longest-Run-of-Ones in a block	sstring_LongestHeadRun
The Binary Matrix Rank test	smarsa_MatrixRank
The Discrete Fourier Transform (Spectral) test	sspectral_Fourier1
The Non-overlapping Template Matching test	smarsa_CATBits
The Overlapping Template Matching test	smultin_MultinomialBitsOver (*)
Maurer's "Universal Statistical" test	svaria_AppearanceSpacings
The Linear Complexity test	scomp_LinearComp
The Serial test	smultin_MultinomialBitsOver with Delta = 1
The Approximate Entropy test	smultin_MultinomialBitsOver with Delta = 0; sentrop_EntropyDiscOver; sentrop_EntropyDiscOver2
The Cumulative Sums (Cusums) test	swalk_RandomWalk1
The Random Excursions test	swalk_RandomWalk1 (*)
The Random Excursions Variant test	swalk_RandomWalk1 (*) based on the times that the excursion returns to 0.

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