CS111 Fall'25 ASSIGNMENT 1 Solutions

Solution 1: (a) The internal loop of the first **for** loop prints $2i - i = i$ words for each $i = 2, 3, ..., 3n+1$, so the total number of words printed will be

$$h_1(n) = \sum_{i=2}^{3n+1} 2i - \sum_{i=2}^{3n+1} i = \sum_{i=2}^{3n+1} (2i - i) = \sum_{i=2}^{3n+1} i$$

The internal loop of the second **for** loop prints $(3i+1)^2$ words for each $i = 1, \ldots, 2n$, so the total number of words printed will be

$$h_2(n) = \sum_{i=1}^{2n} (3i+1)^2.$$

Then

$$h(n) = h_1(n) + h_2(n) = \sum_{i=2}^{3n+1} i + \sum_{i=1}^{2n} (3i+1)^2.$$

(b) Using formulas for the sum of the first k terms of an arithmetic series and the sum of squares of k first integers ($\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$), we can simplify the above formula as follows:

$$\begin{aligned}
h(n) &= \sum_{i=2}^{3n+1} i + \sum_{i=1}^{2n} (3i+1)^2 \\
&= \sum_{i=1}^{3n+1} i - 1 + \sum_{i=1}^{2n} 9i^2 + \sum_{i=1}^{2n} 6n + \sum_{i=1}^{2n} 1 \\
&= \frac{(3n+1)(3n+2)}{2} - 1 + 9 \times \frac{2n(2n+1)(4n+1)}{6} + 6 \times \frac{2n(2n+1)}{2} + 2n \\
&= \frac{9n^2}{2} + \frac{9n}{2} + 1 - 1 + 24n^3 + 18n^2 + 3n + 12n^2 + 6n + 2n \\
&= 24n^3 + \frac{69}{2}n^2 + \frac{31}{2}n.
\end{aligned}$$

(c) $f(n) = \Omega(n^3)$, since $24n^3 + \frac{69}{2}n^2 + \frac{31}{2}n \geq 24n^3, \forall n \geq 1$;

$f(n) = O(n^3)$, since $24n^3 + \frac{69}{2}n^2 + \frac{31}{2}n \leq 24n^3 + \frac{69}{2}n^3 + \frac{31}{2}n^3 = 74n^3, \forall n \geq 1$.

We conclude that $f(n) = \Theta(n^3)$.

Solution 2: (a) The inequality $\frac{5}{4}x \geq (x+1) \Leftrightarrow \frac{1}{4}x \geq 1 \Leftrightarrow x \geq 4$ holds when $x \geq 4$.

(b)$5x^2 \geq (x+1)^2$ is true if and only if $5x^2 - (x+1)^2 \geq 0$. First, we will simplify, $5x^2 - (x+1)^2 = 4x^2 - 2x - 1$. $f(x) = 4x^2 - 2x - 1$ is a quadratic function. Its graph is an upward-facing parabola. So $f(x) \geq 0$ for each value x that is not between its zeros. The roots of this function are:

$$x_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 4 \cdot (-1)}}{8} = \frac{2 \pm \sqrt{20}}{8} < 4$$

Thus, for each real $x \geq 4$, $f(x) \geq 0$. So, $5x^2 \geq (x+1)^2$ for all real $x \geq 4$.

(c) Base case: we check that the inequality holds for $n = 4$.
For $n = 4$, LHS: $2 \cdot 5^4 = 1250$ and RHS: $4 \cdot 4^4 + 10 \cdot 4^2 = 4 \cdot 256 + 160 = 1184$. Since $1250 > 1184$, the base case holds.

Inductive step.
Let k be any integer such that $k \geq 4$. We assume that the inequality holds for $n = k : 2 \cdot 5^k \geq k4^k + 10k^2$.
Now we need to show that it also holds for $n = k+1$, that is $2 \cdot 5^{k+1} \geq (k+1)4^{k+1} + 10(k+1)^2$.
We proceed as follows:

$$\begin{aligned}
2 \cdot 5^{k+1} &= 5 \cdot (2 \cdot 5^k) \\
&\geq 5(k4^k + 10k^2) \qquad \text{(by inductive assumption)} \\
&= 5 \cdot k4^k + 10 \cdot 5k^2 \\
&= \frac{5}{4} \cdot 4 \cdot k4^k + 10 \cdot 5k^2 \\
&= \frac{5}{4} \cdot k4^{k+1} + 10 \cdot 5k^2 \\
&\geq (k+1)4^{k+1} + 10(k+1)^2 \qquad \text{(after applying (a) and (b))}
\end{aligned}$$

Now we can conclude that $2 \cdot 5^n \geq n4^n + 10n^2$ for all integers $n \geq 2$.

(d) Let g(n) and f(n) be two functions defined on $\mathbb{N}$. We say that $f(n) = O(g(n))$ if and only if there exist two constants, c and n_0, such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
Let f(n) $= n4^n + 10n^2$ and $g(n) = 5^n$. In (c), we proved that $2 \cdot 5^n \geq n4^n + 10n^2$ for all integers $n \geq 4$. The last inequality implies that $f(n) \leq 2 \cdot g(n)$ for all $n \geq 4$. Thus there exist two constants, $c = 2$ and $n_0 = 4$, such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. Following the definition of big-O, $f(n) = O(5^n)$.

Solution 3:
(a) $3n^6 + 5n^3 - 3n^2 + 1 \geq 3n^6$ for $n \geq 1$ (since $5n^3 - 3n^2 + 1 \geq 5n^2 - 3n^2 + 1 = 2n^2 + 1 \geq 0, \forall n \geq 1$).
Also, $3n^6 + 5n^3 - 3n^2 + 1 \leq 3n^6 + 5n^6 + 3n^6 + 1 \cdot n^6 = 12n^6$ for $n \geq 1$. Let $c = 12, n_0 = 1$, we have $f(n) = O(n^6)$.
Therefore, $\forall n \geq 1, 3n^6 \leq 3n^6 + 5n^3 - 3n^2 + 1 \leq 12n^6 \Rightarrow 3n^6 + 5n^3 - 3n^2 + 1 = \Theta(n^6)$.

(b) $4n^3 \log_3 n + 4n^2 \log_4 n + 2n^4 \geq 2n^4$ for $n \geq 1 \Rightarrow 4n^3 \log_3 n + 4n^2 \log_4 n + 2n^4 = \Omega(n^4)$.
$4n^3 \log_3 n + 4n^2 \log_4 n + 2n^4 \leq 4n^3 \cdot n + 4n^2 \cdot n^2 + 2n^4 = 10n^4$ for $n \geq 1 \Rightarrow 4n^3 \log_3 n + 4n^2 \log_4 n + 2n^4 = O(n^4)$.
Therefore, $4n^3 \log_3 n + 4n^2 \log_4 n + 2n^4 = \Theta(n^4)$.

(c) $3n^4 \log^5 n + 2n^3\sqrt{n} \cdot \log^2 n + 2\sqrt{n^9} = 3n^4 \log^5 n + 2n^{7/2} \log^2 n + 2n^{9/2} \geq 2n^{9/2}$ for $n \geq 0 \Rightarrow 3n^4 \log^5 n + 2n^3\sqrt{n} \cdot \log^2 n + 2\sqrt{n^9} = \Omega(n^{9/2})$.
$3n^4 \log^5 n + 2n^{7/2} \log^2 n + 2n^{9/2} = 3n^4 O(n^{1/2}) + 2n^{7/2} \cdot O(n) + O(n^{9/2}) = O(n^{9/2}) = O(n^{9/2})$.
Therefore, $3n^4 \log^5 n + 2n^3\sqrt{n} \cdot \log^2 n + 2\sqrt{n^9} = \Theta(n^{9/2})$.

(d) $n^3\sqrt{n^5} + n \cdot (1.2)^n + 4n^5 \log^3 n \geq n \cdot (1.2)^n$,for $n \geq 1 \Rightarrow n^3\sqrt{n^5} + n \cdot (1.2)^n + 4n^5 = \Omega(n(1.2)^n)$.
$n^3\sqrt{n^5} + n \cdot (1.2)^n + 4n^5 \log^3 n = n \cdot n^{\frac{5}{2}} + n \cdot (1.2)^n + 4n \cdot n^4 \log^3 n = n \cdot O((1.2)^n) + O(n \cdot (1.2)^n) + 4n \cdot O((1.2)^n) = O(n \cdot (1.2)^n)$.
Therefore, $n^3\sqrt{n^5} + n \cdot (1.2)^n + 4n^5 \log^3 n = \Theta(n(1.2)^n)$.

(e) $n^7 + n^3 \cdot 4^n + n^2 \cdot \left(\frac{9}{2}\right)^n \geq n^2 \cdot \left(\frac{9}{2}\right)^n$, for $n \geq 1 \Rightarrow$ $n^7 + n^3 \cdot 4^n + n^2 \cdot \left(\frac{9}{2}\right)^n = \Omega(n^2(\frac{9}{2})^n)$.
$n^7 + n^3 \cdot 4^n + n^2 \cdot (\frac{9}{2})^n = n^2 \cdot n^5 + n^2 4^n \cdot n + n^2 \cdot (\frac{9}{2})^n = n^2 \cdot O((\frac{9}{2})^n) + n^2 4^n \cdot O((\frac{9}{8})^n) + O(n^2(\frac{9}{2})^n) = O(n^2(\frac{9}{2})^n)$.
Therefore, $n^7 + n^3 \cdot 4^n + n^2 \cdot \left(\frac{9}{2}\right)^n = \Theta\left(n^2\left(\frac{9}{2}\right)^n\right)$.

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