CS 111 ASSIGNMENT 3

Problem 1: a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

- First, we set $n = 2$ and get $R_2 = 4R_1 - 3R_0$

- Then, we substitute the give values of $R_0 = 1, R_2 = 5$ giving the value $R_1 = 2$

- Now, we set $n = 3$ and get $R_3 = 4R_2 - 3R_1$

- Lastly, we compute for R_3 and get 14.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

- With these roots in mind, we know that the characteristic equation of a recurrence relation has the form $(x - r_1)^{m_k} = 0$. We also know that the general solution form is $R_n = C_1(r_1)^n + ...$

- Then, we see that 1 has a multiplicity (m) of 1, -2 is $m = 2$, 2 is $m = 1$, and 7 is $m = 1$

- Lastly, we write the general solution which is $R_n = C_1 + C_2(-2)^n + C_3n(-2)^n + C_4(2)^n + C_5(7)^n + c_6n(7)^n$

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

- Assuming $A_n = r^n$, that means $r^n = 256r^{n-4}$ and we can divide both side by r^{n-4} giving $r^4 = 256$

- Then, we solve for $r^4 = 256$ and since $256 = 2^8$, that leads for $r = 4, -4, 4i, -4i$ which are the characteristic roots.

- Lastly, we write our general solution that is $A_n = C_1(4)^n + C_2(-4)^n + C_3(4i)^n + C_4(-4i)^n$. If we want to rewrite the complex terms $4i, -4i$, we can us Euler's formula and use $4^n(C_3cos(npi/2) + C_4sin(npi/2))$ for $4i, -4i$

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

- First, after finding the corresponding homogeneous recurrence relation of $B_n - 3B_{n-2} + 2B_{n-3} = 0$, we find the characteristic equation when assuming $B_n = r^n$ which leads to it being $r^3 - 3r^2 + 0r + 2 = 0$

- Then, after factoring the characteristic equation, we see the general solution to the equation is $B_n^h = C_1 + C_2n + C_3(2)^n$.

- Next, we need to try a quadratic approach to find the particular solution. We substitute and expand any squared terms to get $An^2 + Bn + C = 3(A(n^2 - 4n + 4) + B(n-2) + C) - 2(A(n^2 - 6n + 9) + B(n-3) + C) + 2$

- After grouping like terms, We solve for the coefficients, leading for $An^2 = A - 2A + 3A = A = 0$, $Bn = 3B - 2B = B = 0$, and $C = -6A + C = 2$ to have a particular solution of $B_n^p = 2$.

- Thus, the general solution for the particular solution is $B_n = C_1 + C_2n + C_3(2)^n + 2$.

Problem 2: Solve the following recurrence equations:
a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

- First, the characteristic equation with its roots turned out to be $r^3 - r^2 - 4r - 2 = 0$ with roots of $r = -1, 1 + sqrt(3), 1 - sqrt(3)$

- Then, the general solution turned out to be $f_n = c_1(-1)^n + C_2(1 + sqrt(3))^n + C_3(1 - sqrt(3))^n$

- Lastly, after computing for each initial condition, the final solution turned out to be $f_n = (-1)^n + (3 + sqrt(3))/(6)(1 + sqrt(3))^n + (3 - sqrt(3))/6(1 - sqrt(3))^n$

$b)$

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

- First, the associated homogeneous equation would be $t_n - t_{n-1} - 2t_{n-2} = 0$

- Next, the characteristic polynomial and its roots would be $r^2 - r - 2 = 0$ with roots of $r = 2, -1$

- Then, the general solution with these in mind would be $t_n^h = C_1 2^n + C_2(-1)^n$

- Next, computing for a particular solution with a modified form approach using $t_n^p = An2^n$ since 2^n is already a solution to the homogeneous equation, we get $t_n^p = (2/3)n2^n$

- After that, the general solution of the non-homogeneous equation would be $t_n = C_1 2^n + C_2(-1)^n + (2/3)n2^n$

- Lastly, the final solution using the inital conditions would be $t_n = (2/9)(2^n - (-1)^n) + (2/3)n2^n$

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

- In order to find a formula for these conditions, we need to identify a variable, T_n, as the number of the tiles in the $n \times 1$ strip

- Next, we need to define base cases for the last tile conditions. These would include a total of 4 cases.

  - Case 1: If the last tile is Red (R), Contribution T_{n-1}

  - Case 2: If the last tile is Green (G) or Blue (B), Contribution T_{n-2}

  - Case 3: If the last tile is Purple (P), Contribution T_{n-2}

  - Case 4: If the last tile is Orange (O), Contribution T_{n-2}

- After we combine our base cases together, we get $T_n = T_{n-1} + 3T_{n-2}$ with T_{n-1} accounting for cases ending in Red (R) and $3T_{n-2}$ accounting for cases that end in Purple (P), Orange (O), Green (G) or Blue (B).

- Next, we calculate our base cases individually, starting with n=1 and n=2

- For $n = 1$, The only valid tiles are G, B, R, leading to $T_1 = 3$

- For $n = 2$, The only possible tilings are GB, GR, BG, BR, RB, RR, OR, PR, leading to $T_2 = 8$

- With this, the recurrence relation formula would be $T_n = T_{n-1} + 3T_{n-2}$ for $n >= 3$ with inital conditions of $T_1 = 3$ and $T_2 = 8$

Academic integrity declaration. The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.

Submission. To submit the homework, you need to upload the pdf file to Gradescope. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment.