CS 111 Practice Quiz 1

Problem 1:

(a) Use the Θ-notation to determine the rate of growth of the following functions:

Function	Θ estimate
$n^4 + n^3 - 10n + 3$	$\Theta(n^4)$
$7n^3 + 3 \cdot 5^n + 12\log(n^4) + 2\log^7 n$	$\Theta(5^n)$
$\frac{10\log^5 n}{n^2\sqrt{n}} + 5n^2 \log n + 12n\log^7 n$	$\Theta(n^2 \log n)$
$15 \cdot 2^n + 5 \cdot 3^n$	$\Theta(3^n)$
$n3^n + n^3 5^n + 3n^5 \log n^8 + n^3 14^{-n}$	$\Theta(n^3 5^n)$

(b) Justify, using the definition of Θ, that $6n^5 + 2n^3 - 1 = \Theta(n^5)$.

First we will show that $6n^5 + 2n^3 - 1 = \Omega(n^5)$. Note that for $n \geq 1$ we have that $2n^3 - 1$ is positive. Thus, for $n \geq 1$, $6n^5 + 2n^3 - 1 \geq 6n^5$. By definition of Ω, with $n_0 = 1$ and $c = 6$, we have shown that $6n^5 + 2n^3 - 1 = \Omega(n^5)$.

Now, we will show that $6n^5 + 2n^3 - 1 = O(n^5)$. For $n \geq 1$, we have that

$$\begin{aligned} 6n^5 + 2n^3 - 1 &\leq 6n^5 + 2n^3 \\ &\leq 6n^5 + 2n^5 \\ &= 8n^5 \end{aligned}$$

So by definition of O, with $n_0 = 1$ and $c = 8$, we have shown that $6n^5 + 2n^3 - 1 = O(n^5)$.

Since $6n^5 + 2n^3 - 1 = \Omega(n^5)$, and $6n^5 + 2n^3 - 1 = O(n^5)$, by definition of Θ, $6n^5 + 2n^3 - 1 = \Theta(n^5)$.

Problem 2:

For each piece of pseudo-code below, give:

(a) The expression for the exact number of "A"s printed by that pseudo-code as a function of n. (You do not have to simplify the expressions, leaving them in summation notation is fine.)

(b) Its asymptotic running time as a function of n. Express this running time using the Θ notation.

Include a brief explanation for (a) and (b).

Pseudo-code	Exact number	Running time	Explanation

for $i \leftarrow 1$ **to** $2n$ **do**
    for $j \leftarrow 1$ **to** n^2 **do**
        $k \leftarrow x + 7$
        Print A
	$\sum_{i=1}^{2n} n^2$	$\Theta(n^3)$	The outer loop runs $\Theta(n)$ times and the inner loop runs $\Theta(n^2)$ times.

for $i \leftarrow 0$ **to** n^2 **do**
    for $j \leftarrow 1$ **to** 5 **do**
        $x \leftarrow x^4$
        Print A
for $k \leftarrow 1$ **to** $2n$ **do**
    $z \leftarrow x + z$
    Print A
	$\sum_{i=0}^{n^2} 5 + \sum_{k=1}^{2n} 1$	$\Theta(n^2)$	The first loop runs $\Theta(n^2)$ times, and the last loop runs $\Theta(n)$ times.

for $i \leftarrow 0$ **to** n **do**
    for $j \leftarrow 1$ **to** i **do**
        $x \leftarrow x - z$
        Print A
	$\sum_{i=0}^{n} i$	$\Theta(n^2)$	For each i of the outer loop, the inner loop runs i times. This is a well-known series.

for $i \leftarrow 1$ **to** 10 **do**
    $x \leftarrow 3x$
    Print A
	$\sum_{i=1}^{10} 1$	$\Theta(1)$	This loop runs exactly 10 times.

for $i \leftarrow 1$ **to** $2n$ **do**
    $j \leftarrow 1$
    while $j < n$ **do**
        $x \leftarrow 7x$
        $j \leftarrow 3j$
        Print A
	$\sum_{i=1}^{2n} \log(n)$
	Note: this should be log base 3, rounded up to next integer
		$\Theta(n \log n)$	The outer loop runs $\Theta(n)$ times, and the inner loop runs $\Theta(\log n)$ times.

Problem 3: Using mathematical induction, prove that the following identity holds for all integers $n \geq 1$:

$$\sum_{i=1}^{n} i \cdot i\,! = (n+1)\,! - 1.$$

Proof: We first verify the base case, for $n = 1$. If $n = 1$, then the LHS is 1, and the RHS is $(1+1)\,! - 1 = 2 - 1 = 1$ as well. So the equation is true in the base case.

In the inductive step, assume the identity holds for some n, that is $\sum_{i=1}^{n} i \cdot i\,! = (n+1)\,! - 1$. We now want to prove that it also holds for the next integer $n+1$, that is, $\sum_{i=1}^{n+1} i \cdot i\,! = (n+2)\,! - 1$. We start with the LHS and proceed as follows:

$$\begin{aligned}
\sum_{i=1}^{n+1} i \cdot i\,! &= \sum_{i=1}^{n} i \cdot i\,! + (n+1) \cdot (n+1)\,! && \text{separate last term from sum} \\
&= (n+1)\,! - 1 + (n+1) \cdot (n+1)\,! && \text{apply inductive assumption} \\
&= (n+1)\,![(n+1) + 1] - 1 && \text{factor out } (n+1)\,! \\
&= (n+1)\,!(n+2) - 1 = (n+2)\,! - 1 && \text{algebra}
\end{aligned}$$

which gives us the desired equality for $n+1$.

Summarizing, the identity holds for $n = 0$, and we showed that if it holds for some n, then it holds for $n+1$ as well. Thus, the identity holds for all $n \geq 0$.

Academic integrity declaration. Please provide a statement confirming that you completed this assignment all by yourself. (For example, *"Hereby I affirm that I completed this test on my own, without any unauthorized help."*) and sign it. Submissions without the signed statement will not be graded.