CS 111 ASSIGNMENT 3 Solutions

Solution 1: a) From the recurrence, we know that $R_2 = 4R_1 - 3R_0$. So, $5 = 4R_1 - 3 \cdot 1$. Solving for R_1, we get $R_1 = 2$. From the recurrence, we know that $R_3 = 4R_2 - 3R_1 = 4 \cdot 5 - 3 \cdot 2 = 20 - 6 = 14$. So $R_3 = 14$.

b)

$$\alpha_1 \cdot 1^n + \alpha_2 \cdot n \cdot 1^n + \alpha_3 \cdot (-2)^n + \alpha_4 \cdot n \cdot (-2)^n + \alpha_5 \cdot n^2 \cdot (-2)^n + \alpha_6 \cdot 7^n$$

c) We will reconstruct the recurrence using the characteristic equation. We know the roots of the characteristic equation are: $3, 6$, so the characteristic equation is $(x-3)(x-6) = 0$ or $x^2 - 9x + 18 = 0$. We can also rewrite this as: $x^n = 9x^{n-1} - 18x^{n-2}$. This leads to the recurrence equation

$$B_n = 9B_{n-1} - 18B_{n-2}$$

with initial conditions

$$B_0 = 2 \cdot 3^0 + 6^0 = 3$$

$$B_1 = 2 \cdot 3^1 + 6^1 = 12$$

d) The characteristic equation is $x^4 - 256$, which has roots: $4, -4, 4i, -4i$ (where $i = \sqrt{-1}$). The general solution is

$$A_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot (-4)^n + \alpha_3 \cdot (4i)^n + \alpha_4 \cdot (-4i)^n$$

e) The characteristic equation is:

$$x^3 - x^2 - 4x - 2 = 0$$

The roots of the equation are: $r_1 = -1$, $r_2 = 1 - \sqrt{3}$ and $r_3 = 1 + \sqrt{3}$.
Thus, the general solution of the recurrence is:

$$f_n = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot (1 - \sqrt{3})^n + \alpha_3 \cdot (1 + \sqrt{3})^n$$

Using the initial conditions, we obtain:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2\,(1 - \sqrt{3}) + \alpha_3\,(1 + \sqrt{3}) = 1 \\ \alpha_1 + \alpha_2\,(1 - \sqrt{3})^2 + \alpha_3\,(1 + \sqrt{3})^2 = 4 \end{cases} \quad \Rightarrow \begin{cases} \alpha_1 = 2 \\ \alpha_2 = \dfrac{-6 - 5\sqrt{3}}{6} \\ \alpha_3 = \dfrac{-6 + 5\sqrt{3}}{6} \end{cases}$$

The final solution is:

$$f_n = 2 \cdot (-1)^n - \frac{6 + 5\sqrt{3}}{6} \cdot (1 - \sqrt{3})^n + \frac{-6 + 5\sqrt{3}}{6} \cdot (1 + \sqrt{3})^n$$

Solution 2: a) The non-homogeneous term is $q(n) = 2^n$. The associated homogeneous recurrence is $t_n = 4t_{n-1} - 4t_{n-2}$, giving us the characteristic equation $x^2 - 4x + 4 = 0$. Since 2 is a double root of the characteristic equation, the particular solution is in the form $f''_n = \beta n^2 \cdot 2^n$. Plugging this into the equation, we get

$$\begin{aligned}
&\beta \cdot n^2 \cdot 2^n = 4\beta \cdot (n-1)^2 \cdot 2^{n-1} - 4\beta \cdot (n-2)^2 \cdot 2^{n-2} + 2^n, \\
&\Rightarrow 4\beta n^2 = 8\beta(n-1)^2 - 4\beta(n-2)^2 + 4 \\
&\Rightarrow 8\beta = 4 \\
&\Rightarrow \beta = \frac{1}{2}
\end{aligned}$$

Finally,

$$f''_n = \frac{1}{2} \cdot n^2 \cdot 2^n$$

or

$$f''_n = n^2 \cdot 2^{n-1}$$

b) We start with the associated homogeneous equation, $f_n = 13f_{n-2} + 12f_{n-3}$. The characteristic equation is $x^3 - 13x - 12 = 0$. It's a cubic equation. The candidates for integer roots are -12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12. Plugging these in, we obtain that -3, -1 and 4 satisfy this equation, so we have all three roots: $r_1 = -3$, $r_2 = -1$ and $r_3 = 4$. Thus the general solution of the associated homogeneous equation is

$$f'_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n.$$

Next, we want to find a particular solution of the non-homogeneous equation. Since the non-homogeneous term is $q(n) = 2n+1$, a linear function, $f''_n = \beta_1 n + \beta_2$. (Notice, that 1 is not a solution of the characteristic equation.) Plugging this into the equation, we get

$$\beta_1 n + \beta_2 = 13[\beta_1(n-2) + \beta_2] + 12[\beta_1(n-3) + \beta_2] + 2n + 1$$

which reduces to

$$(-24\beta_1 - 2)n + (62\beta_1 - 24\beta_2 - 1) = 0.$$

Setting both, $-24\beta_1 - 2$ and $62\beta_1 - 24\beta_2 - 1$ to 0, and solving these equations simultaneously, yields $\beta_1 = -\frac{1}{12}$ and $\beta_2 = -\frac{37}{144}$. So our particular solution is $f''_n = -\frac{1}{12}n - \frac{37}{144}$.

This gives us the general solution of the original non-homogeneous equation:

$$f_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n - \frac{1}{12}n - \frac{37}{144}.$$

Solution 3: Let T_n be the number of such tilings of an $n \times 1$ strip. We know that an $n \times 1$ tiling must end in one of the G, B, R, P, O, or L tiles. Let us count the number of tilings that end with each type of tile.

- **Tilings ending in G**:
  G cannot be preceded immediately by B, P, or GG, so if a tiling ends in G, it must end in one of RG, OG, LG, RGG, OGG, or LGG. (The final G in each of these endings is the last tile in the tiling.) These endings are lengths 2×1, 3×1, 4×1, 3×1, 4×1, and 5×1, respectively. So, the total number of tilings that end in G is $T_{n-2} + 2T_{n-3} + 2T_{n-4} + T_{n-5}$.

- **Tilings ending in B**: (Very similar to G)
  B cannot be preceded immediately by G, P, or BB, so if a tiling ends in B, it must end in one of RB, OB, LB, RBB, OBB, or LBB. (The final B in each of these endings is the last tile in the tiling.) These endings are lengths 2×1, 3×1, 4×1, 3×1, 4×1, and 5×1, respectively. So the total number of tilings that end in B is $T_{n-2} + 2T_{n-3} + 2T_{n-4} + T_{n-5}$.

- **Tilings ending in R**:
  R can be preceded immediately by any tile. As long as the rest of the tiling is valid, so is this ending. So the number of tilings ending in R is T_{n-1}.
- **Tilings ending in P**:
  P cannot be preceded immediately by G or B or P. So if a tiling ends in P, it must end in one of RP, OP, or LP.(The final P in each of these endings is the last tile in the tiling.) These endings are lengths 3×1, 4×1, and 5×1, respectively. The number of tilings ending in P is $T_{n-3} + T_{n-4} + T_{n-5}$.
- **Tilings ending in O**:
  O can be preceded immediately by any tile. As long as the rest of the tiling is valid, so is this ending. So the number of tilings ending in O is T_{n-2}.
- **Tilings ending in L**: (Very similar to R)
  L can be preceded immediately by any tile. As long as the rest of the tiling is valid, so is this ending. So the number of tilings ending in L is T_{n-3}.

Adding up all of these tilings, we get that the total number of tilings is

$$T_n = T_{n-1} + 3T_{n-2} + 6T_{n-3} + 5T_{n-4} + 3T_{n-5} \text{ for } n \geq 5$$

The initial conditions are:

$$T_0 = 1$$

$$T_1 = 3$$

$$T_2 = 9$$

$$T_3 = 24$$

$$T_4 = 74$$

Academic integrity declaration. The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.

Submission. To submit the homework, you need to upload the pdf file to Gradescope. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment.