NAME: SID:

Date ______________________ **Time** ______________________

CS111 Quiz 1 (B) Solutions

Rules:

- Time: 45 minutes.
- Closed notes, closed book. No electronic devices, including calculators.
- To receive credit, you must show your work and provide justification when appropriate.
- Only the front pages will be scanned! If you need more room, write on the last blank page. If you need scratch space, use the back side of the previous page.

Problem 1: For each piece of pseudo-code below, give the Θ-estimate for its running time. Include a very brief informal explanation for the third piece of pseudo-code.

Pseudo-code	Θ-estimate	Justification
	$\Theta(n^6)$	No explanation is required here.
	$\Theta(n)$	No explanation is required here.
	$\Theta(n)$	For $n \geq 3$, the inner loop makes j iterations for $j = 3, 3 \cdot 2, 3 \cdot 2^2, \cdots, 3 \cdot 2^{\lceil \log_2(n/3) \rceil - 1}$, forming a geometric sequence. So the total is $\Theta(n)$.
	$\Theta(n^3)$	No explanation is required here.

NAME: SID:

Problem 2: (a) Use the Θ-notation to determine the rate of growth of functions below. Justify your answers for the last two functions only, using asymptotic relations between the basic reference functions. (Reminder: $\log x = \log_2 x$.)

Function	Θ estimate	Justification
	$\Theta(n^8)$	Don't need to justify
	$\Theta(n \cdot (1.25^n))$	Don't need to justify
	$\Theta((8^n log^2(n))$	$(8)^n \cdot \log^2 n + (5)^n \cdot n^3 + n^{12}$ $= (5)^n \cdot \log^2 n \left(\left(\frac{8}{5}\right)^n + \frac{n^3}{\log^2 n} \right) + O(5^n)$ $= (5)^n \cdot \log^2 n \left(\left(O(\frac{8}{5}\right)^n + O(\frac{8}{5})^n \right)$ $= (5)^n \cdot \log^2 n \cdot O\left(\left(\frac{8}{5}\right)^n \right) = O(8^n \cdot \log^2 n)$ $(5)^n \cdot n^3 + (8)^n \cdot \log^2 n + n^{12} = \Omega((8)^n \cdot \log^2 n)$
	$\Theta(n^{4.5})$	$5n^4 \log n = n^4 \cdot O(n^{0.5}) = O(n^{4.5})$ $n^3 \log^3 n = n^3 \cdot O(n^{1.5}) = O(n^{4.5})$ and $5\sqrt{n^9} = 5n^{4.5} = O(n^{4.5})$ $5n^4 \log n + n^3 \log^3 n + 5\sqrt{n^9} = \Omega(n^{4.5})$

(b) Justify, using the definition of Θ, that $7n^6 + 5n^4 - 3n^2 + 1 = \Theta(n^6)$.

Let $f(n) = 7n^6 + 5n^4 - 3n^2 + 1$. For all $n \geq 1$, we have $f(n) \leq 7n^6 + 5n^4 \cdot n^2 + 1 \cdot n^6 = 13n^4$. So choosing $c_1 = 13$ and $n_1 = 1$, we have $f(n) \leq c_1 n^6$ for $n \geq n_1$. Thus $f(n) = O(n^6)$.

For all $n \geq 1$, we have $f(n) = n^6 + 5n^4 + 3 \cdot (2n^6 - n^2) + 1 \geq n^6$. So choosing $c_2 = 1$ and $n_2 = 1$, we have $f(n) \geq c_2 n^6$ for $n \geq n_2$. Thus $f(n) = \Omega(n^6)$.

Since $f(n) = O(n^6)$ and $f(n) = \Omega(n^6)$, we conclude that $f(n) = \Theta(n^6)$.

NAME: SID:

Problem 3: (a) Below you are given a sketch of an inductive proof of the following inequality:

$$5^n \ \geq \ 2^{n+3} + 9n, \ \text{for } n \geq 3.$$

Complete this proof by filling in the remaining steps.

Base case: For $n = 3$, the left-hand side is 125 and the right-hand side is 91. Since $125 > 91$, the base case holds.

Inductive step: Assume that the inequality holds when $n = k$, for some $k \geq 3$:
$5^k \ \geq \ 2^{k+3} + 9k$.
We need to prove that it holds for the next integer, that is $5^{k+1} \ \geq \ 2^{k+4} + 9(k+1)$. Starting with the left-hand side of this inequality, we proceed as follows:

$$\begin{aligned} 5^{k+1} &= \quad 5 \cdot 5^k \\ &\geq 5 \cdot (2^{k+3} + 9k) && \text{(from the assumption)} \\ &= 2^{k+4} + 9(k+1) + (3 \cdot 2^{k+3} + 36k - 9) \\ &\geq 2^{k+4} + 9(k+1) && (n \geq 3) \end{aligned}$$

So the inequality holds for $n = k + 1$. From the base case and the inductive step, it holds for all $n \geq 3$.

(b) Prove, directly from definition and using part (a), that $2^{k+3} + 9k = O(5^k)$. (It is sufficient to specify an inequality and two constants from the definition of the big-Oh notation.)
In (a) we proved that $5^n \ \geq \ 2^{n+3} + 9n$, for $n \geq 3$. So, there exist constants $c = 1$ and $n_0 = 3$ such that $2^{n+3} + 9n \leq c \cdot 5^n$ for all $n \geq n_0$. By the definition of Big-O, $2^{k+3} + 9k = O(5^k)$.