An Introduction to Reliability and Maintainability Engineering

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ABOUT THE AUTHOR

Charles E. Ebeling is an associate professor of engineering management and systems in the School of Engineering, University of Dayton, where he has taught for the last eight years. He received his Ph.D. in systems and industrial engineering at the Ohio State University in 1973 while on active duty in the United States Air Force. Professor Ebeling retired as a lieutenant colonel after twenty years of service in the Air Force. His Air Force experience includes communications-electronics, logistics, reliability, simulation modeling, and manpower planning. After retirement he was the director of operations research for BDM Corporation. In this capacity he directed the development of a large computerized management information system to determine spare parts requirements for the Air Force. Prior to his retirement from the Air Force, he was a faculty member for the Air Force Institute of Technology, where he taught courses in logistics, statistics, and operations research. While on active duty he also served as an adjunct faculty member for George Washington University, Golden Gate University, and the College of William and Mary. For the last eight years Professor Ebeling has taught courses in the Department of Engineering Management at the University of Dayton in probability and statistics, operations research, reliability engineering, simulation, optimization, inventory theory, and production engineering. His research interests currently center around a multiyear NASA grant to develop reliability and maintainability models, computer simulation models, and life-cycle costing models to be used during the conceptual design of space transportation systems. Professor Ebeling is a registered engineer in the state of Ohio.

CONTENTS

PREFACE

ABOUT THE BOOK

This is an introductory textbook on reliability and maintainability engineering. It is written at the undergraduate senior and first-year graduate levels. The majority of this text may be covered in a one-semester course, or the entire text may be covered in two academic quarters. The text is divided into three parts. The first part covers reliability and maintainability modeling, the second part addresses the analysis of failure and repair data, and the third part provides examples and applications of reliability engineering and considers implementation of reliability and maintainability programs. Current textbooks on reliability have typically focused on either the modeling or on the statistical analysis of failure data. Many are written at an advanced level that requires extensive background in probability and statistics on the part of the student. It is the intent of this book to provide a broad coverage of the important concepts in reliability and maintainability and to avoid the more formal theory-proof approach. The student interested in additional depth is encouraged to read some of the more advanced texts available as well as appropriate technical journals.

Available with the text is MS-DOS software that may be used for problem-solving. Since failure and repair data analysis is computationally intensive, this software spares the student the burden of performing numerous and tedious calculations. Students fortunate to have one of the many commercial software reliability packages available are encouraged to use it. The practicing reliability or maintainability engineer will need to make extensive use of the computer in performing data analysis. Therefore, computer applications should be included as part of any study in reliability. The available software is intended for educational use only.

OBJECTIVE

The primary objective of this text is to introduce the subject of reliability and maintainability engineering to the engineering student, practicing engineer, or technical manager who has had a very limited formal education in probability and statistics. Hopefully, this is accomplished in part by the introduction of probability and statistics concepts within the context of their use in reliability. Additionally, derivations of many of the formulas are relegated to appendices so that the primary focus is on concepts and applications. Notation has been kept as simple as possible while attempting to adhere to convention. References to more advanced material are provided in appropriate places. Nevertheless, it is certainly true that those students who have had a formal introductory course or courses in probability and statistics will benefit the most from this text. A secondary objective is to prepare the student for more advanced study in reliability and maintainability engineering and to enable the student to access the increasing amount of technical material available in the literature.

It is hoped that the material in this text will enable the student to collect and analyze failure and repair data, derive appropriate reliability and maintainability models, and apply these models in the design of products, components, and systems. The end result should be products and systems having improved reliability and maintainability characteristics.

ACKNOWLEDGMENTS

I wish to thank the several (anonymous) reviewers for their many helpful suggestions. To the extent that I have been able to respond to their criticism, the text has been immeasurably improved. If I fell short of their expectations, the limitation has been mine alone. I would be remiss if I did not acknowledge the help of several graduate students. Ken Beasley and Colleen Donohue have especially helped throughout this effort in numerous ways. The following individuals have contributed examples, exercises, or verified solutions: Wilbur Bhagat, Annette Clayton, David Gels, Ronald Niehaus, John Stahl, and James Wafzig. Special thanks go to the editorial and production staff at McGraw-Hill and at Publication Services. Their suggestions and editorial comments have been invaluable in producing a readable textbook. Finally, a special thanks to my wife, Patricia, who gave up several well-earned vacations and many evenings out while I worked on this manuscript. It is to her that I dedicate this book.

COURSE SOFTWARE

INSTRUCTIONS ON THE USE OF THE SOFTWARE TO ACCOMPANY THE TEXT

Available for use with this text is an MS-DOS executable file (REL.EXE) that may be run under DOS or under Windows as a non-Windows application. This software is available from the instructor and is included as part of the *Solutions Manual.* To execute the file, simply (1) type REL at the DOS prompt while in the directory the file is resident in; or (2) if not in the same directory as the file, include the path to the file (for example, >A:REL); or (3) when operating from Windows, double click on REL.EXE while in the file manager or Windows Explorer (Windows 95).

This software is intended for use in Part II of the text. It performs analysis on failure and repair data. Analysis options include:

Empirical models for ungrouped and grouped complete and singly censored data and for multiply censored data, including life tables (multiply censored grouped data). For multiply censored data the models include the incremental rank method, the Kaplan-Miers product-limit estimator, and an alternative product-limit estimator.

Least-squares analysis for fitting exponential, Weibull, normal, and lognormal distributions to either complete or censored data.

The Duane reliability growth model.

Nonhomogeneous Poisson processes (NHPP) (such as the AMSAA growth model).

Maximum likelihood estimation for exponential, Weibull, normal, and lognormal distributions with complete or censored data.

Goodness-of-fit tests including the chi-square, Bartlett (exponential), Mann (Weibull), Komogorov-Smirnov (normal and lognormal), and Cramer–von Mises (NHPP) with a test for trend.

Upon execution, the following main menu will appear:

```
MAIN MENU
INPUT/SAVE/OUTPUT OPTIONS
EMPIRICAL ANALYSIS
REL GROWTH NHPP POWER/LAW MODELS
LEAST-SQUARES CURVE FIT (PROB-PLOT)
MAXIMUM LIKELIHOOD ESTIMATE (MLE)
GOODNESS-OF-FIT TEST
QUIT
```

Entering data to the program is accomplished by selecting **INPUT/SAVE/OUTPUT OPTIONS** from the main menu. The **DATA/INPUT/DISPLAY MENU** shown below will appear. Initially data must be entered from the keyboard or from a compatible file provided with the text or created by the instructor. However, once you have entered data, they may be saved in a new file for subsequent use. To input data, the number of units at risk (or number of repair observations) is entered followed by the failure and censor (if applicable) times or repair times. Censored times are entered as negative values. Once the data have been entered, they may be displayed and corrections may be made if necessary. With three exceptions, input is accomplished through the input module, in which individual (or cumulative) failure or repair times are inserted. This allows the user to conduct several tests on the same data set without reentering the data. The three exceptions are group data and life tables, the Duane growth curve, and a manual input mode for the chi-square goodness-of-fit test. Each of these requires insertion of interval data at the respective module. Input data files carry the file extension .DAT, which is supplied by the program.

Output is generated and displayed on the screen by invoking the various modules. The output may also be selectively saved in a text file by selecting the **TURN ON/OFF WRITE OPTION** on the input menu and entering an S (SAVE) after each output display. This file can then be read by a word processor, edited, and printed. The output file has the file extension .TXT. The program automatically attaches the extension to the file when the user supplies the file name.

```
DATA INPUT/DISPLAY MENU
INPUT FROM A FILE
INPUT FROM KEYBOARD
UPDATE/DISPLAY DATA
SAVE INPUT DATA TO A FILE
TURN ON/OFF WRITE OPTION
```

Data Sets

Four data sets are included with the software for the student to use in becoming familiar with this software. The data were randomly generated from the distributions shown in the following table.

File (.DAT)	Sample size	Distribution	Parameters	Data type
EX1	50 at risk, 40 failures	Exponential	$\lambda = 0.001$	Multiply censored
EX2	35 at risk, 22 failures	Weibull	$\beta = 2;\ \theta = 500$	Type II censored
EX3	30 failures	Normal	$\mu = 5000;\ \sigma = 250$	Complete
EX4	22 repair times	Lognormal	$t_{\text{med}} = 3.5;\ s = 0.7$	Complete

CHAPTER 1

Introduction

Things fail. During the past two years this author has experienced a lawn-mower casing crack, a washing machine fail, a car battery go dead, a toaster oven electrical plug burn, a water-heater leak, a floppy disk drive go bad, a TV remote control quit functioning, a stereo amplifier quit, an automobile engine starter fail, and a house roof leak. The cracked lawn-mower casing was a result of its aluminum construction having insufficient strength to withstand the stresses placed on it. The car battery, the engine starter, and the washing-machine motor experienced wearout after a "normal" life. The toaster oven plug was a poor design, considering the amount of current passing through it. Corrosion of the hot water tank caused it to leak. The corrosion was partly attributed to the lack of preventive maintenance, which required periodical draining of the bottom of the tank. The failure of the disk drive was a result of an unknown (premature) mechanical failure, and the TV remote control's failure was caused by a "random" electronic component failure. On the other hand, the stereo amplifier failure was caused by an open at a solder joint. Poor construction resulted in the house roof leaking adjacent to the dormers. Some of these failures caused much inconvenience in addition to their economic impact. Several of the failures raised concerns about personal safety, although no injuries resulted from them.

Many failures, however, are much more significant in both their economic and safety effects. For example, in 1946 the entire fleet of Lockheed Constellation aircraft was grounded following a crash killing four of the five crew members. The crash was attributed to a faulty design in an electrical conduit that caused the fuselage to burn. In 1979 the left engine of a DC-10 broke away from the aircraft during takeoff, killing 271 people. Poor maintenance procedures and a bad design led to the crash. Engine removal procedures introduced unacceptable stresses on the pylons. The Ford Pinto, introduced in 1971, was recalled by Ford in 1978 for modifications to the fuel tank to reduce fuel leakage and fires resulting from rear-end collisions. Numerous reported deaths, lawsuits, and the negative publicity eventually contributed to Ford discontinuing production of the Pinto. Firestone's steel-belted radials, introduced in 1972, failed at an abnormal rate as a result of the outer tread coming apart from the main body of the tire. Because of the excessive number of failures, Firestone was forced to recall 7.5 million tires. On November 8, 1940, the Tacoma Narrows Bridge, five months old, collapsed into Puget Sound from vibrations caused by high winds. Metal fatigue induced by several months of oscillations led to the failure. The Manus River bridge (Greenwich, Connecticut) collapsed in 1983, killing three people and injuring three. While there is disagreement on the cause of the disaster, blame has been placed on the original design, on corrosion that had caused undetected displacement of the pin-and-hanger suspension assembly, on poor maintenance, and on inadequate inspections. The Hartford (Connecticut) Civic Center Coliseum roof collapsed in 1978 from structural failure due to the weight of the snow and ice accumulated on the roof. A major shortcoming in the roof frame system was the lack of redundancy of members to carry loads when other individual members failed. An inadequate safety margin may also have contributed. The Three Mile Island disaster in 1979, which resulted in a partial meltdown of a nuclear reactor, was a result of both mechanical and human error. When a backup cooling system was down for routine maintenance, air cut off the flow of cooling water to the reactor. Warning lights were hidden by maintenance tags. An emergency relief valve failed to close, causing additional water to be lost from the cooling system. Operators were either reading gauges that were not working properly or taking the wrong actions on the basis of those that were operating. The 1986 explosion of the space shuttle *Challenger* was a result of the failure of the rubber O-rings that were used to seal the four sections of the booster rockets. The below-freezing temperatures before the launch contributed to the failure by making the rubber brittle.[1]

From the above examples one can conclude that the impact of product and system failures varies from minor inconvenience and costs to personal injury, significant economic loss, and death. Causes of these failures include bad engineering design, faulty construction or manufacturing processes, human error, poor maintenance, inadequate testing and inspection, improper use, and lack of protection against excessive environmental stress. Under current laws and recent court decisions, the manufacturer can be held liable for failing to account properly for product safety and reliability. Engineers responsible for product design must therefore include both reliability and maintainability as design criteria. The objective of this text is to introduce the technical manager and the engineer to the concepts, models, and analysis techniques that form the basis of reliability and maintainability engineering.

This, then, is a book on the failure and repair characteristics of systems, products, and their component parts. Reliability and maintainability engineering attempts to study, characterize, measure, and analyze the failure and repair of systems in order to improve their operational use by increasing their design life, eliminating or reducing the likelihood of failures and safety risks, and reducing downtime, thereby increasing available operating time. Closely associated with reliability and maintainability for systems or components that can be repaired or restored to an operating state once they have failed is the concept of availability. Availability measures the combined

[1]Details of these and other disasters may be found in the highly interesting book *When Technology Fails* [Schlager, 1994].

effect of both the failure and the repair process and is an important characteristic of the system.

1.1 THE STUDY OF RELIABILITY AND MAINTAINABILITY

As engineering disciplines, reliability and maintainability are relatively new. Their growth has been motivated by several factors, which include the increased complexity and sophistication of systems, public awareness of and insistence on product quality, new laws and regulations concerning product liability, government contractual requirements to meet reliability and maintainability performance specifications, and profit considerations resulting from the high cost of failures, their repairs, and warranty programs.

A Gallup poll conducted in 1985 for the American Society for Quality Control interviewed over 1000 individuals to determine what attributes were most important to them in selecting a product. The 10 attributes listed in Table 1.1 were ranked by each individual on a scale from 1 (least important) to 10 (most important); the average scores are as shown in the table. Obviously, both reliability and maintainability are important considerations in consumer purchasing.

Reliability and maintainability are not only an important part of the engineering design process but also necessary functions in life-cycle costing, cost benefit analysis, operational capability studies, repair and facility resourcing, inventory and spare parts requirement determinations, replacement decisions, and the establishment of preventive maintenance programs.

1.1.1 Reliability Improvement

A product has value as a result of its utility or performance in satisfying a need or requirement. Factors that contribute to a high value for a product are its versatility,

TABLE 1.1
Ten most important product attributes

Attribute	Average score
Performance	9.5
Lasts a long time (reliability)	**9.0**
Service	8.9
Easily repaired (maintainability)	**8.8**
Warranty	8.4
Easy to use	8.3
Appearance	7.7
Brand name	6.3
Packaging/display	5.8
Latest model	5.4

Source: *Quality Progress*, vol. 18 (Nov.), pp. 12–17, 1985.

ease of use, safety, aesthetics, and reliability. The primary reason for reliability and maintainability engineering is to improve the reliability and availability of the product or system being developed and thereby add to its value. During the initial design activity, this improvement can be achieved in several ways. For critical and high–failure rate components, redundancy or duplication of functions may be feasible. Designing excess strength into components or careful selection of material or parts will decrease the probability of failure. *Derating,* meaning operating the system below its rated stress level, provides an alternative means of achieving a desired reliability goal. For example, an electronic component may be designed to operate at 200 volts at a specified temperature but normal usage may dictate only 120 volts. Choice of technology, such as mechanical versus electronic switches or transistors versus integrated circuits, can have a significant effect on reliability. Reducing the complexity of the system, particularly as measured by the number of components or subassemblies, will also reduce the failure rate. As will be shown later, as the number of components in a system increases, the reliability of the components must be significantly increased in order to maintain a target system reliability. Once the limits of reliability improvement have been reached, further gain in product availability may be obtained by decreasing downtime through good maintainability design. To a large degree, then, reliability and maintainability must be addressed during system design, and they therefore become an inherent design feature of the system.

Reliability improvement, however, is not limited to the product design itself. For example, during initial product development an aggressive reliability growth program can play a major role in determining final product reliability. During manufacture a good quality control program will maintain product design reliability by ensuring conformance to production specifications and tolerances and by reducing variability in the manufacturing process. Inspection and acceptance sampling procedures ensure that raw material and supplier parts meet agreed standards. Once a product becomes operational, failures may be reduced through preventive maintenance, sound parts replacement policies, engineering modifications, and careful attention to environmental conditions and operating loads. Once the system is operational, downtime can be decreased (maintainability improved) with the proper amount of repair resources, including maintenance technicians, test equipment, and available spare parts. Secondary considerations such as skill levels, resupply lead times, maintenance training, and the ease of use of technical manuals will also improve maintainability. Reliability and maintainability engineering, therefore, must be practiced throughout the product's life cycle.

1.1.2 Random versus Deterministic Failure Phenomena

The traditional approach to safety in engineering is to design a high safety margin or safety factor into the product. This is a deterministic method in which a safety factor of perhaps 4 to 10 times the expected load or stress would be allowed for in the design. Safety factors often result in overdesign, thus increasing costs, or, less frequently, in underdesign, resulting in failure caused by an unanticipated load or a material weakness. On the other hand, the classic point of view taken in developing

reliability is to treat system and component failures as random, or probabilistic, occurrences. In theory, if we were able to comprehend the exact physics and chemistry of a failure process, many internal failures of a component could be predicted with certainty. In practice, however, with limited data on the physical state of a component and an incomplete knowledge of the physical and chemical (and perhaps biological) processes that cause failures, failures will appear to occur at random over time. Even failures caused by events external to the component, for example, environmental conditions such as hurricanes, earthquakes, or excessive heat or vibration, will appear to be random. However, if we had sufficient understanding of the conditions resulting in the event as well as the effect such an event would have on the component, then we would also be able to predict these failures deterministically. This uncertainty, or incomplete information, about a failure process is therefore a result of its complexity, imprecise measurements of the relevant physical constants and variables, and the indeterminate nature of certain future events.

This random process may exhibit a pattern that can be modeled by some probability distribution. Such phenomena are often observed in practice, especially when large numbers of components are involved. We are able to predict the failure (or nonfailure) behavior of these systems statistically.

A currently fashionable alternative view of reliability attempts to analyze the physics of the failure process and, through a mathematical model, determine the time to failure. This approach requires knowledge of the failure mechanisms and the basic causes of failures. Mean times to failure are determined on the basis of known or predicted stresses, environmental factors, operating conditions, material properties, and part geometries. We will return to the physics-of-failure approach later. The definitions and much of the development that follow are based on the probabilistic and statistical view of reliability.

1.2 CONCEPTS, TERMS, AND DEFINITIONS

Reliability *is defined to be the* ***probability*** *that a component or system will perform a required function for a given period of time when used under stated operating conditions.* It is the probability of a nonfailure over time. To determine reliability in an operational sense, the definition must be made specific. First, an unambiguous and observable description of a failure must be established. Failures should be defined relative to the function being performed by the system. Second, the unit of time must be identified. For example, the specified time interval may be based on calendar or clock time, operating hours, or cycles. A cycle, for example, may be the landing of an aircraft, a load reversal, or the turning on of an electric motor. In some cases reliability is not defined over time but over another measurement, such as miles traveled. For production systems failures may be defined in terms of units or batches produced. Third, the system should be observed under normal performance. This would include such factors as design loads (e.g., weight, voltages, pressure), environment (e.g., temperature, humidity, vibration, altitude), and operating conditions (e.g., use, storage, maintenance, transportation).

***Maintainability** is defined to be the probability that a failed component or system will be restored or repaired to a specified condition within a period of time when maintenance is performed in accordance with prescribed procedures.* Maintainability is the probability of repair in a given time. Usually, when maintainability is computed, time is defined to be clock time (although it could, for example, be duty or shift time). It may or may not include such measures as waiting time for maintenance personnel and parts, travel time, and administrative time. Often, however, maintainability refers to the inherent repair time, which includes only the hands-on repair of the failed unit and not any administrative or resource delay times.

Prescribed maintenance procedures include not only the manner in which repair is to be performed but also the availability of maintenance resources (people, spare parts, tools, and manuals), the preventive maintenance program, skill levels of personnel, and the number of people assigned to the maintenance crew.

***Availability** is defined as the probability that a component or system is performing its required function at a given point in time when used under stated operating conditions.* Availability may also be interpreted as the percentage of time a component or system is operating over a specified time interval or the percentage of components operating at a given time. As we will see later, availability can be mathematically defined in several different ways depending on how system uptime and downtime are measured. It differs from reliability in that availability is the probability that the component is currently in a nonfailure state even though it may have previously failed and been restored to its normal operating condition. Therefore system availability can never be less than system reliability. Availability may be the preferred measure when the system or component can be restored since it accounts for both failures (reliability) and repairs (maintainability). These definitions will become more precise in subsequent chapters when these concepts are defined mathematically.

Reliability versus quality

Reliability is closely associated with the quality of a product and is often considered a subset of quality. Quality can be defined qualitatively as the amount by which the product satisfies the users' (customers') requirements. Product quality is in part a function of design and conformance to design specifications. It also depends on the production system and on adherence to manufacturing procedures and tolerances. Quality is achieved through a good quality assurance program. Quality assurance is a planned set of processes and procedures necessary to achieve high product quality.

On the other hand, reliability is concerned with how long the product continues to function once it becomes operational. A poor-quality product will likely have poor reliability, and a high-quality product will have a high reliability. As we have seen, however, reliability may depend on external factors and not just the quality of the product itself. Nevertheless, reliability may be viewed as the quality of the product's operational performance over time, and as such it extends quality into the time domain.

TABLE 1.2
Reliability test data results

	Motor					
	1–100	**101–200**	**201–300**	**301–400**	**401–500**	**Total**
Number tested	12	11	12	12	15	62
Hours on test	2540	2714	2291	1890	2438	11873
Number failed	1	0	1	5	7	14
Failure rate	0.000394	0	0.000436	0.002646	0.002871	0.001179

1.3 APPLICATIONS

The types of problems solved through the use of reliability and maintainability concepts are illustrated by the following examples.

EXAMPLE 1.1. The B. A. Miller Company manufactures small motors for use in household appliances such as washing machines, dryers, refrigerators, and vacuum cleaners. It has recently designed a new motor that has experienced the abnormally high failure rate of 43 failures among the first 1000 motors produced. Several of these failures were observed during final product testing by the appliance manufacturer. These motors were inspected, and it appeared that the bearing case was turning in its seat. The sealed ball bearings, however, appeared to be okay. Possible causes of these failures included faulty design, defective material, and a manufacturing (tolerance) problem. The company initiated an aggressive program of accelerated life testing of motors randomly selected from the production line. As a result of the testing program, it observed that those motors produced near the end of a production run were failing at a higher rate than those at the start of the run. Table 1.2 summarizes the results of the testing program, which are displayed graphically in Fig. 1.1.

The failure rate is computed by dividing the number of failures by the total number of hours on test. Dividing total hours on test by the number of failures provides an

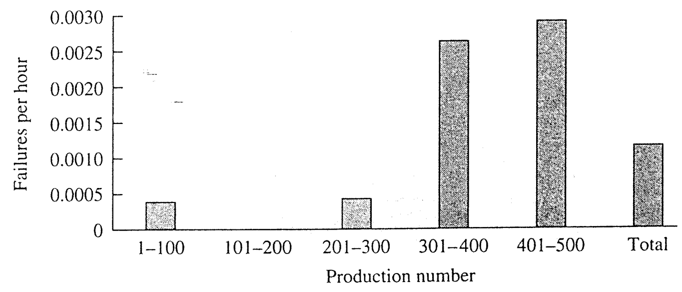


FIGURE 1.1
Failure rates of motors.

estimate for the mean time to failure (MTTF). The MTTF of the first 300 units produced is 3773.5 operating hours (7545/2), and the MTTF of the last 200 units produced[2] is 360.7 operating hours (4328/12). As a result, it was assumed that the production process was going out of control and design tolerances were not being met. The company therefore placed additional emphasis on its quality control program in order to eliminate premature failures of the motors.

EXAMPLE 1.2. Most electronic products sold have a warranty in which the seller or manufacturer will replace or repair with no cost to the consumer a failed unit if the failure occurs within a specified time, such as one year from the date of purchase. In order to estimate the number of failures that will occur over a warranty period and hence the cost of the warranty program, the probability distribution of the time to failure of the product must be established. For a new VCR unit produced by the XYZ Company, the distribution of the time to failure shown in Fig. 1.2 was obtained from a reliability testing program.

From these data, and using the techniques discussed in Part II of the text, the probability $F(t)$ of a VCR failure occurring by time t (in operating hours)[3] was found to be $F(t) = 1 - e^{-t/8750}$. Assuming that the typical consumer will use the VCR an average of 3 hr a day, for the first year 1095 operating hours (3×365) will be observed. Therefore, the probability of a unit failing is

$$F(1095) = 1 - e^{-1095/8750} = 1 - 0.8824 = 0.1176$$

With over 10 percent of the units sold expected to fail during the first year, the company decided to initiate a reliability growth program in order to improve product reliability, reduce warranty costs, and increase customer satisfaction.

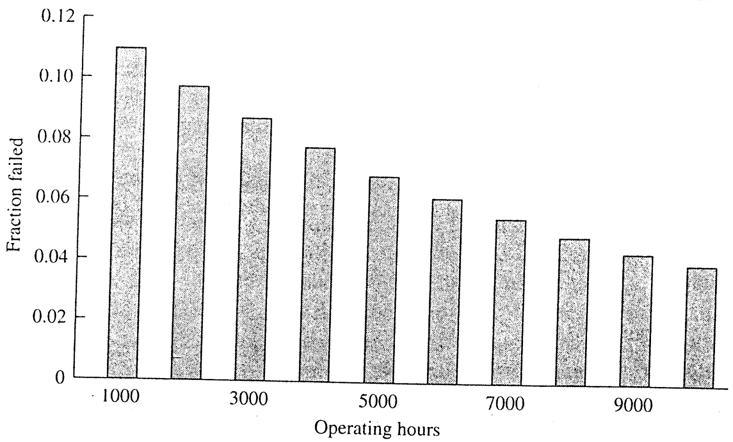


FIGURE 1.2
Distribution of VCR failure times.

[2]The MTTF is defined formally in Chapter 2.

[3]This formula is based on the exponential distribution discussed in Chapter 3, and the constant 8750 is the MTTF.

EXAMPLE 1.3. A continuous-flow production line requires a product to be processed sequentially on 10 different machines. When a machine breaks down, the entire line must be stopped until the failure is repaired. The average downtime of the line is 12 hr. This includes time spent waiting for parts and maintenance personnel as well as time spent troubleshooting and fixing the failed machine. Machine specifications require a 0.99 reliability for each machine over an 8-hr production run. Therefore the reliability of the production line over an 8-hr run is $0.99^{10} = 90$ percent.[4] Assuming a constant failure rate (exponential failure distribution), this is equivalent to a mean time between failures (MTBF) of 75.9 operating hours, found by solving the following for MTBF: $R(8) = e^{-8/\mathrm{MTBF}} = 0.90$, where $R(8)$ is the production line reliability over an 8-hr period.

Under fairly general conditions the steady-state availability of a system is given by

$$A = \frac{\mathrm{MTBF}}{\mathrm{MTBF} + \mathrm{MTTR}}$$

where MTTR = the mean time to repair. Therefore, the steady-state availability of the production line is currently $75.9/(75.9 + 12) = 0.86$. In order to meet production quotas, the line must maintain at least a 0.92 availability. Since improvements in machine reliability above 0.99 did not appear feasible, the company decided to increase availability by improving the maintainability (i.e., decreasing the MTTR).

A minimum MTTR is obtained from Fig. 1.3 or by solving the availability formula $75.9/(75.9 + x) = 0.92$ for x: $x = 6.6$ hr.

By hiring an additional maintenance person, increasing machine spare parts inventory, relocating the inventory closer to the production line, improving diagnostic procedures, and simplifying the removal and replacement of high-failure components, the company was able to reduce the MTTR to under 6 hr and achieve the desired availability goal.

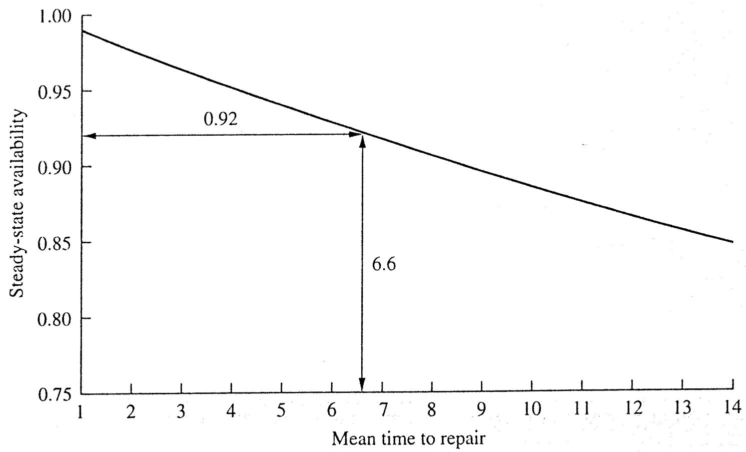


FIGURE 1.3
Availability of the production line.

[4]The logic of computing a system reliability for serial units is discussed in Chapter 5.

1.4 A BRIEF HISTORY

Early foundations of reliability[5] may be found in actuarial concepts used in the insurance industry, particularly in the study of human survival probabilities. In the late 1930s and 1940s Weibull analyzed fatigue life in materials, leading to the probability distribution bearing his name. The study of structural reliability and fatigue failure began in the 1930s and has since continued. Although early (1930s and 1940s) development in such areas as queuing and renewal theory, particularly with the use of the exponential distribution, provided some of the mathematical foundations of reliability, it was not until after World War II that reliability became a subject of study. This came about because of the relatively complex electronic equipment used during the war and the rather high failure rates observed. In particular, vacuum tubes were notoriously unreliable. Following the war, Aeronautical Radio, Inc. (ARINC) was established by the commercial airlines to improve airborne electronic equipment (referred to as *avionics* by the military). In 1950 the U.S. Air Force formed an ad hoc group to improve general equipment reliability, and in 1952 the Defense Department established the Advisory Group on Reliability of Electronic Equipment (AGREE). The requirement of reliability testing and demonstration of new systems emerged from this advisory group.

Work published during the 1950s centered around the use of the exponential distribution to represent failure times, although the interest in and the importance of the Weibull distribution increased by the end of the decade. Initial textbooks on reliability include Bazovsky [1961] and Barlow and Proschan [1967]. These were followed by Smith [1976] and Kapur and Lamberson [1977]; the latter book is still popular today. During the 1970s the focus shifted to fault tree analysis, largely because of concern about nuclear reactor safety. Progressing into the 1980s, reliability networks received considerable attention in the literature. Both reliability and maintainability received renewed emphasis in the mid 1980s with the introduction of the Air Force's Reliability and Maintainability (R&M) 2000 program. Objectives of the R&M 2000 program were to increase system readiness and availability and to reduce maintenance personnel requirements and life-cycle costs through increased reliability and maintainability by the year 2000.

Some of the current literature in reliability and maintainability may be found in the following journals and proceedings:

IEEE Transactions on Reliability
Proceedings Annual Reliability and Maintainability Symposium
Technometrics
Applied Statistics
Operations Research
IIE Transactions
Journal of the American Statistical Association
Reliability Review
Naval Research Logistics
International Journal of Reliability, Quality and Safety Engineering
Microelectronics and Reliability
Reliability Engineering
Journal of Applied Reliability

[5]Students interested in more detail about the development of reliability are encouraged to read "Mathematical Theory of Reliability: A Historical Perspective," by Barlow [1984].

1.5 SCOPE OF THE TEXT

Figure 1.4 shows the organization of this book. The book is divided into three parts. Part I, "Reliability Models," develops mathematical models useful in analyzing component and system reliability, maintainability, and availability. Chapters 2–4

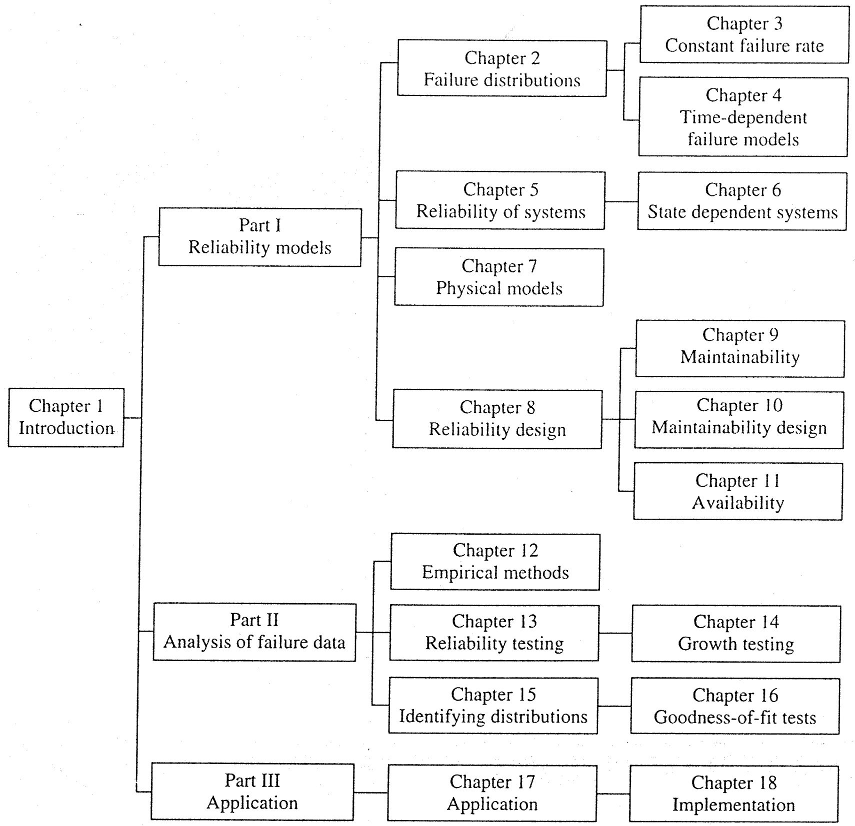


FIGURE 1.4
Organization of the text.

develop the common failure laws based upon the exponential, Weibull, normal, and lognormal distributions. The failure distribution is defined in its three most useful representations: the density function, the reliability function, and the hazard rate function. Conditional reliability and the mean time to failure are also presented. Chapters 5 and 6 develop the mathematics for analyzing complex system reliability given knowledge of the reliability of the components and their configuration within the system. Chapter 5 assumes independence among the components, and Chapter 6 addresses the case in which component failures depend on the state of the system. These dependencies, such as those encountered with standby and shared load systems, are analyzed with Markov processes. Chapter 7 deviates somewhat from the primary development of reliability by introducing models based on the physical processes involved. In Chapter 8 the important task of integrating reliability into the engineering design process is presented. This is further developed in the next three chapters, in which maintainability and availability are defined, thereby allowing for design trade-offs between reliability and maintainability. Specifically, Chapter 9 defines the repair time distribution, which is used to quantify maintainability, and Chapter 11 develops point, interval, and steady-state availabilities. Chapter 10 discusses maintainability from the design point of view.

Part II, "The Analysis of Failure Data," is concerned with statistical techniques for specifying the correct reliability or maintainability model from failure or repair data, respectively. Empirical (nonparametric) reliability distributions are developed in Chapter 12, and Chapter 13 introduces several different types of reliability testing. Reliability testing provides one of the two main sources of failure data, the other being field or operational data. Reliability tests discussed in Chapter 13 include burn-in testing, acceptance testing, sequential tests, and accelerated life testing. Chapter 14 focuses on the important topic of reliability growth testing. Two popular mathematical models are presented: the Duane and the AMSAA reliability growth models. Once the failure data have been collected, a parametric failure or repair distribution must be selected from among those models discussed in Part I. This requires estimation of the distribution parameters, discussed in Chapter 15, and a statistical goodness-of-fit test, discussed in Chapter 16, to either reject or not reject the hypothesized distribution. From the material presented in Part II, the reliability engineer should be able to establish a test plan, collect failure or repair data, and complete a statistical analysis of the data, resulting in an acceptable reliability or maintainability model. The accepted model may then be manipulated using the techniques developed in Part I.

Part III, "Application," concludes the development of reliability engineering by presenting several applications in Chapter 17 and by identifying policies, procedures, issues, and concerns in implementing a reliability program in Chapter 18. Together, these last two chapters attempt to integrate the material from Parts I and II.

Other recent introductory texts on reliability include Bain and Engelhardt [1991], Blanks [1992], Bunday [1991], Crowder et al. [1991], Dai and Wang [1992], Grosh [1989], Kececioglu [1991], Leemis [1995], Ramakumar [1993], Rao [1992], Sundararajan [1991], and Zacks [1992].

APPENDIX 1A
A PROBABILITY PRIMER

There are two general approaches to modeling uncertainty by using probability concepts. The more basic method makes use of the idea of a sample space and events defined over the sample space. It then defines the probability of these events and computes probabilities of more complex events formed from unions and intersections of elementary events. The second method is based on the concept of a random variable and a probability distribution associated with the random variable. A random variable is a variable that takes on certain values in accordance with specified probabilities. By specifying the probability distribution of a random variable, we can completely characterize the random process. We will have occasion throughout this text to utilize both methods. For example, we may define an event to be a component failure and a random variable to be the time to failure of the same component.

1A.1
RANDOM EVENTS

In reliability engineering a failure can be described as a random event. A random event E will occur with some probability denoted by $P(E)$ where $0 \leq P(E) \leq 1$. $P(E) = 0$ describes an impossible event, and $P(E) = 1$ denotes a certain event. The closer $P(E)$ is to 1, the more likely it is that the event (e.g., failure) will occur.

The collection of all possible outcomes (events) relative to a random process is called the sample space S, where $S = \{E_1, E_2, \ldots, E_k\}$ and $P(S) = 1$.

Every event E has associated with it a complementary event E^c, which is the negation of the event E. For example, if E represents a failure, then E^c represents a nonfailure. Since either the event or its complement must occur,

$$P(E) + P(E^c) = 1$$

or

$$P(E^c) = 1 - P(E) \tag{1.1}$$

There are two primary operations performed on events. The first is defined as the intersection of two events A and B and is written $A \cap B$. The intersection is the event consisting of those outcomes common to both events A and B. The second operation is defined as the union of two events A and B and is written $A \cup B$. The union is the event consisting of those outcomes in either event A or in event B (or common to both).

EXAMPLE 1A.1. Let the event A = the failure of component 1 and B = the failure of component 2. Then

$A \cap B$ = the event that both components have failed

$A^c \cap B$ = the event that component 1 did not fail and component 2 failed

$A^c \cup B$ = the event that component 1 did not fail or component 2 failed

Two events A and B are mutually exclusive if the occurrence of one precludes the occurrence of the other. For example, an event and its complement are mutually exclusive. A component cannot both fail and not fail as a result of the same random process. In general, if A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B) \tag{1.2}$$

$$P(A \cap B) = P(\varnothing) = 0 \tag{1.3}$$

where $\varnothing$ is the null, or impossible, event; $P(\varnothing) = 0$.

Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A)\,P(B) \tag{1.4}$$

That is, the probability of their intersection is equal to the product of their probabilities.

EXAMPLE 1A.2. Define the events A and B as in Example 1A.1, where $P(A) = 0.1$ and $P(B) = 0.2$. Then $P(A \cap B) = (0.1)(0.2) = 0.02$ is the probability that both components fail.

If two events are not independent (that is, they are dependent), the probability of their intersection must be defined using conditional probability. Define the conditional probability as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{1.5}$$

which is read as "the probability of A given B." Then

$$P(A \cap B) = P(A \mid B)\,P(B) \tag{1.6}$$

Equation (1.6) provides a general formula for finding a joint (intersection) probability if the conditional and individual probabilities are known. The conditional probability can be viewed as a reduced sample space in which the event B defines the set of all possible outcomes (i.e., the reduced sample space) and the intersection of A and B represents those events that are also part of A. From this perspective, Eq. (1.5) gives the percentage of outcomes in B that are also in A. If A and B are independent,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

This result provides insight into the concept of independent events. If A and B are independent, the occurrence of event B does not alter the probability of A.

EXAMPLE 1A.3. Two identical components operating in parallel share a common load.[6] If one component fails, the probability that the other component fails increases as a result of the increased load placed on it. Let A and B be the event that component 1 fails and the event that component 2 fails, respectively. If $P(A) = P(B) = 0.05$ and $P(A \mid B) = P(B \mid A) = 0.10$, we are interested in the event that both components fail, or using Eq. (1.6),

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A) = 0.05(0.10) = 0.005$$

EXAMPLE 1A.4. A two-component parallel system (repairable) is in a failure state (both components have failed) 3 percent of the time. Component 1 is in a failed state 8 percent of the time, and component 2 is in a failed state 6 percent of the time. If we let A = the event that component 1 is in a failed state and B = the event that component 2 is in a failed state, then $P(A \mid B) = 3/6 = 0.5$ and $P(B \mid A) = 3/8 = 0.375$.

The above discussion provides a general approach to analyzing the intersection of events. The following rule, referred to as the addition rule, does much the same for the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1.7}$$

Equation (1.7) can be explained by observing that $P(A)$ and $P(B)$ both include $P(A \cap B)$. Therefore $P(A \cap B)$ must be subtracted out once. If A and B are independent, Eq. (1.7) becomes

$$P(A \cup B) = P(A) + P(B) - P(A)P(B) \tag{1.8}$$

otherwise

$$P(A \cup B) = P(A) + P(B) - P(A \mid B)P(B) \tag{1.9}$$

EXAMPLE 1A.5. Given the two events A and B in Example 1A.3 and using Eq. (1.9), $P(A \cup B) = 0.05 + 0.05 - 0.005 = 0.095$ is the probability that at least one of the two components fails. Also, $P(A \cup B)^c = 1 - 0.095 = 0.905$ is the probability that neither component fails. In this example the reliability of the system is found from $P(A \cap B)^c = 1 - 0.005 = 0.995$, which is the probability that at least one of the two components does not fail.

1A.2 BAYES' FORMULA

Bayes' formula involves conditional probabilities of two events. It can be derived by observing that

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P[(B \cap A) \cup (B \cap A^c)]} \\ &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} \end{aligned} \tag{1.10}$$

where the events $B \cap A$ and $B \cap A^c$ are mutually exclusive.

EXAMPLE 1A.6. A smoke detector is routinely inspected. Eighty percent of the detectors found inoperative had experienced a power surge, and 10 percent of those found in operating condition had experienced a power surge. Twenty percent of the detectors

[6] Operating in parallel implies a redundant system in which both components must fail for the system to fail.

inspected have failed. What is the probability of a detector failing given it experiences a power surge?

Solution. Let A = the event that a detector has failed and B = the event that the detector had experienced a power surge. Then $P(A) = 0.20$, $P(B \mid A) = 0.80$, and $P(B \mid A^c) = 0.10$. Therefore

$$P(A \mid B) = \frac{(0.80)(0.20)}{(0.80)(0.20) + (0.10)(0.80)} = \frac{0.16}{0.24} = 0.667$$

Bayes' formula is generalized as follows:

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)} \qquad i = 1, \ldots, n \tag{1.11}$$

where the events A_i, $i = 1, \ldots, n$, form a partition. That is, they are mutually exclusive and their intersection comprises the entire sample space (they are collectively exhaustive). As a result the denominator in Eq. (1.11) is equal to $P(B)$.

1A.3 RANDOM VARIABLES

Although the concept of a random event is a useful one in describing random processes, a more useful concept is that of a random variable. A random variable is a variable that takes on numerical values in accordance with some probability distribution. Random variables may be either continuous (taking on real numbers) or discrete (usually taking on nonnegative integer values). The probability distribution that assigns a probability to each value of a discrete random variable or assigns a probability to an interval of values of a continuous random variable can be described in terms of a probability mass function (PMF) $p(x)$ in the discrete case and a probability density function (PDF) $f(x)$ in the continuous case. For both discrete and continuous distributions a cumulative distribution function (CDF) $F(x)$ is defined. The PMF or PDF describes the shape of the probability distribution, whereas the CDF provides a cumulative probability, i.e., $\Pr\{X \le x\} = F(x)$. By convention, capital letters represent the random variable and the corresponding lowercase letters denote particular values the random variable may assume.

EXAMPLE 1A.7. Here are some examples of random variables:

T = the continuous random variable representing the time to failure of a component
Y = the discrete random variable representing the number of failures occurring in some time t
W = the continuous random variable representing the time to repair a failed system
X = the discrete random variable representing the number of cycles until the first failure occurs

1A.4 DISCRETE DISTRIBUTIONS

For any discrete distribution, we will define $p(x) = \Pr\{X = x\}$ to be its probability mass function. Then

$$F(x) = \Pr\{X \le x\} = \sum_{\text{all } \xi}^{x} p(\xi) \tag{1.12}$$

is the CDF. Therefore $F(x)$ is monotonically increasing, with $0 \le F(x) \le 1$ and in particular $F(0) = 0$ and $F(\infty) = 1$. For any discrete distribution,

1. $0 \le p(x) \le 1$
2. $\sum_{\text{all } x} p(x) = 1$
3. $\mu = \sum_{\text{all } x} x p(x)$ (1.13)

is the mean of the distribution.

4. $\sigma^2 = \sum_{\text{all } x} (x - \mu)^2 p(x)$ (1.14)

is the variance of the distribution.

Two discrete distributions are very useful in reliability: the binomial and the Poisson.

1A.5 BINOMIAL DISTRIBUTION

Let X be the discrete random variable representing the number of successes in n independent trials where the probability of success on each trial is a constant p: $X = 0, 1, 2, \ldots, n$. The binomial probability mass function is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \tag{1.15}$$

where
$$\binom{n}{x} = \frac{n!}{(n-x)!x!}$$

The mean, or expected value, is $E(X) = np$, and the variance is $\text{Var}(X) = np(1-p)$.

EXAMPLE 1A.8. Let X be the discrete random variable representing the number of failed components among five independent and identical components of which each component has 1 chance in 100 of failing. Then $X = 0, 1, \ldots, 5$ and has a binomial distribution with $p = 0.01$ and $n = 5$. Therefore the mean number of failures is $(5)(0.01) = 0.05$, the variance is $(0.05)(0.99) = 0.0495$, and the probability of exactly one failure is

$$\Pr\{X = 1\} = \binom{5}{1}(0.01)^1(0.99)^4 = 0.048$$

1A.6 POISSON DISTRIBUTION

Let X be the discrete random variable representing the number of random occurrences (events) in a specified time; $X = 0, 1, 2, \ldots$. The Poisson probability mass function is

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots \tag{1.16}$$

where $\lambda = E(X)$ is the mean number of occurrences in the specified time and $\mathrm{Var}(X) = \lambda$.

EXAMPLE 1A.9. Let X be the discrete random variable representing the number of failures and subsequent repairs of a restorable system over a one-year period. Assuming X has a Poisson distribution with a mean of $\lambda = 2$ failures per year, the probability of no more than one failure a year is

$$\Pr\{X \le 1\} = F(1) = \sum_{x=0}^{1} \frac{e^{-2}2^x}{x!} = 0.406$$

1A.7 CONTINUOUS DISTRIBUTIONS

The following relations hold for any continuous probability distribution:

1. $0 \le F(x) \le 1$

2. $$\Pr\{X \le x\} = F(x) = \int_{-\infty}^{x} f(\xi)\, d\xi \tag{1.17}$$

3. $$f(x) = \frac{dF(x)}{dx} \tag{1.18}$$

4. $$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{1.19}$$

5. $$\Pr\{a \le X \le b\} = \int_a^b f(x)\, dx = F(b) - F(a) \tag{1.20}$$

6. $$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx \tag{1.21}$$

where μ is the mean or expected value of the random variable X.

7. $$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \tag{1.22}$$

where σ^2 is the variance of the distribution.

8. If $Y = \sum_{i=1}^{n} a_i X_i$ where the X_i are independent random variables having means μ_i and variances σ_i^2 and the a_i are constants, then

$$E(Y) = \sum_{i=1}^{n} a_i \mu_i \qquad \text{and} \qquad \mathrm{Var}(Y) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 \tag{1.23}$$

In general we may describe the distribution of a random variable in terms of its PDF, $f(x)$, or its CDF, $F(x)$. Although a random variable may be defined over the open interval $(-\infty, \infty)$, when the random variable represents a failure time or repair time, only nonnegative values are permitted. Therefore the domain of the random variable would normally be $[0, \infty)$, and the lower limits of the integrals in relations 2, 4, 6, and 7 would reflect this change. The concepts of random variables and probability distributions are explored more fully in the next several chapters as the failure probability distribution is developed as the primary focus of Part I. Examples of these distributions will therefore be presented in the context of their use in developing reliability models. More detailed coverage of probability theory may be found in Ross [1987].

PART I

Basic Reliability Models

CHAPTER 2

The Failure Distribution

The initial focus of this book is on the development of mathematical models that describe the reliability of components and systems. This chapter develops four related probability functions: the reliability function, the cumulative distribution function, the probability density function, and the hazard rate function. Each of these functions can be used to compute reliabilities, but they offer four different perspectives. Specifying any one of these functions will uniquely and completely characterize the failure process. Various summary measures of reliability, such as the mean time to failure, the variance of the failure distribution, and the median time to failure, may then be determined. Subsequent chapters will define several common and highly useful theoretical reliability models.

2.1 THE RELIABILITY FUNCTION

Reliability is defined as the probability that a system (component) will function over some time period t. To express this relationship mathematically we define the continuous random variable T to be the time to failure of the system (component); $T \geq 0$. Then reliability can be expressed as

$$R(t) = \Pr\{T \geq t\} \tag{2.1}$$

where $R(t) \geq 0$, $R(0) = 1$, and $\lim_{t\to\infty} R(t) = 0$. For a given value of t, $R(t)$ is the probability that the time to failure is greater than or equal to t.

If we define

$$F(t) = 1 - R(t) = \Pr\{T < t\} \tag{2.2}$$

where
$$F(0) = 0$$
and
$$\lim_{t\to\infty} F(t) = 1$$

then $F(t)$ is the probability that a failure occurs before time t.

We will refer to $R(t)$ as the *reliability function* and $F(t)$ as the *cumulative distribution function* (CDF) of the failure distribution. A third function, defined by

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \tag{2.3}$$

is called the *probability density function* (PDF). This function describes the shape of the failure distribution. These three functions are illustrated in Fig. 2.1.

The PDF, $f(t)$, has these two properties:

$$f(t) \geq 0 \qquad \text{and} \qquad \int_0^\infty f(t)\,dt = 1$$

Given the PDF, $f(t)$, then

$$F(t) = \int_0^t f(t')\,dt' \tag{2.4}$$

and

$$R(t) = \int_t^\infty f(t')\,dt' \tag{2.5}$$

In other words, both the reliability function and the CDF represent areas under the curve defined by $f(t)$. Therefore, since the area beneath the entire curve is equal to one, both the reliability and the failure probability will be defined so that

$$0 \leq R(t) \leq 1 \qquad 0 \leq F(t) \leq 1$$

The function $R(t)$ is normally used when reliabilities are being computed, and the function $F(t)$ is normally used when failure probabilities are being computed. Graphing the PDF, $f(t)$, provides a visual representation of the failure distribution.

EXAMPLE 2.1. Given the following PDF for the random variable T, the time (in operating hours) to failure of a compressor, what is its reliability for a 100-hr operating life?

Solution

$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t+1)^2} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$R(t) = \int_t^\infty \frac{0.001}{(0.001t'+1)^2}\,dt' = \left.\frac{-1}{(0.001t'+1)}\right|_t^\infty = \frac{1}{0.001t+1}$$

and

$$F(t) = 1 - R(t) = 1 - \frac{1}{0.001t+1} = \frac{0.001t}{0.001t+1}$$

Then

$$R(100) = \frac{1}{0.1+1} = 0.909$$

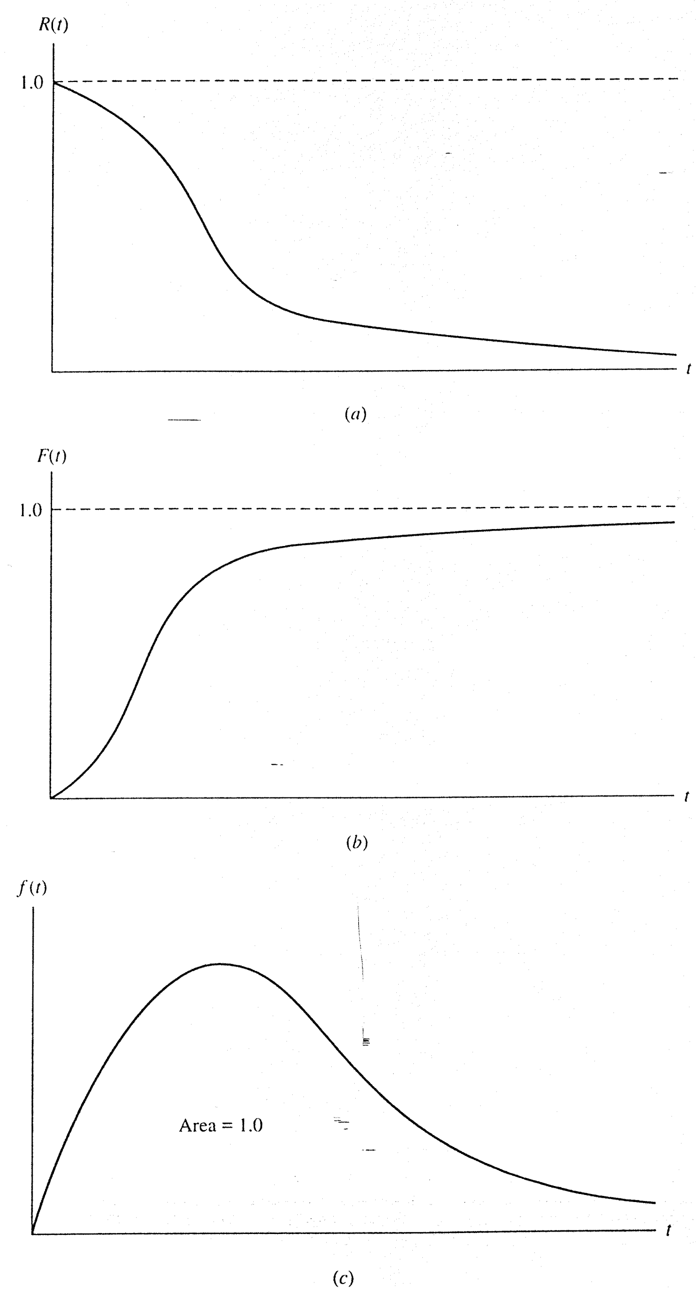


FIGURE 2.1
(*a*) The reliability function. (*b*) The cumulative distribution function. (*c*) The probability density function.

A design life is defined to be the time to failure t_R that corresponds to a specified reliability R. That is $R(t_R) = R$. To find the design life if a reliability of 0.95 is desired, we set

$$R(t_R) = \frac{1}{0.001t_R + 1} = 0.95$$

Solving for t_R,

$$t_R = 1000\left(\frac{1}{0.95} - 1\right) = 52.6 \text{ hr}$$

The probability of a failure occurring within some interval of time $[a, b]$ may be found using any of the three probability functions, since

$$\Pr\{a \le T \le b\} = F(b) - F(a) = R(a) - R(b) = \int_a^b f(t)\,dt \tag{2.6}$$

From the previous example,

$$\Pr\{10 \le T \le 100\} = R(10) - R(100) = \frac{1}{0.01 + 1} - \frac{1}{0.1 + 1} = 0.081$$

2.2 MEAN TIME TO FAILURE

The mean time to failure (MTTF) is defined by

$$\text{MTTF} = E(T) = \int_0^\infty t f(t)\,dt \tag{2.7}$$

which is the mean, or expected value, of the probability distribution defined by $f(t)$.

It can also be shown (Appendix 2A) that

$$\text{MTTF} = \int_0^\infty R(t)\,dt \tag{2.8}$$

Equation (2.8) is often easier to apply than (2.7).

The mean of the failure distribution is only one of several measures of central tendency of the failure distribution. Another is the median time to failure, defined by

$$R(t_{\text{med}}) = 0.5 = \Pr\{T \ge t_{\text{med}}\} \tag{2.9}$$

The median divides the distribution into two halves, with 50 percent of the failures occurring before the median time to failure and 50 percent occurring after the median. The median may be preferred to the mean when the distribution is highly skewed.

A third frequently used average is the mode, or most likely observed failure time, defined by

$$f(t_{\text{mode}}) = \max_{0 \le t < \infty} f(t) \tag{2.10}$$

For a small fixed interval of time centered around the mode, the probability of failure will generally be greater than for an interval of the same size located elsewhere within the distribution. Figure 2.2 shows the locations of the mean, the median, and the mode for a distribution skewed to the right.

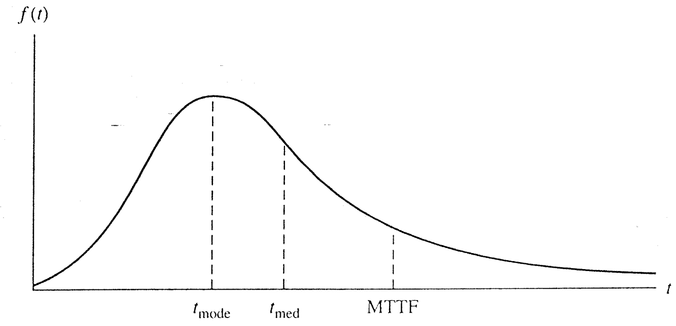


FIGURE 2.2
Comparison of the measures of central tendency.

EXAMPLE 2.2. Consider the probability density function

$$f(t) = \begin{cases} 0.002e^{-0.002t} & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

with t in hours. Then

$$R(t) = \int_t^\infty 0.002e^{-0.002t'}\,dt' = e^{-0.002t}$$

and

$$\text{MTTF} = \int_0^\infty e^{-0.002t}\,dt = \left.\frac{e^{-0.002t}}{-0.002}\right|_0^\infty = \frac{1}{0.002} = 500 \text{ hr}$$

To find the median time to failure, set

$$R(t_{\text{med}}) = e^{-0.002t_{\text{med}}} = 0.5$$

Then solving for t_{med},

$$t_{\text{med}} = \frac{\ln 0.5}{-0.002} = 346.6 \text{ hr}$$

To find the mode, we observe that the function $f(t)$ is monotonically decreasing and positive. Therefore its maximum value occurs at $t = 0$, and $t_{\text{mode}} = 0$.

EXAMPLE 2.3. Even if two reliability functions have the same mean, their reliabilities may be quite different for the same operating time. For example, let

$$R_1(t) = e^{-0.002t} \qquad t \ge 0$$

with $\text{MTTF}_1 = 500$ hr (as shown in Example 2.2),

and

$$R_2(t) = \frac{1000 - t}{1000} \qquad 0 \le t \le 1000$$

where

$$\text{MTTF}_2 = \int_0^{1000}\left(1 - \frac{t}{1000}\right)dt = t - \left.\frac{t^2}{2000}\right|_0^{1000} = 500 \text{ hr}$$

We compute their reliabilities for an operating time of 400 hr. For $R_1(t)$ we obtain

$$R_1(400) = e^{-0.002(400)} = 0.449$$

and for $R_2(t)$,

$$R_2(400) = \frac{1000 - 400}{1000} = 0.60$$

Obviously, the MTTF alone will not uniquely characterize a failure distribution. Other measures are necessary. One measure that is often used to further describe a failure distribution is its variance σ^2, defined by

$$\sigma^2 = \int_0^\infty (t - \text{MTTF})^2 f(t)\,dt \tag{2.11}$$

The variance represents an average squared distance a failure time will be from the MTTF. It is a measure of the spread, or dispersion, of the failure times about the mean. It is shown in Appendix 2B that the variance can also be written as

$$\sigma^2 = \int_0^\infty t^2 f(t)\,dt - (\text{MTTF})^2 \tag{2.12}$$

Equation (2.12) is computationally simpler than the definitional Eq. (2.11). The square root of the variance is the standard deviation. Since the standard deviation will be measured in the same time units as the random variable, T, it is easier to interpret than the variance (which is measured in square units).

EXAMPLE 2.4. From the first failure distribution in Example 2.3, we have

$$\sigma^2 = \int_0^\infty t^2(0.002e^{-0.002t})\,dt - (500)^2 = 250{,}000$$

and $\sigma = 500$.

From the second, with $f(t) = -dR(t)/dt = 1/1000$,

$$\sigma^2 = \int_0^{1000} t^2\left(\frac{1}{1000}\right)dt - (500)^2$$

$$= \frac{t^3}{3000}\Big|_0^{1000} - (500)^2 = 83{,}333\tfrac{1}{3}$$

and $\sigma = 288.67$.

Therefore, although their MTTFs are identical, they have considerably different standard deviations, from which we would conclude that their reliability distributions should be inherently different. We would generally prefer the distribution having the smaller variance. Why?

2.3 HAZARD RATE FUNCTION

In addition to the probability functions defined earlier, another function, called the *failure rate* or *hazard rate function*, is often used in reliability. It provides an instantaneous (at time t) rate of failure. From Eq. (2.6),

$$\Pr\{t \le T \le t + \Delta t\} = R(t) - R(t + \Delta t)$$

and the conditional probability of a failure in the time interval from t to $t + \Delta t$ given that the system has survived to time t is

$$\Pr\{t \le T \le t + \Delta t \mid T \ge t\} = \frac{R(t) - R(t + \Delta t)}{R(t)}$$

Then

$$\frac{R(t) - R(t + \Delta t)}{R(t)\,\Delta t}$$

is the conditional probability of failure per unit of time (failure rate).

Set

$$\begin{aligned}\lambda(t) &= \lim_{\Delta t \to 0} \frac{-[R(t + \Delta t) - R(t)]}{\Delta t} \cdot \frac{1}{R(t)} \\ &= \frac{-dR(t)}{dt} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)}\end{aligned} \tag{2.13}$$

Then $\lambda(t)$ is known as the *instantaneous hazard rate* or *failure rate function*. The failure rate function $\lambda(t)$ provides an alternative way of describing a failure distribution. Failure rates in some cases may be characterized as increasing (IFR), decreasing (DFR), or constant (CFR) when $\lambda(t)$ is an increasing, decreasing, or constant function.

A particular hazard rate function will uniquely determine a reliability function. To see this, let

$$\lambda(t) = \frac{-dR(t)}{dt} \cdot \frac{1}{R(t)}$$

or

$$\lambda(t)\,dt = \frac{-dR(t)}{R(t)}$$

Integrating,

$$\int_0^t \lambda(t')\,dt' = \int_1^{R(t)} \frac{-dR(t')}{R(t')}$$

where $R(0) = 1$ establishes the lower limit in the integral on the right-hand side. Then

$$-\int_0^t \lambda(t')\,dt' = \ln R(t)$$

or

$$R(t) = \exp\left[-\int_0^t \lambda(t')\,dt'\right] \tag{2.14}$$

Equation (2.14) can then be used to derive the reliability function from a known hazard rate function.

EXAMPLE 2.5. Given the linear hazard rate function $\lambda(t) = 5 \times 10^{-6}t$ where t is measured in operating hours, what is the design life if a 0.98 reliability is desired?

Solution

$$R(t) = \exp\left[-\int_0^t 5 \times 10^{-6}t'\,dt'\right] = \exp\left[-2.5 \times 10^{-6}t^2\right] = 0.98$$

or
$$t_{0.98} = \sqrt{\frac{\ln 0.98}{-2.5 \times 10^{-6}}} = 89.89 \approx 90 \quad \text{hr}$$

Cumulative and average failure rate

The cumulative failure rate over a period of time t is defined by

$$L(t) = \int_0^t \lambda(t')\,dt' \tag{2.15}$$

A related and useful concept is the average failure rate, defined between two times t_1 and t_2:

$$\text{AFR}(t_1, t_2) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \lambda(t')\,dt' = \frac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1} \tag{2.16}$$

From Eq. (2.14) and letting $t_1 = 0$ and $t_2 = T$, Eq. (2.16) may be written

$$\text{AFR}(t) = \frac{\ln R(0) - \ln R(t)}{t - 0} = \frac{-\ln R(t)}{t} = \frac{L(t)}{t}$$

since $\ln R(0) = \ln 1 = 0$.

If AFR(t) is a nondecreasing function, the failure distribution is characterized as having an increasing failure rate average (IFRA). If the function is nonincreasing, the distribution has a DFRA. Obviously IFR (DFR) systems are also IFRA (DFRA), but the converse is not necessarily true.

EXAMPLE 2.5 (CONTINUED). The cumulative failure rate is given by

$$L(t) = \int_0^t 5 \times 10^{-6} t'\,dt' = 2.5 \times 10^{-6} t^2$$

and the average failure rate from time 0 to t is

$$\text{AFR}(t) = \frac{2.5 \times 10^{-6} t^2}{t} = 2.5 \times 10^{-6} t$$

EXAMPLE 2.6. A component has a reliability function given by

$$R(t) = 1 - \frac{t^2}{a^2} \qquad \text{for } 0 \le t \le a$$

where a is a parameter of the distribution representing the component's maximum life. Then

$$f(t) = \frac{2t}{a^2} \qquad \text{and} \qquad \lambda(t) = \frac{2t/a^2}{(a^2 - t^2)/a^2} = \frac{2t}{a^2 - t^2} \qquad \text{for } 0 \le t \le a$$

$$\text{MTTF} = \int_0^a \left(1 - \frac{t^2}{a^2}\right) dt = t - \frac{t^3}{3a^2}\bigg|_0^a = \frac{2}{3}a$$

$$1 - \frac{t_{\text{med}}^2}{a^2} = 0.5; \qquad t_{\text{med}} = \sqrt{0.5a^2} = 0.707a$$

$$\text{AFR}(t) = \frac{-\ln\left(1 - t^2/a^2\right)}{t}$$

The average failure rate up to the MTTF is

$$\text{AFR(MTTF)} = \text{AFR}\left(\frac{2}{3}a\right) = -\ln\left(1 - \frac{(4/9)a^2}{a^2}\right) \bigg/ \left(\frac{2}{3}a\right) = \frac{0.8817}{a}$$

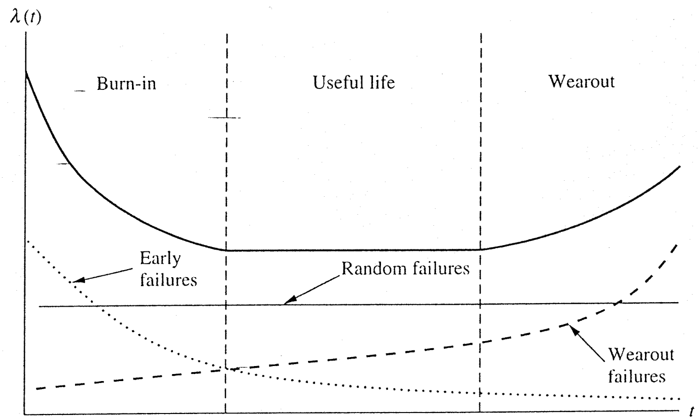


FIGURE 2.3
The bathtub curve.

2.4 BATHTUB CURVE

An important form of the hazard rate function is shown in Fig. 2.3. Because of its shape, it is commonly referred to as the *bathtub curve.* Systems having this hazard rate function experience decreasing failure rates early in their life cycle (infant mortality), followed by a nearly constant failure rate (useful life), followed by an increasing failure rate (wearout). This curve may be obtained, as shown later, as a composite of several failure distributions, or as shown in the following example, as a function of piecewise linear and constant failure rates. Table 2.1 summarizes some of the distinguishing features of the bathtub curve.

EXAMPLE 2.7. A simplified form of the bathtub curve is based upon linear and constant hazard rates:

$$\lambda(t) = \begin{cases} c_0 - c_1 t + \lambda & 0 \le t \le \dfrac{c_0}{c_1} \\ \lambda & \dfrac{c_0}{c_1} < t \le t_0 \\ c_2(t - t_0) + \lambda & t_0 < t \end{cases}$$

Then

$$R(t) = \begin{cases} \exp-\left\{(c_0 + \lambda)t - c_1(t^2/2)\right\} & 0 \le t \le \dfrac{c_0}{c_1} \\ \exp-\left(\lambda t + \dfrac{c_0^2}{2c_1}\right) & \dfrac{c_0}{c_1} < t \le t_0 \\ \exp-\left\{\left(\dfrac{c_2}{2}\right)(t - t_0)^2 + \lambda t + \left(\dfrac{c_0^2}{2c_1}\right)\right\} & t_0 < t \end{cases}$$

where c_0, c_1, c_2, and t_0 are constants to be determined. Figure 2.4 portrays this hazard rate graphically. Appendix 2D derives $R(t)$ from $\lambda(t)$.

TABLE 2.1
The bathtub curve

	Characterized by	Caused by	Reduced by
Burn-in	DFR	Manufacturing defects: welding flaws, cracks, defective parts, poor quality control, contamination, poor workmanship	Burn-in testing Screening Quality control Acceptance testing
Useful life	CFR	Environment Random loads Human error "Acts of God" Chance events	Redundancy Excess strength
Wear-out	IFR	Fatigue Corrosion Aging Friction Cyclical loading	Derating Preventive maintenance Parts replacement Technology

2.5 CONDITIONAL RELIABILITY

Conditional reliability is useful in describing the reliability of a component or system following a burn-in period T_0 or after a warranty period T_0. We define conditional reliability as the reliability of a system given that it has operated for time T_0:

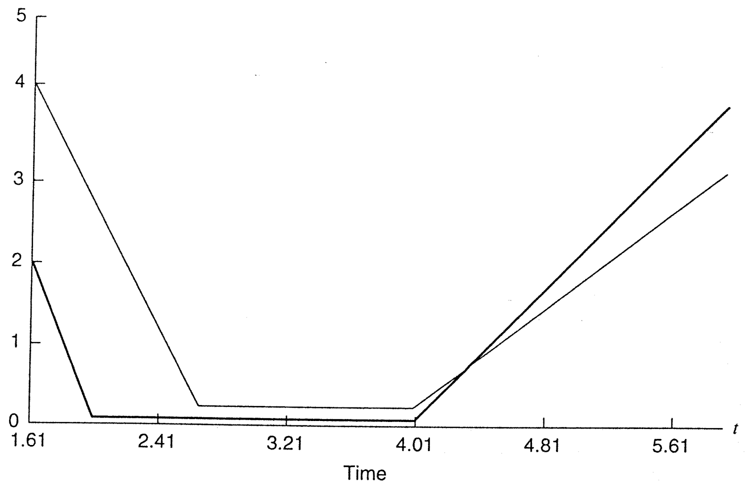


FIGURE 2.4
A piecewise linear bathtub curve.

$$\begin{aligned} R(t \mid T_0) &= \Pr\{T > T_0 + t \mid T > T_0\} \\ &= \frac{\Pr\{T > T_0 + t\}}{\Pr\{T > T_0\}} = \frac{R(T_0 + t)}{R(T_0)} \\ &= \frac{\exp\left[-\int_0^{T_0+t} \lambda(t')\,dt'\right]}{\exp\left[-\int_0^{T_0} \lambda(t')\,dt'\right]} = \exp\left[-\int_{T_0}^{T_0+t} \lambda(t')\,dt'\right] \end{aligned} \tag{2.17}$$

EXAMPLE 2.8. Let

$$\lambda(t) = \frac{0.5}{1000}\left(\frac{t}{1000}\right)^{-0.5} \qquad t \text{ in years}$$

which is a DFR. Then from Eq. (2.14), for a reliability of 0.90

$$R(t) = \exp - \left(\frac{t}{1000}\right)^{1/2} = 0.90$$

and the design life is found from

$$t = 1000(-\ln 0.90)^2 = 11.1 \text{ yr}$$

If we let $T_0 = 0.5$, a six-month burn-in period, then

$$R(t \mid T_0) = \frac{R(t + 0.5)}{R(0.5)}$$

$$= \frac{\exp - [(t + 0.5)/1000]^{0.5}}{\exp - (0.5/1000)^{0.5}} = 0.90$$

and

$$t = 1000\left[-\ln 0.90 + \left(\frac{0.5}{1000}\right)^{0.5}\right]^2 - 0.5 = 15.8 \text{ yr}$$

This is an increase of over 4 years in the design life as a result of a six-month burn-in period. This improvement in reliability resulting from a burn-in period T_0 will only be realized for a DFR as illustrated in the following example and shown in Appendix 2C.

EXAMPLE 2.9. Let $\lambda(t) = \lambda t$, an IFR for $\lambda > 0$. Then

$$R(t) = e^{-(1/2)\lambda t^2}$$

and

$$R(t \mid T_0) = \frac{e^{-(1/2)\lambda(t+T_0)^2}}{e^{-(1/2)\lambda T_0^2}}$$

which can be simplified to

$$R(t \mid T_0) = e^{-\lambda T_0 t} e^{-(1/2)\lambda t^2}$$

Since $\exp(-\lambda T_0 t)$ for $\lambda > 0$ is a decreasing function of T_0, the conditional reliability will decrease as the burn-in period T_0 increases.

EXAMPLE 2.10. For the reliability function given in Example 2.6, the conditional reliability is

$$R(t \mid T_0) = \frac{1-(t+T_0)^2/a^2}{1-T_0^2/a^2} = \frac{a^2-(t+T_0)^2}{a^2-T_0^2}$$

Residual MTTF

Since $R(t \mid T_0)$ is a reliability function, a residual MTTF may be obtained from

$$\mathrm{MTTF}(T_0) = \int_0^\infty R(t \mid T_0)\,dt = \int_{T_0}^\infty \frac{R(t')}{R(T_0)}\,dt' = \frac{1}{R(T_0)}\int_{T_0}^\infty R(t')\,dt' \quad (2.18)$$

where $t' = t + T_0$. For those units having survived to time T_0, Eq. (2.18) determines their mean remaining lifetime. For components having an IFR (DFR), one would expect the MTTF(T_0) to be a decreasing (increasing) function of T_0, as shown in the following examples.

EXAMPLE 2.11. The reliability function $R(t) = (b-t)/b$ for $0 \le t \le b$ and zero elsewhere has an IFR (see Exercise 2.8). Its residual MTTF is given by

$$\mathrm{MTTF}(T_0) = \left(\frac{b-T_0}{b}\right)^{-1}\int_{T_0}^{b} \frac{b-t'}{b}\,dt' = \frac{b}{b-T_0}\,\frac{(b-t')^2}{-2b}\bigg|_{T_0}^{b} = \frac{(b-T_0)}{2}$$

for $0 \le T_0 \le b$.

EXAMPLE 2.12. The reliability function

$$R(t) = \frac{a^2}{(a+t)^2} \qquad t \ge 0$$

where $a > 0$ is a parameter (constant) of the distribution, has the hazard rate function

$$\lambda(t) = \frac{2}{a+t}$$

which is decreasing. The residual MTTF is

$$\mathrm{MTTF}(T_0) = \frac{(a+T_0)^2}{a^2}\int_{T_0}^\infty \frac{a^2}{(a+t')^2}\,dt' = \frac{(a+T_0)^2}{a^2}\,\frac{-a^2}{a+t'}\bigg|_{T_0}^\infty = a + T_0$$

which has the interesting property that the residual mean increases by the amount of the current age. If $T_0 = 0$, the unconditional mean, MTTF $= a$, is obtained.

2.6 SUMMARY

A failure process, represented by the random variable T (time to failure), may be uniquely characterized by any of the following four functions:

1. $f(t)$, the probability density function (PDF)
2. $F(t)$, the cumulative distribution function (CDF)
3. $R(t)$, the reliability function
4. $\lambda(t)$, the hazard rate function

The following relationships hold:

$$F(t) = \int_0^t f(t')\,dt' \qquad R(t) = \int_t^\infty f(t')\,dt'$$

$$R(t) = 1 - F(t) \qquad f(t) = \frac{-dR(t)}{dt} = \frac{dF(t)}{dt}$$

$$\lambda(t) = \frac{f(t)}{R(t)}$$

$$R(t) = \exp\left[-\int_0^t \lambda(t')\,dt'\right]$$

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt$$

$$\sigma^2 = \int_0^\infty t^2 f(t)\,dt - (\mathrm{MTTF})^2$$

$$R(t \mid T_0) = \frac{R(t+T_0)}{R(T_0)}$$

$$L(t) = \int_0^t \lambda(t')\,dt'$$

$$\mathrm{AFR}(t_1, t_2) = \frac{\int_{t_1}^{t_2} \lambda(t)\,dt}{t_2 - t_1} = \frac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1}$$

$$\mathrm{MTTF}(T_0) = \frac{1}{R(T_0)}\int_{T_0}^\infty R(t')\,dt'$$

APPENDIX 2A DERIVATION OF EQUATION (2.8)

By definition,

$$\mathrm{MTTF} = \int_0^\infty t f(t)\,dt$$

From Eq. (2.3),

$$\mathrm{MTTF} = \int_0^\infty -\frac{dR(t)}{dt}\,t\,dt$$

Using integration by parts,

$$\mathrm{MTTF} = -tR(t)\Big|_0^\infty + \int_0^\infty R(t)\,dt$$

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt \qquad (2.8)$$

since $$\lim_{t\to\infty} t\,R(t) = \lim_{t\to\infty} t\exp\left[-\int_0^t \lambda(t')\,dt'\right] = 0$$

and $$0 \cdot R(0) = 0$$

APPENDIX 2B DERIVATION OF EQUATION (2.12)

$$\sigma^2 = \int_0^\infty (t - \text{MTTF})^2 f(t)\,dt$$

$$= \int_0^\infty \left[t^2 - 2t \cdot \text{MTTF} + (\text{MTTF})^2\right] f(t)\,dt$$

$$= \int_0^\infty t^2 f(t)\,dt - 2\text{MTTF}\int_0^\infty t\,f(t)\,dt + (\text{MTTF})^2 \int_0^\infty f(t)\,dt$$

$$= \int_0^\infty t^2 f(t)\,dt - 2(\text{MTTF})^2 + (\text{MTTF})^2$$

since $$\int_0^\infty t f(t)\,dt = \text{MTTF} \qquad \text{and} \qquad \int_0^\infty f(t)\,dt = 1$$

Therefore $$\sigma^2 = \int_0^\infty t^2 f(t)\,dt - (\text{MTTF})^2 \tag{2.12}$$

APPENDIX 2C CONDITIONAL RELIABILITY AND FAILURE RATES

From Eq. (2.17),

$$R(t \mid T_0) = \frac{R(T_0 + t)}{R(T_0)} = \exp\left[-\int_{T_0}^{t+T_0} \lambda(t')\,dt'\right]$$

Therefore $$\frac{dR(t \mid T_0)}{dT_0} = \exp\left[-\int_{T_0}^{t+T_0} \lambda(t')\,dt'\right] \frac{d}{dT_0}\left[\int_{T_0}^{t+T_0} \lambda(t')\,dt'\right]$$

$$= R(t \mid T_0)[-\lambda(t + T_0) + \lambda(T_0)]$$

In order for the reliability to improve as a function of T_0, the derivative, or slope, of $R(t \mid T_0)$ with respect to T_0 must be positive. From the above result, this will occur only if $\lambda(T_0) > \lambda(t + T_0)$. In other words, the failure rate must be decreasing.

APPENDIX 2D INTERMEDIATE CALCULATIONS FOR THE LINEAR BATHTUB CURVE

For $t \le c_0/c_1$:

$$R(t) = \exp\left[-\int_0^t (c_0 - c_1 t' + \lambda)\,dt'\right]$$

$$= \exp\left[-(c_0 t - \frac{c_1 t^2}{2} + \lambda t)\right]$$

For $c_0/c_1 < t \le t_0$:

$$R(t) = R\left(\frac{c_0}{c_1}\right)\exp\left[-\int_{c_0/c_1}^{t} \lambda\,dt'\right]$$

$$= \exp\left(-\frac{c_0^2}{c_1} + c_1\frac{c_0^2}{2c_1^2} - \lambda\frac{c_0}{c_1}\right)\exp -\left(\lambda t - \lambda\frac{c_0}{c_1}\right)$$

$$= \exp -\left(\lambda t + \frac{c_0^2}{2c_1}\right)$$

For $t_0 < t$:

$$R(t) = R(t_0)\exp\left[-\int_{t_0}^{t} [c_2(t' - t_0) + \lambda]\,dt'\right]$$

$$= \exp -\left(\lambda t_0 + \frac{c_0^2}{2c_1}\right)\exp -\left(\frac{c_2}{2}(t - t_0)^2 + \lambda t - \lambda t_0\right)$$

$$= \exp -\left(\frac{c_2}{2}(t - t_0)^2 + \lambda t + \frac{c_0^2}{2c_1}\right)$$

APPENDIX 2E
TABLE OF INTEGRALS

2E.1
INDEFINITE INTEGRALS

1. $$\int a\,dt = at$$

2. $$\int t^n\,dt = \frac{t^{n+1}}{n+1} \qquad n \neq 1$$

3. $$\int \frac{dt}{t} = \ln t$$

4. $$\int e^{at}\,dt = \frac{e^{at}}{a}$$

5. $$\int (a+bt)^n\,dt = \frac{(a+bt)^{n+1}}{(n+1)b} \qquad n \neq -1$$

6. $$\int te^{at}\,dt = \frac{e^{at}}{a^2}(at-1)$$

7. $$\int u\,dv = uv - \int v\,du \qquad \text{(integration by parts)}$$

2E.2
DEFINITE INTEGRALS

8. $$\int_0^\infty te^{-at}\,dt = \frac{1}{a^2} \qquad a > 0$$

9. $$\int_0^\infty t^n e^{-at}\,dt = \frac{n!}{a^{n+1}} \qquad a > 0, \; n \text{ a positive integer}$$

EXERCISES

2.1 Consider the following reliability function, where t is in hours:

$$R(t) = \frac{1}{0.001t + 1} \qquad t \geq 0$$

(a) Find the reliability after 100 operating hours; after 1000 operating hours.
(b) Derive the hazard rate function. Is it an increasing or a decreasing failure rate?

2.2 A component has the following linear hazard rate, where t is in years:

$$\lambda(t) = 0.4t \qquad t \geq 0$$

(a) Find $R(t)$ and determine the probability of a component failing within the first month of its operation.
(b) What is the design life if a reliability of 0.95 is desired?

2.3 The time-to-failure density function (PDF) for a system is

$$f(t) = 0.01 \qquad 0 \leq t \leq 100 \text{ days}$$

Find
(a) $R(t)$
(b) The hazard rate function
(c) The MTTF
(d) The standard deviation
(e) The median time to failure

2.4 The failure distribution is defined by

$$f(t) = \frac{3t^2}{10^9} \qquad 0 \leq t \leq 1000 \text{ hr}$$

(a) What is the probability of failure within a 100-hr warranty period?
(b) Compute the MTTF.
(c) Find the design life for a reliability of 0.99.

2.5 For

$$R(t) = e^{-\sqrt{0.001t}} \qquad t \geq 0$$

(a) Compute the reliability for a 50-hr mission.
(b) Show that the hazard rate is decreasing.
(c) Given a 10-hr burn-in period, compute the reliability for a 50-hr mission.
(d) What is the design life for a reliability of 0.95 given a 10-hr burn-in?

2.6 The PDF for the time to failure in years of the drivetrain on a Regional Transit Authority bus is given by

$$f(t) = 0.2 - 0.02t \qquad 0 \leq t \leq 10 \text{ yr}$$

(a) Show that the hazard rate function is increasing, indicating continuous wearout over time.
(b) Find the MTTF.
(c) Find the median time to failure.
(d) Find the mode of the failure distribution.
(e) Compute the standard deviation.

2.7 The Itwill Fail Company manufactures gizmos for use on widgets. The time to failure, in years, of these gizmos has the following PDF:

$$f(t) = \frac{200}{(t+10)^3} \qquad \text{for } t \geq 0$$

(a) Derive the reliability function and determine the reliability for the first year of operation.
(b) Compute the MTTF.
(c) What is the design life for a reliability of 0.95?

(d) Is the failure rate DFR, CFR, or IFR?
(e) Will a one-year burn-in period improve the reliability in part (a)? If so, what is the new reliability?

2.8 A uniform failure distribution has the characteristic that equal intervals of time have equal failure probabilities. The density function is given by

$$f(t) = \frac{1}{b} \qquad 0 \le t \le b$$

Analyze this general failure distribution by finding $F(t)$, $R(t)$, $\lambda(t)$, MTTF, T_{med}, T_{mode}, and σ.

2.9 Repeat Exercise 2.8 assuming that $b = 20$ yr.

2.10 The reliability of a turbine blade can be represented by the following:

$$R(t) = \left(1 - \frac{t}{t_0}\right)^2 \qquad 0 \le t \le t_0$$

where t_0 is the maximum life of the blade.
(a) Show that the blades are experiencing wearout.
(b) Compute the MTTF as a function of the maximum life.
(c) If the maximum life is 2000 operating hours, what is the design life for a reliability of 0.90?

2.11 A new fuel injection system is experiencing high failure rates. The reliability function has been found to be

$$R(t) = (t + 1)^{-3/2} \qquad t \ge 0$$

where t is measured in years. The reliability over its intended life of 2 yr is 0.19, which is unacceptable. Will a burn-in period of 6 months significantly improve upon this reliability? If so, by how much?

2.12 Using Eq. (2.14), derive the reliability function for a system having a linearly increasing hazard rate function, i.e., $\lambda = at$, $a > 0$, $t \ge 0$. Find the density function, and derive expressions for the median time to failure and the mode.

2.13 Compute the average failure rate for the failure distribution in Exercise 2.4. What is the average failure rate over the first 500 hours?

2.14 A household appliance is advertised as having more than a 10-yr life. If the following is its PDF, determine its reliability for the next 10 yr if it has survived a 1-yr warranty period:

$$f(t) = 0.1(1 + 0.05t)^{-3} \qquad t \ge 0$$

What is its MTTF before the warranty period, and what is its MTTF after the warranty period assuming it has still survived?

2.15 Show that if the hazard rate function is decreasing, the PDF, $f(t)$, is also a decreasing function and its mode must therefore occur at $t = 0$.

2.16 Find $f(t)$ and $R(t)$ when the hazard rate function is exponential. That is, $\lambda(t) = ae^t$, where a is a constant and $a > 0$. This is one form of an extreme value distribution that has been used to model wearout due to corrosion. It is also known as Gumbel's distribution.

CHAPTER 3

Constant Failure Rate Model

These next two chapters will develop several probability models useful in describing a failure process. These models are based upon the exponential, Weibull, normal, and lognormal probability distributions. They are often referred to as theoretical distributions since they are derived mathematically and not empirically. An important question to be answered is that of the ability of a particular theoretical distribution to describe the failures and reliability of a component or a system. The answer to that question is a central theme in Part II of the text. The immediate concern, however, is the development and use of these theoretical models in analyzing a failure process.

A failure distribution that has a constant failure rate is called an *exponential probability distribution.* The exponential distribution is one of the most important reliability distributions. Many systems exhibit constant failure rates, and the exponential distribution is in many respects the simplest reliability distribution to analyze. Another useful concept, failure modes, is also discussed. If the failure rates of all failure modes of a component are constant and independent, then the overall failure rate of the component is also constant.

3.1 THE EXPONENTIAL RELIABILITY FUNCTION

One of the most common failure distributions in reliability engineering is the exponential, or CFR, model. Failures due to completely random or chance events will follow this distribution. It should dominate during the useful life of a system or component. It is also one of the easiest distributions to analyze statistically. For example, good parameter estimators and exact confidence intervals can be easily computed under different testing conditions (e.g., in the presence of censoring).[1]

We begin the development of the CFR model by assuming that $\lambda(t) = \lambda, t \geq 0$, $\lambda > 0$. Then from Eq. (2.14)

$$R(t) = \exp\left[-\int_0^t \lambda\, dt'\right] = e^{-\lambda t}, \quad t \geq 0 \tag{3.1}$$

and $$F(t) = 1 - e^{-\lambda t}$$

Then

$$f(t) = -\frac{dR(t)}{dt} = \lambda e^{-\lambda t}$$

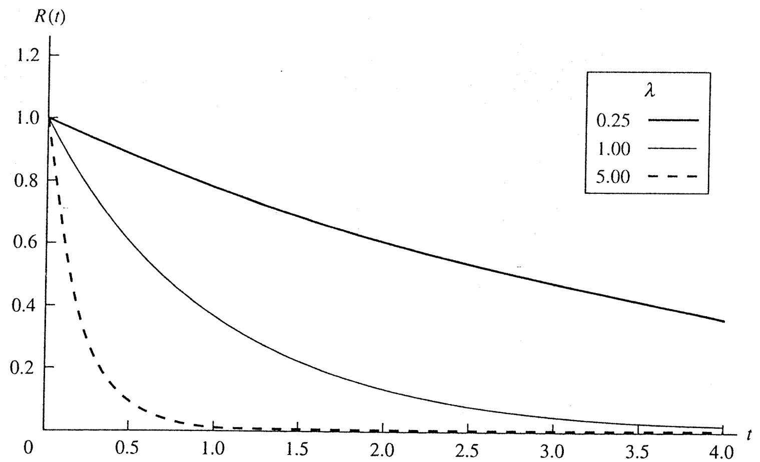
The three probability functions are illustrated graphically in Fig. 3.1 for several different values of λ. To find the MTTF, we use Eq. (2.8):

$$\text{MTTF} = \int_0^\infty e^{-\lambda t}\, dt = \left.\frac{e^{-\lambda t}}{-\lambda}\right|_0^\infty = \frac{1}{\lambda} \tag{3.2}$$

The variance is given by Eq. (2.11), or

$$\sigma^2 = \int_0^\infty \left(t - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda t}\, dt = \frac{1}{\lambda^2} \tag{3.3}$$

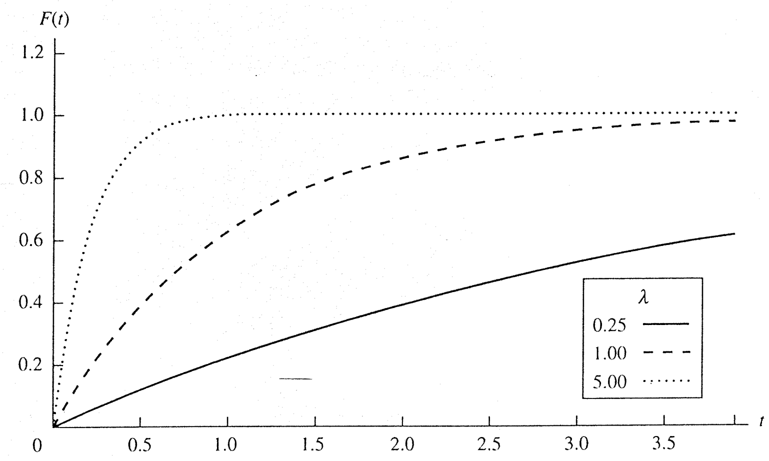
and the standard deviation is $1/\lambda$ = MTTF. This is an interesting result since it implies that the variability of failure time increases as the reliability (MTTF) increases.



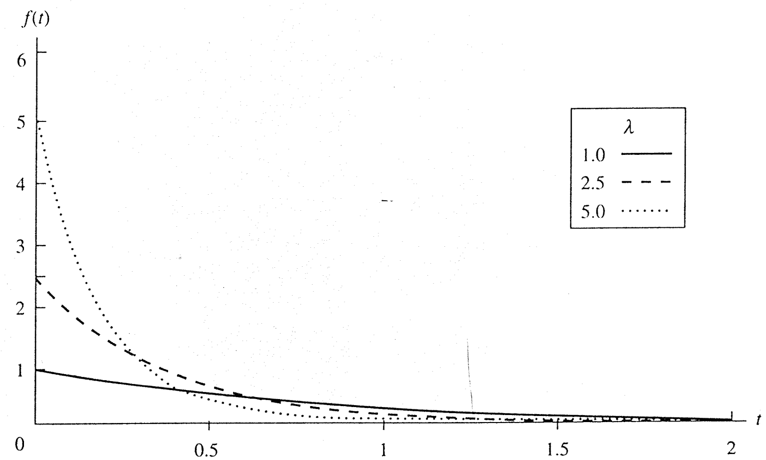
(a)

FIGURE 3.1
(a) The exponential reliability function.

[1]Censoring will be discussed later. It requires the analysis of incomplete failure data (not all items tested have failed).



(b)



(c)

FIGURE 3.1 *(continued)*
(b) The exponential cumulative distribution function. *(c)* The exponential probability density function.

High variability in failure times is often observed in practice. It should also be noted that the mean time to failure is the reciprocal of the failure rate. Although $1/\lambda(t)$ is always in units of time per (between) failures, it is the mean of the failure distribution for the CFR model only.

A second observation concerning this distribution is that

$$R(\text{MTTF}) = e^{-\text{MTTF}/\text{MTTF}} = e^{-1} = 0.368$$

A component having a CFR has a slightly better than one-third chance of surviving to its mean time to failure. The design life of a component having an exponentially distributed failure time may be obtained by solving for the inverse of the reliability function. That is, for a given reliability R,

$$R(t_R) = e^{-\lambda t_R} = R$$

then

$$t_R = -\frac{1}{\lambda} \ln R \tag{3.4}$$

When $R = 0.5$, the median of the distribution is obtained from Eq. (3.4):

$$\begin{aligned} t_{\text{med}} &= -\frac{1}{\lambda} \ln 0.5 = \frac{0.69315}{\lambda} \\ &= 0.69315 \text{ MTTF} \end{aligned} \tag{3.5}$$

The median is always less than the mean since the exponential distribution is skewed to the right.

EXAMPLE 3.1. A microwave transmitter has exhibited a constant failure rate of 0.00034 failure per operating hour. Therefore MTTF = 1/0.00034 = 2941 hr and $t_{\text{med}} = 0.69315 \times 2941 = 2039$ hr. The reliability function is given by $R(t) = e^{-0.00034t}$. The reliability over a 30-day continuous operating period is $R(30 \times 24) = 0.78286$. The design life for a reliability of 0.95 specification is $(-\ln 0.95)/0.00034 = 150.86$ hr.

Memorylessness

A well-known characteristic of the CFR model, one not shared by other failure distributions, is its lack of memory. That is, the time to failure of a component is not dependent on how long the component has been operating. There is no aging or wearout effect. The probability that the component will operate for the next 1000 hr is the same regardless of whether the component is brand new, has been operating for several hundred hours, or has been operating for several thousand hours. This property is consistent with the completely random and independent nature of the failure process. For example, when external, random environmental stresses are the primary cause of failures, the failure or operating history of the component will not be relevant.

This property can be demonstrated mathematically using conditional reliability. From Eq. (2.17),

$$\begin{aligned} R(t \mid T_0) &= \frac{R(t + T_0)}{R(T_0)} = \frac{e^{-\lambda(t+T_0)}}{e^{-\lambda T_0}} \\ &= \frac{e^{-\lambda t} \cdot e^{-\lambda T_0}}{e^{-\lambda T_0}} = e^{-\lambda t} = R(t) \end{aligned}$$

In other words, a burn-in period T_0 has no subsequent effect on reliability and will not improve the component's reliability. Time to failure depends only on the length of the observed operating time (t) and not on its current age (T_0).

3.2 FAILURE MODES

Complex systems will fail through various means resulting from different physical phenomena or different failure characteristics of individual components. A useful analysis approach in reliability engineering is to separate these failures according to the mechanisms or components causing the failures. These categories of failures are then referred to as failure modes.[2]

If $R_i(t)$ is the reliability function for the ith failure mode, then, assuming *independence* among the failure modes, the system reliability $R(t)$ is obtained as

$$R(t) = \prod_{i=1}^{n} R_i(t) \tag{3.6}$$

$R_i(t)$ is the probability that the ith failure mode does not occur before time t, so $R(t)$, defined as the product of these probabilities, is the probability that none of the n failure modes occurs before time t. The system hazard rate function may be derived from Eq. (3.6), or it may be obtained directly by summing the hazard rate functions of all failure modes, as shown below.

Let $\lambda_i(t)$ be the failure rate function for the ith failure mode. Then

$$R_i(t) = \exp\left[-\int_0^t \lambda_i(t')\,dt'\right]$$

and

$$\begin{aligned} R(t) &= \prod_{i=1}^{n} \exp\left[-\int_0^t \lambda_i(t')\,dt'\right] \\ &= \exp\left[-\int_0^t \sum_{i=1}^{n} \lambda_i(t')\,dt'\right] \\ &= \exp\left[-\int_0^t \lambda(t')\,dt'\right] \end{aligned}$$

where

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t) \tag{3.7}$$

This is an important result stating that the hazard rate function for the system is determined by summing the hazard rate functions of the n independent failure modes. Reliability testing and analysis can be conducted for different failure modes and a composite reliability function determined. This result also provides an alternative model of the bathtub curve, as shown in the following example.

[2]Failure modes are an example of a series relationship. Serial relationships are discussed further in Chapter 5.

EXAMPLE 3.2. BATHTUB CURVE. Define three failure modes: (1) a burn-in effect, (2) a constant failure rate, and (3) aging or wearout phenomena. That is,

$$\lambda_1(t) = \frac{1}{20}\left(\frac{t}{10}\right)^{-1/2} \qquad \lambda_2(t) = 0.01 \qquad \lambda_3(t) = \frac{1}{50}\left(\frac{t}{100}\right)$$

Then[3]

$$\lambda(t) = \frac{1}{20}\left(\frac{t}{10}\right)^{-1/2} + 0.01 + \frac{1}{50}\left(\frac{t}{100}\right)$$

and

$$R(t) = \exp - \left[\left(\frac{t}{10}\right)^{1/2} + 0.01t + \left(\frac{t}{100}\right)^2\right]$$

For this system all three effects of the bathtub curve are always present. However, during its early life infant mortality dominates (DFR). This is followed by a period when the failure rate remains relatively constant (its useful life) until the wearout (IFR) failure mode becomes dominant.

3.2.1 Failure Modes with CFR Model

If a system consists of n independent, serially related components[4] each having a constant failure rate λ_i, then from Eq. (3.7),

$$\lambda(t) = \lambda = \sum_{i=1}^{n} \lambda_i$$

and

$$R(t) = \exp\left[-\int_0^t \lambda \, dt'\right] = \exp[-\lambda t]$$

where

$$\text{MTTF} = \frac{1}{\lambda} = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} 1/\text{MTTF}_i}; \qquad \text{MTTF}_i = \frac{1}{\lambda_i} \tag{3.8}$$

In other words, the system itself will have an exponential failure time (CFR model). If the components are also all identical (i.e., $\lambda_i = \lambda_1$ for $i = 2, 3, \ldots, n$), then

$$\lambda = n\lambda_1 \qquad \text{and} \qquad \text{MTTF} = \frac{1}{n\lambda_1} \tag{3.9}$$

EXAMPLE 3.3. An aircraft engine consists of three modules having constant failure rates of $\lambda_1 = 0.002$, $\lambda_2 = 0.015$, and $\lambda_3 = 0.0025$ failure per operating hour. The reliability function for the engine is given by

$$R(t) = e^{-(0.002+0.015+0.0025)t} = e^{-0.0195t}$$

The MTTF is $1/0.0195 = 51.28$ operating hours.

[3] As will be shown later, the three hazard rate terms are from Weibull failure distributions. However, the composite hazard rate $\lambda(t)$ is not Weibull.

[4] Serially related means that the system will fail when the first component fails.

3.2.2 Failures on Demand

Certain components that operate on a cyclical basis may fail on demand. For example, an air conditioning and heating system whose cycle time includes an operating time and an idle time may fail when switching from an idle to an operating mode. Lightbulbs are more likely to fail when being turned on than while operating. If a constant failure rate can be assumed for each failure mode and a constant probability of failure on demand can be assumed, then the component will have an effective constant failure rate on a clock-hour basis. To see this, let

λ_I = the average failure rate while idle (may be zero; e.g., failures per idle hour)
λ_O = the average failure rate while operating (e.g., failures per operating hour)
p = the probability of failure on demand
t_I = average length of the idle time period per cycle
t_O = average length of the operating time per cycle

Then

$$\lambda_{\text{eff}} = \frac{t_I}{t_I + t_O}\lambda_I + \frac{t_O}{t_I + t_O}\lambda_O + \frac{p}{t_I + t_O} \tag{3.10}$$

is the failure rate in failures per clock unit of time.

EXAMPLE 3.4. An air conditioning compressor operates once for an average time of 20 minutes each hour. While operating, it has experienced a failure rate of 0.01 failure per operating hour, and while idle has experienced a dormant failure rate of 0.0002 failure per idle hour. The probability that the compressor fails on demand is 0.03. Therefore,

$$\lambda_{\text{eff}} = \frac{40}{60}(0.0002) + \frac{20}{60}(0.01) + \frac{0.03}{60} = 0.003967$$

The probability that the compressor will not fail over a 24-hr period is

$$R(24) = e^{-0.003967(24)} = 0.9092$$

3.3 APPLICATIONS

There are several interesting physical processes that give rise to the exponential probability distribution. Previously it was shown that an exponential distribution is observed if the failure rate is constant. A constant failure rate implies completely random and independent failures over time and hence results in lack of memory. In fact these three characteristics—randomness, constant failure rates, and memorylessness—are different forms of the same phenomenon. Earlier it was shown that given the exponential distribution, the memoryless property followed. Consider the converse: given $R(t \mid T_0) = R(t)$, does the CFR apply? From Eqs. (2.17) and (2.14) for any T_0,

$$R(t \mid T_0) = \frac{\exp\left[-\int_0^{T_0+t} \lambda(t')\,dt'\right]}{\exp\left[-\int_{0_-}^{T_0} \lambda(t')\,dt'\right]} = \exp\left[-\int_{T_0}^{T_0+t} \lambda(t')\,dt'\right]$$

$$= R(t) = \exp\left[-\int_0^t \lambda(t')\,dt'\right]$$

Then
$$\int_{T_0}^{T_0+t} \lambda(t')\,dt' = \int_0^t \lambda(t')\,dt'$$

which requires $\lambda(t) = \lambda$, a constant. Therefore if failures are determined by completely random independent events not associated with the age of the system, a constant failure rate model will result.

3.3.1 Renewal Process

Assume that a system is comprised of many components acting independently and in such a way that an individual component failure causes a system failure. A renewal process, in which a failed component is immediately replaced with a new one, will cause the system to reach a steady-state, constant number of failures per unit of time. This is best illustrated by a numerical example.

EXAMPLE 3.5. Consider the following discrete failure distribution for each of 1000 identical components.

$$\Pr\{T = n\} = \frac{n}{15} \qquad \text{for } n = 1, 2, 3, 4, 5$$

where T represents the number of operating cycles until failure. The number of failures (and hence replacements each cycle) is shown in Table 3.1. Since each component is identical,

$$\text{MTTF}_i = \tfrac{1}{15} + 2\left(\tfrac{2}{15}\right) + 3\left(\tfrac{3}{15}\right) + 4\left(\tfrac{4}{15}\right) + 5\left(\tfrac{5}{15}\right) = \tfrac{11}{3}$$

where MTTF_i is the mean time (number of operating cycles) to failure for the ith component. The system mean time between failures is $(11/3)/1000 = 0.0036667$, and a steady-state failure rate of $1/0.0036667 = 272.7$ failures per cycle is observed. In general, for a renewal process in which failed components are replaced as they fail, a steady-state constant failure rate λ is obtained such that

$$\lambda = \sum_{i=1}^{n} \frac{1}{\text{MTTF}_i}$$

Renewal processes will be discussed further in Chapter 9.

3.3.2 Repetitive Loading

If there is a small constant probability of failure p as a result of a load or stress being placed on the system and if independent loads are applied at constant, fixed intervals

TABLE 3.1
System renewals

Cycle	Number of failures
1	$\frac{1}{15}(1000) = 67$
2	$\frac{2}{15}(1000) + \frac{1}{15}(67) = 138$
3	$\frac{3}{15}(1000) + \frac{2}{15}(67) + \frac{1}{15}(138) = 218$
4	$\frac{4}{15}(1000) + \frac{3}{15}(67) + \frac{2}{15}(138) + \frac{1}{15}(218) = 313$
5	$\frac{5}{15}(1000) + \frac{4}{15}(67) + \frac{3}{15}(138) + \frac{2}{15}(218) + \frac{1}{15}(313) = 429$
6	$\frac{5}{15}(67) + \frac{4}{15}(138) + \frac{3}{15}(218) + \frac{2}{15}(313) + \frac{1}{15}(429) = 173$
7	$\frac{5}{15}(138) + \frac{4}{15}(218) + \frac{3}{15}(313) + \frac{2}{15}(429) + \frac{1}{15}(173) = 235$

Continuing in this manner:

Cycle	Number of failures	Cycle	Number of failures
8	281	18	277
9	303	19	274
10	294	20	271
11	236	21	272
12	269	22	275
13	283	23	274
14	281	24	273
15	271	25	273
16	263	26	273
17	275		

A steady-state constant rate of 273 failures per cycle is reached by the 24th cycle.

of time, an approximate exponential reliability function results.[5] Let $R = (1 - p)$, the reliability of a single load. Then

$$R_n = (1 - p)^n = e^{n \ln(1-p)}$$

is the reliability given n loads. Since[6] $\ln(1 - p) \approx -p$ for very small p,

$$R_n = e^{-np}$$

Let $n = t/\Delta t$, with Δt being the fixed time between loads. Then

$$R(t) = e^{-(p/\Delta t)t} = e^{-\lambda t}$$

with $\lambda = p/\Delta t$, a constant failure rate.

[5]A similar result is obtained if the loads are applied at random if the number of loads per time period has a Poisson distribution.

[6]This follows from a first-order Taylor series approximation around the origin; i.e., $f(x) = \ln(1 - x) \approx f(0) + xf'(0) = 0 + x(-1) = -x$.

3.3.3 Reliability Bounds

Even when the CFR model does not apply, it can still be used to provide bounds on system reliability provided the system hazard rate can be bounded. Assume that

$$0 < \lambda_L \leq \lambda(t) \leq \lambda_U$$

Then
$$\int_0^t \lambda_L \, dt' \leq \int_0^t \lambda(t')\, dt' \leq \int_0^t \lambda_U \, dt'$$

and
$$\exp\left[-\int_0^t \lambda_L \, dt'\right] \geq \exp\left[-\int_0^t \lambda(t')\, dt'\right] \geq \exp\left[-\int_0^t \lambda_\mu \, dt'\right]$$

or
$$e^{-\lambda_L t} \geq R(t) \geq e^{-\lambda_U t} \tag{3.11}$$

Therefore if the hazard rate function is bounded, the reliability function can also be bounded.

The exponential, because of its constant failure rate, provides a boundary between distributions having decreasing failure rates and those having increasing failure rates. Barlow and Proschan [1967] develop the following upper and lower reliability bounds for when the hazard rate is decreasing and increasing, respectively. In addition to knowing that the process has an IFR or DFR, one must know the mean time to failure. For a DFR process,

$$R(t) \leq \begin{cases} e^{-t/\text{MTTF}} & t \leq \text{MTTF} \\ \text{MTTF}\dfrac{e^{-1}}{t} & t > \text{MTTF} \end{cases} \tag{3.12}$$

For an IFR process,

$$R(t) \geq \begin{cases} e^{-t/\text{MTTF}} & t < \text{MTTF} \\ 0 & t \geq \text{MTTF} \end{cases} \tag{3.13}$$

They also develop the following upper bound for when the hazard rate is increasing.

$$R(t) \leq \begin{cases} 1 & t \leq \text{MTTF} \\ e^{-\omega/\text{MTTF}} & t > \text{MTTF} \end{cases} \tag{3.14}$$

where
$$1 - \omega\text{MTTF} = e^{-\omega t}$$

In order for the above equation to be satisfied, a different ω must be found for each t.

EXAMPLE 3.6. A government specification calls for a mechanical pump to have an MTTF of 1000 operating hours. These pumps exhibit continuous wearout and therefore have an IFR. As a result, lower and upper bounds on the pump's reliability after 200 operating hours are given by $0.8187 \leq R(200) \leq 1$. For 2000 operating hours, $0 \leq R(2000) \leq 0.2032$, where ω was found to be 0.0007968 by solving Eq. (3.14) numerically with $t = 2000$.

EXAMPLE 3.7. The following reliability function[7] has a DFR and has an MTTF of 400.

$$R(t) = e^{-\sqrt{t/200}}$$

Comparing the actual reliability to the upper bound computed using Eq. (3.12):

t	$R(t)$	Upper bound
10	0.7996	0.9753
50	0.6060	0.8825
100	0.4930	0.7780
200	0.3678	0.6065
300	0.2938	0.4723
400	0.243	0.3679
500	0.2057	0.294
600	0.1769	0.245

3.4 THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION

If a failure will never occur prior to some specified time t_0, then t_0 is a minimum, or threshold, time. It is also known as the *guaranteed lifetime.* The parameter t_0 is a location parameter that shifts the distribution an amount equal to t_0 to the right on the time (horizontal) axis. This is equivalent to rewriting the density function by replacing t with $t - t_0$, with the domain of the random variable now $t \geq t_0$. For the exponential distribution, the probability density function becomes

$$f(t) = -\frac{dR(t)}{dt} = \lambda e^{-\lambda(t-t_0)} \qquad 0 < t_0 \leq t < \infty \tag{3.15}$$

and the reliability function will take on the following form:

$$R(t) = e^{-\lambda(t-t_0)} \qquad t \geq t_0 \tag{3.16}$$

From Eqs. (3.15) and (3.16) the failure rate is $\lambda(t) = f(t)/R(t) = \lambda$. However, the mean of the distribution is no longer $1/\lambda$ but is shifted a distance t_0 along the t axis. Using integration by parts or formula 6 in Appendix 2E:

$$\text{MTTF} = \int_{t_0}^{\infty} \lambda t e^{-\lambda(t-t_0)}\, dt = t_0 + \frac{1}{\lambda} \tag{3.17}$$

The median of the distribution is obtained by solving Eq. (3.16) for t_{med},

$$R(t_{\text{med}}) = e^{-\lambda(t_{\text{med}}-t_0)} = 0.5 \tag{3.18}$$

and obtaining

$$t_{\text{med}} = t_0 + \frac{\ln 0.5}{-\lambda} = t_0 + \frac{0.69315}{\lambda} \tag{3.19}$$

[7]This is a Weibull reliability function. The Weibull distribution will be discussed in the following chapter.

The design life t_R for a specified design reliability R can be obtained in the same manner as the median time. Therefore,

$$t_R = t_0 + \frac{\ln R}{-\lambda} \tag{3.20}$$

The variance and standard deviation of the two-parameter exponential distribution are not affected by the location parameter. Therefore $\sigma = 1/\lambda$. The mode occurs at t_0.

EXAMPLE 3.8. Let $\lambda = 0.001$ and $t_0 = 200$. Then

$$R(t) = e^{-0.001(t-200)} \qquad t \geq 200$$

and

$$\text{MTTF} = 200 + \frac{1}{0.001} = 1200 \qquad t_{\text{med}} = 200 + \frac{0.69315}{0.001} = 893.15$$

$$t_{0.95} = 200 - \ln\frac{0.95}{0.001} = 251.3 \qquad \sigma = \frac{1}{0.001} = 1000$$

3.5 POISSON PROCESS

If a component having a constant failure rate λ is immediately repaired or replaced upon failing, the number of failures observed over a time period t has a Poisson distribution. The probability of observing n failures in time t is given by the Poisson probability mass function $p_n(t)$:

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots \tag{3.21}$$

Unlike the failure distribution that is continuous over time, the Poisson distribution is discrete. The mean or expected number of failures over time t is given by λt, and the variance of the distribution is also λt.

Equation (3.21) can be derived by first letting

$$Y_k = \sum_{i=1}^{k} T_i$$

where T_i, a random variable, is the time between failure $i - 1$ and failure i and has an exponential distribution with parameter λ. Therefore Y_k is a random variable, the time of the kth failure. Since the sum of k independent exponential random variables has a gamma distribution[8] with parameters k and λ, the cumulative distribution function for Y_k can be written as

[8]A proof of this result may be found in many statistics texts, e.g., Ross [1987, pp. 114–117]. Since k is an integer, this is a special case of the gamma distribution known as the Erlang distribution. It is only this special case that results in a closed-form expression for the CDF.

$$\Pr\{Y_k \leq t\} = F_{Y_k}(t) = 1 - e^{-\lambda t}\sum_{i=0}^{k-1}\frac{(\lambda t)^i}{i!} \tag{3.22}$$

The cumulative probability given by Eq. (3.22) is the probability that the kth failure will occur by time t. The mean value for Y_k is k/λ, and the variance is k/λ^2. The mode is $(k - 1)/\lambda$. We have

$$P_n(t) = \Pr\{Y_n \leq t\} - \Pr\{Y_{n+1} \leq t\} = F_{Y_n}(t) - F_{Y_{n+1}}(t)$$

Equation (3.21) follows from the above and from Eq. (3.22).

The relationship between the two probability distributions can also be seen by determining the probability of no failures occurring in time t, which is equivalent to $\Pr\{T \geq t\}$. That is,

$$p_0(t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} = R(t)$$

The Poisson process is often used in inventory analysis to determine the number of spare components when the time between failures is exponential. For example, if S spare components are available to support a continuous operation over a time period t, then

$$R_S(t) = \sum_{n=0}^{S} p_n(t) \tag{3.23}$$

is the cumulative probability of S or fewer failures occurring during time t. Equation (3.23) therefore represents the probability of satisfying all demands for spare components during time t. Therefore, $R_S(t)$ is the component reliability if there are S spares available for immediate replacement when a failure occurs.

EXAMPLE 3.9. A specially designed welding machine has a nonrepairable motor with a constant failure rate of 0.05 failure per year. The company has purchased two spare motors. If the design life of the welding machine is 10 yr, what is the probability that the two spares will be adequate?

Solution. The expected number of failures over the life of the machine is $\lambda t = 0.05(10) = 0.5$. From Eq. (3.23),

$$R_2(10) = \sum_{n=0}^{2}\frac{e^{-0.5}0.5^n}{n!} = e^{-0.5}\left(1 + 0.5 + \frac{0.25}{2}\right) = 0.9856$$

is the probability of 2 or fewer failures occurring over the 10 yr.

Let Y_3 be the time of the third failure. Y_3 has a gamma distribution with $k = 3$ and $\lambda = 0.05$. Therefore, the expected, or mean, time to obtain 3 failures is $3/0.05 = 60$ yr. The probability that the third failure will occur within 10 yr is obtained from Eq. (3.22):

$$F_{Y_3}(10) = 1 - e^{-0.05\times 10}\left(1 + 0.05 \times 10 + \frac{(0.05 \times 10)^2}{2!}\right) = 0.0144$$

Observe that $0.0144 = 1 - 0.9856$ since the probability of two or fewer failures in 10 yr is complementary to the event that the third failure occurs within 10 yr.

3.6 REDUNDANCY AND THE CFR MODEL

Although it would appear from the previous applications that the CFR model is quite pervasive, numerous examples lead to other failure laws even when individual components are CFR. Consider the case of two independent and redundant[9] components each having the same constant failure rate λ. In this case a system failure will occur only when both components have failed. Since $(1 - e^{-\lambda t})^2$ is the probability that both components will fail by time t, the system reliability $R(t)$ is given by

$$\begin{aligned} R(t) &= 1 - (1 - e^{-\lambda t})^2 \\ &= 1 - (1 - 2e^{-\lambda t} + e^{-2\lambda t}) \\ &= 2e^{-\lambda t} - e^{-2\lambda t} \end{aligned} \tag{3.24}$$

The hazard rate in this case is

$$\begin{aligned} \lambda(t) &= \frac{f(t)}{R(t)} = \frac{2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}}{2e^{-\lambda t} - e^{-2\lambda t}} \\ &= \frac{\lambda(1 - e^{-\lambda t})}{(1 - 0.5e^{-\lambda t})} \end{aligned} \tag{3.25}$$

By inspection, $\lambda(t)$ is clearly not a constant hazard rate. Therefore the system does not have an exponential failure distribution even though both components are CFR. However, from Eq. (3.25), as $t \to \infty$, $\lambda(t) \to \lambda$ (see Fig. 3.2). Therefore the redundant system failure rate asymptotically approaches the constant failure rate.

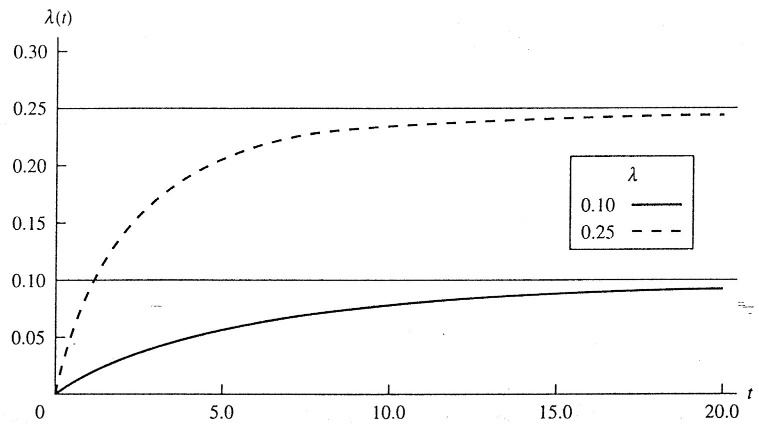


FIGURE 3.2
The failure rate of two CFR components in parallel.

[9]Redundancy will be discussed at length in Chapter 5.

The system MTTF can be determined from

$$\begin{aligned} \text{MTTF} &= \int_0^\infty R(t)\,dt = \int_0^\infty (2e^{-\lambda t} - e^{-2\lambda t})\,dt \\ &= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda} = \frac{1.5}{\lambda} \end{aligned} \tag{3.26}$$

In other words, the two-component redundant system will increase the MTTF by a factor of 1.5 over the single component MTTF.

EXAMPLE 3.10. For the microwave transmitter described in Example 3.1, a second redundant transmitter is added. The reliability function for the parallel system is

$$R(t) = 2e^{-0.00034t} - e^{-0.00068t}$$

Therefore the reliability over a 30-day period is

$$R(720) = 2e^{-0.00034\times 720} - e^{-2\times 0.00034\times 720} = 0.95285$$

This is a significant increase over the single-unit reliability of 0.78286. The redundant system MTTF is $1.5/0.00034 = 4411.76$ hr, compared with 2941.17 hr for a single unit. Its hazard rate function is

$$\lambda(t) = \frac{0.00034(1 - e^{-0.00034t})}{(1 - 0.5e^{-0.00034t})}$$

EXERCISES

3.1 A component experiences chance (CFR) failures with an MTTF of 1100 hr. Find the following:
(a) The reliability for a 200-hr mission
(b) The design life for a 0.90 reliability
(c) The median time to failure
(d) The reliability for a 200-hr mission if a second, redundant (and independent) component is added

3.2 A CFR system with $\lambda = 0.0004$ has been operating for 1000 hr. What is the probability that it will fail in the next 100 hr? The next 1000 hr?

3.3 A gearbox has two independent failure modes: a constant failure rate of 0.0003 and a linearly increasing (wearout) failure rate given by $\lambda(t) = t/(5\times 10^5)$. Find the reliability of the gearbox for 100 hr of operation.

3.4 A hydraulic system is comprised of five components having the following constant failure rates (times are in days): $\lambda_1 = 0.001$, $\lambda_2 = 0.005$, $\lambda_3 = 0.0007$, $\lambda_4 = 0.0025$, and $\lambda_5 = 0.00001$.
(a) Find the system MTTF and standard deviation.
(b) Find the system design life if a 0.99 reliability is desired.

3.5 A complex power system has a nonlinear failure rate that has historically been greater than 0.001 failure per day. Determine an upper bound on the system's reliability for a 60-day operating period.

3.6 A landing gear system has repetitive stresses placed on it twice a day as a result of landings. The probability of a failure during landing is 0.0028. Determine the reliability of the landing gear system over a 30-day contingency operation. What is the probability of a failure occurring between days 10 and 20 of the operation?

3.7 A system contains 20 identical and critical components that will be replaced on failure (renewal process). As a result, a constant failure rate for the system will be observed. If a design life of 10 yr with a reliability of 0.99 is required, what should the system MTTF and median time to failure be? If each component has a CFR, what will the component MTTF and median time to failure be?

3.8 Two identical and CFR computers are placed in a redundant configuration. If the system reliability is to be 0.95 at 3000 operating hours, determine the corresponding MTTF (design specifications) for each computer.

3.9 Specifications for a power unit consisting of three independent and serially related components (failure modes) require a design life of 5 yr with a 0.95 reliability.

(a) Let each component have a constant failure rate such that the first component's rate is twice that of the second and the third component's rate is three times that of the second. What should be the MTTF of each component and the system?

(b) If two identical power units are placed in parallel, what is the system reliability at 5 yr, and what is the system MTTF?

3.10 The time to failure of fluorescent lights in a large office building is exponentially distributed with a failure rate of 0.03125 hr. How many spare tubes must the building custodian maintain to have at least a 0.95 probability of replacing all failures on a given day? Assume continuous 24-hour use of the lights.

3.11 A flashlight contains two batteries each having an MTTF of 5 operating hours (assume CFR).

(a) What is the probability of battery failure occurring within the first 2 hr of operation?

(b) If failed batteries are immediately replaced, what is the probability of more than one failure occurring during the first 5 hr of operation?

(c) Would you expect batteries to have a constant failure rate?

3.12 Consider two redundant components having constant but different failure rates.

(a) Derive the reliability function and the MTTF.

(b) Find the reliability at 1000 hr and the MTTF of a two-component redundant system where $\lambda_1(t) = 0.000356$ per hour and $\lambda_2(t) = 0.00156$ per hour.

3.13 An electronic circuit board with $\lambda(t) = 0.00021$ per hour is replaced on failure. What is the probability that the third failure will occur by 10000 hours?

3.14 In reliability testing it is of interest to know how long the test must run in order to generate a specified number of failures. A new condenser fan motor is believed to have a constant failure rate of 3.4 failures per 100 operating hours. A single test stand is to be used in which a motor is operated until failure and then replaced with a new motor from production. What is the expected test time if 10 failures are desired?

3.15 Consider two identical and redundant CFR components having a guaranteed life of 2 months and a failure rate of 0.15 failure per year. What is the system reliability for 10,000 hr of continuous operation?

3.16 Derive a general expression for $R(t)$ and the MTTF for the two-component system described in Exercise 3.15.

3.17 A microwave link in a communications network has a high failure rate. Although several pieces of equipment have been used in the link, the link always seems to fail at about the same rate regardless of the age of the equipment and its prior maintenance history. In general, microwave transmissions are subject to fading. Selective fading occurs when atmospheric conditions bend a transmission to the extent that signals reach the receiver in slightly different paths. The merging paths can cause interference and create data errors. Other channels in the microwave transmission are not affected by selective fading. Selective fading occurs when there is an electrical storm. Flat fading occurs during fog and when the surrounding ground is very moist. It is more serious since it may last several hours and affect surrounding channels. If during the current season, electrical storms occur about once every week and fog alerts are issued at the rate of one every two months, what is the reliability of the link over a 24-hour period? What assumptions, if any, are necessary?

3.18 A 60-watt outdoor lightbulb is advertised as having an average life (i.e., MTTF) of 1000 (operating) hours. However, experience has shown that it will also fail on demand an average of once every 120 cycles. A particular bulb is turned on once each evening for an average of 10 hr. If it is desired to have a reliability of 90 percent, what is its design life in days?

3.19 A more general exponential reliability model may be defined by

$$R(t) = a^{-bt} \qquad \text{where } a > 1, \qquad b > 0$$

and a and b are parameters to be determined. Find the hazard rate function, and show how this model is equivalent to $R(t) = e^{-\lambda t}$.

3.20 **Repetitive loading.** A packaging machine (cartoner) in a food processing facility will jam with a constant probability of 0.005 per application (per carton). Twelve cans of coffee are combined into a single case for shipment to buyers. The production rate is 30 cans of coffee every minute. What is the probability (reliability) of no jams during a 1-hr production run?

$-(t/2000)^2$

3.21 For the reliability function $R(t) = e^{-\cancel{(t/1000)^2}}$, use Eqs. (3.13) and (3.14) and compare the upper and lower bounds with the actual reliabilities at 100, 200, 500, 800, 1000, 2000, 5000, and 10,000 hr. This failure distribution has an IFR with an MTTF of 1772.46.

3.22 Show for the exponential distribution that the residual mean life is $1/\lambda$ regardless of the length of time the system has been operating.

CHAPTER 4

Time-Dependent Failure Models

The development of theoretical probability distributions useful in analyzing failure processes continues with discussions on the Weibull, normal, and lognormal distributions. These distributions have hazard rate functions that are not constant over time, thus providing a necessary alternative to the exponential failure law. The approach taken is similar to that of the previous chapter: the distribution is defined, its characteristics are explored, and examples of its use are provided.

4.1 THE WEIBULL DISTRIBUTION

One of the most useful probability distributions in reliability is the Weibull. The Weibull failure distribution may be used to model both increasing and decreasing failure rates. It is characterized by a hazard rate function of the form

$$\lambda(t) = at^b$$

which is a power function. The function $\lambda(t)$ is increasing for $a > 0, b > 0$ and is decreasing for $a > 0, b < 0$. For mathematical convenience it is better to express $\lambda(t)$ in the following manner:

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \qquad \theta > 0, \beta > 0, t \geq 0 \tag{4.1}$$

Using Eq. (2.14),

$$R(t) = \exp\left[-\int_0^t \frac{\beta}{\theta}\left(\frac{t'}{\theta}\right)^{\beta-1} dt'\right] \tag{4.2}$$

$$= e^{-(t/\theta)^\beta}$$

and

$$f(t) = -\frac{dR(t)}{dt} = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta} \tag{4.3}$$

Beta (β) is referred to as the *shape parameter.* Its effect on the distribution can be seen in Fig. 4.1(a) for several different values. For $\beta < 1$, the PDF is similar in shape to the exponential, and for large values of β (e.g., $\beta \geq 3$), the PDF is somewhat symmetrical, like the normal distribution. For $1 < \beta < 3$, the density function is skewed. When $\beta = 1$, $\lambda(t)$ is a constant, and the distribution is identical to the exponential with $\lambda = 1/\theta$. As seen in Fig. 4.1(b) and (c), each of the CDF and reliability curves passes through the same point where $t = \theta$, since from Eq. 4.2, $R(\theta) = \exp[-(\theta/\theta)^\beta] = \exp(-1) = 0.368$. Therefore, 63.2 percent of all Weibull failures will occur by time $t = \theta$ regardless of the value of the shape parameter. Figure 4.1(d) shows that the hazard rate function can be increasing or decreasing depending on the value of β.

Theta (θ) is a scale parameter that influences both the mean and the spread, or dispersion, of the distribution. The effect of θ on the spread of the probability density function is illustrated in Fig. 4.2(a) for several different values. As θ increases in Fig. 4.2(b), the reliability increases at a given point in time. In Fig. 4.2(c), the slope of the hazard rate decreases as θ increases. The hazard rate curve is linear in this example since $\beta = 2$. The parameter θ is also called the *characteristic life,* and it has units identical to those of the failure time, T.

The MTTF (see Appendix 4A) and variance of the Weibull distribution are found from

$$\text{MTTF} = \theta\Gamma\left(1 + \frac{1}{\beta}\right) \tag{4.4}$$

and

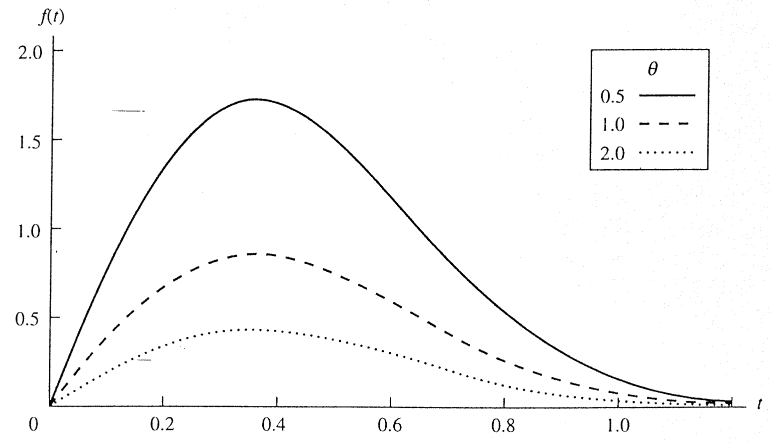
$$\sigma^2 = \theta^2\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\} \tag{4.5}$$

where $\Gamma(x)$ is the gamma function:

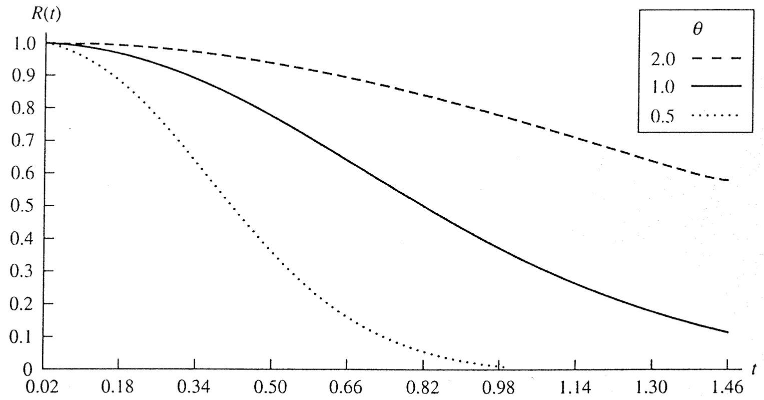
$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}\,dy$$

Table A.9 in the Appendix contains selected values of $\Gamma(x)$. With these tables, along with the fact that, for $x > 0$,

$$\Gamma(x) = (x-1)\Gamma(x-1)$$



(a)



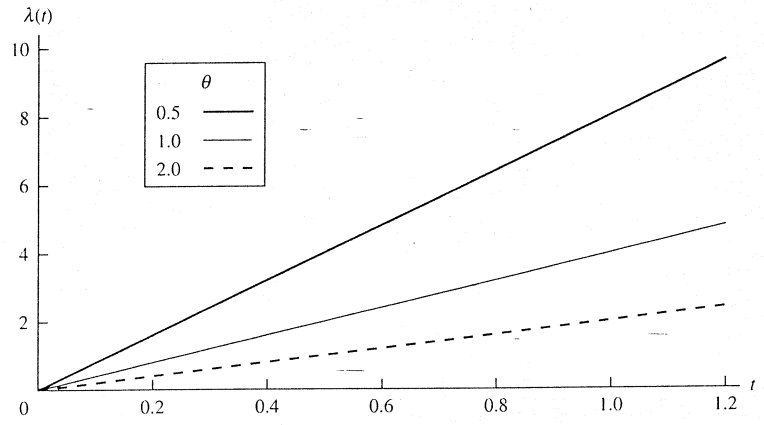
(b)

FIGURE 4.2
The effect of θ *(a)* on the Weibull probability density function; *(b)* on the Weibull reliability function.

and
$$\sigma^2 = 10^6\left\{\Gamma(1+1) - \left[\Gamma\left(1+\tfrac{1}{2}\right)\right]^2\right\}$$
$$= 214{,}601.7$$

or
$$\sigma = 463.25 \text{ hr}$$

where
$$\Gamma\left(1+\tfrac{1}{2}\right) = 0.886227$$



(c)

FIGURE 4.2 *(continued)*
The effect of θ *(c)* on the Weibull hazard rate curve.

from Table A.9 in the Appendix and

$$\Gamma(2) = (2-1)! = 1$$

Whenever $\beta = 2$, the hazard, or failure, rate is linear (LFR), and the Weibull distribution takes the form of the Rayleigh distribution.

The value of the shape parameter β provides insight into the behavior of the failure process. Table 4.1 summarizes this behavior. An increasing hazard rate can increase at a decreasing rate (concave), increase at a constant rate (linear), or increase at an increasing rate (convex), depending on β as shown in Table 4.1. Hazard rate functions that increase at an increasing rate reflect very aggressive wearout phenomena. The hazard rate curve in Fig. 4.1(d) for $\beta = 4$ is convex, and for $\beta = 1.5$, it is concave. It is not surprising that the Weibull provides a good model for much of the failure data found in practice, considering the variety of shapes and properties that are obtainable.

4.1.1 Design Life, Median, and Mode

Given a desired reliability R,

$$R(t) = e^{-(t/\theta)^\beta} = R$$

the design life is found from

$$t_R = \theta(-\ln R)^{1/\beta} \tag{4.6}$$

When $R = 0.99$, $t_{0.99}$ is referred to as the B1 life, i.e., the time at which 1 percent of the population will have failed. Similarly, $t_{0.999}$ is called the B.1 life, the time when

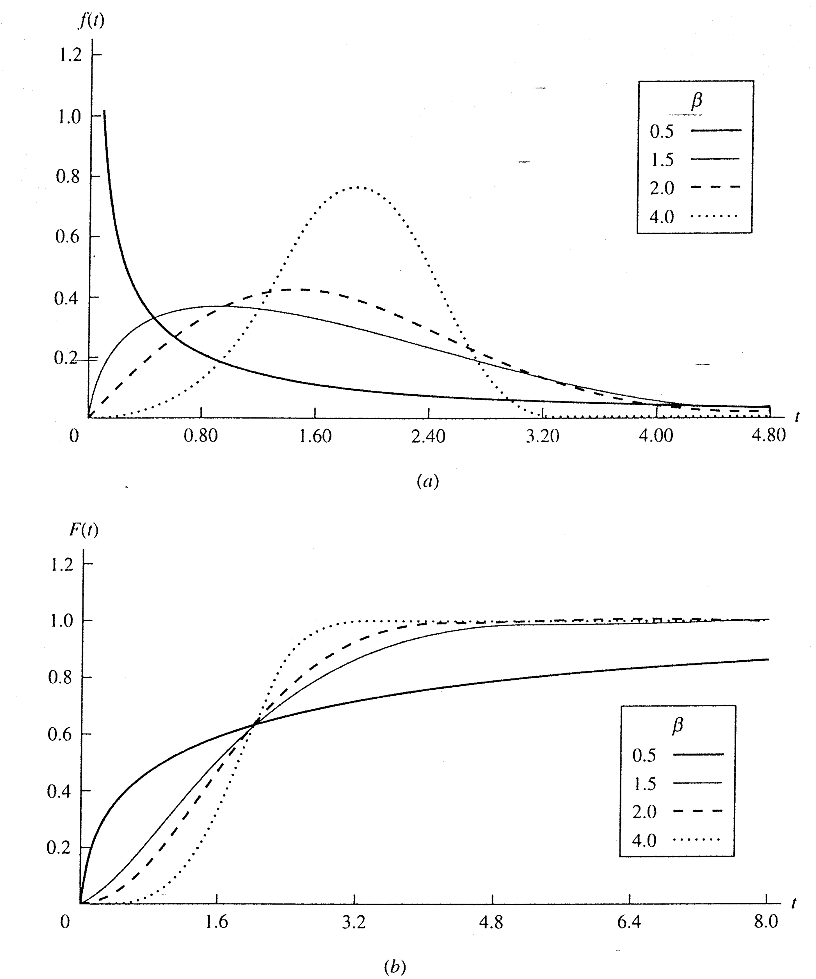


FIGURE 4.1
The effect of β *(a)* on the Weibull probability density function; *(b)* on the Weibull cumulative distribution function.

the mean and variance can easily be computed.[1] Unlike the exponential distribution, there is no direct relationship between the MTTF and $\lambda(t)$.

EXAMPLE 4.1. A compressor experiences wearout with a linear hazard rate function

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right) = 2 \times 10^{-6} t$$

[1] For x a positive integer, $\Gamma(x) = (x-1)!$

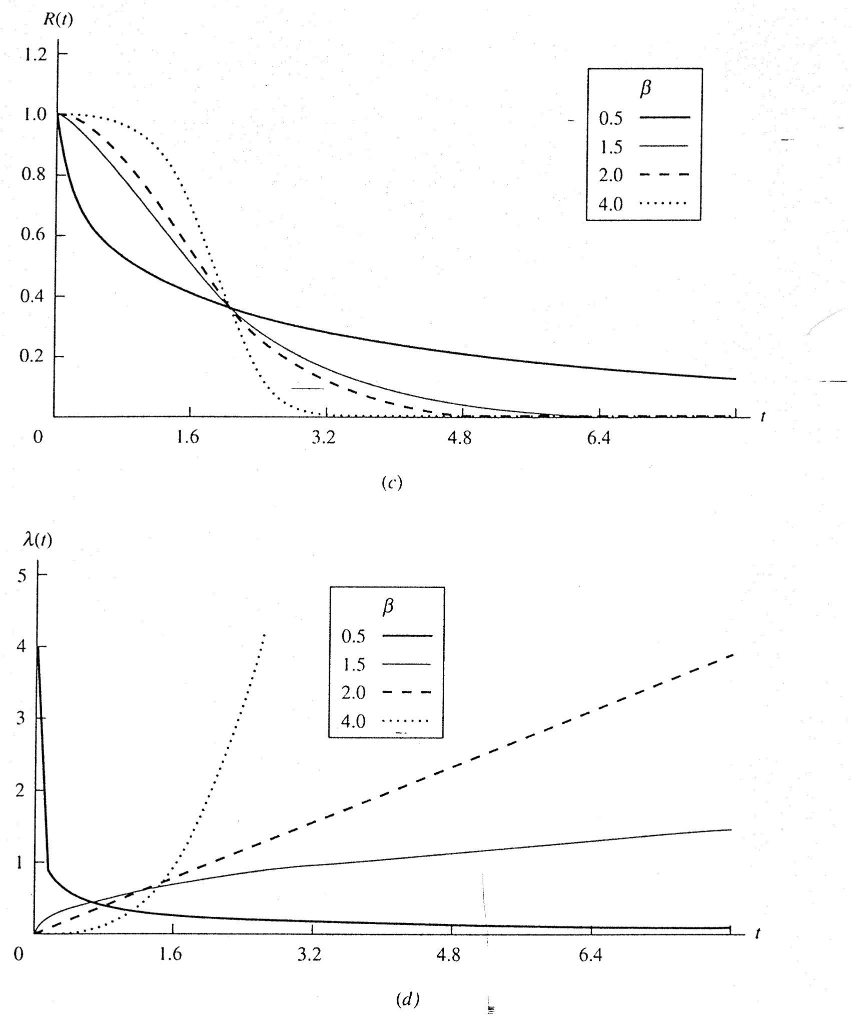


FIGURE 4.1 *(continued)*
The effect of β *(c)* on the Weibull reliability function; *(d)* on the Weibull hazard rate curve.

In this case, $\beta = 2$ and $\theta = 1000$ hr. For a desired 0.99 reliability,

$$R(t) = e^{-(t/1000)^2} = 0.99$$

The design life is given by

$$t_R = 1000\sqrt{-\ln 0.99} = 100.25 \text{ hr}$$

From Eqs. (4.4) and (4.5),

$$\text{MTTF} = 1000\,\Gamma\left(1 + \tfrac{1}{2}\right) = 886.23 \text{ hr}$$

TABLE 4.1
Weibull shape parameter

Value	Property
$0 < \beta < 1$	Decreasing failure rate (DFR)
$\beta = 1$	Exponential distribution (CFR)
$1 < \beta < 2$	IFR, concave
$\beta = 2$	Rayleigh distribution (LFR)
$\beta > 2$	IFR, convex
$3 \le \beta \le 4$	IFR, Approaches normal distribution; symmetrical

0.1 percent of the population will have failed. Let $R = 0.50$; then

$$\begin{aligned} t_{0.50} &= t_{\text{med}} = \theta(-\ln 0.5)^{1/\beta} \\ &= \theta(0.69315)^{1/\beta} \end{aligned} \tag{4.7}$$

is the median time to failure.

Since the median is a more representative measure of central tendency for highly skewed data, it may be preferred to the MTTF when β is small (e.g., less than 3). The mode of the distribution can be found by solving for t^* such that

$$f(t^*) = \max_{t \ge 0} f(t)$$

which results in

$$t_{\text{mode}} = \begin{cases} \theta(1 - 1/\beta)^{1/\beta} & \text{for } \beta > 1 \\ 0 & \text{for } \beta \le 1 \end{cases} \tag{4.8}$$

The details are provided in Appendix 4B.

EXAMPLE 4.2. Given a Weibull failure distribution with a shape parameter of $\frac{1}{3}$ and a scale parameter of 16,000, completely characterize the failure process.

Solution

1. The reliability function is

$$R(t) = \exp\left[-\left(\frac{t}{16{,}000}\right)^{1/3}\right]$$

2. $\beta = \frac{1}{3}$, a decreasing failure rate indicating high infant mortality.
3. $\text{MTTF} = 16{,}000\Gamma\left(1 + \frac{1}{1/3}\right) = 16{,}000 \cdot 3! = 96{,}000$ hr

$$t_{\text{med}} = 16{,}000(0.69315)^3 = 5{,}328 \text{ hr}$$

Since the distribution is highly skewed, the median provides a better average. The mode is zero since $\beta < 1$.

4. $\sigma^2 = (16{,}000)^2\{\Gamma(7) - [\Gamma(4)]^2\} = 175{,}104 \times 10^6$

$$\sigma = 418.4 \times 10^3 \text{ hr}$$

5. The characteristic life is 16,000 hr. Therefore 63 percent of the failures will occur by this time.
6. If a 90 percent reliability is desired, the design life is

$$t_R = 16{,}000(-\ln 0.90)^3 = 18.71 \text{ hr}$$

7. Its B1 life is $(16{,}000)(-\ln 0.99)^3 = 0.0162$ hr, indicating a high percentage of early failures.

4.1.2 Burn-In Screening for Weibull

The component depicted in Example 4.2 would be a good candidate for burn-in screening. Using the conditional reliability as defined by Eq. (2.17),

$$R(t \mid T_0) = \frac{\exp\{-[(t + T_0)/\theta]^\beta\}}{\exp[-(T_0/\theta)^\beta]} = \exp\left[-\left(\frac{t + T_0}{\theta}\right)^\beta + \left(\frac{T_0}{\theta}\right)^\beta\right]$$

for the Weibull model. If, in the previous example, a 10-hr burn-in period is accomplished, then

$$R(t \mid 10) = \exp\left[-\left(\frac{t + 10}{16{,}000}\right)^{1/3} + \left(\frac{10}{16{,}000}\right)^{1/3}\right]$$

For a 90 percent reliability, the design life is found by solving for t_R:

$$R(t_R \mid 10) = 0.90$$

$$\begin{aligned} t_R &= 16{,}000\left[-\ln 0.90 + \left(\frac{10}{16{,}000}\right)^{1/3}\right]^3 - 10 \\ &= 101.24 \text{ hr} \end{aligned}$$

This is a significant increase in the component's design life over the original 18.71 hr.

4.1.3 Failure Modes

For a system comprised of n serially related components or having n independent failure modes, each having an independent Weibull failure distribution with shape parameter β and scale parameter θ_i, the system failure rate function can be determined from Eq. (3.7):

$$\begin{aligned} \lambda(t) &= \sum_{i=1}^{n} \frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta - 1} \\ &= \beta t^{\beta - 1}\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^\beta\right] \end{aligned} \tag{4.9}$$

However, this is a hazard rate for a Weibull distribution having a shape parameter of β and a characteristic life of

$$\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^{\beta}\right]^{-1/\beta}$$

This reproductive property of the Weibull distribution is true only when each component has the same shape parameter. If the failure modes are all Weibull but with differing shape parameters, then the system failure distribution will not be Weibull.

EXAMPLE 4.3. A jet engine consists of five modules each of which was found to have a Weibull failure distribution with a shape parameter of 1.5. Their scale parameters (characteristic life) are (in operating cycles) 3600, 7200, 5850, 4780, and 9300. Find the MTTF and median time to failure of the engine.

Solution. The engine has a Weibull failure distribution with $\beta = 1.5$. The characteristic life is

$$\theta = \left[\left(\frac{1}{3600}\right)^{1.5} + \left(\frac{1}{7200}\right)^{1.5} + \left(\frac{1}{5850}\right)^{1.5} + \left(\frac{1}{4780}\right)^{1.5} + \left(\frac{1}{9300}\right)^{1.5}\right]^{-1/1.5} = 1842.7$$

and
$$\text{MTTF} = 1842.7\,\Gamma\left(1 + \tfrac{2}{3}\right) = 1664.5 \text{ cycles}$$

$$t_{\text{med}} = (1842.7)(0.69315)^{1/1.5} = 1443.2 \text{ cycles}$$

The reliability function for the engine is given by

$$R(t) = \exp\left[-\left(\frac{t}{1842.7}\right)^{1.5}\right]$$

4.1.4 Identical Weibull Components

If a system of n serially related and independent components have identical hazard rate functions

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \tag{4.10}$$

then from Eq. (4.9) the system hazard rate is

$$\lambda(t) = \sum_{i=1}^{n}\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \frac{n\beta}{\theta^{\beta}}(t)^{\beta-1}$$

and
$$R(t) = \exp\left[-n\left(\frac{t}{\theta}\right)^{\beta}\right] \tag{4.11}$$

which is Weibull with shape parameter β and scale parameter $\theta/n^{1/\beta}$.

EXAMPLE 4.4. An electrical system has four series connectors each having a Weibull failure law with $\beta = \frac{3}{4}$ and $\theta = 2000$ hr. Find the reliability of the system of four connectors for 150 hours.

Solution

$$R(150) = \exp\left[-4\left(\frac{150}{2000}\right)^{3/4}\right] = 0.5637$$

4.1.5 The Three-Parameter Weibull

Whenever there is a minimum life t_0 such that $T > t_0$, the three-parameter Weibull may be appropriate. This distribution assumes that no failures will take place prior to time t_0. For this distribution,

$$R(t) = \exp\left[-\left(\frac{t-t_0}{\theta}\right)^{\beta}\right] \qquad t \geq t_0 \tag{4.12}$$

and
$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t-t_0}{\theta}\right)^{\beta-1} \qquad t \geq t_0 \tag{4.13}$$

The parameter t_0 is called the *location parameter*. The variance of this distribution is the same as that in the two-parameter model. However,

$$\text{MTTF} = t_0 + \theta\,\Gamma\left(1 + \frac{1}{\beta}\right) \tag{4.14}$$

$$t_{\text{med}} = t_0 + \theta(0.69315)^{1/\beta} \tag{4.15}$$

and the design life t_R corresponding to a reliability of R is

$$t_R = t_0 + \theta(-\ln R)^{1/\beta} \tag{4.16}$$

It is always possible to transform the three-parameter Weibull into the two-parameter Weibull with the transformation $t' = t - t_0$.

EXAMPLE 4.5. The three parameter Weibull has $\beta = 4$, $t_0 = 100$, and $\theta = 780$. Compute its MTTF, median, standard deviation, and reliability for a 500-hr mission.

Solution

$$\text{MTTF} = 100 + 780\Gamma\left(1 + \tfrac{1}{4}\right) = 806.99 \text{ hr}$$

$$t_{\text{med}} = 100 + 780(0.69315)^{1/4} = 811.7 \text{ hr}$$

$$\sigma^2 = (780)^2\left\{\Gamma\left(1 + \tfrac{2}{4}\right) - \left[\Gamma\left(1 + \tfrac{1}{4}\right)\right]^2\right\} = 39{,}340.6$$

$$\sigma = 198.3 \text{ hr}$$

$$R(500) = \exp\left[-\left(\frac{500-100}{780}\right)^{4}\right] = 0.933$$

4.1.6 Redundancy with Weibull Failures

If two identical (and assumed independent) components are used to form a redundant system (both must fail for the system to fail), then the system reliability is

$$R_s(t) = 1 - [1 - R(t)]^2$$

If
$$R(t) = e^{-(t/\theta)^\beta}$$

then
$$R_s(t) = 1 - \left[1 - e^{-(t/\theta)^\beta}\right]^2 = 2e^{-(t/\theta)^\beta} - e^{-2(t/\theta)^\beta} \tag{4.17}$$

To find the MTTF:

$$\text{MTTF} = \int_0^\infty R_s(t)\,dt = \int_0^\infty \left[2e^{-(t/\theta)^\beta} - e^{-2(t/\theta)^\beta}\right] dt$$
$$= 2\int_0^\infty e^{-(t/\theta)^\beta}\,dt - \int_0^\infty e^{-2(t/\theta)^\beta}\,dt$$

The value of the first integral is given by Eq. (4.4) as $\theta\Gamma(1+1/\beta)$. The second integral is also a Weibull reliability function with parameters β and $\theta' = \theta/2^{1/\beta}$. Therefore its value is

$$\frac{\theta}{2^{1/\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$$

The result is

$$\begin{aligned}\text{MTTF} &= 2\theta\Gamma\left(1+\frac{1}{\beta}\right) - \frac{\theta}{2^{1/\beta}}\Gamma\left(1+\frac{1}{\beta}\right)\\ &= \theta\Gamma\left(1+\frac{1}{\beta}\right)\left(2 - 2^{-1/\beta}\right)\end{aligned} \tag{4.18}$$

The hazard rate for this system is derived in Appendix 4D and is given by

$$\lambda_s(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\frac{2 - 2e^{-(t/\theta)^\beta}}{2 - e^{-(t/\theta)^\beta}} \tag{4.19}$$

From Eq. (4.19) it is apparent that the system formed from two redundant Weibull components is not itself Weibull. Further, for large values of t, $\exp[-(t/\theta)^\beta]$ is small and $\lambda_s(t)$ approximates the hazard rate function of a single Weibull component. This result is similar to that of the two redundant CFR components in which the system failure rate asymptotically approaches the constant failure rate of a single component. Intuitively, the longer the system operates, the more likely it is that one of the components will fail, thereby reducing the system to a single component.

EXAMPLE 4.6. Two fuel pumps, each having a Weibull failure distribution with $\beta = \frac{1}{2}$ and $\theta = 1000$ hr, are configured to provide a redundant system. Find the system reliability for a 100-hr mission and the system MTTF.

Solution. Using Eq. (4.17),

$$R_s(100) = 2\exp\left[-\left(\frac{100}{1000}\right)^{1/2}\right] - \exp\left[-2\left(\frac{100}{1000}\right)^{1/2}\right] = 0.9265$$

Then from Eq. (4.18),

$$\text{MTTF} = 1000\,\Gamma\,(3)(2 - 2^{-2}) = 3500 \text{ hr}$$

As a comparison, a single fuel pump will have a mission reliability of 0.7288 and an MTTF equal to 2000.

4.2 THE NORMAL DISTRIBUTION

The normal distribution has been used successfully to model fatigue and wearout phenomena. Because of its relationship with the lognormal distribution, it is also useful in analyzing lognormal probabilities. The density function of the normal provides the familiar bell-shaped curve shown in Fig. 4.3(a). The formula for the PDF is

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right] \qquad -\infty < t < \infty \tag{4.20}$$

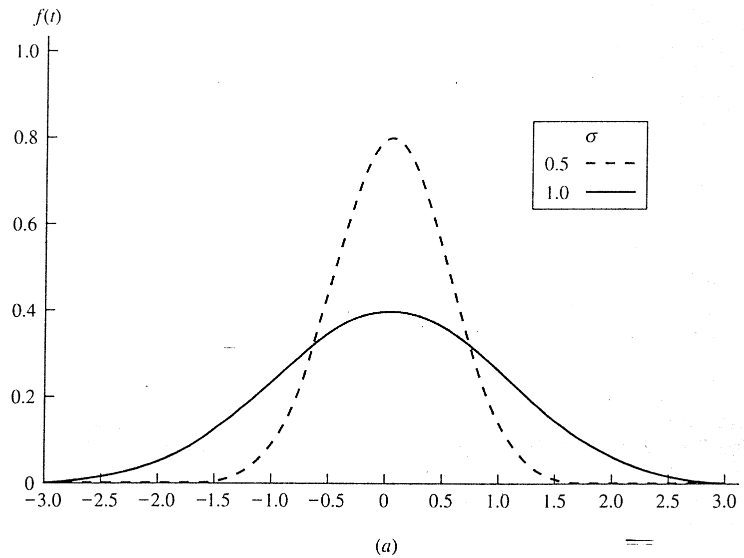
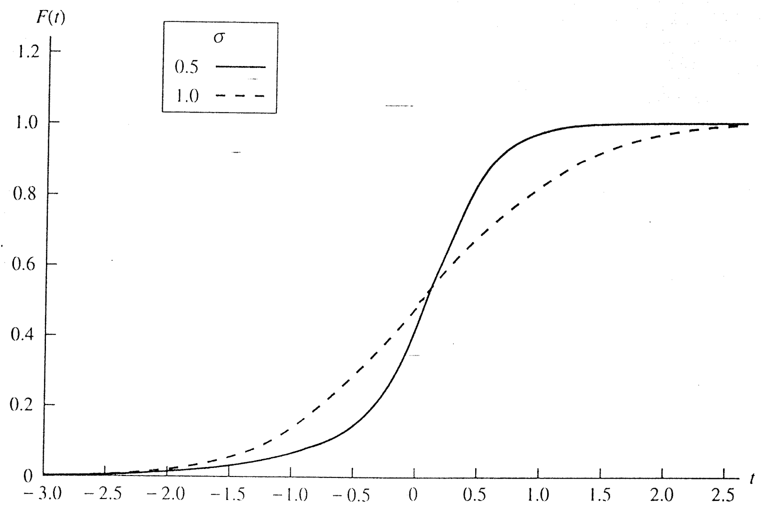
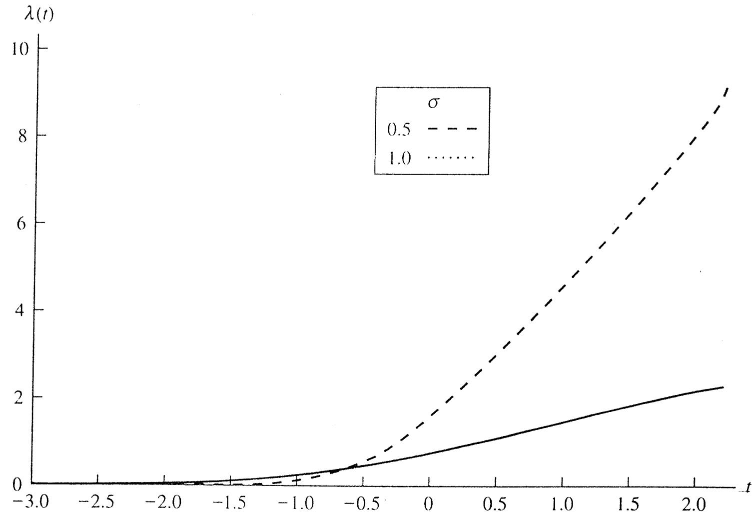


FIGURE 4.3
The effect of the standard deviation σ *(a)* on the normal probability density function.



(b)



(c)

FIGURE 4.3 *(continued)*
The effect of the standard deviation σ *(b)* on the normal cumulative distribution function; *(c)* on the normal hazard rate curve.

where the parameters μ and σ^2 are the mean and variance of the distribution, respectively. The normal is not a true reliability distribution since the random variable ranges from minus infinity to plus infinity. However, for most observed values of μ and σ, the probability that the random variable will take on negative values is negligible, and the normal can therefore be a reasonable approximation to a failure process. The distribution is symmetrical about its mean with the spread of the distribution determined by the standard deviation σ, as seen in Fig. 4.3(a).

The reliability function for this distribution is determined from

$$R(t) = \int_t^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\frac{(t'-\mu)^2}{\sigma^2}\right] dt' \tag{4.21}$$

However, there is no closed-form solution to this integral, and it must be evaluated numerically. If the transformation

$$z = \frac{T-\mu}{\sigma}$$

is made, then z will be normally distributed with a mean of zero and a variance of one. The PDF for z is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{4.22}$$

and z is referred to as the *standardized normal variate*. Its cumulative distribution function is

$$\Phi(z) = \int_{-\infty}^{z} \phi(z')\, dz' \tag{4.23}$$

Table A.1 provides cumulative probabilities of the standardized normal distribution. The table can be used to find the cumulative probabilities of any normally distributed random variable by making use of

$$\begin{aligned} F(t) &= \Pr\{T \le t\} = \Pr\left\{\frac{T-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right\} \\ &= \Pr\left\{z \le \frac{t-\mu}{\sigma}\right\} = \Phi\left(\frac{t-\mu}{\sigma}\right) \end{aligned} \tag{4.24}$$

Therefore, in general,

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{4.25}$$

with

$$F(t) = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

Figure 4.3(b) shows the shape of the CDF for two different values of σ.

The hazard function cannot be written in closed form either. However,

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - \Phi((t-\mu)/\sigma)} \tag{4.26}$$

can be shown to be an increasing function, as seen in Fig. 4.3(c). Therefore, the normal distribution would be used to model wearout (IFR) phenomena only.

EXAMPLE 4.7. Wearout (failure) of an oil-drilling bit is normally distributed with a mean of 120 drilling hours and a standard deviation of 14 drilling hours. Drilling occurs for 12 hr each day. How many days should drilling continue before the operation is stopped in order to replace the drill bit? A 95 percent reliability is desired.

Solution. Find $t_{0.95}$ such that $\Pr\{T \geq t_{0.95}\} = 0.95$. Standardizing,

$$\Pr\left\{z \geq \frac{t_{0.95} - 120}{14}\right\} = 1 - \Phi\left(\frac{t_{0.95} - 120}{14}\right) = 0.95$$

Using the normal tables: $(t_{0.95} - 120)/14 = -1.645$, or $t_{0.95} = 96.97$ hr ≈ 8 (12-hr) days.

EXAMPLE 4.8. Five percent of a certain grade of tires wear out before 25,000 miles, and another 5 percent of the tires exceed 35,000 miles. Determine the tire reliability at 24,000 miles if wearout is normally distributed.

Solution. We are given $\Pr\{25{,}000 \leq T \leq 35{,}000\} = 0.90$. Standardizing,

$$\Pr\left\{\frac{25{,}000 - \mu}{\sigma} \leq z \leq \frac{35{,}000 - \mu}{\sigma}\right\} = 0.90$$

From the normal tables and the symmetry of the distribution,

$$\Pr\{-1.645 \leq z \leq 1.645\} = 0.90$$

or

$$\frac{25{,}000 - \mu}{\sigma} = -1.645 \qquad \frac{35{,}000 - \mu}{\sigma} = 1.645$$

Solving,

$$\mu = 30{,}000 \qquad \sigma = 3039.5$$

Therefore

$$R(24{,}000) = 1 - \Phi\left(\frac{24{,}000 - 30{,}000}{3039.5}\right)$$

$$= 1 - \Phi(-1.97) = 0.9756$$

Central limit theorem

The central limit theorem (CLT) provides strong motivation for many of the statistical applications of the normal distribution. The CLT states that under fairly general conditions the sum of n random variables approaches a normal distribution as n approaches infinity. More formally, if $Y_n = X_1 + X_2 + \cdots + X_n$ where $X_1, X_2, \ldots, X_n$ are n independent random variables with finite means $E(X_i)$ and finite variances $V(X_i)$, then for sufficiently large n, Y_n has an approximately normal distribution with the following mean and variance:

$$E(Y_n) = \sum_{i=1}^{n} E(X_i) \qquad V(Y_n) = \sum_{i=1}^{n} V(X_i)$$

The CLT implies that each random variable contributes a small amount to the total, with no single variable dominating. This result holds regardless of the distribution of the random variables. From a reliability perspective, if a failure is a result of many small cumulative effects, the normal failure process may be appropriate. Later, the CLT will enable us to compute approximate confidence intervals about an estimate of the MTTF obtained from sample failure data. However, our interest now in the normal distribution is also motivated by its use in the development of the lognormal failure process.

4.3 THE LOGNORMAL DISTRIBUTION

If the random variable T, the time to failure, has a lognormal distribution, the logarithm of T has a normal distribution. This is a very useful relationship in working with the lognormal distribution. The density function for the lognormal is

$$f(t) = \frac{1}{\sqrt{2\pi}st} \exp\left[-\frac{1}{2s^2}\left(\ln\frac{t}{t_{\text{med}}}\right)^2\right] \qquad t \geq 0 \tag{4.27}$$

where the parameter s is a shape parameter and t_{med}, the location parameter, is the median time to failure.[2] The distribution is defined for only positive values of t and is therefore more appropriate than the normal as a failure distribution. Examples of the lognormal probability density function for different values of the shape parameter are provided in Fig. 4.4(a). Like the Weibull distribution, the lognormal can take on a variety of shapes. It is frequently the case that data that fit a Weibull distribution will also fit a lognormal distribution.

The mean, variance, and mode of the lognormal are

$$\text{MTTF} = t_{\text{med}} \exp(s^2/2) \tag{4.28}$$

$$\sigma^2 = t_{\text{med}}^2 \exp(s^2)[\exp(s^2) - 1] \tag{4.29}$$

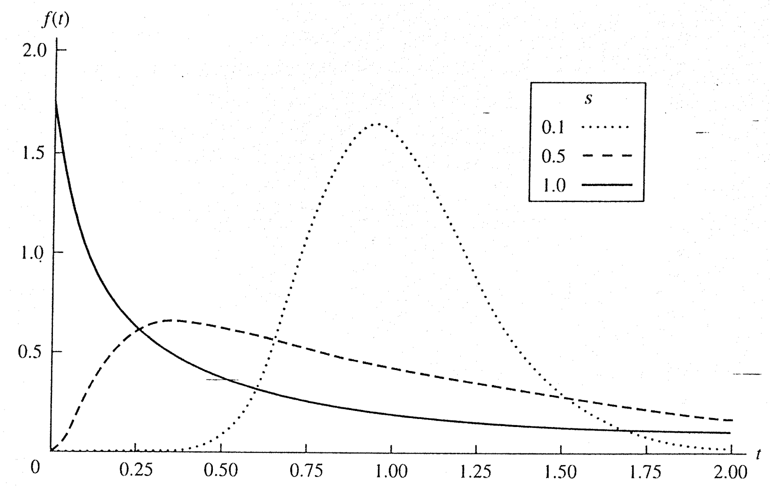
$$t_{\text{mode}} = \frac{t_{\text{med}}}{\exp(s^2)} \tag{4.30}$$

To compute failure probabilities, the lognormal's relationship to the normal is utilized. This relationship between the lognormal and normal distributions is summarized in Table 4.2.

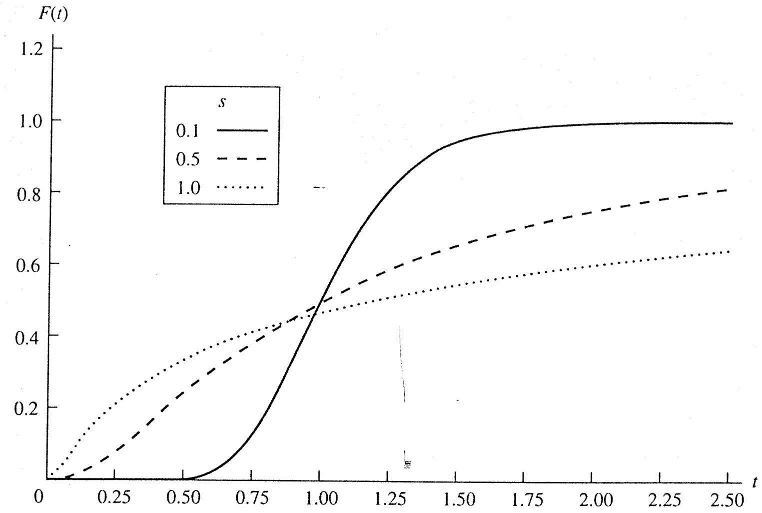
[2] An alternative form of the lognormal density function uses the mean and standard deviation of the logarithm of T as the distribution parameters. In this case,

$$f(t) = \frac{1}{\sqrt{2\pi}t\sigma_n} \exp\left[-\frac{(\ln t - \mu_n)^2}{2\sigma_n^2}\right]$$

where μ_n and σ_n are the mean and standard deviation of $\ln t$.

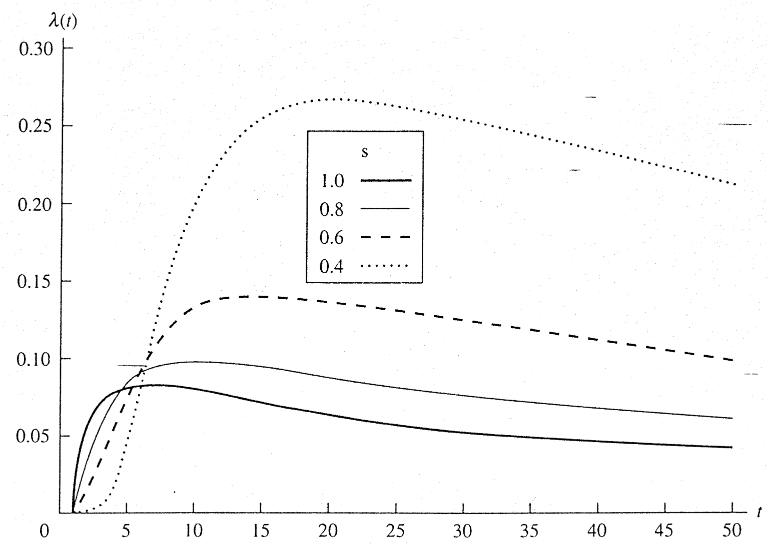


(a)



(b)

FIGURE 4.4
The effect of the shape parameter s (a) on the lognormal probability density function; (b) on the lognormal cumulative distribution function; $t_{\text{med}} = 1.0$.



(c)

FIGURE 4.4 *(continued)*
The effect of the shape parameter s (c) on the lognormal hazard rate curve; $t_{\text{med}} = 10$.

Since the logarithm is a monotonically increasing function,

$$
\begin{aligned}
F(t) &= \Pr\{T \le t\} = \Pr\{\ln T \le \ln t\} \\
&= \Pr\left\{\frac{\ln T - \ln t_{\text{med}}}{s} \le \frac{\ln t - \ln t_{\text{med}}}{s}\right\} \\
&= \Pr\left\{z \le \frac{1}{s}\ln\frac{t}{t_{\text{med}}}\right\} \\
&= \Phi\left(\frac{1}{s}\ln\frac{t}{t_{\text{med}}}\right)
\end{aligned}
$$

TABLE 4.2
Relationship between lognormal and normal distributions†

Distribution	Lognormal	Normal
Mean	$t_{\text{med}}\exp\left(\frac{s^2}{2}\right)$	$\ln t_{\text{med}}$
Variance	$t_{\text{med}}^2\exp(s^2)[\exp(s^2) - 1]$	s^2

† Given that T is a lognormal random variable

Then $$R(t) = 1 - \Phi\left(\frac{1}{s}\ln\frac{t}{t_{\text{med}}}\right) \tag{4.31}$$

Figure 4.4(b) is a graph of the CDF for different shape parameters for $t_{\text{med}} = 1.0$.

The hazard rate for the lognormal distribution, like the normal distribution, cannot be solved for analytically. However, as was done in Fig. 4.4(c), the lognormal hazard rate can be calculated numerically at selected points in time by finding $f(t)/R(t)$ using Eqs. (4.27) and (4.31). The hazard rate function increases until it reaches a peak, and then it slowly decreases. This is an uncommon failure rate behavior for most components. However, if $s < \sqrt{2/\pi} = 0.798$, the maximum is reached beyond the median and mean life; otherwise it peaks between the median and the mode [Kececioglu, 1991]. The smaller the value of s, the greater the time before the peak is reached. Figure 4.4(c) shows this effect for four values of the shape parameter. In each case t_{med} is 10. The following table summarizes the other characteristics of the distributions shown in Fig. 4.4(c):

s	1.0	0.8	0.6	0.4
Mode	3.7	5.3	7.0	8.5
MTTF	16.5	13.8	12.0	10.8
Max $\lambda(t)$	7	10	16	20

Additional discussions on the behavior of the lognormal hazard rate function may be found in Gottfried [1990] and Sweet [1990].

EXAMPLE 4.9. Fatigue wearout of a component has a lognormal distribution with $t_{\text{med}} = 5000$ hr and $s = 0.20$. Then

$$\text{MTTF} = 5000e^{(0.20)^2/2} = 5101 \text{ hr}$$

$$\sigma^2 = 5000^2 e^{(0.20)^2}\,[e^{(0.20)^2} - 1] = 1.0619 \times 10^6$$

$$\sigma = 1030 \text{ hr}$$

$$t_{\text{mode}} = \frac{5000}{e^{(0.20)^2}} = 4804 \text{ hr}$$

$$R(3000) = 1 - \Phi\left(\frac{1}{0.2}\ln\frac{3000}{5000}\right)$$
$$= 1 - \Phi(-2.55) = 0.99461$$

The design life of a component with a lognormal failure distribution may be obtained by finding the inverse function of $R(t)$. Let R represent the desired reliability.

Then $$1 - \Phi\left(\frac{1}{s}\ln\frac{t_R}{t_{\text{med}}}\right) = R$$

or $$\Phi\left(\frac{1}{s}\ln\frac{t_R}{t_{\text{med}}}\right) = 1 - R$$

and $$\frac{1}{s}\ln\frac{t_R}{t_{\text{med}}} = z_{1-R}$$

where z_{1-R} is found in Table A.1 such that

$$\Phi(z_{1-R}) = 1 - R$$

Solving for t_R:

$$t_R = t_{\text{med}}\, e^{s z_{1-R}} \tag{4.32}$$

EXAMPLE 4.10. From the previous example, for a reliability of 0.95,

$$t_{0.95} = 5000\, e^{0.20(-1.64)} = 3602 \text{ hr}$$

APPENDIX 4A
DERIVATION OF THE MTTF FOR THE WEIBULL DISTRIBUTION

By definition,

$$\text{MTTF} = \int_0^\infty \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\, t\, dt$$

Let $$y = \left(\frac{t}{\theta}\right)^\beta$$

then $$dy = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} dt$$

or $$\text{MTTF} = \int_0^\infty t e^{-y} dy$$

Since $$t = \theta y^{1/\beta}$$

we have $$\text{MTTF} = \theta\int_0^\infty y^{1/\beta} e^{-y}\, dy$$
$$= \theta\,\Gamma\left(1 + \frac{1}{\beta}\right)$$

since $$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\, dy$$

APPENDIX 4B
DERIVATION OF THE MODE FOR THE WEIBULL DISTRIBUTION

$$\max_{t \geq 0} f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}$$

$$\frac{df(t)}{dt} = \frac{\beta(\beta-1)}{\theta^2}\left(\frac{t}{\theta}\right)^{\beta-2} e^{-(t/\theta)^\beta} - \frac{\beta^2}{\theta^2}\left(\frac{t}{\theta}\right)^{2\beta-2} e^{-(t/\theta)^\beta}$$

Setting $df(t)/dt = 0$ and solving for t,

$$e^{-(t/\theta)^\beta}\left(\frac{t}{\theta}\right)^{\beta-2}\frac{\beta}{\theta^2}\left[(\beta-1)-\beta\left(\frac{t}{\theta}\right)^\beta\right] = 0$$

Therefore

$$(\beta-1)-\beta\left(\frac{t}{\theta}\right)^\beta = 0$$

and

$$t_{\text{mode}} = \theta\left(\frac{\beta-1}{\beta}\right)^{1/\beta}$$

However, t_{mode} is not defined for $\beta < 1$, in which case the mode occurs at the origin.

APPENDIX 4C
MINIMUM EXTREME-VALUE DISTRIBUTION

The Weibull distribution can also be viewed as a minimum extreme-value distribution. Let the time to failure of a system depend on the minimum failure time of a large number of components each having independent and identical cumulative distribution functions of the form

$$F_i(t_i) = 1 - \exp\left[-\left(\frac{t_i - t_0}{\theta}\right)^\beta\right]$$

Then with $T = \min\{t_1, t_2, \ldots, t_n\}$, where t_i is the failure time of the ith component,

$$\begin{aligned} F(t) &= \Pr\{T \leq t\} = \Pr\{t_1 < t, t_2 < t, \ldots, t_n < t\} \\ &= 1 - \Pr\{t_1 > t, t_2 > t, \ldots, t_n > t\} \\ &= 1 - \Pr\{t_1 > t\}\Pr\{t_2 > t\}\cdots\Pr\{t_n > t\} \\ &= 1 - \left(\exp\left[-\left(\frac{t-t_0}{\theta}\right)^\beta\right]\right)^n \\ &= 1 - \exp\left[-n\left(\frac{t-t_0}{\theta}\right)^\beta\right] \end{aligned}$$

or

$$R(t) = 1 - F(t) = \exp\left[-n\left(\frac{t-t_0}{\theta}\right)^\beta\right]$$

which is the same result as obtained in Eq. (4.11) with $t_0 = 0$.

APPENDIX 4D
HAZARD RATE FOR THE TWO-COMPONENT WEIBULL REDUNDANT SYSTEM

To find $\lambda_s(t)$ for this system, first find, using Eq. (4.17),

$$\begin{aligned} \frac{dR_s(t)}{dt} &= 2e^{-(t/\theta)^\beta}\left[-\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\right] - e^{-2(t/\theta)^\beta}\left[-\frac{2\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\right] \\ &= \frac{-\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\left[2 - 2e^{-(t/\theta)^\beta}\right] \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_s(t) &= -\frac{dR(t)}{dt}\cdot\frac{1}{R(t)} \\ &= \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\frac{2-2e^{-(t/\theta)^\beta}}{2-e^{-(t/\theta)^\beta}} \end{aligned}$$

EXERCISES

4.1. For a system having a Weibull failure distribution with a shape parameter of 1.4 and a scale parameter of 550 days, find the following:
(a) $R(100 \text{ days})$
(b) The B1 life
(c) MTTF
(d) The standard deviation
(e) t_{med}
(f) t_{mode}
(g) The design life for a reliability of 0.90

4.2. A turbine blade has demonstrated a Weibull failure pattern with a decreasing failure rate characterized by a shape parameter of 0.6 and a scale parameter of 800 hr.
(a) Compute its reliability for a 100-hr mission.
(b) If there is a 200-hr burn-in of the blades, what is the reliability for a 100-hr mission?

4.3. A power supply consists of three rectifiers in series. Each rectifier has a Weibull failure distribution with β equal to 2.1. However, they have different characteristic lifetimes given by 12,000 hr, 18,500 hr, and 21,500 hr. Find the MTTF and the design life of the power supply corresponding to a reliability of 0.90.

4.4. For a three-parameter Weibull distribution with $\beta = 1.54$ hr, $\theta = 8500$, and $t_0 = 50$ hr, find the following:
(a) $R(150 \text{ hr})$
(b) MTTF
(c) t_{med}
(d) The standard deviation
(e) The design life for a reliability of 0.98

4.5. For the component defined in Exercise 4.1, determine the reliability for 100 hr if there are two redundant components.

4.6. A system has two failure modes. One failure mode, due to external conditions, has a constant failure rate of 0.07 failure per year. The second, attributed to wearout, has a Weibull distribution with a characteristic life of 10 yr and a shape parameter of 1.8. Determine the system reliability for a design life of 1 yr. Find by trial and error the median time to failure.

4.7. What is the maximum number of identical and independent Weibull components having a scale parameter of 10,000 operating hours and a shape parameter of 1.3 that can be put in series if a reliability of 0.95 at 100 operating hours is desired? What is the resulting system MTTF?

4.8. The time to failure in operating hours of a critical solid-state power unit has the following hazard rate function: $\lambda(t) = 0.003(T/500)^{0.5}$ for $t \geq 0$.
(a) What is the reliability if the power unit must operate continuously for 50 hr?
(b) Determine the design life if a reliability of 0.90 is desired.
(c) Compute the MTTF.
(d) Given that the unit has operated for 50 hr, what is the probability that it will survive a second 50 hr of operation?
(e) The power unit is also subject to chance failures at a rate of 0.002 failure per operating hour because of the external environment. What is the power unit's reliability for the first 50 hr of operation considering both failure modes?

4.9. The wearout of a machine part is normally distributed with 90 percent of the failures occurring between 200 and 270 hr of use (i.e., 5 percent below 200 hr and 5 percent above 270 hr).
(a) Find the MTTF and the standard deviation of failure times.
(b) What is the reliability if the part is to be used for 210 hr and then replaced?
(c) Determine the design life if no more than a 1 percent probability of failure prior to be replacement is to tolerated.
(d) Compute the reliability for a 10-hr use if the part has been operating for 200 hr (i.e., find the conditional reliability).

4.10. After burn-in, the lifetime of a mechanical valve is known to be lognormal with $t_{med} = 2236$ hr and $s = 0.41$.

(a) Determine its design life if specifications call for a reliability of 0.98.
(b) The component is to be used in a pumping device that will require 5 weeks of continuous use. What is the probability of a failure occurring because of the valve?
(c) Compute the MTTF.
(d) Find the standard deviation of the time to failure.
(e) Determine the mode.

4.11. A pressure gauge has been observed to have a Weibull failure distribution with a shape parameter of 2.1 and a characteristic life of 12,000 hr. Find the following:
(a) $R(5000 \text{ hr})$
(b) The B1 and B.1 life
(c) The MTTF and the standard deviation
(d) The median and the mode
(e) The probability of failure in the first year of continuous operation

4.12. A component has a Weibull failure distribution with β equal to 0.86, and its characteristic life is 2450 days. By how many days will the design life for a 0.90 reliability specification be extended as a result of a 30-day burn-in period?

4.13. An automobile engine has four belts each showing the identical wearout effect with a Weibull shape parameter of 1.34. However, their scale parameters are 2500, 3400, 8000, and 6100 operating hours.
(a) If the automobile is new, determine the probability of a belt failure on a 72-hr trip.
(b) If the car (with the belts) has had 4000 hr of use, what is the probability of a belt failure during the next 72 hr of use?

4.14. For the pressure gauge in Exercise 4.11, assume that two identical gauges are used in parallel (redundant) configuration.
(a) Find the system reliability for 5000 hr.
(b) Compute the system MTTF.
(c) Find the probability of failure in the first year.
(d) Determine, by trial and error, the B1 life.

4.15. A cutting tool wears out with a time to failure that is normally distributed with a mean of 10 working days and a standard deviation of 2.5 days.
(a) Determine its design life for a reliability of 0.99.
(b) Find the reliability if the tool is replaced every (i) day; (ii) two days; (iii) five days.
(c) Determine the probability that the cutting tool will last one more day given it has been in use for 5 days.

4.16. A complex machine has a high number of failures. The time to failure was found to be lognormal with $s = 1.25$. Specifications call for a reliability of 0.95 at 1000 cycles.
(a) Determine the median time to failure that must be achieved by engineering modifications to meet the specifications. Assume that design changes do not affect the shape parameter s.
(b) What is the corresponding MTTF and standard deviation?
(c) If the desired median time to failure is obtained, what is the reliability during the next 1000 cycles given it has operated for 1000 cycles?

4.17. The Notso Reliable Company must decide between two AC motors for use in a new household appliance. Motor A has a CFR of 0.000011 failure per operating hour. Motor B has a hazard rate function given by $2 \times 10^{-10}t$.
(a) Specify completely the failure distribution of motor B.
(b) Compare the MTTF of the two motors. On the basis of this comparison, which is preferred?
(c) The company has a 1-yr warranty on its appliances. If the motor is operated two-thirds of the time, which motor would the company prefer in order to reduce its warranty costs? Assume the same replacement cost for both motors.
(d) If the customer expects to operate this appliance over a 10-yr period, which motor would he or she prefer in order to reduce his or her replacement cost? Support your answer. Remember that the appliance is under a warranty the first year.

4.18. Derive a general expression for the average failure rate of a Weibull distribution between two points in time, t_1 and t_2.

4.19. The Cutting Edge Company is planning to manufacture a new lathe cutting tool. This tool has a lifetime that is normal with a standard deviation of 12.0 (cutting) hours. The mean of the distribution is determined by the length of time the material is hardened and its hardening temperature. If a reliability of 0.99 is desired over 100 hr of use, find the corresponding MTTF.

4.20. A rotor used in an AC motor manufactured by the Toole N. Di Company has a time to failure that is lognormal with an MTTF found to be 3600 operating hours and a shape parameter s equal to 2.
(a) Current preventive maintenance practices require the rotor to be replaced every 100 operating hours. Determine the probability that a rotor will survive the 100 hr.
(b) If at the end of 100 operating hours, the maintenance department neglects to replace the rotor, what is the probability that it will survive until the next scheduled replacement (assume that it has not failed at 100 hr)?
(c) If the rotor is still operating after 200 hr, should it be replaced?
(d) From the above analysis, what can you say about the hazard rate and the preventive maintenance replacement policy?

4.21. The failure distribution of your new \$500 VCR is Weibull with $\beta = 2$ and $\theta = 5.6419$ yr. Assume that the unit cannot be repaired but must be replaced on failure.
(a) Find the MTTF.
(b) A 12-month warranty is available. Compute the probability of a failure occurring during the first year.
(c) You got by your first year without a failure. You can now extend your warranty for another 12 months. What is the probability of failure the second year given that your VCR did not fail the first year?
(d) The first-year warranty cost \$25 and the second year follow-on warranty costs \$40. Compute the expected value of having the first- and second-year warranty and compare with the warranty costs to determine whether the warranties should be purchased.
(e) Suppose the second-year warranty can only be purchased if the first-year warranty has been purchased. On the basis of the probability of a failure occurring over the first two years and the total cost of the two warranties, decide whether both should be purchased.

CHAPTER 5

Reliability of Systems

Previous chapters developed several reliability models based on different failure laws. In analyzing a complex system, a particular failure law may be applied to the entire system. However, an alternative approach is to determine an appropriate reliability or reliability model for each component of the system, and by applying the rules of probability according to the configuration of the components within the system, compute a system reliability. This is the topic of this chapter.

5.1 SERIAL CONFIGURATION

Components within a system may be related to one another in two primary ways: in either a serial or a parallel configuration. In series all components must function for the system to function. In a parallel, or redundant, configuration, at least one component must function for the system to function. In the discussion that follows, all components are considered critical in the sense that their function must be performed in order for the system to continue to perform. Under this concept, if either of two serially related components fails, the system will fail. The series relationship is represented by the reliability block diagram of Fig. 5.1.

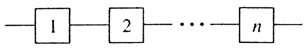


FIGURE 5.1
Reliability block diagram for components in series.

Since reliability is a probability, a system reliability R_s may be determined from the component reliabilities in the following way.[1]

$$E_1 = \text{the event that component 1 does not fail}$$
$$E_2 = \text{the event that component 2 does not fail}$$

Then $\quad P(E_1) = R_1 \quad \text{and} \quad P(E_2) = R_2$

where R_1 = the reliability of component 1
R_2 = the reliability of component 2

Therefore $R_s = P(E_1 \cap E_2) = P(E_1)P(E_2) = R_1(R_2)$ assuming that the two components are independent (i.e., the failure or nonfailure of one component does not change the reliability of the other component). In words, in order for the system to function, both component 1 and component 2 must function.

Generalizing to n mutually independent components in series,

$$R_s(t) = R_1(t) \times R_2(t) \times \cdots \times R_n(t) \le \min\{R_1(t), R_2(t), \ldots, R_n(t)\} \tag{5.1}$$

The inequality results from $0 < R_i(t) < 1$, $i = 1, 2, \ldots, n$, and multiplication. The system reliability can therefore be no greater than the smallest component reliability. Because of Eq. (5.1), it is important for all components to have a high reliability, especially if the system contains a large number of components (see Table 5.1).

If each component has a constant failure rate of λ_i, the system reliability is given by

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} \exp(-\lambda_i t) = \exp\left(-\sum_{i=1}^{n} \lambda_i t\right) = \exp(-\lambda_s t) \tag{5.2}$$

where $\lambda_s = \sum_{i=1}^{n} \lambda_i$. From (5.2) it is apparent that the system also has a constant failure rate. If component failures are governed by the Weibull failure law, then

$$R_s(t) = \prod_{i=1}^{n} \exp\left[-\left(\frac{t}{\theta_i}\right)^{\beta_i}\right] = \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right] \tag{5.3}$$

and
$$\lambda(t) = \exp\left\{-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right\}\left[\sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i-1}\right]\Big/\exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$
$$= \sum_{i=1}^{n}\frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i-1} \tag{5.4}$$

The functional form of Eq. (5.4) indicates that the system does not exhibit Weibull-type failures although every component has a Weibull failure distribution.

[1]To simplify the notation, the argument of $R(t)$ will occasionally be dropped. When this is the case, it is understood that all reliabilities are to be evaluated for the same point in time t.

TABLE 5.1
Serially related system reliability

Component reliability	Number of components: 10	100	1000
0.900	0.3487	0.266×10^{-4}	0.17479×10^{-45}
0.950	0.5987	0.00592	0.52918×10^{-22}
0.990	0.9044	0.3660	0.432×10^{-4}
0.999	0.9900	0.9048	0.3677

EXAMPLE 5.1. Consider a four-component system of which the components are independent and identically distributed with CFR. If $R_s(100) = 0.95$ is the specified reliability, find the individual component MTTF.

Solution

$$R_s(100) = e^{-100\lambda_s} = e^{-100(4)\lambda} = 0.95$$

or
$$\lambda = \frac{-\ln 0.95}{400} = 0.000128$$

and
$$\text{MTTF} = \frac{1}{0.000128} = 7812.5$$

In general, for CFR components

$$\text{MTTF}_s = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} 1/\text{MTTF}_i} \tag{5.5}$$

where MTTF_i = mean time to failure of the ith component.

EXAMPLE 5.2. A system is comprised of four serially related components each having a Weibull time to failure distribution with parameters as shown in the accompanying table.

Component	Scale parameter	Shape parameter
1	100	1.20
2	150	0.87
3	510	1.80
4	720	1.00

The system reliability is therefore given by

$$R_s(t) = \exp\left\{-\left[\left(\frac{t}{100}\right)^{1.2} + \left(\frac{t}{150}\right)^{0.87} + \left(\frac{t}{510}\right)^{1.8} + \left(\frac{t}{720}\right)^{1.0}\right]\right\}$$

and, for example, $R(10) = e^{-0.172627} = 0.8415$.

5.2 PARALLEL CONFIGURATION

Two or more components are in parallel, or redundant, configuration if all units must fail for the system to fail. If one or more units operate, the system continues to operate. Parallel units are represented by the block diagram of Fig. 5.2.

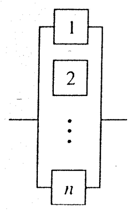


FIGURE 5.2
Reliability block diagram for components in parallel.

System reliability for n parallel and independent components is found by taking 1 minus the probability that all n components fail (i.e., the probability that at least one component does not fail). To see this for two components, consider

$$R_s = P(E_1 \cup E_2) = 1 - P(E_1 \cup E_2)^C = 1 - P(E_1^C \cap E_2^C)$$
$$= 1 - P(E_1^C)P(E_2^C) = 1 - (1 - R_1)(1 - R_2)$$

Generalizing,

$$R_s(t) = 1 - \prod_{i=1}^{n}[1 - R_i(t)] \tag{5.6}$$

It is always true that

$$R_s(t) \geq \max\{R_1(t), R_2(t), \ldots, R_n(t)\}$$

since $\prod_{i=1}^{n}[1 - R_i(t)]$ must be less than the failure probability of the most reliable component. For a redundant system consisting of all CFR components,

$$R_s(t) = 1 - \prod_{i=1}^{n}[1 - e^{-\lambda_i t}] \tag{5.7}$$

where λ_i = the failure rate of the ith component.

EXAMPLE 5.3. For a two-component system in parallel and having CFR,

$$R_s(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$$

and

$$\text{MTTF} = \int_0^\infty R_s(t)\,dt = \int_0^\infty e^{-\lambda_1 t}\,dt + \int_0^\infty e^{-\lambda_2 t}\,dt - \int_0^\infty e^{-(\lambda_1+\lambda_2)t}\,dt$$
$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

EXAMPLE 5.4. Two parallel, identical, and independent components have CFR. If it is desired that $R_s(1000) = 0.95$, find the component and system MTTF.

Solution. We have $R_s(1000) = 2e^{-1000\lambda} - e^{-2000\lambda} = 0.95$. Then by trial and error,

λ	$R_s(1000)$
0.001	0.600
0.0001	0.991
0.0005	0.845
0.0002	0.967
0.00025	0.951
0.000253	0.950

Therefore MTTF = 1/0.000253 = 3952 and MTTF$_s$ = 2/0.000253 − 1/(2[0.000253]) = 5928.9.

5.3 COMBINED SERIES-PARALLEL SYSTEMS

Systems typically contain components in both serial and parallel relationships. Consider, for example, Fig. 5.3. R_i represents the reliability of the ith component. To compute the system reliability, the network may be broken into serial or parallel subsystems. The reliability of each subsystem is found. Then the system reliability may be obtained on the basis of the relationship among the subsystems. In the network of Fig. 5.3, the subsystems have the following reliabilities:

$$R_A = [1 - (1 - R_1)(1 - R_2)]$$
$$R_B = R_A(R_3) \qquad R_C = R_4(R_5)$$

Since R_B and R_C are in parallel with one another and in series with R_6,

$$R_s = [1 - (1 - R_B)(1 - R_C)](R_6)$$

If $R_1 = R_2 = 0.90$, $R_3 = R_6 = 0.98$, and $R_4 = R_5 = 0.99$, then

$$R_B = [1 - (0.10)^2](0.98) = 0.9702$$
$$R_C = (0.99)^2 = 0.9801$$

and

$$R_s = [1 - (1 - 0.9702)(1 - 0.9801)](0.98) = 0.9794$$

5.3.1 High-Level versus Low-Level Redundancy

System redundancy may be obtained in two ways. Each component comprising the system may have one or more parallel components, or the entire system may be placed in parallel with one or more identical systems. The first case is referred to as low-level redundancy, and the second is referred to as high-level redundancy.

As an example, consider a simple system comprised of two serial components, A and B. Figure 5.4 shows the system having low-level redundancy, and Fig. 5.5 depicts high-level redundancy. If it is assumed that both components have the

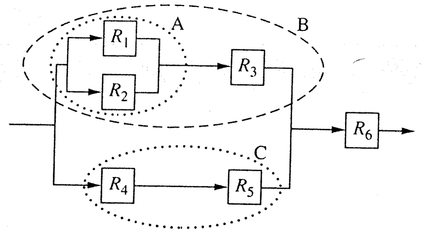


FIGURE 5.3
A system comprised of components in a combined series and parallel relationship.

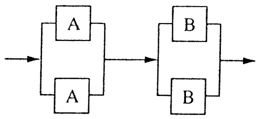


FIGURE 5.4
Two components in low-level redundancy.

same reliability R, the system reliability for the case of low-level redundancy is given by

$$R_{\text{low}} = [1 - (1 - R)^2]^2 = [1 - (1 - 2R + R^2)]^2 = (2R - R^2)^2$$

For the high-level redundancy, system reliability is represented by

$$R_{\text{high}} = 1 - (1 - R^2)^2 = 1 - [1 - 2R^2 + R^4] = 2R^2 - R^4$$

By comparing the two reliabilities, it can be shown that the reliability of the low-level redundancy is greater than the reliability of the high-level redundancy. That is,

$$\begin{aligned} R_{\text{low}} - R_{\text{high}} &= (2R - R^2)^2 - (2R^2 - R^4) \\ &= R^2(2 - R)^2 - R^2(2 - R^2) \\ &= R^2(4 - 4R + R^2 - 2 + R^2) \\ &= 2R^2(R^2 - 2R + 1) = 2R^2(R - 1)^2 \geq 0 \end{aligned}$$

with the equality obtained when $R = 1$. In general this equation will be true if the components' reliabilities are mutually independent and independent of the configuration in which they are placed. Intuitively, this result can also be argued on the basis of the observation that both the low-level and the high-level redundant system will fail if either both components A fail or both components B fail. However, the high-level redundant system may also fail if one A fails and one B fails, provided they fail on separate paths. Therefore, the high-level redundant system has additional failure paths.

EXAMPLE 5.5. A radio set consists of three major components: a power supply, a receiver, and an amplifier, having reliabilities of 0.8, 0.9, and 0.85, respectively. Compute system reliabilities for both high-level and low-level redundancy for systems with two parallel components.

Solution

$$R_{\text{high}} = 1 - [1 - (0.8)(0.9)(0.85)]^2 = 0.849$$

$$\begin{aligned} R_{\text{low}} &= [1 - (1 - 0.8)^2][1 - (1 - 0.9)^2][1 - (1 - 0.85)^2] \\ &= (0.96)(0.99)(0.9775) = 0.929 \end{aligned}$$

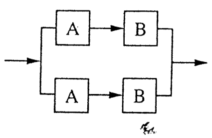


FIGURE 5.5
Two components in high-level redundancy.

5.3.2 k-out-of-n Redundancy

A generalization of n parallel components occurs when a requirement exists for k out of n identical and independent components to function for the system to function. Obviously $k \leq n$. If $k = 1$, complete redundancy occurs, and if $k = n$, the n components are, in effect, in series. The reliability can be obtained from the binomial probability distribution.

If each component is viewed as an independent trial with R (its reliability) as a constant probability of success, then

$$P(x) = \binom{n}{x} R^x (1 - R)^{n-x} \tag{5.8}$$

is the probability of exactly x components operating. This is true since

$$\binom{n}{x} = \frac{n!}{x!(n - x)!}$$

is the number of ways (arrangements) in which x successes (nonfailures) can be obtained from n components. $R^x(1 - R)^{n-x}$ is the probability of x successes and $n - x$ failures for a single arrangement of successes and failures. Therefore

$$R_s = \sum_{x=k}^{n} P(x) \tag{5.9}$$

is the probability of k or more successes from among the n components.

EXAMPLE 5.6. A space vehicle requires three out of its four main engines to operate in order to achieve orbit. If each engine has a reliability of 0.97, determine the reliability of achieving orbit.

Solution

$$\begin{aligned} R_s &= \sum_{x=3}^{4} \binom{4}{x} 0.97^x 0.03^{4-x} \\ &= 4(0.97)^3(0.03) + 0.97^4 = 0.9948 \end{aligned}$$

Exponential failures

If the failure distribution is exponential,

$$R_s(t) = \sum_{x=k}^{n} \binom{n}{x} e^{-\lambda x t}[1 - e^{-\lambda t}]^{n-x}$$

Jumonville and Lesso [1969] have shown that in this case the MTTF can be expressed as

$$\text{MTTF} = \int_0^{\infty} R_s(t)\,dt = \frac{1}{\lambda} \sum_{x=k}^{n} \frac{1}{x} \tag{5.10}$$

EXAMPLE 5.6 (CONTINUED). The main engines in Example 5.6 require an 8-minute burn time. Assume a constant failure rate of 0.0038074 for each engine. Then $R(8) = e^{-0.0038074(8)} = 0.97$, and a single-engine MTTF is 262.65 minutes. Therefore

$$\text{MTTF}_s = 262.65\left(\tfrac{1}{3} + \tfrac{1}{4}\right) = 153.21$$

If $k = 1$, then the MTTF computed by Eq. (5.10) is the mean time to failure of a redundant system consisting of n identical and constant failure rate components.

5.3.3 Complex Configurations

For certain systems, the component configuration is such that the system reliability cannot be simply decomposed into series and parallel relationships. For example, the system shown in Fig. 5.6(a) cannot be analyzed by the previous approach. The problem with this network is that the linkages with component E will not permit definition of a subset of components that are strictly in parallel or in series. However, this network can be analyzed by using either decomposition or enumeration.

Decomposition

Two subnetworks are created: one, shown in Figure 5.6(b), in which component E is assumed to be functioning (with probability R_E), and one, in Figure 5.6(c), in which component E has failed (with probability $1 - R_E$). The reliability of each network is determined separately. Then the system reliability is given by

$$R_s = R_E R_{(b)} + (1 - R_E) R_{(c)}$$

where $R_{(b)} = [1 - (1 - R_A)(1 - R_B)][1 - (1 - R_C)(1 - R_D)]$

$R_{(c)} = 1 - (1 - R_A R_B)(1 - R_C R_D)$

If $R_A = R_B = 0.9$, $R_C = R_D = 0.95$, and $R_E = 0.80$, then

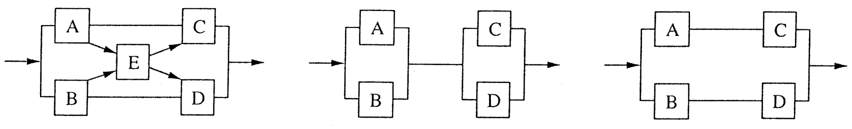
$$R_{(b)} = [1 - (1 - 0.9)^2][1 - (1 - 0.95)^2] = 0.99 \times 0.9975 = 0.9875$$

$$R_{(c)} = 1 - [1 - (0.9)(0.95)]^2 = 0.978975$$

and $R_s = 0.8(0.9875) + 0.2(0.978975) = 0.9858$

Enumeration

For small networks, enumeration may be used to determine the system reliability. The method of enumeration consists in identifying all possible combinations of success (S) or failure (F) of each component and the resulting success or failure of the system. For each possible combination of component successes or failures, the probability of the intersection of these events is computed. The events are assumed to be mutually independent. The sum of the success probabilities or one minus the sum of the failure probabilities is the system reliability. For the linked system in Figure 5.6(a), there are $2^5 = 32$ possible combinations, as shown in Table 5.2.



(a) (b) (c)

FIGURE 5.6
Decomposition of a linked network: (a) a linked network; (b) component E does not fail; (c) component E fails.

TABLE 5.2
Enumeration for system in Fig. 5.6(a)

A	B	C	D	E	System	Probability
S	S	S	S	S	S	0.584820
F	S	S	S	S	S	0.064980
S	F	S	S	S	S	0.064980
S	S	F	S	S	S	0.030780
S	S	S	F	S	S	0.030780
S	S	S	S	F	S	0.146205
F	F	S	S	S	F	
S	F	F	S	S	S	0.003420
S	S	F	F	S	F	
S	S	S	F	F	S	0.007695
F	S	F	S	S	S	0.003420
F	S	S	F	S	S	0.003420
F	S	S	S	F	S	0.016245
S	F	S	F	S	S	0.003420
S	F	S	S	F	S	0.016245
S	S	F	S	F	S	0.007695
F	F	F	S	S	F	
S	F	F	F	S	F	
S	S	F	F	F	F	
F	S	F	F	S	F	
F	S	S	F	F	F	
S	F	S	F	F	S	0.000855
F	F	S	F	S	F	
F	F	S	S	F	F	
S	F	F	S	F	F	
F	S	F	S	F	S	0.000855
F	F	F	F	S	F	
S	F	F	F	F	F	
F	S	F	F	F	F	
F	F	S	F	F	F	
F	F	F	S	F	F	
F	F	F	F	F	F	
					Total	0.985800

The method of decomposition is illustrated again in the following, more complex problem.

EXAMPLE 5.7. An automobile braking system consists of a fluid braking subsystem and a mechanical braking subsystem (parking brake). Both subsystems must fail in order for the system to fail. The fluid braking subsystem will fail if the master cylinder fails (event M) (which includes the hydraulic lines) or all four wheel braking units fail. A wheel braking unit will fail if either the wheel cylinder fails (events WC_1, WC_2, WC_3, WC_4) or the brake pad assembly fails (events BP_1, BP_2, BP_3, BP_4). The mechanical braking system will fail if the cable system fails (event C) or both rear brake pad assemblies fail (events BP_3, BP_4). The reliability block diagram has a linked path to the two rear brake pad assemblies, as shown in Fig. 5.7.

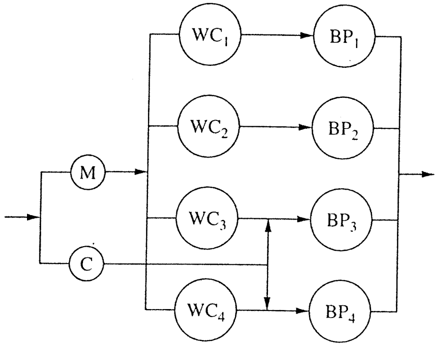


FIGURE 5.7
A reliability block diagram for an automotive braking system.

Solution. Assume that the reliabilities of each wheel cylinder subsystem and brake pad assembly are identical and are represented by $R(\mathrm{WC})$ and $R(\mathrm{BP})$, respectively. To decompose this network, four cases are necessary since there are two linked components.

Case I. Assume for Case I that BP_3 fails and BP_4 is operational. This probability is given by $P_{\mathrm{I}} = [1 - R(\mathrm{BP})]R(\mathrm{BP})$. The reliability for the fluid braking system in Case I is

$$R_f = R(\mathrm{M})\{1 - [1 - R(\mathrm{WC})R(\mathrm{BP})]^2[1 - R(\mathrm{WC})]\}.$$

The reliability for the mechanical (cable) subsystem is $R(\mathrm{C})$ since BP_4 is operational. These two subsystems operate in parallel; therefore, the system reliability for Case I is

$$R_{\mathrm{I}} = 1 - [1 - R_f][1 - R(\mathrm{C})]$$

Case II. Assume for Case II that BP_3 is operational and BP_4 fails. Because each wheel cylinder and brake pad have the same reliability, Case II is identical to Case I and

$$P_{\mathrm{II}} = P_{\mathrm{I}} \quad \text{and} \quad R_{\mathrm{II}} = R_{\mathrm{I}}$$

Case III. Assume BP_3 and BP_4 have failed. This probability is $P_{\mathrm{III}} = [1 - R(\mathrm{BP})]^2$. The cable system will not function, and therefore

$$R_{\mathrm{III}} = R(\mathrm{M})\{1 - [1 - R(\mathrm{WC})R(\mathrm{BP})]^2\}$$

Case IV. Both BP_3 and BP_4 are operational. This probability is $P_{\mathrm{IV}} = [R(\mathrm{BP})]^2$ and the reliability of the fluid system is

$$R_f = R(\mathrm{M})\{1 - [1 - R(\mathrm{WC})R(\mathrm{BP})]^2[1 - R(\mathrm{WC})]^2\}$$

Since the cable system will function as long as C functions,

$$R_{\mathrm{IV}} = 1 - [1 - R_f][1 - R(\mathrm{C})]$$

The overall system reliability can now be found from

$$R_s = P_{\mathrm{I}}R_{\mathrm{I}} + P_{\mathrm{II}}R_{\mathrm{II}} + P_{\mathrm{III}}R_{\mathrm{III}} + P_{\mathrm{IV}}R_{\mathrm{IV}}$$

Observe that $P_{\mathrm{I}} + P_{\mathrm{II}} + P_{\mathrm{III}} + P_{\mathrm{IV}} = 1$.

5.4 SYSTEM STRUCTURE FUNCTION, MINIMAL CUTS, AND MINIMAL PATHS (OPTIONAL)

A very general alternative approach for analyzing the reliability of complex systems is through the use of the system structure function. To define the system structure function, let

$$X_i = \begin{cases} 1 & \text{if component } i \text{ operates} \\ 0 & \text{if component } i \text{ has failed} \end{cases}$$

Then the system structure function is defined by

$$\Psi(X_1, X_2, \ldots, X_n) = \begin{cases} 1 & \text{if system operates} \\ 0 & \text{if system has failed} \end{cases} \tag{5.11}$$

Therefore, for a series system,

$$\Psi(X_1, X_2, \ldots, X_n) = X_1X_2\cdots X_n = \min[X_1, \ldots, X_n] \tag{5.12}$$

and for a parallel system,

$$\Psi(X_1, X_2, \ldots, X_n) = 1 - (1 - X_1)(1 - X_2)\cdots(1 - X_n) = \max[X_1, X_2, \ldots, X_n] \tag{5.13}$$

For a k-out-of-n system, the system structure function is given by

$$\Psi(X_1, X_2, \ldots, X_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0 & \text{if } \sum_{i=1}^{n} X_i < k \end{cases} \tag{5.14}$$

Our interest is in finding $R_s = \Pr\{\Psi(X_1, X_2, \ldots, X_n) = 1\} = E[\Psi(X_1, X_2, \ldots, X_n)]$. The second equality is a result of the binary form of the structure function since

$$E[\Psi(X_1, X_2, \ldots, X_n)] = 0 \cdot \Pr\{\Psi(X_1, X_2, \ldots, X_n) = 0\} + 1\,\Pr\{\Psi(X_1, X_2, \ldots, X_n) = 1\}$$

Assuming independence, for a series system,

$$\begin{aligned} \Pr\{\Psi(X_1, X_2, \ldots, X_n) = 1\} &= \Pr\{X_1 = 1, X_2 = 1, \ldots, X_n = 1\} \\ &= \Pr\{X_1 = 1\}\Pr\{X_2 = 1\}\cdots\Pr\{X_n = 1\} \\ &= R_1R_2\cdots R_n \end{aligned} \tag{5.15}$$

and for a parallel system,

$$\begin{aligned} \Pr\{\Psi(X_1, X_2, \ldots, X_n) = 1\} &= \Pr\{\max(X_1, X_2, \ldots, X_n) = 1\} \\ &= 1 - \Pr\{\text{all } X_i = 0\} \\ &= 1 - \Pr\{X_1 = 0, X_2 = 0, \ldots, X_n = 0\} \\ &= 1 - (1 - R_1)(1 - R_2)\cdots(1 - R_n) \end{aligned} \tag{5.16}$$

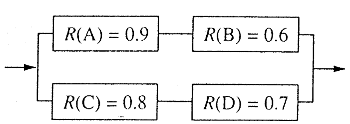


FIGURE 5.8
A four-component reliability block diagram.

To compute the reliability of the k-out-of-n system, we use

$$\Pr\{\Phi(X_1, X_2, \ldots, X_n) = 1\} = \Pr\left\{\sum_{i=1}^{n} X_i \geq k\right\} \tag{5.17}$$

When $R_1 = R_2 = \cdots = R_n$ in Eq. (5.17), the reliability can be determined using the binomial probability distribution and Eq. (5.8) and (5.9).

EXAMPLE 5.8. For the simple system consisting of series components A and B in parallel with series components C and D as shown in Fig. 5.8, the reliability computed using the approach discussed in Section 5.3 is $R_s = 1 - [1 - (0.9)(0.6)][1 - (0.8)(0.7)] = 0.7976$.

The system structure function is

$$\begin{aligned}\Psi(X_A, X_B, X_C, X_D) &= 1 - (1 - X_A X_B)(1 - X_C X_D)\\ &= X_A X_B + X_C X_D - X_A X_B X_C X_D\end{aligned}$$

In general, $E(X_1, X_2, \ldots, X_k) = R_1 R_2 \cdots R_k$ since X_i is binary with $\Pr\{X_i = 1\} = R_i$. Therefore,

$$\begin{aligned}R_s &= E[\Psi(X_A, X_B, X_C, X_D)]\\ &= E(X_A X_B) + E(X_C X_D) - E(X_A X_B X_C X_D)\\ &= R_A R_B + R_C R_D - R_A R_B R_C R_D\\ &= 0.9(0.6) + 0.8(0.7) - 0.9(0.6)(0.8)(0.7) = 0.7976\end{aligned}$$

5.4.1 Coherent Systems

A system is *coherent* when a component reliability improvement does not degrade the system reliability. A coherent system has a structure function that is monotonically increasing. That is, if $Y_i \geq X_i$, then $\Psi(Y_1, Y_2, \ldots, Y_n) \geq \Psi(X_1, X_2, \ldots, X_n)$. If the inequality is strict for a given component i, then that component is said to be *relevant*. Obviously, if $\Psi(1, 1, 1, 1) = \Psi(1, 1, 1, 0)$, then component 4 is not relevant to the operation of the system.

5.4.2 Minimal Path and Cut Sets

The system reliability may also be determined using minimal path sets or minimal cut sets. A path is a set of components whose functioning ensures that the system functions. A minimal path is one in which **all** the components within the set must

function for the system to function. A cut is a set of components whose failure will result in a system failure. A minimal cut is one in which **all** the components must fail in order for the system to fail.

EXAMPLE 5.8 (CONTINUED). For the network in Example 5.8, the minimal path and cut sets are as given here:

Minimal paths	Minimal cuts
A, B	A, C
C, D	A, D
	B, C
	B, D

For the system to function, at least one of the minimal paths must function. Therefore

$$\Psi(X_A, X_B, X_C, X_D) = [1 - (1 - X_A X_B)(1 - X_C X_D)] = X_A X_B + X_C X_D - X_A X_B X_C X_D$$

$$R_s = E[\Psi(X_A, X_B, X_C, X_D)] = E(X_A X_B) + E(X_C X_D) - E(X_A X_B X_C X_D)$$

which gives the same result as above.

Using minimal cut sets, at least one component in each cut must function for the system to function. Therefore the structure function may be written as

$$\begin{aligned}&\Psi(X_A, X_B, X_C, X_D)\\ &\quad= [1 - (1 - X_A)(1 - X_C)] \times [1 - (1 - X_A)(1 - X_D)]\\ &\qquad\times [1 - (1 - X_B)(1 - X_C)] \times [1 - (1 - X_B)(1 - X_D)]\\ &\quad= (X_A + X_C - X_A X_C)(X_A + X_D - X_A X_D)(X_B + X_C - X_B X_C)(X_B + X_D - X_B X_D)\end{aligned}$$

Then using the fact that $X_i^2 = X_i$,

$$\begin{aligned}\Psi(X_A, X_B, X_C, X_D) &= (X_A + X_C X_D - X_A X_C X_D)(X_B + X_C X_D - X_B X_C X_D)\\ &= X_A X_B + X_C X_D - X_A X_B X_C X_D\end{aligned}$$

Setting $R_s = E[\Psi(X_A X_B X_C X_D)]$ provides the same result as before.

5.4.3 System Bounds

In general it is possible to construct a very crude lower bound on the system reliability by considering all the components in series. Likewise an upper bound on the system reliability is obtained by considering all the components in parallel. Unfortunately, in most cases the difference between these two bounds is quite large and, as a result, they are not very useful. However, using minimal sets, one can construct upper and lower bounds that are much closer to one another. An upper-bound network is determined by placing all the minimal path sets in parallel, and a lower-bound network is obtained by placing each of the minimal cut sets in series. If two or more paths

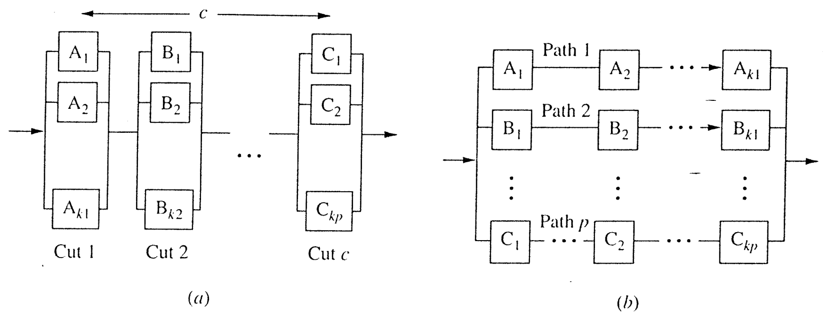


FIGURE 5.9
Alternative networks formed from cut sets and path sets: (*a*) lower-bound network formed from minimal cuts in series; (*b*) upper-bound network formed by minimal paths in parallel.

share common components, the paths are not independent. As a result, if a common component fails, each path containing that component will fail as well. Therefore, a network formed by minimal path sets in parallel will overestimate the system reliability when the paths are assumed to be independent. A similar argument holds for cut sets in series. If two or more cut sets contain a common component, the cut sets are no longer independent. Therefore, if a common component operates, every cut set containing that component will operate as well. Therefore, series cut sets will underestimate the system reliability when the cut sets are assumed to be independent. The upper (lower) bound will be attained if the minimal path sets (minimal cut sets) share no common components (i.e., they are independent).

The lower-bound reliability R_l, is found from

$$R_l = \prod_{i=1}^{c}\left[1 - \prod_{k \in S_i}(1 - R_k)\right] \tag{5.18}$$

where c is the number of minimal cut sets and S_i is the set of indices of those components composing the ith minimal cut set. The expression in brackets is the probability that at least one component in cut set i is operating. See Fig. 5.9(a).

The upper-bound reliability R_u is given by

$$R_u = 1 - \prod_{i=1}^{p}\left[1 - \prod_{k \in S_i'} R_k\right] \tag{5.19}$$

where p is the number of minimal path sets and S_i' is the set of indices of those components composing the ith minimal path set. The expression in brackets is the probability that the ith minimal path has failed. See Fig. 5.9(b).

EXAMPLE 5.9. For the linked network in Fig. 5.6(a), the minimal cut sets and their probabilities are as given here:

Minimal cuts	Probability $= \prod(1 - R_k)$
A, B	$0.1(0.1) = 0.01$
C, D	$0.05(0.05) = 0.0025$
A, E, D	$0.1(0.2)(0.05) = 0.001$
B, E, C	$0.1(0.2)(0.05) = 0.001$

By configuring the set of minimal cuts in series, a lower bound on the system reliability is obtained since the system reliability cannot be any worse than the reliability that at least one component in each cut set operates. Therefore, from Eq. (5.18),

$$R_l = (1 - 0.01)(1 - 0.0025)(1 - 0.001)^2 = 0.98555$$

The minimal path sets and their reliabilities are as given here:

Minimal paths	Reliability $= \prod(R_k)$
A, C	$0.9(0.95) = 0.855$
B, D	$0.9(0.95) = 0.855$
A, E, D	$0.9(0.8)(0.95) = 0.684$
B, E, C	$0.9(0.8)(0.95) = 0.684$

By configuring the set of minimal paths in parallel, an upper bound on the system is obtained since the system cannot be more reliable than the union of all of its paths. Therefore, from Eq. (5.19),

$$R_u = 1 - (1 - 0.855)^2(1 - 0.684)^2 = 0.9979$$

5.5 COMMON-MODE FAILURES

The assumption of independence of failures among n components within a system may be easily violated. For example, several components may share the same power source, or external environmental conditions such as excessive heat or vibration may affect several components in the same manner. Operations or maintenance errors, design flaws, and substandard material or parts may also contribute to a common-mode failure. A common-mode failure can be depicted in series with those components sharing the failure mode. Figure 5.10 illustrates a common-mode failure associated with a three-component redundant system. The system reliability is given by $R_s = [1 - (1 - R_1)(1 - R_2)(1 - R_3)]R'$. In order to represent the system in this way, it must be possible to separate independent failures from the common-mode failures. In order for the redundancy network to have an effect, the common-mode failure must have a high reliability.

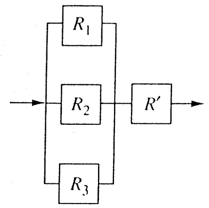


FIGURE 5.10
A reliability block diagram showing the effect of a common failure mode on redundant components.

EXAMPLE 5.10. For the system of two CFR components discussed in Example 5.4, assume a common-mode CFR of 0.00001 in addition to the components' independent failure rates. Then

$$R_s(t) = (2e^{-0.000253t} - e^{-0.000506t})e^{-0.00001t}$$

and
$$R_s(1000) = 0.95e^{-0.00001\times 1000} = 0.94$$

To find the MTTF,

$$\text{MTTF} = \int_0^\infty \left[2e^{-0.000263t} - e^{-0.000516t}\right] dt = \frac{2}{0.000263} - \frac{1}{0.000516} = 5666.6$$

EXAMPLE 5.11. Consider the same two components in series; the individual failure rates are composed of both the independent and common-mode components. That is,

$$\lambda = 0.000253 + 0.00001 = 0.000263$$

Then the system failure rate is given by

$$\lambda_s = 2 \times 0.000253 + 0.00001 = 0.000516,$$

since the common-mode failure rate is shared by both components. Therefore

$$R_s(t) = (e^{-0.00253t})^2 e^{-0.00001t} = e^{-0.000516t}$$

The common-mode system failure rate is less than what would be observed if both components had independent failure rates given by λ.

5.6 THREE-STATE DEVICES

Three-state devices are components that have both an *open* and *short* failure mode and an operating state. Examples are diodes, electrical circuits, and flow valves. The interesting problem with respect to systems comprising these components is that redundancy may either increase or decrease system reliability. For example, if the short failure mode is present, then adding additional parallel units will increase the likelihood of the system experiencing a short. The system reliability in this case depends on the dominant failure mode, the system configuration, and the number of redundant components. Another example of a three-state device is an alarm system that may fail safe (false alarm) or may fail to danger (failure to function when needed). Redundancy, in this case, will increase the likelihood of false alarms and decrease the likelihood of a fail-to-danger event.

Two primary assumptions are made in analyzing these types of systems. First, it is assumed the failure modes are mutually exclusive (only one of the failure modes can occur), and second, all the components composing the system are independent.

5.6.1 Series Structure

To illustrate the methodology for computing system reliability, a simple network consisting of two switches in series is considered. For the system to fail short, both of the switches must fail short. For the system to fail open, at least one switch must fail open. Therefore, we can define the following two failure events:

$$E_1 = \text{the event that both switches fail short}$$
$$E_2 = \text{the event that at least one switch fails open}$$

Let $Q = 1 - R$ equal the probability of a system failure. Then

$$Q = P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

since the events are mutually exclusive. Let

$$q_{oi} = \text{the probability that component } i \text{ fails open}$$
$$q_{si} = \text{the probability that component } i \text{ fails short}$$

Then
$$Q = (q_{s1}q_{s2}) + (q_{o1} + q_{o2} - q_{o1}q_{o2})$$

and
$$R = 1 - [(q_{s1}q_{s2}) + (q_{o1} + q_{o2} - q_{o1}q_{o2})]$$
$$= (1 - q_{o1})(1 - q_{o2}) - q_{s1}q_{s2}$$

In words, the system reliability is the probability that neither component fails open minus the probability that both components fail short. This result can be generalized. For n components in series,

$$R = \prod_{i=1}^{n}(1 - q_{oi}) - \prod_{i=1}^{n} q_{si} \tag{5.20}$$

System reliability is the probability that there are no opens minus the probability that all components short.

5.6.2 Parallel Structure

Two parallel switches are now considered. For the system to fail short, one or both of the switches must fail short. For the system to fail open, both switches must fail open. Therefore, we can define the following two failure events:

$$E_1 = \text{the event that both switches fail open}$$
$$E_2 = \text{the event that at least one switch fails short.}$$

Then
$$Q = P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
$$= (q_{o1}q_{o2}) + (q_{s1} + q_{s2} - q_{s1}q_{s2})$$
and
$$R = 1 - [(q_{o1}q_{o2}) + (q_{s1} + q_{s2} - q_{s1}q_{s2})]$$
$$= (1 - q_{s1})(1 - q_{s2}) - q_{o1}q_{o2}$$

In words, the system reliability is the probability that neither component fails short minus the probability that both components fail open. This result can be generalized. For n redundant components,

$$R = \prod_{i=1}^{n}(1 - q_{si}) - \prod_{i=1}^{n} q_{oi} \tag{5.21}$$

Reliability is the probability that there are no shorts minus the probability that all components are open.

EXAMPLE 5.12. A mechanical valve fails to open (fails open) 5 percent of the time and fails to close (fails short) 10 percent of the time. Compute the system reliability for three valves (1) in series and (2) in parallel.

Solution. For the three valves in series,

$$R = (1 - 0.05)^3 - 0.10^3 = 0.856375$$

For the parallel configuration:

$$R = (1 - 0.10)^3 - 0.05^3 = 0.728875$$

Since the dominant failure mode is to fail short, the series configuration has the greater reliability.

5.6.3 Low-Level Redundancy

Figure 5.11 is an example of a low-level redundant system having m serial components each having n redundant units. The system reliability is a generalization of the series configuration (Eq. 5.20) and is found by computing the probability of at least one unopen path in each redundant string minus the probability of at least one short in each path. Mathematically, this reliability can be written as

$$R_L = \prod_{i=1}^{m}(1 - q_{oi}^n) - \prod_{i=1}^{m}[1 - (1 - q_{si})^n] \tag{5.22}$$

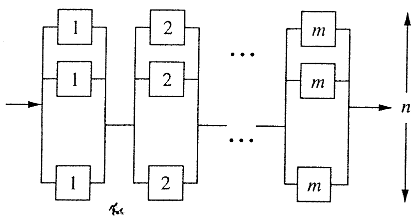


FIGURE 5.11
Block diagram of m three-state devices in a low-level redundant configuration.

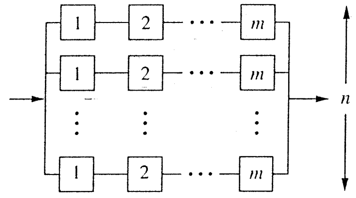


FIGURE 5.12
Block diagram of m three-state devices in a high-level redundant configuration.

5.6.4 High-Level Redundancy

Figure 5.12 represents a high-level redundant configuration having n redundant systems each with m serial components. The system reliability is a generalization of the parallel configuration (Eq. 5.21) and is found by computing the probability that no path is completely shorted minus the probability that there is at least one open on each path. Mathematically,

$$R_H = \left(1 - \prod_{i=1}^{m} q_{si}\right)^n - \left[1 - \prod_{i=1}^{m}(1 - q_{oi})\right]^n \tag{5.23}$$

EXAMPLE 5.13. With three-state devices, it is not necessarily true that low-level redundancy provides a greater reliability than high-level redundancy. Consider a system composed of the following three components with two redundant units available: $m = 3$ and $n = 2$.

Component	q_s	q_o
1	0.15	0.05
2	0.10	0.06
3	0.20	0.01

Then

$$R_L = (1 - 0.05^2)(1 - 0.06^2)(1 - 0.01^2)$$
$$- [1 - (1 - 0.15)^2][1 - (1 - 0.10)^2[1 - (1 - 0.20)^2]$$
$$= 0.9748$$
$$R_H = [1 - (0.15)(0.10)(0.20)]^2 - [1 - (1 - 0.05)(1 - 0.06)(1 - 0.01)]^2 = 0.9806$$

The reliability of the above networks could have been determined from an opposite viewpoint. For example, the low-level redundant network could have been analyzed by finding the system reliability with respect to a short minus the system probability of failing open. This approach is equivalent to Eq. (5.22) since

$$R\{\text{short}\} - \Pr\{\text{failing open}\} = 1 - \Pr\{\text{failing short}\} - [1 - R\{\text{failing open}\}]$$
$$= R\{\text{failing open}\} - \Pr\{\text{failing short}\}$$

Because the events "failing open" and "failing short" are mutually exclusive, this approach works.

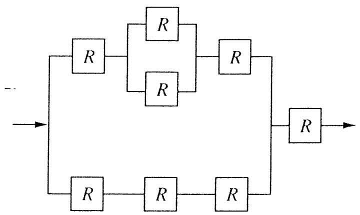
EXERCISES

5.1 The time to failure (in years) of a Cyclone 365 computer has the probability density function $f(t) = 1/(t+1)^2$, $t \geq 0$.
(a) If three of these computers are placed in parallel aboard the proposed space station, what is the system reliability for the first 6 months of operation?
(b) What is the system design life in days if reliability of a 0.999 is required?
(c) What is the system reliability for 6 months if two out of the three computers must function?

5.2 Find the minimum number of redundant components, each having a reliability of 0.4, necessary to achieve a system reliability of 0.95. There is a common-mode failure probability of 0.03.

5.3 Which system, (i) or (ii), has the higher reliability at the end of 100 operating hours?
(i) Two CFR components in parallel each having an MTTF of 1000 hr.
(ii) A Weibull component with a shape parameter of 2 and a characteristic life of 10,000 hr in series with a CFR component with a failure rate of 0.00005.

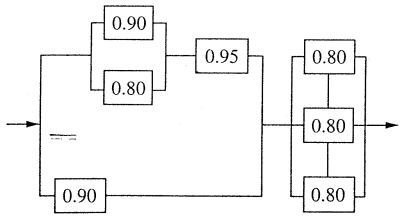
5.4 (a) For the following network, derive an expression for the system reliability in terms of the component reliabilities. Assume that each component has a reliability of R.
(b) Compute the system reliability if $R = 0.9$.



5.5 Three communications channels in parallel have independent failure modes of 0.1 failure per hour. These components must share a common transceiver. Determine the MTTF of the transceiver in order that the system has a reliability of 0.85 to support a 5-hr mission. Assume constant failure rates.

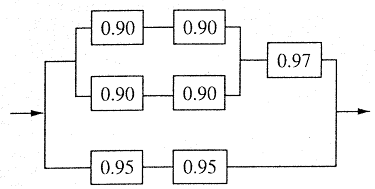
5.6 Ten Weibull components, each having a shape parameter of 0.80, must operate in series. Determine a common characteristic life in order that they have a design life of 1 yr with a reliability of 0.99.

5.7 Find the system reliability of the configurations in (a) and (b).
(a)



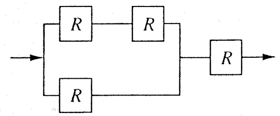
(*a*)

(b)



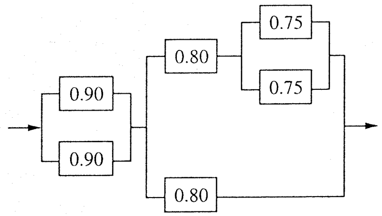
(*b*)

For $R_s = 0.99$, find R in (c). (*Hint:* Find R_s in the simplest terms of R and use trial and error).

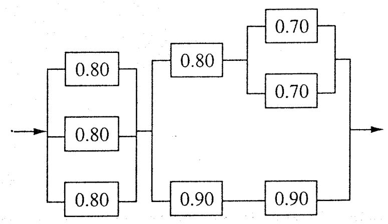


(*c*)

5.8 Find the system reliability of the following series-parallel configurations. Component reliabilities are given.

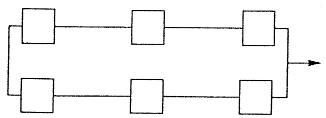


(*a*)

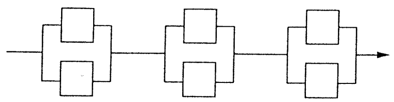


(*b*)

5.9 For each of the following redundant systems, determine the component MTTF necessary to provide a system reliability of 0.90 after 100 hr of operation. The components have the same constant failure rate.



(*a*) High-level redundancy



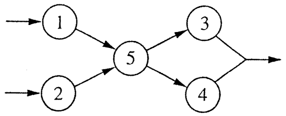
(*b*) Low-level redundancy

5.10 Itsa Failing, a reliability engineer, has determined that the reliability function for a critical solid-state power unit for use in a communications satellite is $R(t) = 10/(10 + t)$, $t \geq 0$ and t measured in years.
(a) How many units must be placed in parallel in order to achieve a reliability of 0.98 for 5 yr of operation?
(b) If there is an additional common mode, constant failure rate of 0.002 as a result of environmental factors, how many units should be placed in parallel? Compute the achieved system reliability.

5.11 Derive the reliability function and MTTF for three CFR components in parallel. If three components, each with CFR, are placed in parallel, determine the system reliability for 0.1 yr and the MTTF. Their failure rates are 5 per year, 10 per year, and 15 per year.

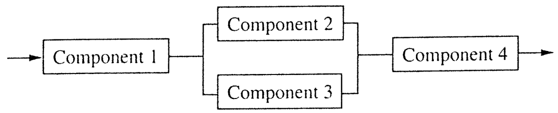
5.12 A signal processor has a reliability of 0.90. Because of the low reliability a redundant signal processor is to be added. However, a signal splitter must be added before the signal processors, and a comparator must be added after the signal processors. Each of these new components has a reliability of 0.95. Does adding a redundant signal processor increase the system reliability?

5.13 The following natural gas distribution network contains five shut-off valves configured as shown. Valves 1–4 have a probability of 0.02 of failing open and a probability of 0.15 of failing short. Valve 5 has a probability of 0.05 of failing open and a probability of 0.20 of failing short. Find the system reliability.



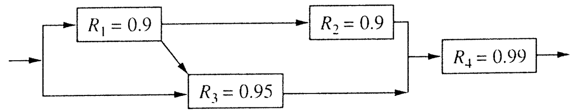
5.14 Compare the reliability between a high-level redundant network and a low-level redundant network comprised of two serially related components with three redundant units of each available. The probability of any component failing open is 0.05, and the probability of any component failing short is 0.1.

5.15 Determine the reliability of the following series-parallel configuration, in which the ith component has a probability q_{oi} of failing open and a probability q_{si} of failing short.



5.16 A system is designed to operate for 100 days. The system consists of three components in series. Their failure distributions are (1) Weibull with shape parameter 1.2 and scale parameter 840 days; (2) lognormal with shape parameter (s) 0.7 and median 435 days; (3) constant failure rate of 0.0001.
(a) Compute the system reliability.
(b) If two units of components 1 and 2 are available, determine the high-level redundancy reliability. Assume that components 1 and 2 can be configured as a subassembly.
(c) If two units of components 1 and 2 are available, determine the low-level redundancy reliability.

5.17 Determine the reliability of the following linked system using the decomposition method.



5.18 For k-out-of-n systems there may be a crossover point, where the single-component reliability is greater than the k-out-of-n system reliability. If R is the single-component reliability, then the crossover point is found from solving

$$R = \sum_{x=k}^{n} \binom{n}{x} R^x (1 - R)^{n-x}$$

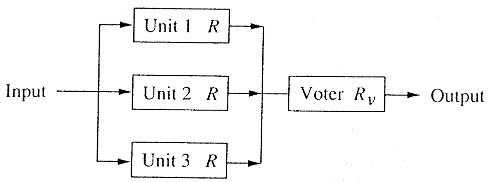
(a) Find the crossover point for a 2-out-of-3 redundant system.
(b) Repeat for a 3-out-of-4 system.

5.19 An alternative approach for analyzing reliability networks is to identify all possible combinations of failure events of the components, compute the probabilities of those outcomes resulting either in system failure or in a system success (nonfailure), and sum the individual probabilities to obtain the system failure probability or reliability. Since each component has two states, success (S) or failure (F), there are 2^n mutually exclusive combinations to be considered, where n is the number of components composing the network. For the system of Exercise 5.17, the analysis will take the following form:

R_1	R_2	R_3	R_4	System	Probability
S	S	S	S	S	$(0.9)^2(0.95)(0.99) = 0.761805$
S	S	S	F	F	
S	S	F	S	S	$(0.9)^2(0.05)(0.99) = 0.040095$
			...		

Complete this problem by identifying all possible combinations and summing the probabilities of those resulting in a system success. Compare your result with that obtained from the decomposition approach used in Exercise 5.17.

5.20 In the design of computer systems, increased reliability may be achieved through the use of triple modular redundancy. This consists of three identical units (logic or binary variables) feeding into a common voting system, as shown in the figure. The value of the output variable is determined by majority voting. If the single-unit reliability is R and the reliability of the voting system is R_v, show that the system reliability is given by $R^2(3 - 2R)R_v$. If the reliability of the voting system is 0.95, what is the crossover reliability (see Exercise 5.18) of a single unit?



5.21 Show that

$$\text{MTTF} = \frac{1}{\lambda}\sum_{i=1}^{n}\binom{n}{i}\frac{(-1)^{i-1}}{i}$$

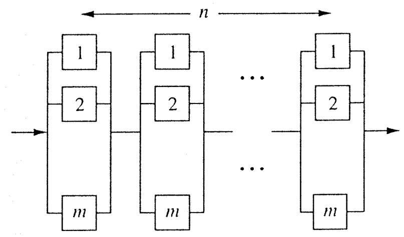
for n redundant and independent components each with $\lambda(t) = \lambda$. *Hint:* From the binomial theorem,

$$(p+q)^n = \sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^i$$

5.22 (a) For three-state devices, write an expression for the reliability of a low-level redundant system as the system reliability with respect to a short minus the system probability of an open. Show that this is equivalent to Eq. (5.22).

(b) Repeat for a high-level redundant system by writing an expression for the system reliability with respect to an open minus the system probability of a short. Show that this expression is equivalent to Eq. (5.23).

5.23 Write an expression for the system reliability of the following series-parallel network, where q_{oi} is the probability that the ith component fails open and q_{si} is the probability that the ith component fails short.



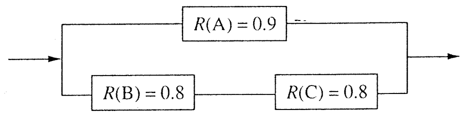
5.24 Show that the MTTF for n serially related and independent components each having a linear hazard rate function is

$$\text{MTTF} = \left(\frac{\pi}{2\sum_{i=1}^{n} a_i}\right)^{1/2}$$

where $\lambda_i(t) = a_i t$ and $a_i > 0$.

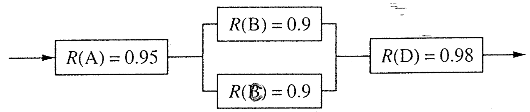
Hint:
$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$$

5.25 Structure function (optional). Consider the network shown below.



(a) Write the structure function.
(b) Find the system reliability using the structure function.
(c) Identify the minimal path sets and minimal cut sets.
(d) Find the lower- and upper-bound reliabilities using the minimal sets in (c).

5.26 Structure function (optional). Consider the network shown below.



(a) Write the structure function.
(b) Find the system reliability using the structure function.
(c) Identify the minimal path sets and minimal cut sets.
(d) Find the lower- and upper-bound reliabilities using the minimal sets in (c).

CHAPTER 6

State-Dependent Systems

As discussed in the previous chapter, a fundamental computation in reliability engineering is the determination of system reliability from a knowledge of component reliabilities and their system configuration. On the basis of the critical assumption of independent failures among components, Eqs. (5.1) and (5.6) were easily derived using the basic rules of probability. However, when component failures are in some way dependent, more powerful methods, such as Markov analysis, may be needed.

6.1 MARKOV ANALYSIS

Markov analysis looks at a system as being in one of several states. One possible state, for example, is that in which all the components composing the system are operating. Another possible state is that in which one component has failed but the other components continue to operate. The fundamental assumption in a Markov process is that the probability that a system will undergo a transition from one state to another state depends only on the current state of the system and not on any previous states the system may have experienced. In other words, the transition probability is not dependent on the past (state) history of the system. This is equivalent to the memorylessness of the exponential distribution, and it is therefore not surprising that exponential times to failure satisfy this Markovian property. We will express the transition from one state to another as an instantaneous (failure) rate. Assuming the process is also stationary (i.e., the transition probabilities do not change over time), the transition rates will be constant. Again, this is equivalent to assuming exponential failure times.

The methodology is presented through the use of an example. We begin by using Markov analysis to derive Eqs. (5.2) and (5.7) for the two-component system. Although independence among components is assumed here, this derivation

provides a good example of the technique. In applying Markov analysis to this problem, we assume that each of n components will be in one of two states—operating or failed. The system state is then defined to be one of the 2^n possible combinations of operating and failed components. For our two-component system we define the following four system states:

State	Component 1	Component 2
1	operating	operating
2	failed	operating
3	operating	failed
4	failed	failed

If the two components are in parallel (redundant), only state 4 results in a system failure. On the other hand, if the two components are in series, then states 2, 3, and 4 would each constitute a failure state. The objective is to find the probability of the system being in each state as a function of time. We denote the probability of being in state i at time t as $P_i(t)$. Then for a two-component series system,

$$R_s(t) = P_1(t)$$

and for a two-component parallel system,

$$R_p(t) = P_1(t) + P_2(t) + P_3(t)$$

Observe that the system must be one of the four states at any given time. Therefore,

$$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1 \tag{6.1}$$

What remains is to find $P_i(t)$, $i = 1, 2, 3, 4$.

If we assume that individual components have constant failure rates λ_i, we can represent the two-component system using the rate diagram in Fig. 6.1. The nodes in Fig. 6.1 represent the four system states, and the branches show the transition rate (λ_i) from one node to another. From the rate diagram we can derive the following equation:

$$P_1(t + \Delta t) = P_1(t) - \lambda_1 \Delta t P_1(t) - \lambda_2 \Delta t P_1(t) \tag{6.2}$$

Equation 6.2 states that the probability of the system being in state 1 at time $t + \Delta t$ is equal to the probability of it being in state 1 at time t minus the probability of it

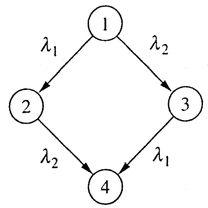


FIGURE 6.1
Rate diagram for a two-component system with independence.

being in state 1 at time t times the probability of transitioning ($\lambda_i \Delta t$) to either state 2 or 3. Observe that $\lambda_1 \Delta t$ is the conditional probability of a transition to state 2 occurring during time Δt given that the system is currently in state 1. Therefore, $\lambda_1 \Delta t P_1(t)$ is the joint probability of the system being in state 1 at time t and making a transition to state 2 during time Δt. A similar argument holds for $\lambda_2 \Delta t P_1(t)$ for transitioning to state 3.

A second equation is obtained from state 2.

$$P_2(t + \Delta t) = P_2(t) + \lambda_1 \Delta t P_1(t) - \lambda_2 \Delta t P_2(t) \tag{6.3}$$

is the probability of the system being in state 2 at time $t + \Delta t$ and is equal to the probability of being in state 2 at time t plus the probability of being in state 1 at time t and making a transition ($\lambda_1 \Delta t$) to state 2 in time Δt minus the probability of being in state 2 at time t and making a transition to state 4 ($\lambda_2 \Delta t$) in time Δt. Similarly for state 3,

$$P_3(t + \Delta t) = P_3(t) + \lambda_2 \Delta t P_1(t) - \lambda_1 \Delta t P_3(t) \tag{6.4}$$

and for state 4,

$$P_4(t + \Delta t) = P_4(t) + \lambda_2 \Delta t P_2(t) + \lambda_1 \Delta t P_3(t) \tag{6.5}$$

Rewriting Eq. 6.2,

$$\frac{P_1(t + \Delta t) - P_1(t)}{\Delta t} = -(\lambda_1 + \lambda_2) P_1(t)$$

Then

$$\lim_{\Delta t \to 0} \frac{P_1(t + \Delta t) - P_1(t)}{\Delta t} = \frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2) P_1(t) \tag{6.6}$$

In a similar fashion, Eqs. (6.3) and (6.4) lead to the following differential equations:

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t) \tag{6.7}$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1 P_3(t) \tag{6.8}$$

Equations (6.6), (6.7), and (6.8) along with equation (6.1) can be solved simultaneously (see Appendix 6A for details). Their solution is

$$P_1(t) = e^{-(\lambda_1 + \lambda_2)t} \tag{6.9}$$

$$P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t} \tag{6.10}$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1 + \lambda_2)t} \tag{6.11}$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t) \tag{6.12}$$

Then for a series system we have

$$R_s(t) = P_1(t) = e^{-(\lambda_1 + \lambda_2)t}$$

and for a parallel system,

$$\begin{aligned} R_p(t) &= P_1(t) + P_2(t) + P_3(t) \\ &= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t} \end{aligned}$$

which are equivalent to Eqs. (5.2) and (5.7) for a two-component ($n = 2$) system.

6.2 LOAD-SHARING SYSTEM

A rather straightforward application of Markov analysis is to a load-sharing system. We are given two components in parallel as before except there is now a dependency between the two components. If one component fails, the failure rate of the other component increases as a result of the additional load placed on it. Because of this dependency, the reliability block diagram techniques of Chapter 5 cannot be applied. Instead, we must use Markov analysis to determine the system reliability.

Define the four states of the system as before. The rate diagram is shown in Fig. 6.2, where λ_1^+ and λ_2^+ represent the increased failure rates of components 1 and 2, respectively, as a result of the increased load. The resulting differential equations are

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2) P_1(t) \tag{6.13}$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2^+ P_2(t) \tag{6.14}$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1^+ P_3(t) \tag{6.15}$$

The solution to these equations are found using the same approach as before (see Appendix 6B) and is as follows:

$$P_1(t) = e^{-(\lambda_1 + \lambda_2)t} \tag{6.16}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_2^+}\left[e^{-\lambda_2^+ t} - e^{-(\lambda_1 + \lambda_2)t}\right] \tag{6.17}$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_1^+}\left[e^{-\lambda_1^+ t} - e^{-(\lambda_1 + \lambda_2)t}\right] \tag{6.18}$$

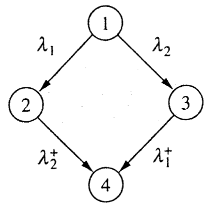


FIGURE 6.2
Rate diagram for a two-component load-sharing system.

and $$R(t) = P_1(t) + P_2(t) + P_3(t)$$

If we let $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^+ = \lambda_2^+ = \lambda^+$, then

$$R(t) = e^{-2\lambda t} + \frac{2\lambda}{2\lambda - \lambda^+}\left[e^{-\lambda^+ t} - e^{-2\lambda t}\right] \tag{6.19}$$

and $$\text{MTTF} = \int_0^\infty R(t)\,dt = \frac{1}{2\lambda} + \frac{2\lambda}{2\lambda - \lambda^+}\left[\frac{1}{\lambda^+} - \frac{1}{2\lambda}\right] \tag{6.20}$$

EXAMPLE 6.1. Two generators provide needed electrical power. If either fails, the other can continue to provide electrical power. However, the increased load results in a higher failure rate for the remaining generator. If $\lambda = 0.01$ failure per day and $\lambda^+ = 0.10$ failure per day, determine the system reliability for a 10-day contingency operation and determine the system MTTF.

Solution. From Eq. 6.19,

$$R(t) = e^{-2(0.01)t} + \frac{2(0.01)}{2(0.01) - 0.10}\left[e^{-0.10t} - e^{-2(0.01)t}\right]$$

and $$R(10) = e^{-0.2} + \frac{0.02}{-(0.08)}\left[e^{-1} - e^{-0.2}\right] = 0.9314$$

From Eq. (6.20),

$$\text{MTTF} = \frac{1}{0.02} + \frac{0.02}{-0.08}\left[\frac{1}{0.10} - \frac{1}{0.02}\right] = 60 \text{ days}$$

6.3 STANDBY SYSTEMS

Standby systems are an important area of study within reliability. Depending on the probability of a failure occurring when switching to a standby unit, these systems are generally much more reliable than an active redundant system. The two-component standby system differs from the active redundant system discussed earlier in that the standby unit will have no failures or a reduced failure rate while in its standby mode. Once active, the backup unit may experience the same failure rate as the on-line (primary) system (if they are identical units) or may have a different failure rate. The dependency arises because the failure rate of the standby unit depends on the state of the primary unit.

The rate diagram is shown in Fig. 6.3, where state 3 represents a failure (perhaps undetected) of the standby unit while in standby with λ_2^- being the corresponding failure rate. The resulting system of equations is

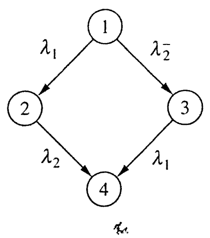


FIGURE 6.3
Rate diagram for a two-component standby system with failures in standby.

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2^-)P_1(t) \tag{6.21}$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t) \tag{6.22}$$

$$\frac{dP_3(t)}{dt} = \lambda_2^- P_1(t) - \lambda_1 P_3(t) \tag{6.23}$$

having solution (see Appendix 6C)

$$P_1(t) = e^{-(\lambda_1 + \lambda_2^-)t} \tag{6.24}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right] \tag{6.25}$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1 + \lambda_2^-)t} \tag{6.26}$$

with $$R(t) = P_1(t) + P_2(t) + P_3(t)$$

$$= e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right] \tag{6.27}$$

and

$$\text{MTTF} = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2^-}\right] = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_2(\lambda_1 + \lambda_2^-)} \tag{6.28}$$

If there are no failures of the standby unit, set $\lambda_2^- = 0$ in Eqs. (6.27) and (6.28). If $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_2^- = \lambda^-$, then Eqs. (6.27) and (6.28) simplify to

$$R(t) = e^{-\lambda t} + \frac{\lambda}{\lambda^-}\left[e^{-\lambda t} - e^{-(\lambda + \lambda^-)t}\right] \tag{6.29}$$

and $$\text{MTTF} = \frac{1}{\lambda} + \frac{\lambda}{\lambda^-}\left[\frac{1}{\lambda} - \frac{1}{\lambda + \lambda^-}\right]$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda^-} - \frac{\lambda}{(\lambda + \lambda^-)\lambda^-} = \frac{1}{\lambda} + \frac{1}{\lambda + \lambda^-} \tag{6.30}$$

EXAMPLE 6.2. An active generator has a failure rate (failures per day) of 0.01. An older standby generator has a failure rate of 0.001 while in standby and a failure rate of 0.10 when on-line. Determine the system reliability for a planned 30-day use and compute the system MTTF.

Solution

$$R(t) = e^{-0.01t} + \frac{0.01}{0.01 + 0.001 - 0.1}\left[e^{-0.1t} - e^{-0.011t}\right]$$

Therefore $$R(30) = 0.741 - 0.11236[0.04978 - 0.7189] = 0.8162$$

$$\text{MTTF} = \frac{1}{0.01} + \frac{0.01}{0.1(0.01 + 0.001)} = 109.09 \text{ days}$$

EXAMPLE 6.3. Both units of a two-component standby system are identical with $\lambda = 0.002$ failure per hour and $\lambda^- = 0.0001$ failure per hour. Determine the design life on the basis of a 95 percent reliability.

Solution. Using Eq. (6.29),

$$0.95 = R(t) = e^{-0.002t} + \frac{0.002}{0.0001}\left[e^{-0.002t} - e^{-0.0021t}\right]$$

Solving for t by trial and error:

$$R(100) = 0.982$$
$$R(200) = 0.935$$
$$R(150) = 0.961$$
$$R(175) = 0.949$$
$$R(173) = 0.950$$

The answer is 173 hours.

6.3.1 Identical Standby Units

If the primary and backup units have identical constant failure rates with no failures in the standby mode, then Eq. 6.27 is undefined. This is a special case in which the system of differential equations must be solved for separately. However, a simpler and more general approach is possible under this condition. Assume that there are k identical units of which one is on-line and the remaining are backup. When the on-line component fails, the first backup component is placed on-line. When it fails, the next is placed on-line, and so on. Therefore, the time in which the kth failure is observed is the sum of k identical and independent exponential distributions. As discussed in Section 3.5, the time of the kth failure has a gamma distribution with parameters λ and k. Therefore

$$R_k(t) = e^{-\lambda t}\sum_{i=0}^{k-1}\frac{(\lambda t)^i}{i!} \tag{6.31}$$

and MTTF $= k/\lambda$.

EXAMPLE 6.4. The Rey Lie Able Printing Company has four presses: one operating and three in standby. Each press has an identical constant failure rate where the MTTF is 50 operating hours. The company has received a rush order requiring 75 hr of continuous time on a press. If a standby is utilized whenever the on-line press fails, determine the probability of there being continuous printing support while the order is being processed.

Solution. The time to failure of the four-unit standby system has a gamma distribution with $\lambda = \frac{1}{50}$ and $k = 4$. Therefore

$$R_4(75) = e^{-75/50}\sum_{i=0}^{3}\frac{(75/50)^i}{i!} = e^{-1.5}\left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{48}\right] = 0.9344$$

and MTTF $= 4/(1/50) = 200$ hr.

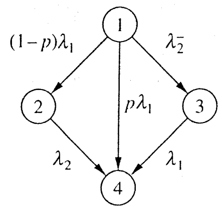


FIGURE 6.4
Rate diagram for a two-component standby system with failures in standby and with switching failures.

6.3.2 Standby System with Switching Failures

It is not uncommon in a standby system to have some probability p of there being an on-demand failure of a switching device designed to place the standby system on-line. The rate diagram in Fig. 6.3 is modified as shown in Fig. 6.4.

The resulting differential equation for state 1 does not change since

$$\frac{dP_1(t)}{dt} = -[(1-p)\lambda_1 + p\lambda_1 + \lambda_2^-]P_1(t) = -(\lambda_1 + \lambda_2^-)P_1(t)$$

It is also apparent by comparing Fig. 6.3 and Fig. 6.4 that Eq. (6.23) will not change either. Equation (6.22) is modified as follows:

$$\frac{dP_2(t)}{dt} = (1-p)\lambda_1 P_1(t) - \lambda_2 P_2(t)$$

The solution to these three equations is given by Eqs. (6.24) and (6.26) and by

$$P_2(t) = \frac{(1-p)\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2^-)t}\right] \tag{6.32}$$

Therefore the reliability function is

$$R(t) = e^{-\lambda_1 t} + \frac{(1-p)\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2^-)t}\right] \tag{6.33}$$

Observe that if $p = 1$, the standby system has no effect and the overall system reliability is that of the primary unit only.

EXAMPLE 6.5. Consider the standby system described in Example 6.2. If there is a 10 percent probability of a switching failure, the system reliability becomes

$$R(30) = 0.741 + \frac{(0.90)(0.01)}{0.01 + 0.001 - 0.1}[0.04978 - 0.7189] = 0.8087$$

This is a slight decrease from the perfect switching case.

6.3.3 Three-Component Standby System

Consider a system with one active unit and two standby units. For simplicity assume that no units fail while in standby and that all three systems have the same constant failure rate when on-line. Define the following states:

State	Unit 1	Unit 2	Unit 3
1	on-line	standby	standby
2	failed	on-line	standby
3	failed	failed	on-line
4	failed	failed	failed

This leads to the following differential equations:

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) \tag{6.34}$$

$$\frac{dP_2(t)}{dt} = \lambda P_1(t) - \lambda P_2(t) \tag{6.35}$$

$$\frac{dP_3(t)}{dt} = \lambda P_2(t) - \lambda P_3(t) \tag{6.36}$$

with initial conditions $P_1(0) = 1$, $P_2(0) = 0$, and $P_3(0) = 0$. The solution follows the methodology given in Appendix 6A, resulting in

$$P_1(t) = e^{-\lambda t} \tag{6.37}$$

$$P_2(t) = \lambda t e^{-\lambda t} \tag{6.38}$$

$$P_3(t) = \frac{\lambda^2 t^2}{2} e^{-\lambda t} \tag{6.39}$$

Since the system is functioning while in any of the first three states,

$$R(t) = e^{-\lambda t}\left[1 + \lambda t + \frac{\lambda^2 t^2}{2}\right] \tag{6.40}$$

A comparison of Eq. 6.40 with the reliability function computed for the case of identical standby units based on the gamma distribution (Eq. 6.31) will reveal that the two are equivalent. The MTTF may be found from[1]

$$\text{MTTF} = \int_0^\infty e^{-\lambda t}\,dt + \int_0^\infty \lambda t e^{-\lambda t}\,dt + \int_0^\infty \frac{\lambda^2 t^2}{2} e^{-\lambda t}\,dt = \frac{3}{\lambda} \tag{6.41}$$

It is not surprising that the MTTF is three times a single-unit MTTF since each unit will have an MTTF of $1/\lambda$ while active. In general, if there are k identical and independent units with $k-1$ on standby, then the system MTTF is k/λ (see Exercise 6.15).

[1]The definite integrals may be solved from the following:

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

for $a > 0$ and n a positive integer.

EXAMPLE 6.6. Three identical transmitters are available each having a constant failure rate of 0.0035 per operating hour. A mission requires 500 hr of continuous transmission. Determine the reliability of the system.

Solution

$$R(500) = e^{-0.0035\times 500}\left[1 + 0.0035 \times 500 + \frac{(0.0035 \times 500)^2}{2}\right] = 0.744$$

In comparison, a single unit has a mission reliability of 0.174.

6.4 DEGRADED SYSTEMS

Some systems may continue to operate in a degraded mode following certain types of failures. The system may continue to perform its function but not at a specified operating level. For example, a computer system may not be able to access all of its direct access storage devices, a copying machine may not be able to automatically feed originals and may thereby require manual operation, or a multi-engine aircraft may experience a problem in one of its engines. Whether the degraded mode is considered a failure or not must be determined as part of the reliability specification. However, if it is important to distinguish the degraded state from that of a complete failure, then Markov analysis can be utilized if constant failure rates are assumed.

Defining the states of a system as fully operational (state 1), degraded (state 2), and failed (state 3), the rate diagram of Fig. 6.5 is constructed. The differential equations are

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t) \tag{6.42}$$

$$\frac{dP_2(t)}{dt} = \lambda_2 P_1(t) - \lambda_3 P_2(t) \tag{6.43}$$

The solution to Eq. (6.42) is straightforward and is given by

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t} \tag{6.44}$$

An integrating factor must be used in solving Eq. (6.43); however, the solution procedure is similar to that of the standby system (Eq. 6.22):

$$P_2(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right] \tag{6.45}$$

Finally, $P_3(t) = 1 - P_1(t) - P_2(t)$. The mean time to a complete failure is found from

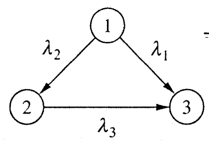


FIGURE 6.5
Rate diagram for modeling a degraded system.

$$\text{MTTF} = \int_0^\infty [P_1(t) + P_2(t)]\,dt = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[\frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2}\right]$$

EXAMPLE 6.7. A machine used in a manufacturing process experiences complete failures at a constant rate of 0.01 per day. However, the machine may degrade randomly, producing substandard parts (out of tolerances) at a constant rate of 0.05 per day. Once it has degraded, it will fail completely at a constant rate of 0.07 per day. Therefore,

$$P_1(t) = e^{-0.06t} \qquad P_2(t) = \frac{0.05}{0.06 - 0.07}\left[e^{-0.07t} - e^{-0.06t}\right]$$

and over a one-day operation, $P_1(1) = 0.942$, $P_2(1) = 0.047$, and $P_3(1) = 0.011$. The MTTF is found to be 28.6 days. Of interest perhaps is the mean number of days the machine may operate in a degraded mode until it fails, which is given by $1/0.07 = 14.3$ days. Predictive maintenance attempts to determine when a machine will fail and, as a result, perform corrective maintenance just prior to a failure. From this analysis, once the machine begins to produce substandard parts, maintenance of the machine should be accomplished on the average within 14 days. Notice that

$$\int_0^\infty P_1(t)\,dt = \frac{1}{0.01 + 0.05} = 16.67 \text{ days}$$

is the mean number of days the machine will spend in state 1 prior to degrading or experiencing a complete failure.

6.5 THREE-STATE DEVICES

Components having three states, or two dependent failure modes, were introduced in Chapter 5. These components are in either an operating state, a failed open state, or a failed short state. Since there is a dependency between the two failure states (i.e., they are mutually exclusive), Markov analysis can be used to determine the component reliability assuming each failure state has a constant failure rate.

In the rate diagram of Fig. 6.6, state 1 is the operating state, state 2 is the failed open state, and state 3 is the failed short state. The differential equations are

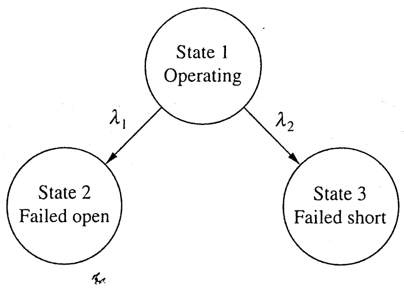


FIGURE 6.6
Rate diagram for a three-state device.

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t)$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t)$$

The solution is straightforward and given by

$$R(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t} \tag{6.46}$$

with

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}[1 - e^{-(\lambda_1+\lambda_2)t}]$$

and

$$P_3(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2}[1 - e^{-(\lambda_1+\lambda_2)t}]$$

Equation (6.46) may also be interpreted as the reliability of a component having two failure modes both with constant failure rates.

APPENDIX 6A SOLUTION TO TWO-COMPONENT REDUNDANT SYSTEM

From Eq. (6.6),

$$\frac{dP_1(t)}{P_1(t)} = -(\lambda_1 + \lambda_2)\,dt$$

Integrating both sides,

$$\ln P_1(t) = -(\lambda_1 + \lambda_2)t$$

or

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

From Eq. (6.7),

$$\frac{dP_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2 P_2(t)$$

with $e^{+\lambda_2 t}$ as an integrating factor,

$$P_2(t)e^{+\lambda_2 t} = +\lambda_1 \int e^{-(\lambda_1+\lambda_2)t}e^{+\lambda_2 t}\,dt + C$$

or

$$P_2(t) = -e^{-(\lambda_1+\lambda_2)t} + ce^{-\lambda_2 t}$$

The initial conditions are $P_1(0) = 1$, $P_2(0) = 0$, and $P_3(0) = 0$. Therefore, $c = 1$. $P_3(t)$ is derived in a similar manner.

APPENDIX 6B
SOLUTION TO LOAD-SHARING SYSTEM

$P_1(t)$ is solved for as shown in Appendix 6A. Then

$$\frac{dP_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2^+ P_2(t)$$

With $e^{\lambda_2^+ t}$ as an integrating factor, the solution is

$$e^{\lambda_2^+ t} P_2(t) = \int \lambda_1 e^{-(\lambda_1+\lambda_2)t} e^{\lambda_2^+ t}\, dt + C$$

or

$$P_2(t) = e^{-\lambda_2^+ t}\left\{\int \lambda_1 e^{[\lambda_2^+ - (\lambda_1+\lambda_2)]t}\, dt + C\right\}$$

$$= e^{-\lambda_2^+ t}\left[\frac{\lambda_1 e^{[\lambda_2^+ - (\lambda_1+\lambda_2)]t}}{\lambda_2^+ - (\lambda_1+\lambda_2)} + c\right]$$

$$= \frac{\lambda_1 e^{-(\lambda_1+\lambda_2)]t}}{\lambda_2^+ - (\lambda_1+\lambda_2)} + ce^{-\lambda_2^+ t}$$

Since $P_2(0) = 0$, then

$$c = \frac{-\lambda_1}{\lambda_2^+ - (\lambda_1+\lambda_2)}$$

and

$$P_2(t) = \frac{\lambda_1}{\lambda_2^+ - (\lambda_1+\lambda_2)}\left[e^{-(\lambda_1+\lambda_2)t} - e^{-\lambda_2^+ t}\right]$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_2^+}\left[e^{-\lambda_2^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$P_3(t)$ is solved for in a similar manner.

APPENDIX 6C
SOLUTION TO STANDBY SYSTEM MODEL

Equation (6.21) has as its solution

$$P_1(t) = e^{-(\lambda_1+\lambda_2^-)}$$

by the same procedure as that of Appendix 6A. Substituting this solution into Eq. (6.22),

$$\frac{dP_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2^-)t} - \lambda_2 P_2(t)$$

With $e^{\lambda_2 t}$ as an integrating factor, the solution is

$$e^{\lambda_2 t} P_2(t) = \int \lambda_1 e^{-(\lambda_1+\lambda_2^-)t} e^{\lambda_2 t}\, dt + C$$

$$= \frac{-\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2} e^{-(\lambda_1+\lambda_2^- - \lambda_2)t} + c$$

or

$$P_2(t) = \frac{-\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2} e^{-(\lambda_1+\lambda_2^-)t} + ce^{-\lambda_2 t}$$

With the initial condition $P_2(0) = 0$,

$$c = \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}$$

giving Eq. (6.25).

To obtain $P_3(t)$, use Eq. (6.23) and the solution to $P_1(t)$:

$$\frac{dP_3(t)}{dt} = \lambda_2^- e^{-(\lambda_1+\lambda_2^-)t} - \lambda_1 P_3(t)$$

With $e^{\lambda_1 t}$ as the integrating factor,

$$e^{\lambda_1 t} P_3(t) = \int \lambda_2^- e^{-(\lambda_1+\lambda_2^-)t} e^{\lambda_1 t}\, dt + C$$

$$= \frac{-\lambda_2^-}{\lambda_2^-} e^{-(\lambda_2)t} + c$$

or

$$P_3(t) = -e^{-(\lambda_1+\lambda_2^-)t} + ce^{-\lambda_1 t}$$

With $P_3(0) = 0$, we know $c = 1$, giving Eq. (6.26).

EXERCISES

6.1 Two nickel-cadmium batteries provide electrical power to operate a satellite transceiver. If both batteries are operating in parallel, they have an individual failure rate of 0.1 per year. If one fails, the other can operate the transceiver (at a reduced power output). However, the increased electrical demand will triple the failure rate of the remaining battery. Determine the system reliability at 1, 2, 3, 4, and 5 yr. What is the system MTTF?

6.2 An engine health monitoring system consists of a primary unit and a standby unit. The MTTF of the primary unit is 1000 operating hours, and the MTTF of the standby unit is 333 hr when in operation and 2000 hr while in standby status. Estimate the design life of the system if specifications require a reliability of 0.90. What is the system MTTF?

6.3 A computerized airline reservation system has a main computer on-line and a secondary standby computer. The on-line computer fails at a constant rate of 0.001 failure per hour, and the standby unit fails when on-line at the constant rate of 0.005 failure per hour. There are no failures while the unit is in the standby mode.

(a) Determine the system reliability over a 72-hr period.
(b) The airline desires to have a system MTTF of 2000 hr. Determine the (minimum) MTTF of the main computer to achieve this goal assuming that the standby computer MTTF does not change.

6.4 Repeat Exercise 6.3 assuming a 0.005 probability of a switching failure.

6.5 An alarm system may fail safe (false alarm) or may fail to danger (failure on demand). If the fail-safe failure rate is a constant 0.00034 failure per operating hour and the fail-to-danger failure rate is a constant 0.000021 failure per operating hour, what is the design life of the alarm if an operating reliability of 0.99 is desired? What is the probability of a fail-to-danger failure occurring over 1000 operating hours?

6.6 Consider a three-component standby system in which two units are normally on-line. Both on-line units must fail before the standby unit is placed on-line. Compute the system reliability function and the MTTF. Assume no failures in standby and a constant failure rate of λ when the unit is on-line.

6.7 The Brake A. Bac Trucking Company requires two workers to unload a truck in 5 hr. Working together, the two workers have a mortality rate (due to heart failure) of 0.05 per hour. However, if one experiences a heart attack and dies while unloading a truck, the other's mortality rate increases to 0.2 per hour as a result of the increase in the unloading rate necessary in order to complete the task on time. Compute the probability (reliability) that a truck will be successfully unloaded by a two-person crew. What is the expected time to failure (in working hours) of the two-person crew?

6.8 A contractor must decide between two different sump pump systems to be installed in a new housing development. The option is to install a single 1000 gallon per minute (gpm) system or two 500-gpm pumps. If the two-pump system is used, one pump can carry most of the load in the event the other pump fails. Both of the 500-gpm pumps have an MTTF of 800 hr when working together. Their individual MTTF is 200 hr. The 1000-gpm system has a rated MTTF of 700 hr. Which system is preferred on the basis of system MTTF? Which system has the best design life for a reliability of 0.80?

6.9 The contractor in Exercise 6.8 can obtain used 1000-gpm sump pumps to back up the primary unit. Because of their age, these pumps could fail while in standby with a failure rate of 0.001 failure per hour. On-line, they have experienced an MTTF of 200 hr. Would this (standby) system be preferred to that in Exercise 6.8?

6.10 Determine which of the following systems is the most reliable at 100 hr.
(a) Two parallel and CFR units with $\lambda_1 = 0.0034$ and $\lambda_2 = 0.0105$
(b) A standby system with $\lambda_1 = 0.0034$, $\lambda_2 = 0.0105$, $\lambda_2^- = 0.0005$, and a switching failure probability of 15 percent
(c) A load-sharing system with $\lambda_1 = 0.0034$ and $\lambda_2 = 0.0105$ in which the single-component failure rate increases by a factor of 1.5

Compare the MTTF of all three systems.

6.11 A hospital has three identical generators, one on-line and the other two in standby. In the primary failure mode, which is due to external weather conditions, the failure rate is constant and equal to 0.02 failure per hour during bad weather. Assuming that no failures occur in standby and that the average duration of a storm is 10 hr, what is the reliability of the generator system?

6.12 Derive the reliability function for the case of a three-component standby system (i.e., one active and two standby units) in which the standby units have a different failure rate (λ_s) than the active unit (λ_a). What is the system MTTF? Assume that there are no switching failures and no failures while in standby.

6.13 Derive the expression for the MTTF for the load-sharing system defined by Eqs. (6.13) through (6.15).

6.14 A manufacturing company operates two production lines. When both lines are operating, the production rate on each line is 500 units per hour. At this production rate the failure rate of line 1 is 3 failures per 8-hr day (CFR) and the failure rate of line 2 is 2 failures per 8-hr day. When one line fails, the production rate of the second line must be increased in order to make production quotas. At the increased rate of 800 units per hour, the failure rate of line 1 is 6 per 8-hr day and the failure rate of line 2 is 3 per 8-hr day. Find the MTTF and the reliability of the production system over a 1-hr and over an 8-hr production run.

6.15 By integrating the reliability function given by Eq. 6.31, show that the MTTF of the standby system having k identical and independent units of which $k - 1$ are on standby is k/λ. *Hint:* Use integration formula 9, Appendix 2E.

6.16 Find the general solution for a degraded system that fails completely from the degraded state only. In other words, the system must degrade before it can fail. Also show that the MTTF for this system is given by $1/\lambda_1 + 1/\lambda_2$ where λ_1 is the rate at which the system degrades and λ_2 is the rate at which the system fails from the degraded state.

6.17 Define the states, construct the rate diagram, and write the corresponding system of equations for a three-component standby system in which two units must always be operating on-line. If either unit fails, the standby unit will replace the failed unit. When a second component failure occurs, a system failure is observed. Assume no switching failures and no failures in standby. Assume that each component has the same failure rate.

CHAPTER 7

Physical Reliability Models

The previous chapters have focused on the development of reliability models in which system or component reliability was considered as a function of time only. In many applications other factors may be equally important. For example, electronic component failures may depend on the applied voltage or on the operating temperature of the equipment. The strength (and therefore the reliability) of a precast concrete support beam may depend on the impurities found in the water and in other materials used in the mixture. In general, a more accurate reliability model may be one in which the inherent characteristics or the external operating conditions of a component are included. Covariate models incorporate these additional factors into the failure distribution generally by expressing one or more of the distribution parameters as a function of the covariates. This chapter begins with a discussion of several covariate models. Static stress-strength models, in which time is not a factor in determining reliability, are then introduced. In these models only the physical (internal) strength and applied loads are involved. This is followed with the development of dynamic stress-strength models, in which the application of loads over time is also considered. The chapter concludes with the physics-of-failure modeling approach, in which the (mean) time to failure is estimated by determining the root causes of failures through analysis of the physical properties and operating conditions of the components.

7.1 COVARIATE MODELS

Our interest is in developing failure distributions involving one or more covariates or explanatory variables. To do this, we will build on the previous failure distributions in an obvious manner. The approach is to define one or more of the distribution

parameters as a function of the explanatory variables. In general, if α is a distribution parameter, such as a percentile, representing a nominal lifetime, then

$$\alpha(\mathbf{x}) = f(x_1, x_2, \ldots, x_k)$$

where $\mathbf{x} = (x_1, \ldots, x_k)$ and x_i = the ith covariate. A covariate may be a voltage, current, temperature, humidity, or other measure of stress or environment. There should be an obvious correlation between the covariates and the parameter value, although there may not necessarily be a cause-and-effect relationship. The functional form of $f(\mathbf{x})$ may be determined by the physical process relating the covariates to the parameters. However, if these relationships are unknown, a simple functional form (e.g., linear) is assumed.

7.1.1 Proportional Hazards Models

Models having the property that individual component hazard rate functions are proportional to each other are referred to as *proportional hazards models*.

Exponential case

For the constant failure rate model, the simplest covariate model is given by

$$\lambda(\mathbf{x}) = \sum_{i=0}^{k} a_i x_i \tag{7.1}$$

where the a_i are unknown parameters to be determined and by convention $x_0 = 1$. The x_i may be transformed variables (e.g., squares or reciprocals), thus allowing, for example, polynomials to be used. The failure rate remains constant over time but does depend on the particular covariate values. For example, the failure rate of a circuit board may be linearly related to the operating temperature of the equipment and the ambient relative humidity. Clearly, other functional forms could be assumed. One popular model is that in which the covariates affect the parameter (failure rate) multiplicatively. It has provided good correlation with observed data.

EXAMPLE 7.1. Ploe and Skewis [1990] provide the following multiplicative model for predicting the failure rate of ball bearings:

$$\lambda = \lambda_b \left(\frac{L_a}{L_s}\right)^y \left(\frac{A_e}{0.006}\right)^{2.36} \left(\frac{\nu_0}{\nu_l}\right)^{0.54} \left(\frac{C_l}{60}\right)^{0.67} \left(\frac{M_b}{M_f}\right) C_w$$

where λ_b = base failure rate of a bearing per 10^6 hr of operation (obtained from the manufacturer)
L_a = actual radial load in pounds
L_s = specification radial load in pounds
y = 3.33 for roller bearings; 3.0 for ball bearings
A_e = alignment error in radians
ν_0 = specification lubricant viscosity
ν_l = operating lubricant viscosity
C_l = actual contamination level (μg/m^3)

M_b = material factor of base material in PSI (yield strength)
M_f = material factor of operating material in PSI (yield strength)
C_w = water contamination factor (leakage of water into the oil lubricant)

$$= \begin{cases} 1 + 460x & \text{for } x < 0.002 \\ 2.036 + 1.029x - 0.0647x^2 & \text{for } x \geq 0.002 \end{cases}$$

where x is the percentage of water present in the oil

A popular form of the multiplicative model is obtained by letting

$$\lambda(\mathbf{x}) = \prod_{i=0}^{k} \exp(a_i x_i) = \exp\left(\sum_{i=0}^{k} a_i x_i\right) \tag{7.2}$$

This model has the desirable property that $\lambda(\mathbf{x}) > 0$ and is linear in the logarithm of $\lambda(\mathbf{x})$. Regardless of the model used,

$$R(t) = e^{-\lambda(\mathbf{x})t}$$

EXAMPLE 7.2. To account for voltage stress, *Military Handbook: Reliability Prediction of Electronic Equipment* [1986] provides the following multiplicative factor for certain types of monolithic microelectronic devices:

$$g(x) = 0.11e^{0.916+0.0005638x_1x_2}$$

where x_1 is the operating supply voltage and x_2 is the worst-case junction temperature (in degrees centigrade). This factor is multiplied by other similar factors in arriving at a (constant) failure rate.

EXAMPLE 7.3. A popular approach in the aerospace industry for estimating life-cycle costs and failure rates or mean failure times is to use parametric estimating equations. Parametric equations relate the mean failure time (the dependent variable) to one or more independent variables (parameters), which are often surrogate variables for factors that may cause failures. There is not necessarily a cause-and-effect relationship present. Instead, the independent variables are useful in explaining the dependent variable. Examples include the use of component weight as a substitute for complexity, or surface area as a substitute for the number of parts. The following parametric regression equation was derived from mean time between maintenance (MTBM) data from 33 aircraft over a two-year period; the dependent variable is the mean number of flying hours between maintenance on the propulsion subsystem of the aircraft, and the engine weight is in pounds.

$$\text{MTBM} = 34.104 + 0.0009853 \times (\text{engine weight}) - 0.31223\sqrt{\text{engine weight}}$$

Other aircraft parametric equations include the electrical subsystem MTBM as a function of the subsystem weight and the maximum power output of the generators, and landing gear subsystem MTBM as a function of the number of wheels and the vehicle weight. Generally, exponential failure distributions are then assumed.

Weibull case

For the Weibull distribution it is common to assume that only the characteristic lifetime and not the shape parameter depends on the covariates. For the multiplicative model, let

$$\theta(\mathbf{x}) = \exp\left(\sum_{i=0}^{k} a_i x_i\right) \tag{7.3}$$

Then

$$R(t) = \exp\left[-\left(\frac{t}{\theta(\mathbf{x})}\right)^{\beta}\right]$$

and

$$\lambda(t \mid \mathbf{x}) = \frac{\beta t^{\beta-1}}{\theta(\mathbf{x})^{\beta}} = \beta t^{\beta-1}\left[\exp\left(\sum_{i=0}^{k} a_i x_i\right)\right]^{-\beta} \tag{7.4}$$

The ratio of two Weibull hazard rates having different covariate vectors is

$$\frac{\lambda(t \mid \mathbf{x}_1)}{\lambda(t \mid \mathbf{x}_2)} = \left[\frac{\theta(\mathbf{x}_2)}{\theta(\mathbf{x}_1)}\right]^{\beta} \tag{7.5}$$

which does not depend on time. Therefore, this model is a proportional hazards model since the component hazard rate functions are proportional to one another. This also suggests that a general form of the hazard rate function may be

$$\lambda(t \mid \mathbf{x}) = \lambda_0(t)g(\mathbf{x}) \qquad \text{with } g(\mathbf{x}) = \exp\left(\sum_{i=1}^{k} a_i x_i\right) \tag{7.6}$$

where $\lambda_0(t)$ is a baseline hazard rate function when $g(\mathbf{x}) = 1$. For example, the exponential baseline failure rate in Eq. (7.2) is $\lambda_0(t) = e^{a_0}$.

EXAMPLE 7.4. An AC motor is known to have a Weibull failure distribution with a shape parameter of 1.5. Reliability test results have shown that the characteristic life in operating hours depends on the load placed on the motor in the following manner:

$$\theta(x) = e^{23.2-0.134x}$$

Find the design life for a reliability of 0.95 of a particular motor that is to be placed under a load of 115. If the load is reduced to 100, how much improvement in design life should be expected?

Solution

$$\theta(115) = 2416.3 \qquad t_{0.95} = 2416.3(-\ln 0.95)^{0.6667} = 333.5 \text{ hr}$$
$$\theta(100) = 18{,}033.7 \qquad 18{,}033.7(-\ln 0.95)^{0.6667} = 2489.3 \text{ hr}$$

7.1.2 Location-Scale Models

A separate family of covariate models referred to as *location-scale models* is obtained by setting

$$\mu(\mathbf{x}) = \sum_{i=0}^{k} a_i x_i$$

and letting

$$Y = \mu(\mathbf{x}) + \sigma z \tag{7.7}$$

where $\sigma > 0$ and z has a specified probability distribution not dependent on the covariate vector $\mathbf{x}$.

Normal case

The most common example of this model assumes that z is normally distributed with a mean of 0 and a variance of 1. Therefore Y is normally distributed with mean $\mu(\mathbf{x})$ and standard deviation σ.

Lognormal case

By setting $T = e^Y$, we obtain

$$R(t) = 1 - \Phi\left(\frac{\ln t - \sum_{i=0}^{k} a_i x_i}{s}\right) \tag{7.8}$$

T is lognormal with shape parameter $s = \sigma$ and $t_{\text{med}} = e^{u(\mathbf{x})}$.

In the location-scale model, the covariates act in a linear manner on the (mean) failure time when failures are normal, and they act in a multiplicative manner on the (median) failure times when failures are lognormal.

EXAMPLE 7.5. The time to failure of an electrical connector is lognormal with a shape parameter of 0.73. Since failures, measured in operating hours, were observed to be related to the operating temperature of the connector and the number of electrical contacts, the following covariate model was derived:

$$u(\mathbf{x}) = -3.86 + 0.1213x_1 + 0.2886x_2$$

where x_1 is the operating temperature in degrees centigrade and x_2 is the number of contacts. A particular connector used in a personal computer will operate at 80°C and has 16 contact pins. Therefore its reliability over a 5000-hr life is found from the following:

$$u(80, 16) = -0.386 + 0.1213(80) + 0.2886(16) = 10.46$$

$$R(5000) = 1 - \Phi\left(\frac{\ln(5000) - 10.46}{0.73}\right) = 1 - \Phi(-2.66) = 0.996$$

In this case $t_{\text{med}} = e^{10.46} = 34{,}891.55$ hr.

The more difficult task of estimating the parameters of these models will be covered in Chapter 15. Lawless [1982] provides additional detail on the use of covariate models. These covariate models are also useful in accelerated life testing, discussed in Chapter 13.

7.2 STATIC MODELS

In many situations it is not appropriate to assume that reliability is a function of time. This section considers a single stress placed on a system during a relatively

short time period. A stress is any load that may produce a failure. A failure occurs if the stress exceeds the strength of the system. The strength is the highest stress value that the system can endure without failing. Reliability is therefore viewed as static and not as a function of time, as it was previously defined. Either the system will bear the load or it will not, resulting in a failure. Loads may be electrical, thermal, chemical, or mechanical. A load may be measured in volts, degrees, pounds, operations per second, miles per hour, or any other units. Examples of a static stress being applied to a system include the landing gear of an aircraft on landing, a rocket being fired, and a building withstanding a hurricane. After the development of a static reliability, we will return to the dynamic case, in which loads are applied either periodically or randomly over time.

Static models are a result of failures due to (nearly) instantaneous stress placed on a system and not a result of any prior effects or history. To quantify stress and strength, let X be the random variable representing the stress placed on a system such that $f_X(x)$ is the probability density function, and let Y be the random variable representing the capacity of the system such that $f_Y(y)$ is the probability density function. Then the probability that the stress does not exceed the value x is given by

$$\Pr\{X \le x\} = F_X(x) = \int_0^x f_{x'}(x')\,dx' \tag{7.9}$$

and the probability that the capacity does not exceed the value y is given by

$$\Pr\{Y \le y\} = F_Y(y) = \int_0^y f_{y'}(y')\,dy' \tag{7.10}$$

7.2.1 Random Stress and Constant Strength

If the system strength is a known constant k and the stress is a random variable with PDF as defined above, then the system (static) reliability can be defined as the probability that stress does not exceed strength. That is,

$$R = \int_0^k f_X(x)\,dx = F_X(k) \tag{7.11}$$

Figure 7.1 illustrates the reliability graphically.

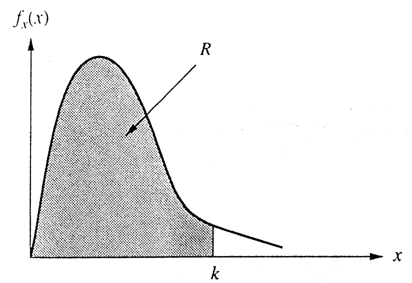


FIGURE 7.1
Reliability for a component under a random load with fixed strength.

EXAMPLE 7.6. The stress placed on a motor mount has the following PDF:

$$f_x(x) = \begin{cases} \dfrac{x^2}{1125} & \text{for } 0 \le x \le 15 \text{ lb} \\ 0 & \text{otherwise} \end{cases}$$

The motor mount has been found through laboratory testing to have a fixed tolerance of 14 lb. Therefore, its static reliability is given by

$$R = \Pr\{X \le 14\} = \int_0^{14} \frac{x^2}{1125}\,dx = \frac{14^3}{3375} = 0.813$$

7.2.2 Constant Stress and Random Strength

If the stress, or load, is fixed at a known constant s and the strength is a random variable having a PDF as given above, then the system (static) reliability is the probability that the strength exceeds the fixed load, as shown in Fig. 7.2. That is,

$$R = \Pr\{Y \ge s\} = \int_s^{\infty} f_Y(y)\,dy = 1 - F_Y(s) \tag{7.12}$$

EXAMPLE 7.7. The strength of a new superglue has been found to be random, its value depending on the exact mixture of the compounds used in the manufacturing process. It has the following PDF:

$$f_Y(y) = \begin{cases} 10/y^2 & \text{for } y \ge 10 \text{ lb} \\ 0 & \text{otherwise} \end{cases}$$

A fixed load of 12 lb is to be applied. What is the reliability?

$$R = \Pr\{Y \ge 12\} = \int_{12}^{\infty} \frac{10}{y^2}\,dy = \frac{10}{12} = 0.833$$

7.2.3 Random Stress and Random Strength

If both stress and strength are random variables, the reliability remains the probability that stress is less than strength (or equivalently, that strength is greater than stress). However, to compute the reliability, the following double integral must be solved:

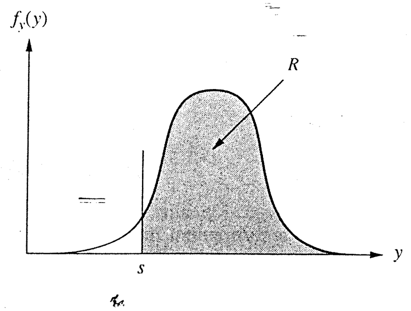


FIGURE 7.2
Reliability for a component under a fixed load with random strength.

$$\begin{aligned} R = \Pr\{X \le Y\} &= \int_0^{\infty} \left[\int_0^{y} f_X(x)\,dx\right] f_Y(y)\,dy \\ &= \int_0^{\infty} F_X(y) f_Y(y)\,dy \end{aligned} \tag{7.13}$$

since for a given y,

$$R(y) = \int_0^{y} f_X(x)\,dx = F_X(y)$$

The static reliability can then be obtained from

$$R = \int_0^{\infty} R(y) f_Y(y)\,dy$$

The reliability depends on the region of the two curves in which the tails overlap, or interfere with one another, as shown in Fig. 7.3. For this reason the analysis of stress versus strength is sometimes referred to as *interference theory*.

EXAMPLE 7.8. Let $f_X(x) = \frac{1}{50}$, $0 \le x \le 50$, and let $f_Y(y) = 0.0008y$, $0 \le y \le 50$. Then

$$F_X(y) = \int_0^{y} \frac{1}{50}\,dx = \frac{y}{50}$$

$$R = \int_0^{50} \frac{y}{50} 0.0008y\,dy = \int_0^{50} 0.000016y^2\,dy = 0.0000053y^3 \Big|_0^{50} = 0.667$$

An equivalent method for finding R is given by

$$R = \Pr\{Y > X\} = \int_0^{\infty} \left[\int_x^{\infty} f_Y(y)\,dy\right] f_X(x)\,dx \tag{7.14}$$

For some problems, performing the integration in the order given by the above expression may be easier than solving Eq. (7.13). If the stress and strength are not defined over similar intervals, the integration may have to be performed over disjoint intervals, as shown in the following example.

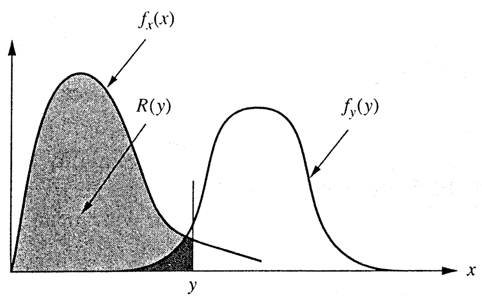


FIGURE 7.3
Reliability for a component under a random load with random strength.

EXAMPLE 7.9. Let $f_x(x) = 3x^2/10^9$, $0 \le x \le 1000$ lb, and let $f_y(y) = (5 \times 10^{-7})y$, $0 \le y \le 2000$ lb. Then

$$R = \int_0^{1000}\left[\int_0^y \frac{3x^2}{10^9}\,dx\right](5 \times 10^{-7})y\,dy + (1)\int_{1000}^{2000}(5 \times 10^{-7})y\,dy = 0.85$$

where the second integral is simply the probability that Y is greater than 1000 ($\Pr\{Y > 1000\}$) in which case $\Pr\{X < Y\} = 1$. However, if the order of the integration is reversed, as suggested by the above form for R, then

$$R = \int_0^{1000}\left[\int_x^{2000}(5 \times 10^{-7})y\,dy\right]\frac{3x^2}{10^9}\,dx = 0.85$$

Exponential case

One of the simplest cases involving both random stress and random strength occurs when both distributions are exponential. In this case their PDFs are given by

$$f_x(x) = \frac{1}{\mu_x}e^{-x/\mu_x} \qquad \text{and} \qquad f_y(y) = \frac{1}{\mu_y}e^{-y/\mu_y}$$

where μ_x is the mean stress and μ_y is the mean strength. Therefore,

$$R = \int_0^\infty\left[\int_0^y \frac{1}{\mu_x}e^{-x/\mu_x}\,dx\right]\frac{1}{\mu_y}e^{-y/\mu_y}\,dy = \int_0^\infty\left[1 - e^{-y/\mu_x}\right]\frac{1}{\mu_y}e^{-y/\mu_y}\,dy$$

$$= \int_0^\infty \frac{1}{\mu_y}e^{-y/\mu_y}\,dy - \int_0^\infty \frac{1}{\mu_y}\exp\left[-y\left(\frac{1}{\mu_x} + \frac{1}{\mu_y}\right)\right]dy$$

$$= 1 - \frac{1}{\mu_y}\frac{\exp[-y(1/\mu_x + 1/\mu_y)]}{-(1/\mu_x + 1/\mu_y)}\bigg|_0^\infty$$

where the first integral is the total area under the density function. Then

$$R = 1 - \frac{1}{\mu_y}\left(\frac{\mu_x\mu_y}{\mu_x + \mu_y}\right) = 1 - \frac{\mu_x}{\mu_x + \mu_y} = \frac{\mu_y}{\mu_x + \mu_y} = \frac{1}{1 + \mu_x/\mu_y} \tag{7.15}$$

Selected values of the ratio of the mean stress to the mean strength and the corresponding reliabilities calculated from Eq. 7.15 are given in Table 7.1. It is obvious from Table 7.1 that when both distributions are exponential, the mean strength must be *more* than 10 times the mean stress in order to achieve an acceptable reliability.

Normal case

When both stress and strength are normally distributed, system reliability may be obtained as follows. Let X be a normally distributed stress with mean μ_x and standard deviation σ_x, and let Y be a normally distributed strength with mean μ_y and standard deviation σ_y. Then

$$R = \Pr\{Y \ge X\} = \Pr\{Y - X \ge 0\} = \Pr\{W \ge 0\}$$

TABLE 7.1
Ratio of stress to strength for two exponentials

μ_x/μ_y	Reliability
1.0	0.50
0.9	0.53
0.8	0.56
0.7	0.59
0.6	0.63
0.5	0.67
0.4	0.71
0.3	0.77
0.2	0.83
0.1	0.91

where $W = Y - X$, $E(W) = E(Y - X) = \mu_y - \mu_x$, and $\operatorname{Var}(W) = \operatorname{Var}(Y - X) = \sigma_y^2 + \sigma_x^2$ (assuming Y and X are independent). Then

$$R = \Pr\{W \ge 0\} = \Pr\left\{\frac{W - \mu_W}{\sigma_W} \ge \frac{-(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right\}$$

$$= \Pr\left\{\frac{W - \mu_W}{\sigma_W} \le \frac{(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right\} = \Phi\left(\frac{\mu_y - \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right) \tag{7.16}$$

EXAMPLE 7.10. If stress is normally distributed with a mean of 10.3 and a standard deviation of 2.1, and strength is normally distributed with a mean of 25.8 and a standard deviation of 8.2, determine the system reliability.

Solution

$$R = \Phi\left(\frac{25.8 - 10.3}{\sqrt{67.24 + 4.41}}\right) = \Phi(1.83) = 0.96638$$

using the normal table in the Appendix, Table A.1.

Lognormal case

Because of the relationship between the normal and lognormal distributions, similar results may be obtained when both stress and strength follow lognormal distributions. Let X be a lognormally distributed stress with median m_x and shape parameter s_x, and let Y be a lognormally distributed strength with median m_y and shape parameter s_y. Then

$$R = \Pr\{Y \ge X\} = \Pr\left\{\frac{Y}{X} \ge 1\right\}$$

Let $W = \ln(Y/X) = \ln Y - \ln X$; W is normal with mean $\mu_W = \ln(m_y/m_x)$ and variance $\mathrm{Var}(W) = s_y^2 + s_x^2$, assuming Y and X are independent. Then

$$
\begin{aligned}
R &= \Pr\{W \ge \ln 1\} = \Pr\{W \ge 0\} = \Pr\left\{\frac{0-\mu_W}{\sigma_W} \ge \frac{0-\mu_W}{\sigma_W}\right\} \\
&= \Pr\left\{\frac{W-\mu_W}{\sigma_W} \le \frac{\mu_W}{\sigma_W}\right\} = \Phi\left(\frac{\ln(m_y/m_x)}{\sqrt{s_y^2+s_x^2}}\right)
\end{aligned}
\tag{7.17}
$$

EXAMPLE 7.11. A structure has a capacity to withstand earthquakes, which are lognormally distributed with a median of 8.1 on the Richter scale and with $s_y = 0.07$. Historically, the magnitude of earthquakes in this region has been lognormal with a median of 5.5 and $s_x = 0.15$. Therefore the static reliability to withstand a single random occurrence of an earthquake is found from Eq. (7.17):

$$R = \Phi\left(\frac{\ln(8.1/5.5)}{\sqrt{0.15^2+0.07^2}}\right) = \Phi(2.33) = 0.99$$

Table 7.2 provides formulas for calculating the static reliability when stress or strength is modeled using the theoretical distributions. Other stress-strength models may be developed by assuming other combinations of probability distributions for stress and strength. However, the integrals resulting from applying Eq. (7.13) or Eq. (7.14) generally cannot be solved in closed form. Therefore, some form of numerical integration must be performed or use made of tables in which the integration has already been accomplished. Kapur and Lamberson [1977] provide reliability expressions for normal and exponential combinations, gamma stress and strength, and normal stress and Weibull strength distributions. Evaluating the gamma reliability requires the use of tables of the incomplete beta function. For the normal-Weibull combination, Kapur and Lamberson provide table values for selected parameter values. Other discussions on stress-strength analysis may be found in Carter [1972] and Dhillon and Singh [1981].

TABLE 7.2
Static reliability for specified distributions

Distribution	Constant strength k	Constant stress s	Random stress and strength
Exponential	$R = 1 - \exp\left(-\frac{k}{\mu_x}\right)$	$R = \exp\left(-\frac{s}{\mu_y}\right)$	$R = \frac{\mu_y}{\mu_x+\mu_y}$
Weibull	$R = 1 - \exp\left[-\left(\frac{k}{\theta_x}\right)^{\beta_x}\right]$	$R = \exp\left[-\left(\frac{s}{\theta_y}\right)^{\beta_y}\right]$	Solve numerically
Normal	$R = \Phi\left(\frac{k-\mu_x}{\sigma_x}\right)$	$R = 1 - \Phi\left(\frac{s-\mu_y}{\sigma_y}\right)$	$R = \Phi\left(\frac{\mu_y-\mu_x}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)$
Lognormal	$R = \Phi\left(\frac{1}{s_x}\ln\frac{k}{m_x}\right)$	$R = 1 - \Phi\left(\frac{1}{s_y}\ln\frac{s}{m_y}\right)$	$R = \Phi\left(\frac{\ln(m_y/m_x)}{\sqrt{s_x^2+s_y^2}}\right)$

7.3 DYNAMIC MODELS

If a load is placed repetitively over time on a system, then, under certain conditions, a dynamic reliability may be derived. Two cases will be discussed. The first case occurs when loads occur at regular (or known) intervals of time. The second case corresponds to loads occurring at completely random times as characterized by the Poisson probability distribution. In both cases the assumption is made that the distribution of the strength or capacity of the system does not change over time (a stationary process). This precludes those situations in which aging or wearout occurs.

7.3.1 Periodic Loads

For the first case assume that n loads (cycles) occur at times $t_1, t_2, \ldots, t_n$, that each load has an identical and independent distribution represented by Eq. (7.9), and that the strength of the system at each cycle has an identical and independent probability distribution represented by Eq. (7.10). Either the load or the strength may also be a known constant rather than a random variable. Let X_i be the load and Y_i be the strength observed on the ith cycle. After n cycles the reliability R_n is found from

$$
\begin{aligned}
R_n &= \Pr\{X_1 < Y_1, X_2 < Y_2, \ldots, X_n < Y_n\} \\
&= \Pr\{X_1 < Y_1\}\Pr\{X_2 < Y_2\}\cdots\Pr\{X_n < Y_n\}
\end{aligned}
\tag{7.18}
$$

assuming independent load and strength applications each cycle. If the distributions of X and Y are identical for each cycle (a stationary process), then $\Pr\{X_i < Y_i\} = R$ where R is the static reliability for a single load-versus-strength application and may be computed using Eq. (7.11), (7.12), (7.13), or (7.14) as appropriate. Therefore, for n independent loads applied to a system, $R_n = R^n$.

EXAMPLE 7.12. If the strength of a system is a constant k and the load can be represented by an exponential distribution with parameter α, then

$$R_n = \left(1 - e^{-\alpha k}\right)^n$$

where $1/\alpha$ is the mean load per application.

EXAMPLE 7.13. If the load is a constant s and the strength of the system can be described by a Weibull distribution with parameters θ and β, then

$$R_n = e^{-n(s/\theta)^\beta}$$

EXAMPLE 7.14. The breaking strength of a support beam has a Weibull distribution with $\beta = 2.1$ and $\theta = 1200$ lb. Four beams are used to support a structure that places 100 pounds on each beam. What is the reliability of the structure?

$$R_4 = e^{-4(100/1200)^{2.1}} = 0.9785$$

If load application times are known constants, a dynamic reliability may be found from

$$R(t) = R^n \qquad \text{for } t_n \leq t < t_{n+1} \tag{7.19}$$

with $t_0 = 0$. If the cycle (load) time is uniformly spaced with $\Delta t = t_{i+1} - t_i$, then

$$R(t) = R^{\lfloor t/\Delta t \rfloor} \tag{7.20}$$

where $\lfloor z \rfloor$ is the integer portion of z.

EXAMPLE 7.15. A die is designed to withstand a force of 10,000 lb. A hydraulic forge has been found to exert a force that has an exponential distribution with a mean of 1000 lb. If castings are made at the fixed rate of one every two minutes ($\Delta t = 1/30$ hr), what is the reliability of completing an 8-hr shift without the die failing under the load?

Solution

$$R = \Pr\{X < Y\} = F_x(10{,}000) = 1 - e^{-10{,}000/1{,}000} = 0.9999546$$

$R(t) = 0.9999546^{\lfloor 30t \rfloor}$ where t is measured in hours. Then $R(8) = 0.9999546^{\lfloor 240 \rfloor} = 0.98916$.

7.3.2 Random Loads

If loads are applied at random so that the number of loads per unit of time has a Poisson distribution, then

$$P_n(t) = (\alpha t)^n e^{(-\alpha t)}/n! \qquad n = 0, 1, 2, \ldots$$

is the probability of n loads occurring during time t. Alpha (α) is the mean number of loads per unit of time, and αt is therefore the mean number of loads during time t. The reliability can be found from

$$\begin{aligned} R(t) &= \sum_{n=0}^{\infty} R^n P_n(t) = \sum_{n=0}^{\infty} R^n \left[\frac{(\alpha t)^n e^{-\alpha t}}{n!}\right] \\ &= e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha t R)^n}{n!} \\ &= e^{-(1-R)\alpha t} \end{aligned} \tag{7.21}$$

This last expression results from the infinite series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

R is again computed using Eq. (7.11), (7.12), (7.13), or (7.14).

EXAMPLE 7.16. A building is designed to withstand winds of speeds up to 120 mph. Hurricane winds are normally distributed with a mean of 86 mph and a standard deviation of 9 mph. In this region hurricanes occur at random (Poisson process) at the mean rate of 2 per year. Derive the reliability function.

Solution. Using Eq. (7.11) with $F_x(x)$ normally distributed,

$$R = \Phi\left(\frac{120 - 86}{9}\right) = \Phi(3.78) = 0.99992 \qquad \text{and} \qquad R(t) = e^{-(0.00008)2t}$$

If a reliability of 0.99 is desired, one would expect buildings of this type to last

$$t = \frac{\ln 0.99}{-0.00016} = 62.8 \text{ yr}$$

7.3.3 Random Fixed Stress and Strength

A different result is obtained if stress and strength are randomly determined once and then fixed for each cycle. In the case of random cycle times (e.g., Poisson),

$$R(t) = \sum_{n=0}^{\infty} R_n P_n(t) = P_0(t) + R \sum_{n=1}^{\infty} P_n(t)$$

since $R_0 = 1$ and $R_n = R = \Pr\{x \leq y\}$ for $n = 1, 2, \ldots$. Thus

$$R(t) = P_0(t) + R(1 - P_0(t))$$

using the fact that $\sum_{i=0}^{\infty} P_i(t) = 1$. Since $P_0(t) = e^{-\alpha t}$ for the Poisson process,

$$\begin{aligned} R(t) &= e^{-\alpha t} + R(1 - e^{-\alpha t}) \\ &= R + (1 - R)e^{-\alpha t} \end{aligned} \tag{7.22}$$

Equation (7.22) is simply the weighted average of a reliability of 1 times the probability that no load is applied and the static reliability times the probability that one or more loads are applied.

EXAMPLE 7.17. A single emergency shut-off valve has an exponentially distributed strength having a mean of 3700 lb. The force, or load, is also exponentially distributed with a mean of 740 lb. Once applied, the load will then remain constant. Emergency shut-off procedures occur randomly at the rate of 1 per year. We have $\mu_x/\mu_y = 740/3700 = 0.20$, so from Table 7.1, $R = 0.83$ and $R(t) = 0.83 + 0.17e^{-t}$ with t measured in years. Therefore, the reliability for 1 yr is obtained from $R(1) = 0.83 + 0.17e^{-1} = 0.8925$.

7.4 PHYSICS-OF-FAILURE MODELS

The primary approach taken so far is to treat the occurrence of failures as a random process. As discussed in Chapter 1, this view results from our lack of knowledge of the physical processes resulting in a failure. As a consequence, we must develop reliability models statistically. Through the collection and analysis of failure data, we can estimate parameters and perform goodness-of-fit tests in deriving an acceptable reliability model. Indeed, this is the focus of the second part of this book. However, in taking this approach, we can only infer from our sample of failure data to the general population. Our reliability estimates are valid predictions for the general population but say very little concerning an individual component or failure occurrence. In fact, if failures are distributed exponentially (or have almost any distribution, for that matter), then the time to failure of a single occurrence can be any $t \geq 0$. It is only over a large number of failures that we can begin to see the exponential pattern. Therefore, it is only over a large number of failures that we can make reliable reliability predictions. A second limitation of this approach is that it does not consider the effect of individual stresses and operating conditions on components. Not all components will experience the same voltage, ambient temperature, vibration, shock, or humidity. Therefore we would not expect all of the components necessarily to exhibit the same failure pattern. The covariate models presented earlier address this problem to some degree, but they are still statistical models developed from samples and applied over an entire population.

An alternative approach to reliability estimation is called physics of failure. We define the physics-of-failure models as mathematically derived, usually deterministic models based on knowledge of the failure mechanisms and the root causes of failures. A failure is not viewed as a stochastic event. Instead, a time to failure is found for each component failure mode and failure site based on the stresses, material properties, geometry, environmental conditions, and conditions of use. The failure times can then be ranked, and the most dominant one (i.e., the shortest time) provides the component time to failure estimate. Once the model has been developed, individual use and environmental conditions can be considered to develop a reliability estimate tailored to a particular application. The disadvantages of this approach are that the models are very specific to the failure mechanism and failure site, a detailed understanding of the failure processes is required, and experimental data as well as engineering analysis may be required in order to derive the equations. As a result, only a limited number of useful models are available.

Although there is no well-defined approach for developing physical failure models, several general steps can be identified:

1. Identify failure sites and mechanisms.
2. Construct mathematical models.
3. Estimate reliability for a given operating and environmental profile and for given component characteristics.
4. Determine dominant service life.
5. Redesign to increase service (design) life.

The significant activity in this methodology is the development of the physical model. Figure 7.4 identifies the generic inputs to the model.

Ideally the effect of the variables that collectively describe the failure mechanism can be mathematically modeled on the basis of known physical principles and

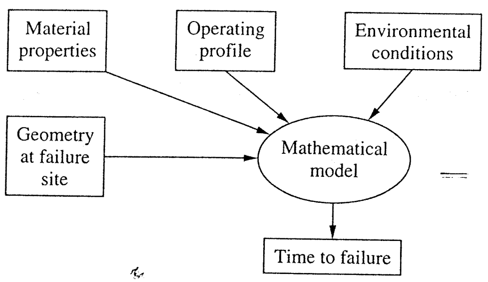


FIGURE 7.4
A physics-of-failure conceptual model.

constants or empirically derived through experimentation and observation. Typical failure mechanisms that have been modeled include fatigue, friction, corrosion, dielectric breakdown, electromigration, contamination, molecular migration, temperature cycling, and mechanical stress. Several simple examples in the spirit of this approach follow.

EXAMPLE 7.18. The useful life of cutting tools, such as drill bits or saw blades, may be modeled on the basis of the geometry and operating characteristics of the cut and the hardness of the material. There are several failure modes that may be identified, including fracture, plastic deformation, and gradual wear. The life of a cutting tool with respect to gradual wear was first expressed mathematically by Frederick Taylor in 1907 and has been enhanced since then. A typical form of this model is given by

$$t = \frac{c(B_{hn})^m}{v^\alpha f^\beta d^\gamma}$$

where t = the life of a tool in minutes
B_{hn} = the Brinell hardness number of the work material
v = cutting speed in feet per minute
f = the feed in inches per revolution or inches per sawtooth
d = depth of the cut in inches
c, m, α, β, and γ are constants to be determined empirically

Usually $\alpha > \beta > \gamma > m$, indicating that tool life is most sensitive to cutting speed, then feed, depth of cut, and finally material hardness.

Consider a milling operation performed on cast iron in which the cutting tool advances at a feed rate of 0.02 inch per revolution with a cutting speed of 40 feet per minute. The depth of the cut is 0.011 inch. For the particular operation and cutting tool used, the following model parameters were determined from a least-squares fit to data generated in a laboratory:

$$t = \frac{0.023(B_{hn})^{1.54}}{v^{7.1} f^{4.53} d^{2.1}}$$

The cast iron has a Brinell hardness number of 180. Therefore

$$t = \frac{0.023(180)^{1.54}}{(40)^{7.1}(0.02)^{4.53}(0.011)^{2.1}} = 186 \text{ minutes}$$

EXAMPLE 7.19. Ploe and Skewis [1990] report the following model, developed by T. P. Newcomb for estimating brake-pad wear and used in determining the pad life:

$$W = \left(\frac{10^4 W_b}{2A}\right)\left(\frac{\mathrm{Wt}(\Delta v^2 N)y}{4g}\right)$$

where W = pad wear per mile in inches
W_b = specific wear rate of friction material (in^3/ft-lb)
A = lining area (in^2)
Δv = average change in velocity per brake action (ft/sec)
Wt = weight of vehicle (lb)
g = acceleration due to gravity (32.2 ft/sec^2)
N = frequency of brake applications per mile
y = proportion of total braking effort transmitted through the lining

This model assumes that pad wear is proportional to the energy absorbed and to the wear rate of the friction material. If the thickness of the pad is d inches, then

$$\text{Pad life} = \frac{d}{W_{-}} \text{ miles}$$

EXAMPLE 7.20. An empirical formula by Andrade [1914] measures the strain (deformation) as a function of time at a specified temperature on a material under constant stress resulting eventually in a fracture. Let

$$\varepsilon = \varepsilon_0(1 + \beta t^{1/3})e^{kt}$$

where ε = the strain at time t
ε_0 = the initial elastic strain
β and k are constants depending on the particular material and temperature

If ε_{max} is the fracture stress of the material, then setting $\varepsilon = \varepsilon_{max}$ in the above formula and solving for t provides the design life with respect to creep (progressive deformation of the material).

EXAMPLE 7.21. Continued miniaturization of electronic microcircuits has increased current densities in electrical conductors. This in turn increases the probability of electromigration failures. The following is an equation for electromigration [Bhagat, 1992]:

$$t = \frac{bA}{J^2} e^{E_a/(kT)}$$

where t = mean time to failure (hr)
b = empirically derived constant (2.85×10^{15} for an aluminum conductor)
A = cross-sectional area of the conductor (cm^2)
J = current density (A/cm^2)
E_a = activation energy (eV)
k = Boltzmann's constant (8.62×10^{-5} eV/k)
T = temperature (K)

For a particular aluminum conductor having a cross-sectional area of 10^{-7}cm^2, an activation energy of 0.5 eV, and a stress temperature of 95° C (368 K), a design life of 20,000 hr is desired. Setting $t = 20{,}000$ and solving for J results in $J = 316{,}000$ A/cm^2. Therefore the current density must be limited to 316,000 A/cm^2.

Other examples of physical reliability models for electronic components may be found in Rawicz [1994].

Comparison of covariate and physics-of-failure models

Both covariate models and physics-of-failure models in some manner relate reliability to environmental, physical, and operational characteristics of the failure process. However, there are some noticeable differences between the two approaches. The parametric covariate models that have been presented here explicitly retain an underlying failure distribution, the parameters of which are determined by the explanatory variables. Physics-of-failure models treat the time to failure as deterministic, although in some applications it is interpreted to be a mean or median of a probability distribution. Covariate models are not physical models and do not necessarily show cause and effect, although the covariate and the parameter may be related as

cause and effect. Physics-of-failure models, on the other hand, attempt to capture the relevant causal variables and their relationships among one another in order to model the failure mechanism. Covariate models generally do not include physical constants (such as the acceleration due to gravity or Boltzmann's constant), whereas physics-of-failure models normally include them. Finally, simple functional forms are assumed for covariate models, and therefore they are usually linear or loglinear (although this is not a necessary condition). The functional form of a physics-of-failure model is strongly determined by the physical processes generating the failures. Both kinds of models are based on empirical data obtained from testing or laboratory experimentation, and both may use least-squares techniques for estimation. It is anticipated that reliability models will continue to become more sophisticated, with new models being developed that may combine features of both approaches.

EXERCISES

7.1 Reliability testing indicates that the failure rate of a ceramic capacitor rated at 160 V is constant over time but depends on the temperature and the ratio of the operating voltage to the rated voltage (derating fraction). The following hazard rate function (in operating hours) was derived from test data:

$$\lambda(t \mid \mathbf{x}) = e^{-9.48+0.01759x_1+7.017x_2} \times 10^{-3}$$

where x_1 is the temperature in degrees centigrade and x_2 is the ratio operating voltage / rated voltage. If the capacitor is to be used in an electronic circuit where the temperature is 45°C and the voltage is 120 V, determine the MTTF and the reliability at the end of the first 1000 operating hours. Through redesign, the equipment can be modified to operate at a temperature of 30°C or a capacitor having a rating of 200 V can be used. Which alternative increases the reliability the most? What is the percentage increase?

7.2 The time to failure of a drill bit is lognormal with $s = 1.43$. From field data the median life of the bit has been found to depend on the Brinell hardness and the density of the material being drilled. The following function was empirically derived:

$$u(x) = 12.31 - 0.0157x_1 - 0.35x_2$$

where x_1 is the Brinell hardness and x_2 is the density (mg/m^3). If a steel plate has been heat-treated to a Brinell hardness of 200 and has a density of 7.3 mg/m^3, determine its reliability over 20 cutting hours.

7.3 A component has a fixed capacity of 1000 psi. Determine the static reliability of the component if a single load having the following distribution is applied.
(a) Exponential with a mean of 500 psi
(b) Normal with a mean of 500 psi and a standard deviation of 165 psi
(c) Lognormal with a median of 500 psi and a shape parameter (s) of 0.30

7.4 A fixed load of 250 lb is applied to a support beam having the following strength. Determine the static reliability.
(a) Exponential with a mean of 2600 lb
(b) Weibull with a scale parameter of 2600 lb and a shape parameter of 0.80
(c) Lognormal with a median of 2600 lb and a shape parameter (s) of 0.90

7.5 A load is exponential with a mean of 25. The strength is exponential. Determine the minimum value of the mean strength in order to achieve a reliability of 0.95.

7.6 Consider the following load distribution:

$$f_x(x) = \begin{cases} 3x^2/10^9 & 0 \le x \le 1000 \quad \text{kg} \\ 0 & \text{otherwise} \end{cases}$$

If the strength is a constant 950 kg, determine the static reliability.

7.7 A brake assembly has been tested and found to have a capacity that is normally distributed with a mean of 275 lb and a standard deviation of 25 lb. If a normally distributed force having a mean of 180 lb and a standard deviation of 30 lb is applied, what is the static reliability? As the variability (variance) of either the strength or load increases, what happens to the reliability?

7.8 A dam has a fixed water capacity of 20 feet. The distribution of flood levels is given by the following probability density function with x measured in feet:

$$f_x(x) = 0.25e^{-0.25x} \qquad x \ge 0$$

Floods occur at random following a Poisson process with a mean rate of 1 every 2 yr. Compute the reliability of the dam over:
(a) A 10-yr period
(b) A 20-yr period

7.9 A capacitor for use in an emergency signal transmitter is selected at random from a shipment in which the rated strength of the capacitors as measured in applied voltage is normal with a mean of 200 V and a standard deviation of 45 V. If the transmitter is needed, a fixed load of 120 V will be applied. Frequency of use of the transmitter is random with a Poisson distribution having a mean rate of once every 48 months. What is the reliability over a 5-yr design life? *Hint:* Does this problem fit the conditions of the model discussed in Section 7.3.3?

7.10 Determine the expected reliability (static) of a system having the following load and capacity distributions: $f_x(x) = \frac{1}{2}$, $15 \le x \le 17$; $f_y(y) = 0.04(y-15)$, $15 \le y \le 20$.

7.11 Each day during the peak demand period, an extra electrical generator powers up. It has a maximum output of 1200 watts. The demand for this additional power is random, having an exponential distribution with a mean of 300 watts. What is the reliability of the generator over a week's time (7 days) in meeting the peak demands?

7.12 The strength of a concrete structure designed to support a fixed load of 464 lb has the following probability density function:

$$f_y(y) = \frac{3y^2}{10^9} \qquad 0 \le y \le 1000 \quad \text{lb}$$

(a) Compute the static reliability.
(b) If the load is also random having the probability density function below, find the static reliability.

$$f_x(x) = 2(0.001 - 0.000001x) \qquad 0 \le x \le 1000 \quad \text{lb}$$

(c) If the load in part (a) is applied at random according to a Poisson process with a mean occurrence rate of 0.01 per year, compute the design life for a reliability of 0.99.

7.13 The Weibull Building, a marvel in engineering design, is subject to random (Poisson-distributed) wind gusts at an average rate of two per day. The strength of the building is such that it can withstand winds of up to 100 mph (deterministic). Wind speed, however, during gusts, is random with a Weibull distribution having a shape parameter of 2 and a scale parameter of 50 mph. Determine the building's reliability function and the mean number of days to failure.

7.14 The peak daily load in megawatt-hours on an electric utility substation is normally distributed with a mean of 10,000 and a standard deviation of 1000. The capacity of the system is 13,500 megawatt hours. What is the reliability of the station over a 100-day period?

7.15 The breaking strength of a cutting tool is a constant 25 lb. If the load being placed on the tool has the following probability density function, compute the tool's static reliability.

$$f_x(x) = \frac{200}{(x+10)^3} \qquad x \ge 0$$

7.16 The Fawlty Construction Company has built a condo for Mr. Herr A. Cane on the Outer Banks of North Carolina having a static reliability to withstand a major storm in this area of 0.992. If major storms hit the Outer Banks with an average frequency of 1 per year, what is the probability that the condo will incur major damage during Mr. Cane's 25-yr mortgage on the condo? If a reliability of 0.95 is desired, what is the condo's design life?

7.17 Determine the reliability of a system having a capacity that is lognormal with a median of 100 and $s = 0.6$ subject to a load that is lognormal with a median of 20 and $s = 0.8$.

7.18 A remote sensor communicates with a central processing facility using a 2400-bit-per-second modem. Each event detected by the sensor requires one bit to relay the information. The sensor must report 100 percent of all events. Events occur according to the following probability density function:

$$f_x(x) = \frac{x}{3.125 \times 10^6} \qquad 0 \le x \le 2500 \quad \text{events/sec}$$

What is the static reliability of the system?

7.19 Refer back to Exercise 3.17. The moisture content in the air necessary to cause flat fading is 100 PPM. When fog occurs, the moisture content of the air is normally distributed with a mean of 75 PPM and a standard deviation of 25 PPM. Fog occurs at the rate of once every two months. The charge density of the air necessary to cause selective fading is 5×10^{12} electrons per cubic centimeter. The charge density of an electrical storm is lognormal with a median of 3×10^{12} electrons per cubic centimeter and a mean of 4×10^{12} electrons per cubic centimeter. Electrical storms occur at the rate of 4 per month. Recompute the reliability for a 24-hr period.

7.20 The buckling force under an axial compressive load for a column made from timber and hinged at both ends is given by Euler's formula:

$$F_c = \frac{\pi^2 EI}{L^2}$$

where F_c = critical load in pounds
E = modulus of elasticity = 1.6×10^6 pounds per square inch for timber
I = moment of inertia of the cross section of the column = 5.359 inches^4 for a 2-inch by 4-inch column
L = the length of the column in inches

A 10-foot column is used to support a load that has a Weibull distribution with a shape parameter of 1.76 and a scale parameter of 2,500 lb.
(a) What is the static reliability?
(b) If a material stronger than timber is used that has a safety factor of 3 (i.e., F_c is three times the mean load), what is the static reliability?

7.21 The thermal protection system on the space shuttle consists in part of approximately 30,000 ceramic and reinforced carbon–carbon tiles. These tiles are designed to withstand reentry temperatures of up to 1260°C. Assume that the temperature of the tile system on reentry is lognormally distributed with a shape parameter of 0.15 and a median value of 835°C. If a failure occurs when the reentry temperature exceeds the design capacity of the tile system, determine the reliability of the tile system over a 5-yr period in which the shuttle averages 12 flights a year. A failure in this case requires the removal and replacement of one or more tiles during the ground recovery and restoration process.

7.22 A manufacturing lathe operation requires cutting a malleable iron rod having a Brinell hardness number 125. For this particular cutting operation, the following tool life equation was derived:

$$t = \frac{7.4B_{\text{hn}}^{1.1}}{v^{4.76} f^{2.11} d^{1.84}}$$

The depth of the cut is 0.02 inch, and the feed rate is 0.03 inch per revolution. Because of production requirements, the cutting tool can only be replaced once every 4 hr. What is the maximum cutting speed v permissible?

CHAPTER 8

Design for Reliability

To a large degree, reliability is an inherent attribute of a system, component, or product. As such, it is an important consideration in the engineering design process. When the life-cycle costs of a system are being analyzed, reliability plays an important role as a major driver of these costs and has considerable influence on system performance. As we will see, decisions made during conceptual and preliminary design will affect total life-cycle costs. The objective of this chapter is to describe a reliability design process that will establish and then achieve realistic reliability goals.

Reliability design is an iterative process that begins with the specification of reliability goals consistent with cost and performance objectives. This requires consideration of the life-cycle costs of the system and the effect that reliability has on overall costs and system effectiveness. Figure 8.1 outlines this process. Once the reliability goals have been established, these goals must be translated into individual component, subcomponent, and part specifications. This is not necessarily an easy task, and it generally requires a reliability block analysis. After individual component and part requirements have been determined, various design methods can be applied in order to meet the goals. These methods include the proper selection of parts and material, stress-strength analysis, derating, simplification, identification of technologies, and use of redundancy. Following completion of preliminary and detailed design and along with initial development and prototyping, a failure analysis may be performed to determine whether specifications are being met and to provide a systematic approach for identifying, ranking, and eliminating failure modes. This requires the use of reliability testing, including, perhaps, a formalized reliability growth testing program. Once reliability goals have been achieved, verification that safety margins are also being met must be made. Fault tree analysis can be a

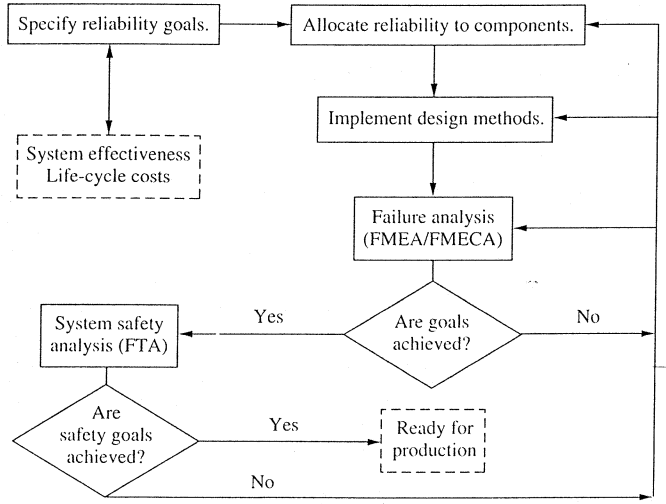


FIGURE 8.1
The reliability design process.

useful tool in identifying critical (catastrophic) failure modes. If either the reliability or safety goals are not met, the design process must continue. This may require reallocating reliability goals among the components if it is not possible to achieve a desired component reliability. More often it may require a redesign through the use of the design methods presented in Section 8.3. The effect of design changes should then be verified through continued use of failure analysis and reliability testing.

Although we are considering reliability as an inherent system or component attribute that can largely be determined during design, we cannot ignore the fact that reliability is influenced throughout a product life cycle by factors external to the product itself. Table 8.1 shows the typical product life cycle and the major activities

TABLE 8.1
Reliability activities and product life cycle

Development phase	**Conceptual preliminary design**	**Detailed design, development, and prototyping**	**Production and manufacture**	**Product use and support**
	Specification Allocation Design methods	Design methods Failure analysis Growth testing Safety analysis	Acceptance testing Quality control Burn-in and screen testing	Preventive and predictive maintenance Modifications Parts replacement

influencing reliability. During the initial design activity, reliability goals should be specified and allocated to individual components. Early in the design process, various reliability methods should be considered. Once the detailed design has been completed, it will be much more difficult to change parts, add redundancy, incorporate new technology, or simplify the overall design in order to achieve a desired reliability. As the process goes from design to development, a formal failure analysis program should be established to identify and eliminate critical failure modes. It is at this time also that a reliability growth program designed to track reliability improvement during initial development should be implemented. Reliability growth testing is the subject of Chapter 14. Once the design has been completed and the reliability goals have been met, the specifications arrived at must be maintained during production. This requires conformance to design and manufacturing specifications through a good quality control program. Supplier parts must also meet agreed specifications, and their compliance may be verified through acceptance testing procedures. Products having high infant mortality rates may benefit from a burn-in testing program designed to eliminate substandard and marginal parts before the products are fielded. Burn-in and screen testing will be discussed in Chapter 13. Once the product has entered into operation, an active preventive maintenance program, continuous engineering improvements and field modifications, and routine replacement of worn parts can significantly improve reliability. Preventive maintenance will be discussed in the next two chapters. The remainder of this chapter will expand on the design process as outlined in Fig. 8.1.

8.1 RELIABILITY SPECIFICATION AND SYSTEM MEASUREMENTS

Historically the mean time to failure (MTTF) or the mean time between failures (MTBF) has been the primary and often only measure of reliability. Unless the failure distribution is exponential, this parameter alone is insufficient to specify reliability. Section 2.2 showed how two components having the same MTTF will have considerably different reliabilities at a specified time. A better measure would be to specify the reliability at specific points in time. For example, to state that a 99 percent reliability is required after 2 years of operation and a 95 percent reliability is required at the end of 5 years of operation is much more precise than stating that an MTTF of 10 years is required. The ideal situation would be to completely specify the failure distribution or hazard rate function including all of its parameters. Unfortunately, there is no known way to design a desired failure distribution into a system.[1] If the MTTF is to be the only reliability specification, a constant failure rate should be assumed as part of the specification and subsequently demonstrated. Otherwise,

[1]I have actually seen a proposal to do this very thing: "If the underlying failure mechanism can be shifted to produce normally distributed failures, what would be the effect on operations and on logistics support?"

a wide range of failure distributions (and therefore reliabilities) are attainable by selecting various distributions and variances each having the same MTTF.

As discussed in Chapter 1, in specifying a reliability, it is necessary to also define what constitutes a failure. Many systems degrade before failing completely. At what point is performance no longer acceptable? It may also be necessary to exclude certain failure modes in the specification. For example, failures due to natural catastrophes may be difficult to eliminate as part of the inherent design process. On the other hand, certain systems may be designed specifically to operate in these environments, and their effect should then be an integral part of the design process. The measurement of time is equally important, especially if time to failure is not being measured in clock or calendar time. Jet engine failures, for example, have often been measured in terms of cycles to maximum power rather than operating hours. Finally, the normal operating and environmental conditions in which the product is to be evaluated should be clearly stated as part of the specification. This includes not only external stress factors such as temperature and humidity, but also policy and procedural conditions such as frequency and type of preventive maintenance and experience and training levels of the users or operators.

8.1.1 System Effectiveness

In establishing the system-level reliability requirements, both system performance (effectiveness) and life-cycle cost must be considered. *System effectiveness* is the probability that the system can successfully perform its intended purpose or mission when operated under specified conditions. System effectiveness includes reliability, as shown in Fig. 8.2. *Operational readiness* is the probability that the system is operational when first used or at the start of a mission. Recall that $R(0) = 1$. However, at time $t = 0$, when a system is first powered up, turned on, put on-line, etc., it may or may not work. For example, it may have been defective coming from the factory or experienced a dormant failure. *Mission availability* will be defined in Chapter 11. It is the percentage of time the system will be operating during the mission. Maintainability and availability will be discussed in the following three chapters. However, note here that if the system is not repairable, availability is the same as the system reliability. Design adequacy is the probability that the system will accomplish its

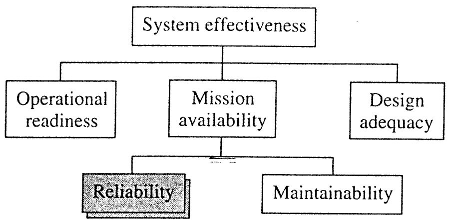


FIGURE 8.2
System effectiveness.

mission or fulfill its purpose given that the system is operating within its design parameters. For example, assume that a copying machine is working on demand and continues to work over the period of time in which copies are to be produced. However, in order to complete the job on time, it must operate at a speed of 45 copies a minute. If it is unable to maintain this rate, then the copier is inadequately designed for the job. Design adequacy in this example may be the probability or percentage of jobs in which the reproduction rate required is within the machine capability. Since all three components of system effectiveness are defined to be probabilities, then assuming independence among all three,

$$\text{System effectiveness} = \text{operational readiness} \times \text{availability} \times \text{design adequacy} \tag{8.1}$$

Obviously, all three probabilities must be high in order for system effectiveness to be at an acceptable level.

In selecting from among design alternatives, ideally we would want to either maximize system effectiveness subject to an upper bound on life-cycle cost or minimize life-cycle cost subject to a lower bound on system effectiveness.

8.1.2 Economic Analysis and Life-Cycle Costs

The other major consideration in establishing a system reliability is life-cycle cost. Life-cycle costing is the process of determining all relevant costs from conceptual development through production, utilization, and phase-out. It is the total cost of ownership. Our interest in discussing life-cycle cost is to ensure that those costs affected by our choice of design variables, especially reliability (and later maintainability), are properly accounted for. There are many different ways to establish life-cycle cost categories; a typical cost element structure is shown in Table 8.2.

TABLE 8.2
Cost categories

Acquisition costs	Operations and support costs	Phase-out
Research and development	Operations	Salvage value
Management	Facilities	Disposal costs
Engineering	Operators	
Design and prototyping	Consumables (energy and fuel)	
Engineering design	Unavailable time or downtime	
Fabrication	Support	
Testing and evaluation	Repair resources	
Production	Supply resources:	
Manufacturing	Repairables	
Plant facilities and overhead	Expendables	
Marketing and distribution	Tools, test, and support equipment	
	Failure costs	
	Training	
	Technical data	

In performing design trade-offs, total life cycle cost of each alternative design should be estimated and compared. At the highest level, a life-cycle cost model may take on the following form:

$$\begin{aligned}\text{Life-cycle cost} &= \text{acquisition costs} + \text{operations costs} + \text{failure cost}\\ &\quad + \text{support costs} - \text{net salvage value} \qquad (8.2)\end{aligned}$$

where Net salvage value = salvage value − disposal cost

Since the system will normally be operated over an extended period of time corresponding to its design life or economic life, the time value of money must be taken into account. The economic life is the number of years beyond which it is no longer economical to operate or maintain the system and replacement or discontinuance is justified on a cost basis. To discount monetary values over time, all revenues and costs can be expressed in present-day equivalent dollars.[2] Therefore, the following adjustments must be made. P is the present value, and i is the real, or effective, discount rate. If we assume a constant annual inflation rate of f and an annual return on investment rate of e, then $i \approx e - f$ for small values[3] of f and e.

Let $P_F(i, d) = 1/(1 + i)^d$ where F is a future amount at the end of year d, and $P_A(i, d) = [(1 + i)^d - 1]/[i(1 + i)^d]$, where A is an equal annual amount observed over d years. The term $P_A(i, d)$ is an annuity factor, which converts equal annual payments over d years to a single present-day equivalent amount. Writing Eq. (8.2) more explicitly,

$$\begin{aligned}\text{Life-cycle cost} &= C_u N + [F_o + P_A(i, t_d)C_o N] + \left[P_A(i, t_d)C_f \frac{t_0}{\text{MTTF}} N\right] \qquad (8.3)\\ &\quad + [F_s + P_A(i, t_d)C_s N] - [P_F(i, t_d)SN]\end{aligned}$$

where C_u = unit acquisition cost
N = number of identical units to be procured
F_o = fixed cost of operating
C_o = annual operating cost per unit
F_s = fixed support cost
C_s = annual support cost per unit
C_f = cost per failure
t_o = operating hours per year per unit
t_d = design life (in years)
S = unit salvage value (a negative value is interpreted as a disposal cost)

The expression t_0/MTTF in Eq. (8.3) is the expected number of failures per year assuming replacement or repair to "as good as new" condition of the failed unit (a renewal process which is discussed in the next chapter). The cost per failure, C_f, may be a repair cost, replacement cost, or a warranty cost. The unit acquisition cost includes the design, development, and production costs allocated over the total number produced. As the reliability goal increases, these costs will increase because of additional reliability growth testing, improved manufacturing quality control, more expensive parts and material, increased use of redundancy, and additional resources committed to reliability improvement.

[2]Alternative methods of comparing two or more cash flows may be used, such as annual equivalent amounts and internal rate of return. Present value is used here since it is the simplest and most convenient for the problem at hand.

[3]A more precise relationship is given by $1/(1 + i) = (1 + f)/(1 + e)$.

Assuming that only the unit acquisition cost and failure cost are sensitive to the design reliability, we may wish to compare the following expected present equivalent unit cost for each alternative:

$$C_u + (P/A, i, t_d)C_f \frac{t_0}{\text{MTTF}}$$

EXAMPLE 8.1 Two designs are being considered for a new product operating throughout the year and having the following characteristics:

Design 1: $C_u = 1200$, $\lambda_1 = 0.02$/yr
Design 2: $C_u = 1300$, $\lambda_2 = 0.01$/yr

The design life of the product is 10 yr. A failure results in a replacement at the unit acquisition cost (i.e., $C_u = C_f$). Assuming an interest rate of 5 percent, $P/A(0.05, 10) = 7.7217$, and

$$\text{Design 1: } 1200[1 + (P/A, 0.05, 10)0.02] = 1385.32$$

$$\text{Design 2: } 1300[1 + (P/A, 0.05, 10)0.01] = 1400.38$$

Therefore design 1 is preferred.

The acquisition cost may be difficult to estimate during the early design of a product. The reader is encouraged to see Fabrycky and Blanchard [1991] for a detailed development of life-cycle costing. Discussions on life-cycle costs of repairable items will be deferred until Chapter 11. The reader is cautioned not to assume that the minimum-cost design that meets the system effectiveness goal is always the preferred one. Other constraints, for example, may be minimum acceptable reliability goals and product safety requirements.

8.2 RELIABILITY ALLOCATION

Once the system reliability goals have been defined, reliability must then be allocated to the components and possibly subcomponents in a manner that will support these goals. Reliability block diagrams and the relationships developed in Chapter 5 are useful tools in accomplishing this. In general, we want the following inequality to hold:

$$h(R_1(t), R_2(t), \ldots, R_n(t)) \geq R^*(t) \tag{8.4}$$

where $R_i(t)$ is the reliability at time t of the ith component, $R^*(t)$ is the system reliability goal at time t, and h is a function that relates component reliabilities to system reliability. If MTTF* is a system reliability goal and component failures are exponential, then we desire

$$g(\text{MTTF}_1, \text{MTTF}_2, \ldots, \text{MTTF}_n) \geq \text{MTTF}^* \tag{8.5}$$

where MTTF_i is the MTTF of component i and g is a function that provides the system MTTF. The form of the functions h and g depend on the component serial-parallel configuration. For example, if all the components are serially related and their failures are independent of one another, then

$$\prod_{i=1}^{n} R_i(t) \geq R^*(t) \tag{8.6}$$

8.2.1 Exponential Case

If all the components have constant failure rates, Eq. (8.6) can be written as

$$\prod_{i=1}^{n} e^{-\lambda_i t} \geq R^*(t) \tag{8.7}$$

or equivalently,

$$\sum_{i=1}^{n} \lambda_i \leq \lambda_s \tag{8.8}$$

based on Eq. (5.2) where λ_s is the system failure rate goal.

8.2.2 Optimal Allocations

Ideally, reliability allocation should be accomplished in a least-cost manner. For example, assume that Eq. (8.6) applies and that R* has been specified for some time t (we will drop the argument t as being understood). If each component has a current reliability R_i where $\prod R_i < R^*$, we may be interested in solving the following problem:

$$\min z = \sum_{i=1}^{n} C_i(x_i) \tag{8.9}$$

subject to

$$\prod_{i=1}^{n} (R_i + x_i) \geq R^* \tag{8.10}$$

$$0 < R_i + x_i \leq B_i < 1 \qquad i = 1, 2, \ldots, n \tag{8.11}$$

where x_i is the increase in reliability of the ith component, $C_i(x_i)$ is the corresponding cost of achieving this growth, and B_i is an upper bound on the attainable component reliability. In practice, specifying the cost functions may be quite difficult. However, let us assume that the cost of component i is determined by $c_i x_i^2$ over the range in which we are interested. A quadratic cost function is the simplest nonlinear function, and since it is convex, it shows reliability growth cost increasing at an increasing rate—a reasonable assumption. Equation (8.9) then becomes

$$\min z = \sum_{i=1}^{n} c_i x_i^2 \tag{8.12}$$

To solve our problem, we assume that Eq. (8.10) is a strict equality, relax (ignore) the inequalities (8.11) for the moment, and form the Lagrangian function:

$$L(x_i, \theta) = \sum_{i=1}^{n} c_i x_i^2 - \theta \left[\prod_{i=1}^{n} (R_i + x_i) - R^* \right] \tag{8.13}$$

where θ is the Lagrangian multiplier. Necessary conditions for a minimum are then found by taking the partial derivatives of Eq. (8.13) and setting them equal to zero:

$$\frac{\partial L(x_i, \theta)}{\partial x_i} = 2c_i x_i - \theta \prod_{\substack{j=1 \\ j \neq i}}^{n} (R_j + x_j) = 0 \qquad i = 1, 2, \ldots, n \tag{8.14}$$

$$\frac{\partial L(x_i, \theta)}{\partial \theta} = \prod_{i=1}^{n} (R_i + x_i) - R^* = 0 \tag{8.15}$$

Multiplying the ith Eq. (8.14) by $(R_i + x_i)$ and rearranging terms:

$$\theta \prod_{i=1}^{n} (R_i + x_i) = 2c_i x_i (R_i + x_i) = \theta R^*$$

where the last equality resulted from Eq. (8.15). Rearranging terms again:

$$2c_i x_i^2 + 2c_i x_i R_i - \theta R^* = 0$$

Solving for x_i by using the quadratic formula with the positive root results in

$$x_i = \frac{-2c_i R_i + \sqrt{4c_i^2 R_i^2 + 8c_i \theta R^*}}{4c_i} = -0.5R_i + \sqrt{0.25R_i^2 + \frac{0.5\theta R^*}{c_i}} \tag{8.16}$$

In order to solve for x_i in Eq. (8.16), θ must be known. The generalized Lagrangian multiplier technique can be used to search for the correct value of θ. For a given value of θ, Eq. (8.16) solves a related optimization problem in which the $R*$ value in Eq. (8.15) is equal to the product in the equation. In other words, if we guess at a value for θ and compute the x_i using Eq. (8.16), then we would have solved an optimization problem in which our reliability goal was equal to the product in Eq. (8.15). Therefore, if we systematically update θ and compute the resulting reliability, we should eventually converge to the desired solution. This is illustrated in the following example.

EXAMPLE 8.2 A system consists of three components in series having the following parameters:

Component	Current reliability	Cost coefficient
1	0.85	$25
2	0.80	20
3	0.90	40

The upper bound on component reliability is 0.99. Current system reliability is (0.85)(0.80)(0.90) = 0.612. The system reliability goal is $R^* = 0.90$. Varying θ in Eq. (8.16) and then computing the left-hand side of Eq. (8.10) results in the following system reliability values:

θ	System reliability	Total cost
4.00	0.791	$5.840
5.00	0.835	7.168
6.00	0.880	8.453
7.00	0.924	9.701
6.5	0.902	9.081
6.48	0.901	9.057
6.46	0.900	9.031

The final solution is as follows:

Component	Reliability increase x_i	Component reliability	Component cost
1	0.1199	0.9699	2.997
2	0.1526	0.9526	3.051
3	0.0746	0.9746	2.983

The solution obtained by this method may not satisfy the upper-bound constraints given by Eq. (8.11). If this is the case, the problem must be resolved with the upper-bound constraints included. The solution to the inequality-constrained problem must satisfy more general conditions (e.g., the Kuhn-Tucker conditions). To find a solution, however, will generally require a numerical procedure. Since θ must be positive, the x_i will always be positive, automatically satisfying the lower-bound constraint.

8.2.3 ARINC Method

There are several popular reliability allocation strategies discussed in the literature that do not require optimization. One of the earliest proposed and the simplest is the ARINC method. This method assumes that the components are in series, are

independent, and have constant failure rates. If λ_i is the current estimated failure rate for the ith component and $\lambda*$ is the target system failure rate, then

$$\text{new } \lambda_i = w_i \lambda*$$

where

$$w_i = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} \qquad i = 1, 2, \ldots, n$$

Therefore, allocated failure rates are proportional to the current failure rates. This method could be adopted if failure rates are not constant by using an average failure rate (see Exercise 8.4).

8.2.4 AGREE Method

The AGREE (Advisory Group on Reliability of Electronic Equipment) method assumes that a system is comprised of n components each having n_i modules, or subcomponents. This method is somewhat more sophisticated than the ARINC method, for it allows component operating times to be less than the system operating times and it allows the inclusion of an importance index. Let

t = system operating time
$R^*(t)$ = system reliability goal at time t
n = number of components
n_i = a complexity number, e.g., the number of modules within component i
$N = \sum n_i$ = total number of modules in system
t_i = operating time of the ith component ($t_i \leq t$)
λ_i = failure rate of the ith component
w_i = probability that the system will fail given component i has failed (importance index)

The approach is to allocate an equal share of the reliability to each module in the system. Therefore the ith component's contribution to system reliability is given by $[R^*(t)]^{n_i/N}$. This allows us, then, to establish the following relationship for the ith component:

$$w_i(1 - e^{-\lambda_i t_i}) = 1 - [R^*(t)]^{n_i/N} \tag{8.17}$$

The left side of Eq. (8.17) is the joint probability that the ith component fails and results in a system failure. The right side of the equation is the failure probability allocated to the ith component. Solving Eq. (8.17) for λ_i results in

$$\lambda_i = -\frac{1}{t_i} \ln\left(1 - \frac{1 - R^*(t)^{n_i/N}}{w_i}\right) \qquad i = 1, 2, \ldots, n \tag{8.18}$$

Observe that $\prod_{i=1}^{n} e^{-\lambda_i t_i} \leq R^*(t)$ since not all component failures result in system failures (if $w_i < 1$).

EXAMPLE 8.3 A transceiver has four components: a receiver, a power supply, a transmitter, and an antenna system. Reliability specifications require the transceiver to operate 1000 hr with a probability of 0.99. The following component data are available:

Component	Importance index w_i	Operating time t_i, hr	Number of modules, n_i
Receiver	0.8	1000	25
Antenna	1.0	1000	15
Transmitter	0.7	500	23
Power supply	1.0	1000	70

The total module count is 133. Therefore, $0.99^{n_i/133}$ is the reliability to be allocated to the ith component. From Eq. (8.18) the following results are obtained:

Component	Failure rate	MTTF	Reliability	System reliability
Receiver	2.362×10^{-6}	423,369	0.9976	0.9981
Antenna	1.1335×10^{-6}	882,227	0.9989	0.9989
Transmitter	4.9676×10^{-6}	201,303	0.9975	0.9983
Power supply	5.2896×10^{-6}	189,048	0.9947	0.9947
System	1.3753×10^{-5}	72,713	0.9888	0.9900

From the above table, the probability of a component failure is $1 - 0.9888$ and the probability of a system failure is $1 - 0.99$. The difference is that some component failures will not cause a system failure (i.e., $w_i < 1$).

8.2.5 Redundancies

A general strategy to follow in allocating reliability among the components is to consider only serially related components. Redundancy can then be used to achieve the allocated component reliability if necessary, as discussed in the next section. However, there may be instances in which redundancy is an integral part of the design. It may therefore be desirable to account for this redundancy as part of the reliability allocation process. Reliability block diagrams can be a useful tool in this case. For example, consider the block diagram of Fig. 8.3. If R^* is the system reliability goal, we can write $R^* = R_1 \times R' \times R_4$ and use one of the above allocation methods based on serially related components. Components 2 and 3 would initially be treated as a single component. Then, assuming R' is the allocated reliability, we have

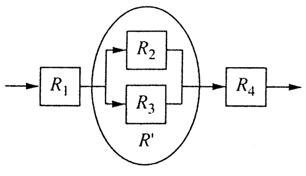


FIGURE 8.3
Reliability block diagram for reliability allocation.

$$R' = 1 - (1 - R_2)(1 - R_3) = R_2 + R_3 - R_2R_3$$

We can assign a reliability to one of the components, say component 3, such that $R_3 < R'$. (Why?) Then solving for the other,

$$R_2 = \frac{R' - R_3}{1 - R_3}$$

If both components receive the same reliability R, we have $R' = 2R - R^2$, which has the solution $R = 1 - (1 - R')^{0.5}$ from the quadratic formula. Obviously, if the system configuration is more complex, the allocation process is more difficult. However, generally it will be possible to reduce the system initially to serially related components and then further decompose as necessary.

8.3 DESIGN METHODS

In order to design reliability into a product, reasons for product failures must be considered. Generally, a product fails prematurely because of inadequate design features, manufacturing and parts defects, abnormal stresses induced during packaging or distribution, operator or maintenance error, or external conditions (environmental or operating) that exceed the design parameters. The reliability engineer should ensure that reliability is a primary consideration in the product design and during manufacturing. Several methods are available to the engineer to accomplish this, including parts and material selection, derating, stress-strength analysis, use of technology, simplification, and redundancy. Generally combinations of these methods will be utilized in the design process as trade-offs are made between performance and costs. Each of these methods is discussed below.

8.3.1 Parts and Material Selection

Often the designer can choose between selecting standard parts and manufacturing specialized parts having perhaps greater tolerances and reliability. The trade-off is usually in cost, but ease of repair, parts availability, energy requirements, weight, and size may also be considerations. Historical databases can assist in determining relative reliabilities among competing parts. For example, *Military Handbook: Reliability Prediction of Electronic Equipment* [1986] provides detailed information on various electronic parts and their failure rates.

Knowledge of material properties and the external stresses the system will experience is important. Stress is typically measured in pounds per square inch (psi) or megapascals (MPa) where 1 MPa $= 10^6$ Newtons per meter squared (N/m^2). The mechanical properties of materials such as metals, polymers, ceramics, and composites, include tensile strength, hardness, impact value, fatigue life, and creep. They will be discussed in the following.

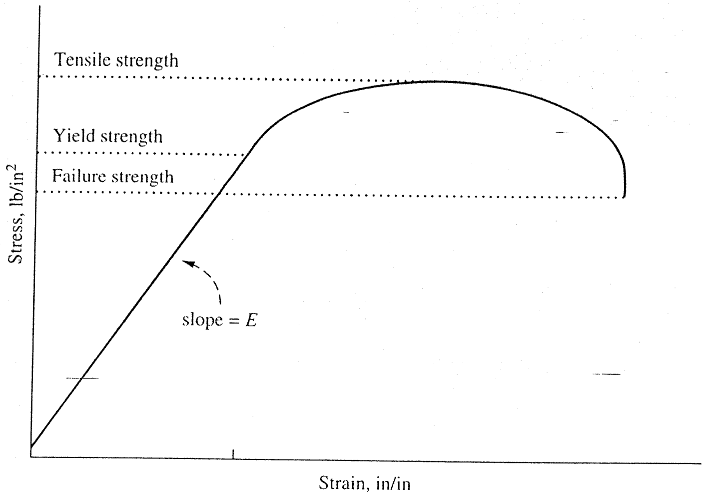


FIGURE 8.4
A stress-strain curve.

Tensile strength

Tensile strength is the ability to withstand a tensile or compressive load. Material will typically deform first elastically and then plastically. The deformation is elastic if after the load has been removed, the material returns to its original shape. When a material is stressed beyond its elastic limit, a permanent, or plastic, deformation results. See Fig. 8.4. The elastic range is represented by Hooke's law: stress $= E \times$ strain, where E, the slope of the stress-versus-strain line, is the modulus of elasticity (in psi or MPa) of the material. Beyond the elastic limit (yield strength) this relationship is nonlinear. At some point beyond the maximum stress level (tensile strength), the material will fracture. Empirically derived stress-strain diagrams may be utilized for specific materials. Strain is dimensionless since it represents the change in length per original length resulting from the load application. Yield strength is therefore the stress level at which plastic deformation occurs on a stress-strain curve.

Hardness

Hardness is the resistance of material to the penetration of an indenter. Hardness measurements are useful in analyzing the service wear of material. Hardness may be measured by several different relative scales, including Brinell (B_{hn})(kg/mm^2), Rockwell (R), and Vickers (V_{hn}). Specific tests are used in obtaining numerical values that are made readily available in published handbooks.

Impact value

Impact value is a measure of the toughness of the material under sudden impact. *Toughness* refers to the amount of energy absorbed before fracture. Impact testing will determine the ability of the material to withstand a sudden dynamic force. Since temperature may have a significant effect on the fracture behavior of the material, graphs of impact energy versus temperature have been established for certain types of material. For example, steel will become very brittle in very cold temperatures.

Fatigue life

Fatigue life is the number of cycles until failure for a part subjected to repeated stresses over an extended period of time. The fatigue strength is generally less than would be observed under a static load. Fatigue testing results in experimental data relating the number of cycles to failure (N) to the magnitude of the cyclical stress (S). Certain materials, such as steel, have an infinite life below a specific stress level (endurance limit). The fatigue strength is the maximum stress amplitude for a specified number of cycles until failure. Mathematically, an S-N curve (Fig. 8.5) may take the form $N = cS^{-m}$, where $c > 0$ and $m > 0$ are constants determined experimentally in laboratory tests that duplicate the amplitude, frequency, and pattern of specific stresses.

Creep

Creep is the progressive deformation of material under a constant stress. At temperatures within 40 percent of its absolute melting point, a metal or alloy begins to elongate continuously under a constant load until a fracture occurs. Creep is a design consideration when the component will be operating at moderate or high

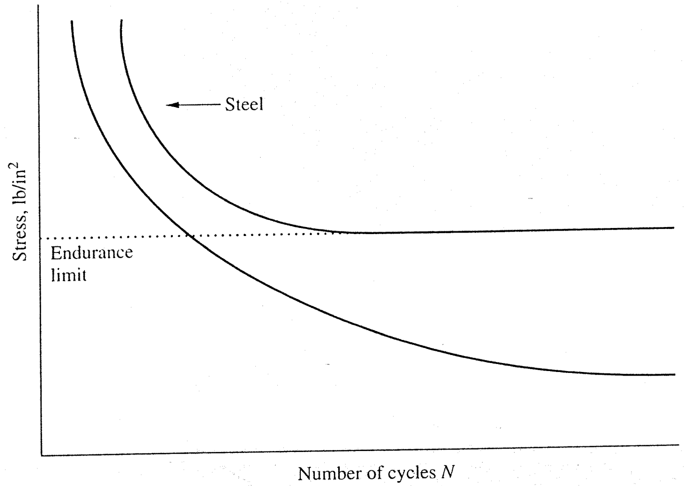


FIGURE 8.5
An S-N curve.

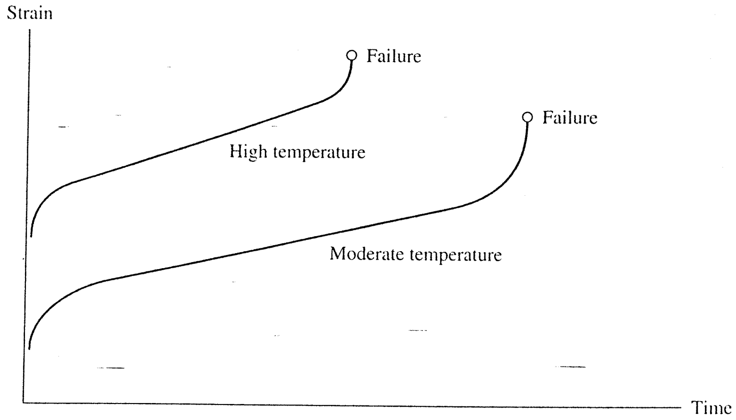


FIGURE 8.6
Typical creep curves at constant load.

temperatures. Andrade's formula, presented in Example 7.20, relates the strain to the time under the load for a particular material as a function of temperature. See Fig. 8.6.

The physical design of a part can affect its fatigue properties. For example, any irregularities or discontinuities such as seams, grooves, holes, and sharp corners may have weaker stress points and therefore become the site of a fatigue crack. Increasing the amount of material subject to wear or fatigue, improving the properties of the material used in a part, replacing one material with another, and redesigning the geometry of the part are options available for improving part reliability. Other nonmechanical properties of material that may be important design considerations include heat capacity, thermal conductivity, electrical resistivity, and magnetic permeability. Additional details on the properties of materials may be found in Callister [1994] and Dieter [1991].

8.3.2 Derating

Derating consists of using a component under stress significantly below its rated value. It has been most beneficial when applied to electronics, in which case the designed voltage or current strength of the part is well above the normal operating level. *Military Handbook: Reliability Prediction of Electronic Equipment* [1986] provides derating curves similar to the example in Fig. 8.7. The values given in the legend are the ratios of the applied voltage to the rated voltage. The covariate models discussed in the previous chapter may be useful for establishing curves such as these.

Voltage and temperature are common derating stresses for electrical components. Other types of derating stresses can be found in mechanical systems. For example, Ploe and Skewis [1990] present the following general formula for estimating the failure rate of a gear:

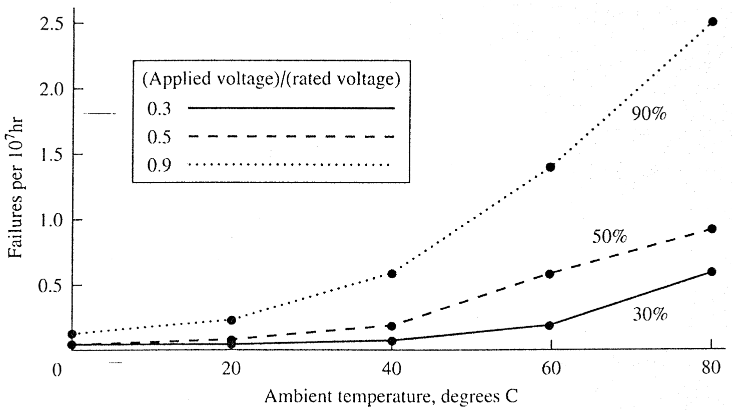


FIGURE 8.7
Examples of derating curves.

$$\lambda = \lambda_b \left(\frac{s}{s_d}\right)^{0.7} \left(\frac{L}{L_d}\right)^{4.69} \left(\frac{\nu_s}{\nu}\right)^{0.54} \left(\frac{c}{c_s}\right)^{0.67} \left(\frac{T}{T_s}\right)^{3}$$

where λ_b = base failure rate specified by the manufacturer
s = operating speed
s_d = design speed
L = operating load
L_d = design load
ν = viscosity of lubricant used
ν_s = viscosity of specification lubricant
c = concentration of contaminants
c_s = standard contamination level
T = operating temperature
T_s = specification temperature

In addition to the above, other factors that measure the effect of misalignment, vibration, and shock on the failure rate may be included.

8.3.3 Stress-Strength Analysis

When abnormal loads that may cause a failure are a possibility and a concern, then a probabilistic evaluation of the magnitude of the stress in comparison with the designed strength may be required. This method makes use of the techniques discussed in Chapter 7. Often this may be in response to a system safety analysis to ensure there are adequate safety margins designed into the system. There are four major categories of stress: electrical, thermal, mechanical, and chemical. Stresses can be environmental or operating; they include electrical loads, temperature, vibration, and humidity. There are two general design approaches: select parts and material with sufficient strength to withstand the maximum possible load, or protect the part against excessive stresses. Examples of the latter approach are the use of fuses and surge protectors in electrical circuits, shielding of electromagnetic radiation, overflow tanks in fluid systems, relief valves in pressurized systems, fans, insulation and shielding against heat and thermal shock, mountings to protect against mechanical shock and vibration, control rods in nuclear reactors, and modularization and sealing against dirt, salt air, or contaminants. The control rods are an example of a fail-safe design, in which a failure results in a safe condition (shutdown of the reactor). Other examples of fail-safe designs are street crossing signals that blink red and yellow when there is a failure (other than a complete loss of power) and automatic shutoff throttles on locomotives and lawnmowers.

The traditional engineering approach is to design safety margins, or safety factors, into the equipment. The safety factor (SF) is defined to be the ratio of the capacity of the system to the load placed on the system. The safety margin (SM) is the difference between the system capacity and the load. Failure will occur if the safety factor is less than 1 or the safety margin becomes negative. This is often a deterministic approach that ignores the variability present in both the loads placed on a system and the system's ability to react to the load. Variability may occur as a result of differences in the composition of the material, manufacturing, handling, use, and environment. Laboratory tests designed to measure yield strength, tensile strength, and fatigue life often report "average" values or "best-fit" curves drawn through data points. Occasionally both the average and the variability (standard deviation) may be recorded. Without an understanding of the variability present, the safety factors and safety margins designed into a system have to be quite large. However, it is not unusual to see safety factors in the range of 1.2 to 4 recommended (with perhaps 2 being a good "average") in traditional engineering texts. This may or may not be adequate, depending on the distribution of the strength or the load. To see this, let X and Y be random variables representing the stress placed on the system and the strength of the system, respectively. Then

$$\text{SF} = \frac{Y}{X} \qquad \text{and} \qquad \text{SM} = Y - X$$

Assuming that stress and strength are lognormal and the safety factor is defined as the ratio of the median strength m_y to the median stress m_x, the probability of a system failure as a function of the safety factor is given by Eq. (7.17), where

$$\Pr\{\text{SF} < 1\} = 1 - \Phi\left(\frac{\ln \text{SF}}{\sqrt{s_y^2 + s_x^2}}\right) \qquad \text{where SF} = \frac{m_y}{m_x}$$

Figure 8.8 graphs the probability of a failure as a function of the safety factor for several values of the radical in the denominator of the above expression. If a safety factor approach is to be used in the design process, these probabilities along with the accuracy of the measurements of stress and strength, the severity of a failure, and the economics involved should be considered.

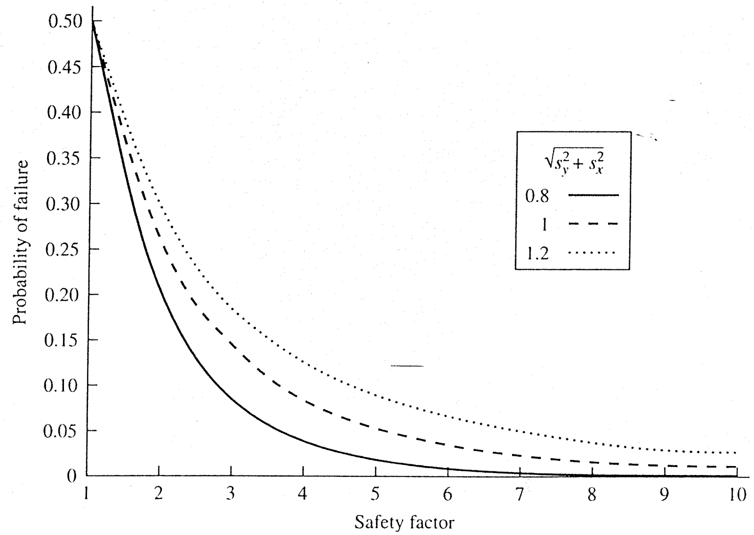


FIGURE 8.8
The probability of a failure versus the safety factor (lognormal distribution).

8.3.4 Complexity and Technology

The number of parts in a system is one measure of system complexity. Since most parts will be serially related with respect to the system reliability, each part is required to have a very high reliability. Any design alternative that can reduce the number of parts can lead to a significant improvement in reliability. As illustrated in Table 5.1, component reliabilities above 0.999 are necessary for systems having as few as 10 components in order to achieve a 0.99 system reliability. Component reliability increases to 0.9999 for a 100-component system. Part counts may be reduced by designing parts to serve more than one function or use. For example, a single switch may apply power to an entire system instead of a separate switch applying power to each powered component. It may also be possible to eliminate or reduce some function along with its components. Closely related to parts minimization is minimization of part variation. By using common parts and components, material quality and manufacturing tolerances can be better controlled.

When alternative technologies are available, the design engineer has additional flexibility in meeting the design objectives. Examples include electromechanical devices versus solid state devices, ink-jet versus laser versus impact printing, light-emitting diodes versus filament devices versus liquid crystals, discrete electronics components versus integrated circuits, and digital display versus analog display.

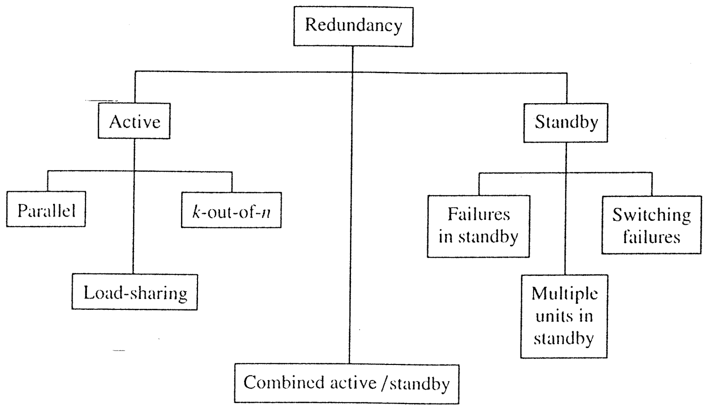


FIGURE 8.9
A classification of redundancy.

8.3.5 Redundancy

Redundancy may play an important role in the design process, especially when individual component reliabilities have already been established through an existing design or as a result of various uncontrollable failure modes. When it is impossible to achieve the desired component reliability through inherent component design, redundancy may provide the only alternative. In addition, redundancy may allow for increased reliability against external environmental stresses. Design trade-off analysis should consider the increased costs of additional components, the size or weight added to the system, and possibly the increase in repair and preventive maintenance necessary to maintain two or more components rather than one.

Redundancy includes both active and standby units. There may be duplicate active units with all operating and only one required to survive, or there may be the more general k-out-of-n redundancy. See Fig. 8.9. Failed units may also be repairable while the redundant unit or units are operating. Because the analysis of these various configurations was addressed in Chapters 5 and 6, they will not be discussed further here. Instead, we will look at active redundancy as a cost-performance trade-off by considering the following optimization problem.

Redundancy optimization

We will assume that the component reliabilities have been determined and that further improvement in system reliability is desired. The number of active redundant components is the relevant design variable. If a unit cost is associated with each component, an optimization scheme for allocating redundancy among the various components is possible. Assuming a serial relationship among a set of m independent components within a system, let

$R_i(t)$ = (known) reliability of component i at time t
n_i = number of parallel components i (decision variables)
c_i = unit cost of component i
B = budget available for additional units (redundancy)

The problem is to find values for n_i so that

$$\max \prod_{i=1}^{M} [1 - (1 - R_i(t))^{n_i}] \tag{8.19}$$

$$\text{subject to } \sum_{i=1}^{m} c_i n_i \le B + \sum_{i=1}^{m} c_i$$

The summation on the right side of the inequality accounts for the sunk costs necessary to have at least one of each component in the design. Marginal analysis may be used to solve this problem if the (natural) logarithm of the reliability function is maximized rather than the function itself.[4] Therefore the objective function becomes

$$\max \sum_{i=1}^{m} \ln[1 - (1 - R_i(t))^{n_i}] \tag{8.20}$$

Eliminate the argument from $R_i(t)$, since the analysis is performed for a specified time t, and let

$$\Delta_i = \frac{\ln[1 - (1 - R_i)^{n_i+1}] - \ln[1 - (1 - R_i)^{n_i}]}{c_i}$$

Then the marginal analysis consists of the following steps:

1. Set $n_i = 1, i = 1, 2, \ldots, m$, and set cost = 0.
2. Compute $\Delta_i, i = 1, 2, \ldots, m$.
3. Find $\max\{\Delta_1, \Delta_2, \ldots, \Delta_m\}$; call it Δ_k.
4. Set cost = cost + c_k.
5. If cost $< B$, then set $n_k = n_k + 1$, recompute Δ_k, and go to step 3; otherwise, stop.

The marginal values Δ_i represent the increase in the logarithm of the component reliability per dollar investment in the ith component. At each iteration the component with the largest (current) marginal value is selected for an additional redundant unit. The process is repeated until the budget target is met. Since the budget may not be precisely met by the last component to be added, the final iteration may require selecting an alternate component having a smaller unit cost. The component having the largest marginal value with a unit cost that will satisfy the budget is selected.

EXAMPLE 8.4 An engineer has an \$850 budget (per system) to be used to increase the reliability of a four-component series system. The reliability of each component at the

[4]Marginal analysis requires separability of the variables to ensure an optimal solution. If logarithms are used, the terms are additive rather than multiplicative. Since the logarithmic transformation is monotonically increasing, the optimal solution is not affected.

desired system lifetime has been established through a reliability growth testing program with the following results:

Component	Reliability	Unit cost
1	0.90	\$100
2	0.85	150
3	0.90	50
4	0.95	300

The objective is to maximize the system reliability through redundancy subject to the budget of \$850.

Solution. The initial system reliability is $0.90 \times 0.85 \times 0.90 \times 0.95 = 0.654$.

Step 1: $n_1 = n_2 = n_3 = n_4 = 1$, and cost $= 0$.

Step 2: $\Delta_1 = 0.0009531$, $\Delta_2 = 0.000931746$, $\Delta_3 = 0.001906$, and $\Delta_4 = 0.00016263$.

Steps 3–5:

Iteration	Max Δ_i	k	Cumulative cost	New Δ_k	System reliability
1	0.00190600	3	50	0.0001810	0.719
2	0.00095310	1	150	0.0000905	0.791
3	0.00093175	2	300	0.0001292	0.910
4	0.00018100	3	350	0.0000180	0.918
5	0.00016263	4	650	0.0000079	0.964
6	0.00012920	2	800	0.0000192	0.983

The next component to be included is component 1. However, the unit cost will result in the budget being exceeded. The only component that will keep the cost within budget is component 3. Therefore, the final solution is $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, and $n_4 = 2$. The final system reliability is

$$R = [1 - (1 - 0.90)^2][1 - (1 - 0.85)^3][1 - (1 - 0.90)^4][1 - (1 - 0.95)^2] = 0.984$$

This last step resulted in a very small increase in reliability for the dollars invested.

Because of the decreasing returns, it may be desirable to terminate the algorithm before the budget is reached. Although the above problem maximized reliability subject to a cost constraint, the constraint could have just as easily been a weight or volume constraint. The algorithm would not have changed. In addition to using marginal analysis, optimal redundancy problems may be solved using a generalized Lagrangian multiplier technique or dynamic programming. The theory behind the use of marginal analysis in optimization may be found in Fox [1966; 1970], Lawler and Bell [1966], and Proll [1976].

8.4 FAILURE ANALYSIS

One formalized design process with an objective of improving the inherent reliability is failure mode and effect analysis (FMEA), or failure mode, effect, and criticality analysis (FMECA). This is an iterative process that influences design by identifying failure modes, assessing their probabilities of occurrence and their effects on the system, isolating their causes, and determining corrective action or preventive measures. It is often performed as a bottom-up analysis, though it may be applied at any level in which there is sufficient definition to provide the necessary data. Therefore, an FMECA program should be initiated early in the design phase. In the reliability design process, FMECA provides a design tool that measures progress toward the reliability goals and indicates areas for redesign. It is an inductive process in which individual failures are generalized into possible failure modes. Through early (in the design process) identification of significant failure modes, they can be eliminated or their probability reduced. Additional information on FMECA to that given below may be found in *Military Standards: Procedures for Performing a Failure Mode, Effects, and Criticality Analysis* (1980). Typical steps in conducting an FMECA include system definition, identification of failure modes, determination of cause, assessment of effect, classification of severity, estimation of probability of occurrence, computation of criticality index, and determination of corrective action. They will be discussed in what follows.

8.4.1 System Definition

The objective of this first step is to identify those system components that will be subject to failure. A functional and physical (hardware) description of the system provides the definitions and boundaries for performing the analysis. A functional description can be represented as a functional flow diagram consisting of blocks defining *what* is to be done along with the interfaces between the blocks. A functional analysis provides the initial description of the system without regard to *how* the system will operate and be maintained. It should be part of the preliminary system design. The reader interested in further detail on the general systems engineering process is referred to Blanchard and Fabrycky [1990]. The physical description of the system is represented by an indenture diagram showing subassemblies, components, and parts along with their hierarchical relationships. The level of detail available in defining the system will depend upon how early in the design phase the FMECA is initiated. As the system evolves from preliminary design to detailed design and development, more detailed functional analysis, schematics, and hardware specifications will be utilized. In order to define failures, acceptable performance specifications under expected operating and environmental conditions must be determined. Using the functional flow and indenture diagrams, a reliability block diagram will be constructed and used as the basis for performing the analysis. The reliability block diagram may be based upon the hardware configuration, the functional analysis, or a combination of the two. The hardware approach is usually a bottom-up analysis, whereas the functional approach requires a top-down analysis of the system.

8.4.2 Identification of Failure Modes

Failure modes will be identified either by component (hardware approach) or function. Through development and reliability testing and the analysis of the reliability block diagrams, observed and predicted failure modes are identified and described. Failure modes are the observable manners in which a component fails. Examples include shorts, opens, ruptures, power losses, fractures, being out of tolerance, and loss of output. Failures may also occur as a result of a premature event, failure to operate or cease operation at a prescribed time, intermittent operations, or degraded performance. The failure mode description should include the operational and environmental conditions present when the failure occurs.

8.4.3 Determination of Cause

For each failure mode an assessment is made as to the probable cause or causes. Here are some specific examples of causes:

Abnormal stress. This is usually external or environmental, but it could be an internal power surge, for example.
Mechanical stress. Continued vibration may loosen fittings, for example.
Contamination. Dirt can cause electrical failure.
Evaporation. Filaments age because of filament molecules evaporating.
Fatigue. Physical changes in material may result in fracture.
Friction. This is a common cause of failures in belts, gears, and machinery in general.
Temperature cycling. Repeated expansion and contraction will weaken material.
Aging and wearout. This is not a prime cause, but it reflects prolonged exposure to other causes.
Substandard or defective parts. These reflect poor quality control during manufacture.
Operator-or maintenance-induced error. This is also known as human error.
Corrosion. This is chemical change that weakens material.

A failure mode may have more than one cause, and each cause should be attributed to the lowest component in the indenture, or level in the parts hierarchy, at which the failure occurs. To provide corrective action and eliminate critical failures, an understanding of the failure mechanism is necessary. Table 8.3 illustrates an analysis of several failure modes for their cause. Cause in these examples is defined by category (electrical, thermal, chemical, or mechanical), the cause itself, and the failure mechanism.

8.4.4 Assessment of Effect

The impact each failure has on the operation or status of the system is assessed. Effects may range from complete system failure to partial degradation to no impact on performance. When a failure occurs in a redundant unit, system performance will not be immediately affected, but system reliability will be reduced. Maintenance capability and system safety may also be affected.

TABLE 8.3
Analysis of failure mode causes

Failure mode	Category	Cause	Failure mechanism	Possible corrective action
Capacitor short	Electrical	High voltage	Dielectric breakdown	Derating
Failure of metal contacts	Chemical	Humid and salty atmosphere	Corrosion	Use of a protective casing
Connector fractures	Mechanical	Excessive vibration	Fatigue	Redesign of mountings

The last three steps discussed—identification of failure modes, determination of cause, and assessment of effect—are related as illustrated in Table 8.4.

8.4.5 Classification of Severity

Various severity classifications may be used. A severity classification is assigned to each failure mode to be used as a basis for ranking corrective actions. One of the more common classifications places failures in one of the following four categories:

Category I: Catastrophic. Significant system failure occurs that can result in injury, loss of life, or major damage.
Category II: Critical. Complete loss of system occurs; performance is unacceptable.
Category III: Marginal. System is degraded, with partial loss in performance.
Category IV: Negligible. Minor failure occurs, with no effect on acceptable system performance.

TABLE 8.4
Failure mechanisms, modes, and effects

Failure mechanism*	Failure mode	Failure effect
Corrosion	Failure in tank wall seam	Tank rupture
Manufacturing defect in casing	Leaking battery	Failure of flashlight to light
Prolonged excessive vibration and fatigue	Break in a motor mount	Loss of engine power and excessive noise
Friction and excessive wear	Drive belt break	Shutdown of production line
Contamination (dust and dirt)	Loss of contact	Circuit board failure
Evaporation	Filament breaking	Light bulb burnout
Prolonged low temperatures	Brittle seals	Leakage in hydraulic system

*The failure mechanism causes the failure mode, which causes the failure effect.

8.4.6 Estimation of Probability of Occurrence

Initial probability estimates will be based on the reliability specification and allocation, experience (past history), existing databases, *Military Handbook: Reliability Prediction of Electronic Equipment* [1986] (for electronic parts), and comparability with components and parts having known reliabilities. As development progresses, functional and reliability testing will provide an alternative source for these probabilities. Reliability testing will be discussed at length in Chapters 13 and 14. The probability of occurrence will be based on the expected number of occurrences of each failure mode over a specific time interval. This interval may be a mission time, a scheduled maintenance interval, or the system design life. Using the reliability block diagram, one may group these probability estimates by component and roll them up to higher component levels, including the system level. This will then provide an assessment as to whether the design meets the reliability specifications.

When sufficient data does not exist for quantifying the probability of occurrence, *Military Standard: Procedures for Performing a Failure Mode, Effects, and Criticality Analysis* [1980] provides the following qualitative grouping of failure mode frequencies over the operating time interval:

Level A: Frequent. High probability of failure ($p \geq 0.20$)
Level B: Probable. Moderate probability of failure ($0.10 \leq p < 0.20$)
Level C: Occasional. Marginal probability of failure. ($0.01 \leq p < 0.10$)
Level D: Remote. Unlikely probability of failure ($0.001 \leq p < 0.01$)
Level E: Extremely unlikely. Rare event ($p < 0.001$)

8.4.7 Computation of Criticality Index

This is a quantitative measure of the criticality of the failure mode that combines the probability of the failure mode's occurrence with its severity ranking. For each severity classification, the criticality index is computed for each of the corresponding failure modes. The result is a rank ordering of failure modes within each severity classification. The index may be defined as follows:

$$C_k = \alpha_{kp}\beta_k\lambda_p t \tag{8.21}$$

where C_k = the critical index for failure mode k
α_{kp} = the fraction of the component p's failures having failure mode k (that is, the conditional probability of failure mode k given component p has failed)
β_k = the conditional probability that failure mode k will result in the identified failure effect
λ_p = the failure rate of component p
t = duration of time used in the analysis

The conditional probability, β_k, is a subjective estimate that may be quantified within the following guidelines [*Military Standards: Procedures for Performing a Failure Mode, Effects, and Criticality Analysis, 1980*]:

Failure effect	β
Certain	$\beta = 1.00$
Probable	$0.10 < \beta < 1.00$
Possible	$0 < \beta \leq 0.10$
No effect	$\beta = 0$

For a given p, the sum of α_{kp} over all its failure modes would normally equal 1. This probability may be derived from the estimated probability of occurrences. Component failure rates are obtained from existing data sources or aggregations from the failure mode probability of occurrences. In the later case, $\alpha_{kp}\lambda_p$ is the probability of occurrence of failure mode k and the distinction between the two probabilities is not necessary. The rationale for this distinction is to allow for the component failure rate, λ_p, to be estimated from other sources (such as *Military Handbook: Reliability Prediction of Electronic Equipment* [1986]).

Criticality numbers may be computed for each component by summing all of the component's failure mode criticality indices. A separate number is computed for each component and severity combination. A criticality matrix is then constructed, as shown in Fig. 8.10. The cells will contain the corresponding failure modes or

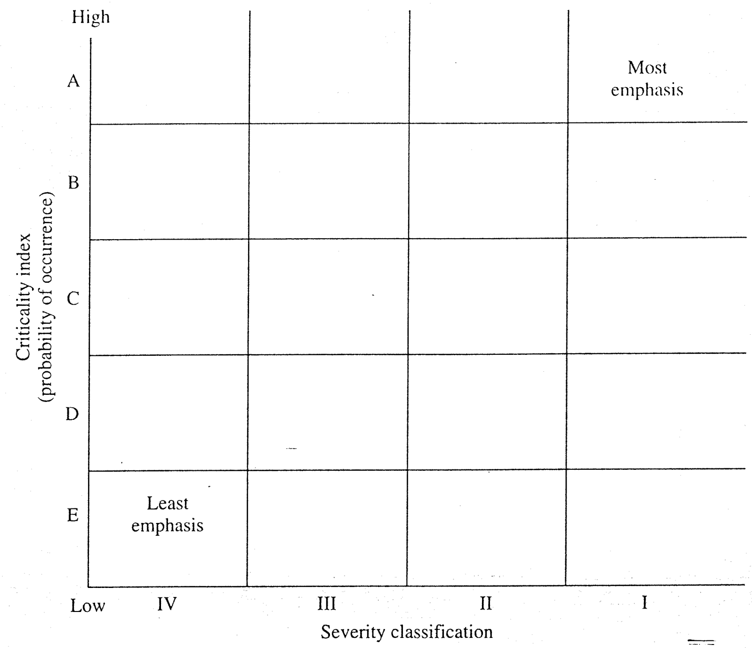


FIGURE 8.10
Failure mode classification matrix.

components. In a qualitative analysis the critical index on the vertical axis could be replaced with the qualitative assessment of the probability of occurrence. As we move from the lower left corner of the matrix to the upper right corner, the criticality and severity of the failure mode become greater. The matrix then provides a guide to be used during design for eliminating or mitigating failures.

8.4.8 Determination of Corrective Action

This is very dependent on the problem. Many times a solution will suggest itself as a result of a thorough analysis of the failure mode along with the identification of the cause and the failure mechanism. Obviously, those failure modes having a high criticality index and severity classification should receive the most attention. Design activity should be oriented toward removing the cause of the failure, decreasing the probability of occurrence, and reducing the severity of the failure.

The above analysis can be summarized in a worksheet, as shown in Table 8.5, in which each component has an operating time of 1000 hr in the computation of the criticality index. The worksheet is completed for all failure modes and all components, and a subtotal is computed for each combination of component and severity class.

An FMECA can be accomplished from several perspectives. The primary focus may be to improve reliability during design, but the technique may also be used in addressing system safety, availability, maintainability, or logistics support. Additional detail on FMECA can be found in the *Handbook of Reliability Engineering and Management* [Ireson and Coombs, 1988].

8.5 SYSTEM SAFETY AND FAULT TREE ANALYSIS

Reliability and product safety are obviously related. Safety can be broadly defined as the avoidance of conditions that can cause injury, loss of life, or severe damage to equipment and possibly the surrounding environment. Therefore the focus here is on failures that may create safety hazards. The objective is to determine during design how these failures are likely to occur, to estimate their probability of occurrence, and to take corrective action. Often safety-related failure modes have a low probability of occurrence and are therefore difficult to estimate. Reliability testing at the system level may fail to generate an unsafe condition. Additionally, because of designed safety features with backup or redundancy, a system safety failure is usually caused by a combination of events. For example, a combination of equipment failure, human error, and an alarm failure may be necessary before a boiler begins to overheat, causing pressure buildup.

TABLE 8.5
FMECA worksheet

Component	Failure mode	Level of probability of occurrence	Failure cause	Failure effect	Severity class	Failure rate λ_p	β_k	α_{kp}	Criticality index
Receiver	Open circuit	C	Shock	Signal loss	II	$1.2 \times 10 - 6$	1.0	0.05	60.0×10^{-6}
	Short circuit	D	Contaminant leak in battery	Component damage	II		0.9	0.01	10.8×10^{-6}
	Loss of power	B		Signal loss	II		0.8	0.12	115.2×10^{-6}
					Subtotal				186.0×10^{-6}
	Bad speaker	B	Vibration	Distortion	III		0.8	0.07	67.2×10^{-6}
	. . .	. . .	. . .	. . .			. . .	. . .	. . .
					III				
					Subtotal	3.4×10^{-5}			
Transmitter	. . .								

8.5.1 Fault Tree Analysis

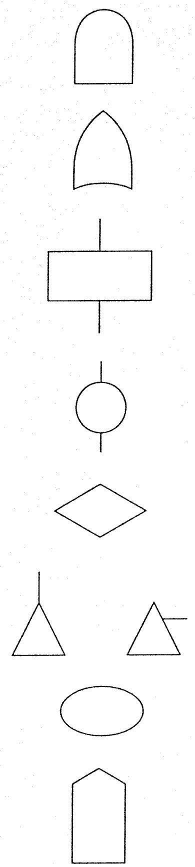
A useful tool in performing a system safety analysis is fault tree analysis. A fault tree analysis is a graphical design technique that provides an alternative to reliability block diagrams. It is broader in scope than a reliability block diagram and differs from reliability block diagrams in several respects. It is a top-down, deductive analysis structured in terms of events rather than components. The perspective is on faults rather than reliability. All failures are faults, but not all faults may be considered failures. For example, a human error resulting in an incorrect switch being set would be treated as a fault although it would not normally be an inherent equipment failure mode. An advantage of focusing on failures is that failures are usually easier to define than nonfailures and there may be far fewer ways in which a failure can occur, as opposed to the numerous ways in which nonfailures can occur. The focus is usually on a significant failure or a catastrophic event, which is referred to as the *top event* and appears at the top of the fault tree diagram. The qualitative analysis consists of identifying the various combinations of events that will cause the top event to occur. This may be followed by a quantitative analysis to estimate the probability of occurrence of the top event.

There are four major steps to a fault tree analysis:

1. Define the system, its boundaries, and the top event.
2. Construct the fault tree, which symbolically represents the system and its relevant events.
3. Perform a qualitative evaluation by identifying those combinations of events that will cause the top event.
4. Perform a quantitative evaluation by assigning failure probabilities or unavailabilities to the basic events and computing the probability of the top event.

Symbols frequently used in fault trees include those presented in Fig. 8.11. A typical fault tree has the structure shown in Fig. 8.12. In construction of a fault tree, the two logic gates, the OR gate and the AND gate, are used to relate the resultant, basic, and intermediate events, or faults, to the top event. Lower events are input to a gate, and a higher event is the gate's output. The type of gate determines whether all input events must occur for the output event to occur (AND gate) or whether only one of the input events must occur for the output event to occur (OR gate). The following example illustrates the use of the two gate types. For clarity, all fault trees will use the words AND and OR in place of the symbols shown in Fig. 8.11.

EXAMPLE 8.5 In Fig. 8.13(a) the OR gate indicates that a tank rupture is caused either by overpressure in a hot water tank or by an inherent fatigue failure in the wall of the tank. The fatigue failure is not developed further since it is represented as a basic event. On the other hand, overpressure is depicted as a resultant event. Overpressure is further developed in Fig. 8.13(b) through the use of the AND gate. If both excessive temperature and failure of a relief valve occur, then a tank rupture will result. In completion of the fault tree, both gates would be utilized as shown in Figure 8.13(c).



AND gate. A logic gate where an output event occurs only when all the input events have occurred. In Boolean algebra the output is the intersection ($\cap$) of the input events.

OR gate. A logic gate where an output event occurs if at least one of the input events has occurred. In Boolean algebra the output is the union ($\cup$) of the input events.

Resultant event. A fault event resulting from the logical combination of other fault events and usually an output to a logic gate.

Basic event. An independent elementary event representing a basic fault or component. The analysis ends with a basic event (there are no events below a basic event).

Incomplete event. An event that has not been fully developed because of lack of knowledge concerning its cause or because of its unimportance in the analysis.

Transfer-in and transfer-out. Used to link sections of the fault tree that are not contiguous or appear on separate pages.

Conditional event. A condition or restriction tied to a logic gate.

Normal event. A normally occurring event that is not a fault.

FIGURE 8.11
Fault tree symbols.

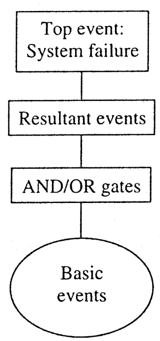


FIGURE 8.12
The general structure of a fault tree.

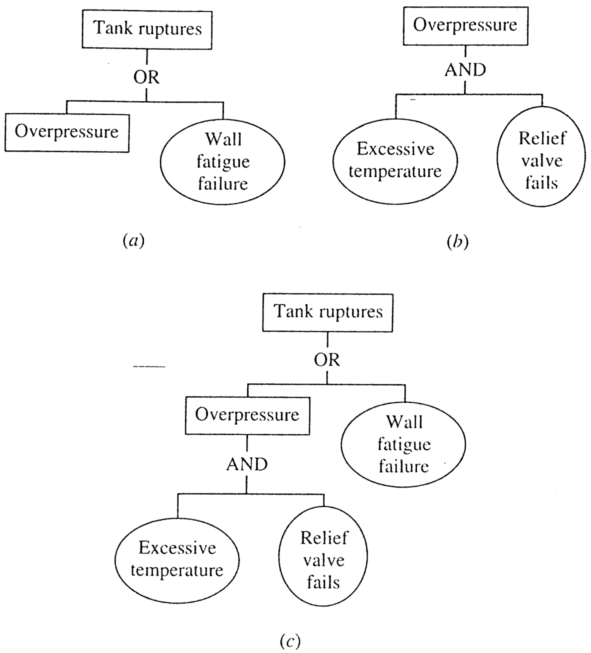


FIGURE 8.13
An example of the use of AND gates and OR gates.

Faults can be classified as primary, secondary, and command. A fault is primary if the component or part is functioning within its design parameters when an inherent failure occurs. A secondary failure occurs when an environmental stress or an excessive operational stress causes the failure (that is, it is not an inherent failure but an external event). A command fault is one that occurs as a result of a correct action being accomplished at a wrong time or place. For example, a command fault may occur when turning power on prematurely or turning off a cooling subsystem before the system has been shut down. Faults may also be classified as active or passive. Passive faults or events are typically related to a static transmitter of energy, material, loads, or signals such as pipes, bearings, structural beams, and wires. Active faults are related to dynamic events or components in operation, such as valves regulating flow, electrical switches, mechanical pumps, and relays or actuators. A component such as a valve may have a static failure mode in which a rupture occurs or an active failure mode in which the valve opens when it should remain closed. Active faults typically have probabilities of occurrence that are two to three times those of passive faults.

EXAMPLE 8.6 Construct a fault tree for the redundant manual and automated alarm system shown in Fig. 8.14.

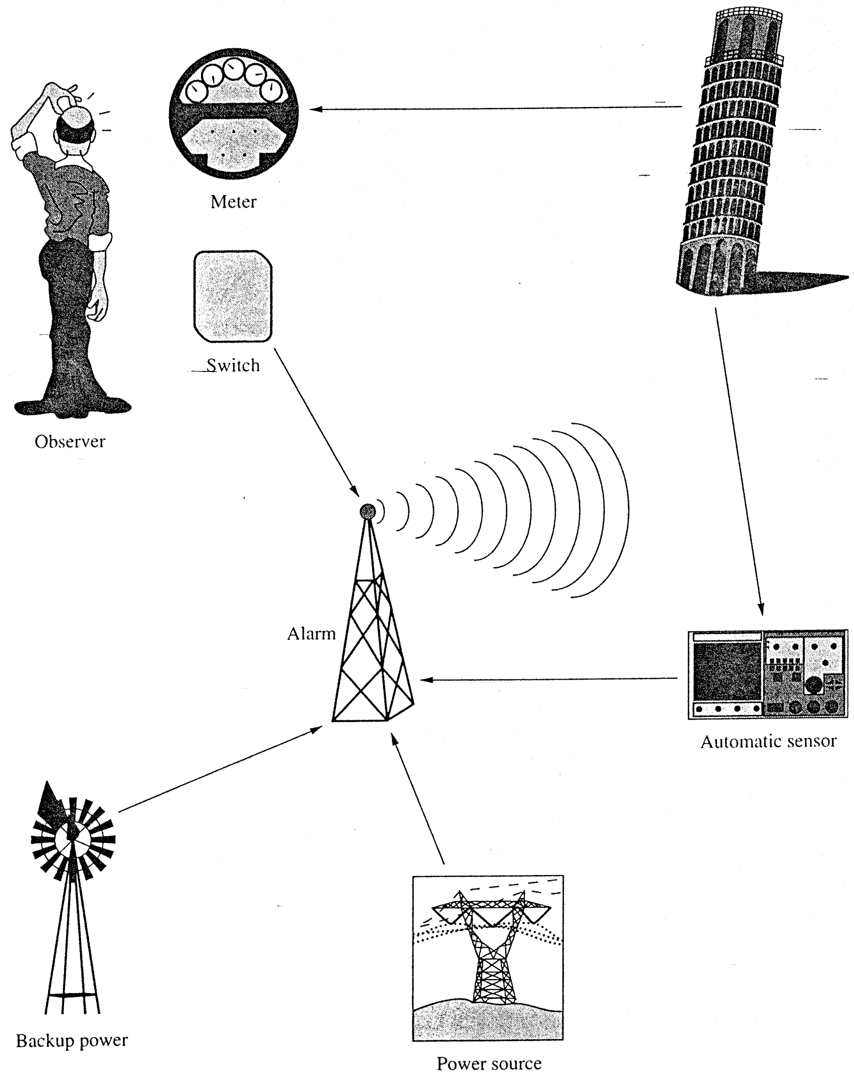


FIGURE 8.14
An example of an alarm system.

Solution. The fault tree may take the form shown in Fig. 8.15. From the fault tree, the top event, an alarm system failure, will occur if the alarm fails from either a basic (inherent) failure or a secondary failure or there is a power or sensor failure. The event D, secondary alarm failure, represents external failures to the alarm, perhaps resulting from a natural disaster. Both events D and I are diagrammed as incomplete events, presumably because

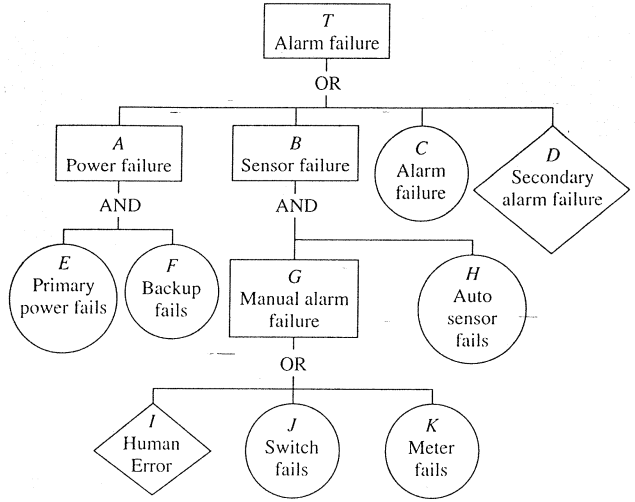


FIGURE 8.15
The alarm system fault tree.

there is incomplete data available to further define the events. The top event, T, can be written as

$$\begin{aligned} T &= A \cup B \cup C \cup D \\ &= (E \cap F) \cup (G \cap H) \cup C \cup D \\ &= (E \cap F) \cup [(I \cup J \cup K) \cap H] \cup C \cup D \end{aligned}$$

The objective of the Boolean algebra representation is to express the top event in terms of nonredundant basic events. Eliminating redundancies results in a simplified fault tree and should be accomplished prior to the quantitative analysis. In the above example there are no redundant (duplicate) events. Redundancy exists when the same event occurs more than once in the fault tree or when one event is a subset of another event. The following example illustrates the procedure.

EXAMPLE 8.7 For the fault tree of Fig. 8.16,

$$\begin{aligned} T = A \cup B &= A \cup (C \cap D) \\ &= A \cup [(E \cup F) \cap (E \cup A)] \\ &= A \cup [E \cup (F \cap A)] = A \cup E \end{aligned}$$

since $A \cup (F \cap A) = A$. Therefore the top event will occur if either the event A occurs or the event E occurs. At this point we can construct an equivalent fault tree, as shown in Fig. 8.17, that provides a much simpler analysis.

8.5.2 Minimal Cut Sets

Another form of qualitative analysis utilizes minimal cut sets. A cut set is a collection of basic events that will cause the top event. A minimal cut set is one

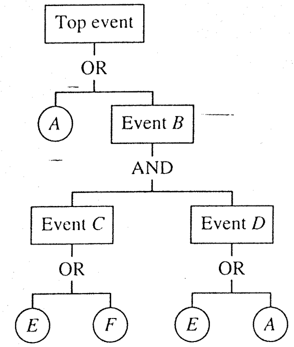


FIGURE 8.16
A fault tree with redundant events.

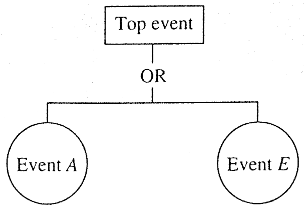


FIGURE 8.17
The equivalent fault tree.

with no unnecessary events. That is, all the events within the cut set must occur to cause the top event. Every fault tree has a finite number of minimal cut sets (since there are a finite number of events). A cut set can be characterized by the number of basic events comprising it. For example, a single event that will cause the top event is a singlet, and a two-event minimal cut set is a doublet. If M_i, $i = 1, 2, \ldots, k$, are all possible minimal cut sets, then

$$T = M_1 \cup M_2 \cup \cdots \cup M_k \tag{8.22}$$

where $M_i = E_1 \cap E_2 \cap \cdots \cap E_{ni}$ and E_i are basic events.

Minimal cut sets may be generated by a top-down expansion of events. Events input to OR gates are expanded by generating new rows (row-wise), and events input to AND gates are expanded by generating new columns (columnwise). This expansion continues until only basic events (or incomplete events) remain. At the completion, each row results in a cut set. Redundant cut sets, if present, may then be eliminated.

EXAMPLE 8.8 Consider the fault tree in Example 8.6 (Fig. 8.15):

Iteration		
1	2	3
A	E, F	E, F
B	G, H	I, H
C	C	J, H
D	D	K, H
		C
		D

Therefore $T = (E \cap F) \cup (I \cap H) \cup (J \cap H) \cup (K \cap H) \cup C \cup D$. An equivalent fault tree may be constructed using the minimal cut sets. The tree would have the general structure shown in Fig. 8.18.

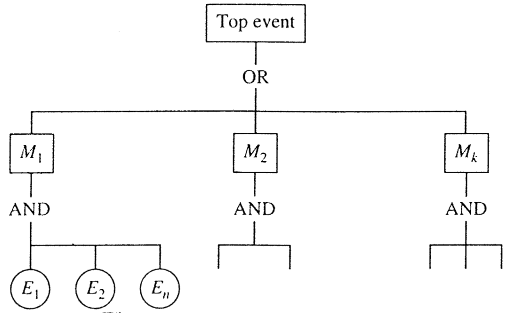


FIGURE 8.18
The minimal-cut-set fault tree.

EXAMPLE 8.9 For the fault tree shown in Fig. 8.16, the minimal cut set is found from the following:

Iteration				
1	2	3	4	5
A	A	A	A	A
B	C, D	E, D	E, E	E
		F, D	E, A	
			F, E	
			F, A	

Redundancies are eliminated in iteration 5 since $E \cap E = E, E \cap A \subseteq A, F \cap E \subseteq E$, and $F \cap A \subseteq A$. Therefore $T = A \cup E$, as found earlier when we applied Boolean algebra to eliminate repeated events.

In performing a qualitative evaluation of the minimal cut sets, one should place more importance on those sets having a single or double event since they are more likely to occur than cut sets having multiple events. For example, if a single event has a probability of occurrence on the order of 10^{-2}, then a double-event cut set could be expected to have a probability of occurrence on the order of 10^{-4}, and a triple-event cut set, a probability on the order of magnitude 10^{-6}. Therefore an objective would be to eliminate or minimize the failure probability of single-event cut sets. In general, ordering cut sets according to size will provide a qualitative measure of their importance. Events appearing in more than one cut set would also be candidates for elimination, especially if the cut sets were doublets or triplets. If a quantitative analysis is performed, those events in cut sets having the greatest probability of occurrence should receive the most attention. For simple systems such as those in the above examples, cut sets may be apparent from a system or fault tree diagram. However, for complex systems combinations of events leading to the top event are not as obvious. For such systems computer algorithms are available for performing this type of analysis.

8.5.3 Quantitative Analysis

Quantitative analysis consists of assigning failure probabilities, unavailabilities, failure-on-demand probabilities, or other measures to each basic event and then computing the corresponding measure for the top event. Obviously, this is easiest when the repeated events have been eliminated, since repeated events generate dependencies. If the top event has been defined in terms of the minimal cut sets (Eq. 8.22), then

$$P(T) = P(M_1 \cup M_2 \cup \cdots \cup M_k) = P(M_1) + P(M_2) + \cdots + P(M_k) \quad (8.23)$$

if the cut sets are mutually exclusive. Generally the events will not be mutually exclusive, and terms involving the probability of the intersection of two or more events must be included. Therefore, from Example 8.9,

$$P(T) = P(A \cup E) = P(A) + P(E) - P(A \cap E)$$

In general,

$$P(M_i) = P(E_1 \cap E_2 \cap \cdots \cap E_{ni}) = P(E_1)P(E_2) \cdots P(E_{n1}) \quad (8.24)$$

if the basic events are mutually independent. Therefore if the events A and E are independent,

$$P(T) = P(A) + P(E) - P(A)P(E)$$

and it is sufficient to estimate the failure probabilities of the basic events in order to find the probability of the top event. Equation 8.23 may be a useful approximation if the events are not mutually exclusive but their individual failure probabilities are quite small. This is because the remaining terms will consist (under independence) of products of the individual probabilities. If the probability of a basic event is on the order of 10^{-6}, the probability of the intersection of two or more events will be of order of magnitude 10^{-12} or smaller.

EXAMPLE 8.10 Using Eq. 8.23 as an approximation, we obtain the following for the fault tree in Example 8.6:

$$\begin{aligned} P(T) &= P\{(E \cap F) \cup [(I \cup J \cup K) \cap H] \cup C \cup D\} \\ &\approx P(E \cap F) + P[(I \cup J \cup K) \cap H] + P(C) + P(D) \\ &\approx P(E)P(F) + [P(I) + P(J) + P(K)]P(H) + P(C) + P(D) \end{aligned}$$

If each basic event has a probability of 0.01, then

$$P(T) \approx (0.01)^2 + (0.01 + 0.01 + 0.01)(0.01) + 0.01 + 0.01 = 0.0204$$

From both the qualitative and quantitative analysis, it is apparent that events C and D are the most important in causing an alarm system failure, and they should be considered for improved reliability. Although this conclusion may be obvious since the remainder of the basic events (sensor and power failures) have redundancy, for more complicated systems the insight provided by a detailed fault tree analysis can be considerable.

EXAMPLE 8.11 (EXAMPLE 5.7 REVISITED). An automobile braking system consists of a fluid braking subsystem (foot brake) and a mechanical braking subsystem (parking brake). Both subsystems must fail in order for the system to fail. The fluid braking subsystem will fail if the master cylinder or a hydraulic line fails (event M) or all four wheel braking units fail. A wheel braking unit will fail if either the wheel cylinder fails (events WC_1, WC_2, WC_3, WC_4) or the brake pad assembly fails (events BP_1, BP_2, BP_3, BP_4). The mechanical braking system will fail if the cable system fails (event C) or both rear brake pad assemblies fail (events BP_3, BP_4).

The fault tree may be constructed as shown in Fig. 8.19. The top event may be expressed in Boolean algebra terms of the basic events as

$$T = \{M \cup [(WC_1 \cup BP_1) \cap (WC_2 \cup BP_2) \cap (WC_3 \cup BP_3) \cap (WC_4 \cup BP_4)]\} \cap \{C \cup (BP_3 \cap BP_4)\}$$

Minimal cut sets can be formed from $\{M, C\}$, $\{M, BP_3, BP_4\}$, {wheel subsystem failure, C}, {wheel subsystem failure, BP_3, BP_4}. Wheel subsystem failure can be decomposed into 16 combinations of wheel cylinder and brake-pad assembly failures. In the latter case, four of these decompositions include failure of both BP_3 and BP_4. Therefore, only 12 unique cut sets are formed. Assuming that each basic event has approximately the same probability of occurrence, the cut sets involving wheel subsystem failures are less likely since they contain four or five events. Therefore, the focus

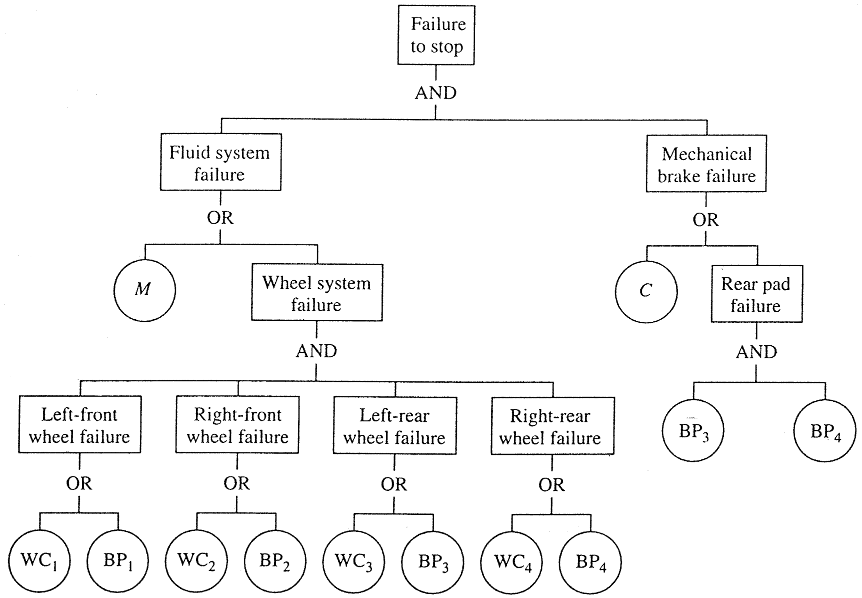


FIGURE 8.19
Fault tree for the automotive braking system.

of reliability improvement would be on the master cylinder subsystem and the cable subsystem.

The reader interested in more detail on fault tree analysis is referred to Dhillon and Singh [1981] or Roberts et al. [1981].

EXERCISES

8.1 Two alternative components are being considered for use in a fiber-optic cable communications system. One component has a unit cost of \$840 and a Weibull failure distribution with $\beta = 2$ and $\theta = 10{,}000$ hr. The other component has a unit cost of \$870 and an exponential failure distribution with an MTTF of 10,000 hr. It is estimated that the selected component will experience 2000 operating hours a year, and the system is being designed for a 20-yr life. Any component failures will require a replacement at the current unit cost. Assuming a 3 percent interest rate, which is the preferred component?

8.2 In the design of a space station, four major subsystems have been identified, each having a Weibull failure distribution with parameter values as given here:

Subsystem	Scale parameter, θ, yr	Shape parameter, β
Computer	3.5	0.91
Avionics	4.0	0.80
Structures	5.0	1.80
Life support	6.0	1.00

The reliability of the station must be 0.995 at the end of the first year. Determine the percentage increase in reliability for each of the major subsystems needed in order to reach the system goal. Assign equal reliability goals to all subsystems.

8.3 If redundancy is the only means of achieving further reliability growth for each of the subsystems in Exercise 8.2, what is the minimum number of redundant units of each necessary to achieve the component reliability goals?

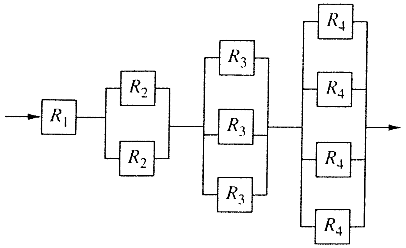
8.4 On the basis of the current average failure rate of each subsystem over the first year of the space station, apply the ARINC method to determine an average failure rate goal for each subsystem of Exercise 8.2. Assuming that reliability growth will improve only the characteristic life and not the shape parameter, what is the characteristic life goal for each subsystem?

8.5 Apply the AGREE allocation method to a personal computer system containing the following components:

Component	Parts count	Importance index	Operating time, hr/yr
System board	153	0.95	2000
Hard drive	28	0.90	1000
DC power pack	34	1.00	2000

The warranty program requires a reliability of 0.99 over the first year of use.

8.6 Allocate the system reliability goal of 0.95 to the components of the following reliability block diagram. Assume an equal allocation to each redundant subset. Each redundant subset contains identical components.



8.7 The following table (from *Military Handbook: Reliability Prediction of Electronic Equipment* [1986]) shows the effect of derating at various temperatures for a wire-wound resistor. The values are the number of failures per 10^6 hr.

Temperature, °C	Operating wattage/rated wattage					
	0.4	0.5	0.6	0.7	0.8	0.9
30	0.015	0.017	0.019	0.021	0.023	0.026
40	0.016	0.018	0.021	0.023	0.026	0.029
50	0.018	0.02	0.023	0.026	0.029	0.033
60	0.02	0.023	0.026	0.029	0.033	0.037
70	0.023	0.026	0.03	0.033	0.038	0.043

A system containing 73 resistors is being designed. With the selection of a system cooling fan, the operating temperature can be controlled:

Fan size	Unit cost	Operating temperature, °C
Small	$ 50	60
Medium	90	50
Large	160	30

Operating power consumption is 180 watts. The following resistors may be used in the design of the system:

Resistor	Rating, W	Unit cost
1	200	$1.00
2	225	1.20
3	300	2.00

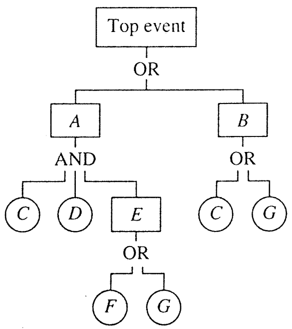
Select the fan size and resistor type that will provide the largest MTTF per dollar cost. The system of 73 resistors in series must have at least a 0.95 reliability after 10,000 operating hours. Assume constant failure rates.

8.8 Given a budget of $700 and the following data on three components that must operate in series, determine, using marginal analysis, the optimum number of redundant units. Compute the achieved reliability.

Component	Reliability	Unit cost
1	0.80	$200
2	0.90	100
3	0.95	75

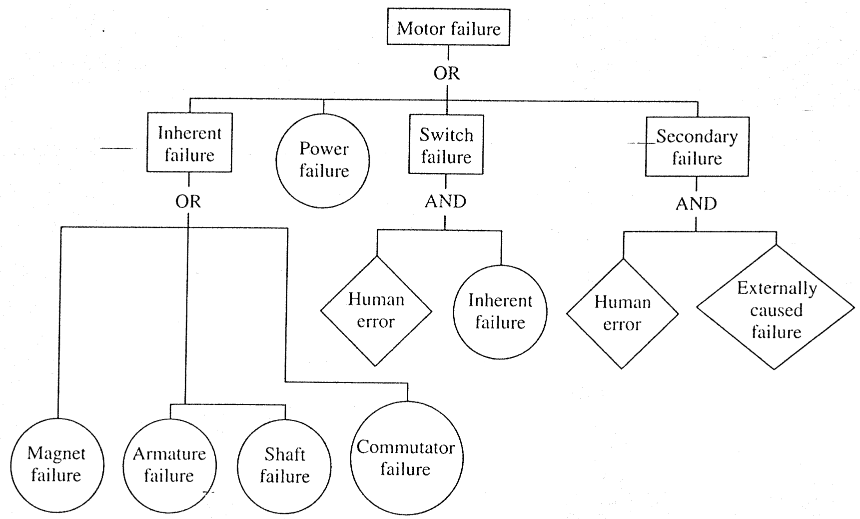
8.9 Establish a criticality index for the components in Exercise 8.5 assuming that the conditional probability β_k is equal to the importance index. Assume that failure modes equate to components.

8.10 Perform a qualitative analysis on the following fault tree by expressing the top event in terms of nonredundant basic events using Boolean algebra. If the probability of each basic event is 0.005 and events are independent, what is the probability of the top event?



8.11 A space probe to be launched by NASA will contain a module of three scientific experiments. Because of the high cost of the launch vehicle and the probe and the importance of the experiments, there will be redundant experimental packages within the module. Environmental stress testing has resulted in a (nonredundant) reliability estimate over the duration of the mission of 0.85, 0.90, and 0.95 for the respective experiments. There is a weight limitation of 350 pounds allocated to the module. How many redundant experiments of each type should be included in the module if their respective weights are 40, 30, and 50 pounds? Since a minimum of one unit of each experiment is required, 120 pounds of the 350 pounds allocated are already accounted for. The objective is to maximize the overall reliability of the module.

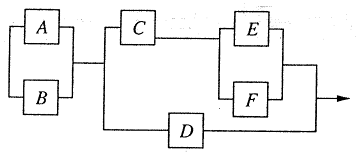
8.12 Consider the following fault tree.



(a) Express the failure of the top event in Boolean algebra terms consisting of basic events and incomplete events.
(b) Find the minimal cut set.
(c) Estimate the probability of the top event given that the probability of each basic event is 0.01 and that of each incomplete event is 0.02.

8.13 Construct a fault tree of a gas water heating system such that the top event is a safety-related failure.

8.14 Construct a fault tree from the following reliability block diagram such that the top event is a system failure and component failures are basic events. Express the top event in Boolean algebra terms of the basic events. If Pr{A} = Pr{B} = 0.9, Pr{C} = Pr{D} = 0.8, and Pr{E} = Pr{F} = 0.75, compute the probability of the top event. Compare your answer to that of Exercise 5.8(a).



8.15 Nether Fales, a reliability engineer, must decide which of two components to use in the design of a new product. According to the supplier, component A has a unit cost of \$225 and a Weibull failure distribution with a shape parameter of 1.7 and a characteristic life of 12 yr. Component B has a unit cost of \$245 and a constant failure rate of 0.11 failure per year according to the manufacturer's specifications. A failure of either component results primarily in the replacement of a comparable (same age) part. The average part cost of component A is \$40, and that of component B is \$35. In addition, if component A is selected, a special test unit must be purchased at a cost of \$4300 to be used in identifying the failed part. The product will have a 10-yr design life, at the end of which component A has a salvage value of \$60 and component B has a salvage value of \$40. Operating costs and other support costs are the same for either component. If 100 units of the product are to be manufactured, determine which component to use in the design of the product by comparing the life cycle costs. Assume a 5 percent effective discount rate.

8.16 Assuming that the load applied to a part is fixed at some value K and the design strength of the material used in the part has a Weibull distribution with a shape parameter $\beta = 0.8$, determine the probability of the part failing if a safety factor of 1.2 is used for establishing the part's strength. The safety factor is defined to be SF $= \mu/K$, where μ is the mean strength of the material used in manufacturing the part. Repeat your analysis for safety factors of 2 and 4. What conclusion would you make concerning the deterministic approach to using safety factors?

8.17 Fatigue testing of a steel alloy resulted in the following *S-N* curve:

$$N = (1.23 \times 10^{28})S^{-14.85}$$

where N is in 1000 cycles to failure and S is the stress amplitude in 1000 psi. The steel alloy is to be used in the design of an automobile engine that under normal use will experience 20 cycles per second under a stress amplitude of 35,000 psi. It is estimated that the typical driver will operate the vehicle 350 hr/yr. If the engine is being designed for a 10-yr life, is the selected material adequate?

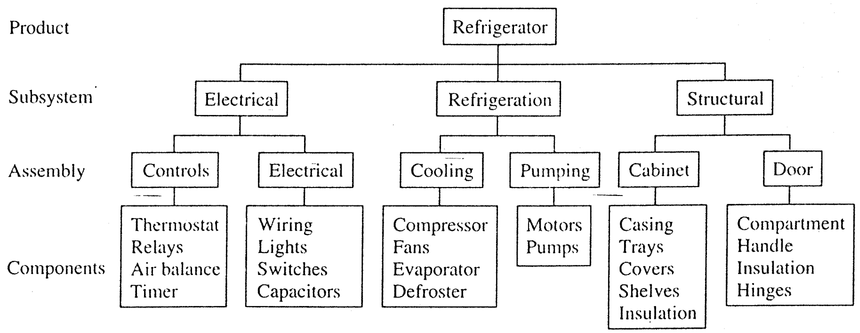
8.18 In an optimal reliability allocation based on Eqs. 8.9 and 8.10 where $C_i(x_i) = cx_i$, show that the solution is $x_i = \sqrt[n]{R^*} - R_i$.

8.19 A system is comprised of three components in series each having a Weibull failure distribution with the parameter as follows:

Component	Scale parameter, θ	Shape parameter, β
1	7,000	1.7
2	10,000	1.0
3	12,000	2.0

The cost of improving the current reliability, $C_i(x_i)$, is assumed to linear and equal for each of the components within the current (somewhat narrow) design range (see Exercise 8.18). The system reliability goal is 0.99 at 1000 hours of operation. Determine the percentage improvement required for each component in order to achieve the system goal.

8.20 The following is a hierarchical breakdown of a new refrigerator model. Discuss a scheme for establishing reliability goals at each indenture level assuming that the reliability is known for some components that are common to other products and models while other components are new with no previous failure history.



CHAPTER 9

Maintainability

The objective of this chapter is to characterize and quantify the repair or restoration of a failed item. We can distinguish between two general types of maintenance: reactive and proactive maintenance. Reactive maintenance is performed in response to unplanned or unscheduled downtime of the unit, usually as a result of a failure, whether it be internal (inherent) or external (for example, operator-induced). Proactive maintenance may be either preventive maintenance or predictive maintenance. *Preventive maintenance* is scheduled downtime, usually periodical, in which a well-defined set of tasks, such as inspection and repair, replacement, cleaning, lubrication, adjustment, and alignment, are performed. *Predictive maintenance* estimates, through diagnostic tools and measurements, when a part is near failure and should be repaired or replaced, thereby eliminating a presumably more costly unscheduled maintenance action. Proactive maintenance should be performed only when and to the extent it is cost-effective. It must reduce the number of unscheduled failures or extend the life of the item or both. It is generally assumed that a proactive maintenance action is less costly than a reactive maintenance action.

9.1 ANALYSIS OF DOWNTIME

When an item fails, it enters the repair process. The repair process itself can be decomposed into a number of different subtasks and delay times, as shown in Fig. 9.1.

Supply delay consists of the total delay time in obtaining necessary spare parts or components in order to complete the repair process. This time may consist of administrative lead times, production or procurement lead times, repair of the failed subcomponents themselves, and transportation times. To a large extent this time is influenced by the breadth and depth of the selection of spare parts and components

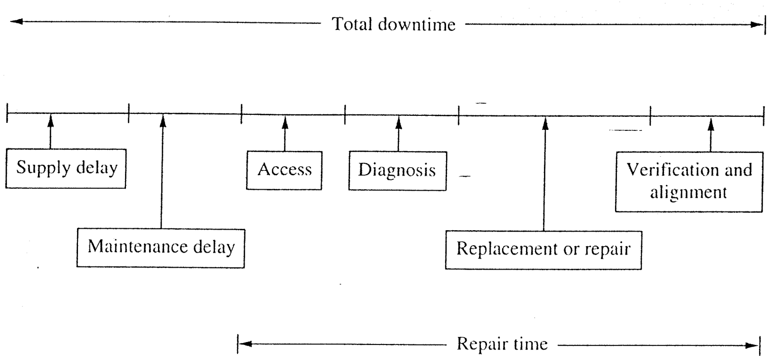


FIGURE 9.1
Maintenance downtime.

available at the repair facility. *Breadth* refers to the range of different components and parts that are stocked; *depth* refers to the number of spares of a given component or part. The supply delay time will not necessarily occur at the beginning of the repair cycle, but may occur after the diagnosis subtask has identified the failed component to be replaced. However, it will be advantageous to keep this time separate from the other categories. Obviously, the supply delay time will be zero if the needed replacement part is immediately available.

Maintenance delay time is the time spent waiting for maintenance resources or facilities. It may also include administrative (notification) time and travel time. Resources may be personnel, test equipment, support equipment, tools, and manuals or other technical data. Facilities may be a repair dock such as an aircraft hangar, a service bay in an automotive repair shop, or a fixed test stand. Maintenance delay time is influenced by the number of assigned parallel repair channels. A repair channel is defined as all the necessary maintenance resources and facilities needed to initiate and complete the repair process (other than spare components and parts). If a repair channel is immediately available on failure of the item, the maintenance delay time is zero.

Because supply delay times and maintenance delay times are influenced by external parameters (such as resource levels) that are not part of the system itself, they are not considered part of the inherent repair time of the item. The inherent repair time of the item is defined to be the sum of the durations of the following subtasks: access, diagnosis, repair or replacement, validation, and alignment. Access time is the amount of time required to gain access to the failed component. It may, for example, require removal of panels or covers. Diagnosis, or troubleshoot, time is the amount of time required to determine the cause of the failure. It is also referred to as fault isolation time. The repair time or replacement time includes only the actual hands-on time to complete the restoration process once the problem has been identified and access to the failed component obtained. Any delay in waiting for spares, additional personnel, test equipment, and so on, is either supply or maintenance delay time and is not considered part of the task of replacement or repair. Following restoration, some failures may require validating the restoration or alignment check

to ensure that the unit has been returned to an operational condition. If this check is required, it is considered part of the repair time.

Whenever we talk about repair time, we mean the inherent repair time unless stated otherwise. The repair time is considered to be an inherent design feature of the item and reflects the maintainability of the item. Maintainability is defined as the probability that a failed system or component will be restored or repaired to a specified condition within a period of time when maintenance is performed in accordance with prescribed procedures.

In order to specify the maintainability of an item, the acceptable operational condition of the unit and the conditions under which maintenance is to be performed must be defined. Like reliability, maintainability is defined to be a probability and will be characterized by specifying a repair-time probability distribution. Equipment design activities that directly affect maintainability include use of built-in test equipment, modularization and part layout, type of component (for example, electrical versus mechanical), ergonomic factors, labeling and coding, displays and indicators, standardization and interchangeability. Secondary considerations affecting maintainability include maintenance skill levels, repair crew size, level of repair, and use and clarity of maintenance procedures and diagrams.

There are several measures of maintainability. The most popular, and the one we will focus on, is the mean time to repair (MTTR) as defined in the next section. Other possible measures include the median time to repair; the mode, or most likely repair time; the time in which a specified percentage of the failures must be repaired; the mean repair time plus the mean preventive maintenance downtime; and the number of maintenance hours per operating hour. This last measure quantifies the total maintenance workload, whereas the others focus strictly on downtime.

References that provide a general discussion on maintainability include Blanchard and Lowery [1969], Dhillon and Reiche [1985], Smith [1985], Blanchard [1992], and Smith [1993].

9.2 THE REPAIR-TIME DISTRIBUTION

In order to quantify maintainability, the repair-time distribution must first be defined. Generally, repair times can be treated as random variables since repeated repair actions will result in different repair times. This is partly a result of different failure modes or the failure of different components and parts, and the variation in skill levels, experience, and training on the part of the maintenance personnel. The troubleshooting task itself can vary considerably in length, depending, perhaps, on luck or on the order in which potential problems are eliminated. Alignment or calibration times may vary from one unit to another depending on their respective design tolerances. The time required to replace a part may depend upon the compatibility of the replacement part.

To quantify the repair time, let T be the continuous random variable representing the time to repair a failed unit, having a probability density function of $h(t)$. Then the cumulative distribution function is

$$\Pr\{T \le t\} = H(t) = \int_0^t h(t')\,dt' \tag{9.1}$$

Equation (9.1) is the probability that a repair will be accomplished within time t. The mean time to repair may be found from

$$\text{MTTR} = \int_0^\infty t\,h(t)\,dt = \int_0^\infty (1 - H(t))\,dt \tag{9.2}$$

and the variance of the repair distribution is found from

$$\sigma^2 = \int_0^\infty (t - \text{MTTR})^2 h(t)\,dt \tag{9.3}$$

EXAMPLE 9.1. The time to repair a failed widget has the following probability density function:

$$h(t) = 0.08333t \qquad 1 \le t \le 5 \text{ hr}$$

where $$H(t) = \int_1^t 0.08333t'\,d't = 0.041665t^2 - 0.041665$$

The probability of completing a repair, in less than 3 hr, for example, is

$$H(3) = 0.041665 \times 9 - 0.041665 = 0.333$$

The mean time to repair is

$$\text{MTTR} = \int_1^5 0.08333t^2\,dt = \left.\frac{0.08333t^3}{3}\right|_1^5 = 3.44 \text{ hr}$$

9.2.1 Exponential Repair Times

If the repair distribution is exponential, then

$$H(t) = \int_0^t \frac{e^{-t'/\text{MTTR}}}{\text{MTTR}}\,dt' = 1 - e^{-t/\text{MTTR}} \tag{9.4}$$

where the parameter of the distribution is the MTTR. For this distribution, $r = 1/\text{MTTR}$ is the rate of repair (number of repairs per unit of time). The repair rate is constant only for the exponential distribution.

EXAMPLE 9.2. A component can be repaired at the constant rate of 10 per 8-hr day. What is the probability of a single repair exceeding 1 hr?

Solution. MTTR = 0.1 day = 0.8 hr. Therefore

$$\Pr\{T > 1\} = 1 - H(1) = e^{-1/0.8} = e^{-1.25} = 0.2865$$

9.2.2 Lognormal Repair Times

The lognormal distribution is often used to represent the repair distribution. For the lognormal distribution,

$$h(t) = \frac{1}{\sqrt{2\pi}ts}\exp\left\{-\frac{1}{2}\frac{[\ln(t/t_{\text{med}})]^2}{s^2}\right\} \qquad t \ge 0 \tag{9.5}$$

is the probability density function for the repair time. The lognormal distribution is a two-parameter distribution such that t_{med} is the median time to repair and s is a shape parameter. The probability of a repair being completed in time t is found by utilizing the relationship between the normal and lognormal distributions:

$$\Pr\{T \le t\} = H(t) = \Phi\left(\frac{1}{s}\ln\frac{t}{t_{\text{med}}}\right) \tag{9.6}$$

$H(t)$ is the lognormal cumulative distribution function for repair and $\Phi(\,)$ is the standardized normal cumulative distribution as defined by Eq. 4.23. The shape of the lognormal is asymmetric; its being skewed right indicates that most repair times will be distributed around the center (mode) of the distribution with a relatively few long repair times in the right-hand tail of the distribution. The mean time to repair is the mean of the lognormal distribution, which is related to the median time to repair:

$$\text{MTTR} = t_{\text{med}}e^{s^2/2} \tag{9.7}$$

EXAMPLE 9.3. A requirement exists for an engine fuel pump to be repaired (or replaced) within 3 hr 90 percent of the time. If the repair distribution is lognormal with $s = 0.45$, what MTTR should be achieved to meet this goal?

Solution. We must first find t_{med}. Given $H(3) = 0.90$,

$$\Phi\left(\frac{1}{0.45}\ln\frac{3}{t_{\text{med}}}\right) = 0.90 \qquad \text{or} \qquad \frac{1}{0.45}\ln\frac{3}{t_{\text{med}}} = 1.28$$

where 1.28 is the normal deviate (z value) corresponding to a cumulative probability of 0.90. Solving for t_{med}:

$$t_{\text{med}} = \frac{3}{e^{1.28(0.45)}} = 1.686 \text{ hr}$$

Therefore $$\text{MTTR} = 1.686e^{(0.45)^2/2} = 1.866 \text{ hr}$$

The mode, or most likely repair time, may be found by

$$t_{\text{mode}} = \frac{t_{\text{med}}}{e^{s^2}} = \frac{1.686}{e^{(0.45)^2}} = 1.377 \text{ hr}$$

If the repair distribution was exponential, rather than lognormal, and had the same mean, then a repair will be completed within 3 hr only 80 percent of the time, in contrast to the 90 percent under the lognormal repair time. That is,

$$H(3) = 1 - e^{-3/1.865} = 0.80$$

This is illustrative of the importance of determining the correct distribution when quantifying the repair time.

9.3 STOCHASTIC POINT PROCESSES

With the ability to repair or restore a failed system, a failure–repair–failure–repair cycle is generated. Depending on the nature of the repair process, the failure distribution representing the time to the first failure may not be the same as the time between successive failures. A stochastic point process is characterized by isolated events occurring at instants distributed randomly over a time continuum. The events in this case are failures, with T_1, T_2, ... representing their occurrence times. Assuming that repair or restoration time is negligible (or, equivalently, limiting our interest to operating intervals only) and considering a single system, let T_k be the random variable representing the (operating) time to the kth failure, where $T_0 = 0$, and $X_k = T_k - T_{k-1}$ be the random variable representing the (operating) time between failure $(k-1)$ and failure k. Then $T_k = \sum_{i=1}^{k} X_i$ and $E(T_k) = \sum_{i=1}^{k} E(X_i)$. $E(X_i)$ is referred to as the *mean time between failures* (MTBF). Of interest is the probability distribution of T_k, which will depend upon the distribution of the X_i. This distribution can in some cases easily be determined when the time between failures form a renewal process.

9.3.1 Renewal Process

A renewal process is defined to be one in which the random variables X_i are independent and identically distributed. This is consistent with the notion that the system is restored to its original condition, or "as good as new."[1] Under a renewal process

$$E(T_k) = kE(X_1) = k\mu_1 \qquad \text{and} \qquad \mathrm{Var}(T_k) = k\mathrm{Var}(X_1) = k\sigma_1^2$$

where μ_1 is the mean time to the first failure and σ_1 is the standard deviation. Since T_k is the sum of independent and identically distributed random variables, under the central limit theorem for large k, T_k has an approximate normal distribution, and therefore

$$\Pr\{T_k \le t\} \approx \Phi\left(\frac{t - k\mu_1}{\sigma_1\sqrt{k}}\right) \tag{9.8}$$

The approximation is good for $k \ge 30$ although smaller values of k are acceptable if the distribution of X_k is somewhat symmetrical. This result is perhaps not as useful as we would like since we are often interested in the distribution of T_k for small values of k. However, if the distribution of X_i is normal, T_k is normal and the probability in (9.8) is exact for any k.

EXAMPLE 9.4. A continuous-flow production line experiences numerous operational failures such as jams, spills, splicing problems, and misalignments. The time to failure

[1] Ascher and Feingold[1984] point out that "good as new" may be a poor choice of words if the system has a decreasing failure rate.

has been found to be Weibull with $\beta = 0.5$ and $\theta = 1.5$ hr. The decreasing failure rate is consistent with the observation that if the line is to fail, it is likely to fail soon after startup. The longer the line is operating, the more likely it will continue to operate without an operational failure occurring. Assuming that restoration of the line results in a renewal process, the probability of the kth failure occurring by time t is approximated by

$$\Pr\{T_k \le t\} \approx \Phi\left(\frac{t-3}{6.71\sqrt{k}}\right)$$

where the MTTF $= 3$ hr with $\sigma = 6.71$.

EXAMPLE 9.5. A cutting tool has a time-to-failure distribution that is normal with a mean of 5 operating hours and a standard deviation of 1 hr. Nine replacement tools are available with which to complete a production run requiring 40 hr of operation. Determine the probability (reliability) of completing the production run with the available tools.

Solution

$$\Pr\{T_9 \ge 40\} = 1 - \Phi\left(\frac{40-45}{\sqrt{9}}\right) = 1 - \Phi(-1.67) = 0.95254$$

A stochastic point process can equivalently be defined by the number of failures in the interval $(0, t)$. Let $N(t)$ be the discrete random variable representing the cumulative number of failures in the interval $(0, t)$. Then

$$\begin{aligned}
\Pr\{N(t) = 0\} &= \Pr\{T_1 > t\} \\
\Pr\{N(t) = j\} &= \Pr\{T_j \le t < T_{j+1}\} \\
&= \Pr\{T_{j+1} \ge t\} - \Pr(T_j > t\} \qquad \text{for } j = 1, 2, \ldots
\end{aligned} \tag{9.9}$$

The second line of Eq. (9.9) holds since the event that exactly j failures occur by time t is equivalent to failure j occurring by time t and failure $j+1$ occurring after time t.

EXAMPLE 9.5. (CONTINUED). The probability distribution for the number of failures during the first 12 hr of operation may be found as follows:

$$\Pr\{N(12) = 0\} = \Pr\{T_1 > 12\} = 1 - \Phi\left(\frac{12-5}{1}\right) = 1 - \Phi(+7) = 0$$

$$\Pr\{N(12) = 1\} = \Pr\{T_2 \ge 12\} - \Pr\{T_1 > 12\} = 1 - \Phi\left(\frac{12-10}{\sqrt{2}}\right) - 0$$

$$= 1 - \Phi(1.414) = 0.07927$$

$$\Pr\{N(12) = 2\} = \Pr\{T_3 \ge 12\} - \Pr\{T_2 > 12\}$$

$$= 1 - \Phi\left(\frac{12-15}{\sqrt{3}}\right) - \left[1 - \Phi\left(\frac{12-10}{\sqrt{2}}\right)\right]$$

$$= -\Phi(-1.732) + \Phi(1.414) = 0.87891$$

Continuing:

$$\Pr\{N(12) = 3\} = 0.04179$$
$$\Pr\{N(12) = 4\} = 0.00003$$
$$\Pr\{N(12) \geq 5\} \approx 0$$

The mean number of failures is approximately 1.96.

Homogeneous Poisson process

If the distribution of the time between failures is exponential with parameter λ, the distribution of T_k is gamma (Erlang) with parameters k and λ, as discussed in Section 3.5. The cumulative distribution function is given by Eq. (3.22) with T_k replacing Y_k. To find the probability distribution for the number of failures by time t, we use Eq. (3.22) and (9.9), resulting in

$$\Pr\{N(t) = j\} = e^{-\lambda t}\left[\sum_{i=0}^{j}\frac{(\lambda t)^i}{i!} - \sum_{i=0}^{j-1}\frac{(\lambda t)^i}{i!}\right] = \frac{(\lambda t)^j e^{-\lambda t}}{j!} \tag{9.10}$$

Equation (9.10) is a Poisson probability distribution having a mean of λt, and this renewal process is called a homogeneous Poisson process. A homogeneous Poisson process can be viewed as a point process in which the times between successive events are independent and identically distributed exponential random variables. The expected number of failures (renewals) in time t is just the mean of the Poisson distribution, given by $E[N(t)] = \lambda t$. The distribution given in Example 3.9 is that of a homogeneous Poisson process.

Renewal function

For any renewal process, $m(t) = E[N(t)] = \sum_{j=1}^{\infty} j \Pr\{N(t) = j\}$ is called the *renewal function*. An important theorem in renewal theory is the *elementary renewal theorem:*

$$\lim_{t\to\infty}\frac{m(t)}{t} = \frac{1}{\mu} \tag{9.11}$$

where μ is the mean length of the renewal cycle. Under our assumption of negligible repair time, $\mu = E(X_i) = \text{MTBF} = \text{MTTF}$, where $\text{MTTF} = E[T_1]$, the common mean of the failure distribution.[2] This theorem provides for an asymptotic approximation. Therefore, when t is large relative to the cycle time (that is, the MTBF), the expected number of renewals (failures) in the interval $(0,t)$ is t/MTBF. A corollary to the elementary renewal theorem is

$$\lim_{t\to\infty}[m(t+T) - m(t)] = \frac{T}{\text{MTBF}} \qquad T > 0 \tag{9.12}$$

[2] Some authors will use the term MTTFF (mean time to first failure) in place of MTTF when the system is repairable.

EXAMPLE 9.6. A motor has a (first) time to failure distribution that is Weibull with $\theta = 2400$ hr and $\beta = 1.8$. Determine its reliability at 500 hr if the motor has failed at 200 hr and is restored to as good as new condition. How many failures are expected in the first 10,000 hr of use?

Solution. As a renewal process, the time to the next failure will also have a Weibull distribution. We find the probability that the motor will operate for 300 additional hours following restoration:

$$R(300) = e^{-(300/2400)^{1.8}} = 0.9766$$

Since $\text{MTTF} = 2400\Gamma(1 + 1/1.8) = 2135$ hr, the expected number of failures, $m(t)$, is approximated by $10{,}000/2135 = 4.68$.

Superimposed renewal process

The union of all events (failures) from k independent renewal processes is called a *superimposed renewal process*. The superimposed renewal process will be itself a homogeneous Poisson process with a parameter equal to $\sum \lambda_i$ only if each of the k renewal processes is a homogeneous Poisson process with parameters λ_i. However, the superimposed renewal process will be an approximate homogeneous Poisson process if it is formed from the union of an infinite number of independent renewal processes in equilibrium (steady state). This latter case was discussed in Chapter 3 as an example of a system having a large number of components of which each component was replaced on failure. More formally, if a system consists of k different components with component i having q_i identical units each of which is an independent renewal process, then for sufficiently large t the total expected number of failures in the interval $(0, t)$ is

$$m(t) \approx f = \sum_{i=1}^{k}\frac{q_i t}{\text{MTBF}_i} \tag{9.13}$$

and the system mean time between failures is approximated by $\text{MTBF}_s = t/f$.

Although each unit may have a nonconstant failure rate, eventually a steady-state constant failure rate for the system is observed. However, as pointed out by Ascher and Feingold [1984], each unit must reach equilibrium, and therefore each unit must fail at least once in order for their ages to be randomized. Therefore, it may require an exceedingly long time for a such a system to reach equilibrium and justify the use of the homogeneous Poisson process. The study of renewal processes is primarily concerned with determining the number of renewals (failures) as a function of time. Barlow and Proschan [1967] and Goldberg [1981] provide detailed mathematical treatments of renewal processes.

EXAMPLE 9.7. It is not uncommon to consider the distribution of the number of failures of a military aircraft as a homogeneous Poisson process. An aircraft, consisting of thousands of components, will experience numerous failures and repairs as a result of its flying activity. Generally, a relatively small number of components will repeatedly experience the majority of the failures. As a result, an aircraft after accumulating hundreds of flying hours may attain the equilibrium conditions necessary to approximate a homogeneous Poisson process. This, in turn, justifies the common practice of computing an aircraft MTBF from

$$\text{MTBF}_{\text{aircraft}} = \left(\sum_{i=1}^{n} \frac{1}{\text{MTBF}_i}\right)^{-1} \quad \text{where MTBF}_i \text{ refers to component } i.$$

9.3.2 Minimal Repair Process

It is frequently the case that repair consists of replacing or restoring only a small percentage of the parts or components composing the system. This will leave the system in approximately the same state (age) it was in just prior to the failure. Therefore, as a result of minimal repair, the times between failures may no longer be independent and identically distributed. The system may continue to deteriorate over time, and therefore successive values of the X_i will be correlated and will display a trend. A useful and somewhat natural way to model this situation is to treat it also as a stochastic point process. To model this point process, we define an intensity function, $\rho(t)$, as the rate of change of the expected number of failures with respect to time, or

$$\rho(t) = \frac{dE[N(t)]}{dt} \tag{9.14}$$

A natural estimate for $\rho(t)$ is

$$\rho(t) \approx \frac{N(t + \Delta t) - N(t)}{\Delta t} \tag{9.15}$$

The intensity function may also be referred to as the renewal rate, failure intensity, peril rate, or the rate of occurrence of failure (ROCOF). It should not be confused with the hazard rate function, $\lambda(t)$, since $\lambda(t)\,\Delta t$ is the conditional probability of a failure in time Δt given that the unit has survived to time t, whereas $\rho(t)\,\Delta t$ is the unconditional probability of a failure in time Δt. The hazard rate function is a relative rate pertaining only to the first failure, whereas the intensity function is an absolute rate of failure for repairable systems. Ascher and Feingold [1984] provide additional discussion concerning the hazard rate and the intensity function and the confusion generated by the similar terminology. From the intensity function the expected number of failures is

$$E[N(t)] = \int_0^t \rho(t')\,dt' \tag{9.16}$$

An instantaneous MTBF is defined by

$$\text{MTBF} = \frac{1}{\rho(t)} \tag{9.17}$$

and an interval MTBF is given by

$$\text{MTBF}(t_1, t_2) = \frac{t_2 - t_1}{m(t_1, t_2)} \tag{9.18}$$

where $m(t_1, t_2) = E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \rho(t)\,dt =$ the expected number of failures in the interval (t_1, t_2).

EXAMPLE 9.8. A manufacturing machine has an intensity function given by $\rho(t) = e^{-6.5+0.0002t}$ with t measured in operating hours. After 1 yr of operation (3000 hr), $\rho(3000) = 0.0027394$ and the instantaneous MTBF is $1/\rho(3000) = 365$ hr. The expected number of failures over the second year is

$$m(3000, 6000) = \int_{3000}^{6000} e^{-6.5+0.0002t}\,dt = 11.26$$

Nonhomogeneous Poisson process

A somewhat simple model of a stochastic point process is the *nonhomogeneous Poisson process,* in which the probability distribution of the number of failures in the interval (t_1, t_2) is given by

$$\Pr\{N(t_2) - N(t_1) = j\} = \frac{m(t_1, t_2)^j e^{-m(t_1, t_2)}}{j!} \tag{9.19}$$

where $N(0) = 0$. The reliability at time t with respect to the first failure is equivalent to the probability of no failures in time t, or

$$R(t) = \Pr\{N(t) = 0\} = e^{-m(0,t)} \tag{9.20}$$

The reliability with respect to the next failure can be found given that the system is restored at time T,

$$R(t \mid T) = \Pr\{N(T + t) - N(T) = 0\} = e^{-m(T, T+t)}$$

Observe that $m(T, T + t)$ is the expected number of failures occurring from time T to $T + t$. Note that the homogeneous Poisson process is the special case of the nonhomogeneous Poisson process in which $\rho(t) = \lambda$, a constant, since then $m(0, t) = \lambda t$. If several independent nonhomogeneous Poisson processes are superimposed (that is, if one forms the sequence of renewals from the union of all the nonhomogeneous Poisson process renewals), the intensity function of the superimposition is $\rho(t) = \sum \rho_i(t)$.

A common form for the intensity function is

$$\rho(t) = a\,b\,t^{b-1} \qquad a, b > 0 \tag{9.21}$$

which is called the power law process or the Weibull process.[3] The latter name is the result of the functional form of Eq. (9.21) being identical to the Weibull hazard rate function. However, except for the time to the first failure, the times between failures do not have Weibull distributions. For the intensity function given in Eq. (9.21), if $b < 1$, the system is improving over time, as experienced during reliability growth testing (see Section 14.4). If $b > 1$, the system is deteriorating over time, as might be observed under minimal system repair. Estimation techniques for determining the parameters a and b are presented in Chapter 14 in the context of the U.S. Army Material Systems Analysis reliability growth model. A statistical test for trend and a goodness-of-fit test for the power-law process are provided in Chapter 16. Bain and Engelhardt[1991] provide details concerning other inferential statistics, including

[3] Another common form is $\rho(t) = e^{(a+bt)}$ (see Crowder et al. [1991]).

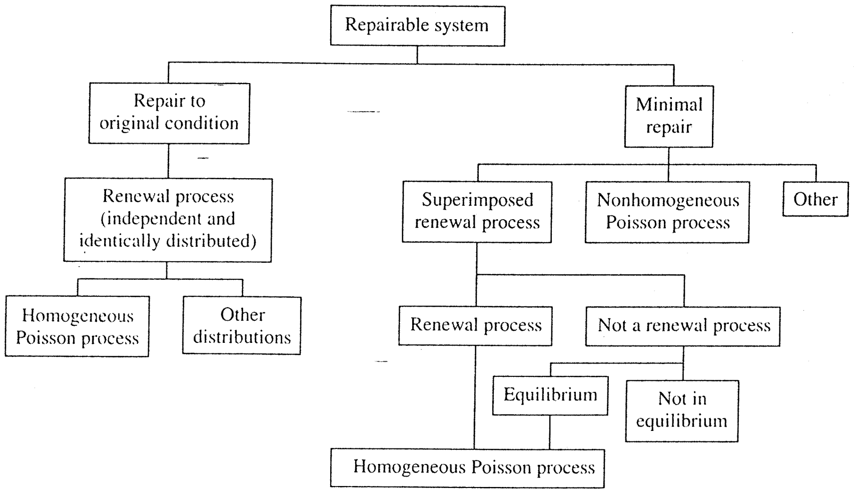


FIGURE 9.2
Stochastic point processes in modeling a repairable system.

estimation and hypothesis testing for the power-law process. Figure 9.2 summarizes the various conditions presented.

EXAMPLE 9.9. A six-year-old regional transit bus experiences minimal repair upon failure. It was found to have an intensity function given by $\rho(t) = 0.0464t^{2.1}$ with t measured in years. Then MTBF (instantaneous) $= 1/[(0.0464)(6)^{2.1}] = 0.5$ yr, and $m(6,7) = \int_6^7 0.0464t^{2.1}\,dt = 2.35$ is the expected number of failures over the coming year. Therefore, $R(1 \mid 6) = \Pr\{\text{No failure occurs in the seventh year}\} = \Pr\{N(7) - N(6) = 0\} = e^{-2.35} = 0.095$, and $\Pr\{\text{Exactly one failure occurs in the seventh year}\} = \Pr\{N(7) - N(6) = 1\} = 2.35e^{-2.35} = 0.224$. The reliability function can be written as $R(t) = \exp[-\int_0^t 0.0464y^{2.1}\,dy] = e^{-0.0149677t^{3.1}}$. This is a Weibull distribution with $\beta = 3.1$ and $\theta = 3.878$ hr. Of course, $R(t)$ represents the reliability relative to the first failure only.

The reader desiring additional information concerning the nonhomogeneous Poisson process should see Ascher and Feingold [1984], Crowder et al. [1991], or Lawless [1982]. As discussed in the first reference, sample values of the times between failures should be tested for trends prior to assuming a renewal process. If trends are present, the nonhomogeneous Poisson process will be a more appropriate model to consider.

9.3.3 Overhaul and Cycle Time

A particular type of renewal process is observed if a system undergoes a complete overhaul either every T_0 time periods or on failure, whichever occurs first. Assume that the overhaul restores to the system to as good as new condition. Then letting T_{ov}

be the random variable representing the time between overhauls, and letting $R(t)$ and $f(t)$ be the system reliability and failure density function, respectively, the mean time between overhauls can be found from

$$E(T_{ov}) = T_0R(T_0) + \int_0^{T_0} tf(t)\,dt = \int_0^{T_0} R(t)\,dt \tag{9.22}$$

Equation (9.22) shows that the mean time to the next overhaul will be T_0 if no failure occurs before T_0 (a probability of $R(T_0)$), or equal to the partial expectation of the failure distribution from 0 to T_0. The second equality in Eq. (9.22) is obtained from setting $\int_0^{T_0} tf(t)\,dt = -tR(t)\big|_0^{T_0} + \int_0^{T_0} R(t)\,dt$ on the basis of integration by parts. If the failure distribution is exponential, then

$$E(T_{ov}) = \int_0^{T_0} e^{-\lambda t} = \frac{1}{\lambda}(1 - e^{-\lambda T_0})$$

EXAMPLE 9.10. An aircraft engine is scheduled for a complete overhaul every 10,000 flying (operating) hours or on a failure requiring removal of the engine from the aircraft. Assuming a constant failure rate of 10^{-5} failure per flying hour for engine-removal failure modes, the mean time between overhauls will be

$$E(T_{ov}) = 10^5(1 - e^{-0.1}) = 9516 \text{ flying hours}$$

Cycle time

In the discussion on renewals and point processes, the repair time was considered to be negligible. In this discussion the repair time is explicitly included. If X_i is the random variable representing the time to the ith failure (following restoration) and S_i random variable representing the repair time of the ith failure, then $Y_i = X_i + S_i$ is the random variable representing the length of the ith renewal cycle. The actual time of the ith renewal, T_i, can then be found from

$$T_i = T_{i-1} + Y_i \qquad i = 1, 2, \ldots$$

with $T_0 = 0$. Y_i is a random variable with cumulative distribution function, $G(t) = \Pr\{Y_i \le t\}$ and probability density function $g(t)$. If X_i has probability density function $f(x)$, S_i has probability density function $h(s)$, and S_i and X_i are independent, then

$$g(t) = \int_0^t f(x)h(t - x)\,dx$$

For most distributions this convolution must be solved numerically. However, if both the failure and the repair distribution are exponential, a closed-form solution can be obtained.

EXAMPLE 9.11. Let $f(t) = \lambda e^{-\lambda t}$ be the probability density function of the failure distribution and $h(t) = re^{-rt}$ be the probability density function of the repair distribution. Then with $\lambda \neq r$,

$$g(t) = \lambda r \int_0^t e^{-\lambda x - r(t-x)}\, dx = \lambda r \left[\frac{e^{-\lambda x - r(t-x)}}{-\lambda + r} \right]_0^t = \frac{\lambda r}{r - \lambda} \left[e^{-\lambda t} - e^{-rt} \right]$$

and $$G(t) = 1 - \frac{re^{-\lambda t} - \lambda e^{-rt}}{r - \lambda}$$

Although both the failure time and the repair time are exponential, the cycle time is not.

When repair time is a significant part of the cycle time, the MTBF in Eq. (9.11) and (9.12) must be replaced by MTBF + MTTR, the expected cycle length.

EXAMPLE 9.12. A system comprises four identical modules each having a time-to-failure distribution that is Weibull with $\beta = 1.4$ and $\theta = 200$ hr. Repair time is lognormal with $t_{med} = 2.5$ hr and $s = 0.87$. Assuming that a failure results in the failed unit's replacement with an identical unit (renewal process) and assuming that an equilibrium is achieved over a 5-yr period, find the expected number of repairs (failures).

Solution

$$\text{MTBF} = 200\Gamma\left(1 + \frac{1}{1.4}\right) = 182.1 \text{ hr} \quad \text{and} \quad \text{MTTR} = 2.5e^{0.37845} = 3.65 \text{ hr}$$

Therefore, the number of failures is $f = 4(365)(24)(5)/(182.1 + 3.65) = 943.2$ and the system MTBF is $(365)(24)(5)/943.2 = 46.4$ hr.

9.4 SYSTEM REPAIR TIME

We will frequently want to express system repair time as a function of the repair times of the components. To do this, we will compute an average (mean) system repair time from knowledge of the mean subsystem or component repair times. For example, the mean time to repair an aircraft depends on the repair distribution of each of the subsystems, such as the electrical, hydraulic, and environmental subsystems. The system MTTR may be computed as a weighted average of the subsystem MTTRs in which the weights are based on the relative number of failures. Let MTTR_i be the mean time to repair the ith unique subsystem, f_i be the expected number of failures of the ith unique subsystem over the system design life, and q_i be the number of identical subsystems of type i. Then the system mean time to repair is

$$\text{MTTR} = \frac{\sum_{i=1}^{n} q_i f_i \,\text{MTTR}_i}{\sum_{i=1}^{n} q_i f_i} \tag{9.23}$$

The expected number of failures of the ith subsystem can be computed from

$$f_i = \begin{cases} \dfrac{t_{oi}}{\text{MTTF}_i} & \text{for renewal process} \\ \int_0^{t_{oi}} \rho_i(t)\, dt & \text{for minimal repair} \end{cases}$$

where t_{oi} is the total number of operating hours of the ith component over the system design life. If all of the components have constant failure rates and the same number of operating hours, f_i can be replaced by λ_i. Identical subsystems may be serially related or active redundant. From Eq. (9.23) the role reliability plays in directly influencing repair times can be seen. Obviously, if a component with a high MTTR also generates a high number of failures, it will affect the system MTTR much more than a component with a high MTTR and a low number of failures. From a design trade-off perspective, high reliability should be a design objective for those components having long repair times, and short repair times should be the design objective of those components having high failure rates.

EXAMPLE 9.13. A radio consists of the following three subsystems:

Subsystem	λ_i	MTTR$_i$, hr
Power supply	0.00045	2.3
Amplifier	0.00130	3.7
Tuner	0.00007	4.6
Total	0.00182	

Then

$$\text{MTTR}_s = \frac{0.00045(2.3) + 0.00130(3.7) + 0.00007(4.6)}{0.00182} = 3.388 \text{ hr}$$

Systems having redundant components

When k out of n active redundant and identical components are present, repair may occur when any one of the n components fails or when $n - k + 1$ components fail (a system failure). If a failure is repaired when it occurs, the component mean repair time is MTTR. In the second case, however, the system may be restored when one is repaired or when all $n - k + 1$ are repaired. If restoration occurs when any one of the redundant components is repaired and only one component can be repaired at a time, MTTR is the system mean repair time. However, if all failures can be repaired at the same time, then under the assumption of a constant repair rate (exponential repair time), the system repair rate is $(n - k + 1)$ / MTTR and the system mean time to repair is MTTR / $(n - k + 1)$. On the other hand, if all units must be repaired before the system is restored and only one can be repaired at a time, the mean time to repair is $(n - k + 1)$MTTR. If all failed units can be repaired simultaneously (each having an exponential distribution) and all must be repaired, the repair time is given by

$$\text{MTTR} \sum_{i=1}^{n-k+1} \frac{1}{i} \tag{9.24}$$

This last expression holds because the repair rate is initially $(n - k + 1)$ / MTTR for the first repair, then it is $(n - k)$ / MTTR for the second repair, $(n - k - 1)$/ MTTR for the third, and so on, and finally 1/MTTR for the last. The sum of the reciprocals of these terms is the desired solution.

TABLE 9.1
System repair time

	Repair one at a time	Repair simultaneously
Restore when one is repaired	MTTR	MTTR/2
Restore when both are repaired	2 MTTR	1.5 MTTR

EXAMPLE 9.14. For a 2-out-of-3 system with each component having a constant repair rate equal to 1/MTTR, the system will fail when two units have failed. Assuming that repair begins when both units have failed, Table 9.1 provides the system mean repair time for each of the four possible cases.

9.5 RELIABILITY UNDER PREVENTIVE MAINTENANCE

For complex systems increased reliability can often be achieved through a preventive maintenance program. Such a program can reduce the effect of aging or wearout and have a significant impact on the life of the system. The following reliability model assumes that a system is restored to its original condition following preventive maintenance. Let $R(t)$ be the system reliability without maintenance, T be the interval of time between preventive maintenance, and $R_m(t)$ be the reliability of the system with preventive maintenance. Then

$$R_m(t) = R(t) \qquad \text{for } 0 \le t < T$$

and

$$R_m(t) = R(T)R(t-T) \qquad \text{for } T \le t < 2T$$

where $R(T)$ is the probability of survival until the first preventive maintenance and $R(t-T)$ is the probability of surviving the additional time $t-T$ given that the system was restored to its original condition at time T. Continuing, in general we have

$$R_m(t) = R(T)^n R(t-nT) \qquad \begin{array}{l} nT \le t < (n+1)T \\ n = 0, 1, 2, \ldots \end{array} \tag{9.25}$$

where $R(t)^n$ is the probability of surviving n maintenance intervals and $R(t-nT)$ is the probability of surviving $t - nt$ time units past the last preventive maintenance. The MTTF under preventive maintenance may be found using Eq. (2.8). The details of this derivation are found in Appendix 9A. The result is

$$\text{MTTF} = \int_0^\infty R_m(t)\,dt = \frac{\int_0^T R(t)\,dt}{1 - R(T)} \tag{9.26}$$

EXAMPLE 9.15. For the constant failure rate model,

$$R(t) = e^{-\lambda t}$$

and

$$R_m(t) = (e^{-\lambda T})^n e^{-\lambda(t-nT)}$$
$$= e^{-\lambda nT} e^{-\lambda t} e^{\lambda nT} = e^{-\lambda t} = R(t)$$

This again reflects the memorylessness of the exponential distribution. Under a constant failure rate, preventive maintenance has no effect!

EXAMPLE 9.16. For the Weibull failure distribution,

$$R_m(t) = \exp\left[-n\left(\frac{T}{\theta}\right)^\beta\right]\exp\left[-\left(\frac{t-nT}{\theta}\right)^\beta\right] \qquad nT \le t \le (n+1)T$$

A compressor has a Weibull failure process with $\beta = 2$ and $\theta = 100$ days. If we assume a 20-day preventive maintenance program ($T = 20$), then

$$R_m(t) = \exp\left[-n\left(\frac{20}{100}\right)^2\right]\exp\left[-\left(\frac{t-20n}{100}\right)^2\right] \qquad 20n \le t \le 20(n+1)$$

In Figure 9.3 are plots of $R(t)$ (no PM) and $R_m(t)$ (cumulative PM) showing the improvement in reliability over time as a result of preventive maintenance. The curve at the top (PM) shows preventive maintenance restoring the system to as good as new at the end of each preventive maintenance cycle, $R(t-nT)$, but does not take into account the cumulative effect over time, that is, $R(T)^n$. The reliability function, $R_m(t)$, must be a monotonically decreasing function. On the other hand, the conditional reliability $R_m(t \mid nT)$ equals $R(t-nT)$ for $nT \le t < (n+1)T$. The reliability for 90 days is found by first observing that $n = 4$. Then

$$R_m(90) = \exp\left[-4\left(\frac{20}{100}\right)^2\right]\exp\left[-\left(\frac{90-80}{100}\right)^2\right] = 0.8437$$

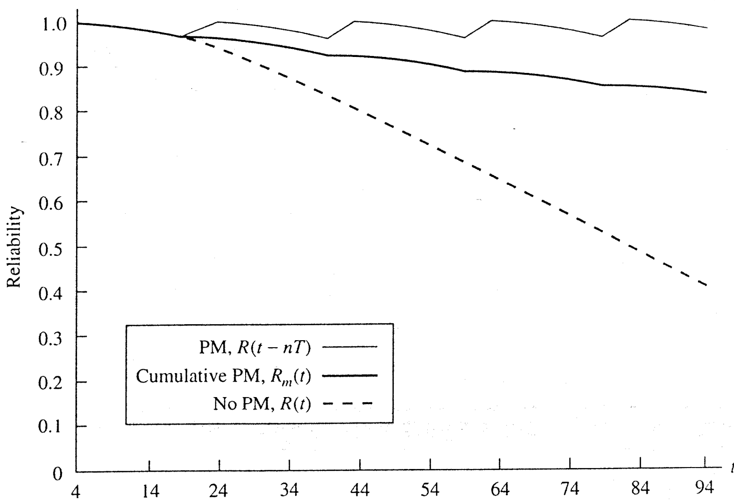


FIGURE 9.3
A periodic maintenance reliability curve for an increasing failure rate.

The design life at 0.90 reliability with no preventive maintenance is 32.5 days. Under preventive maintenance, if we consider the compressor reliability at the end of a maintenance interval, then we desire

$$\exp\left[-n\left(\frac{20}{100}\right)^2\right] \approx 0.90$$

Solving for n,

$$n = \frac{(-\ln 0.90)}{(20/100)^2} = 2.63$$

Letting $n = 2$,

$$R_m(t) = \exp\left[-2\left(\frac{20}{100}\right)^2\right]\exp\left[\left(-\frac{t-40}{100}\right)^2\right] \qquad 40 \le t < 60$$

$$= 0.9231\exp\left[-\left(\frac{t-40}{100}\right)^2\right] = 0.90$$

Now solving for t:

$$t = 100\left[-\ln\left(\frac{0.90}{0.9231}\right)\right]^{1/2} + 40 = 55.9 \text{ days}$$

This is a 32 percent increase in the component's design life as a result of preventive maintenance.

In some preventive maintenance situations there is the possibility of a maintenance-induced failure. The model represented by Eq. (9.25) may be modified to account for these failures by letting p be the probability of a maintenance-induced failure during an individual preventive maintenance. Then

$$R_m(t) = R(T)^n(1-p)^n R(t-nT) \qquad \begin{matrix} nT \le t < (n+1)T \\ n = 0, 1, 2, \ldots \end{matrix} \qquad (9.27)$$

EXAMPLE 9.17. For a component having a lognormal failure distribution,

$$R(T)^n = \left[1 - \Phi\left(\frac{1}{s}\ln\frac{T}{t_{med}}\right)\right]^n$$

$$R(t-nT) = 1 - \Phi\left(\frac{1}{s}\ln\frac{t-nT}{t_{med}}\right)$$

with $s = 1.00$ and $t_{med} = 5{,}000$ hr, the reliability at 5000 hr without preventive maintenance is

$$R(5000) = 1 - \Phi\left(\ln\frac{5000}{5000}\right) = 1 - 0.5 = 0.50$$

Assuming that $T = 500$ hr and $p = 0.005$,

$$R_m(5000) = \left[1 - \Phi\left(\ln\frac{500}{5000}\right)\right]^{10}(1-0.005)^{10} = 0.854$$

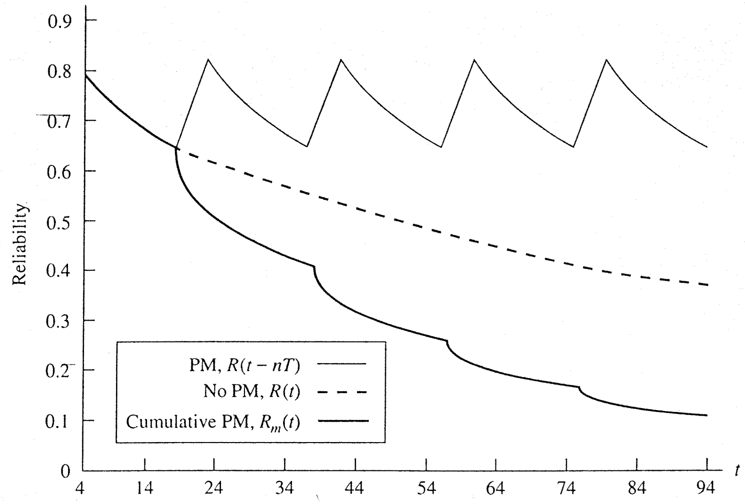


FIGURE 9.4
A periodic maintenance reliability curve for a decreasing failure rate.

EXAMPLE 9.18. If the failure rate is decreasing, preventive maintenance becomes counterproductive since restoring a system to as good as new condition introduces new wear-in failure modes (for example, manufacturing defects and marginal or substandard parts). This can be seen by changing the shape parameter of the Weibull failure distribution in Example 9.16 from 2 to 0.5. Figure 9.4 depicts the reliability curve with and without preventive maintenance.

9.6 STATE-DEPENDENT SYSTEMS WITH REPAIR

In Chapter 6 component dependencies were modeled using Markov analysis. This analysis procedure is here extended to include repairable components assuming that both the failure rate and the repair rate are constant (exponential distributions). Consider two active redundant components such that repair may be completed for a failed unit before the other unit has failed. As a result, no system failure is observed and the system reliability is improved. Assume that both units have the same failure rate, λ, and repair rate, r. The rate diagram appears in Fig. 9.5. State 1 is that of both units operating, state 2 is that of one operating and one being in repair, and state 3 is that of both units having failed. The resulting differential equations are as follows:

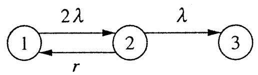


FIGURE 9.5
Rate diagram for a two-component system under repair.

$$
\begin{aligned}
\frac{dP_1(t)}{dt} &= -2\lambda P_1(t) + rP_2(t) \\
\frac{dP_2(t)}{dt} &= 2\lambda P_1(t) - (r + \lambda)P_2(t) \\
\frac{dP_3(t)}{dt} &= \lambda P_2(t)
\end{aligned}
\tag{9.28}
$$

They have the following solution (see Appendix 9B):[4]

$$P_1(t) = \frac{\lambda + r + x_1}{x_1 - x_2} e^{x_1 t} - \frac{\lambda + r + x_2}{x_1 - x_2} e^{x_2 t} \tag{9.29}$$

$$P_2(t) = \frac{2\lambda}{x_1 - x_2} e^{x_1 t} - \frac{2\lambda}{x_1 - x_2} e^{x_2 t} \tag{9.30}$$

$$P_3(t) = 1 + \frac{x_2}{x_1 - x_2} e^{x_1 t} - \frac{x_1}{x_1 - x_2} e^{x_2 t} \tag{9.31}$$

where

$$x_1, x_2 = \frac{1}{2}\left[-(3\lambda + r) \pm \sqrt{\lambda^2 + 6\lambda r + r^2}\right]$$

Therefore

$$R(t) = 1 - P_3(t) = \frac{x_1}{x_1 - x_2} e^{x_2 t} - \frac{x_2}{x_1 - x_2} e^{x_1 t} \tag{9.32}$$

and

$$
\begin{aligned}
\text{MTTF} &= \int_0^\infty \left(\frac{x_1}{x_1 - x_2} e^{x_2 t} - \frac{x_2}{x_1 - x_2} e^{x_1 t} \right) dt \\
&= \frac{-1}{x_1 - x_2}\left[\frac{x_1}{x_2} - \frac{x_2}{x_1}\right] = \frac{-(x_1 + x_2)}{x_1 x_2} = \frac{3\lambda + r}{2\lambda^2}
\end{aligned}
\tag{9.33}
$$

Observe that if $r = 0$ (no repair capability), the MTTF equals $1.5/\lambda$, as was shown in Chapter 3 for the two-component active redundant system. Equation (9.33) can also be written as

$$\text{MTTF} = \left(1.5 + 0.5\frac{\text{MTTF}_c}{\text{MTTR}_c}\right)\text{MTTF}_c$$

where MTTF_c and MTTR_c are the individual unit MTTF and MTTR, respectively. For this particular case, improving component maintainability will also increase the system reliability.

[4]This solution is more difficult than those of previous Markov processes since the two differential equations must be solved simultaneously. Laplace transforms provide the best solution technique.

EXAMPLE 9.19. A computer system consists of two active parallel processors each having a constant failure rate of 0.5 failure per day. Repair of a failed processor requires an average of one-half of a day (exponential distribution). From Eq. (9.33), the system MTTF is $[3(0.5) + 2]/[(2)(0.25)] = 7$ days. The system reliability for a single day is

$$R(1) = \frac{-0.149}{3.201} e^{-3.35} - \frac{-3.35}{3.201} e^{-0.149} = 0.90$$

where $x_1 = -0.149$ and $x_2 = -3.35$. Without repair, the system MTTF is $1.5/0.5 = 3$ days and $R(1) = 2e^{-0.5} - e^{-1} = 0.845$.

Standby system with repair

The two-component standby system with the primary unit on-line and the other in standby was analyzed in Section 6.3 assuming no repair capability. If repair of the primary unit is feasible when it is in a failed state, the system will continue to operate as long as the backup unit has not failed. If the primary is restored before the backup has failed, then from the system's perspective no failure has occurred and the system returns to its initial state. The state transition diagram for this situation is given in Fig. 9.6 where r is the constant repair rate.[5]

The system of equations is as follows:

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2^-)P_1(t) + rP_2(t) \tag{9.34}$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\lambda_2 + r)P_2(t) \tag{9.35}$$

$$\frac{dP_3(t)}{dt} = \lambda_2^- P_1(t) - \lambda_1 P_3(t) \tag{9.36}$$

A somewhat simpler solution exists if the assumption is made that there are no failures while in a standby mode. In this case, the rate diagram is as shown in Fig. 9.7. The system of equations becomes

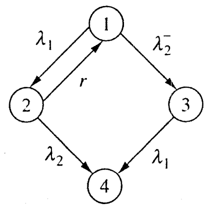


FIGURE 9.6
Rate diagram for a standby system with primary system repair.

[5]It is assumed in this case that failure of the secondary unit while in standby remains undetected and therefore no repair is possible.

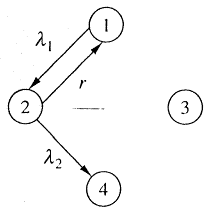


FIGURE 9.7
Rate diagram for a standby system with primary system repair and no failures in standby.

$$\frac{dP_1(t)}{dt} = -\lambda_1 P_1(t) + rP_2(t) \tag{9.37}$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\lambda_2 + r)P_2(t) \tag{9.38}$$

with a solution (Appendix 9C)

$$P_1(t) = \frac{\lambda_2 + r + x_1}{x_1 - x_2}e^{x_1 t} + \frac{\lambda_2 + r + x_2}{x_2 - x_1}d^{x_2 t} \tag{9.39}$$

$$P_2(t) = \frac{\lambda_1}{x_1 - x_2}e^{x_1 t} + \frac{\lambda_1}{x_2 - x_1}e^{x_2 t} \tag{9.40}$$

$$R(t) = P_1(t) + P_2(t) = \frac{(k_1 + x_1)e^{x_1 t} - (k_1 + x_2)e^{x_2 t}}{x_1 - x_2} \tag{9.41}$$

where

$$x_1, x_2 = \frac{-k_1 \pm \sqrt{k_1^2 - 4k_2}}{2} \tag{9.42}$$

and

$$k_1 = \lambda_1 + \lambda_2 + r \qquad k_2 = \lambda_1\lambda_2 \tag{9.43}$$

EXAMPLE 9.20. An on-board computer system has, through the use of built-in test equipment the capability of being restored when a failure occurs. A standby computer is available for use whenever the primary fails. Assuming $\lambda_1 = 0.0005$, $r = 0.1$, and $\lambda_2 = 0.002$, determine system reliability at 1000-hr intervals. All rates are expressed in units per hour.

Solution

$$k_1 = 0.0005 + 0.002 + 0.1 = 0.1025 \qquad k_2 = (0.0005)(0.002) = 10^{-6}$$

$$x_1, x_2 = \frac{-0.1025 \pm \sqrt{(0.1025)^2 - 4 \times 10^{-6}}}{2} = -9.757 \times 10^{-6}, \ -0.10249$$

$$R(t) = \frac{0.10249e^{-9.757\times 10^{-6}t} - (9.757 \times 10^{-6})e^{-0.10249t}}{0.1024852}$$

Then evaluating the reliability function at 1000-hr intervals, we obtain $R(1000) = 0.99039$, $R(2000) = 0.98077$, $R(3000) = 0.97125$, $R(4000) = 0.96182$, and $R(5000) = 0.95248$.

APPENDIX 9A
THE MTTF FOR THE PREVENTIVE MAINTENANCE MODEL

$$\begin{aligned}
\text{MTTF} &= \int_0^\infty R_m(t)\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} R_m(t)\,dt \\
&= \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} R(T)^n R(t - nT)\,dt \\
&= \sum_{n=0}^{\infty} R(T)^n \int_{nT}^{(n+1)T} R(t - nT)\,dt \\
&= \sum_{n=0}^{\infty} R(T)^n \int_0^T R(t')\,d't \qquad \text{where } t' = t - nT
\end{aligned}$$

$\sum_{n=0}^{\infty} R(T)^n$ is an infinite geometric series having as its sum $1/[1 - R(T)]$. Therefore

$$\text{MTTF} = \frac{\int_0^T R(t)\,dt}{1 - R(T)} \tag{9.26}$$

APPENDIX 9B
SOLUTION TO THE ACTIVE REDUNDANT SYSTEM WITH REPAIR

Beginning with Eqs. (9.28), the second equation can be rewritten by substituting for $P_1(t)$ by using the relationship $P_1(t) + P_2(t) + P_3(t) = 1$:

$$\frac{dP_2(t)}{dt} = 2\lambda[1 - P_2(t) - P_3(t)] - (r + \lambda)P_2(t)$$

Then by use of Laplace transforms with $\mathcal{L}\{P_i(t)\} = z_i(s) = z_i$, the last two equations written in the transformed domain become

$$sz_2 = 2\lambda/s - (3\lambda + r)z_2 - 2\lambda z_3 \qquad sz_3 = \lambda z_2$$

It is sufficient to find $P_3(t)$ in order to obtain $R(t)$. Therefore, solving for z_3 by using Cramer's rule,

$$z_3 = \frac{\begin{vmatrix} s + 3\lambda + r & 2\lambda/s \\ -\lambda & 0 \end{vmatrix}}{\begin{vmatrix} s + 3\lambda + r & 2\lambda \\ -\lambda & s \end{vmatrix}}$$

$$= \frac{2\lambda^2}{s[s(s + 3\lambda + r) + 2\lambda^2]} = \frac{2\lambda^2}{s(s - x_1)(s - x_2)} = \frac{A}{s} + \frac{B}{(s - x_1)} + \frac{C}{(s - x_2)}$$

where $$x_1,\ x_2 = \frac{-(3\lambda + r) \pm \sqrt{\lambda^2 + 6\lambda r + r^2}}{2}$$

From the theory of partial fractions,

$$A = \frac{2\lambda^2}{x_1 x_2} \qquad B = \frac{2\lambda^2}{x_1(x_1 - x_2)} \qquad C = \frac{2\lambda^2}{x_2(x_2 - x_1)}$$

Transforming back to the time domain:

$$P_3(t) = \frac{2\lambda^2}{x_1 x_2} + \frac{2\lambda^2}{x_1 - x_2}\left[\frac{e^{x_1 t}}{x_1} - \frac{e^{x_2 t}}{x_2}\right]$$

Some algebra and recognizing that $x_1 x_2 = 2\lambda^2$ results in Eq. 9.31.

APPENDIX 9C SOLUTION TO STANDBY SYSTEM WITH REPAIR

We start with the following:

$$\frac{dP_1(t)}{dt} = -\lambda_1 P_1(t) + rP_2(t) \tag{9.37}$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\lambda_2 + r)P_2(t) \tag{9.38}$$

The initial conditions are $P_1(0) = 1$ and $P_2(0) = 0$. Using Laplace transforms, let $\mathscr{L}\{P_i(t)\} = z_i(s) = z_i$. Then

$$sz_1 - 1 = -\lambda_1 z_1 + rz_2$$
$$sz_2 = +\lambda_1 z_1 - (\lambda_2 + r)z_2$$

Solving for z_1 by use of Cramer's rule,

$$z_1 = \frac{\begin{vmatrix} 1 & -r \\ 0 & \lambda_2 + r + s \end{vmatrix}}{\begin{vmatrix} s + \lambda_1 & -r \\ -\lambda_1 & \lambda_2 + r + s \end{vmatrix}} = \frac{\lambda_2 + r + s}{(s + \lambda_1)(\lambda_2 + r + s) - \lambda_1 r}$$

Rewriting the denominator for z_1,

$$z_1 = \frac{\lambda_2 + r + s}{s^2 + (\lambda_1 + \lambda_2 + r)s + \lambda_1\lambda_2}$$

Let $k_1 = \lambda_1 + \lambda_2 + r$ and $k_2 = \lambda_1\lambda_2$. Then

$$z_1 = \frac{\lambda_2 + r + s}{s^2 + k_1 s + k_2}$$

From the quadratic formula,

$$x_1,\ x_2 = \frac{-k_1 \pm \sqrt{k_1^2 - 4k_2}}{2}$$

Therefore $$z_1 = \frac{\lambda_2 + r + s}{(s - x_1)(s - x_2)} = \frac{A}{s - x_1} + \frac{B}{s - x_2}$$

From partial fractions,

$$A = \frac{\lambda_2 + r + x_1}{x_1 - x_2} \qquad B = \frac{\lambda_2 + r + x_2}{x_2 - x_1}$$

Transforming back to the time domain we get Eq. (9.39). Equation (9.40) is derived in the same way.

EXERCISES

9.1 The time to repair a power generator is best described by the following probability density function:

$$h(t) = \frac{t^2}{333} \qquad 1 \le t \le 10 \text{ hr}$$

Determine the probability that a repair will be completed in 6 hr. What is the MTTR? What is the median time to repair?

9.2 Norm L. Logg, a graduate assistant, must design a coffee pot for the faculty that has a 90 percent chance of operating for 5 yr.
(a) If the coffee pot failure times are lognormal, what should be the MTTF? Assume that the shape parameter is 0.7.
(b) If the coffee pot fails, it must be repaired within 4 hr or the faculty will suffer from caffeine withdrawal. If repair time is lognormal with a mean of 2 hr and a shape parameter of 1, what is the probability that the faculty will have their coffee on time? What is the most likely repair time (mode)?

9.3 The time to repair an engine module is lognormal with $s = 1.21$. Specifications require 90 percent of the repairs to be accomplished within 10 hr. Determine the necessary median and mean time to repair.

9.4 A flange bolt wears out because of fatigue in accordance with the lognormal distribution with MTTF = 10,000 hr and $s = 2.00$. If preventive maintenance consists of periodically replacing the bolt, what is the reliability at 550 hr with and without preventive maintenance? Assume that the bolts are replaced every 100 operating hours.

9.5 A system has two failure modes. One mode, due to external environmental events, has a constant failure rate of 0.008 failure per year. The second failure mode, attributed

to wearout, has a Weibull distribution with a characteristic life of 10 yr and a shape parameter of 1.8. The system has a design life of 5 yr.
(a) Determine the system reliability.
(b) If preventive maintenance will restore the wearout failure mode to as good as new condition, determine, by trial and error, a maintenance interval that will provide a required 0.95 system reliability (over both failure modes). Assume no maintenance-induced failures.
(c) If there is a 0.005 probability of maintenance causing a failure, will the maintenance interval need to be adjusted to provide the desired 0.95 reliability?

9.6 The time to failure of a piece of equipment (in operating hours) is uniform from 0 hr to 1000 hr. That is,

$$f(t) = 0.001 \qquad \text{for } 0 \le t \le 1000$$

(a) Determine the MTTF.
(b) Determine the MTTF if preventive maintenance will restore the system to as good as new and is performed every 100 operating hours.
(c) Compare the reliability with and without preventive maintenance at 225 operating hours. Assume the 100-hr maintenance interval and a maintenance-induced failure probability of 0.01 each time preventive maintenance is performed.
(d) Is there a significant improvement in reliability if a 50-hr preventive maintenance interval is assumed?

9.7 Norm L. Fail can repair just about anything in accordance with a lognormal repair distribution with a MTTR of 2 hr and a shape parameter of 0.2.
(a) Find the median time to repair and the mode.
(b) Determine the repair time such that 95 percent of the repairs will be accomplished within the specified time.
(c) Determine the probability that Norm will complete a repair within 100 minutes.

9.8 Y. Bull, a reliability engineer with the Maken Brake Company, has determined that the hazard rate function for its milling machine is the following:

$$\lambda(t) = 0.0004521t^{0.8} \qquad t \ge 0$$

where t is measured in years. Determine which of the following options will provide the greatest reliability over the machine's 20-yr operating life.

Option A. Do nothing—operate the machine until it fails.
Option B. an annual preventive maintenance program (assume no maintenance-induced failures).
Option C. Operate a second machine in parallel with the first (active redundant).

9.9 The probability density function for the time to failure in years of the drive train on a Rapid Transit Authority bus is given by

$$f(t) = 0.2 - 0.02t \qquad 0 \le t \le 10 \text{ yr}$$

(a) If the bus undergoes preventive maintenance every 6 months that restores it to as good as new condition, determine its reliability at the end of a 15-month warranty.
(b) Compute the MTTF under the preventive maintenance plan in part (a).

9.10 An aircraft consists of the following subsystems having the reliability and maintainability parameters shown:

Subsystem	Failure distribution	Parameters	MTTR
Propulsion	Weibull	$\theta = 1000;\ \beta = 1.7$	6.8
Avionics	Exponential	$\lambda = 0.003$	3.2
Structures	Weibull	$\theta = 2000;\ \beta = 2.1$	5.2
Electrical	Weibull	$\theta = 870;\ \beta = 1.8$	2.0
Environmental	Exponential	$\lambda = 0.001$	4.8

(a) Determine the system MTTR on the basis of the expected number of failures (assume a renewal process) over the 50,000-flying-hour design life of the aircraft.
(b) Repeat the analysis assuming minimal repair with the Weibull failure distribution replaced with the power-law intensity function having the same parameter values. Is the system MTTR sensitive to the repair philosophy in this case? What about the number of failures generated over the system life?

9.11 On failure a part can either be replaced (and thus restored to as good as new) or be minimally repaired (and restored to operating condition). If it is replaced, the time-between-failure distribution is lognormal with a median time to failure of 1150 operating hours and $s = 0.9$. Minimal repair results in a nonhomogeneous Poisson process with $\rho(t) = (0.4 \times 10^{-8})(1.8)t^{1.8}$. If a failure occurs at 400 operating hours, should the part be replaced if another 300 hr of use is required? Base your answer on the reliability over the next 300 hr.

9.12 Determine the MTTF for a repairable equipment item under preventive maintenance when the failure distribution is uniform $(0, b)$. (See Exercise 2.8.)

9.13 Find the MTTF for an active two-component redundant system with constant failure rate under preventive maintenance (use Eq. (3.24)). Compare the MTTF with that of Eq. (3.26).

9.14 Derive the reliability function for the two-unit standby system with repair in Section 9.6 when both units have the same failure rate (identical units). Find the MTTF.

9.15 Assume that repair time can be represented by a rectangular distribution with the following probability density function:

$$h(t) = \begin{cases} \dfrac{1}{b-a} & a \le t \le b \\ 0 & \text{elsewhere} \end{cases}$$

The minimum repair time is a, and the maximum repair time is b. Find $H(t)$ and MTTR.

9.16 A system having a rectangular failure distribution defined over the interval $(0, b)$ is to be overhauled on failure or at time T_0, whichever comes first. Overhaul will restore the system to as good as new condition. Determine the expected time between overhauls.

9.17 Preventive maintenance is to be performed every 5 days on a system having a rectangular failure distribution (that is, uniformly distributed from 0 to 100 days). Derive the reliability function for the system under preventive maintenance. Compare the MTTF and the reliability at 17 days with and without preventive maintenance.

9.18 A power system requires four out of six generators to be operating. The MTTR of a single generator on failure is 10 hr. Repair can be initiated on a generator only when the system has failed. Determine the mean time to restore the system if

(a) Only one generator can be repaired at a time and the system is restored when the first one is repaired.

(b) Only one generator can be repaired at a time and the system is restored when all of the generators have been repaired.

(c) All generators can be repaired simultaneously and the system is restored when the first one is repaired.

(d) All generators can be repaired simultaneously and the system is restored when all of the generators have been repaired.

9.19 Assume that in the system described in Exercise 6.2, the primary unit may be repaired while the standby unit is operating at a repair rate of 0.01. If under this repair condition there are no failures while the backup unit is in standby, estimate the design life for a reliability of 0.90.

9.20 The Fly-by-Nite (sometimes days) Airlines maintains an on-line reservation system with a standby computer available if the primary fails. The on-line system fails at the constant rate of once per day while the standby fails (when on-line) at the constant rate of twice per day. If the primary unit may be repaired at a constant rate with an MTTR of 0.5 of a day, what is the single-day reliability? Assume that there are no failures while in standby.

9.21 A mechanical pumping device has a constant failure rate of 0.023 failure per hour and an exponential repair time with a mean of 10 hr. If two pumps operate in an active redundant configuration, determine the system MTTF and the reliability that the system will operate without failure for 72 hr.

9.22 Show that Eq. 3.24 is a special case of the active redundant system with repair (Eq. 9.32) when the repair rate is zero.

9.23 An automobile undergoing repeated failure and repairs exhibits a nonhomogeneous Poisson process with an intensity function $\rho(t) = e^{-7.5+0.003t}$ where t is measured in operating hours.

(a) Determine the expected number of failures in the first 1000 operating hours.

(b) Find $R(100)$.

(c) Is the automobile showing deterioration or improvement over time?

9.24 A computer system has a constant failure rate of 0.1 per day and a constant repair rate of 1 per day. Determine the probability that a failure and repair cycle will be less than 1 day.

9.25 The Striket Riche Company has begun drilling for oil. It anticipates 130 hr of drilling time before reaching the oil. Experience has shown that the drill bits it uses in this type of rock formation have a lifetime that is lognormal with $t_{med} = 2$ hr and $s = 0.91$. The company currently has 35 drill bits available. What is the probability of striking oil before it runs out of bits?

CHAPTER 10

Design for Maintainability

In this chapter the application of maintainability to the design process is addressed. Maintainability has usually taken a backseat to reliability, particularly in the amount of effort and resources expended. This perhaps may be justified in that there is often a larger payoff, for example in life-cycle costs, through improved reliability designs. It is also justified since the demand for maintenance is driven by the failure distribution. Nevertheless, there may be practical limits to what can be achieved through reliability improvement alone, and it is therefore only through improved maintainability designs that further increases in system availability and readiness can be achieved. The next chapter will address both reliability and maintainability and the resulting system availability. It is through the availability measure that design trade-offs between reliability and maintainability can best be performed. Like reliability, maintainability is to a large degree an inherent design feature of the system. The objective of maintainability is to reduce system downtime by facilitating the repair effort.

The maintainability design process is similar to that of reliability design. Figure 10.1 illustrates this process. It begins by defining the maintainability goals. The determination of these goals coincides with the reliability specifications. Using the availability measure defined in the next chapter, one can make trade-offs between reliability and maintainability. The overall maintainability goal must then be allocated to the repairable subassemblies and components. Maintainability design methods, such as fault isolation techniques, parts standardization, and modularization, are applied to achieve the specified component maintainability parameters. Secondary considerations, such as the quantity and quality of repair resources and the level of spares support, are included in the analysis. Following an assessment of the achieved maintainability, additional design activity may be necessary if the goals are not attained. This may require a revision to the reliability goals as well.

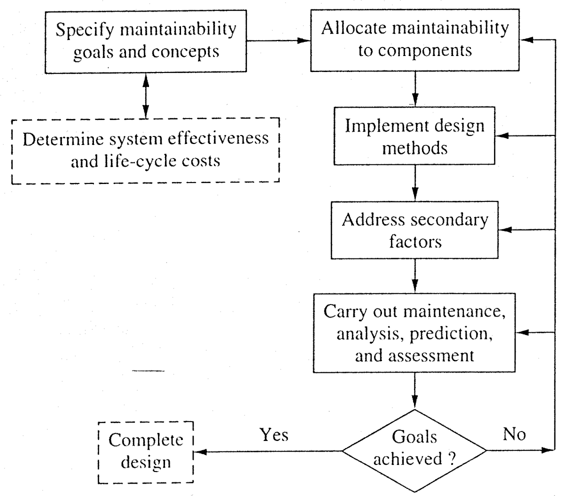


FIGURE 10.1
The maintainability design process.

10.1 MAINTENANCE REQUIREMENTS

The design process should begin by defining system maintainability objectives and specifications. These must include a quantifiable measure of the repair process as well as a qualitative description of the manner in which repair is to be accomplished (that is, under what prescribed conditions). Once this has been accomplished for the system, the maintainability criteria can be allocated to the lower-level components.

10.1.1 Measurements and Specifications

Quantifiable measures of maintainability include the following:

The mean time to repair (MTTR). As an average, this measure has the disadvantage of attempting to summarize the repair distribution with a single value. Two distributions having the same mean can provide a considerably different range of repair times. An improvement would be to specify an upper bound on the variance (or equivalently, the standard deviation) of the repair times along with the mean. A small variance will ensure more consistent repair times that are closer to the MTTR.

Median time to repair. The median is also an average and has the same disadvantage as the MTTR. It is preferred over the MTTR if the repair times are highly skewed. For example, a few very large repair times would influence the MTTR more than the median. In addition to the 50th percentile (that is, the median time), the other quartiles could be specified as well. That is, the repair time at which 25, 75, and 100 percent of the repairs would be accomplished would be part of the specification.

Maximum time, t_p, in which a certain percentage p of the failures must be repaired. Generally this measure is preferred over the MTTR and median time since it identifies an acceptable (maximum) repair time for the majority of the failures. Mathematically,

$$\Pr\{T \leq t_p\} = H(t_p) \leq p \tag{10.1}$$

Mean system downtime ($\overline{M}$). Mean system downtime is the average downtime including scheduled maintenance but not including supply or maintenance delay times. Since the requirement for scheduled maintenance is partly driven by design, it is appropriate to include scheduled downtime as part of the design criteria. Mathematically,

$$\overline{M} = \frac{m(t_d)\text{MTTR} + (t_d/T_{pm})\text{MPMT}}{m(t_d) + t_d/T_{pm}} \tag{10.2}$$

where T_{pm} = the (mean) time between performances of preventive maintenance
t_d = the system design or (economic) life
MPMT = the mean preventive maintenance time
$m(t_d)$ = the expected number of failures in the interval $(0, t_d)$ as defined in Section 9.3 for either a steady-state renewal process or minimal repair

For a constant failure rate, $m(t_d) = \lambda t_d$ and therefore t_d can be factored out of Eq. (10.2). Once failure data is collected, $m(t)$ is the number of observed failures over the time, t.

Mean time to restore (MTR). Mean time to restore is the average unscheduled system downtime including delays for maintenance and supply resources. This is an appropriate measure when maintenance and supply resources are included within the system design specifications.

$$\text{MTR} = \text{MTTR} + \text{MDT} + \text{SDT}$$

where MDT is the mean delay for maintenance, and SDT is the mean delay for supply resources.

Maintenance work hours per operating hour (MH/OH). The number of maintenance work hours per operating hour combines reliability and repair time with the number of maintenance personnel (crew size) necessary to complete repair. It is a measure of the maintenance work generated. Mathematically,

$$\frac{\text{MH}}{\text{OH}} = \frac{m(t) \times \text{MTTR} \times \text{CREW}}{t} \tag{10.3}$$

where $m(t)$ is defined above and CREW is the average crew size. For a constant failure rate, $m(t) = \lambda t$, Eq. (10.3) becomes

$$\frac{\text{MH}}{\text{OH}} = \lambda \times \text{MTTR} \times \text{CREW} \tag{10.4}$$

If mean preventive maintenance time (MPMT) is to be included in the work-hour calculation, then

$$\frac{\text{MH}}{\text{OH}} = \frac{m(t) \times \text{MTTR} \times \text{CREW} + (t/t_{pm}) \times \text{MPMT} \times \text{CREW}_{pm}}{t} \tag{10.5}$$

where CREW_{pm} is the average crew size for preventive maintenance. The time period t could be a specified period of time or could reflect the design life of the system. An improved maintainability measure may be the maintenance cost per operating hour, obtained by multiplying the above by a labor rate. Since the labor cost per hour reflects to some degree the skill level, experience, and education necessary to perform the repair tasks, this measure combines several aspects of maintainability and contributes directly to a life-cycle cost analysis.

EXAMPLE 10.1. A system has a lognormal repair distribution with a median time to repair of 3.5 hr with $s = 0.18$. Specifications call for 95 percent of the maintenance to be completed within 5 hr and the number of maintenance hours to be less than 3 hr for every 100 operating hours. To maintain warranty, a 2-hr preventive maintenance must be performed every 200 operating hours. The crew size is always two for safety reasons. The failure distribution is exponential with an MTBF of 1000 operating hours. Are the specifications being met?

Solution. Since

$$H(5) = \Phi\left(\frac{1}{0.18} \ln \frac{5}{3.5}\right) = \Phi(1.98) = 0.976$$

the initial specification is met. Since MTTR $= 3.5 \exp[0.5(0.18)^2] = 3.557$, then letting $t = 200$ hr and $m(t) = \lambda t = 200/1000$, we obtain

$$\frac{\text{MH}}{\text{OH}} = \frac{200/1000 \times 3.557 \times 2 + 2 \times 2}{200} = \frac{5.4228}{200} = 0.027 < 0.03$$

Therefore the specifications are met. The mean system downtime, $\overline{M}$, is

$$\overline{M} = \frac{0.001(3.557) + (1/200)(2)}{0.001 + 1/200} = 2.2595 \text{ hr}$$

10.1.2 Maintenance Concepts and Procedures

In addition to identifying quantifiable measures of maintainability, one must determine, as part of the design process, the conditions under which maintenance is to be accomplished. For a complex system consisting of subassemblies, components, and parts, it is necessary to determine the following:

1. Which units are to be repaired rather than discarded and replaced
2. The preventive maintenance schedule and associated tasks
3. For repairable units, the level of repair (such as local, service center, or factory) for each failure mode
4. For each repair task, the required skill levels, tools, test equipment, and technical manuals
5. The number of repair channels and spare parts (inventory)

The economic trade-off between repairing and discarding a unit is discussed in Section 10.2.4. For each component and part there are three alternatives: fully repairable, partially repairable, and nonrepairable. Partial repairability implies that there are certain failure modes in which it is not economical to initiate a repair action. Instead, the unit is said to be *condemned* and a replacement unit must be obtained. For a repairable unit experiencing wearout, at some point in its life the continuing cost of repair may exceed the replacement cost. Preventive maintenance may extend the useful life of the unit, in which case the frequency of performing preventive maintenance must be established.

The determination of the level of repair is often an economical decision that considers the maintenance skill levels, special tools and test equipment requirements, spare parts stock levels, and the economy of scale of maintaining centralized repair organizations. At the lowest level repair is often accomplished by the using organization on site with operating personnel frequently performing the maintenance tasks. In this case repair may consist of minor maintenance, removal and replacement, and routine servicing and adjustments.

At the intermediate level a centralized repair center is usually established to service a given geographical region. Maintenance personnel are employed specifically to perform repair, and they have higher skill levels than those observed at the lower level. Repair may be performed on removable components or on the system itself. For nonmobile systems, maintenance personnel may travel to the using site in order to perform repair. In some cases it may be justified to have dedicated maintenance on location in order to eliminate travel time from the total system downtime.

The highest level of repair may be performed at the manufacturer's or contractor's factory or, as in the case of the military, at a specialized depot. Normally this level of repair is only performed on costly, complex components requiring very specialized skills, repair equipment and tools, or critical alignments. Overhauls consisting of complete tear-down and rebuilding of units are performed at this level.

Secondary factors affecting maintainability focus on the maintenance and supply resources necessary to support the repair process. Establishing and maintaining the proper levels of these resources is often considered part of the *logistics* process. For each level of maintenance and failure mode, the necessary facilities, skill levels, spares, tools, test equipment, and service manuals must be identified. The trade-off between maintenance resources and spares levels is discussed in Section 10.4. The inner circle in Fig. 10.2 identifies inherent maintainability design features, and the outer circle lists secondary features that affect the determination of the total system downtime.

Cost model

In establishing an acceptable maintainability level, cost trade-offs among reliability and maintainability may be necessary. In this example a simple cost model captures the effect of the MTBF and the MTTR on costs. Let

$$
\begin{aligned}
t_d &= \text{design or economic life in operating hours} \\
C_u &= \text{unit acquisition cost} \\
C_f &= \text{fixed cost of a failure (such as spare parts)} \\
C_v &= \text{variable cost per hour of downtime} \\
&\quad \text{(such as labor rate} \times \text{crew size and loss of production)}
\end{aligned}
\tag{10.6}
$$

Then $$\text{Cost} = C_u + \frac{t_d}{\text{MTBF}}(C_f + C_v\,\text{MTTR})$$

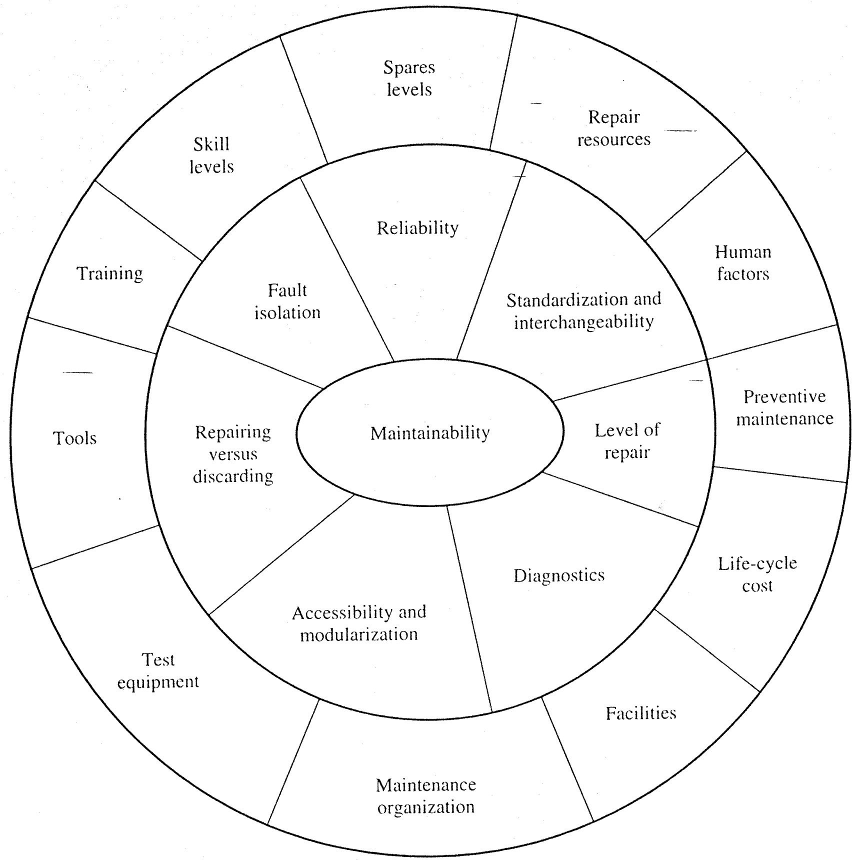


FIGURE 10.2
Inherent and secondary maintainability design features.

This cost equation assumes only one level of repair and no discounting. If there are several alternative designs that vary in unit acquisition costs, support costs (C_f and C_v), and reliability and maintainability parameters, then a direct cost comparison may be possible. If the time between failures does not generate a steady-state renewal process but rather reflects a minimal repair philosophy, then the expression T_d/MTBF can be replaced by $\int_0^{t_d} \rho(t)\,dt$ where $\rho(t)$ is the intensity function for a stochastic point process.

EXAMPLE 10.2. A machine has a purchase cost of \$3100. A machine failure incurs a fixed travel cost of \$75 and a variable repair cost based on a crew of two technicians having a labor rate of \$50 per hour. The manufacturer's warranty requires repair of all failures for the first 1000 operating hours. Current design and reliability testing shows an MTBF (constant failure rate) of 40 operating hours and an MTTR of 6 hr. In order to reduce warranty costs, the manufacturer can increase reliability (MTBF) by 20 percent or increase maintainability (that is, decrease MTTR) by 20 percent. Which is the better option, assuming that both options increase the acquisition cost by $1,000?

Solution. To establish a baseline with no changes:

$$\text{Cost} = 3100 + \frac{1000}{40}(75 + 2 \times 50 \times 6) = \$19{,}975$$

Increasing MTBF by 20 percent:

$$\text{Cost} = 4100 + \frac{1000}{48}(75 + 2 \times 50 \times 6) = \$18{,}162.50$$

Decreasing MTTR by 20 percent:

$$\text{Cost} = 4100 + \frac{1000}{40}(75 + 2 \times 50 \times 4.8) = \$17{,}975$$

Increasing maintainability is the better option.

10.1.3 Component Reliability and Maintainability

Once the system maintainability goals have been defined, they must be translated into lower-level design goals. This can be accomplished through an allocation process similar to that of the reliability allocation. For example, if the system MTTR is one of the specified maintainability requirements, Eq. (9.23) forms the basis for establishing compatible component MTTRs. Since the system MTTR is a weighted average of the component MTTRs with the weights based on the relative numbers of failures, one way of accomplishing this is to set

$$\text{MTTR}_i = \frac{\text{MTTR}_s \sum_{i=1}^{n} q_i f_i}{n q_i f_i} \qquad i = 1, 2, \ldots, n \qquad (10.7)$$

where MTTR_s is the system MTTR goal and f_i is the expected number of failures of the ith component over the life of the system . Those components having the most failures will have the smallest MTTRs. There is no guarantee, however, that these MTTR goals are attainable, and as shown in the following example, this method can produce a wide range of repair times if the component number of failures differ considerably. Therefore, the allocation may need to be adjusted as the design progresses or, as suggested by Eq. (10.7), an improvement may need to be made in the component reliability. If all components have constant failure rates with equal operating hours, then f_i can be replaced with λ_i.

EXAMPLE 10.3 A copying machine consists of the following components:

	Control unit	Input tray system	Camera unit	Feeder system (input/output)	Power unit
Quantity (q_i)	1	2	1	2	1
Failure rate (λ_i)	0.0016	0.008	0.001	0.021	0.005

Specifications require the average repair time to be 4 hr. Determine a suitable allocation of the MTTR to each component. Assume equal operating hours for each component. Since each component has a constant failure rate, replace f_i with λ_i in Eq. (10.7).

Solution. $\sum q_i \lambda_i = 0.0656$

$$\text{MTTR}_{\text{control unit}} = \frac{4(0.0656)}{5(0.0016)} = 32.8 \text{ hr}$$

$$\text{MTTR}_{\text{input tray}} = \frac{4(0.0656)}{5(0.008)(2)} = 3.28 \text{ hr}$$

$$\text{MTTR}_{\text{camera}} = \frac{4(0.0656)}{5(0.001)} = 52.48 \text{ hr}$$

$$\text{MTTR}_{\text{feeder}} = \frac{4(0.0656)}{5(0.021)(2)} = 1.25 \text{ hr}$$

$$\text{MTTR}_{\text{power}} = \frac{4(0.0656)}{5(0.005)} = 10.5 \text{ hr}$$

Of course, an alternative to Eq. (10.7) is to assign $\text{MTTR}_i = \text{MTTR}_s$ for all components. In practice, Eq. (9.23) can be computed by spreadsheet allowing component MTTRs to be determined by trial and error. An alternative approach to allocating maintainability is through an availability goal. This approach is discussed in Chapter 11.

10.2 DESIGN METHODS

In order to increase maintainability, in some manner the repair time must be reduced. There are several key concepts that should be followed as part of any design activity that supports this reduction.

10.2.1 Fault Isolation and Self-Diagnostics

Diagnosis of a failure with the identification of the fault is a major task in the repair process. It is often the longest task and the one having the greatest variability in task times. According to Greenman [1976], "By analyzing the effect of the design trends on the roots of maintainability, it can be concluded that for modern electronic systems, diagnostics is probably more significant to maintainability than all the other roots combined, both from an MTTR and logistics resource requirements standpoint." The potential for improvement may therefore be greatest in this area. *Diagnostics* refers to the process of locating the fault at the level in which restoration may be accomplished. Restoration may consist of removal and replacement of the failed component or its direct repair. Often referred to as *troubleshooting*, diagnostics can take several forms:

Manual. To some degree a trial-and-error procedure, possibly with the aid of meters, oscilloscopes, gauges, test sets, or technical drawings. Typically the malfunction is isolated through the process of elimination.

Automatic. The failed unit is generally removed from the rest of the system and connected to a computerized test station, or a test set is transported to the system and the connection made. The computer then executes one or more diagnostic programs in isolating the failed part.

Self-diagnostic. On failure the system switches to a diagnostic mode and internally isolates and identifies the failed part. Often this is referred to as built-in-test (BIT) or built-in-test equipment (BITE). It is critical that the BITE have a reliability an order of magnitude greater than the system it is supporting; otherwise the added complexity may reduce reliability. There are two functions performed by BITE. It must detect a fault and isolate the fault. Therefore the automatic fault isolation capability of self-test equipment is measured by the product of the percentage of all faults detected and the percentage of detected faults that are correctly isolated to a specified component. In addition, self-test equipment may exhibit a false alarm rate, which is the percentage of self-test alarms that on subsequent testing turn out not to be faults or turn out to be "cannot duplicate" maintenance actions. With respect to a BITE alarm, two types of BITE failures may occur: the BITE may falsely identify a fault, or it may fail to identify a true fault. The following example, which makes use of Bayes' theorem in probability, is somewhat enlightening.

EXAMPLE 10.4. Tests performed on a self-diagnostic module for a complex electronic system resulted in correct diagnostics of a known fault 98 percent of the time with only a 1 percent false reading when it was known there were no faults present. The probability of a failure (fault) occurring over the test period is 0.005. How reliable is the self-diagnostic module?

Solution. Let A be the random event that a fault is present and B be the random event that the self-diagnostic module identifies the fault as present. Then $P(A) = 0.005$, $P(B \mid A) = 0.98$, and $P(B \mid A^c) = 0.01$. From Bayes' theorem,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(0.995)} = 0.33$$

and therefore $P(A^c \mid B) = 1 - 0.33 = 0.67$.

$$P(A^c \mid B^c) = \frac{P(B^c \mid A^c)P(A^c)}{P(B^c \mid A^c)P(A^c) + P(B^c \mid A)P(A)} = \frac{(0.99)(0.995)}{(0.99)(0.995) + (0.01)(0.005)}$$
$$= 0.99995$$

and $P(A \mid B^c) = 1 - 0.99995 = 0.00005$.

The diagnostic module is not very reliable when it indicates that a fault is present. On the other hand, the module is highly reliable when it indicates that no fault is present.

10.2.2 Parts Standardization and Interchangeability

Standardization results in reducing to a minimum the range of parts that must be maintained and stocked. The amount of training and skill required to perform maintenance may therefore also be reduced. It simplifies the coding and labeling of parts and allows for a smaller number of tools, test equipment, and technical manuals. Interchangeability is a design policy that allows specified parts to be substituted within an assembly for any like part. Interchangeability requires both functional and physical substitution. Physical substitution requires standardization in mountings, connectors, pins, and so on, as well as compatibility in size and available space.

For example, personal computers can accept a variety of hard drives and expansion boards as a result of industry standards in cables, system board slots, and connectors. Functional substitution may be supported by the software and dual in-line package switches. Parts standardization also affects reliability design since there are fewer unique components. Therefore any reliability improvement in a standardized component will have an effect wherever that component is used. There is also a type of converse to interchangeability. If two components are not intended to be functionally interchangeable, they should *not* be physically interchangeable.

10.2.3 Modularization and Accessibility

The packaging of components in self-contained functional units, or modularization, facilitates maintenance. Typically a failure can be identified by the failure of a specific function. Under modularization, this isolates the problem to a physical unit (black box). Diagnostics can then be applied to further isolate the fault. Modularization also allows for the removal and replacement of the failed unit, provided a spare module is available, with minimum downtime. Therefore, system availability may be dramatically improved. The failed module may then be repaired separately from the operating system. Modularization permits packaging against known environmental hazards, thus decreasing the chance of failure due to an environmental stress. The amount of interaction, such as that through connectors and cables, required among modules should be minimized.

There may be several levels of modularization. For example, *line-replaceable units* refers to the highest level of modularization, in which a unit may be removed and replaced from its higher assembly. The line-replaceable unit itself may be modularized into *shop-replaceable units* which are removal components of the line-replaceable unit. Lower levels of modularization may reduce spares cost. For example, let

C = unit cost of a module

λ = its failure rate (assumed constant)

N = number of like modules

T = total repair or resupply time (including MTTR and maintenance delay times)

Then λNT is the mean number of modules in repair or resupply. Therefore the expected cost of the number of units in resupply is $C\lambda TN$. If the module is redesigned into two smaller modules each with unit cost of $C/2$ and failure rate of $\lambda/2$, and if T does not change, the expected total cost of the number of units in resupply is $2(C/2)(\lambda/2)(TN) = C\lambda TN/2$. This is half the cost required of the larger module. If the spares policy is to stock the mean number in resupply, a reduction in spares cost will be realized. On the other hand, if safety stock is to be added, the total number of spares (but not necessarily cost) may increase since safety-level stock will be needed for both module types. Sparing will be discussed more fully in Section 10.4.

Design for accessibility is concerned with the configuration of the hardware down to the discard (replacement) level. High-failure components should be more accessible than those that seldom or never fail. Access requirements may vary from complete removal of the unit to the use of certain tools, adjustments, or servicing. Size, weight, and clearances must be considered in the physical design of any removal unit. Ideally, removal of any failed unit should be possible without requiring removal of a unit that has not failed. Controls and indicators, such as meters and displays, should be placed where they can be read or operated from a normal working position in close proximity to the location of the repair or adjustment requiring their use.

10.2.4 Repair versus Replacement

Figure 10.3 shows the *indenture levels* of a system. For most systems there is an indenture level, at which it is no longer economical to repair the failed unit. Instead the failed unit is discarded and replaced with a new one. The obvious benefit in maintainability is the reduction in repair time, assuming a replacement is immediately available. It also reduces documentation, maintenance skill requirements, and amount of test equipment. Reliability advantages of discarded items are that they can be packaged (sealed) to prevent corrosion or dust, internally mounted to protect against shock and vibration, and insulated against thermal stress or temperature variation. Replacement of failed units supports the good-as-new maintenance concept at the replacement level and a steady-state renewal process. Parts within a discarded unit should have approximately the same reliability so that a single weak part does not generate a high replacement rate for the more expensive unit. The sparing policy becomes critical for discarded items. Spares provisioning is covered in Section 10.4. Here we will focus on the economic decision to repair or discard.

The primary criterion for deciding to discard rather than repair is an economical one. A simple cost model will be presented here to illustrate the approach. However, each situation has its own unique maintenance concepts and cost elements that must be considered within the model. Let

f = number of part failures over the life of the system

c = unit cost of the part

a_r = fixed cost of repair (such as test and support equipment, facilities, technical manuals, training, initial spares to fill the repair pipeline)

a_d = fixed cost of discarding (such as inventory management, warehousing, and distribution of spares); assume $a_d < a_r$

b_r = cost to repair a failure (for example, labor rate × crew size × MTTR)

b_d = cost to remove and replace a discarded part (for example, labor rate ×crew size × removal and replacement time); assume $b_d < b_r$

k = condemnation fraction (percentage of failures that cannot be repaired and must be discarded and replaced); $0 \le k < 1$

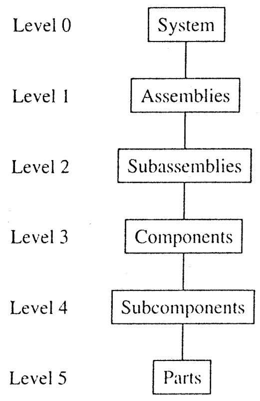


FIGURE 10.3
System indenture level structure.

Then
$$\text{Repair cost} = a_r + b_r f + ckf \tag{10.8}$$
and
$$\text{Discard cost} = a_d + (c + b_d)f$$

If the following inequality is satisfied, it is cheaper to discard than repair:

$$a_d + (c + b_d)f \le a_r + b_r f + ckf \tag{10.9}$$

An indifference trade-off curve between c and f can be generated by assuming a strict equality and rearranging terms:

$$c = \frac{a_r - a_d}{f(1-k)} + \frac{b_r - b_d}{1-k} \tag{10.10}$$

This curve has a general hyperbolic shape, as shown in Fig. 10.4. Discarding is the preferred alternative for any combination of number of failures and unit cost lying below the curve. As the number of failures increases, c approaches $(b_r - b_d)/(1 - k)$. Therefore it is never economical to repair when the unit cost is less than the difference between the variable cost to repair and the variable cost to discard. For a fixed unit cost, as the part becomes more reliable (that is, as f decreases) repair becomes less attractive. For a fixed number of failures, as the unit cost increases, repair becomes more attractive. Although these conclusions are somewhat intuitive, quantitative analysis is necessary to determine the proper decision.

EXAMPLE 10.5. A circuit board under consideration for discarding has the following cost and failure data:

a_r = \$20,000 (primarily test equipment and facilities)

a_d = \$1200 (warehouse overhead for the spares)

b_r = \$768 per failure (\$48 per hour of labor × 8 hr MTTR × 2 crew members)

b_d = \$24 / failure (\$48 per hour of labor × 0.5 hr removal and replacement × 1 crew member)

k = 0.05

The trade-off curve is $c = 783.2 + 19789.5/f$. It is shown in Fig. 10.5.

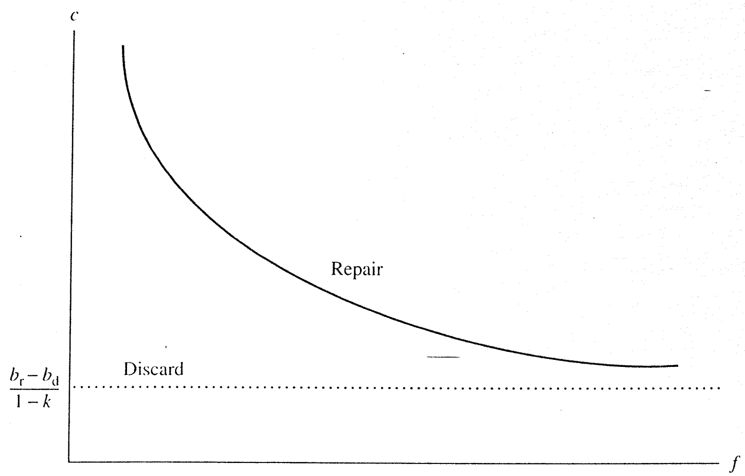


FIGURE 10.4
A repair-versus-discard trade-off curve.

Replacement model

If a component is to be minimally repaired (that is, it continues to age), then a follow-on decision to determine when to replace the component must also be made. Assuming that the component has an increasing intensity function (nonhomogeneous Poisson process), then at some time in the future, it will no longer be economical to continue to maintain it. The following model considers the affect of aging on the acquisition and support costs and assumes no salvage value, no discounting, and an

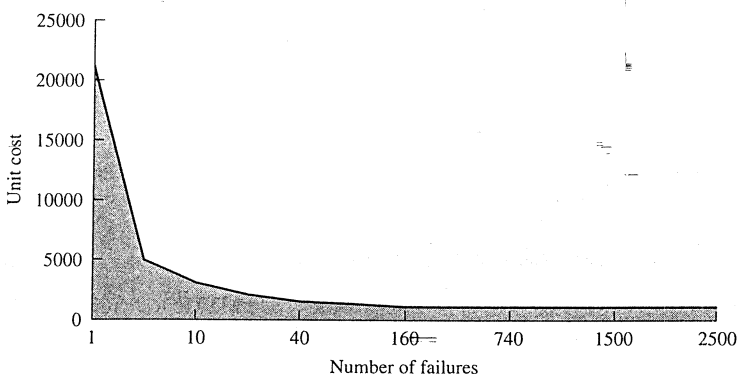


FIGURE 10.5
An indifference curve based on the circuit board cost and failure data.

infinite planning horizon. Let

$$\begin{aligned} C_u &= \text{unit cost} \\ C_o &= \text{operating cost per unit of time} \\ C_f &= \text{cost of a failure} \\ t &= \text{replacement time} \end{aligned} \tag{10.11}$$

Then
$$C(t) = C_u + C_o t + C_f \int_0^t \rho(t)\,dt$$

and the cost per unit of time is

$$C = \frac{C_u}{t} + C_o + \frac{C_f}{t}\int_0^t \rho(t)\,dt \tag{10.12}$$

For the power law process (Section 9.3) with $\rho(t) = abt^{b-1}$, Eq. (10.12) becomes

$$C = \frac{C_u}{t} + C_o + C_f a t^{b-1} \tag{10.13}$$

To minimize the cost per time unit, set $dC/dt = 0$ and solve for t:

$$\begin{aligned} \frac{dC}{dt} &= \frac{-C_u}{t^2} + (b-1)C_f a t^{b-2} = 0 \\ t* &= \left[\frac{C_u}{C_f a(b-1)}\right]^{1/b} \end{aligned} \tag{10.14}$$

If $b \le 1$, then Eq. (10.14) has no solution. We wish to replace an aging unit only.

EXAMPLE 10.6. A repairable machine has a nonhomogeneous Poisson process with an intensity function of $\rho(t) = 2 \times 10^{-6} t$ with t measured in operating hours. If the cost of a failure (repair) is \$500 and the unit cost is \$21,000, then

$$t* = \left[\frac{21{,}000}{(500)(10^{-6})}\right]^{0.5} = 6481 \text{ operating hours}$$

Therefore the machine should be replaced after 6481 operating hours.

10.2.5 Proactive Maintenance

Chapter 9 defined proactive maintenance as either preventive maintenance or predictive maintenance. In the following discussion, a preventive maintenance cost model is derived to establish an optimum preventive maintenance interval. This is followed by an example of predictive maintenance based on a vibration analysis of machines.

Preventive maintenance

Establishing the correct level of preventive maintenance can be an important economic consideration. Quite often a significant cost in manufacturing is machine repair costs coupled with costs due to lost production. These costs may be avoided in part through an aggressive preventive maintenance program. Preventive maintenance may also improve the quality of manufactured products by ensuring that machinery is producing parts within design tolerances and specifications. On the other hand, there is ample evidence suggesting that preventive maintenance is performed too frequently, often in response to the manufacturer's warranty requirements.

In Chapter 9 it was shown that under certain conditions (primarily increasing failure rates), preventive maintenance can increase reliability. However, from a maintainability point of view, preventive maintenance should be avoided since it increases system downtime. It also allows for maintenance-induced failures, which can, if the probability of an induced failure is large, decrease reliability. As will be shown in the next chapter, preventive maintenance has a direct and negative effect on the achieved availability. The frequency of preventive maintenance can be determined by comparing the cost of unscheduled maintenance with the cost and benefits of preventive maintenance. Achieving the correct level of preventive maintenance should be an objective of the maintainability design effort.

The following model determines the fixed time between consecutive preventive maintenance actions that will minimize the maintenance costs of operating the system. It assumes that preventive maintenance restores the system to as good as new condition but repair of a failed unit restores it to its condition at the time of the failure (minimal repair). Let

C_r = cost of a repair or replacement action

C_s = cost of a scheduled (preventive) maintenance activity, where $C_r > C_s$

T = time in hours between preventive maintenance activities

$\rho(t)$ = intensity function of a nonhomogeneous Poisson process

Then the expected number of failures between preventive maintenance activities is found by

$$E[N(T)] = \int_0^T \rho(t)\,dt$$

and the expected hourly cost of unscheduled maintenance is

$$\frac{C_r}{T}\int_0^T \rho(t)\,dt$$

The cost of a single scheduled maintenance activity is C_s, and the total expected hourly maintenance cost is therefore

$$TC = \frac{C_r}{T}\int_0^T \rho(t)\,dt + \frac{C_s}{T} \tag{10.15}$$

If the intensity function is the power-law process $\rho(t) = abt^{b-1}$, Eq. (10.15) becomes

$$TC = \frac{C_r}{T}\int_0^T abt^{b-1}\,dt + \frac{C_s}{T} = C_r aT^{b-1} + \frac{C_s}{T} \tag{10.16}$$

Taking the first derivative of Eq. (10.16) with respect to T and setting the result equal to zero establishes a necessary condition for a minimum:

$$\frac{dTC}{dT} = (b-1)C_r aT^{b-2} - \frac{C_s}{T^2} = 0$$

Solving for the minimum cost maintenance interval, T^*,

$$T^* = \left[\frac{C_s}{C_r a(b-1)}\right]^{1/b} \tag{10.17}$$

T^* is defined for b greater than 1. This is consistent with our earlier results, which showed that preventive maintenance should not be performed when the system is improving. T^* is a minimum when the second derivative is positive. The reader may wish to show that the second derivative is positive for all $b > 1$. The similarity between Eq. (10.17) and Eq. (10.14) should be apparent.

EXAMPLE 10.7. A grinding machine has a power law process (nonhomogeneous Poisson process) with $b = 2.4$ and $a = 2.55 \times 10^{-5}$ hr. The cost of a scheduled maintenance is $20 per hour, and the cost of an unscheduled repair is $80 per hour (production must be halted, causing additional worker idle time). Find the least-cost preventive maintenance interval.

Solution

$$T^* = \left(\frac{20}{80(2.55 \times 10^{-5})(2.4-1)}\right)^{1/2.4} = 40 \text{ hr}$$

Therefore if the machine is operated during one shift, a weekly preventive maintenance schedule should be followed.

Predictive maintenance

Although preventive maintenance, when it is appropriate, can reduce the frequency of failures, it is often difficult to determine the proper maintenance interval and scope of a preventive maintenance program. In addition to the expense of operating a preventive maintenance program, maintenance-induced failures can more than offset the reduction in inherent failures. Preventive maintenance action performed too frequently may actually decrease availability. An alternative proactive maintenance concept is predictive, or diagnostic, maintenance. In predictive maintenance, equipment or components are monitored or evaluated in order to predict when a failure is going to occur. As a result, the proper corrective maintenance can be applied prior to the failure. By avoiding failures, one may reduce damage and wear and tear on the equipment, thus prolonging its life. Maintenance can be scheduled (as is the case with preventive maintenance) so that lost production time or the impact of the nonavailability of the equipment is minimized. Often the nature of the failure can be identified so that the proper tools, test equipment, manuals, and spare parts are available at the time maintenance begins, thereby reducing the repair time. An example of diagnostic maintenance is testing an engine oil sample for contaminants prior to replacing the oil rather than replacing the oil on a scheduled basis.

Candidates for predictive maintenance are machines and equipment for which normal operating conditions can be measured and monitored over time. When these conditions deviate from the norm, it is a result of a problem that may eventually lead to a failure. The nature of the problem and its solution must then be determined. Examples of predictive maintenance techniques include thermographic monitoring, nondestructive inspection, oil analysis, calibration and alignment measurements, and vibration analysis. Oil analysis of jet engines will detect small amounts of metallic particles caused by abrasion. One can measure the amount of particles in the oil and schedule engine overhauls accordingly. Indicators on automotive brakes measure the amount of pad wear, allowing for determination of the remaining useful life before replacement is necessary.

EXAMPLE 10.8. VIBRATION ANALYSIS.[1] ~~A~~ useful predictive analysis technique for rotating and reciprocating machines is the measurement of vibration or the motion of the machine or its parts back and forth from their position of rest. Once mechanical problems develop, they will usually cause vibration. Examples of problems causing vibrations include imbalance of parts, torque variations, bent shafts, misalignment of couplings and bearings, loose fittings, bad bearings, worn or eccentric gears, worn drive belts or chains, improper tension on drive belts or chains, and resonance. Electromagnetic and aerodynamic forces may also induce abnormal vibration. By measuring a machine's vibration characteristics, its condition and any mechanical problems can be determined. The characteristics which can be measured include frequency; displacement, or the distance traveled by a vibrating part from one extreme to the other; peak velocity; acceleration; phase, or the position of a vibrating part with respect to some reference point; and spikes, caused by short-duration, high-frequency pulses of energy. Spikes may be a result of surface flaws in bearings or gears, metal-to-metal contact in rotating machines, high-pressure steam or air leaks, and cavitation of fluids.

Rotating machines experience normal vibration at the same frequency they rotate at. A defective part can often be isolated by comparing its vibration frequency with the rotational speeds of the various parts of the machine. Therefore detection of excessive

TABLE 10.1
Vibration causes

Frequency (ω)	Most likely cause	Other possible causes
$1 \times \omega$	Unbalance	Eccentric gears or pulleys Resonance Bad belts Electrical problems Reciprocating forces
$2 \times \omega$	Mechanical looseness	Misalignment Resonance Bad belts Reciprocating forces
$3 \times \omega$	Misalignment	Shaft misalignment
$4 \times \omega$	Oil whirl	Bad drive belts Resonance Background vibration

[1]This example ~~is~~ based on an engineering case study by Annette Clayton [1992].

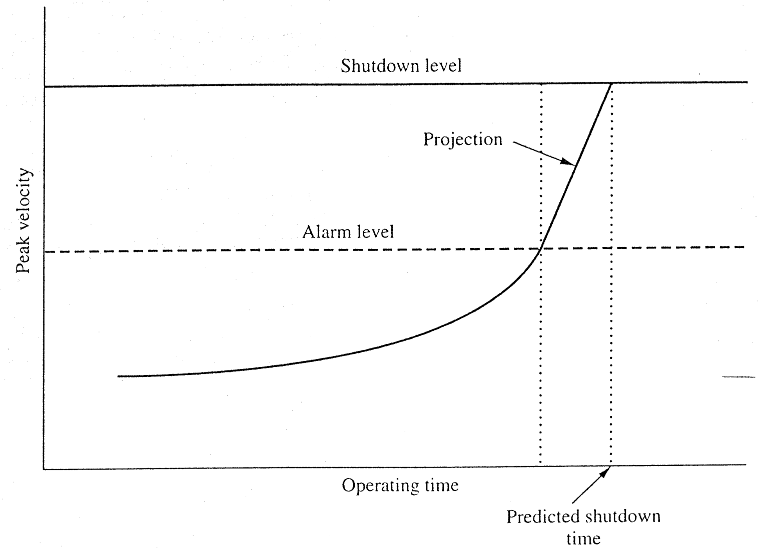


FIGURE 10.6
A vibration trend chart for projecting time to failure.

vibration at 2000 RPM would eliminate a motor running at 3600 RPM as a possible source but not a fan that the motor drives at 2000 RPM. Most machine vibrations will consist of many different frequencies and harmonics (multiples of fundamental frequencies). The fundamental frequency may be $1 \times \omega$ with $2 \times \omega$ and $3 \times \omega$ as harmonics. If there is a dominant frequency, that is, one having the largest amplitude (displacement, velocity, or acceleration), different from the fundamental frequency, it will be more indicative of a problem. Table 10.1 relates problem areas to the observed dominant frequency. Once the normal vibration characteristics have been established, charts similar to Fig. 10.6 may be used to identify problems and determine when to initiate corrective maintenance.

Predictive maintenance, by attempting to anticipate failures, is a possible solution to the problem of establishing proper preventive maintenance intervals. However, predictive maintenance often requires sophisticated monitoring and diagnostic methods and has a more limited applicability.

10.3 HUMAN FACTORS AND ERGONOMICS

Repair is essentially a human activity. Therefore designing for maintainability must include consideration for the capabilities and limitations of maintenance personnel, the environment in which repair must be accomplished, the ~~physical~~ design of the equipment, and safety. Human factors, or ergonomics, is concerned with the interaction of humans with the physical objects they encounter in performing their activities. It is the study of fitting the individual to a physical process such as system repair. The objective is to maintain productivity without compromising safety and while minimizing the mental and physical demands placed on the individual. In maintenance system design, we are concerned with the ergonomic subtopics of anthropometry, biomechanics, physiology, psychophysics, human perception, and human performance. *Anthropometry* is the study of human body dimensions. Static anthropometry deals with such attributes as height and arm length, and dynamic anthropometry deals with physical limitations of individual movements, such as reaching for a tool from a sitting position. *Biomechanics* is interested in how movements are performed, and *physiology* studies the basic processes of the human being, such as the movement of the muscle and joint system. Of concern here are the physiological effects of doing work, such as fatigue, stress, and strain. Both *psychophysics*, which studies human perceptual capability, and *human perception* are concerned with how information is received through the senses, stored (remembered), and interpreted. *Human performance* is concerned with how humans interface with various components of the workplace. In designing for maintainability, five human factor areas are to be considered. These are controls, displays, tool and equipment design, the workplace, and the environment. A recommended source for additional information concerning human factors is Sanders and McCormick [1993].

The physical characteristics of individuals have been extensively studied and documented. Averages (means), standard deviations, and percentiles of weight, height, arm reach, hand size, and so on, are readily available and should be utilized by design engineers. In designing for a given population, minimum and maximum values (such as arm reach) should include a specified percentage of the population. Ninety-five percent is often used, and tables of anthropometric data showing the 5th and 95th percentiles have been published. Examples of typical data are given in Table 10.2. Extensive tables have been compiled by Woodson, Tillman, and Tillman in the *Human Factors Design Handbook* [1992]. In addition to anthropometric data, the handbook discusses the human factors considerations in system planning and layout, control panel arrangements, component and product design, design standards, tools, controls, handles, and displays. Weight-lifting and weight-carrying limitations are often an important consideration. As a general rule, individuals should not carry more than 35 percent of their body weight, weights in excess of 45 pounds should be carried by two persons; and for weights of more than 90 pounds, a mechanical lifting device should be utilized. Labor skills should be identified with specific tasks. For example, fault isolation and repair normally require higher skill levels and

TABLE 10.2
Selected human factors data

Measurement	5th percentile	50th percentile	95th percentile
Eye height—males	60.8 in	64.7 in	68.7 in
Eye height—females	57.3	60.3	65.3
Forward reach—males	31.9	34.6	37.3
Forward reach—females	29.7	31.8	34.1
Standing height—males	63.6	68.3	72.8
Standing height—females	59.0	62.9	67.1
Weight—males	124 lb	168 lb	224 lb
Weight—females	104	139	208

experience than preventive maintenance or simple removal and replacement tasks. Other factors such as vision and illumination requirements and hearing and noise levels should also be considered.

In hostile working environments personnel efficiency decreases. To compensate somewhat for this decrease, workloads can be reduced, personnel can be rotated more frequently, or protection against the environment provided for. There are four primary environmental factors to be considered. These are illumination, noise, vibration, and the ambient air, consisting of temperature and humidity.

The physical design of the equipment must provide for personnel access to all failed components. This access may require inspection, testing, aligning or adjusting, repair, or complete removal of the failed unit. In decreasing preference, access may be direct or by means of an opening with no cover, pull-out shelves or drawers, hinged doors, or removal panels. In the last case, quick-opening fasteners are preferred over screws or nuts and bolts. The type and location of gauges and other instruments must recognize personnel heights and vision distances. Indicator devices can be analog or digital, with analog generally preferred. Measurement scales should be clearly marked, and increasing values should be in an upward or clockwise direction. When color-coding parts and connectors, remember that 5 percent of the population is color-blind.

Safety requires consideration of human error. Human error is a failure to complete a task as specified. This includes actions resulting from misinterpretation as well as errors in timing. Causes of human error in maintenance may include poor training, inadequate procedures and guidelines, a poor working environment, and improper motivation. Errors may occur when the stress level is either too high or too low. Examples of high stress include time-constrained repairs, high-risk (hazard) environments, and repairs having an important consequence if not completed successfully. When the stress level is too low, boredom and carelessness may result. Human reliability may be defined simply as

$$R_h = 1 - \frac{e}{n}$$

where n is the number of times a task has been performed and e is the number of unsuccessful completions. Errors may not be independent, since incorrectly performed tasks may have a tendency to be repeated.

10.4 MAINTENANCE AND SPARES PROVISIONING

In Chapter 9 total maintenance downtime was defined to include maintenance delay time and supply delay time. Maintenance delay occurs when there are no available repair resources, and supply delay occurs when a replacement part or component is not immediately available. These delays are not inherent characteristics of the equipment but instead are dependent on the number of maintenance resources and the number of available spares. Spares may include the number of spare systems (that is, standby redundancy) as well as the number of removable components and part spares. Establishing adequate spares support can have as much impact on system availability as the inherent reliability and maintainability. For example, Hall and Clark [1987] describe a case in which the reliability of a radar system was doubled, the spares were decreased accordingly, and a decrease in availability was observed. This decrease was attributed to an oversimplified spares inventory model.

10.4.1 Finite Population Queuing Model with Spares

When there are a limited number of repair channels available, some queuing (waiting) for a repair channel may occur. If both the failure distribution and the repair distribution are exponential, the pure birth-death finite-server queuing model may be used to determine the maintenance delay time as a function of the number of repair channels (servers). This model assumes all repair channels are identical (that is, have the same MTTR). The steady-state solution to the general birth-death queuing model is given in Appendix 10A. The model presented here is often referred to as the *machine repair problem*. It assumes that a finite (small) number of repairable units are to be supported and that there is no supply delay for components or parts. Let

m = number of operating units
k = number of parallel repair channels (servers)
s = number of spare (standby redundant) units
λ = single-unit failure rate (assumed constant) = 1/MTBF
μ = repair rate (assumed constant) = 1/MTTR
P_n = steady-state probability of n units being in repair; $n = 0, 1, \ldots, m + s$

Figure 10.7 illustrates the repair cycle process being modeled. From the pure birth-death queuing model:

$$P_n = C_n P_0 \qquad n = 1, 2, \ldots, m + s \tag{10.18}$$

where
$$P_0 = \frac{1}{1 + \sum_{n=1}^{m+s} C_n}$$

and for $k > s$:
$$C_n = \begin{cases} \dfrac{(m\lambda)^n}{n!\mu^n} & \text{for } n = 1, \ldots, s \\[2ex] \dfrac{m^s \lambda^n m!}{n!\mu^n (m + s - n)!} & \text{for } n = s + 1, \ldots, k \\[2ex] \dfrac{m^s \lambda^n m!}{\mu^n k! k^{n-k} (m + s - n)!} & \text{for } n = k + 1, \ldots, m + s \end{cases}$$

For $k \le s$:
$$C_n = \begin{cases} \dfrac{(m\lambda)^n}{n!\mu^n} & \text{for } n = 1, \ldots, k \\[2ex] \dfrac{(m\lambda)^n}{k!\mu^n k^{n-k}} & \text{for } n = k + 1, \ldots, s \\[2ex] \dfrac{m^s \lambda^n m!}{\mu^n k! k^{n-k} (m + s - n)!} & \text{for } n = s + 1, \ldots, m + s \end{cases}$$

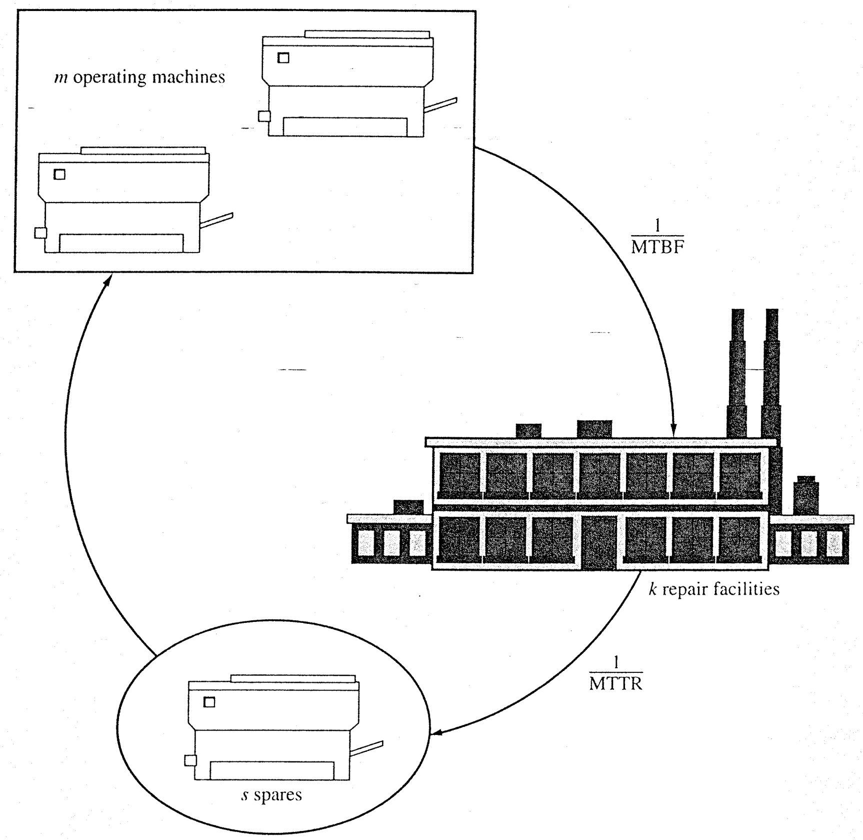


FIGURE 10.7
The repair cycle process.

The expected number of units in repair is given by

$$L_r = \sum_{n=1}^{m+s} n P_n \tag{10.19}$$

and the expected number of units operating is

$$L_o = m \sum_{n=0}^{s} P_n + \sum_{n=s+1}^{m+s} (m + s - n) P_n \tag{10.20}$$

The total downtime (repair time plus maintenance delay time) is

$$W = \frac{L_r}{\lambda} \tag{10.21}$$

where
$$\bar{\lambda} = m\lambda \sum_{n=0}^{s} P_n + \lambda \sum_{n=s+1}^{m+s} (m+s-n)P_n \tag{10.22}$$

is the average arrival rate into the repair facility. The average percentage of the m units that are operating is $100\,L_o/m$.

EXAMPLE 10.9. A company operates three machines with one spare machine available. There are two maintenance technicians (that is, repair channels). Each machine fails at a constant rate of twice per week. The mean time to repair is one and two-thirds days. Assuming a five-day work week, determine the percentage of operating machines and the average downtime.

Solution. Given are $\lambda = 2$ per week, $\mu = 3$ per week, $m = 3$, $k = 2$, and $s = 1$. Then

$$C_1 = C_2 = 2 \qquad C_3 = \frac{4}{3} \qquad C_4 = \frac{4}{9}$$

$$P_0 = \left(1 + 2 + 2 + \frac{4}{3} + \frac{4}{9}\right)^{-1} = \frac{9}{61} \qquad P_1 = P_2 = (2)\left(\frac{9}{61}\right) = \frac{18}{61}$$

$$P_3 = \left(\frac{4}{3}\right)\left(\frac{9}{61}\right) = \frac{12}{61} \qquad P_4 = \left(\frac{4}{9}\right)\left(\frac{9}{61}\right) = \frac{4}{61}$$

$$L_r = \frac{18}{61} + \frac{36}{61} + \frac{36}{61} + \frac{16}{61} = \frac{106}{61} = 1.74$$

$$L_o = 3\left(\frac{9}{61} + \frac{18}{61}\right) + 2\left(\frac{18}{61}\right) + \left(\frac{12}{61}\right) = \frac{129}{61} = 2.11$$

$$\text{Percentage operating} = 100\left(\frac{2.11}{3}\right) = 70.3$$

$$\bar{\lambda} = 6\left(\frac{27}{61}\right) + 4\left(\frac{18}{61}\right) + 2\left(\frac{12}{61}\right) = \frac{258}{61}$$

and
$$W = \text{MDT} + \text{MTTR} = \frac{106/61}{258/61} = 0.41 \text{ week}$$

or 2.05 days. Since MTTR $= \frac{5}{3}$ days, MDT $= W - \text{MTTR} = 0.383$ day, or a little over one-third of a day is spent, on the average, waiting for a maintenance technician.

Using this model, trade-offs between the number of repair channels, k, and the number of spares, s, can be made. For example, if C_f is the fixed cost of repair, C_k is the present-value equivalent unit cost of a repair channel, C_s is the cost of a spare unit, and B is a total investment and operating budget allocated for maintenance and supply resources, we may wish to solve the following problem:

$$\text{Max } L_o$$

$$\text{Subject to} \quad C_k k + C_s s \le B - C_f$$

$$k = 1, 2, \ldots, \left\langle \frac{B - C_f}{C_k} \right\rangle$$

$$s = 0, 1, 2, \ldots, \left\langle \frac{B - C_f}{C_s} \right\rangle$$

where $\langle x \rangle$ is the integer portion of x. Since this is only a two-variable problem, a direct search technique can be used to find the solution.

EXAMPLE 10.10. For the machine in Example 10.9, C_k = \$40,000, C_s = \$60,000, and $B - C_f$ = \$150,000. Steady-state queuing solutions were found for the following alternatives:

s	k	E [number operating]	Percentage operating	Variable cost (\$1000)
0	3	1.8	60	120
0	4	1.8	60	160
1	**2**	**2.11**	**70.3**	**140**
1	3	2.28	76	180
2	1	1.45	48.3	160

The solution of $s = 1$, $k = 2$ is preferred if the budget of \$150,000 must not be exceeded. The next increase in the percentage operating will occur at a variable operating cost budget of \$180,000 since an additional repair channel can then be obtained. Observe that two spares and one repair channel, although costing \$160,000, result in a smaller operating percentage than the cheaper solution obtained with two repair channels and one spare.

If the assumptions concerning constant failure and repair rates cannot be made, computer simulation can be used to estimate the percentage operating and the mean downtime as functions of the number of repair channels and spares. Both Law and Kelton [1991] and Banks, Carlson, and Nelson [1996] are excellent sources on computer simulation modeling and include interesting discussions on machine downtimes. The above queuing model, however, may still provide a reasonable approximation to the expected number in repair and expected number operating even if the assumptions are not precisely met.

10.4.2 Component Sparing

The previous example quantified the effect that redundant units have on system performance but did not include the effect of component spares. In order to establish proper levels of replacement spares for components and parts, these levels must be somehow related to system performance. This is accomplished by measuring the effect these spares have on the mean supply delay time (SDT). Obviously, if there are ample spares, then SDT ≈ 0. In many cases this alternative is too costly and trade-offs must be considered among spare components, repair channels, and (standby) redundant units. In general, we can set

$$\frac{1}{\mu} = \text{MTTR} + \text{SDT} \tag{10.23}$$

and apply the queuing equations (10.18–10.22). What remains is to compute SDT as a function of the various component and spare part levels. If the mean supply delay time for each component can be found, a weighted average of the components' expected delay times can be used where the weights are based on the components' average failure rates as computed by Eq. (10.22). Therefore, assuming a total of p unique repairable components and replaceable parts,

$$\mathrm{SDT}(s_1, s_2, \ldots, s_p) = \frac{\sum_{i=1}^{p} \overline{\lambda}_i \mathrm{SDT}_i(s_i)}{\sum_{i=1}^{p} \overline{\lambda}_i} \tag{10.24}$$

where the subscript i refers to the ith component or part. To find $\mathrm{SDT}_i(s_i)$, consider that the only time there will be waiting for a part is when there are backorders for that part. Therefore, the expected supply delay time should be related to the expected number of backorders. If the part never has backorders (ample spares), there will never be any waiting incurred. At the other extreme, if there are no spares in the system, every demand will be backordered and the total delay time will be MTR_i, the mean time to restore (or resupply) the ith part. For the ith component (part), let $p_i(x)$ be the steady-state probability of x units being in repair (resupply) and let s_i be the number of spare units in the system. Then the expected number of backorders (EBO) is

$$\mathrm{EBO}_i(s_i) = \sum_{x=s_i+1}^{m+s} (x - s_i)p_i(x) \tag{10.25}$$

Then let

$$\mathrm{SDT}_i(s_i) = \frac{\mathrm{EBO}_i(s_i)}{\overline{\lambda}_i} \tag{10.26}$$

If the average failure rate in Eq. (10.26) is in failures (units) per day, the supply delay time will be measured in units/(units/day), or in days, and represents the average time the system repair process is waiting on the ith spare part to repair a failed system unit. To verify that Eq. (10.26) behaves as expected, from Eqs. (10.19), (10.21), and (10.25) we have

$$\mathrm{SDT}_i(0) = \frac{\mathrm{EBO}_i(0)}{\overline{\lambda}_i} = \frac{L_{r,i}}{\overline{\lambda}_i} = \frac{\overline{\lambda}_i W_i}{\overline{\lambda}_i} = W_i \tag{10.27}$$

where $$W_i = \mathrm{MTTR}_i + \mathrm{MDT}_i \tag{10.28}$$

That is, if there are no spares in the system for component i, the expected delay time for replacing (in this case restoring) the failed component is the sum of the mean repair time and the mean time spent waiting for a repair channel.

If $s \to \infty$, then $\mathrm{EBO}_i(s_i)$ and $\mathrm{SDT}_i(s_i) \to 0$ as expected. From Eq. (10.26), it is apparent that finding a value of s that minimizes expected backorders is equivalent to minimizing SDT. Given a finite (small) number of repair channels, the queuing model developed previously can be applied to each component to find $p_i(x)$. If there are ample (infinite) repair facilities or the number of components operating is large (that is, an infinite population), other variations of the birth-death queuing model may be used. Excellent discussions on queuing models may be found in Hillier and Lieberman [1990], Kleinrock [1975], or Gross and Harris [1985].

Palm's theorem

If both infinite repair channels (no queuing) and infinite population assumptions can be made, Palm's theorem may be used to determine the steady-state number in repair (or resupply):

> If failures occur according to a stationary Poisson process and repair times are independent, identically distributed random variables with a finite mean (MTTR), the steady-state probability distribution of the number of units in repair is Poisson and is independent of the particular service distribution.

The significance of Palm's theorem is that regardless of the repair-time distribution, the distribution of the number of items in repair is Poisson, having the following probability mass function:

$$\Pr_i\{X = x\} = p_i(x) = \frac{e^{-\lambda_i T_i}(\lambda_i T_i)^x}{x!} \qquad x = 0, 1, \ldots \tag{10.29}$$

where $E[X] = \lambda_i T_i$ is the mean of the distribution (the mean number of components in repair) and $T_i = \mathrm{MTTR}_i$ for repairable components. If the component (or part) is nonrepairable, T_i represents the mean reorder, resupply, or production lead-time. As a result of the infinite population assumption, λ_i is the (constant) system failure rate into repair. If component i occurs q_i times on each of m operating units, then

$$\lambda_i \approx \frac{mq_i}{\mathrm{MTBF}_i} \tag{10.30}$$

The product mq_i must be large enough that the failure rate, λ_i, is not dependent on the number of components currently in repair. A more accurate failure rate may be obtained by replacing m with L_o (Eq. (10.20)). However, the value of L_o depends in part on the component spare levels through Eq. (10.23).

Further discussion on Palm's theorem and repairable spares inventory modeling may be found in Carrillo [1991], Gross [1982], and Sherbrooke [1968; 1992]. Other relevant queuing and inventory models are presented by Berg and Posner [1990]; Dhakar, Schmidt, and Miller [1994]; Ebeling [1991]; Graves [1985]; Gross and Ince [1981]; Gross, Miller, and Soland [1985]; Hall and Clark [1987]; and Mani and Sarma [1984].

EXAMPLE 10.11. A particular computer consists of three relatively high-failure components: a power supply, the system board, and a hard drive, having MTBFs of 16,800, 35,000, and 13,440 hr, respectively. Replacement parts must be ordered and shipped from the distributor. Normal delivery time is 3 weeks except for the hard drive, which must be ordered from a different distributor and has an expected delivery time of 5 weeks. A company has purchased 100 of these computers along with three spare power supplies and four hard drives. Determine the expected number of backorders for each component and the expected supply delay time when a computer fails because of one of these components. Assume exponential failure times.

Solution. The results, for which Palm's theorem has been utilized, are summarized in the following table. The expected backorders are based on Eq. (10.25), and supply delay times were computed from Eq. (10.26).

Component	λ = 100/MTBF	Resupply time, T, hr	Expected number in resupply, λT	Expected number of backorders	Supply delay time, hrs
Power supply	1/168	504	3	0.6721	113
System board	1/350	504	1.44	1.44	504
Hard drive	1/134.4	840	6.25	2.448	329
System	0.01625				280.6

Therefore the average supply delay time for a computer replacement part is 280.6 hr, or 11.7 days.

10.5 MAINTAINABILITY PREDICTION AND DEMONSTRATION

Once the initial product design has been established, an early assessment as to the degree to which the design is meeting the maintainability goals must be made. Those components needing improvement in their design are identified, and further refinements and trade-offs are made. During the later part of the development cycle or during the initial fielding of the system, a maintainability demonstration is performed on prototypes or early production units. A bottoms-up approach and a work measurement approach may be used for maintainability prediction and demonstration, respectively. An alternative demonstration approach to work measurement is based on sequential sampling theory, discussed in Chapter 13. The statistical techniques for fitting probability distributions presented in Chapters 15 and 16 can be used to determine the repair-time distribution and its parameters from repair-time data obtained during a maintainability demonstration.

10.5.1 Maintainability Prediction

In order to predict maintainability parameters, a bottoms-up analysis may be used early in the design phase before prototypes are available. It consists of estimating, at the lowest practical level, the relevant component, part, or failure-mode maintainability measures and then combining these values to obtain higher assembly values and a system value. For example, if the MTTR is the primary measure of maintainability, then Eq. (9.23) can be used to determine the MTTR for the higher-level assemblies and for the system. To obtain estimates of component MTTRs, each component or failure-mode repair activity is broken into individual subtasks such as those identified in Fig. 9.1. Individual subtask times can be estimated directly, predicted from past experience, or obtained from published databases or work standards. For example, *Military Handbook: Maintainability Prediction* (MIL-HDBK-472) [1984] provides maintenance task time standards. Jager and Krause [1987] describe procedures for automating the calculations specified in *Military Handbook* 472. If the maintenance design and concepts have established a crew size for each repair task, then the number of maintenance hours per operating hour can also be estimated. Preventive maintenance tasks may be included in the analysis. Spreadsheets are ideal tools for performing the calculations, as shown in the next example.

EXAMPLE 10.12. A maintenance prediction of a two-module assembly consisted of the bottoms-up analysis listed in Table 10.3. Individual subtask times were either obtained from published time standards or were measured on comparable fielded equipment.

10.5.2 Maintainability Demonstration

As part of system testing during the detailed design phase, a maintainability demonstration is conducted to verify that the stated goals are being met. The demonstration should be conducted under operating and environmental conditions similar to those to be encountered once the system is operational. The test consists of simulating system failures and then requiring the maintenance technician(s) to check out the system and perform necessary corrective maintenance. Task times are recorded along with any difficulties or problems encountered in performing the repair tasks. The type and frequency of the simulated failures should be in proportion to the failure rates of the various failure modes. If n is the total number of failures and repairs to be demonstrated, then the integer portion of $\lambda_i q_i n / \sum \lambda_i q_i$ is the number of repair tasks to be observed for the ith failure mode. The order of the simulated failures should be determined randomly.

When the maintainability goal is the MTTR or the percentage of repairs to be accomplished within a specified time, a confidence interval based on the recorded repair times may be constructed. A confidence interval consists of a lower and an upper bound (two-sided) or an upper bound (one-sided) for a probability distribution parameter. The degree of confidence is the probability of capturing the true population parameter within the bounds computed from the sample times. A $100(1-\alpha)$ percent upper-bound confidence interval for the MTTR consists of finding an MTTR_U such that

$$\Pr\{\text{MTTR} \le \text{MTTR}_\text{U}\} = 1 - \alpha$$

If $t_1, t_2, \ldots, t_n$ are the observed repair times, then

$$\bar{t} = \frac{\sum_{i=1}^{n} t_i}{n} \qquad \text{and} \qquad s^2 = \frac{\sum_{i=1}^{n} (t_i - \bar{t})^2}{n-1}$$

are the sample mean time to repair and sample variance, respectively. From sampling theory and the central limit theorem (see, for example, Ross [1987]), the statistic

$$t = \frac{\bar{t} - \text{MTTR}}{s/\sqrt{n}}$$

has an approximate t-distribution with $n-1$ degrees of freedom (the parameter of the t-distribution). Therefore, since the t-distribution is symmetrical about zero,

$$\Pr\{t < t_{\alpha,n-1}\} = \Pr\{t > -t_{\alpha,n-1}\} = 1 - \alpha$$

where $t_{\alpha,n-1}$ is obtained from the table of selected values from the t-distribution, which is Table A.2 of the Appendix. Substituting,

$$\Pr\left\{\frac{\bar{t} - \text{MTTR}}{s/\sqrt{n}} > -t_{\alpha,n-1}\right\} = 1 - \alpha$$

TABLE 10.3
Bottom-up analysis using spreadsheet for Example 10.12

	Failure mode	Failure rate	Number q_i	Access	Diagnosis	Replacement	Repair	Alignment	Verification	MTTR_i	Failure rate × q × MTTR
				Maintainance subtask times, hr							
Module 1			2								
	A	0.121		0.20	0.50	0.05	2.15	0.00	0.03	2.93	0.71
	B	0.023		0.30	0.60	0.10	0.00	0.08	0.45	1.53	0.07
	C	0.008		0.30	1.00	0.08	0.33	0.05	0.00	1.76	0.03
	D	0.21		0.20	0.40	0.00	1.10	0.21	0.12	2.03	0.85
Subtotal		0.362									1.66
Module 2			3								
	E	0.301		0.31	0.80	0.30	0.00	0.20	0.30	1.91	1.72
	F	0.076		0.11	0.60	0.00	1.33	0.05	0.20	2.29	0.52
	G	0.045		0.05	0.75	0.00	1.50	0.00	0.20	2.50	0.34
Subtotal		0.422									2.58
Total		0.784									4.24

$1.99 = q_i \times \text{failure rate} = 2(0.362) + 3(0.422)$

MTTR = 4.24/1.99 = 2.13

and rearranging terms,

$$\Pr\left\{\text{MTTR} < \bar{t} + \frac{s}{\sqrt{n}} t_{\alpha,n-1}\right\} = 1 - \alpha$$

Therefore $\text{MTTR}_U = \bar{t} + (s/\sqrt{n})t_{\alpha,n-1}$ provides a $100(1-\alpha)$ percent upper bound on the system MTTR. As a result, if MTTR_g is the maintainability goal, then the goal is being met if $\text{MTTR}_U \le \text{MTTR}_g$.

Note that the random variable in the above probability statements is MTTR_U and not the system MTTR. The system MTTR is a constant, albeit unknown, parameter to be estimated. The above confidence interval is exact if the repair-time distribution is normal; otherwise, a minimum sample size of 20 to 30 repair tasks is necessary in order to obtain a good approximation under the central limit theorem. The more skewed the underlying distribution, the larger is the required sample size.

EXAMPLE 10.13. The MTTR goal of a machine used in the production of automobile brake parts is 5.5 hr. During a maintainability demonstration, 30 repair tasks were simulated, with the following results in hours:

2.4	5.7	7.3	3.2	13.0	2.1	3.7	5.0	2.3	3.7	9.0	3.7	6.4	4.6	3.2
2.1	3.5	2.4	6.1	3.7	5.6	2.1	7.5	8.6	5.1	6.8	11.7	7.2	6.2	5.2

From the data, $\bar{t} = 5.3$ and $s = 2.76$. Therefore, with $t_{0.05,29} = 1.699$, a 95 percent upper bound confidence interval is

$$\text{MTTR}_U = 5.3 + \frac{1.699(2.76)}{5.477} = 6.156$$

Since $\text{MTTR}_U > 5.5$, we cannot conclude that the MTTR goal has been met. At best, we can state that we are 95 percent confident that the MTTR is 6.156 hr or less.

If the maintainability goal is expressed in terms of a percentage repaired, p, in time t_p as given by Eq. (10.1), then the sample proportion, $\bar{p} = X/n$, can be used in constructing a confidence interval for p where X is the number of repairs accomplished within time t_p from among the n repairs attempted. Although X has a binomial distribution, for large sample sizes it may be approximated by the normal distribution. Therefore a one-sided $100(1-\alpha)$ percent *lower* bound confidence interval for p is given by

$$p_L = \bar{p} - z_\alpha \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

where $\Pr\{z > z_\alpha\} = \alpha$ and z is the standardized normal variate.

EXAMPLE 10.14 From a sample of 85 repair times collected on the corrective maintenance of office reproduction machines, 78 of the repairs were completed within 5 hr. Is the contractor's maintenance agreement to complete 90 percent of the repairs within 5 hr being met?

Solution. On the basis of a 95 percent confidence interval, $\bar{p} = 78/85 = 0.9176$ and

$$p_L = 0.9176 - 1.65\sqrt{\frac{0.9176(0.0824)}{85}} = 0.8684$$

We are 95 percent confident that the percentage of repairs completed in 5 hr is no less than 86.83 percent. Therefore, we cannot conclude that the contractor is meeting the agreement.

APPENDIX 10A BIRTH-DEATH QUEUING MODEL

The birth-death process assumes an exponential failure distribution and an exponential repair time such that the failure rate and the repair rate depend on the state of the system. Therefore Markov analysis as presented in Chapter 6 can be applied. System states are defined to be the number of units in repair (or resupply). Let

$$\lambda_i = \text{the failure rate with } i \text{ units in repair}$$
$$\mu_i = \text{the repair rate with } i \text{ units in repair}$$
$$P_i(t) = \text{the probability of } i \text{ units in repair at time } t$$

In the rate diagram of Fig. 10.8, the state of the system is the number in repair, with the failure rates and repair rates being dependent on the state. For state $i = 0$ the following differential equation is constructed:

$$\frac{dP_0(t)}{dt} = \mu_1 P_1(t) - \lambda_0 P_0(t)$$

For the ith state, with $i \neq 0$,

$$\frac{dP_i(t)}{dt} = \lambda_{i-1}P_{i-1}(t) + \mu_{i+1}P_{i+1}(t) - (\lambda_i + \mu_i)P_i(t)$$

To obtain the steady-state solution, let $t \to \infty$ and therefore $dP_i(t)/dt \to 0$. Rewriting $P_i(t)$ as P_i, the steady-state equations become

$$0 = \mu_1 P_1 - \lambda_0 P_0$$
$$0 = \lambda_{i-1}P_{i-1} + \mu_{i+1}P_{i+1} - (\lambda_i + \mu_i)P_i \qquad i = 1, 2, \ldots$$

Rewriting the first equation,

$$P_1 = \frac{\lambda_0}{\mu_1}P_0$$

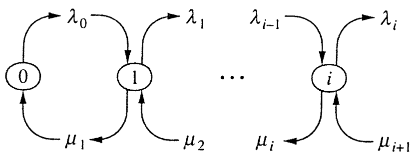


FIGURE 10.8
Rate diagram for the birth-death queuing process.

From the second equation when $i = 1$,

$$P_2 = \frac{(\lambda_1 + \mu_1)P_1}{\mu_2} - \frac{\lambda_0}{\mu_2}P_0 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0$$

where the second equality is obtained by substitution for P_1. Solving recursively,

$$P_3 = \frac{(\lambda_2 + \mu_2)}{\mu_3}P_2 - \frac{\lambda_1}{\mu_3}P_1 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0$$

The pattern repeats. By using induction and letting $C_i = \lambda_0\lambda_1 \cdots \lambda_{i-1}/\mu_1\mu_2 \cdots \mu_i$, it can be shown that

$$P_i = C_i P_0 \qquad i = 1, 2, \ldots$$

What remains is to find P_0. Since the P_i form a probability distribution, we know that $\sum_{i=0}^{\infty} P_i = 1$, or

$$P_0 + \sum_{i=1}^{\infty} C_i P_0 = P_0(1 + \sum_{i=1}^{\infty} C_i) = 1$$

Therefore

$$P_0 = \left[1 + \sum_{i=1}^{\infty} C_i\right]^{-1}$$

The mean of the probability distribution is the average number in repair and is found from

$$L = \sum_{i=1}^{\infty} iP_i$$

The average failure (arrival) rate into repair is given by

$$\bar{\lambda} = \sum_{i=0}^{\infty} \lambda_i P_i$$

The average time in repair, W, is related to the mean number in the system by Little's formula,

$$W = \frac{L}{\bar{\lambda}}$$

It is also true that $W = \text{MTTR} + \text{WQ}$ where WQ is the mean time spent waiting for repair. The mean waiting time can be a function of the number of (parallel) repair channels, k, as a result of the effect k has on the repair rate, as shown below:

$$\mu_i = \begin{cases} i\mu & \text{if } i \leq k \\ k\mu & \text{if } i > k \end{cases}$$

where $\mu = 1/\text{MTTR}$. In most of the applications of interest in this text, the system state is a finite number since the number of systems or components being considered is finite. In this case,

$$\lambda_i = \begin{cases} n\lambda & \text{for } i \le s \\ (n+s-i)\lambda & \text{for } i > s \end{cases}$$

for $i = 0, 1, \ldots, n+s$, where n is the number of operating units and s is the number of spare (standby redundant) units.

EXERCISES

10.1 ~~A~~ system has a repair time that has been demonstrated to be normally distributed with a mean of 5.6 hr and a standard deviation of 1.4 hr. Contract specifications require 95 percent of the repairs to be completed within 8 hr. The average crew size is two. In addition, the number of maintenance work hours per operating hour is not to exceed 1 hr for every 100 operating hours (failure-based maintenance only). A reliability demonstration resulted in time between failures being Weibull with $\theta = 3000$ hr and $\beta = 1.3$. Are the specifications being met? Assume a steady-state renewal process.

10.2 Determine the mean system downtime for a system having a constant failure rate of 0.0521 failure per hour, a lognormal repair time with a median value of 8.6 hr and $s = 1.5$, and a scheduled 4-hr preventive maintenance every 10 days.

10.3 Two alternative mechanical pumps are being considered for use in a cooling system. Their relevant parameters are as follows:

Pump	Failure distribution, days	Repair distribution, hrs	Unit cost	Repair crew size
1	Nonhomogeneous Poisson process: $\rho(t) = 9.2 \times 10^{-6}(2.1)t^{1.1}$	Lognormal ($t_{med} = 5.1, s = 0.81$)	\$450	3
2	Exponential ($\lambda = 0.06$)	Normal ($\mu = 6.2, \sigma = 1.5$)	\$525	2

The labor rate for a maintenance technician is \$50 per hour. The design life of the system is 5 yr. On the basis of the available information, which pump is preferred?

10.4 An aircraft (Exercise 9.10) consists of the following subsystems having the reliability parameters shown:

Subsystem	Failure distribution	Parameters
Propulsion	Weibull	$\theta = 1000$; $\beta = 1.7$
Avionics	Exponential	$\lambda = 0.003$
Structures	Weibull	$\theta = 2000$; $\beta = 2.1$
Electrical	Weibull	$\theta = 870$; $\beta = 1.8$
Environmental	Exponential	$\lambda = 0.001$

A system MTTR of 5 hr based on the average number of failures over 50,000 hr of operation is desired. Allocate the MTTR to the above subsystems. Assume that a failure will restore the subsystem to as good as new.

10.5 Ray Pear fixes broken things. Things break according to a nonhomogeneous Poisson process with $\rho(t) = 9.4 \times 10^{-6} t^{0.5}$. Ray currently follows a 100-hr preventive maintenance program. It costs Ray \$100 to perform a preventive maintenance action, whereas a failure costs \$2000 to repair. How much is it costing Ray per operating hour by not following an optimum preventive maintenance schedule?

10.6 A fuel pump used in a jet engine has a nonhomogeneous Poisson failure process with a power-law intensity function where $a = 6 \times 10^{-9}$ operating hours and $b = 2.5$. If the pump is to be repaired, it will cost \$500 for a special set of tools, and the MTTR will be 6 hr. If the pump is discarded and replaced, there is no fixed cost, and the removal and replacement time is 1 hr. Labor cost is \$55 per hour. The engine is to be overhauled at 10,000 hours at which time the fuel pump is to be replaced. Determine the unit cost for which it is no longer economical to discard the pump on failure and a repair policy should be implemented. A crew size of one is needed for either repair or discard tasks. A condemnation rate of 5 percent should be assumed.

10.7 For the fuel pump in Exercise 10.6, determine the optimum replacement time if the unit cost is \$400 and the pump is repaired on failure. Assume that it has an intensity function of $\rho(t) = (6 \times 10^{-9})(2.5)t^{1.5}$. Compare this solution with the current policy to overhaul the engine and replace the pump at 10,000 hr. Equate the cost per failure to the variable cost of repair in Exercise 10.6.

10.8 Determine the optimum time between preventive maintenance activities for a machine that has a power-law intensity function (nonhomogeneous Poisson process) with $b = 1.5$ and $a = 2.47 \times 10^{-4}$ hr. The cost of performing the scheduled maintenance is \$50 per hour, and the cost of repairing the failed machine is \$200 per hour as a result of lost production time. Express the preventive maintenance interval in months if the machine is operated 8 hr per day. There are 20 working days in a month.

10.9 Construct a cost model to be used in performing a level-of-repair analysis. Assume that a component can be repaired locally or at a central repair facility. Identify and include the relevant factors and costs in your model.

10.10 The Power Broker Company, a moderate-size midwest electric utility company, maintains two turbine generators on-line with two spare generators on standby. A generator will fail on the average of once every three days (MTBF), at which time a single repair crew (if available) will restore the generator to as good as new condition. The mean time to repair is one day. The component inventory levels are poor, resulting in a system expected supply delay time (SDT) of one-half of a day. Failures are Poisson, and the restoration time is exponential.

(a) Determine the probability distribution for the number of generators in repair.
(b) Find the expected number in repair and the expected number operating.
(c) Determine the percentage operating.
(d) In order to placate the irate consumer, the percentage operating in part (c) must be improved by adding either another spare generator or a second repair crew. The discounted present-value cost (investment plus operating cost) of a spare generator is \$175,000, and the cost of a second repair crew is \$60,000. Which alternative

should the utility company implement, and what would be the new operating percentage?

10.11 Derive the preventive maintenance cost model (Eq. (10.15)) for the constant failure rate case (homogeneous Poisson process). What conclusion is reached concerning the minimum-cost preventive maintenance interval?

10.12 Two redundant components having a constant failure rate have a system failure rate that increases (see Section 3.6). Treating the increasing failure rate function as an intensity function, derive the preventive maintenance total cost equation (Eq. (10.15)) assuming that both components have identical failure rates. From this equation determine from a direct search a minimum-cost preventive maintenance interval assuming that $C_r = \$1000$, $C_s = \$500$, and $\lambda = 0.1$ per operating hour. What do you suppose a preventive maintenance program would consist of in this case given the conclusion reached in the previous problem? Hint:

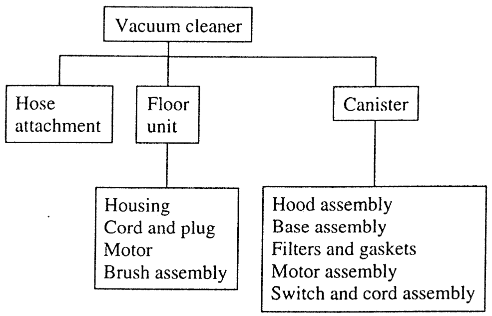
$$\int \frac{dx}{a + be^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln(a + be^{cx})$$

10.13 (a) The following are the numbers of days required to perform a depot-level overhaul of an aircraft engine. The contractor must demonstrate an MTTR of 5 days in order to satisfy contractual agreements. On the basis of a 95 percent one-sided confidence interval, is the goal being met?

4.0	5.3	4.7	4.2	3.5	2.7	3.3	4.7	4.7	5.4	5.7	5.1	4.1
5.3	4.4	4.5	4.6	2.9	5.1	4.5	5.0	5.2	5.8	4.4	5.0	

(b) Fifty percent (the median) of the repairs must be accomplished within 5 days. On the basis of a 90 percent one-sided confidence interval, is this specification being met?

10.14 A canister-type vacuum cleaner consists of the following components:



The following components are repairable on failure: brush assembly ($\lambda(t) = 0.01$), motor assembly ($\lambda(t) = 0.0078$), and switch and cord ($\lambda(t) = 0.0017$). All other components are discarded and replaced on failure ($\lambda(t) = 0.05$). All times are in operating hours.

(a) If the length of time required to repair the brush assembly is lognormal with $s = 0.70$ and MTTR $= 1.5$ hr, 90 percent of all repairs will be completed in what time?

(b) What is the system mean repair time if the motor assembly repair time is normal with a mean of 3 hr and a standard deviation of 0.5 hr, the MTTR for the switch and cord is 2 hr, and discard-and-replace repair time has a lognormal distribution with $s = 1$ and $t_{\text{med}} = 0.5$ hr?

(c) The motor assembly is a candidate for being discarded rather than repaired. It has a unit cost of \$43, and the cost of repair is \$35 × MTTR, while the cost to replace it is \$35 × mean replacement time. The fixed costs are assumed to be negligible. Ten percent of all motor assemblies must be condemned. Should the motor be discarded or repaired? At what unit cost of the motor is it cheaper to discard?

(d) If the brush assembly deteriorates according to the intensity function $\rho(t) = 0.00001(2.5)t^{1.5}$, has a unit cost of \$72, and has a failure (repair) cost of \$52.50 (\$35 × 1.5 hr), when is it economical to replace it?

CHAPTER 11

Availability

With the introduction of a repair capability that will restore a system to an operative state, an alternative measure of system performance is availability. Availability depends on both reliability and maintainability. To predict system availability, both the failure and repair probability distributions must be considered. We begin with the general observation that

$$\text{Availability} = \frac{\text{uptime}}{\text{uptime} + \text{downtime}} \tag{11.1}$$

Equation (11.1) is useful only from a historical point of view, in which, over an elapsed time period, total uptime and total downtime will provide the percentage of time the system was available. However, our primary interest is in predicting availability.

11.1 CONCEPTS AND DEFINITIONS

Availability is the probability that a system or component is performing its required function at a given point in time or over a stated period of time when operated and maintained in a prescribed manner. Like reliability and maintainability, availability is a probability. Therefore the rules of probability theory can be applied to availability when it is being quantified. Availability may be interpreted as the probability that a system is operational at a given point in time or as the percentage of time, over some interval, in which the system is operational. This is made clearer with the following definitions:

1. $A(t)$ is the availability at time t, referred to as the *point availability*.
2. $$A(T) = (1/T)\int_0^T A(t)\,dt \tag{11.2}$$

   is the average availability over the interval $[0,T]$.

The average availability can be generalized into what is often called a *mission* or *interval* availability,

$$A_{t_2-t_1} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} A(t)\,dt \tag{11.3}$$

which represents the average availability over the interval (for example, mission time) from t_1 to t_2.

3. $$A = \lim_{T\to\infty} A(T) \tag{11.4}$$

   is the steady-state or long-run equilibrium availability.

There are several different forms of the steady-state availability depending on the definitions of uptime and downtime. These are discussed in the following.

11.1.1 Inherent Availability

The inherent availability, A_{inh}, is defined as follows:

$$A_{\text{inh}} = \lim_{T\to\infty} A(T) = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR}} \tag{11.5}$$

Inherent availability is based solely on the failure distribution and repair-time distribution. It can therefore be viewed as an equipment design parameter, and reliability-maintainability trade-offs can be based on this interpretation.

EXAMPLE 11.1. An office machine has a time-between-failure distribution that is lognormal with a shape parameter $s = 0.86$ and a scale parameter $t_{\text{med}} = 40$ operating hours. The repair distribution is normal with a mean of 3.5 hr and a standard deviation of 1.8 hr. Therefore MTBF $= 40e^{0.7396/2} = 57.9$ and $A_{\text{inh}} = 57.9/(57.9 + 3.5) = 0.943$.

11.1.2 Achieved Availability

The achieved availability, A_{a}, is defined as

$$A_{\text{a}} = \frac{\text{MTBM}}{\text{MTBM} + \overline{M}} \tag{11.6}$$

where the mean time between maintenance (MTBM) includes both unscheduled and preventive maintenance and is computed from

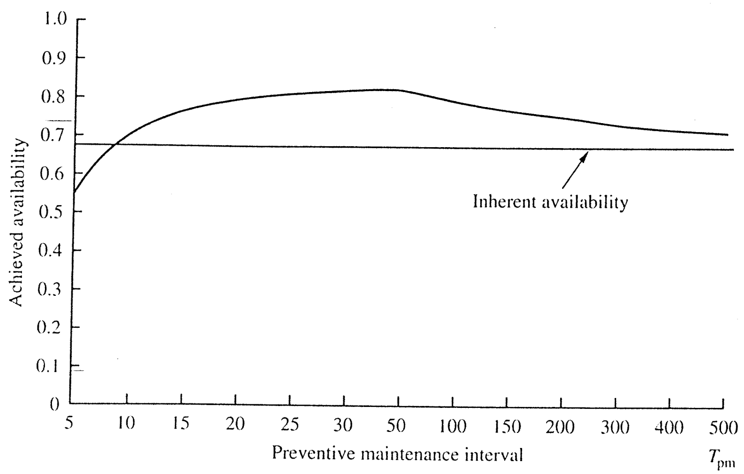


FIGURE 11.1
Achieved availability versus the preventive maintenance interval.

$$\text{MTBM} = \frac{t_d}{m(t_d) + t_d/T_{pm}} \tag{11.7}$$

and $\overline{M}$ is the mean system downtime as defined by Eq. (10.2), T_{pm} is the preventive maintenance interval, t_d is the design life, and $m(t_d)$ is the cumulative average number of failures over the design life. For constant failure rates, $m(t_d) = \lambda t_d$, and t_d can be factored out of Eq. (11.7).

If it is performed too frequently, preventive maintenance can have a negative impact on the achieved availability even though it may increase the MTBF. For example, the graph in Fig. 11.1 shows the change in achieved availability (Eq. (11.6)) as a function of the preventive maintenance interval, T_{pm}. In generating this curve, the assumption was made that MTBF $= a + b/T_{pm}$ where $a, b > 0$. Therefore preventive maintenance has a positive effect on the time between failures. However, the longer the preventive maintenance interval, the less is the effect of preventive maintenance on the MTBF. Very short preventive maintenance intervals resulting in frequent downtimes have an availability less than the (inherent) availability. As the preventive maintenance interval increases, the achieved availability will reach a maximum point and then gradually approach the inherent availability.

11.1.3 Operational Availability

The operational availability, A_o, is defined as

$$A_o = \frac{\text{MTBM}}{\text{MTBM} + \overline{M}'} \tag{11.8}$$

where $\overline{M}'$ is determined by replacing MTTR with MTR = MTTR + SDT + MDT in Eq. (10.2). This definition includes all supply and maintenance delays as part of

the unscheduled downtime. It is useful when there is queuing for maintenance and backorders for replacement parts. Therefore it is a useful definition when making trade-offs concerning the number of spares and the numbers of repair channels. From a product design point of view, the inherent or achieved availability is of more interest since spares and repair capability involve resources and trade-offs external to the product design.

11.1.4 Generalized Operational Availability

The generalized operational availability, A_G, is

$$A_G = \frac{\text{MTBM} + \text{ready time}}{\text{MTBM} + \text{ready time} + \overline{M}'} \tag{11.9}$$

When the system is not operating continuously and the time to failure and preventive maintenance interval time are measured in operating time, the nonoperating time must be accounted for. This definition assumes that there are no failures during the ready, standby, or idle time. An alternative approach is to define time to failure and time between preventive maintenance in clock or calendar time and use Eq. (11.8).

Additional discussions concerning availability definitions may be found in Aven [1990]; Lie, Hwang, and Tillman [1977]; and Bryant and Murphy [1980].

11.2 EXPONENTIAL AVAILABILITY MODEL

The simplest case for determining point, interval, and steady-state availability is a single component or system having both a constant failure rate, λ, and a constant repair rate, r (that is, exponential distributions). Assume that the system will be in one of two possible states: operating or under repair. The transition rate diagram is given by Fig. 11.2. Treating this as a Markov process, the corresponding equations are

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + rP_2(t) \tag{11.10}$$

$$P_1(t) + P_2(t) = 1 \tag{11.11}$$

with the solution (see Appendix 11A for its derivation)

$$P_1(\text{t}) = \frac{r}{\lambda + r} + \frac{\lambda}{\lambda + r} e^{-(\lambda + r)t} \tag{11.12}$$

Since state 1 is the available state, then $A(t) = P_1(t)$ is the point availability for this system and provides the probability that the system is operating at time t. The interval, or mission, availability is

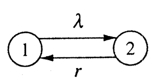


FIGURE 11.2
Rate diagram for a single component with repair.

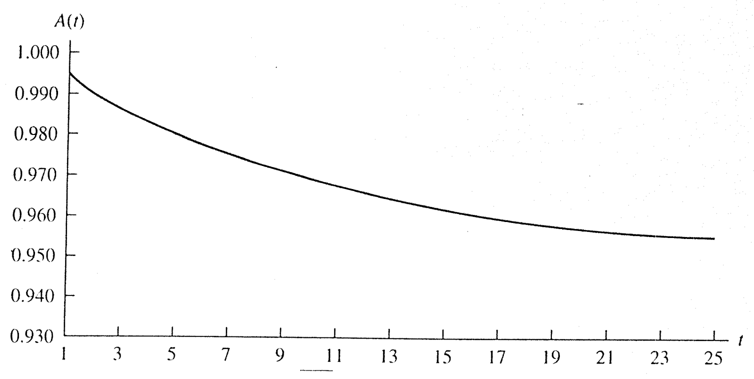


FIGURE 11.3
Point availability of a constant failure rate and constant repair rate component.

$$
\begin{aligned}
A_{t_2-t_1} &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left(\frac{r}{r+\lambda}+\frac{\lambda}{r+\lambda}e^{-(\lambda+r)t}\right)dt \\
&= \frac{r}{r+\lambda}+\frac{\lambda}{(r+\lambda)^2(t_2-t_1)}\left[e^{-(\lambda+r)t_1}-e^{-(\lambda+r)t_2}\right]
\end{aligned}
\tag{11.13}
$$

The steady-state availability is found by $A_{\text{inh}} = \lim_{t\to\infty} A_{t-0}$. By the mission availability given by Eq. (11.13), this becomes

$$
A_{\text{inh}} = \frac{r}{r+\lambda} = \frac{\text{MTBF}}{\text{MTBF}+\text{MTTR}} \tag{11.14}
$$

EXAMPLE 11.2. A component has MTBF = 200 hr and MTTR = 10 hr with both the failure and repair distributions exponential. Then

$$
\begin{aligned}
A(t) &= \frac{0.1}{0.1+0.005}+\frac{0.005}{0.1+0.005}e^{-0.105t} \\
&= 0.952+0.048e^{-0.105t}
\end{aligned}
$$

For a specific point in time, $A(10) = 0.952 + 0.048e^{-0.105(10)} = 0.969$.

The interval availability for the first 10 time units is

$$
A_{10-0} = 0.952 + \frac{0.005}{(0.105)^2(10)}\left[1-e^{-0.105(10)}\right] = 0.981
$$

The graph of $A(t)$ for this example, shown in Fig. 11.3, approaches the steady-state availability of $A = 0.952$.

11.3 SYSTEM AVAILABILITY

Since availability is a probability, the rules of probability theory may be applied to compute system availability from knowledge of component or subsystem availability. The procedure is much the same as described in Chapter 5 for determining system reliability. Therefore, for n independent components in series, each having a component availability of $A_i(t)$, the system availability is given by

$$
A_s(t) = \prod_{i=1}^{n} A_i(t) \tag{11.15}
$$

and for n independent components in parallel, the system availability is given by

$$
A_s(t) = 1 - \prod_{i=1}^{n}(1 - A_i(t)) \tag{11.16}
$$

The system and component availability may be point, interval, or steady-state availability. For more complex configurations, the system can be analyzed through use of reliability block diagrams.

EXAMPLE 11.3. Given two components, each having a constant failure rate of 0.10 failure per hour and a constant repair rate of 0.20 repair per hour, compute point and interval availability for a 10-hr mission, and steady-state availability for both series and parallel configurations.

Solution

$$
A_i(10) = 0.684 \qquad A_{i,0-10} = 0.772 \qquad A_i = 0.667
$$

Then for series,

$$
A_s(10) = 0.684^2 = 0.468 \qquad A_{s,0-10} = 0.772^2 = 0.596 \qquad A_s = 0.667^2 = 0.445
$$

and for parallel,

$$
A_s(10) = 1-(1-0.684)^2 = 0.900 \qquad A_{s,0-10} = 1-(1-0.772)^2 = 0.948
$$
$$
A_s = 1-(1-0.667)^2 = 0.889
$$

11.3.1 Availability with Standby Systems

The concept of availability can be extended to include standby systems. For example, given a backup system with repair permitted for either component with a single repair crew and no failures while in standby, the transition rate diagram is as shown in Fig. 11.4. The resulting equations are

$$
\begin{aligned}
\frac{dP_1(t)}{dt} &= -\lambda_1 P_1(t) + rP_2(t) \\
\frac{dP_2(t)}{dt} &= \lambda_1 P_1(t) + rP_4(t) - (\lambda_2 + r)P_2(t) \\
P_1(t) + P_2(t) + P_4(t) &= 1
\end{aligned}
\tag{11.17}
$$

If we are only interested in a steady-state solution, then

$$
\lim_{t\to\infty}\frac{dP_i(t)}{dt} = 0
$$

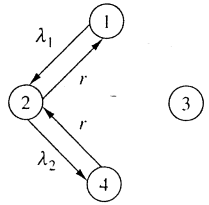


FIGURE 11.4
Rate diagram for a standby system with repair and no failures in standby.

and we can set $P_i(t) = P_i$ since the steady-state probabilities, by definition, are not functions of time. Therefore, we have from Eqs. (11.17)

$$\begin{aligned} -\lambda_1 P_1 + rP_2 &= 0 \\ \lambda_1 P_1 + rP_4 - (\lambda_2 + r)P_2 &= 0 \\ P_1 + P_2 + P_4 &= 1 \end{aligned} \tag{11.18}$$

Solving (11.18) simultaneously results in

$$\begin{aligned} P_1 &= \left[1 + \frac{\lambda_1}{r} + \frac{\lambda_1\lambda_2}{r^2}\right]^{-1} \\ P_2 &= \frac{\lambda_1}{r}P_1 \\ P_4 &= \frac{\lambda_1\lambda_2}{r^2}P_1 \end{aligned} \tag{11.19}$$

Availability, defined as the probability that the system is operating at a given time, corresponds to the system being in state 1 or state 2 and is determined from $A = P_1 + P_2$.

EXAMPLE 11.4. A two-component standby system has the following parameters: $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, and $r = 0.01$. Then

$$P_1 = \left[1 + \frac{0.002}{0.01} + \frac{0.002(0.001)}{(0.01)^2}\right]^{-1} = 0.8196$$

$$P_2 = \frac{0.002}{0.01}(0.8196) = 0.1639$$

$$P_4 = \frac{0.002(0.001)}{(0.01)^2}(0.8196) = 0.01639$$

$$A = P_1 + P_2 = 0.9835$$

When one is computing reliability, the steady-state solution is of no interest since $\lim_{t\to\infty} R(t) = 0$. It is only when repair is introduced that a nonzero steady-state availability exists. From a reliability point of view, the standby system described here has failed when both units have failed even though repair and restoration are possible.

11.3.2 Steady-State Availability

Steady-state equations may be derived directly from rate diagrams under a variety of situations. The key assumptions in these models, however, are the assumptions of

constant failure rates and constant repair rates (exponential distributions). Steady-state equations may be written for each state i on the basis of the general relationship

$$\sum_j (\text{Rate into state } i \text{ from state } j) \times P_j = (\text{rate out of state } i) \times P_i$$

The approach is illustrated with the following two examples.

EXAMPLE 11.5. A system will be in one of three states. In state 1 the system is fully operational. In state 2 it operates in a degraded mode, and in state 3 it is in a failed mode. The system can be repaired to a fully operational status only once it has failed. The rate diagram and transition rates are shown in Fig. 11.5.

Letting P_i be the probability of being in state i, the steady-state equations are

$$\begin{aligned} -\lambda_1 P_1 - \lambda_3 P_1 + rP_3 &= 0 \\ \lambda_1 P_1 - \lambda_2 P_2 &= 0 \\ P_1 + P_2 + P_3 &= 1 \end{aligned} \tag{11.20}$$

having as a solution

$$\begin{aligned} P_1 &= \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1 + \lambda_3}{r}\right)^{-1} \\ P_2 &= \frac{\lambda_1}{\lambda_2}P_1 \\ P_3 &= \frac{\lambda_1 + \lambda_3}{r}P_1 \\ A &= P_1 + P_2 \end{aligned} \tag{11.21}$$

If $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$, and $r = 10$, then $P_1 = \frac{10}{28}$, $P_2 = \frac{15}{28}$, and $P_3 = \frac{3}{28}$. If the system is available in either state 1 or state 2, the steady-state availability is given by $A = \frac{25}{28} = 0.893$.

EXAMPLE 11.6. A two-component active redundant system can be repaired only after both units have failed. Only one unit can then be repaired at a time. The units have constant failure rates of λ_1 and λ_2, respectively, and the mean repair time to complete both repairs is MTTR. The rate diagram is shown in Fig. 11.6.

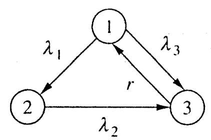


FIGURE 11.5
Rate diagram for a three-state degraded system with repair.

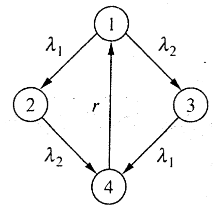


FIGURE 11.6
Rate diagram for an active redundant system with repair.

The system repair rate, r, is assumed to be constant and is obtained from

$$r = \frac{1}{\text{MTTR}}$$

The steady-state equations are

$$\begin{aligned}
&1. \quad (\lambda_1 + \lambda_2)P_1 = rP_4 \\
&2. \quad \lambda_2 P_2 = \lambda_1 P_1 \\
&3. \quad \lambda_1 P_3 = \lambda_2 P_1 \\
&4. \quad rP_4 = \lambda_2 P_2 + \lambda_1 P_3 \\
&5. \quad P_1 + P_2 + P_3 + P_4 = 1
\end{aligned} \tag{11.22}$$

From (1), $P_4 = [(\lambda_1 + \lambda_2)/r]P_1$; and from (2) and (3), $P_2 = (\lambda_1/\lambda_2)P_1$ and $P_3 = (\lambda_2/\lambda_1)P_1$. Therefore, substituting into (5),

$$P_1\left[1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 + \lambda_2}{r}\right] = 1$$

and

$$P_1 = \left[1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 + \lambda_2}{r}\right]^{-1} \tag{11.23}$$

with $A = P_1 + P_2 + P_3$.

11.3.3 Matrix Approach

The previous Markov models can be extended to address multicomponent systems, although the resulting system of differential equations may be quite difficult to solve. In many applications numerical techniques must be used rather than an analytical solution. However, the steady-state solutions can be easily obtained. The transition rates can be expressed more conveniently in the form of a transition matrix, in which element i, j is the transition rate into state i from state j, $i \neq j$. For $i = j$ the element is the negative of the rate out of state i. For the previous standby system, Eqs. (11.17) can be written as

$$\begin{bmatrix} dP_1(t)/dt \\ dP_2(t)/dt \\ dP_4(t)/dt \end{bmatrix} = \begin{bmatrix} -\lambda_1 & r & 0 \\ \lambda_1 & -\lambda_2 - r & r \\ 0 & \lambda_2 & -r \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_4(t) \end{bmatrix} \tag{11.24}$$

Since the above equations include a redundancy, the additional equation

$$P_1(t) + P_2(t) + P_4(t) = 1$$

must be used in place of one of the equations in Eq. (11.24). For a steady-state solution, the above system can be written as

$$\mathbf{TP} = \begin{bmatrix} -\lambda_1 & r & 0 \\ \lambda_1 & -\lambda_2 - r & r \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{11.25}$$

since $\dfrac{dP_i(t)}{dt} = 0 \quad \text{for } i = 1, 2$

Therefore the solution can be expressed in matrix notation as $\mathbf{P} = \mathbf{T}^{-1}\mathbf{b}$ where

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{T}$ is referred to as the *transition matrix.*

EXAMPLE 11.7. Consider a three-component system with two primary units in series and one backup unit that can replace either of the primary units. Define the $2^3 = 8$ states as in Table 11.1. From the state definitions, $R(t) = P_1(t) + P_2(t) + P_3(t) + P_4(t)$. Assuming that the failure rates are λ_1, λ_2, λ_3^-, and λ_3 for components 1, 2, 3 on standby, and 3 on-line, respectively, then the failure rate diagram is given by Fig. 11.7.

If a repair capability is involved, the transition matrix is modified as follows, with the last row replacing a redundant equation and representing the equation $\sum_{i=1}^{8} P_i(t) = 1$.

$$\mathbf{T} = \begin{bmatrix}
-(\lambda_1 + \lambda_2 + \lambda_3^-) & r & r & r & \cdot & \cdot & \cdot & \cdot \\
\lambda_1 & -(\lambda_2 + \lambda_3) - r & \cdot & \cdot & r & \cdot & r & \cdot \\
\lambda_2 & \cdot & -(\lambda_1 + \lambda_3) - r & \cdot & r & r & \cdot & \cdot \\
\lambda_3^- & \cdot & \cdot & -(\lambda_1 + \lambda_2) - r & \cdot & r & r & \cdot \\
\cdot & \lambda_2 & \lambda_1 & \cdot & -\lambda_3 - 2r & \cdot & \cdot & r \\
\cdot & \cdot & \lambda_3 & \lambda_2 & \cdot & -\lambda_1 - 2r & \cdot & r \\
\cdot & \lambda_3 & \cdot & \lambda_1 & \cdot & \cdot & -\lambda_2 - 2r & r \\
\cdot & \cdot & \cdot & \cdot & \lambda_3 & \lambda_1 & \lambda_2 & -3r \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

TABLE 11.1
State definitions for three-component system of Example 11.7

State	Unit 1	Unit 2	Unit 3
1	On-line	On-line	Standby
2	Failed	On-line	On-line
3	On-line	Failed	On-line
4	On-line	On-line	Failed
5	Failed	Failed	On-line
6	On-line	Failed	Failed
7	Failed	On-line	Failed
8	Failed	Failed	Failed

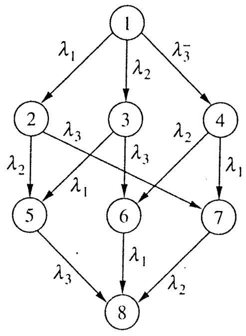


FIGURE 11.7
Rate diagram for a three-component system with one component on standby.

Let $\lambda_1 = 0.01$, $\lambda_2 = 0.02$, $\lambda_3^- = 0.003$, $\lambda_3 = 0.03$, and $r = 0.1$. Then

$$\mathbf{T} = \begin{bmatrix} -0.033 & 0.1 & 0.1 & 0.1 & 0 & 0 & 0 & 0 \\ 0.01 & -0.15 & 0 & 0 & 0.1 & 0 & 0.1 & 0 \\ 0.02 & 0 & -0.14 & 0 & 0.1 & 0.1 & 0 & 0 \\ 0.003 & 0 & 0 & -0.13 & 0 & 0.1 & 0.1 & 0 \\ 0 & 0.02 & 0.01 & 0 & -0.23 & 0 & 0 & 0.1 \\ 0 & 0 & 0.03 & 0.02 & 0 & -0.21 & 0 & 0.1 \\ 0 & 0.03 & 0 & 0.01 & 0 & 0 & -0.22 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

where $\mathbf{P} = \mathbf{T}^{-1}\mathbf{b}$ with $\mathbf{P}^t = [P_1, P_2, \ldots, P_8]$ and $\mathbf{b}^t = [0,0,0,0,0,0,0,1]$. Solving for the steady-state solution:

$$P_1 = 0.7136 \qquad P_2 = 0.0638$$
$$P_3 = 0.1277 \qquad P_4 = 0.0440$$
$$P_5 = 0.0123 \qquad P_6 = 0.0238$$
$$P_7 = 0.0120 \qquad P_8 = 0.0028$$

The steady-state availability is

$$A = P_1 + P_2 + P_3 + P_4 = 0.9491$$

11.4 INSPECTION AND REPAIR AVAILABILITY MODEL

For some restorable systems, such as alarm systems and standby systems, failures are not observed until the system is required to operate. If dormant (undetected) failures may occur while the system is in a nonoperating state, the system availability can be influenced by the frequency at which the system is inspected and, if found inoperable, repaired. Dormant failures are often caused by corrosion or mechanical fracture, but the dominant cause of dormant failures is latent manufacturing defects. If inspection requires some downtime, there is a trade-off between inspection downtime and the detection and restoration of failures. The following model assumes idealized repair that restores the system to as good as new condition. It is based on maximizing a steady-state availability. Note that inspections cannot improve reliability, but can only improve availability.

Let $R(t)$ represent the reliability of the dormant failure distribution, t_1 be the inspection time, t_2 be the repair time, if necessary, and T be the time between inspections (the decision variable). Then $T + t_1 + t_2\,[1 - R(T)]$ is the expected cycle time, that is, the time from completion of one inspection to the start of the next one. The expected available time (uptime) during a cycle is given by

$$\int_0^T R(t)\,dt = R(T)T + \int_0^T t f(t)\,dt \tag{11.26}$$

where the equality comes from the fact that

$$\int_0^T t f(t)\,dt = -tR(t)|_0^T + \int_0^T R(t)\,dt$$

by integration by parts. In Eq. (11.26) the uptime will be T if no failure has occurred (probability $R(T)$) since the last inspection. This is added to the partial expectation, giving the expected time to failure up to the next inspection. A steady-state availability can be expressed as the ratio of the uptime to the cycle time:

$$A(T) = \frac{\int_0^T R(t)\,dt}{T + t_1 + t_2[1 - R(T)]} = \frac{R(T)T + \int_0^T t f(t)\,dt}{T + t_1 + t_2[1 - R(T)]} \tag{11.27}$$

We are interested in finding the inspection interval T that will maximize $A(T)$. The simplest approach is to numerically search uniformly over the interval $0 < T \leq T_{max}$. Numerical optimization techniques, such as the methods of golden sections or interval halving, may be used as well (for example, see Reklaitis, Ravindran, and Ragsdell [1983]).

Exponential case

If $R(t) = e^{-\lambda t}$, then

$$A(T) = \frac{1 - e^{-\lambda T}}{\lambda[T + t_1 + t_2(1 - e^{-\lambda T})]} \tag{11.28}$$

If inspection time and repair time are negligible, Eq. (11.28) simplifies to

$$A(T) \approx \frac{1 - e^{-\lambda T}}{\lambda T}$$

EXAMPLE 11.8. A nuclear power plant has several sophisticated radiation detectors throughout the facility. These detectors historically have had a constant failure rate of 0.0002 failure per hour. Periodically, these detectors can be removed and brought into a lab for inspection. If found defective, they are replaced. If inspection time is 16 hr and it takes 48 hr to obtain a replacement for a failed unit, what is the optimum time between inspections?

Solution. The table below shows the results of a uniform numerical search of the function

$$A(T) = \frac{1 - e^{-0.0002T}}{0.0002[T + 16 + 48(1 - e^{-0.0002T})]}$$

T	100	200	300	**400**	500	600	700	800	900	1000
$A(T)$	0.847	0.900	0.913	**0.916**	0.914	0.910	0.904	0.898	0.891	0.884

From the table, an inspection interval of 400 hr is optimal. Since availability appears to be insensitive to changes in T in the neighborhood of 400 hr, no further refinement to T is made.

11.5 DESIGN TRADE-OFF ANALYSIS

In some situations we may specify or solve for an availability directly. In these cases there are obvious trade-offs between reliability and maintainability. For example, if A_{inh} is specified, then from Eq. (11.5),

$$\text{MTTR} = \frac{1 - A_{\text{inh}}}{A_{\text{inh}}}\text{MTBF} \tag{11.29}$$

The slope of Eq. (11.29)—the MTTR factor—is displayed in Figure 11.8 as a function of the inherent availability. For a specified availability, the MTTR will be a

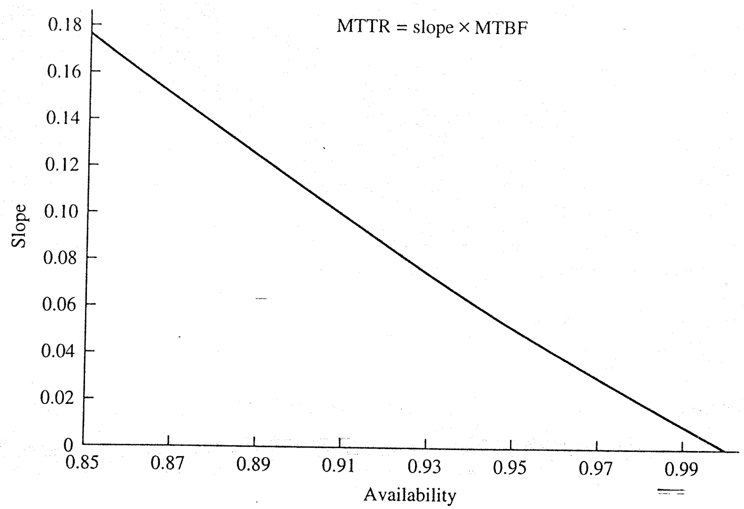


FIGURE 11.8
Availability versus the slope of the MTBF-MTTR line.

fixed percentage of the MTBF. In many cases our primary interest may be in A_{inh} and not the combination of MTTR and MTBF that provides this availability. However, if specifications of the reliability (such as MTBF) and maintainability (such as MTTR) are part of the design process, an opportunity may exist for trade-offs between these two parameters based on an availability specification.

11.5.1 Maintainability Allocation

If system availability is part of the design specification and reliability allocation has been performed, then upper bounds on component MTTRs can be established through use of Eq. (11.29). If a system is composed of n serially related components, then, using Eq. (H.15) and assigning equal availability to each component,

$$A_i = \sqrt[n]{A_s} \qquad \text{and} \qquad \text{MTTR}_i \le \frac{1 - A_i}{A_i}\text{MTBF}_i$$

EXAMPLE 11.9. A four-component (constant failure rate) system having MTBFs of 2100, 3200, 5000, and 1700 hr has an availability specification of 0.99. Then

$$A_i = \sqrt[4]{0.99} = 0.9975 \qquad \text{and} \qquad \frac{1 - A_i}{A_i} = 0.0025$$

Therefore,

$$\text{MTTR}_1 \le 5.25 \text{ hr} \qquad \text{MTTR}_2 \le 8 \text{ hr}$$
$$\text{MTTR}_3 \le 12.5 \text{ hr} \qquad \text{MTTR}_4 \le 4.25 \text{ hr}$$

Assuming that the component MTTRs equal their upper bounds and using Eq. (9.23), the system MTTR is 6.34 hr. Since the system MTBF is 63.4 hr, system availability as computed by Eq. (11.5) is 0.99, as expected.

11.5.2 Economic Analysis

Figure 11.9 illustrates the feasible design region in terms of the MTBF and the MTTR. It assumes that there is a minimum and a maximum MTTR and a minimum MTBF that can be specified. An additional bound on the design region is based on a minimum acceptable availability goal. From Fig. 11.8 it can be seen that as the availability goal increases, the slope of the line decreases thus reducing the area of the design region. Ideally, we would like to find the minimum-cost solution within the design region. In practice it may be difficult to obtain valid cost (or other performance measures) relationships necessary to trade off between reliability and maintainability. However, some insight into the solution can be obtained using somewhat general cost expressions.

If cost relationships can be established as functions of the reliability and maintainability, then optimal cost trade-offs may be found. Letting x = MTBF and y = MTTR be measures of the reliability and maintainability, respectively, and $C_x(x)$ and $C_y(y)$ be the corresponding costs, the following is a meaningful optimization problem:

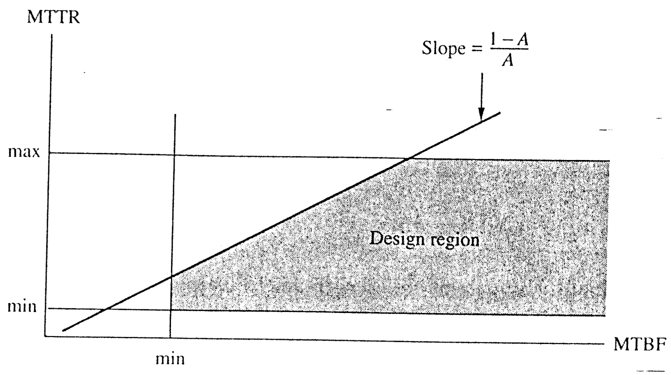


FIGURE 11.9
MTTR versus MTBF design trade-off region.

$$
\begin{aligned}
\min z &= C_x(x) + C_y(y) \\
\text{subject to } (1 - A)x - Ay &\geq 0 \\
\text{MTBF}_{\min} &< x \\
\text{MTTR}_{\min} \leq y &\leq \text{MTTR}_{\max}
\end{aligned}
\tag{11.30}
$$

where A is the specified availability goal and the first inequality results from Eq. (11.29).

11.5.3 Concave Costs

The approach to solving the above problem depends on the nature of the cost functions. Figure 11.10 illustrates the general functional form for linear, concave, and convex functions, recognizing that $C_x(x)$ must be an increasing function and $C_y(y)$ must be a decreasing function.[1]

If the cost functions are linear or concave, optimization theory tells us that the solution to the above problem must occur at one of the extreme points of the feasible design region.[2] Depending on where the boundary equations intersect, three cases are possible, as shown in Fig. 11.11 with the extreme points labeled. In order to solve the above optimization problem, it is sufficient to evaluate the cost function z at each of the extreme points. The solution is the least-cost extreme point.

[1] For concave functions the second derivative is negative, indicating that these functions have decreasing slopes. Convex functions have positive second derivatives and increasing slopes.

[2] An extreme point is a point in the design region that can only be an endpoint of any line segment contained entirely within the design region. In this case, it is defined by the intersection of two of the boundary equations.

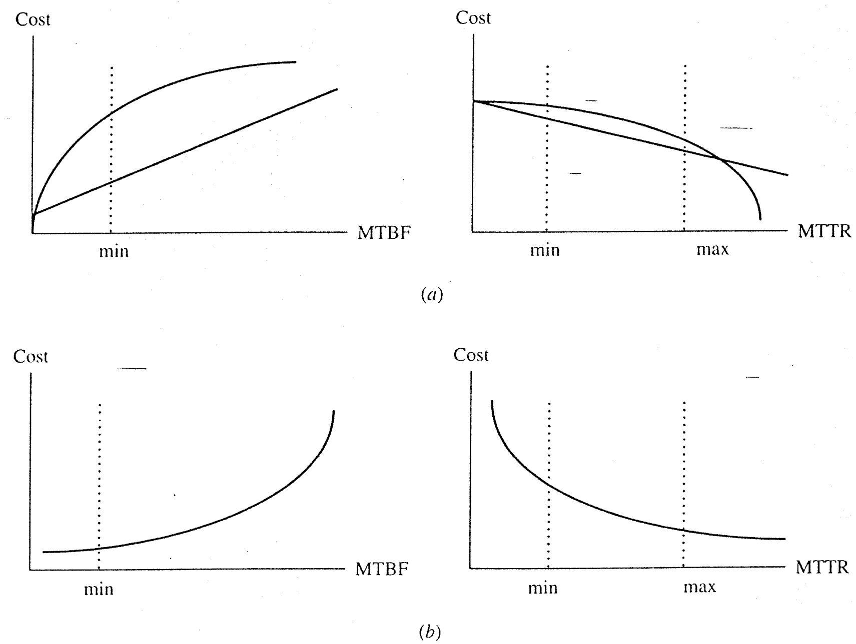


FIGURE 11.10
Cost functions: *(a)* concave and *(b)* convex.

EXAMPLE 11.10. Assuming linear cost functions with times in hours and an availability goal of 0.95, solve the following problem:

$$
\begin{aligned}
\min z = C_x(x) + C_y(y) &= 10x + 5000 - 200y \\
\text{subject to} \quad 0.05x - 0.95y &\geq 0 \\
200 &\leq x \\
5 \leq y &\leq 24
\end{aligned}
$$

The graph of the design region is shown in Fig. 11.12. Evaluating the cost function at each of the extreme points:

a: $z = 10(200) + [5000 - 200(5)] = 6000$

b: $z = 10(200) + [5000 - 200(10.5)] = 4900$

c: $z = 10(456) + [5000 - 200(24)] = 4760$

Therefore the least-cost solution is given by MTBF = 456 hr and MTTR = 24 hr.

Observe that the optimum MTBF will occur either at its minimum value or at the intersection of the minimum or maximum MTTR and the availability equation. Unfortunately, matters are not this simple when the cost functions are not linear or concave.

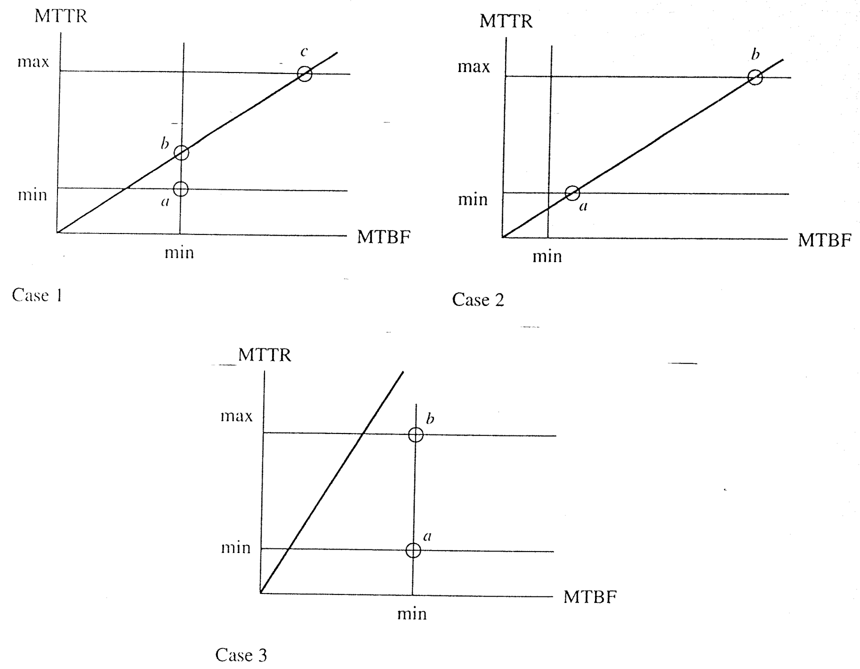


FIGURE 11.11
Feasible solution sets to the concave cost problem.

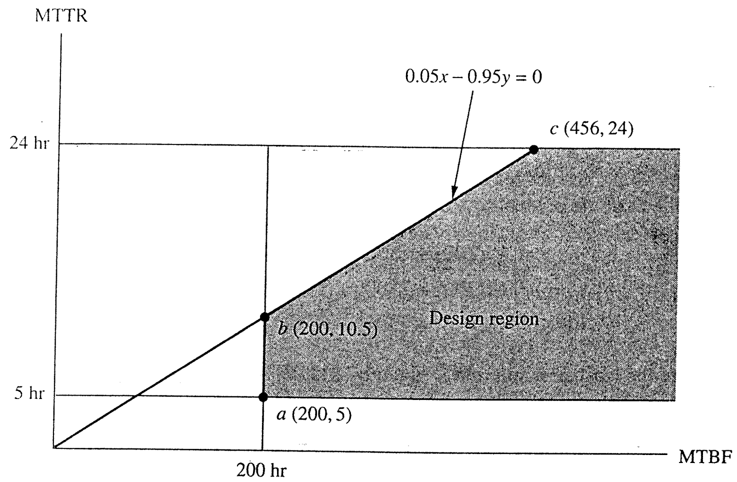


FIGURE 11.12
The feasible design region.

11.5.4 Convex Cost Functions

If the cost functions are convex, the solution may not occur at an extreme point. It is possible for an optimal solution to occur in the interior of the design region. To attempt to solve for this case, we will initially relax the lower- and upper-bound constraints and form the following Lagrangian function:

$$L(x, y, \lambda) = C_x(x) + C_y(y) + \lambda[(1 - A)x - Ay] \tag{11.31}$$

Necessary and sufficient conditions for a minimum for this relaxed problem are given by the following (Kuhn-Tucker) conditions (see Reklaitis, Ravindran, and Ragsdell [1983]):

$$\frac{\partial L(x, y, \lambda)}{\partial x} = \frac{\partial C_x(x)}{\partial x} + (1 - A)\lambda = 0 \tag{11.32}$$

$$\frac{\partial L(x, y, \lambda)}{\partial y} = \frac{\partial C_y(y)}{\partial y} - A\lambda = 0 \tag{11.33}$$

$$(1 - A)x - Ay \geq 0 \tag{11.34}$$

$$\lambda[(1 - A)x - Ay] = 0 \quad \text{and} \quad \lambda \leq 0 \tag{11.35}$$

In order to carry the analysis further, the form of the cost functions must be specified. Let us assume the following:

$$C_x(x) = ax^2, \quad a > 0 \qquad C_y(y) = \frac{b}{y}, \quad b > 0$$

Then the necessary conditions for a minimum are given by

1. $2ax + (1 - A)\lambda = 0$
2. $-\dfrac{b}{y^2} - A\lambda = 0$
3. $(1 - A)x - Ay \geq 0$
4. $\lambda[(1 - A)x - Ay] = 0$
5. $\lambda \leq 0$

Assume that $\lambda < 0$. Solving (1) and (2) along with $[(1 - A)x - Ay] = 0$ from (4) results in

$$x^* = \sqrt[3]{\frac{bA}{2a(1 - A)}} \quad \text{and} \quad y^* = \left(\frac{1 - A}{A}\right)x$$

In implementing the above model, it is only necessary to determine the ratio b/a along with the minimum acceptable availability A. If the costs are known for some value of x and y, the constants can be determined. More desirably, if the costs are known for several different values of the MTBF and the MTTF, regression analysis can be used to derive empirical cost models. Alternatively, a convex cost function may be obtained for the MTBF based on the reliability growth models discussed in Chapter 14. For example, the Duane growth model establishes the following relation between the MTBF and the cumulative number of hours of reliability growth testing (Eq. (14.13)):

$$T = \left[\frac{(1-b)\text{MTBF}}{k}\right]^{1/b}$$

where $0 < b < 1$ and $k > 0$ are constants determined from the test data and T is the cumulative number of hours of reliability growth test time. If c is the cost per hour of reliability growth testing, then

$$C_x(\text{MTBF}) = cT = c\left[\frac{(1-b)\text{MTBF}}{k}\right]^{1/b} = c'\,\text{MTBF}^{1/b} \qquad (11.36)$$

Equation (11.36) is convex since its second derivative is positive for $b < 1$. A value of $b \approx 0.5$ motivates the use of the quadratic cost function in the above example. If the lower bounds for the MTBF and the MTTR or the upper bound for the MTTR are not satisfied, the problem should be resolved with the bounds included. Depending on the nature of the cost functions, this may require a numerical solution.

EXAMPLE 11.11. Let $a = 0.1$ and $b = 10{,}000$. Then the following least-cost solutions are obtained as A varies:

A	0.725	0.750	0.775	0.800	0.825
MTBF	50.9	511.1	55.6	58.5	61.7
MTTR	19.3	17.7	16.1	14.6	13.1
Cost	777	847	929	1026	1145
A	0.850	0.875	0.900	0.925	0.950
MTBF	65.7	70.4	76.6	85.1	98.3
MTTR	11.6	10.1	8.5	6.7	5.2
Cost	1294	1490	1762	2173	2899

An alternative formulation is to maximize availability subject to a budget constraint. Assuming the same cost functions, the optimization problem takes the following form:

$$\max z = \frac{x}{(x+y)}$$

$$\text{subject to } ax^2 + \frac{b}{y} = B$$

The Lagrangian function is $L(x, y, \lambda) = x/(x+y) + \lambda[ax^2 + b/y - B]$, and the necessary conditions for a maximum are

$$\frac{\partial L}{\partial x} = \frac{1}{x+y} - \frac{x}{(x+y)^2} + 2a\lambda x = 0$$

$$\frac{\partial L}{\partial y} = \frac{-x}{(x+y)^2} - \frac{b\lambda}{y^2} = 0$$

$$ax^2 + \frac{b}{y} = B$$

Solving these three equations simultaneously results in $x^* = \sqrt{B/(3a)}$ and $y^* = 3b/(2B)$.

EXAMPLE 11.10 (CONTINUED). Using the same cost parameters as before, maximizing availability results in the following:

Budget	750	1000	1250	1500	1750
MTBF	50	57.7	64.5	70.7	76.4
MTTR	20	15	12	10	8.6
Availability	0.714	0.794	0.843	0.876	0.899
Budget	2000	2250	2500	2750	3000
MTBF	81.6	86.6	91.3	95.7	100
MTTR	7.5	6.7	6	5.5	5
Availability	0.916	0.929	0.938	0.946	0.952

There is very little difference when comparing the two sets of solutions at similar costs or availability. Therefore, either optimization problem could be used for determining reliability and maintainability trade-offs. The choice depends on whether an availability goal or a budget constraint is specified. Obviously, other cost relationships may be observed. However, the resulting analysis would be similar.

11.5.5 Profit and Life-Cycle Cost Trade-Offs

With the introduction of the concept of availability, more general economic models can be developed for use in determining reliability, maintainability, and supportability trade-offs. The example presented in the following is representative only, and variations of this example may well be appropriate for a given application.

Our objective is profit maximization. Define profit to be revenue minus costs, and consider only those costs affected by the system reliability and maintainability. Then

$$\text{Profit} = \text{revenue} - \text{acquisition cost} - \text{repair costs} - \text{supply costs}$$

or

$$\text{Profit} = P_A(i, t_d)RA_{\text{sys}}m - C_u(m+s) - P_A(i, t_d)C_{\text{rep}}k - \sum C_i s_i \qquad (11.37)$$

where m = number of operating units
s = number of spare (standby redundant) units
k = number of repair channels
s_i = number of spare units of component i
R = revenue per year generated by a single unit
C_u = unit acquisition cost
C_{rep} = annual unit cost of a repair channel
C_i = unit cost of component i
A_{sys} = the steady-state operational availability of the m operating units with s spares

$A_{\text{sys}}m$ is the expected number of machines operating. If there are no spare units in the system, this steady-state availability will depend on the individual unit MTBF and MTTR as well as on the maintenance and supply delay times:

$$A_{\text{sys}} = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR} + \text{MDT}(k) + \text{SDT}(s_i)} \tag{11.38}$$

where MDT(k) is the expected maintenance delay time as a function of the number of (system) repair channels, and SDT(s_i) is the expected supply delay time with s_i spares for the ith component. If there are spare units in the system and queuing occurs for repair channels, then $A_{\text{sys}} = L_0/m$, where L_0 is the expected number of units operating as defined by Eq. (10.20).

The objective is to find values for the design variables MTBF, MTTR, m, s, k, and s_i, that will maximize profit. In most cases lower and upper limits for these variables can be determined. In addition, other constraints may be required. For example, there may be an initial investment constraint for purchasing units, spare components, and repair channels. There may also be an annual maintenance cost constraint. A complete formulation of the optimization problem may therefore take the following form:

$$\text{max profit} = P_A(i, t_d)RA_{\text{sys}}m - C_u(m+s) - P_A(i, t_d)C_{\text{rep}}k - \sum C_i s_i$$

subject to

$$\begin{aligned}
&A_{\text{sys}} = h(\text{MTBF}, \text{MTTR}, k, s, s_i)\\
&C_u(m+s) + \sum C_i s_i \le \text{investment budget}\\
&C_{\text{rep}}k \le \text{annual operating budget, first year}\\
&\text{MTBF}_L \le \text{MTBF} \le \text{MTBF}_U\\
&\text{MTTR}_L \le \text{MTTR} \le \text{MTTR}_U\\
&m = 1, 2, 3, \ldots, M \qquad k = 1, 2, \ldots, K \qquad s = 0, 1, 2, \ldots, S\\
&s_i = 0, 1, 2, \ldots, S_i
\end{aligned} \tag{11.39}$$

The functional form of h is unspecified. In general this relationship may be determined by using appropriate queuing or simulation models. This problem is a nonlinear mixed integer program. The design reliability (MTBF) and maintainability (MTTR) parameters are continuous variables; however, the numbers of operating and spare units, repair channels, and spare components are integer variables. Therefore, this is a difficult optimization problem to solve. However, it is relatively easy to compare the expected profits of a fixed number of alternative designs with their parameters specified on the basis of Eqs. (11.39). Life-cycle costing is discussed further in Chapter 18, where a more general cost model is formulated.

Other examples of optimal reliability and maintainability trade-off analysis may be found in Kabak [1969] and Dhinga [1992].

APPENDIX 11A
SOLUTION TO SINGLE UNIT WITH REPAIR MODEL

To solve

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + rP_2(t)$$

Substitute $P_2(t) = 1 - P_1(t)$, giving

$$\frac{dP_1(t)}{dt} = -(\lambda + r)P_1(t) + r$$

With $e^{(\lambda+r)t}$ as an integrating factor,

$$\begin{aligned}
P_1(t)e^{(\lambda+r)t} &= \int re^{(\lambda+r)t}\,dt + C\\
&= \frac{r}{\lambda+r}e^{(\lambda+r)t} + C
\end{aligned}$$

or

$$P_1(t) = \frac{r}{\lambda+r} + Ce^{-(\lambda+r)t}$$

Since $P_1(0) = 1$,

$$C = 1 - \frac{r}{\lambda+r} = \frac{\lambda}{\lambda+r}$$

which results in Eq. (11.12).

EXERCISES

11.1 The Comm Pewter Company is marketing a new computer. Mr. N. Jenair, a reliability engineer, has designed the computer to have a constant failure rate of 0.02 per day (assuming continuous use) and a constant repair rate of 0.1 per day.

(a) Compute the interval availability for the first 30 days and the steady-state availability.

(b) Ms. Needmore Uptyme, a primary user of this computer, feels the availability computed in part (a) is too low. Determine the steady-state availability if a standby unit is purchased. Assume no failures in standby.
(c) If both units are active, what is the steady-state availability?

11.2 Determine an upper bound for each of the following aircraft subsystem MTTRs if a system availability goal of 0.95 is desired. Assume that repair restores the subsystem to as good as new condition and each subsystem has the same availability.

Subsystem	Time between failures	Parameters
Propulsion	Weibull	$\theta = 1000$; $\beta = 1.7$
Avionics	Exponential	$\lambda = 0.003$
Structures	Weibull	$\theta = 2000$; $\beta = 2.1$
Electrical	Weibull	$\theta = 870$; $\beta = 1.8$
Environmental	Exponential	$\lambda = 0.001$

11.3 A replaceable and repairable engine starting unit having a high failure rate has an MTBF of 10 operating hours. A backup unit is even less reliable, with MTBF of 5 hr. If repair averages 2 hr, determine the steady-state availability of the engine starting system.

11.4 The Fly-by-Nite Airlines (see Exercise 9.20) maintains an on-line reservation system with a standby computer available if the primary fails. The on-line system fails at the constant rate of 1 per day while the standby fails (when on-line) at the constant rate of 2 per day. Repair occurs at a constant rate with an MTTR of 0.5 days. Find the steady-state availability for the above system with no failures in the standby mode and one repair crew.

11.5 A critical communications relay has a constant failure rate of 0.1 per day. Once it has failed, the mean time to repair is 2.5 days (the repair rate is constant).
(a) Compute the steady-state availability.
(b) Compute the interval availability over a 2-day mission (starting at time zero).
(c) What is the point availability at the end of the 2 days?
(d) If two communications relays must operate in series, compute the availability in parts (a)–(c).
(e) If two communications relays operate in parallel, compute the availability in parts (a)–(c).
(f) If one communications relay operates in a standby mode with no failure in standby, what is the steady-state availability?

11.6 A generator system consists of a primary and a standby unit. The primary fails at a constant rate of 2 per month, and the standby unit fails only when on-line at a constant rate of 4 per month. Repair can begin only when both units have failed. Both units are repaired at the same time with an MTTR of 20 days ($\frac{2}{3}$ month). Derive the steady-state equations for the state probabilities and solve for the system availability.

11.7 A system may be found in one of three states: operating, degraded, or failed. When operating, it fails at the constant rate of 1 per day and becomes degraded at the rate of 1 per day. If degraded, its failure rate increases to 2 per day. Repair occurs only in the failed mode and is to the operating state with a repair rate of 4 per day. If the operating and degraded states are considered the available states, determine the steady-state availability.

11.8 An emergency backup generator experiences standby (dormant) failures at a constant rate of 0.00314 per day. It takes 6 hr to inspect and test the generator. If it is found to be inoperative, it takes 24 hr to repair it. Determine an inspection interval that will maximize the availability of the generator.

11.9 Find the steady-state availability for two active, identical components having a constant failure rate and sharing a single repair crew with a component repair rate of r.

11.10 Solve for the steady-state availability of the two-component standby system (see Section 11.3.1) with two repair crews available and identical failure rates when on-line. That is, both units may be repaired simultaneously. There are no failures in standby.

11.11 Determine the steady-state availability for the system described in Section 11.3.1 when the repair rate is different for the primary and backup units.

11.12 For the communications relay described in Exercise 11.5, determine how many days the system must be operating in order for the point availability to be within 0.001 of the steady-state availability.

11.13 A radar unit consists of two primary components: a power supply and a transceiver. The power supply has a constant failure rate of 0.0031 failure per operating hour and takes an average of 5 hr (mean) to repair (constant repair rate). The transceiver fails an average (mean) of every 100 operating hours (constant failure rate) and takes 3 hr to repair (constant repair rate). What is the point availability of the radar unit after 10 hr of use? How much does this differ from the steady-state availability? If two radar units are operating in parallel (high-level active redundancy), what are the point, interval (after 10 hr), and steady-state availability? Repeat if low-level redundancy is utilized.

11.14 The distribution of the time between failures of a component is Weibull with $\beta = 2.4$ and $\theta = 400$ hr, and the repair distribution is lognormal with $t_{\text{med}} = 4.8$ hr and $s = 1.2$.
(a) Determine the inherent steady-state availability.
(b) If a 6-hr preventative maintenance task is performed every 200 hr, what is the achieved steady-state availability? Assume a steady-state failure rate of 1/MTBF.

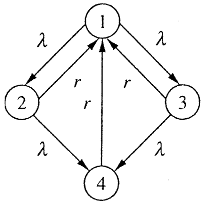
11.15 For the uniform, or rectangular, failure distribution (see Exercise 2.8), derive the function $A(T)$ (Eq. (11.27)). If the parameter b equals 5000 hr, inspection time is 24 hr, and repair time is 72 hr, find the optimum inspection interval.

11.16 The following formula may be used to find system nonavailability. Use this formula to derive Eq. (11.12):

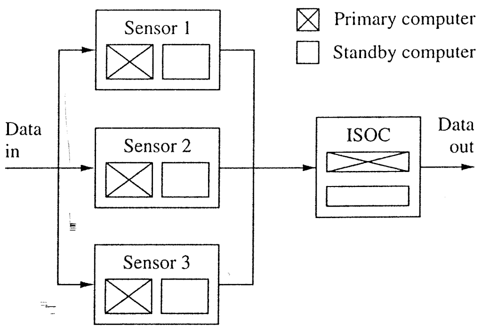
$$A' = 1 - A = \int_o^t [1 - H(t')]f(t')\,dt'$$

where $f(t)$ is the probability density function of the failure distribution and $H(t)$ is the cumulative distribution function of the repair distribution.

11.17 Derive the steady-state availability solution to the two-component active redundant system (constant failure rate) when repair can be accomplished as shown in the following rate diagram.

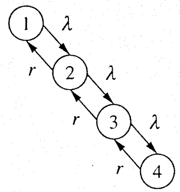


11.18 A ground-based electrooptical deep-space surveillance system consists of an integrated space operations center (ISOC) controlling three sensor sites. Each site and the ISOC have one on-line and one standby computer. The failure rate of the primary computers is $\lambda_1 = 0.004$ per day, and the failure rate of the standby computers when operating is $\lambda_2 = 0.002$ per day. No failures occur while in standby, and the repair rate is $r = 0.01$ per day. Determine the steady-state availability of
(a) A single sensor and the ISOC
(b) A fully operational system
(c) A degraded system with at least two sensors operating



11.19 An alternative configuration for the above system is to combine the sensor at site 1 with the ISOC by having them share a single standby computer. The ISOC function has priority for the standby computer in the event that both the sensor's primary computer and the ISOC's primary computer have failed. If the shared standby computer and the ISOC's primary computer have failed, the sensor's primary computer will perform the ISOC function. Determine the steady-state availability of
(a) The ISOC
(b) A fully operational system

11.20 Determine the steady-state availability for a system consisting of one active and two standby components. All units have identical failure rates and repair rates with only a single repair crew available and no failures in standby, as implied by the following rate diagram:



In state 1 the active component is on-line, in state 2 the first backup is on-line, in state 3 the second backup is on-line, and in state 4 all three components have failed. Determine the steady-state availability if the failure rate is 2 per week and the repair rate is 5 per week.

11.21 Derive the steady-state availability formula for a system composed of two identical active redundant components each having a constant failure rate λ and an individual repair rate r with two repair crews available. Compare your result with that of Eq. (11.16), where $A_i = r/(r + \lambda)$ and $n = 2$.

11.22 Determine the minimum-cost trade-off between the MTBF and the MTTR for a system requiring a 0.98 availability. The MTTR must be between 10 and 30 hr, and the MTBF must be at least 1000 hr. Assume that the cost functions are
(a) $C_x(x) = 2x$; $C_y(y) = 2000 - y^2$
(b) $C_x(x) = 0.0002x^2$; $C_y(y) = 12{,}500/y$.

PART II

The Analysis of Failure Data

CHAPTER 12

Data Collection and Empirical Methods

In Part I the basic reliability and maintainability models were derived and their applications illustrated in numerous examples. The primary problem addressed in Part II is the selection and specification of the most appropriate reliability and maintainability model. This requires the collection and analysis of failure and repair data in order to empirically fit the model to the observed failure or repair process. The derivation of the reliability and maintainability models in Part I is an application of probability theory, whereas the collection and analysis of failure and repair data in Part II are primarily an application of descriptive and inferential statistics.

There are two general approaches to fitting reliability distributions to failure data. The first, and usually preferred, method is to fit a theoretical distribution, such as the exponential, Weibull, normal, or lognormal distributions. The second is to derive, directly from the data, an empirical reliability function or hazard rate function. The first approach is addressed in Chapters 15 and 16, and the second method will be discussed here. Chapters 13 and 14 are concerned with the methods and procedures for collecting and analyzing failure data through controlled testing. Although the emphasis in Part II is on the analysis of failure data, many of the techniques presented can be applied to repair data as well. The analysis of repair data will be illustrated where appropriate by examples. First, however, we address the general problem of data collection and sampling.

12.1 DATA COLLECTION

The generation or observation of failure (or repair) times can be represented by $t_1, t_2, \ldots, t_n$ where t_i represents the time of failure of the ith unit[1] (or in the case of repair data, the ith observed repair time). It is assumed that each failure represents

[1]Elsewhere in this chapter, it is assumed that the sample $t_1, t_2, \ldots, t_n$ is an ordered sample, that is, $t_i \leq t_{i+1}$. We could use the convention of representing the ith ordered sample by $t_{(i)}$. To simplify the notation, however, we will refer to samples as being *ordered* when this is the case.

an independent sample from the same population. The population is the distribution of all possible failure times and may be represented by $f(t)$, $R(t)$, $F(t)$, or $\lambda(t)$. The basic problem is to determine the best failure distribution implied by the n failure times comprised in the sample.

In all cases the sample is assumed to be a simple random (or probability) sample. A simple random sample is one in which the failure or repair times are independent observations from a common population. If $f(t)$ is the probability density function of the underlying population, then $f(t_i)$ is the probability density function of the ith sample value. Therefore, since the sample comprises n independent values, the joint probability distribution of the sample is the product of n identical and independent probability distributions, or

$$f_{t1,t2,\ldots,tn}(t_1, t_2, \ldots, t_n) = f(t_1)f(t_2)\cdots f(t_n) \tag{12.1}$$

We will return to this relationship in Chapter 15 when the maximum likelihood estimator is discussed.

A taxonomy of data

Failure data may be classified in several ways:

Operational versus test-generated failures
Grouped versus ungrouped data
Large samples versus small samples
Complete versus censored data

Sources of failure times are generally either (1) operational or field data reflecting normal use of the component, or (2) failures observed from some form of reliability testing. Reliability testing may include screening or burn-in testing, life or accelerated life testing, and reliability growth testing. Often data received from the field, because of the method of collecting and recording failures, may be grouped into intervals in which individual failure times are not preserved. For large sample sizes, grouping data into intervals may be preferred. Testing may result in small sample sizes because of time and resource limitations. Data generated from testing are likely to be more precise and timely than field data. Field data, in addition to providing larger samples, will reflect the actual operating environment.

A common problem in generating reliability data is censoring. Censoring occurs when the data are incomplete because units are removed from consideration prior to their failure or because the test is completed prior to all units failing. Units may be removed, for example, when they fail because of other failure modes than the one being measured. Censoring may be further categorized as follows:

1. *Singly censored data.* All units have the same test time, and the test is concluded before all units have failed.
   a. *Censored on the left.* Failure times for some units are known to occur only before some specified time.
   b. *Censored on the right.* Failure times for some units are known only to be after some specified time.

   i. *Type I censoring:* Testing is terminated after a fixed length of time, t^*, has elapsed.
   ii. *Type II censoring.* Testing is terminated after a fixed number of failures, r, has occurred. The test time is then given by t_r, the failure time of the rth failure.
2. *Multiply censored data.* Test times or operating times differ among the censored (removed but operating) units. Censored units are removed at various times from the sample, or units have gone into service at different times.

Figure 12.1 graphically compares the operating times of each unit on test under complete, singly censored, and multiply censored conditions. For complete data, Fig. 12.1(a) shows all units operating until failure. For singly censored data on the right, Fig. 12.1(b) implies that the test was terminated at the fourth failure (Type II testing)—with two units still operating. For the multiply censored case, Fig. 12.1(c) reflects two units removed without failing and the other units operating until failure.

Recording failure data by failure mode will result in multiply censored data since units will be removed from a particular sample depending on the nature of their fail-

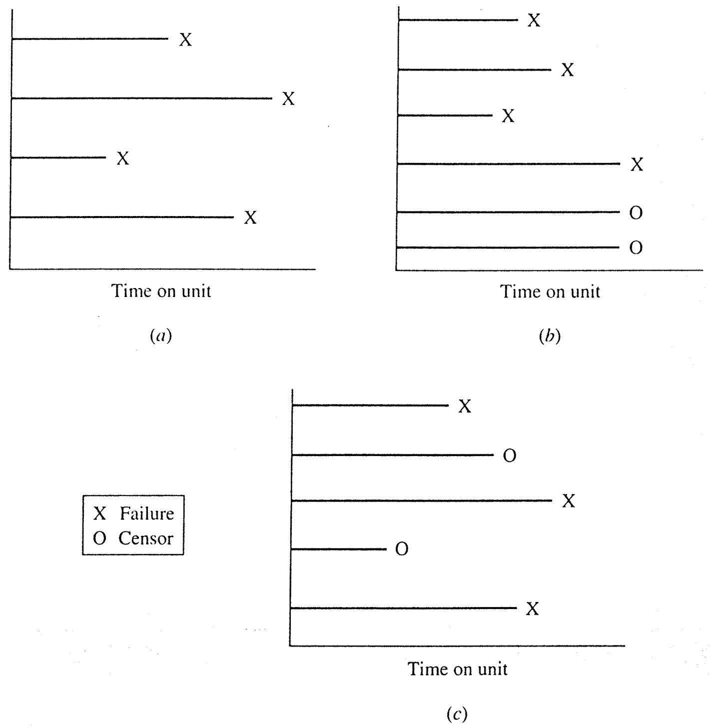


FIGURE 12.1
Illustrative examples of complete and censored data: *(a)* complete data; *(b)* singly censored data; *(c)* multiply censored data.

ure. Data not having any censored units are referred to as complete data. Censoring introduces additional difficulties in the statistical analysis of the failure times. To ignore censored units in the analysis would eliminate valuable information and would bias the results. For example, if the remaining operating units from Type I testing were ignored, only the weakest units having the earliest failure times would be treated in the analysis and the reliability of the component would be seriously underestimated. The empirical methods discussed will address both complete and censored data.

12.2 EMPIRICAL METHODS

Empirical methods of analysis are also referred to as *nonparametric methods* or *distribution-free methods*. The objective is to derive, directly from the failure times, the failure distribution, reliability function, and hazard rate function. For reasons discussed later, the parametric approach consisting of fitting a theoretical distribution is preferred. However, there are occasions when no theoretical distribution adequately fits the data and the only recourse is to apply the following methodology.

12.2.1 Ungrouped Complete Data

Given that $t_1, t_2, \ldots, t_n$, where $t_i \leq t_{i+1}$, are n ordered failure times comprised in a random sample, the number of units surviving at time t_i is $n - i$. Therefore, a possible estimate for the reliability function, $R(t)$, is simply the fraction of units surviving at time t_i, or[2]

$$\hat{R}(t_i) = \frac{n-i}{n} = 1 - \frac{i}{n} \tag{12.2}$$

However, Eq. (12.2) implies that the estimate for the cumulative failure distribution is

$$\hat{F}(t_i) = 1 - \hat{R}(t_i) = \frac{i}{n} \tag{12.3}$$

Therefore $F(t_n) = n/n = 1$ and there is a zero probability of any units surviving beyond t_n. Since it is unlikely that any sample would include the longest survival time, Eq. (12.2) tends to underestimate the component reliability. It is also reasonable to expect the first and last observations, on the average, to be the same distance from the 0 percent and 100 percent observations, respectively. That is, they are symmetrical with respect to the 0 percent, 50 percent, and 100 percent points.

[2]The symbol ˆ is used to indicate an estimate obtained from sample data, or more precisely, a sample statistic. In the narrow sense, a statistic is a function of the random sample. Therefore, it is a random variable having a probability distribution.

TABLE 12.1
Cumulative failure probabilities for selected sample sizes obtained from Eq. (12.4)

Sample size	Cumulative probabilities					
1	0	0.50	1			
2	0	0.33	0.67	1		
3	0	0.25	0.50	0.75	1	
4	0	0.20	0.40	0.60	0.80	1
4	t_0	t_1	t_2	t_3	t_4	t_∞

An improved estimate of the cumulative failure distribution is

$$\hat{F}(t_i) = \frac{i}{n+1} \tag{12.4}$$

Then

$$\hat{R}(t_i) = 1 - \frac{i}{n+1} = \frac{n+1-i}{n+1} \tag{12.5}$$

From Table 12.1 it can be seen that Eq. (12.4) implies that an equal number of failures will occur in the intervals $(0, t_1)$, $(t_1, t_2), \ldots, (t_{n-1}, t_n)$, (t_n, t_∞). This is a reasonable assumption because the sample is completely random.

Plotting positions

Equations (12.3) and (12.4) are only two of several possible estimates for $F(t)$. These estimates are sometimes referred to as *plotting positions* since they provide the ordinate values in plotting the cumulative distribution function. That is, the points $(t_i, \hat{F}(t_i))$ provide a graph of the estimate of $F(t)$. These same ordinate values are used in probability plots, which will be discussed later.

Equation (12.4) provides the mean plotting position for the ith ordered failure. An alternative plotting position is based on the median. The median is often preferred because the distribution of $\hat{F}(t_i)$ is skewed for values of i close to zero and close to n.[3] The median positions are functions of both i and n, and they must be computed numerically. Tables, such as Table A.5 in the Appendix, provide plotting positions for $F(t)$ for selected values of i and n. The formula

$$\hat{F}(t_i) = \frac{i - 0.3}{n + 0.4} \tag{12.6}$$

is often used as an approximation of the median positions. For our estimation of $F(t_i)$, we will primarily use Eqs. (12.4) and (12.6). For relatively large sample sizes, the differences among these plotting positions are insignificant.

EXAMPLE 12.1. On the basis of each of the above approaches, determine the plotting positions for a sample of eight failures.

[3]$\hat{F}(t_i)$, the fraction of observations below the ith sample observation, has a beta probability distribution where $E[\hat{F}(t_i)] = i/(n+1)$.

Solution

i	i/n	$i/(n+1)$	Median	$(i-0.3)/(n+0.4)$
1	0.125	0.111	0.083	0.083
2	0.250	0.222	0.201	0.202
3	0.375	0.333	0.321	0.321
4	0.500	0.444	0.440	0.440
5	0.625	0.555	0.560	0.560
6	0.750	0.666	0.680	0.679
7	0.875	0.777	0.799	0.798
8	1.000	0.888	0.917	0.917

Probability density function and hazard rate function

An estimate of the probability density function may be obtained using Eq. (12.5) and the relationship between $f(t)$ and $R(t)$ given by Eq. (2.3).

$$\hat{f}(t) = -\frac{\hat{R}(t_{i+1}) - \hat{R}(t_i)}{t_{i+1} - t_i} = \frac{1}{(t_{i+1} - t_i)(n+1)} \quad \text{for } t_i < t < t_{i+1} \tag{12.7}$$

Therefore

$$\hat{\lambda}(t) = \frac{\hat{f}(t)}{\hat{R}(t)} = \frac{1}{(t_{i+1} - t_i)(n+1-i)} \quad \text{for } t_i < t < t_{i+1} \tag{12.8}$$

An estimate of the mean time to failure is obtained directly from the sample mean:

$$\widehat{\text{MTTF}} = \sum_{i=1}^{n} \frac{t_i}{n} \tag{12.9}$$

and an estimate of the variance of the failure distribution may be obtained from the sample variance:

$$s^2 = \sum_{i=1}^{n} \frac{(t_i - \widehat{\text{MTTF}})^2}{n-1} \tag{12.10}$$

or

$$s^2 = \frac{\sum_{i=1}^{n} t_i^2 - n\widehat{\text{MTTF}}^2}{n-1} \tag{12.11}$$

Equation (12.10) defines the sample variance, and Eq. (12.11) is the computational form of the sample variance. The square root of the sample variance, s, is the sample standard deviation.

If the sample of n failure times is large, an approximate $100(1-\alpha)$ percent confidence interval for the underlying MTTF may be obtained using

$$\widehat{\text{MTTF}} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}} \tag{12.12}$$

where $t_{\alpha/2,n-1}$ is found from a table of values for the Student's t distribution (Appendix Table A.2) based on $n-1$ degrees of freedom (the parameter of the t distribution) and the desired confidence level $(1-\alpha)$ such that

$$\Pr\{T > t_{\alpha/2,n-1}\} = \frac{\alpha}{2}$$

The derivation of this formula may be found in any introductory statistics text (for example, see Ross [1987]), and its application here assumes that the sample size is large enough to invoke the central limit theorem or the failure distribution itself is normal. Therefore this formula is independent of the precise nature (distribution) of the failure process and may be used in general. Equations (12.9), (12.10), (12.11), and (12.12) may also be used with repair times, with MTTR replacing MTTF. An estimate for the repair cumulative distribution function, H(t), is

$$\hat{H}(t) = \frac{i}{n+1}$$

EXAMPLE 12.2. Given the following 10 failure times in hours, estimate $R(t)$, $F(t)$, $f(t)$, and $\lambda(t)$ and compute a 90 percent confidence interval for the MTTF: 24.5, 18.9, 54.7, 48.2, 20.1, 29.3, 15.4, 33.9, 72.0, 86.1.

Solution: After rank-ordering the data:

Time	Reliability	Density	Hazard rate
0.0	1.00	0.0059	0.0059
15.4	0.9090	0.0260	0.0286
18.9	0.8182	0.0757	0.0926
20.1	0.7273	0.0207	0.0284
24.5	0.6364	0.0189	0.0298
29.3	0.5455	0.0198	0.0362
33.9	0.4546	0.0064	0.0140
48.2	0.3636	0.0140	0.0385
54.7	0.2727	0.0053	0.0193
72.0	0.1818	0.0064	0.0355
86.1	0.0909		

For example,

$$\hat{R}(15.4) = \frac{10+1-1}{11} = 0.9090$$

$$\hat{f}(t) = \frac{1}{(18.9-15.4)\cdot 11} = 0.0260 \quad \text{for } 15.4 < t < 18.9$$

$$\hat{\lambda}(t) = \frac{1}{(18.9-15.4)\cdot 10} = 0.0286 \quad \text{for } 15.4 < t < 18.9$$

A 90 percent confidence interval may be found from

$$\widehat{\text{MTTF}} = \frac{15.4 + 18.9 + 20.1 + \cdots + 86.1}{10} = 40.31$$

and

$$s^2 = \frac{15.4^2 + 18.9^2 + \cdots + 86.1^2 - 10(40.31^2)}{9} = 585.5454$$

or

$$s = 24.198$$

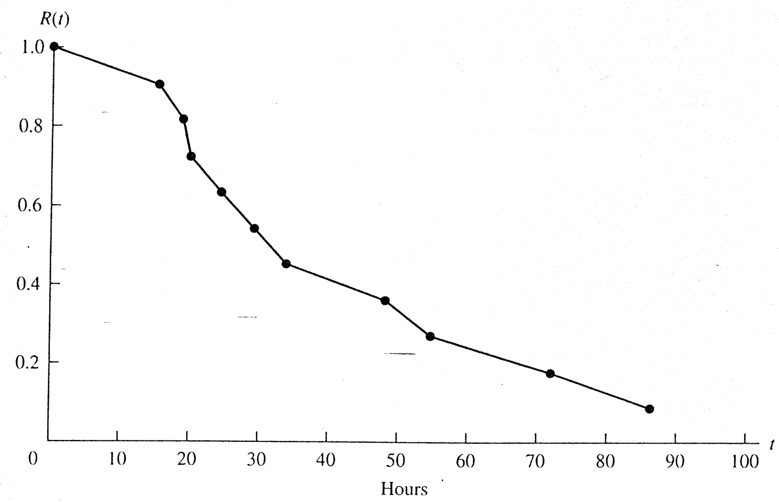


FIGURE 12.2
Empirical reliability curve for ungrouped, complete data.

we know that $t_{0.05,9} = 1.833$ from Table A.2 in the Appendix, so $40.31 \pm 1.833 \times (24.198)/\sqrt{10} = [26.284, 54.34]$ is the desired confidence interval. Graphs of the empirically derived reliability, density, and hazard rate functions are given in Figs. 12.2, 12.3, and 12.4, respectively. $\hat{R}(t)$ is a step function that decreases by $1/(n+1)$ just after each observed failure time. Some authors will therefore graph the reliability function in Fig. 12.2 as a step function. Here the convention of connecting the points with line segments is used for visual clarity in approximating the function $R(t)$.

EXAMPLE 12.3. The following repair times, in hours, were observed as part of a maintainability demonstration on a new packaging machine: 5, 6.2, 2.3, 3.5, 2.7, 8.9, 5.4, 4.6. Estimate the cumulative repair-time distribution and construct a 90 percent confidence interval for the MTTR. If the MTTR is to be 4 hr and 90 percent of the repairs are to be completed within 8 hr, are the maintainability goals being met?

Solution

i	Repair time	$i/(8+1)$
1	2.3	0.111
2	2.7	0.222
3	3.5	0.333
4	4.6	0.444
5	5.0	0.556
6	5.4	0.667
7	6.2	0.777
8	8.9	0.889

These are plotted in Fig. 12.5. The 90 percent confidence interval is

$$\text{MTTR} \pm t_{0.05,7}\frac{s}{\sqrt{n}} = 4.825 \pm 1.894\frac{2.123}{\sqrt{8}} = [3.4, 6.2]$$

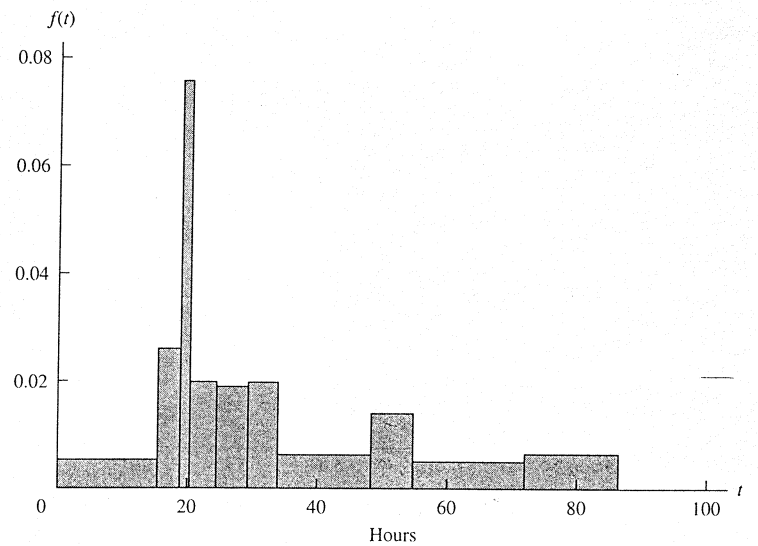


FIGURE 12.3
Empirical failure density curve for ungrouped, complete data.

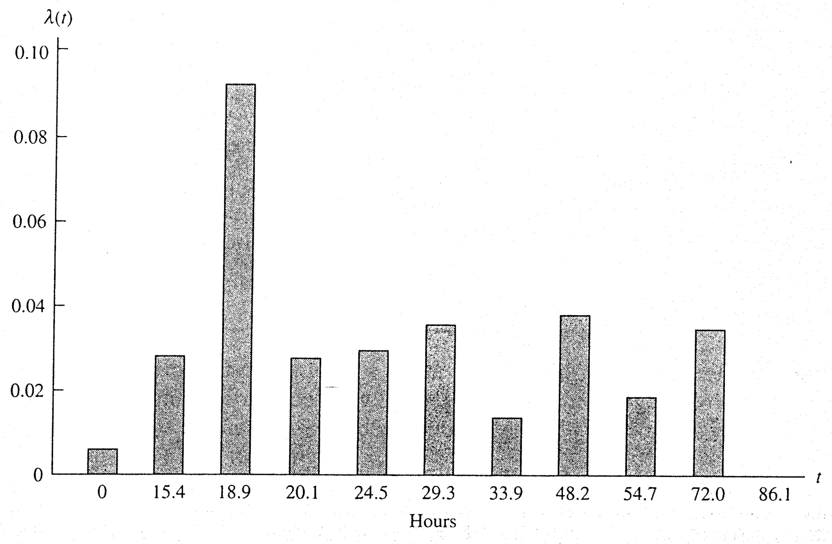


FIGURE 12.4
Empirical hazard curve for ungrouped, complete data.

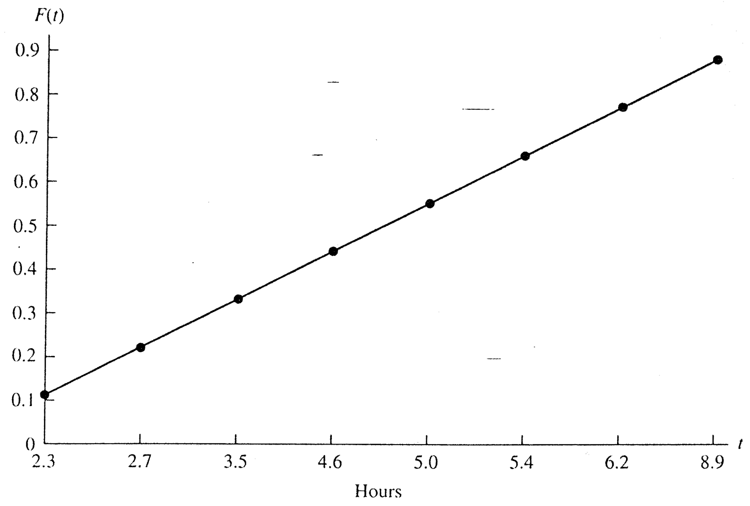


FIGURE 12.5
The cumulative probability of completing repair by time t.

Since the MTTR goal falls within the confidence interval, we accept that the goal is being met. From the empirical cumulative distribution function, it appears we are falling somewhat short of the goal to accomplish 90 percent of the repairs within 8 hr.

12.2.2 Grouped Complete Data

Failure times that have been placed into time intervals, their original values no longer being retained, are referred to as *grouped data.* Since the individual observations are no longer available, let $n_1, n_2, \ldots, n_k$ be the number of units having survived at ordered times $t_1, t_2, \ldots, t_k$, respectively. Then a logical estimate for $R(t)$ is

$$\hat{R}(t_i) = \frac{n_i}{n} \qquad i = 1, 2, \ldots, k \tag{12.13}$$

where n is the number of units at risk at the start of the test. Because of the larger sample size of the grouped data, it is generally unnecessary to obtain more precise estimates by considering plotting positions as before. Therefore

$$\hat{f}(t) = -\frac{\hat{R}(t_{i+1}) - \hat{R}(t_i)}{t_{i+1} - t_i} \qquad \text{for } t_i < t < t_{i+1}$$

$$= \frac{n_i - n_{i+1}}{(t_{i+1} - t_i) \cdot n} \tag{12.14}$$

and

$$\hat{\lambda}(t) = \frac{\hat{f}(t)}{\hat{R}(t)} = \frac{n_i - n_{i+1}}{(t_{i+1} - t_i) \cdot n_i} \qquad \text{for } t_i < t < t_{i+1} \tag{12.15}$$

The MTTF is estimated on the basis of the midpoint of each interval. That is,

$$\widehat{\text{MTTF}} = \sum_{i=0}^{k-1} \bar{t}_i \frac{(n_i - n_{i+1})}{n} \tag{12.16}$$

where

$$\bar{t}_i = \frac{t_i + t_{i+1}}{2} \qquad t_0 = 0 \qquad n_0 = n$$

and $(n_i - n_{i+1})/n$ is the fraction that failed in interval $i + 1$. The sample variance is

$$s^2 = \sum_{i=0}^{k-1} \bar{t}_i^2 \frac{(n_i - n_{i+1})}{n} - \widehat{\text{MTTF}}^2 \tag{12.17}$$

For repair-time data

$$\hat{H}(t) = 1 - \frac{n_i}{n}$$

where n_i is the number of observations exceeding t_i. Then Eqs. (12.16) and (12.17) can be applied with MTTR replacing MTTF.

EXAMPLE 12.4. Seventy compressors are observed at 5-month intervals with the following number of failures: 3, 7, 8, 9, 13, 18, and 12. Estimate $R(t)$, $f(t)$, and $\lambda(t)$ and determine the sample mean time to failure and sample standard deviation.

Solution: Complete the following table:

Upper bound, months	Number failing	Number surviving	Reliability	Failure density	Hazard rate
0	0	70	1.000	0.0086	0.0086
5	3	67	0.957	0.0200	0.0209
10	7	60	0.857	0.0229	0.0267
15	8	52	0.743	0.0257	0.0346
20	9	43	0.614	0.0371	0.0605
25	13	30	0.429	0.0514	0.1200
30	18	12	0.171	0.0343	0.2000
35	12	0	0.000		

For example, $\hat{R}(5) = 67/70 = 0.957$. Therefore

$$\hat{f}(t) = \frac{67 - 60}{(10 - 5) \cdot 70} = 0.0200 \qquad \text{for } 5 < t < 10$$

$$\hat{\lambda}(t) = \frac{67 - 60}{(10 - 5) \cdot 67} = 0.0209 \qquad \text{for } 5 < t < 10$$

$$\widehat{\text{MTTF}} = \frac{2.5 \cdot 3 + 7.5 \cdot 7 + \cdots + 32.5 \cdot 12}{70} = 21.357$$

$$s^2 = \frac{[2.5^2 \cdot 3 + 7.5^2 \cdot 7 + \cdots + 32.5^2 \cdot 12]}{70} - 21.357^2 = 76.551$$

or

$$s = 8.75$$

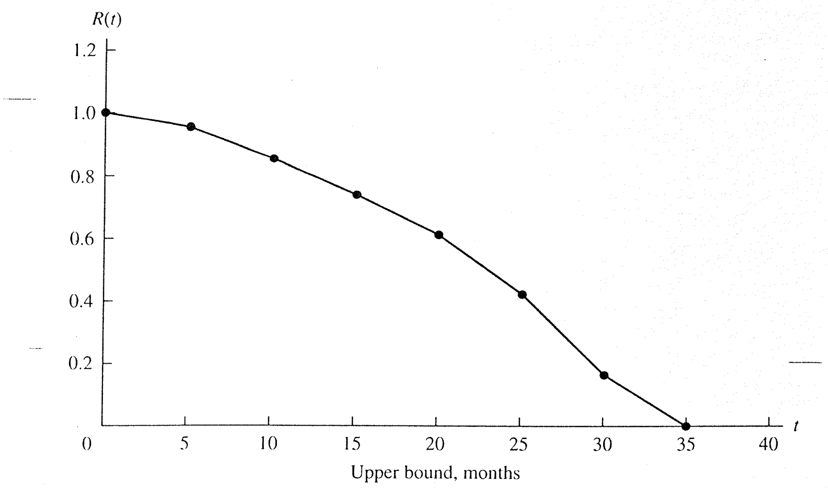


FIGURE 12.6
Empirical reliability curve for grouped, complete data.

Figures 12.6, 12.7, and 12.8 plot the reliability, density, and hazard fate functions, respectively, for this example.

EXAMPLE 12.5. The following aircraft repair data reported by the maintenance organization show the number of days aircraft were out of service because of unscheduled maintenance.

Number of days	Number of aircraft
1–2	4
3–4	7
5–6	9
7–8	6
9–10	4

Derive an empirical cumulative repair time distribution.

Solution

i	Upper bound days	n_i	$1 - n_i/30$
1	2	26	0.133
2	4	19	0.367
3	6	10	0.667
4	8	4	0.867
5	10	0	1.00

From Eq. (12.16) the estimated MTTR is 4.9 days, and from Eq. (12.17) the estimated standard deviation of the repair time is 2.44 days.

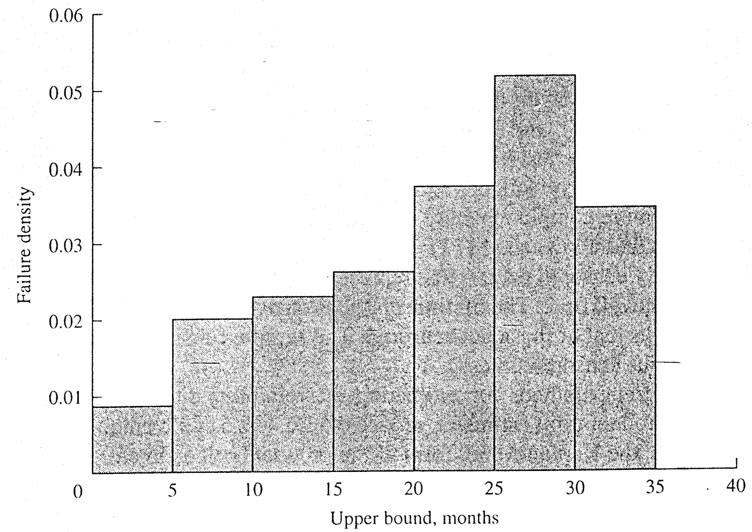


FIGURE 12.7
Empirical failure density curve for grouped, complete data.

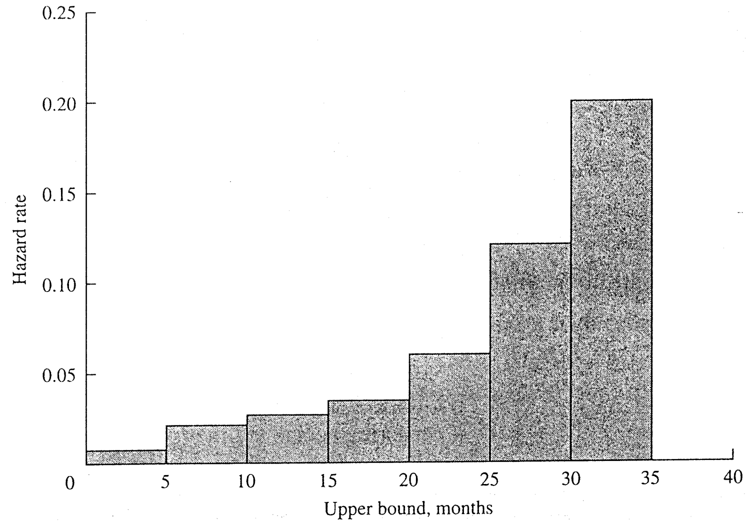


FIGURE 12.8
Empirical hazard rate curve for grouped, complete data.

12.2.3 Ungrouped Censored Data

Assume that n units are placed on test with r failures occurring ($r < n$). For data singly censored on the right, the estimates of $R(t)$, $f(t)$, and $\lambda(t)$ may be computed from Eqs. (12.5), (12.7), and (12.8). The estimated reliability curve is truncated on the right at the time the test is terminated. The formulas for computing the sample mean and variance are no longer valid. In this case fitting a theoretical distribution may provide a more complete picture of the failure process in the right-hand tail of the distribution and allows the MTTF to be computed.

For multiply censored data, t_i will represent a failure time and t_i^+ will represent a censored (removal) time. The lifetime distribution of the censored units is assumed to be the same as that of those not censored. The sample consists of a set of ordered failure times plus censored times: $t_1, t_2, t_3^+, \ldots, t_i, t_{i+1}^+, \ldots, t_n$.

Three different methods for estimating the reliability function are discussed. The first, the product limit estimator, reduces to Eq. (12.5) with complete data. The second method, the Kaplan-Meier form of the product limit estimator, is equivalent to Eq. (12.2) with complete data. The rank adjustment method is presented last.

Product limit estimator

Following Lewis [1987], an estimate of the reliability function without censoring is based on Eq. (12.5). Therefore we can write

$$\hat{R}(t_{i-1}) = \frac{n+2-i}{n+1}$$

and

$$\frac{R(t_i)}{\hat{R}(t_{i-1})} = \frac{n+1-i}{n+2-i} \qquad \text{or} \qquad \hat{R}(t_i) = \frac{n+1-i}{n+2-i}\hat{R}(t_{i-1})$$

That is,

$$\begin{aligned}\hat{R}(t_i) &= \Pr\{\text{Unit survives to time } t_i\}\\ &= \Pr\{\text{Unit will not fail from time } t_i \text{ to } t_{i+1} \text{ given that it has survived to time } t_i\}\\ &\quad \times \Pr\{\text{unit survives to time } t_{i-1}\}\end{aligned}$$

If censoring rather than a failure takes place at time t_i, the reliability should not change and $\hat{R}(t_i^+) = \hat{R}(t_{i-1})$. Let

$$\delta_i = \begin{cases} 1 & \text{if failure occurs at time } t_i \\ 0 & \text{if censoring occurs at time } t_i \end{cases}$$

Then

$$\hat{R}(t_i) = \left(\frac{n+1-i}{n+2-i}\right)^{\delta_i} \hat{R}(t_{i-1}) \tag{12.18}$$

with $\hat{R}(0) = 1$. The estimates for $f(t)$ and $\lambda(t)$ may be derived from Eqs. (12.7) and (12.8) using only the t_i's corresponding to failure times.

EXAMPLE 12.6. The following failure and censor times (in operating hours) were recorded on 10 turbine vanes: 150, 340$^+$, 560, 800, 1130$^+$, 1720, 2470$^+$, 4210$^+$, 5230, 6890. Censoring was a result of failure modes other than fatigue or wearout. Determine an empirical reliability curve.

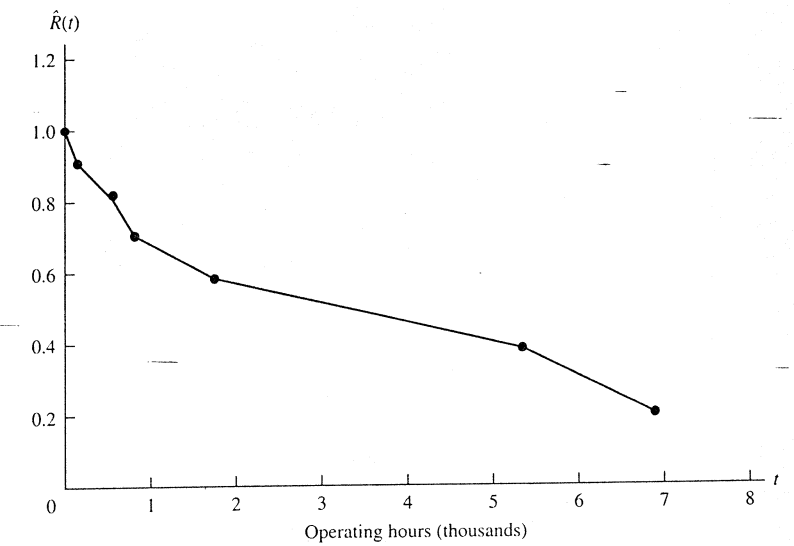


FIGURE 12.9
Empirical reliability curve for ungrouped, multiply censored data using the product limit estimator.

Solution

i	t_i	$(11-i)/(12-i)$	$\hat{R}(t_i)$
1	150	10/11	$R(150) = (10/11)(1) = 0.9090$
2	340$^+$	9/10	
3	560	8/9	$R(560) = (8/9)(0.9090) = 0.8081$
4	800	7/8	$R(800) = (7/8)(0.8081) = 0.7071$
5	1130$^+$	6/7	
6	1720	5/6	$R(1720) = (5/6)(0.7071) = 0.5892$
7	2470$^+$	4/5	
8	4210$^+$	3/4	
9	5230	2/3	$R(5230) = (2/3)(0.5892) = 0.3928$
10	6890	1/2	$R(6890) = (1/2)(0.3928) = 0.1964$

$\hat{R}(t_i)$ is plotted in Fig. 12.9.

Kaplan-Meier form of product limit estimator

A popular method for deriving an empirical reliability function is the Kaplan-Meier product limit estimator, which under complete data is equivalent to Eq. (12.2). Let t_j be the *ordered* failure times and n_j be the number remaining at risk just prior to the jth failure. Assuming that there are no ties in failure times and that censoring times do not coincide with failure times, the Kaplan-Meier product limit estimator is given by:

$$\hat{R}(t) = \prod_{\{j:t_j \le t\}} \left(1 - \frac{1}{n_j}\right) \tag{12.19}$$

For $0 \le t < t_1$, $R(t) = 1$. Each term in Eq. (12.19) represents an estimate of the conditional probability of surviving past time t_j given survival just prior to time t_j. The product of these conditional probabilities is then the unconditional probability of surviving[4] past time t. Lawless [1982] discusses a number of the properties of the Kaplan-Meier product limit estimator and provides the following estimate of its variance. The variance, or its square root, the standard deviation, accounts for the variation in the sampling process and provides a measure of the resulting uncertainty in the estimated reliability.

$$\widehat{\text{Var}}[\hat{R}(t)] = \hat{R}(t)^2 \sum_{\{j:t_j<t\}} \frac{1}{n_j(n_j - 1)} \tag{12.20}$$

EXAMPLE 12.7. Using the multiply censored data from Example 12.6 with $R(t_i + 0)$ representing the reliability immediately following the ith failure, computation of an empirical reliability function by means of the Kaplan-Meier product limit estimator is as follows:

i	t_j	n_j	$1 - 1/n_j$	$\hat{R}(t_i + 0)$	Standard deviation
1	150	10	9/10	$R(150) = (9/10)(1.0) = 0.90$	0.095
2	340⁺				
3	560	8	7/8	$R(560) = (7/8)(0.90) = 0.7875$	0.134
4	800	7	6/7	$R(800) = (6/7)(0.7875) = 0.675$	0.155
5	1130⁺				
6	1720	5	4/5	$R(1720) = (4/5)(0.675) = 0.54$	0.173
7	2470⁺				
8	4210⁺				
9	5230	2	1/2	$R(5230) = (1/2)(0.54) = 0.27$	0.210
10	6890	1	0	$R(6890) = (0)(0.3928) = 0$	

Rank adjustment method

An alternative approach due to Johnson [1959] for estimating $F(t_i)$ and $R(t_i)$ with multiply censored data present makes use of Eq. (12.6) while adjusting the rank order, if necessary, of the ith failure to account for censored times occurring prior to the ith failure. Since a censored unit has some probability of failing before or after the next failure (or failures), it will influence the rank of subsequent failures. For example, suppose the following data were obtained: (1) failure at 50 hr; (2) censor at 80 hr; (3) failure at 160 hr. Then the first failure will have rank 1; however, the second failure could have rank 2 if the censored unit fails after 160 hr, or it could have rank 3 if the censored unit failed before 160 hr. Therefore the second failed unit will be assigned a rank order between 2 and 3 on the basis of the following formula, derived from considering all possible rank positions of the censored unit:

[4]If two or more failures occur at time t_j, the corresponding term in Eq. (12.19) can be replaced with $1 - d_j/n_j$ where d_j is the number of failures occurring at time t_j.

$$\text{Rank increment} = \frac{(n + 1) - i_{t_{i-1}}}{1 + \text{number of units beyond present censored unit}} \tag{12.21}$$

where n is the total number of units at risk and $i_{t_{i-1}}$ is the rank order of failure time $i - 1$. The rank increment is recomputed for the next failure following a censored unit. Its adjusted rank then becomes

$$i_{t_i} = i_{t_{i-1}} + \text{rank increment}$$

and

$$\hat{R}(t_i) = \frac{i_{t_i} - 0.3}{n + 0.4}$$

The rank increment then remains the same until the next censor takes place. This method may also be used when singly censored data are present on the right.

EXAMPLE 12.8. From the failure and censor times given in Example 12.6, $R(t_i)$ can be estimated by the rank adjustment method as follows:

i	t_i	Rank increment	i_{t_i}	$\hat{R}(t_i) = 1 - \frac{i_{t_i} - 0.3}{n + 0.4}$
1	150	1		0.933
2	340⁺			
3	560	$(11 - 1)/(1 + 8) = 1.111$	$1 + 1.111 = 2.111$	0.826
4	800		$2.111 + 1.111 = 3.222$	0.719
5	1130⁺			
6	1720	$(11 - 3.222)/(1 + 5) = 1.2963$	$3.222 + 1.2963 = 4.518$	0.594
7	2470⁺			
8	4210⁺			
9	5230	$(11 - 4.518)/(1 + 2) = 2.16$	$4.518 + 2.160 = 6.679$	0.387
10	6890		$6.679 + 2.160 = 8.839$	0.179

The rank adjustment approach will be used later for determining plotting positions for fitting theoretical distributions when multiply censored data are present. In principle $\hat{R}(t_i)$ could have been computed using Eq. (12.5) in place of Eq. (12.6).

EXAMPLE 12.9. Censoring will occur when failure times of a system comprising two or more components in series are being observed. When the system fails, one component will yield a failure time for that component and censoring times for all other components. For example, the following 10 failure times were observed for a three-component series system with 10 units operating until failure:

Unit	Failed component	Failure time, hr
1	C_1	352
2	C_2	521
3	C_1	177
4	C_1	67
5	C_3	411
6	C_2	125
7	C_1	139
8	C_1	587
9	C_3	211
10	C_1	379

An Introduction to Reliability and Maintainability Engineering
Charles E. Ebeling

Errata

page

xv 4th sentence from bottom: course or course~~s~~

14 Example 1A.2. Define the events A and B as in Example ~~1~~A.1

14 3rd para, 2nd sentence: Define the conditional ~~reliability~~ **probability**

26 top of page: replace t_d with $\mathbf{t_R}$

50 Eq. 3.14: replace $e^{-at/MTTF}$ with e^{-at}

57 Problem 3.21: replace $R(t) = e^{-(t/1000)^2}$ with $R(t) = e^{-(t/2000)^2}$

67 Eq 4.16 and the line above: replace t_d with $\mathbf{t_R}$

75 Table 4.2: mean of lognormal: $t_{med} \exp\left(\dfrac{s^2}{2}\right)$

80 Problem 4.9 (c) should read in part .... " prior to replacement is to be tolerated."

90 middle of page: replace formula for $R_{(c)}$ with: $R_{(c)} = 1 - (1 - R_A R_C)(1 - R_B R_D)$

100 Eq 5.21: replace m with **n**
Example 5.12 reword: A mechanical **valve** fails to ~~close~~ **open** (fails open) 5 percent of the time and fails to ~~open~~ **close** (fails short) 10 percent of the time.

107 Problem 5.26, diagram: second component in parallel should read R() = 0.9.

122 Problem 6.9 Assume the primary unit has an MTTF = 700 hr. Compare both the design life and the system MTTF.

137 line 7 should read: since $R_0 = 1$ and $R_n = \mathbf{R} = \Pr\{x<y\}$ for n=1,2,.... Thus

178 next to last sentence: Another form of qualitative analysis ~~is that~~ utili**zes** minimal cut sets.

195 6th line from the bottom: $\Pr\{N(12)=0\} = \Pr\{T_1>12\} = 1 - F(12-5) = 1 - F(7) = 0$

196 line 8: The cumulative ~~density~~ **distribution** function is given by Eq. (3.22)

196 line 17: The distribution given in Example **9**.is that

197 middle of page: replace m_i with $\mathbf{q_i}$ in both places

199 middle of page: replace m(T+t,T) with **m(T,T+t)** both places

200 line 4 in Example 9.9: the value of the integral is 2.37 not 2.35

203 line 4: From Eq. (9.2**3**)

207 Figure 9.5: the first lambda should have a 2 in front of it That is **2**l

232 9th line from bottom should read: $E[N(T)] = \int_0^T \rho(t)dt$

233 Example 10.7, 2nd sentence should read: The cost of a scheduled maintenance is \$20 ~~per hour~~, and the cost of an unscheduled maintenance repair is \$80 ~~per hour~~ (...).

250 in formula for l_i replace n + s - i with **(n + s - i)** l

251 problem 10.8 a = 2.47 x 10^{-4} / hr. and the cost of the scheduled maintenance is \$50 ~~per hour~~, and the cost repairing a failed machine is \$200 ~~per hour~~ (...).

262 2nd column vector: $P_3(t)$ should be replaced with $\mathbf{P_4}$ (t)

263 first column vector: P_3 should be replaced with $\mathbf{P_4}$

274 new paragraph: replace n with **m**
5th line from the bottom: ~~n~~ **m** = 1,2,3,... ~~N~~ **M**

276 Problem 11.4 change 4th line .... With an MTTR of .5 ~~per~~ day**s**.

293 2nd line from the bottom should read:

$$s^2 = \frac{2.5^2 \cdot 3 + 7.5^2 \cdot 7 + \cdots + 32.5^2 \cdot 12}{70} - 21.357^2 = 76.551$$

299 3rd equation should read: $\hat{R}(t_i) = 1 - \dfrac{i_i - 0.3}{n + 0.4}$

299 2/3 down page: change "using computed" with "**computed using**"

345 line 3: t_i should be $\mathbf{t_1}$

348 Eq. 14.14 should read $r^2 = 1 - \dfrac{\sum_{i=1}^{n}(y_i - \hat{a} - \hat{b}x_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$

350 the 3rd line in the definition of l(t) should read: $l_1 s_1 + l_2 s_2 + l_3 (t - \mathbf{s_2})$ for $s_2 \le t < s_3$

360 Example 15.2: in last row middle column of data set replace 1467 with **476**

364 3rd line from the bottom should read From the Weibull cumulative ~~density~~ **distribution** function,

373 Figure 15.8, legend should read A lognormal least-squares plot of ~~failure~~ **repair** data.

379 Eq 15.21 replace $\sum_{all\,i}\left(t_i^{\hat{\beta}}\right)^{-1}$ with $\left[\sum_{all\,i}\left(t_i^{\hat{\beta}}\right)\right]^{-1}$

$$\prod_{i \in F} \lambda \exp(-\lambda t_i)$$

379 Eq. 15.19: first product should read

387 Last line of example 15.20: set $\hat{\theta}(x) = \exp(8.88 - .0165x)$

388 replace $\frac{\partial L}{\partial \theta}$ *and* $\frac{\partial L}{\partial \beta}$ *with* $\frac{\partial \ln L}{\partial \theta}$ *and* $\frac{\partial \ln L}{\partial \beta}$ in both appendices 14A and 15B.

402 5th line from the bottom: ~~sample~~ **MLE for the** standard deviation is $\hat{\sigma} = 7.041$.
2nd and 3rd lines from the bottom, replace $s = 7.286$ with $s =$ **7.041**

417 13th line from the bottom should read **number** of failures: 16

424 16th line from the bottom: replace $ 750 ~~an hour~~ with $ 750 **per failure**

In order to estimate the reliability of component 1, the failure times of components 2 and 3 are treated as censored times.Therefore, after rank-ordering the failure times, the product limit estimator may be computed as shown below:

	Time, hr	Factor	Reliability
1	67	0.09091	0.9091
2	125⁺	1	
3	139	0.8889	0.8081
4	177	0.8750	0.7071
5	211⁺	1	
6	352	0.8333	0.5892
7	379	0.8	0.4714
8	411⁺	1	
9	521⁺	1	
10	587	0.5	0.2357

12.2.4 Grouped Censored Data

Grouped censored data may be analyzed by constructing a life table. Life tables summarize the survival experiences of the units that are placed at risk (subject to failure). Life tables have been used by medical researchers for estimating the survival probabilities of patients having certain illnesses along with their corresponding medical or surgical treatments. Assume that the failure and censor times have been grouped into $k + 1$ intervals of the form $[t_{i-1}, t_i)$, for $i = 1, 2, \ldots, k + 1$, where $t_0 = 0$ and $t_{k+1} = \infty$. The intervals do not need to be of equal width. Then let

F_i = number of failures in the ith interval

C_i = number of removals (censored) in the ith interval

H_i = number at risk at time t_{i-1}: $H_i = H_{i-1} - F_{i-1} - C_{i-1}$

$H_i' = H_i - \dfrac{C_i}{2}$ = adjusted number at risk assuming that the censored times occur uniformly over the interval

Then $\dfrac{F_i}{H_i'}$ = conditional probability of a failure in the ith interval given survival to time t_{i-1}

and $p_i = 1 - \dfrac{F_i}{H_i'}$ = conditional probability of surviving the ith interval given survival to time t_{i-1}

The reliability of a unit surviving beyond the ith interval can therefore be written as

$$\hat{R}_i = \Pr\{\text{unit survives to } t_i \text{ given it has survived to } t_{i-1}\} \times \Pr\{\text{unit survived to } t_{i-1}\}$$

$$= \left[1 - \frac{F_i}{H_i'}\right] \times \hat{R}_{i-1}$$

The life table then takes the following form:

Interval	Number of failures	Number censored	At risk	Adjusted number at risk	Probability of survival	Reliability
$[t_{i-1}, t_i)$	F_i	C_i	H_i	H_i'	p_i	R_i

EXAMPLE 12.10. Construct a life table for the engines of a fleet of 200 single-engine aircraft having the following annual failures and removals (censors). Removals resulted from aircraft eliminated from the inventory for various reasons other than engine failure.

Year	Number of failures	Number of removals
1981	5	0
1982	10	1
1983	12	5
1984	8	2
1985	10	0
1986	15	6
1987	9	3
1988	8	1
1989	4	0
1990	3	1

Solution

Year	F_i	C_i	H_i	H_i'	p_i	R_i	Standard deviation
1	5	0	200	200	0.975	0.975	0.011
2	10	1	195	194.5	0.949	0.925	0.019
3	12	5	184	181.5	0.934	0.864	0.024
4	8	2	167	166	0.952	0.822	0.027
5	10	0	157	157	0.936	0.770	0.030
6	15	6	147	144	0.896	0.690	0.033
7	9	3	126	124.5	0.928	0.640	0.035
8	8	1	114	113.5	0.930	0.595	0.036
9	4	0	105	105	0.962	0.572	0.036
10	3	1	101	100.5	0.970	0.555	0.036

The reliability function is shown graphically in Fig. 12.10.

As was the case for the Kaplan-Meier product limit estimator, an estimate of the variance of estimated reliabilities, which provides a measure of the precision of the estimate, is available. The following variance estimator, which is based on the work of M. Greenwood [1926], is discussed further in Lawless [1982]. The estimate itself is an approximation. Lawless also discusses the properties of life tables and provides an alternative method for their construction.

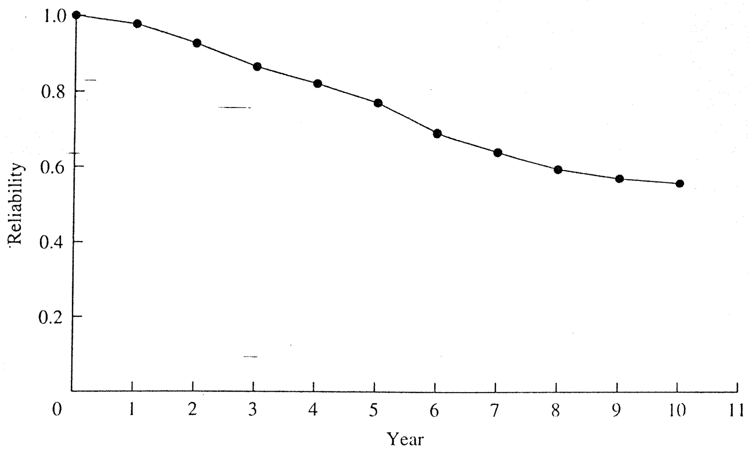


FIGURE 12.10
Empirical reliability curve for grouped, multiply censored data using the life table approach.

$$\widehat{\mathrm{Var}}(\hat{R}_i) = \hat{R}_i^2 \sum_{k=1}^{i} \frac{1 - p_k}{H'_k p_k} \tag{12.22}$$

12.3 STATIC LIFE ESTIMATION

If a reliability estimate is required for a single specified point in time, t_0, then n units may be placed on test for a time t_0 and the number of failures, r, recorded. For the static reliability cases discussed in Section 7.2 in which an event of short duration is observed, t_0 may be omitted and the point reliability estimate is based simply on the number of failures resulting from the application of static loads. A point estimate for the reliability is given by

$$R(t_0) = 1 - \frac{r}{n} \tag{12.23}$$

An interval estimate is obtained such that

$$\Pr\{R_L \leq R(t_0) \leq R_U\} = 1 - \alpha$$

where

$$R_L = \left(1 + \frac{r+1}{n-r} F_2\right)^{-1} \tag{12.24}$$

$$R_U = F_1 \cdot \left(F_1 + \frac{r}{n-r+1}\right)^{-1} \tag{12.25}$$

$$F_1 = F_{\alpha/2,2n-2r+2,2r} \qquad F_2 = F_{\alpha/2,2r+2,2n-2r}$$

and $F_{\alpha/2,n1,n2}$ is a value from the F-distribution having n_1 and n_2 degrees of freedom and having an upper-tail probability of $\alpha/2$.

EXAMPLE 12.11. Specifications call for an engine to have 0.95 reliability at 1000 operating hours. The oldest 50 engines in the fleet have just passed 1000 hr with one failure observed. Is the specification being met?

Solution: $R(1000) = 1 - 1/50 = 0.98$. For a 95 percent lower-bound confidence interval, F_2 is computed with $\alpha = 0.05$ replacing $\alpha/2$; therefore

$$F_2 = F_{0.05,4,98} = 2.48$$

and

$$R_L = \frac{1}{1 + (2/49)(2.48)} = 0.908$$

We are 95 percent confident that the reliability is at least 90.8 percent. Therefore we cannot say for certain that the specification is being met. We would accept the specification if R_L were 0.95 or larger.

EXAMPLE 12.12. It is desired to estimate the launch reliability of a booster rocket used to launch communication satellites into orbit. Twenty launches have been completed to date with one failure observed. Compute a 90 percent confidence interval for the rocket launch reliability.

Solution. With $n = 20$ and $r = 1$,

$$R = 1 - \frac{1}{20} = 0.95$$

$$F_1 = F_{.05,40,2} = 19.47$$

$$F_2 = F_{.05,4,38} = 2.625$$

$$R_L = \frac{1}{1 + (2/19)(2.625)} = 0.7835$$

$$R_U = \frac{19.47}{19.47 + 1/(20 - 1 + 1)} = 0.9974$$

Derivations of the confidence limits for a static reliability based on the binomial distribution may be found in Gibra [1973].

EXERCISES

Note: In solving the following problems, initial values should be obtained using only a calculator. Computer software may then be used to verify your results and to complete the analysis.

12.1 For the following failure times estimate $(t_i, F(t_i))$ using each of the discussed plotting positions: 12, 243, 318, 502, 771.

12.2 From the following failure times, obtained from testing 15 new fuel pumps until failure, derive empirical estimates of the reliability function, the density function, and the hazard rate function. Also compute a 95 percent confidence interval for the MTTF.

130.3	160.4	178.9	131.8	89.7	104.2	87.9	111.9	244.1	31.7
437.1	171.8	187.1	159.0	173.5					

12.3 Three hundred AC motors were originally installed in 1984 as part of a fan assembly. They have all failed. The following data were collected over their operating history:

Year	Number of failures
1985	15
1986	20
1987	18
1988	27
1989	35
1990	31
1991	45
1992	43
1993	66

Derive an empirical reliability function, density function, and hazard rate function for this motor. Estimate the MTTF and the standard deviation of the failure times. Would you conclude that the failure rate is decreasing, constant, or increasing? Which would you expect it to be if the dominant failure mode were due to mechanical wearout?

12.4 Derive an empirical reliability function using Eq. 12.18 and the adjusted rank method based on the following multiply censored data: 5, 12, 15^+, 22, 27, 35^+, 49, 71^+, 73, 81, 112^+, 117.

(a) Assume that 12 units are at risk.

(b) Assume that 15 units were originally placed on test and the test was terminated at the time of the last failure.

12.5 Complete a jet engine life table based on the annual number of failures (replacements) due to the compressor failure and the number of engine removals (censors) for reasons other than compressor failure given in the following table. Five hundred engines (compressors) were at risk initially.

Year	Number of failures	Number of removals
1983	5	15
1984	6	26
1985	12	14
1986	20	23
1987	18	27
1988	25	32
1989	27	46
1990	33	38
1991	31	34
1992	38	30

If engines are now to be overhauled every 2 years (and as a result restored to as good as new condition), what is the reliability estimate over a 5-year period?

12.6 Thirty units were placed on test in order to estimate the reliability of the shift driver over a 200 operating hour design life. Two failures were recorded at the end of the 200 operating hours.

(a) Determine a 90 percent, two-sided confidence interval for $R(200)$.

(b) Determine a 90 percent lower-bound confidence interval for $R(200)$.

12.7 One hundred AIDS patients were given a new drug to test. The results were as follows:

Years on drug	Number of deaths	Number of withdrawals (censors)
1	5	2
2	8	4
3	12	2
4	18	10
5	24	12

Withdrawals occurred when patients left the test area or died from causes not related to the AIDS disease. Construct a life table to estimate the probability (reliability) that a patient will survive at least 5 years.

12.8 Complete the table below. The grouped data reflect failures, in operating hours, of an air conditioning unit (n_i = number surviving).

i	t_i	n_i	$R(t_i)$	$\lambda(t_i)$
0	0	44		
1	100	41		
2	200	36		
3	300	28		
4	400	18		
5	500	6		

Is the hazard rate increasing or decreasing? Can you estimate the MTTF?

12.9 The following multiply censored data reflect failure times, in months, of a new laser printer. Censored times resulted from removals of the printer due to upgrades. Determine the reliability of this printer over its 2-year warranty period. Use Eq. (12.18), the adjusted rank method, and the Kaplan-Meier method.

8, 33, 15^+, 27, 18, 24^+, 13^+, 12, 37, 29^+, 25, 30.

12.10 A 72-hr test was carried out on 25 gizmos, resulting in the following failure times (in hours): 10, 33, 36, 42, 55, 59, 61, 62, 65, 68, 71. Three other units were removed from the test at times 15, 42, and 50 to satisfy customer demands for gizmos. Determine an empirical reliability function and estimate the reliability at the end of the 72-hr test.

12.11 Specifications call for a power transistor to have a reliability of 0.95 at 2000 hr. Five hundred transistors are placed on test for 2000 hr with 15 failures observed. Is the specification being met?

12.12 Will I. Fail, a reliability engineer for Major Motors, has been tasked to test 20 alternators based on a new design in order to estimate their reliability. He has decided to terminate the test after 10 failures with the following failure times (in operating hours) observed: 251, 365, 286, 752, 465, 134, 832, 543, 912, 220. Derive an empirical reliability distribution. On the basis of this distribution, estimate, from a total of 5000 alternators placed in Major Motors' new Zazoom sedan, the number that will fail within the 12-month warranty period. Assume that the typical driver averages 1.0 driving hour per day.

12.13 Fifteen units each of two different deadbolt locking mechanisms were tested under accelerated conditions until 10 failures of each were observed. The following failure times in thousands of cycles were recorded:

Design A: 44, 77, 218, 251, 317, 380, 438, 739, 758, 1115
Design B: 32, 63, 211, 248, 327, 404, 476, 877, 903, 1416

Which design appears to provide the best reliability?

12.14 The following repair times were obtained during product testing as part of a maintainability assessment. If the maintainability goals include an MTTR of 4 hr and 90 percent of the repairs are to be completed within 10 hr, are the goals being achieved? Answer by constructing a 95 percent confidence interval for the MTTR and an empirical cumulative distribution function. Times are in hours: 6.0, 7.5, 5.0, 4.0, 4.5, 5.1, 14, 8.5, 10.2, 5.5, 5.8, 11.5, 8.9, 10.0, 5.7, 4.4, 6.5, 7.0, 8.0, 7.7.

12.15 The Allways Fail Company maintains repair data on the number of hours its production line is down for unscheduled maintenance. Over the past six months the following data have been collected:

Hours	Number of occurrences
0–1	7
1–2	5
2–3	7
3–4	6
4–5	8
5–6	3
6–7	3
7–8	1

Construct an empirical cumulative distribution function for the repair distribution. Estimate the MTTR. If the production line is down for more than 6 hr at a time, the maintenance crew will be penalized. What is an estimate of the probability that the crew will be penalized during a given downtime?

12.16 An electric dryer experiences two failure modes—one with the motor subsystem and the other with the heating subsystem. The following failures have occurred on nine machines that have been put on accelerated life tests for 1500 operating hours.

Machine	Failure time, hr	Failure mode
1	250	Motor
2	780	Motor
3	673	Heating
4	891	Motor
5	190	Heating
6	1020	Motor
7	Did not fail	
8	922	Motor
9	432	Heating

The supplier of the motor claims that it has $R(500) = 0.95$. Is this claim supported by the data?

CHAPTER 13

Reliability Testing

13.1 PRODUCT TESTING

An integrated product test program may consist of several types of tests each having different objectives. For example, with new product design, functional or operational tests will determine whether performance requirements are being achieved; their objective is to evaluate design adequacy. Environmental stress testing will establish the capability of the product to perform under various operating conditions. Reliability qualification tests, in general, obtain various measures of product reliability. Safety testing attempts to generate and correct serious faults, which may result in hazardous or catastrophic occurrences that could cause injury, loss of life, or significant economic loss. Reliability growth testing, on the other hand, consists of repeated reliability testing of prototypes, followed by determination of the causes of failures and elimination of those failure modes through design changes. This cycle of test–fix–test–fix is referred to as reliability growth testing because it has as its objective increased reliability for the end product. As a result of the design changes, each cycle produces a new component or system that has a different (hopefully, improved) failure distribution. Specific models have been developed for estimating and predicting this growth in reliability over time. Other types of product testing may include maintainability demonstration (discussed in Chapter 10), system integration testing, and operational test and evaluation. All product testing may provide useful reliability information, and an aggressive failure mode, effect, and criticality analysis program will capture any relevant failure data. Reliability testing and (to some degree) safety testing are distinguished from other tests in that they attempt to generate failures in order to identify failure modes and eliminate them.

13.2 RELIABILITY LIFE TESTING

The primary objective of reliability life testing is to obtain information concerning failures in order to quantify reliability, to determine whether reliability and safety goals are being met, and to improve product reliability. Typically, the result of a reliability test on a product or part will be a set of failure times $t_1, t_2, \ldots, t_r$. These times will then be analyzed using either the empirical methods discussed in the previous chapter or the parametric methods presented in Chapters 15 and 16. Reliability improvement may result from burn-in or screen testing and reliability growth testing. Chapter 14 will address growth testing; this chapter will address the following types of reliability tests:

Burn-in and screen testing is designed to eliminate or reduce "infant mortality" failures by accumulating initial equipment operating hours and resulting failures prior to user acceptance.

Acceptance and qualification testing demonstrates through life testing that the reliability goals or specifications have been met or determines whether parts or components are within acceptable standards.

Sequential tests are an efficient test for demonstrating that a reliability or maintainability goal is met or not met.

Accelerated life testing comprises techniques for reducing the length of the test period by accelerating failures of highly reliable products.

Experimental design involves statistical methods that are useful in isolating causes of failures in order to eliminate them.

Several important factors must be addressed before any reliability test is conducted. These include the *objective* of the test, the *type of test* to be performed (such as sequential or accelerated), the *operating and environmental conditions* under which the test is to be conducted, the *number of units to be tested* (sample size), the *duration of the test,* and an unequivocal *definition of a failure.* The type of test will depend, in part, on the objectives. If reliability improvement is the objective, then reliability growth testing should be conducted. If the objective is to demonstrate that reliability goals or specifications have been met, then acceptance testing or sequential testing may be used. The test environment should closely simulate the operating environment, particularly with respect to such variables as temperature, humidity, and vibration, including extreme conditions that may be encountered (stress testing). More important than extreme values of environmental factors may be the rates of change in environmental conditions, such as the changes experienced with temperature cycling. The effect of maintenance-induced failures (if applicable) should also be considered. Often, a combination (interaction) of conditions such as temperature and humidity may be needed to induce failures. Systems experiencing dormant failures should be tested accordingly, in order to account for the effect of cycling on and off as well as the impact that dormant time periods have on failures. For example, some hydraulic systems exhibit higher failure rates when used less frequently. Duration of testing is random if the test duration is

TABLE 13.1
Calculation of total test time

Data	T
Complete	$\sum_{i=1}^{n} t_i;\quad r = n$
Type I censor	$\sum_{i=1}^{r} t_i + (n - r)t_*$
Type II censor	$\sum_{i=1}^{r} t_i + (n - r)t_r$
Type I k multiply	$\sum_{i=1}^{r+k} t_i^+ + (n - r - k)t_*$
Type II k multiply	$\sum_{i=1}^{r+k} t_i^+ + (n - r - k)t_r$
Type I replacement	nt_*
Type II replacement	nt_r

t_i = failure time
t_i^+ = failure time or censor time
t_* = test time (Type I testing)
t_r = time of the rth failure (Type II testing)
n = total number of units at risk
r = number of failures
k = number of multiply censors

based on obtaining a specified number of failures. On the other hand, if test duration is defined in terms of hours or days "on test," then the number of failures will be random. The precision by which reliability parameters are estimated depends on the number of failures generated from the sample and not just the number at risk. Therefore, in planning a reliability test, sample size and test duration must be considered together, as discussed further in the next section.

13.3 TEST TIME CALCULATIONS

If a constant failure rate is assumed, then the cumulative test time, T, may be obtained using Table 13.1. Cumulative test time is the total operating time that all units experienced "on test". Once T has been obtained, an estimate for the MTTF (for a CFR model) is given by

$$\text{MTTF} = \frac{T}{r} \tag{13.1}$$

where r is the total number of failures.[1]

EXAMPLE 13.1. During a testing cycle, 20 units were tested for 50 hr with the following failure times and censor times observed: 10.8, 12.6$^+$, 15.7, 28.1, 30.5, 36.0$^+$, 42.1, 48.2. Determine the total test time and estimate the MTTF for this particular cycle, assuming a CFR.

[1] In Chapter 15 it is shown that Eq. (13.1) is the maximum likelihood estimator for the MTTF.

Solution. For Type I testing with $t^* = 50$ hr as the test termination time,

$$\begin{aligned} T &= 10.8 + 12.6 + 15.7 + 28.1 + 30.5 + 36.0 + 42.1 + 48.2 + (20 - 6 - 2)50 \\ &= 824 \text{ hr} \end{aligned}$$

Then MTTF = 824/6 = 137.3 hr

EXAMPLE 13.2. Ten units were placed on test, with a failed unit immediately replaced. The test was terminated after the eighth failure, which occurred at 20 hr. Estimate T and MTTF.

Solution. This is Type II testing with replacement. Therefore,

$$T = (10)(20) = 200 \text{ hr}$$

$$\text{MTTF} = \frac{200}{8} = 25 \text{ hr}$$

13.3.1 Length of Test

For Type II testing, the length of time to complete the test will depend on the number of units being tested, the number of failures to be observed, and the time-to-failure distribution. If only one unit is tested to failure at a time and then replaced with a new unit, the expected test time to generate r failures is $r \times$ MTTF. Under the CFR model, if n units are placed on test until r failures are observed, then the expected test time is given by

$$E(\text{test time}) = \text{MTTF} \times \text{TTF}_{r,n} = \text{MTTF}\left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}\right] \tag{13.2}$$

where $\text{TTF}_{r,n}$ is the test time factor for r failures with n units at risk. Equation (13.2) is derived in Appendix 13A, with selected values of $\text{TTF}_{r,n}$ in Appendix 13B. These values may then be multiplied by an estimated MTTF to determine the expected test time. If failed units are immediately replaced, so that there are always n units on test, then the expected test time to observe r failures is given by

$$E(\text{test time}) = \text{MTTF} \times \text{TTR}_{r,n} = \frac{r\,\text{MTTF}}{n} \tag{13.3}$$

where $\text{TTR}_{r,n}$ is the test time factor with replacement of failed units. The number of units needed to complete the test is $n + r - 1$, since the last failure need not be replaced. It is apparent from Eqs. (13.2) and (13.3) that putting more units on test (increasing n) will decrease the expected test time.

For Type I testing, the length of time is specified as t^*. The number of failures, r, is random. For the CFR model, with n units on test,

$$E(r) = n(1 - e^{-t^*/\text{MTTF}}) \tag{13.4}$$

since $p = 1 - e^{-t^*/\text{MTTF}}$ is the probability of a single unit failing by time t^*. Therefore, the number of failures among n units on test may be viewed as a binomial process with mean np.

With replacement of failed units,

$$E(r) = \frac{nt^*}{\text{MTTF}} \tag{13.5}$$

since the number of failures will have a Poisson distribution with the above mean.[2]

EXAMPLE 13.3. To support the current cycle in a reliability growth testing program, a total of 8 failures need to be generated. The current estimate of the MTTF is 55 hr. The test department is scheduled to complete testing within 72 hr. How many units should be placed on test?

Solution. This is Type II testing. Since the length of the test is MTTF × TTF then the $\text{TTF}_{8,n} = 72/55 = 1.31$. From the table in Appendix 13B,

$$\text{TTF}_{8,10} = 1.429$$
$$\text{TTF}_{8,11} = 1.187$$

Then 11 units should be placed on test.

EXAMPLE 13.4. For the problem in Example 13.3, the test department is told it must complete the testing within 48 hr. How many failures would it expect to generate?

Solution. From Eq. (13.4), $E(r) = 11(1 - e^{-48/55}) = 6.4$ units.

13.4 BURN-IN TESTING

A primary objective of burn-in testing is to increase the mean residual life of components as a result of having survived the burn-in period. Those items that have survived will have a MTTF greater than the MTTF of the original items because the early failures would have been eliminated. The mean residual lifetime can be found from Eq. (2.18). The probability of a failure occurring over a fixed length of time is also reduced for the same reason. Costs are an important consideration in determining whether to utilize burn-in testing or not, and if so, to what degree. There is the cost of the testing, warranty costs, items lost due to burn-in failures, and the cost of failures during operation to consider. As shown in Chapter 2, the item must have a decreasing failure rate (DFR) if burn-in testing is to have any merit. Burn-in testing requires testing of all units produced for the designated time; therefore, it increases production lead time as well as costs. However, accelerated life testing techniques, as discussed later in this chapter, may be applied to reduce the length of time required for burn-in. Burn-in testing may allow contract specifications to be met where they otherwise could not.

Items that have failed during burn-in may be discarded and replaced or be repaired. If a failed item is replaced, it may be replaced with a new item from the

[2]Since there are always n units on test, the time to the next failure is exponential with a mean of $1/n\lambda$. As a result of the relationship between the exponential and Poisson distributions, the number of failures in time t is Poisson with a mean of $n\lambda t = nt/\text{MTTF}$.

same parent population, which may or may not have had some burn-in time accumulated. If a failed item is repaired, it may be repaired to its original condition or it may be minimally repaired, as discussed in Chapter 9. In the latter case, if the intensity function is decreasing, then improved reliability will result from the burn-in. How the burn-in period is modeled mathematically depends on the manner in which failures are disposed of. Often, the primary determination for burn-in testing is the length of the test. The following model to determine the length of the burn-in period assumes that only the surviving units are utilized following burn-in. The model is based on Fig. 13.1.

Given a reliability goal at time t_0 of R_0, where $R(t_0) < R_0$ and $R(t)$ has a DFR, a burn-in period, T, is desired such that $R(t_0 \mid T) = R_0$. For the Weibull distribution this conditional reliability results in the following nonlinear equation (see Section 4.1.1), which must be solved numerically:

$$\exp\left[-\left(\frac{t_0 + T}{\theta}\right)^{\beta}\right] - R_0 \exp\left[-\left(\frac{T}{\theta}\right)^{\beta}\right] = 0 \tag{13.6}$$

EXAMPLE 13.5. Reliability testing has shown that a ground power unit used to supply DC power to aircraft has a Weibull distribution with $\beta = 0.5$ and $\theta = 45{,}000$ operating hours. Determine a burn-in period necessary to obtain a required reliability specification of $R(1000) = 0.90$.

Solution. Observe that $R(1000) = 0.86$ and $\beta < 1$. Therefore, a burn-in period is necessary. Numerically solving

$$\exp\left[-\left(\frac{1000 + T}{45000}\right)^{0.5}\right] - 0.90 \exp\left[-\left(\frac{T}{45000}\right)^{0.5}\right] = 0$$

yields $T = 126$ hr. Therefore $R(1000 \mid 126) = 0.90$. The actual clock time for burn-in may be reduced through the use of accelerated test methods.

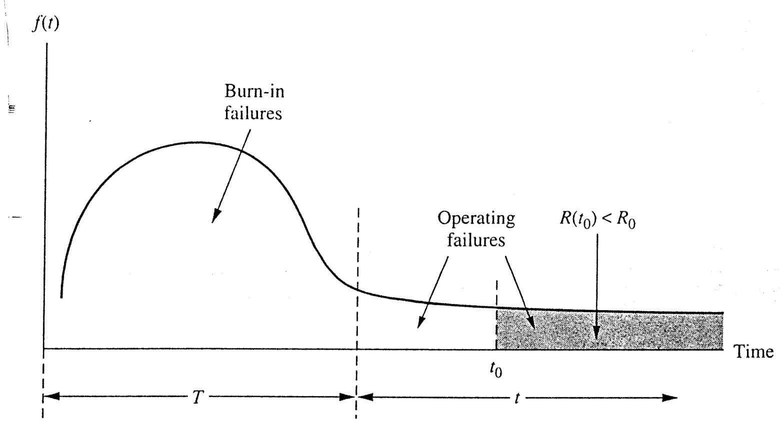


FIGURE 13.1
The failure distribution and burn-in testing.

The length of the burn-in period can also depend on costs. The following expected cost model addresses the trade-off between the costs of conducting the burn-in and the cost of failures following burn-in. Let

C_b = cost per unit time for burn-in testing
C_f = cost per failure during burn-in
C_o = cost per failure when operational
T = length of burn-in testing
t = operational life of the units

Assume that n units are to be produced, each having a reliability function $R(t)$ and each undergoing burn-in testing. Those that fail during burn-in are discarded, and the survivors become operational. The expected number of failures during burn-in is $n[1 - R(T)]$. The expected number of operational failures is

$$nR(T)[1 - R(t \mid T)] = nR(T)[1 - R(t + T)/R(T)] = n[R(T) - R(t + T)]$$

Therefore the expected total cost is

$$E[\text{TC}] = nC_bT + C_fn[1 - R(T)] + C_on[R(T) - R(t + T)]$$

and the expected cost per unit is

$$E[C] = E(\text{TC})/n = C_bT + C_f[1 - R(T)] + C_o[R(T) - R(t + T)] \quad (13.7)$$

For the Weibull distribution,

$$E[C] = C_bT + C_f\left\{1 - \exp\left[-\left(\frac{T}{\theta}\right)^\beta\right]\right\} + C_o\left\{\exp\left[-\left(\frac{T}{\theta}\right)^\beta\right] - \exp\left[-\left(\frac{T + t}{\theta}\right)^\beta\right]\right\} \quad (13.8)$$

Numerical search procedures can be used to find the value of T that minimizes Equation (13.8).

EXAMPLE 13.6. The replacement cost on a new product, if it fails during its operational life of 10 years (3650 days), is \$6200. It will cost the company \$70 a day per unit tested to operate a burn-in program, and any failures during burn-in will cost \$500. Reliability testing has established that the life distribution of the product is Weibull with $\beta = 0.35$ and $\theta = 3500$ days. What is the minimum-cost time period for the burn-in?

Solution. The expected cost equation to be minimized is

$$E(C) = 70T + 500\left\{1 - \exp\left[-\left(\frac{T}{3500}\right)^{0.35}\right]\right\} + 6200\left\{\exp\left[-\left(\frac{T}{3500}\right)^{0.35}\right] - \exp\left[-\left(\frac{T + 3650}{3500}\right)^{0.35}\right]\right\}$$

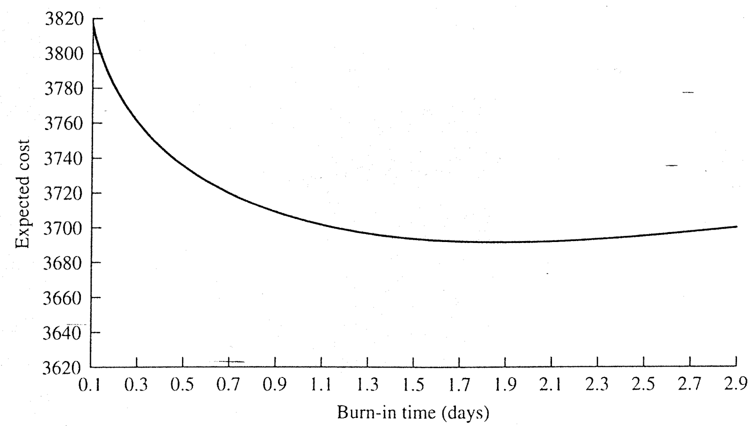


FIGURE 13.2
Burn-in testing time versus expected total costs.

A direct search resulted in the curve in Figure 13.2, in which the minimum-cost burn-in time $T^* = 1.9$ days, resulting in an expected cost per unit of \$3,690. With no burn-in, the expected unit cost is \$3952. It may be desirable to operate further up on the curve from the least-cost solution. For example, a burn-in time of 1 day results in an expected cost of \$3704—a difference of only \$14 per unit.

The number of units produced and tested (n) may depend on the number required to survive the operational life. The expected number surviving to time t is $nR(t)$. Therefore, if k units are required to be operating at the end of t time units, then $n = k/R(t)$. For Example 13.6, $R(3650) = 0.362$. Therefore, if 100 units must survive, then $n = 100/0.362 = 276$ units must be produced and tested. Notice that burn-in testing does not reduce the number of failures. It simply moves failures from operations to manufacturing, presumably on the premise that the cost of failures during burn-in is less than the cost of operational failures. Costs in this case may also include considerations for safety. Considering the large number of expected failures in the foregoing example, improved quality control and reliability redesign may have a greater economic impact.

For further discussion on burn-in testing, the reader is referred to the text *Burn-In,* by Jensen and Peterson [1982], and the survey on burn-in models and methods by Leemis and Beneke [1990]. Jacobowitz [1987] describes an automated process for designing cost-effective burn-in programs.

13.5 ACCEPTANCE TESTING

The objective of acceptance or qualification testing is to demonstrate that the system design meets performance and reliability requirements under specified operating and environmental conditions. Acceptance testing may be based on a predetermined sample size or on an unspecified sample size resulting from a sequential test as described subsequently. Units from the production line should be randomly selected for testing.

13.5.1 Binomial Acceptance Testing

One of the simplest reliability acceptance test plans is based on the binomial process. The objective is to demonstrate that the system reliability at time T is R_1 (that is, $R(T) = R_1$). A total of n units are placed on test, and X failures are observed by time T. If $X \leq r$, then the desired reliability is demonstrated; otherwise, it is concluded that $R(T) < R_1$. The test plan is based on specifying the sample size n and the maximum number of failures, r, for acceptance.

Observe that X, the number of failures by time T among n independent units at risk, is a random variable. Then X has a binomial probability distribution with parameters n and $p = (1 - R)$, where R is the "true" system reliability at time T. Clearly, the randomness or uncertainty associated with the sampling and testing of the n units may result in incorrectly accepting or rejecting the reliability specification. What is desired is to find values for n and r that will result in a high probability of acceptance if $R(T) = R_1$ and a low probability of acceptance if $R(T) = R_2 < R_1$. To state this requirement more formally,

$$\Pr\{X \leq r \mid R = R_1\} = 1 - \alpha \qquad \text{and} \qquad \Pr\{X \leq r \mid R = R_2\} = \beta$$

Figure 13.3 shows the relationship between the system failure probability $(1 - R)$ and the probability of acceptance. Observe that α is the probability of incorrectly rejecting the reliability specification and β is the probability of incorrectly accepting the reliability specification.[3] The curve in Fig. 13.3 is called an *operating characteristic curve*. The shape of the curve depends on the values specified for n and r. The region, $R_1 < R < R_2$ is referred to as the *indifference zone*. Since X is binomial, the foregoing probability statements can be written in terms of n and r:

$$\sum_{i=0}^{r} \binom{n}{i} (1 - R_1)^i R_1^{n-i} = 1 - \alpha$$
$$\sum_{i=0}^{r} \binom{n}{i} (1 - R_2)^i R_2^{n-i} = \beta \qquad (13.9)$$

By specifying R_1, R_2, α, and β, the problem is to find values for n and r that will satisfy Eqs. (13.9). (Since n and r must be integer-valued, Eqs. (13.9) can be converted to inequalities.) In practice, it is easier to specify R_1, R_2, n, and r, solve Eqs. (13.9) for $1 - \alpha$ and β, and repeat until, through trial and error, acceptable values for n and r are found. The result is a reliability demonstration or acceptance plan that will discriminate between an acceptable reliability and an unacceptable reliability at specified risk levels. Additional discussion on binomial acceptance sampling may be found in Kolarik [1995].

[3] Alpha (α) is often called the *producer's risk* and beta (β) the *consumer's risk*.

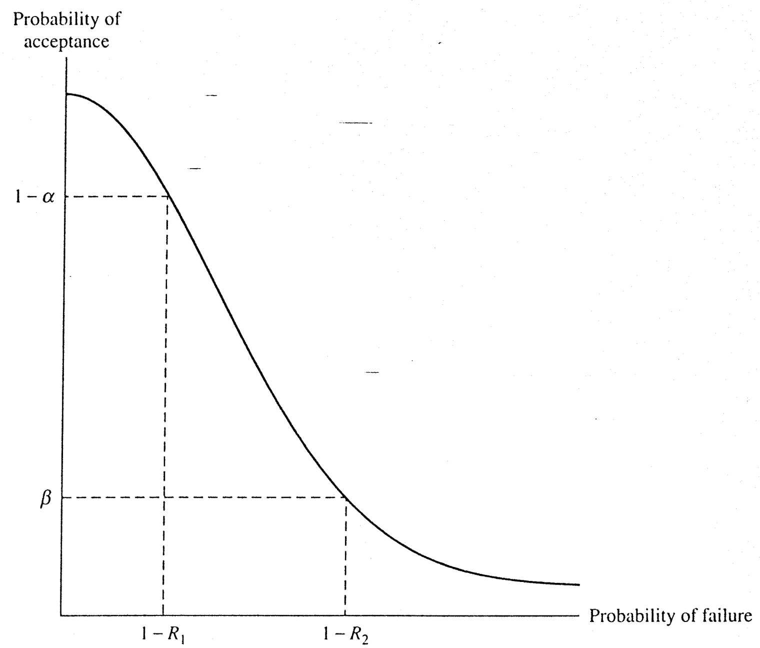


FIGURE 13.3
The operating characteristic curve.

EXAMPLE 13.7. Equations (13.9) were solved for $1 - \alpha$ and β for various combinations of R_1, R_2, n, and r in order to generate representative reliability acceptance plans. Plans for which both $\alpha < 0.10$ and $\beta < 0.10$ are displayed in Table 13.2. A number of more comprehensive sampling plans have been published, such as those found in Military Standard 105 (MIL-STD-105) [1963].

TABLE 13.2
Selected reliability acceptance plans

R_1	R_2	n	r	$1 - \alpha$	β
0.99	0.90	50	1	0.911	0.034
0.99	0.90	60	2	0.978	0.053
0.99	0.90	70	3	0.995	0.071
0.95	0.89	150	11	0.926	0.091
0.95	0.89	175	13	0.942	0.077
0.98	0.92	100	4	0.949	0.090
0.98	0.92	120	5	0.966	0.075
0.95	0.85	75	6	0.919	0.054
0.95	0.85	100	9	0.972	0.055
0.96	0.92	250	14	0.921	0.095
0.96	0.92	275	15	0.912	0.069
0.995	0.95	90	1	0.925	0.057
0.995	0.95	120	2	0.977	0.058

13.5.2 Sequential Tests

Sequential testing provides an efficient method for accepting or rejecting a statistical hypothesis when the evidence (sample) is highly favorable to one of the two decisions. Since the sample size required depends on the observed times, fewer failures may need to be generated than would be the case under a fixed-sample-size test. This test, based on the sequential probability ratio test developed by Wald [1947], would be used in a reliability or maintainability demonstration or in acceptance and qualification testing; it would not be used for estimating a reliability parameter.

Assume that a reliability parameter (such as MTTF, failure rate, failure probability, or a characteristic life) represented in general by ϕ has a *specification* ϕ_0. Assume as well that we can state an *unacceptable value* for this parameter, denoted by ϕ_1. Then we can state a hypothesis that the product being tested meets (or exceeds) the specification against an alternative hypothesis that the specification is not met. Formally, we define a null (H_0) and alternate (H_1) hypothesis as follows:

$$H_0: \quad \phi = \phi_0$$
$$H_1: \quad \phi = \phi_1 > \phi_0$$

The general approach is to generate failure or repair times, $t_1, t_2, \ldots, t_r$, sequentially. With each new time a test statistic, $y_r = h(t_1, t_2, \ldots, t_r)$, is computed. Depending on the value of the test statistic, we accept the null hypothesis, reject the null hypothesis, or reserve judgment. If we reserve judgment, another sample time is generated, y_r is recomputed, and the test is repeated. This process continues until the null hypothesis is either accepted or rejected.

The criterion to accept, reject, or continue sampling is based on the probability of making an incorrect decision. There are two ways in which an incorrect decision can be made. We may reject a correct null hypothesis (called a type I error), or we may accept a false null hypothesis (called a type II error). Mathematically,

$$\Pr\{\text{reject } H_0 \mid \phi_0\} = \alpha \qquad \text{and} \qquad \Pr\{\text{accept } H_0 \mid \phi_1\} = \beta$$

Alpha (α), the producer's risk, is the probability of rejecting an acceptable product, whereas beta (β), the consumer's risk, is the probability of not rejecting an unacceptable product.

From Equation (12.1), the joint probability distribution for the sample $t_1, \ldots, t_r$ is $\prod_{i=1}^{r} f(t_i \mid \phi)$. The joint distribution formed from an independent random sample taken from the identical population having a parameter ϕ is called the likelihood function. In the case of a discrete distribution, the likelihood function is the probability of generating a sample that has the observed failure or repair times. It would seem reasonable, therefore, to select a value for ϕ that will maximize the likelihood function. Therefore, a test statistic y can be formed from the ratio of the likelihood function formed under H_1 to that formed under H_0. If the null hypothesis is correct, the denominator of this ratio will be larger than the numerator, and y will be small. Therefore, we accept H_0 if $y_r \leq A$, where y_r is defined as

$$y_r = \frac{\prod_{i=1}^{r} f(t_i \mid \phi_1)}{\prod_{i=1}^{r} f(t_i \mid \phi_0)} \approx \frac{\Pr\{\text{accept } H_0 \mid \phi_1\}}{\Pr\{\text{accept } H_0 \mid \phi_0\}} = A \tag{13.10}$$

If the alternate hypothesis is correct, then the numerator will be larger than the denominator, and y will be large. Therefore, we will reject the null hypothesis if $y_r \geq B$, where

$$B \approx \frac{\Pr\{\text{reject } H_0 \mid \phi_1\}}{\Pr\{\text{reject } H_0 \mid \phi_0\}}$$

The values for A and B are computed so that the specified probabilities of making a Type I and Type II error are approximated. Therefore

$$A = \frac{\beta}{1-\alpha} \qquad \text{and} \qquad B = \frac{1-\beta}{\alpha}$$

In conducting a sequential test, α, β, ϕ_0, and ϕ_1 must be specified. Then A and B are computed as shown. If $A < y_r < B$, then the test continues by generating another sample.

Exponential case

For the exponential distribution $f(t) = \lambda e^{-\lambda t}$. The hypotheses are

$$H_0: \quad \lambda \leq \lambda_0$$
$$H_1: \quad \lambda = \lambda_1 > \lambda_0$$

Assuming that the data are complete and that t_i is the time to failure of the ith unit tested, then the continuation region is represented by

$$A < y_r = \prod_{i=1}^{r} \frac{\lambda_1 e^{-\lambda_1 t_i}}{\lambda_0 e^{-\lambda_0 t_i}} < B$$

Taking logs and rearranging terms,

$$\frac{-\ln B + r\ln(\lambda_1/\lambda_0)}{\lambda_1 - \lambda_0} \leq \sum_{i=1}^{r} t_i \leq \frac{-\ln A + r\ln(\lambda_1/\lambda_0)}{\lambda_1 - \lambda_0} \tag{13.11}$$

Therefore, the total test time generated by r failures forms the basis for the test.

EXAMPLE 13.8. Develop an exponential sequential ratio test where $\lambda_0 = 0.00125$ ($\text{MTTF}_0 = 800$), $\lambda_1 = 0.0014286$ ($\text{MTTF}_1 = 700$), $\alpha = 0.05$, and $\beta = 0.10$. Then

$$A = 0.10/(1 - 0.05) = 0.1052632 \qquad \text{and} \qquad B = (1 - 0.10)/0.05 = 18$$

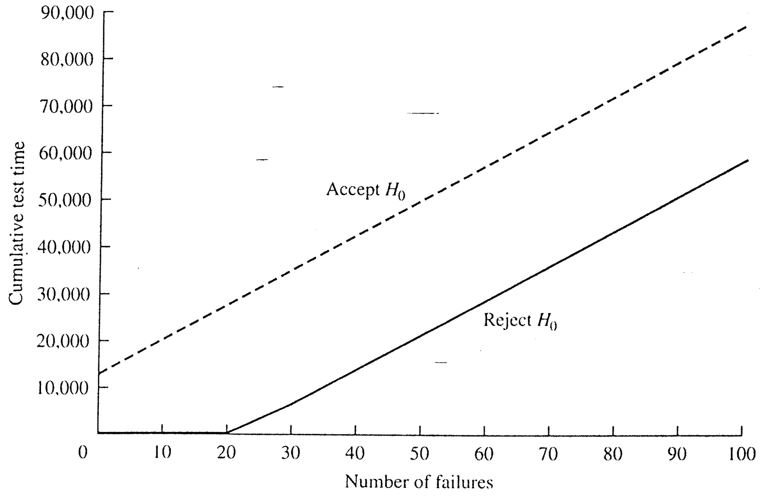


FIGURE 13.4
Sequential test based on the exponential distribution. The solid line indicates the lower bound rejection of H_0; the dashed line indicates the upper bound for acceptance of H_0.

The continuation region is given by

$$-16183 + 747.8r = \frac{-\ln 18 + r\ln(0.0014286/0.00125)}{0.0014286 - 0.00125}$$
$$< \sum_{i=1}^{r} t_i$$
$$< \frac{-\ln 0.10526 + r\ln(0.0014286/0.00125)}{0.0014286 - 0.00125}$$
$$= 12607 + 747.8r$$

Figure 13.4 shows a graph of the lower and upper bounds as a function of the total time on test versus number of failures generated. Therefore, testing continues until the sum of the failure times either exceeds the upper bound for r, in which case H_0 is accepted, or falls below the lower bound for r, in which case H_0 is rejected. A minimum of 21 failures must be generated before H_0 can be rejected, and a minimum of 12,607 units of test time is needed before H_0 can be accepted.

Binomial testing

An alternative acceptance or qualification criterion is based on a reliability demonstration. In this case, no assumption concerning the failure distribution is necessary. The test is based on a binomial process. The hypotheses to test are

$$H_0: \quad R(t_0) = R_0$$
$$H_1: \quad R(t_0) = R_1 < R_0$$

Assume that n units are currently tested until time t_0 and that y survivors are observed. Then the likelihood functions under H_0 and H_1 are

$$p(y) = \binom{n}{y} R_0^y (1 - R_0)^{n-y} \quad \text{and} \quad p(y) = \binom{n}{y} R_1^y (1 - R_1)^{n-y} \quad \text{respectively.}$$

Therefore, the continuation region is found from

$$A < \left(\frac{R_1}{R_0}\right)^y \left(\frac{1 - R_1}{1 - R_0}\right)^{n-y} < B \tag{13.12}$$

After taking logarithms and performing some algebra, we obtain

$$\frac{\ln B - n\ln\dfrac{1 - R_1}{1 - R_0}}{D} < y_n < \frac{\ln A - n\ln\dfrac{1 - R_1}{1 - R_0}}{D} \tag{13.13}$$

where
$$D = \ln\left(\frac{R_1(1 - R_0)}{R_0(1 - R_1)}\right)$$

The test consists of observing y_n, which is the number of survivors from among n units at risk. If y_n is less than or equal to the lower bound, then H_0 is rejected. If y_n is equal to or greater than the upper bound, then H_0 is accepted and the reliability specification has been demonstrated. If y_n falls within the continuation region, another unit is tested until time t_0.

EXAMPLE 13.9. Test the hypothesis

$$H_0: R_0 = 0.90$$
$$H_1: R_1 = 0.85$$

with $\alpha = 0.05$ and $\beta = 0.10$. Therefore

$$A = 0.10526$$
$$B = 18$$
$$D = \ln\{(0.85)(0.10)/[(0.90)(0.15)]\} = -0.4626$$

and the slope of the accept/reject lines is

$$-\ln(0.15/0.1)/D = 0.876$$

Then $\ln(B)/D = -6.2478$ and $\ln(A)/D = 4.866$. The acceptance region is $-6.2478 + 0.876n < y_n < 4.866 + 0.876n$, and H_0 will be rejected if y_n falls below the lower bound and accepted if y_n exceeds the upper bound. A graph of the regions is shown in Figure 13.5. The minimum number of test cases to reject H_0 is 8 (where the reject line first crosses the horizontal axis), whereas the minimum number needed to accept H_0 is 40. Below 40, the number of survivors needed to accept H_0 is more than the number at risk.

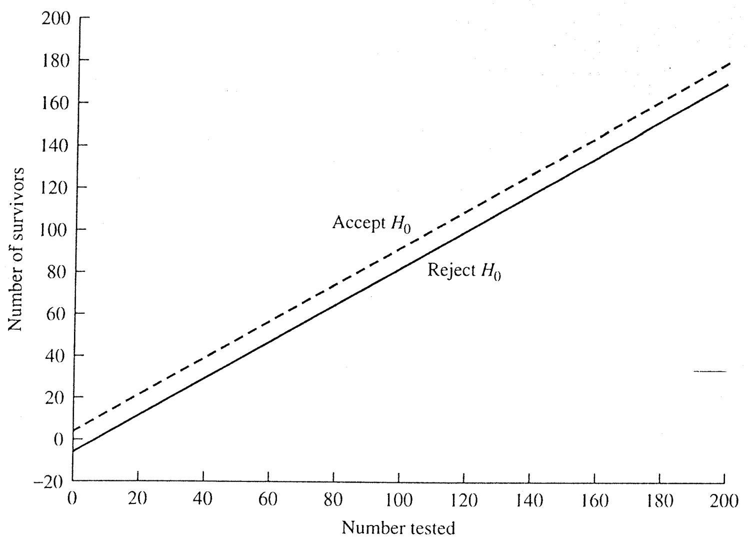


FIGURE 13.5
Sequential test based on binomial sampling.

Maintainability demonstration

The binomial sequential test can be used in performing a maintainability demonstration. The hypotheses are

$$H_0:\quad H(t_0) = P_0$$
$$H_1:\quad H(t_0) = P_1 < P_0$$

where $H(t)$ is the cumulative distribution function of the repair distribution and P_0 is the fraction of repairs to be completed within t_0 time units. The P_1 in the alternate hypothesis is an unacceptable fraction of repairs to be completed within time t_0. By defining y_n to be the number of repairs from among n attempts completed within time t_0, the acceptance and rejection regions are computed using Eq. (13.13), with P_0 replacing R_0 and P_1 replacing R_1. If y_n equals or exceeds the upper bound, then H_0 is accepted and the maintainability goal has been demonstrated. If y_n is less than or equal to the lower bound, then H_0 is rejected.

In a hypothesis test the parameter under the alternative hypothesis may take on a range of values. The farther these values are from the hypothesized value ϕ_0, the smaller will be the probability of a Type II error, β. A plot of the probability of a Type II error versus the value of ϕ under the alternate hypothesis generates the operating characteristic (OC) curve such as the one shown in Figure 13.6. The reader is referred to Kapur and Lamberson [1977] for details on computing OC curves. Other sequential tests may be developed on the basis of Weibull or normal failure or repair distributions. Additional discussions on acceptance sampling and sequential sampling may be found in Gibra [1973].

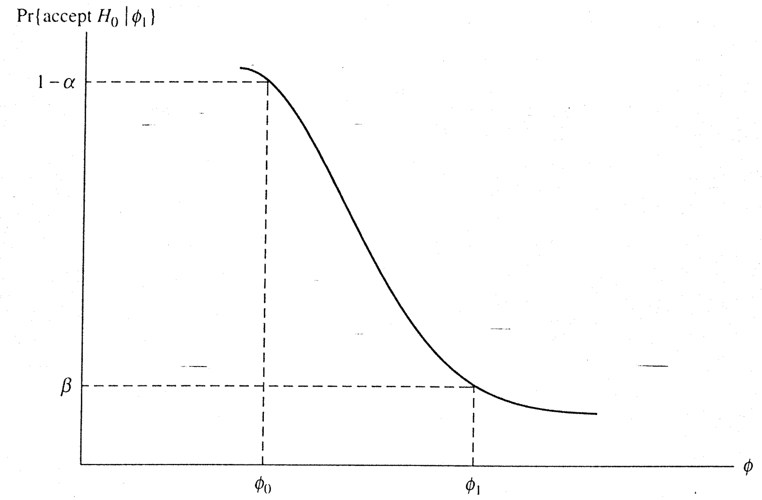


FIGURE 13.6
The operating characteristic curve for a sequential test.

13.6 ACCELERATED LIFE TESTING

The amount of time available for testing is often considerably less than the expected lifetime of the component. This is certainly true for highly reliable components, for which testing under normal conditions would generate few if any failures within a reasonable time period. In order to identify design weaknesses during growth testing, burn-in testing, or reliability testing, one or more of the following may be necessary:

1. Increase the number of units on test
2. Accelerate the number of cycles per unit of time
3. Increase the stresses that generate failures (accelerated stress testing).

For example, additional units may be placed on test, thus increasing the number of failures within a given time. Motors that are expected to operate for only a few hours a day in the field can be operated continuously with intermittent starting and stopping during testing. On the other hand, some wearout failure modes, such as corrosion, can be accelerated by operating the system under elevated stress levels, such as higher temperature and humidity. Increased mechanical stress, higher voltage or current, and increased radiation may accelerate other failure modes. If time is measured in cycles, then time compression may simply require increasing the number of cycles per unit of time. For example, a mechanical switch may fail on demand (such as by being cycled on/off), in which case the frequency of use (such as cycles per day) can be significantly increased under accelerated test conditions.

13.6.1 Number of Units on Test

For Type II testing, the effect of adding additional units on test was discussed at length in Section 13.3 for the CFR model. By using the expected-test timetable in Appendix 13B, we can find the fraction savings in test time that result from having n units, rather than r units, at risk when r failures are desired. Let

$$f_{r,n} = \frac{\mathrm{TTF}_{n,r}}{\mathrm{TTF}_{r,r}} \tag{13.14}$$

Then the percent savings is $100(1 - f_{r,n})$. If failed units are replaced, then $f_{r,n} = r/n$. For the Weibull failure distribution, Kapur and Lamberson [1977] suggest the following approximation:

$$f_{r,n} = \left(\frac{\mathrm{TTF}_{n,r}}{\mathrm{TTF}_{r,r}}\right)^{1/\beta} \tag{13.15}$$

To apply this formula, the shape parameter β must be specified. If failures are replaced, then $f_{r,n} = (r/n)^{1/\beta}$.

EXAMPLE 13.10. For the case in which $n = 15$ and $r = 8$,

$$\begin{aligned}
f_{8,15} &= \mathrm{TTF}_{8,15}/\mathrm{TTF}_{8,8} \\
&= 0.725/2.718 = 0.2667 \text{ for the CFR model without replacement} \\
f_{8,15} &= 8/15 \\
&= 0.533 \text{ for the CFR model with replacement} \\
f_{8,15} &= (0.725/2.718)^{1/2} \\
&= 0.516 \text{ for a Weibull distribution with } \beta = 2 \text{ without replacement} \\
f_{8,15} &= (8/15)^{1/2} = 0.730 \text{ for a Weibull distribution with } \beta = 2 \text{ with replacement}
\end{aligned}$$

The relative savings of replacing failed components versus not replacing them can also be established by forming the ratio

$$\frac{r}{n\mathrm{TTF}_{r,n}}$$

where $\mathrm{TTF}_{r,n}$ is a value from Appendix 13B. Therefore, $8/[15(0.725)] = 0.7356$ is the fraction of test time obtained by replacing failed units with 15 units on test and 8 failures generated. For CFR components, the additional $n - r$ units on test will not be affected by the test hours accumulated against them. However, for Weibull components with $\beta > 1$, the effect of wearout must be considered.

13.6.2 Accelerated Cycling

Assume that no new failure modes are introduced as a result of increasing the number of cycles per unit of time and that failures occur due to cycling only. Define

x_n = number of cycles per unit of time under normal operating conditions
x_s = number of cycles per unit of time under accelerated conditions
t_n = time to failure under x_n cycles per unit of time
t_s = time to failure under x_s cycles per unit of time

Since the number of cycles to failure is the same for both the normal and accelerated conditions, then $x_n t_n = x_s t_s$, or

$$t_s = \frac{x_n}{x_s} t_n \qquad \text{and} \qquad R_n(t_n) = R_s(t_s) = R_s\left(\frac{x_n}{x_s} t_n\right)$$

For the Weibull distribution (as well as the exponential),

$$R_n(t_n) = \exp\left[-\left(\frac{t_n}{\theta_n}\right)^{\beta_n}\right] = \exp\left[-\left(\frac{t_s}{\theta_s}\right)^{\beta_s}\right] = \exp\left[-\left(\frac{x_n t_n}{x_s \theta_s}\right)^{\beta_s}\right] \tag{13.16}$$

Therefore $\beta_s = \beta_n = \beta$, and

$$\theta_n = \frac{x_s}{x_n}\theta_s$$

Under accelerated cycling, only the characteristic life changes, and the Weibull retains its shape parameter. For the exponential distribution the MTTF replaces θ, and $\mathrm{MTTF}_n = x_s \mathrm{MTTF}_s / x_n$.

EXAMPLE 13.11. An automotive part was tested at an accelerated cycling level of 100 cycles per hour. The resulting failure data were found to have a Weibull distribution, with $\beta = 2.5$ and $\theta_s = 1000$ hr. If the normal cycle time is 5 per hour, then

$$\theta_n = (100/5)1000 = 20{,}000 \text{ hr} \qquad \text{and} \qquad R_n(t) = \exp\left[-\left(\frac{t}{20000}\right)^{2.5}\right]$$

13.6.3 Constant-Stress Models

The basic assumption of accelerated stress testing is that at the higher stress levels the same failure mechanism will be present and act in the same manner as at normal stress levels. Failures will happen more quickly; only a transformation of the time scale is observed; no new failure modes are introduced. Under these assumptions, accelerated stress testing can be modeled mathematically. The simplest case assumes a linear (constant) acceleration effect over time. That is, letting

t_n = time to failure under normal stress
t_s = time to failure at high stress level

then

$$t_n = \mathrm{AF} \times t_s$$

where AF is an acceleration factor to be determined. Therefore,

$$\Pr\{T_n < t_n\} = F_n(t_n) = \Pr\{T_s < t_s\} = F_s(t_n/\text{AF}) \tag{13.17}$$

is the CDF of the failure distribution,

$$f_n(t) = \frac{d}{dt}F_s\left(\frac{t}{\text{AF}}\right) = \frac{1}{\text{AF}}f_s\left(\frac{t}{\text{AF}}\right) \tag{13.18}$$

is the PDF, and

$$\lambda_n(t) = \frac{\dfrac{1}{\text{AF}}f_s\left(\dfrac{t}{\text{AF}}\right)}{1 - F_s\left(\dfrac{t}{\text{AF}}\right)} = \frac{1}{\text{AF}}\lambda_s\left(\frac{t}{\text{AF}}\right) \tag{13.19}$$

is the hazard rate function. Equation (13.19) suggests that if the failure rate at the accelerated stress level is constant, then the failure rate under normal stress will also be constant. Thus, the exponential failure distribution is preserved under constant acceleration.

EXAMPLE 13.12. For the CFR model, a component is tested at 120°C and found to have an MTTF = 500 hr. Normal use is at 25°C. Assuming AF = 15, determine the component's MTTF and reliability function at normal stress levels.

Solution.

$$F_n(t) = F_s\left(\frac{t}{\text{AF}}\right) = 1 - e^{-\lambda_s(t/\text{AF})} = 1 - e^{-t/(500\times 15)}$$

or

$$R(t) = e^{-t/7500}$$

and MTTF = 7500 hr.

For the Weibull failure distribution,

$$F_s(t) = 1 - \exp\left[-\left(\frac{t}{\theta_s}\right)^{\beta_s}\right]$$

$$F_n(t) = 1 - \exp\left[-\left(\frac{t}{\text{AF}\theta_s}\right)^{\beta_s}\right]$$

or

$$\theta_n = \text{AF} \times \theta_s \quad \text{and} \quad \beta_n = \beta_s$$

Therefore, only the characteristic life is affected by the linear accelerated stress testing. The acceleration factor, AF, can be estimated by AF = $\hat{\theta}_n/\hat{\theta}_s$. Methods for estimating the characteristic life will be discussed in the following chapter. In general, the characteristic life can be estimated at two different stress levels, and their ratios will provide the desired value for AF. Using Eq. (13.19), for the Weibull failure law,

$$\lambda_n(t) = \frac{1}{\text{AF}}\frac{\beta}{\theta_s}\left(\frac{t}{\text{AF}\theta_s}\right)^{\beta-1} = \frac{1}{\text{AF}^{\beta}}\lambda_s(t) \tag{13.20}$$

EXAMPLE 13.13. Consider the following set of data collected at an accelerated stress level:[4]

39.4	40.8	47.1	66.8	69.3	71.0
77.7	81.2	83.3	84.3	142.8	146.2

Using the procedure discussed in the next chapter, β = 2.556 and θ = 89.4. A second sample is obtained at a normal stress level:

118.3	122.4	141.2	200.3	208.0	213.1
233.0	243.7	249.9	253.0	428.5	438.6

For this sample,

$$\beta = 2.556 \quad \text{and} \quad \theta = 268$$

Therefore, AF = 268/89.4 = 2.9977 ≈ 3.0. Then, from a larger sample at an accelerated stress level, the following data are recorded:

19.8	21.8	29.6	39.4	44.9
57.8	60.0	62.7	66.9	70.3
71.3	76.8	76.8	83.2	83.5
84.9	89.7	92.7	106.4	115.6
119.5	125.2	132.0	140.7	142.7
143.0	172.5	186.2	209.8	237.7

The β = 1.96 and θ = 111.7. Therefore, $R_n(t) = \exp[-(t/335.1)^{1.96}]$.

13.6.4 Other Acceleration Models

Arrhenius model

When failures are accelerated primarily as a result of an increase in temperature, a common approach is based on the Arrhenius model,

$$r = Ae^{-B/T} \tag{13.21}$$

where r is the reaction or process rate, A and B are constants, and T is temperature measured in kelvins.[5] Therefore, the acceleration factor may be determined from

$$\text{AF} = \frac{Ae^{-B/T_2}}{Ae^{-B/T_1}} = \exp\left[B\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right] \tag{13.22}$$

[4]Data is generated from a Weibull distribution with β = 1.75 and θ = 100 (high-stress) and θ = 300 (low-stress).

[5]B can be expressed as $\Delta E/8.6171 \times 10^{-5}$, where ΔE is the activation energy in electron volts and the constant is the Boltzmann constant in electron volts per kelvin (Kelvin temperature = 273.16 + temperature in °C). It is referred to as the coefficient of reaction.

The constant B can be estimated by testing at two different stress temperatures and computing the acceleration factor on the basis of the fitted distributions. In that case

$$B = \frac{\ln \text{AF}}{1/T_1 - 1/T_2} \tag{13.23}$$

where $\text{AF} = \theta_1/\theta_2$, with θ_i representing a scale parameter or a percentile at the stress level corresponding to T_i.

EXAMPLE 13.14. An electronic component has a normal operating temperature of 294 K (about 21° C). Under stress testing at 430 K a Weibull distribution was obtained with $\theta = 254$ hr, and at 450 K, a Weibull distribution was obtained with $\theta = 183$ hours. The shape parameter did not change with $\beta = 1.72$. Therefore, the constant B is estimated to be

$$B = \frac{\ln(254/183)}{1/430 - 1/450} = 3172$$

and the acceleration factor to be applied at the normal stress temperature is found from

$$\text{AF} = \exp\left[3172\left(\frac{1}{294} - \frac{1}{450}\right)\right] = 42.1$$

Therefore, the time to failure of the component at normal operating temperatures is estimated to be Weibull with a shape parameter of 1.72 and $\theta = 42.1 \times 183 = 7704.3$ hr.

Eyring model

The Eyring model as presented here follows the discussion by Tobias and Trindade [1986]. This model allows for additional stresses and can be derived from quantum mechanics. In its simplest form it can be written as

$$r = AT^a e^{-B/T} e^{CS}$$

where r is the process rate; A, a, B, and C are constants; T is temperature (in kelvins); and S is a second stress. The first exponential factor and its coefficient account for the temperature and, except for T^a, behave as in the Arrhenius model. The second exponential factor involves a second, nonthermal stress. Additional factors like the second can be included (with constants different from C) if additional stresses are present.

If a is close to zero, then the T^a factor will be close to 1 at all temperatures, and its effect can be included as part of the constant A. In the absence of a second stress, the similarity with the Arrhenius model is apparent and explains why the Arrhenius model works as well as it does although it is strictly an empirical model and the Eyring model is derived from theoretical considerations.

To apply this model, the constants must be estimated from test data. Estimating the four constants in this model will require at least four data points at two different temperature levels and two different stress levels. The acceleration factor for this model is

$$\text{AF} = \left(\frac{T_2}{T_1}\right)^a \exp\left[B\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right] e^{C(S_2 - S_1)} \tag{13.24}$$

Degradation models

If a product has an observable performance measure that changes with time, it may be possible to predict the time to failure by extrapolating degradation in performance over time. Performance may be measured at either stressed or normal operating conditions, and a critical level of performance that will result in a failure must be specified. Examples of degradation processes include corrosion, crack propagation, and the shelf life of pharmaceutical products. Regression analysis can be used to develop empirical models that relate degradation in performance to time. The simplest relationship is linear with $y = a - bt$, where y is the performance measure (or frequently the log of the performance measure), a and b are constants to be determined experimentally, and t is the amount of time the product is exposed at a constant stress level. If y_f is the level at which a failure occurs, then the time to failure, t_f, is given by

$$t_f = \frac{a - y_f}{b} \tag{13.25}$$

This time to failure is treated as a "typical" value; therefore, it may be interpreted as the mean or median of the failure distribution. Alternatively, n units may be tested, performance measured several times for each unit, and separate regression lines fitted to each unit. This will then generate a sample of n predicted failure times.

EXAMPLE 13.15. For material subject to corrosion, the length of time before degradation becomes unacceptable may be very lengthy. However, a corrosion penetration rate (CPR), which measures the thickness loss of material per unit of time, can be computed as

$$\text{CPR} = \frac{kw(t)}{\rho A t}$$

where
t = exposure time in hours
$w(t)$ = weight loss due to corrosion after t hr exposure, in mg
ρ = density of the material, in g/cm^3
A = exposed surface area, in cm^2
k = 87.6, a constant that converts CPR to mm/year

Through laboratory testing, material specimens are subject to normal environmental conditions leading to corrosion. After some time t_0, the weight loss $w(t_0)$ is measured and the CPR is computed using the above formula. If l_f is the maximum allowable loss in mm, after which the material is no longer structurally sound, then the time to failure is projected to be

$$t_f = l_f/\text{CPR}$$

Each specimen may result in somewhat different CPRs, thereby generating a sample of projected failure times.

EXAMPLE 13.16. When an acceleration factor is available, degradation modeling can be performed at high stress levels as well as at normal levels. For example, consider the potency of a particular drug that degrades continuously over time. This degradation can be represented mathematically by

$$p = e^{-rt} \tag{13.26}$$

where p = potency of the drug
r = rate of chemical reaction
t = drug exposure time

Then

$$t = -\ln \frac{p}{r}$$

If the rate of the chemical reaction depends on the temperature at which the drug is stored, then the Arrhenius model may be used to introduce temperature as a stress factor. With $r = Ae^{-B/T}$, then

$$t = \frac{-\ln p}{Ae^{-B/T}} \tag{13.27}$$

By specifying a critical potency level p_f, then the "typical" time to failure can be determined from the foregoing relationships. The constants A and B can be determined experimentally at high temperatures, and this model will allow prediction of the degradation rate and time to failure at normal storage temperatures.

Cumulative damage models

If component damage that will lead to failure accumulates continuously, and if the damage rate depends only on the amount of damage and not on any past history, then the following generalization of Miner's rule may be used:[6]

$$\sum_{i=1}^{n} \frac{t_i}{L_i} = 1 \tag{13.28}$$

where t_i = the amount of time at stress level i
L_i = the expected lifetime at stress level i

To apply this model, consider two stress levels—one normal (t_1) and the other high (t_2). Then

$$\frac{t_1}{L_1} + \frac{t_2}{L_2} = 1 \quad \text{or} \quad t_2 = L_2 - \frac{L_2}{L_1}t_1 \tag{13.29}$$

The line represented by Eq. (13.29) and shown in Fig. 13.7 is called the failure line, since any combination of stress times (t_1, t_2) that lie on the line will result in a failure.

To determine the value for L_2, test the component at the high stress level until failure (L_2). Then, to determine a second point on the line, test the component first for some time t_1 at the normal stress level and then at the high level until failure occurs at time t_2. Then L_1, the time to failure under normal stress, is found from

$$L_1 = \frac{t_1}{1 - (t_2/L_2)} \tag{13.30}$$

[6]Miner's rule has the form $\sum(n_i/N_i) = 1$ where n_i is the number of cycles at stress level i and N_i is the number of cycles to failure at the same stress level, determined from the S-N fatigue curve discussed in Chapter 8.

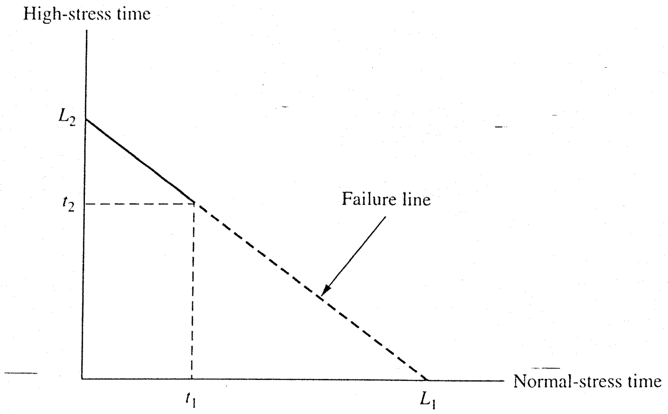


FIGURE 13.7
The failure line in a cumulative damage model.

Step stress models

In a step stress accelerated life test, testing begins with normal stress. After a period of time, the stress is increased. Such stepwise increases are then continued until all the test units fail. The primary assumption in developing the step stress model is that the increase in stress is equivalent to a linear change in the time scale. These models are more complex than the constant-stress models. Nelson [1990] discusses several step stress models and the resulting data analysis and provides an in-depth treatment of accelerated life testing.

13.7 Experimental Design

Experimental design is concerned with the efficient collection and analysis of data in ways that will maximize the information obtained. It consists of the identification of the *factors* and their values (referred to as *levels*) that are to be investigated with respect to their effect on a response or dependent variable. A particular experimental design is selected that consists of a statistical model for the collection and analysis of the data. A given design will identify the factors, their levels, the number of *replications* (repeat experiments) at the specified levels, randomization of the experimental units, and the use of *blocking*. Blocking reduces variation in an experiment by comparing homogeneous units. The objective of the experiment may be to identify critical factors, to estimate the effect selected factors have on the response variable, or both.

It is not the intent here to cover the entire field of experimental design. Our objective is merely to illustrate the use of experimental design techniques in reliability engineering. The student is encouraged to take a course in the design of experiments, because it has many practical applications in engineering beyond those discussed here. Hicks [1993] or Montgomery [1991] are excellent texts for those desiring more information on the design of experiments.

The discussion here will be limited to the use of factorial designs in identifying factors that significantly affect a reliability or maintainability parameter. For example, we may be interested in conducting a screening experiment in order to determine which factors are affecting component failures. A factorial experiment consists of the collection of data at all combinations of the levels of the factors being investigated and thereby allows the simultaneous evaluation of the factors. Therefore, if k factors are being considered, each at m different levels, then a single replication will consist of m^k experiments. Obviously, if k or m is large, then a prohibitively large number of experiments may be required. To overcome this difficulty, the number of levels and factors must be kept small. Alternatively, *fractional factorial* designs, which use a subset of the full factorial experiments, may be used. However, as a result, some information is lost, and certain effects are confounded (or indistinguishable from one another). We will address only the full factorial design in its use as a screening technique for determining which factors significantly affect failures or repair times. An advantage of factorial designs is the ability to measure the effect the interaction two or more factors have on the response variable.

The mathematical model for a two-factor factorial experiment is

$$Y_{ijk} = \mu_{...} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

where
$\mu_{...}$ = overall mean effect
α_i = the (main) effect of factor A at level i
β_j = the (main) effect of factor B at level j
$(\alpha\beta)_{ij}$ = the interaction effect with factor A at level i and factor B at level j
ε_{ijk} = random error of the kth replication with factor A at level i and factor B at level j
Y_{ijk} = the value of the response variable at the kth replication with factor A at level i and factor B at level j

The factor effects are assumed to be deviations from the overall mean; therefore

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_{ij} (\alpha\beta)_{ij} = 0$$

The statistical hypotheses of interest are

$$\begin{aligned}
H_0&: \alpha_i = 0 \quad \text{for all } i\\
H_0&: \beta_j = 0 \quad \text{for all } j\\
H_0&: (\alpha\beta)_{ij} = 0 \quad \text{for all } i, j\\
H_1&: \alpha_i \neq 0 \quad \text{for at least one } i\\
H_1&: \beta_j \neq 0 \quad \text{for at least one } j\\
H_1&: (\alpha\beta)_{ij} \neq 0 \quad \text{for at least one } i, j
\end{aligned}$$

To test these hypotheses an analysis of variance (ANOVA) is performed. ANOVA consists of computing independent estimates of the population variance (referred to as factor mean squares) from the data. If a factor is not significant, its variance estimate should not differ significantly from a pure population mean square (the mean square for error). A significant factor would have a larger mean square than the mean square for error. The ratio of the factor mean square over the mean square for

TABLE 13.3
Two-factor ANOVA for the fixed-effects model

Source of variation	Sum of squares	Degrees of freedom	Mean square	F statistic
Factor A	SS_A	$a-1$	$MS_A = SS_A/(a-1)$	MS_A/MSE
Factor B	SS_B	$b-1$	$MS_B = SS_B/(b-1)$	MS_B/MSE
AB Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = SS_{AB}/[(a-1)(b-1)]$	MS_{AB}/MSE
Error	SS_E	$ab(n-1)$	$MSE = SS_E/[ab(n-1)]$	
Total	SS_T	$abn-1$		

error forms an F distribution. The larger the computed F statistic, the more likely the factor is significant. A comparison with a tabulated F distribution will establish the critical value at a given level of significance. Table 13.3 summarizes the results of the analysis when the factor levels are determined by the experimenter (a fixed-effects model) rather than being randomly selected from a parent population (a random-effects model). For the fixed-effects model, conclusions are valid only for the factor levels considered. In Table 13.3,

$$\begin{aligned}
a &= \text{the number of levels of factor A}\\
b &= \text{the number of levels of factor B}\\
n &= \text{the number of replications}
\end{aligned}$$

and

$$SS_A = \sum_{i=1}^{a} \frac{Y_{i..}^2}{bn} - \frac{Y_{...}^2}{abn}$$

$$SS_B = \sum_{j=1}^{b} \frac{Y_{.j.}^2}{an} - \frac{Y_{...}^2}{abn}$$

$$SS_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{Y_{ij.}^2}{n} - \frac{Y_{...}^2}{abn} - SS_A - SS_B$$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} Y_{ijk}^2 - \frac{Y_{...}^2}{abn}$$

and

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B$$

where the notations

$$Y_{i..} = \sum_j \sum_k Y_{ijk}$$

$$Y_{.j.} = \sum_i \sum_k Y_{ijk}$$

$$Y_{ij.} = \sum_k Y_{ijk} \qquad Y_{...} = \sum_i \sum_j \sum_k Y_{ijk}$$

EXAMPLE 13.17.[7] An aircraft manufacturer is concerned with the large number of failures of the auxiliary power unit (APU) aboard a particular model of its aircraft. The APU is a gas turbine engine mounted internally in the lower rear of the fuselage. It provides the aircraft with a source of power, independent of the main engines, for ground operations, main engine starting, and in-flight emergencies. Its reliability is measured by the number of unscheduled removals from the aircraft. The manufacturer is interested in establishing whether there are significant differences in the removal rate that depend on carrier type (factor A) and fleet size (factor B). Carrier type was defined to be either domestic or foreign, and fleet size was categorized as small, medium, and large. The company's maintenance data collection system provided the following information over a three-year period. Each year's worth of data constitutes a single replication. The response variable is the number of removals per 100 flying hours.

	Factor B (fleet size)		
Factor A (type)	**Small (1–10)**	**Medium (11–22)**	**Large (over 22)**
Domestic	0.82/1.267/0.9	0.80/0.56/0.7867	0.74/0.74/0.76
Foreign	0.7865/0.57/0.74	0.545/0.41/0.63	0.63/0.54/0.58

Therefore, with $a = 2$, $b = 3$, and $n = 3$,

$$SS_A = \frac{83.8727}{9} - \frac{163.9731}{18} = 0.20957$$

$$SS_B = \frac{55.6877}{6} - \frac{163.9731}{18} = 0.17167$$

$$SS_T = 9.711 - \frac{163.9731}{18} = 0.60138$$

$$SS_{AB} = \frac{28.5181}{3} - \frac{163.9731}{18} - 0.20957 - 0.17167 = 0.01519$$

$$SS_E = 0.60138 - 0.20957 - 0.17167 - 0.01519 = 0.20495$$

Source of variation	Sum of squares	Degrees of freedom	Mean square	*F* statistic
Operator	0.20957	1	0.20957	12.2699
Fleet Size	0.17167	2	0.08583	5.0252
Interaction	0.01519	2	0.00759	0.44438
Error	0.20495	12	0.01708	
Total	0.60138	17		

At the 5 percent level of significance, critical F table values are $F_{1,12,.05} = 4.75$ and $F_{2,12,.05} = 3.89$. Therefore both carrier type and fleet size are significant, but the interaction between carrier type and fleet size is not significant. From a practical point of view, this means that the removal (failure) rate differs among operators and among carrier fleet sizes. Further investigation yields an estimate for each factor level. The formulae are

$$\hat{\alpha}_i = \frac{Y_{i\cdot\cdot}}{bn} - \frac{Y_{\cdot\cdot\cdot}}{abn}$$

$$\hat{\beta}_j = \frac{Y_{\cdot j\cdot}}{an} - \frac{Y_{\cdot\cdot\cdot}}{abn}$$

$$(\widehat{\alpha\beta})_{ij} = \frac{Y_{ij\cdot}}{n} - \frac{Y_{i\cdot\cdot}}{bn} - \frac{Y_{\cdot j\cdot}}{an} + \frac{Y_{\cdot\cdot\cdot}}{abn}$$

Therefore, $Y_{\cdot\cdot\cdot}/18 = 12.8052/18 = 0.7114$, and

$$\hat{\alpha}_1 = \frac{7.3737}{9} - 0.7114 = 0.1079$$

$$\hat{\alpha}_2 = \frac{5.4315}{9} - 0.7114 = -0.1079$$

$$\hat{\beta}_1 = \frac{5.0835}{6} - 0.7114 = 0.13585$$

$$\hat{\beta}_2 = \frac{3.7317}{6} - 0.7114 = -0.08945$$

$$\hat{\beta}_3 = \frac{3.99}{6} - 0.7114 = -0.0464$$

The interactions were not significantly different, so we will not estimate their effect. From the foregoing analysis, it can be concluded that domestic carriers have a significantly greater removal (failure) rate than foreign carriers and that small carriers have a significantly greater removal (failure) rate than median or large carriers. Individual comparisons among factor levels can be made more precise through the use of multiple comparison tests that will identify where the statistical significance will be found among the possible level comparisons. If the interaction effect had been significant, then the removal rate would depend on the carrier type and the fleet size working together. In other words, the effect of the fleet size on the removal rate would differ depending on whether the carrier is domestic or foreign. For example, the removal rate may increase as fleet size decreases for domestic carriers but remain relatively constant for foreign carriers. In this case, of course, that effect was not observed. Further investigation would be necessary to determine the reason for the higher failure rates with the domestic carriers and with the smaller fleet sizes.

[7]James Wafzig also contributed to this problem.

13.8 COMPETING FAILURE MODES

When it is important to distinguish among failure modes during reliability testing, then the test is described as involving *competing failure modes.* If the failure modes are mutually independent, they can be separately analyzed by treating them as multiply censored data. The failure times of all failure modes except the failure mode under investigation would be considered censored times. Then the empirical techniques discussed in the previous chapter for multiply censored data would be applied, or the techniques discussed in Chapter 15 would be used to determine an acceptable theoretical reliability model.

APPENDIX 13A
DERIVATION OF EXPECTED TEST TIME

Assume that n CFR units are placed on test. Then $R(t) = e^{-\lambda t}$ for each unit. Let

$$Y_i = t_i - t_{i-1}$$

where t_i is the time of the ith failure. Then

$$t_r = \sum_{i=1}^{r} Y_i$$

is the time of the rth failure, and

$$E(t_r) = \sum_{i=1}^{r} E(Y_i)$$

is the expected time of the rth failure.

According to Chapter 3 (Eq. (3.9)), when there are n identical units operating in a system,

$$\text{MTTF} = \frac{1}{n\lambda}$$

Therefore

$$E(Y_1) = \frac{1}{n\lambda}$$

After the first failure there are $(n-1)$ units operating, and

$$\text{MTTF} = E(Y_2) = \frac{1}{(n-1)\lambda}$$

The derivation continues for the first r failures, where with $r-1$ failures

$$E(Y_r) = \frac{1}{[n-(r-1)]\lambda}$$

Therefore

$$E(t_r) = \frac{1}{\lambda}\left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}\right]$$

If failed units are replaced immediately, then

$$E(Y_i) = \frac{1}{n\lambda}$$

for all i, and

$$E(t_r) = \frac{1}{\lambda}\frac{r}{n}$$

Observing that

$$\text{MTTF} = \frac{1}{\lambda}.$$

we have Eqs. (13.2) and (13.3).

APPENDIX 13B
EXPECTED TEST TIME (TYPE II TESTING)

Length of Test = MTTF × $\text{TTF}_{r,n}$

n	r	$\text{TTF}_{r,n}$
1	1	1.000
2	1	0.500
2	2	1.500
3	1	0.333
3	2	0.833
3	3	1.833
4	1	0.250
4	2	0.583
4	3	1.083
4	4	2.083
5	1	0.200
5	2	0.450
5	3	0.783
5	4	1.283
5	5	2.283
6	1	0.167
6	2	0.367
6	3	0.617
6	4	0.950
6	5	1.450
6	6	2.450
7	1	0.143
7	2	0.310
7	3	0.510
7	4	0.760
7	5	1.093
7	6	1.593
7	7	2.593
8	1	0.125
8	2	0.268
8	3	0.435
8	4	0.635
8	5	0.885
8	6	1.218
8	7	1.718
8	8	2.718
9	1	0.111
9	2	0.236
9	3	0.379
9	4	0.546
9	5	0.746
9	6	0.996
9	7	1.329
9	8	1.829
9	9	2.829
10	1	0.100
10	2	0.211
10	3	0.336
10	4	0.479
10	5	0.646
10	6	0.846
10	7	1.096
10	8	1.429
10	9	1.929
10	10	2.929
11	1	0.091
11	2	0.191
11	3	0.302
11	4	0.427
11	5	0.570
11	6	0.737
11	7	0.937
11	8	1.187
11	9	1.520
11	10	2.020
11	11	3.020
12	1	0.083
12	2	0.174
12	3	0.274
12	4	0.385
12	5	0.510
12	6	0.653
12	7	0.820
12	8	1.020
12	9	1.270
12	10	1.603
12	11	2.103
12	12	3.103
13	1	0.077
13	2	0.160
13	3	0.251
13	4	0.351
13	5	0.462
13	6	0.587
13	7	0.730
13	8	0.897
13	9	1.097
13	10	1.347
13	11	1.680
13	12	2.180
13	13	3.180
14	1	0.071
14	2	0.148
14	3	0.232
14	4	0.323
14	5	0.423
14	6	0.534
14	7	0.659
14	8	0.802
14	9	0.968
14	10	1.168
14	11	1.418
14	12	1.752
14	13	2.252
14	14	3.252
15	1	0.067
15	2	0.138
15	3	0.215
15	4	0.298
15	5	0.389
15	6	0.489
15	7	0.600
15	8	0.725
15	9	0.868
15	10	1.035
15	11	1.235
15	12	1.485
15	13	1.818
15	14	2.318
15	15	3.318
16	1	0.063
16	2	0.129
16	3	0.201
16	4	0.278
16	5	0.361
16	6	0.452
16	7	0.552
16	8	0.663
16	9	0.788
16	10	0.931
16	11	1.097
16	12	1.297
16	13	1.547
16	14	1.881
16	15	2.381
16	16	3.381
17	1	0.059
17	2	0.121
17	3	0.188
17	4	0.259
17	5	0.336
17	6	0.420
17	7	0.511
17	8	0.611
17	9	0.722
17	10	0.847
17	11	0.990
17	12	1.156
17	13	1.356
17	14	1.606
17	15	1.940
17	16	2.440
17	17	3.440
18	1	0.056
18	2	0.114
18	3	0.177
18	4	0.244
18	5	0.315
18	6	0.392
18	7	0.475
18	8	0.566
18	9	0.666
18	10	0.777
18	11	0.902
18	12	1.045
18	13	1.212
18	14	1.412
18	15	1.662
18	16	1.995
18	17	2.495
18	18	3.495
19	1	0.053
19	2	0.108
19	3	0.167
19	4	0.230
19	5	0.296
19	6	0.368
19	7	0.445
19	8	0.528
19	9	0.619
19	10	0.719
19	11	0.830
19	12	0.955
19	13	1.098
19	14	1.264
19	15	1.464
19	16	1.714
19	17	2.048
19	18	2.548
19	19	3.548
20	1	0.050
20	2	0.103
20	3	0.158
20	4	0.217
20	5	0.280
20	6	0.346
20	7	0.418
20	8	0.495
20	9	0.578
20	10	0.669
20	11	0.769
20	12	0.880
20	13	1.005
20	14	1.148
20	15	1.314
20	16	1.514
20	17	1.764
20	18	2.098
20	19	2.598
20	20	3.598

EXERCISES

13.1 On the basis of an estimated MTTF of 1800 hr, find the expected test time required to generate 8 failures (Type II testing) if 15 units are placed on test. Assume CFR. If the testing were to continue for 500 hours (Type I testing) with 15 units on test, how many failures would be expected?

13.2 Wil I. Fail, a reliability engineer for Major Motors, has the task of testing 20 alternators of a new design in order to estimate their reliability. He terminated the test after 10 failures with the following failure times (in operating hours):

Alternator:	2	3	6	7	10	12	13	16	17	19
Failure time:	251	365	286	752	465	134	832	543	912	220

(a) Assuming a CFR model, estimate the MTTF.
(b) On the basis of (a), what is the expected test time if Wil conducts a second test with 25 items placed on test and stops after observing 50 failures? He will immediately replace failed units on test.
(c) What is the expected number of failures in the first 700 hours of testing?

13.3 In order to measure the reliability of a high-failure item, 50 units were placed on test. The following failure and censor times (in hours) were recorded: 3, 10, 12, 17, 22, 28^+, 30, 32, 32, 45, 53, 59^+, 71, 77, 79, 90, 91, 101, 129, 131. The test was terminated by management after 150 hours. Assume a CFR model.
(a) Estimate the MTTF from the test data.
(b) Based on the estimated MTTF, estimate the number of units to be placed on test if management desires to generate 5 additional failures with 200 hr of additional test time.
(c) If the test in (b) is to be terminated after 100 hr, what is the expected number of failures generated without replacement of failed units? With replacement of failed units?

13.4 Determine the burn-in test time for a new product. The product after reliability growth testing has a Weibull failure distribution with $\beta = 0.3$ and $\theta = 3{,}750{,}000$ hr. Contract specifications require a 0.95 reliability at 1000 operating hours.

13.5 For the following reliability function, determine the mean residual life after a burn-in period of T_0. Compare results for several values of T_0 with the MTTF without a burn-in period.

$$R(t) = \frac{100}{(t + 10)^2} \qquad t \geq 0$$

13.6 Develop a sequential test for the CFR model to test the null hypothesis that the MTTF = 100 hr versus the alternate hypothesis that the MTTF = 50 hr. Set $\alpha = 0.1$ and $\beta = 0.15$. What is the minimum number of failures necessary to reject the null hypothesis, and what is the minimum time on test before the null hypothesis may be accepted?

13.7 Testing of an electric starter switch was accelerated from a normal rate of 5 cycles per hour to an accelerated rate of 1 cycle every 20 seconds. At the accelerated rate, failure times were Weibull with an estimated shape parameter of 1.8 and an estimated characteristic life of 5000 hours. What is the reliability of the switch under normal use over a 1-year period (24 hours a day)?

13.8 Referring to Problem 13.7, if 20 switches are to be tested for 12 hr at an accelerated level of one cycle every 15 seconds, how many are expected to fail at the conclusion of the test period?

13.9 Show that the lognormal distribution is preserved under the assumption of a linear acceleration factor with the shape parameter unchanged. Determine the effect on the median time to failure.

13.10 A CFR item is tested at two elevated temperatures. At 341 K the MTTF is estimated to be 250 hr; at 415 K the MTTF is estimated to be 143 hr. If the normal operating temperature is 200 K, what is the reliability of the item over 500 operating hours?

13.11 An electronic component underwent accelerated life testing and the following Eyring model was empirically derived from high stress–generated data:

$$R = 153T^{0.9}e^{-283/T}e^{0.015V}$$

where T is the operating temperature in degrees C and V is the applied voltage. At a high stress level of 85° C and 200 volts, a Weibull distribution was observed with $\theta = 87$ hr and $\beta = 2.3$. The normal operating environment is 35° C at 120 volts. Determine the design life of the component if a 0.99 reliability is required.

13.12 A new product is tested at two elevated temperatures: 450 K and 500 K. A Weibull distribution was found with $\beta = 1.18$ and a characteristic life of 1450 hr and 1280 hr at the two temperatures respectively. Based upon the Arrhenius model, what will be the product reliability at 500 hours if normal usage is at 35 °C?

13.13 Derive the sequential test for the following hypotheses when sampling from an exponential distribution. Define the continuation region in terms of the cumulative failure times. Assume complete data.

$$H_0: \text{ MTTF} = \mu_0$$
$$H_1: \text{ MTTF} = \mu_1 < \mu_0$$

13.14 (a) Derive the sequential test to perform a reliability demonstration based upon the following hypotheses:

$$H_0: \; R(1000) = 0.95$$
$$H_1: \; R(1000) = 0.90$$

The probabilities of a Type I and Type II error are 0.10 and 0.15, respectively.

(b) Determine the minimum number to be tested in order to reject the null hypothesis and to accept the null hypothesis.
(c) If after 70 units were tested there were 6 failures, what is the decision? What if there are 9 failures after 80 units have been tested?

13.15 Determine the least-cost hours of burn-in for a unit having a Weibull distribution with a shape parameter of 0.53 and a characteristic life of 476 hours. The cost of conducting the burn-in is $30/hr, and each failure costs $175. It is estimated that operational failures will cost $8,300 each. The operational life is 40,000 hours.

13.16 Under accelerated life testing, a component has a Weibull distribution but with the following nonlinear acceleration factor:

$$t_n = (ct_s)^a$$

where c and a are constants to be determined. Determine the proper relationships between β_n and β_s and between θ_n and θ_s.

13.17 Twenty (20) units are placed on test for 200 hr (Type I testing). If the units are believed to have a lognormal distribution with $s = 1.21$ and $t_{med} = 480$ hours, what is the expected number of failures?

13.18 Five specimens of a new corrosion-resistant material are tested for 240 hours in a highly corrosive environment. The density of the material is 7.6 g/cm^3, and the exposed surface area of each specimen is 4.3 cm^2. At the end of the test period, the measured weight losses in mg were 11.1, 10.4, 12.1, 11.4, and 9.8. If a degradation of 1 mm or more results in a structural failure, predict the failure times for the five specimens.

13.19 A cumulative damage model is applied to the failure of ball bearings under both a high-stress and a normal (specification) radial load. At the high load, a failure was observed at 45.3 hours. A second bearing had been tested at the normal load level for 67 hours and at a high load level for 40 hours when it failed. Predict the failure of the bearing under normal operating conditions.

13.20 A maintainability goal of 90 percent restoration on all automotive transmission failures within 8 hours has been established for a repair shop. If 80 percent is unacceptable, determine the accept and reject region for a maintainability demonstration using the sequential binomial test. Set the probability of both a Type I and Type II error to 10 percent. If after observing 30 repairs, 27 were completed within 8 hours, what is the decision? If after 60 repairs, 55 were completed within 8 hours?

13.21 Find a binomial acceptance testing plan to demonstrate a reliability of 0.98. An unacceptable reliability is 0.90. The risk of incorrectly accepting or incorrectly rejecting should be less than 10 percent. What is the minimum sampling size for which both

risks are less than 5 percent? *Hint:* Binomial probabilities can be computed recursively using

$$\Pr\{X = i + 1\} = \frac{1 - R}{R}\frac{n - i}{i + 1}\Pr\{X = i\}$$

where X is the number of failures, $1 - R$ is the probability of a failure and n is the number on test. Numerical problems encountered with large factorials can therefore be avoided. You are encouraged to prove the foregoing relationship before using it.

CHAPTER 14

Reliability Growth Testing

14.1 RELIABILITY GROWTH PROCESS

The objective of reliability growth testing is to improve reliability over time through changes in product design and in manufacturing processes and procedures. This is accomplished through the test–fix–test–fix cycle illustrated in Fig. 14.1. Reliability tests and assessments are conducted on prototypes to determine whether reliability goals are being met. If not, a failure analysis will determine the high-failure modes and the corresponding fixes. The failure modes are eliminated (or their effects are reduced) through engineering redesign, and the cycle is repeated. The failure data generated from the test program are summarized in the form of a growth curve. These growth curves are used to monitor the progress of the development program and to predict the time required to achieve a desired reliability target. A formal failure mode, effect, and criticality analysis (FMECA) will support the collection and analysis of the reliability data by identifying and categorizing failure modes. Actions taken during growth testing include the correction of design weaknesses and manufacturing flaws and the elimination of inferior parts or components. Candidates for redundancy may also be identified at this time.

Reliability growth testing is often a required task under government contracts. However, even if not required, reliability growth testing will identify product deficiencies and areas of improvement that would otherwise be overlooked until the final reliability demonstration was performed or until the product was fielded. Reliability growth models provide a means of assessing current reliability parameters, measuring progress toward stated goals, and estimating the time required to reach these goals.

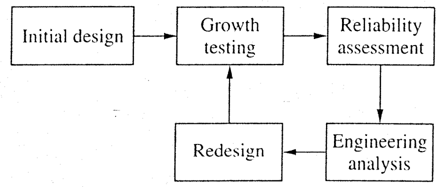


FIGURE 14.1
The reliability growth cycle.

14.2 IDEALIZED GROWTH CURVE

Reliability growth is achieved through a continuous test, evaluation, and redesign activity. A realistic reliability growth curve should be developed at the start of the test program; it will identify the reliability goals and provide a target for evaluating progress toward the goals. The continuous growth curve in Fig. 14.2 represents the idealized growth curve. In an idealized curve, reliability growth, as measured by the MTTF, increases monotonically as a function of the test time. Presumably, the more testing is performed, the greater the reliability improvement will be. In reality, growth occurs during the fix phase of the cycle and is only measured during the test phase. However, when reliability is plotted versus test time data, strong functional relationships are suggested; as a result, test time is the basis for constructing many of the growth models. Increased testing generates additional failure modes, thereby providing new information for improving the design.

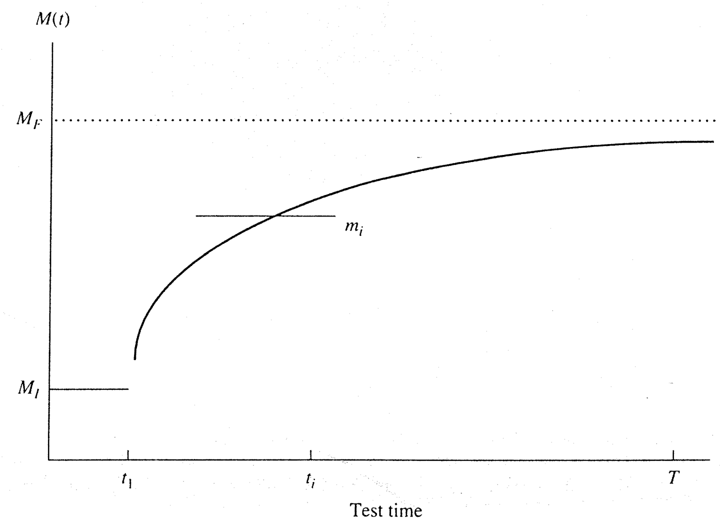


FIGURE 14.2
An idealized reliability growth curve.

Military Handbook: Reliability Growth Management [1981] defines the idealized growth curve in the following manner:

$$M(t) = \begin{cases} M_I & 0 < t \le t_1 \\ \dfrac{M_I}{1-\alpha}\left(\dfrac{t}{t_1}\right)^{\alpha} & t > t_1 \end{cases} \tag{14.1}$$

where $M(t)$ = instantaneous MTTF at time t
t = cumulative test time
M_I = average MTTF over the initial test cycle
t_1 = length of initial test cycle in cumulative test time
α = growth parameter

Equation (14.1) is based on a learning curve effect, where the plot of $M(t)$ versus t is linear on a log-log scale with a slope of α. During any test cycle, the average MTTF, m_i, is computed from

$$m_i = \frac{t_i - t_{i-1}}{n(t_i) - n(t_{i-1})} \tag{14.2}$$

where t_i is the cumulative test time at the end of i test cycles, and $n(t_i)$ is the cumulative number of failures after i test cycles. It is assumed that the failure rate, $\lambda_i = 1/m_i$, is constant over the ith cycle. An approximate value for the growth parameter is given by

$$\alpha = -\ln\left(\frac{T}{t_1}\right) - 1 + \left\{\left[1 + \ln\left(\frac{T}{t_1}\right)\right]^2 + 2\ln\left(\frac{M_F}{M_I}\right)\right\}^{0.5} \tag{14.3}$$

where M_F is the final (goal) MTTF at the end of the growth program having a cumulative test time of T. To find an expression for $n(t)$, consider that

$$n(t) - n(t_1) = \lambda(t - t_1),$$

where λ is the average failure rate over the interval (t_1, t); or

$$\lambda = \frac{1}{t - t_1}\int_{t_1}^{t} \frac{1}{M(t')}dt' = \frac{1}{t-t_1}\int_{t_1}^{t} \frac{1-\alpha}{M_I}\left(\frac{t'}{t_1}\right)^{-\alpha} dt'$$

Integrating yields

$$\lambda = \frac{t_1}{M_I(t - t_1)}\left[\left(\frac{t}{t_1}\right)^{1-\alpha} - 1\right]$$

or

$$n(t) = \lambda(t - t_1) + n(t_1) = \frac{t_1}{M_I}\left(\frac{t}{t_1}\right)^{1-\alpha} = \lambda_1 t_1\left(\frac{t}{t_1}\right)^{1-\alpha} \tag{14.4}$$

using $n(t_1) = t_1/M_I$.

EXAMPLE 14.1. An initial 100 hr of reliability testing has resulted in a product MTTF of 50 hr. An MTTF goal of 500 hr has been set, and resources are available for about 4000 cumulative hours of testing. Therefore $T = 4000$, $t_1 = 100$, $M_I = 50$, and $M_F = 500$. From Eq. (14.3), the growth parameter is estimated to be 0.46. Therefore the ideal growth curve is

$$M(t) = \begin{cases} 50 & 0 < t \le 100 \\ \dfrac{50}{1-0.46}\left(\dfrac{t}{100}\right)^{0.46} & t > 100 \end{cases}$$

After an additional 1000 hr of testing, the instantaneous MTTF should be $M(1100) = 279$, and the cumulative number of failures should be

$$n(1100) = 2\left(\frac{1100}{100}\right)^{1-0.46} = 7.3$$

Therefore, the average MTTF over the additional 1000 hr of testing is

$$m = \frac{1100 - 100}{7.3 - 2} = 188.6$$

After 2100 hr of testing, the target MTTF is

$$M(2100) = 375.6$$

with $n(2100) = 10.4$.

14.3 DUANE GROWTH MODEL

The earliest developed and most frequently used reliability growth model was first proposed by Duane [1964], who observed that a plot of the logarithm of the cumulative number of failures per test time versus the logarithm of test time during growth testing was approximately linear (Fig. 14.3). This observation can be expressed mathematically and then extrapolated to predict the growth in MTTF while the test–fix–test–fix cycle continues. This model assumes the underlying failure process is exponential (constant failure rate).

Let

T = total test time accumulated on all prototypes
$n(T)$ = accumulated failures through time T

Then $n(T)/T$ is the cumulative failure rate, and $T/n(T)$ is the cumulative MTTF. If the graph in Fig. 14.3 is linear, then we can write

$$\ln\left[\frac{T}{n(T)}\right] = a + b\ln T \tag{14.5}$$

and

$$\text{MTTF}_c = \frac{T}{n(T)} = e^{a+b\ln T} = e^a T^b = kT^b \tag{14.6}$$

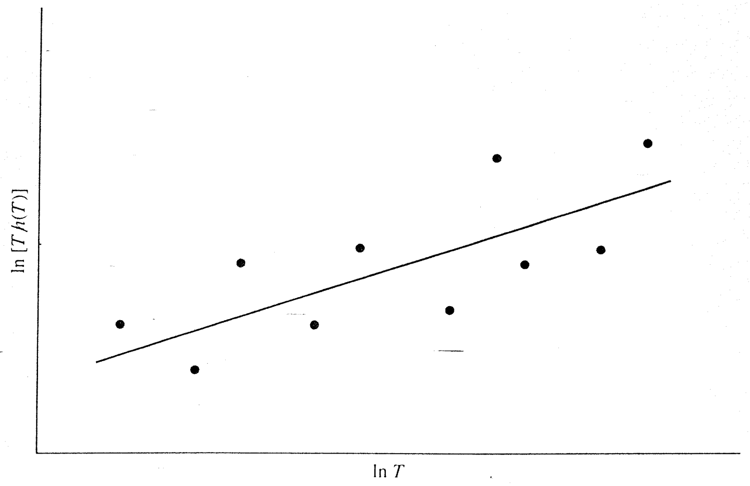


FIGURE 14.3
The Duane growth curve.

is the cumulative mean time to failure. Observe that b is the rate of growth, or the slope of the fitted straight line, and a is the vertical intercept. Typical growth rates for b range from 0.3 to 0.6. Since, from Eq. (14.6),

$$n(T) = \left(\frac{1}{k}\right) \times T^{1-b} \tag{14.7}$$

and $n(T)$ is the accumulated failures through time T,

$$\frac{dn(T)}{dT} = \lambda(T) = \frac{(1-b)}{k} T^{-b} \tag{14.8}$$

is the instantaneous failure rate. Assuming a constant failure rate, if growth testing were to stop at time T, the reciprocal would be the instantaneous MTTF, or

$$\text{MTTF}_i = k\frac{T^b}{1-b} = \frac{\text{MTTF}_c}{1-b} \tag{14.9}$$

To use this model, it is necessary to estimate the parameters a and b. This can be done by plotting $T/n(T)$ versus T on log-log graph paper or plotting ($\ln T$, $\ln[T/n(T)]$) directly. A more accurate method is to fit a straight line to the points ($\ln T$, $\ln[T/n(T)]$) using the method of least squares. The least-squares equations for estimating a and b are

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i - \overline{x}\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} \tag{14.10}$$

$$\hat{a} = \overline{y} - b\overline{x} \tag{14.11}$$

where $\overline{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

$$\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$x_i = \ln(t_i)$$

$$y_i = \ln\left[\frac{t_i}{n(t_i)}\right]$$

t_i = cumulative test time associated with $n(t_i)$ failures

From the least-squares estimates of a, b, $\hat{a}$ and $\hat{b}$,

$$k = e^{\hat{a}}$$

$$\text{MTTF}_i = \frac{kT^{\hat{b}}}{1-\hat{b}} \tag{14.12}$$

Given an MTTF_i goal, say M_f, then by solving Eq. (14.12) for T,

$$T_m = \left[\frac{(1-\hat{b})M_f}{k}\right]^{1/\hat{b}} \tag{14.13}$$

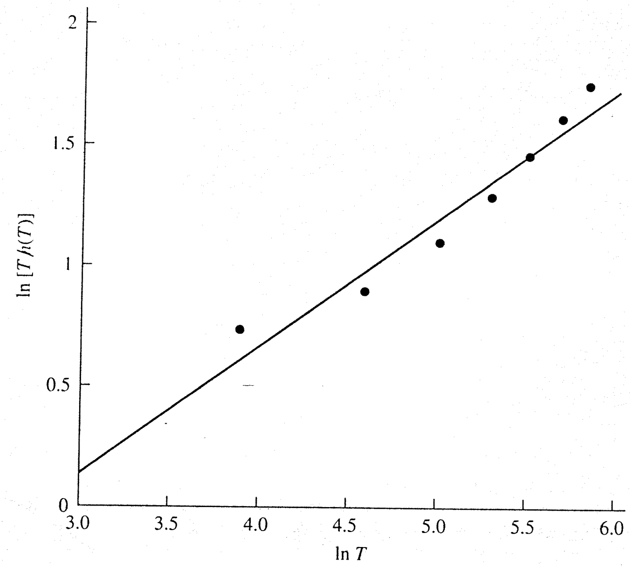


FIGURE 14.4
A product growth curve using the Duane model.

an estimate for the required time to complete the reliability growth testing may be obtained. The *coefficient of determination*, r^2, can be computed as

$$r^2 = \frac{\sum_{i=1}^{n}(y_i - \hat{a} - \hat{b}x_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \tag{14.14}$$

The coefficient of determination measures the strength of the fit of the regression curve and can be interpreted as the proportion of the variation in the y's explained by the x variables. It will have a value between 0 and 1; a value of 1 is a perfect fit. The square root, r, is called the *index of fit*. If both y and x are random variables, the index of fit would have the same value as the correlation between the two variables.

EXAMPLE 14.2. A new product while in the development stage undergoes reliability growth testing in which each test–fix cycle consists of 50 hr of testing. The following numbers of failures per cycle were observed in the following order: 24, 17, 9, 5, 3, 2, 1. Estimate the current MTTF and the additional test time required to obtain an MTTF goal of 20 hr.

Solution. Complete the following table:

T	$n(T)$	$T/n(T)$	$x_i = \ln T$	$y_i = \ln[T/n(T)]$	$x_i y_i$	x_i^2
50.0	24.0	2.0833	3.9120	0.7340	2.8713	15.3039
100.0	41.0	2.4390	4.6052	0.8916	4.1060	21.2076
150.0	50.0	3.0000	5.0106	1.0986	5.5047	25.1065
200.0	55.0	3.6364	5.2983	1.2910	6.8400	28.0722
250.0	58.0	4.3103	5.5215	1.4610	8.0670	30.4865
300.0	60.0	5.0000	5.7038	1.6094	9.1799	32.5331
350.0	61.0	5.7377	5.8579	1.7471	10.2342	34.3154
Total			35.9093	8.8327	46.8030	187.0252

(See Fig. 14.4.) Then

$$\bar{x} = 5.1299 \qquad \text{and} \qquad \bar{y} = 1.261811.$$

Therefore,

$$\hat{b} = \frac{46.803 - 5.1299(8.8327)}{187.0252 - 7(5.1299)^2} = 0.53$$

and

$$\hat{a} = 1.261811 - 0.53(5.1299) = -1.457 \qquad \text{and} \qquad k = e^{-1.457} = 0.233$$

At the end of the last test cycle, 350 hours, the cumulative MTTF is given by

$$\text{MTTF}_c = 0.233(350)^{0.53} = 5.196 \qquad \text{and} \qquad \text{MTTF}_i = 5.196/(1 - 0.53) = 11.0$$

The index of fit was computed to be 0.97, indicating that the estimated model is a good fit.

If an MTTF goal of 20 hr is specified, then

$$T_{20} = \left[\frac{(1 - 0.53)20}{0.233}\right]^{1/0.53} = 1071 \text{ hr of test time}$$

or $1071 - 350 = 721$ additional hours.

14.4 AMSAA MODEL

The U.S. Army Material Systems Analysis Activity (AMSAA) model was developed by Crow [1984]. This model attempts to track reliability within a series of growth testing cycles, referred to as phases. At the conclusion of each design change (cycle), the failure rate decreases. However, during the subsequent testing, the failure rate remains constant, as shown in Fig. 14.5. The staircase behavior of the failure rates is then approximated with a continuous curve of the form at^b. This also leads to a linear relationship between cumulative failure rate and time on a log-log scale. As a result, the AMSAA model has the same mathematical form as the Duane model. However, the AMSAA model is often applied to a single test phase, whereas the Duane model attempts to account for the global change in failure rates and MTTFs over the entire program. In addition, the underlying assumptions of the AMSAA model differ considerably from those of the Duane model, which is primarily empirically based. This can be seen from the mathematical development of the AMSAA model.

We begin by letting $0 < s_1 < s_2 < \cdots < s_k$ denote cumulative test times at which design changes are made. Assuming that the failure rates are constant between design changes, and letting N_i (the number of failures during the ith testing period) be a random variable, then N_i has a Poisson probability distribution with a probability function

$$\Pr\{N_i = n\} = \frac{[\lambda_i(s_i - s_{i-1})]^n e^{-\lambda_i(s_i - s_{i-1})}}{n!} \tag{14.15}$$

The mean of this distribution is $\lambda_i(s_i - s_{i-1})$. As a result of the relationship between the Poisson distribution and the exponential distribution, the time to failure during

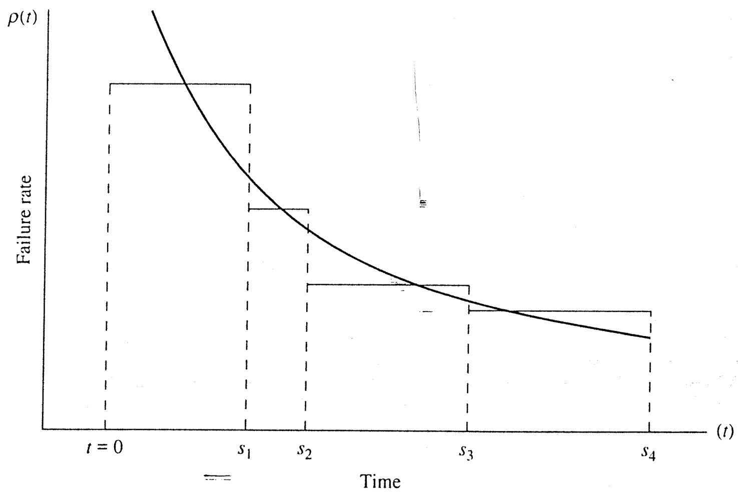


FIGURE 14.5
The AMSAA reliability growth model.

the ith test cycle is exponential with parameter λ_i. If t = the cumulative test time and $n(t)$ = the cumulative number of failures through t hours of testing, then

$$\Pr\{n(t) = n\} = \frac{\lambda(t)^n e^{-\lambda(t)}}{n!} \tag{14.16}$$

where the cumulative failure rate is

$$\lambda(t) = \begin{cases} \lambda_1 t & \text{for } 0 \le t < s_1 \\ \lambda_1 s_1 + \lambda_2(t - s_1) & \text{for } s_1 \le t < s_2 \\ \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3(t - s_2) & \text{for } s_2 \le t < s_3 \end{cases}$$

This failure law is the nonhomogeneous Poisson process discussed in Chapter 9 and having an intensity function

$$\rho(t) = \lambda_i \qquad \text{for } s_{i-1} < t < s_i \tag{14.17}$$

As long as $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ (that is, the failure rates are monotonically decreasing) reliability growth is observed.

For the practical implementation of the model, the intensity function is approximated by the power law process as

$$\rho(t) = abt^{b-1} \qquad t > 0; \qquad a, b > 0 \tag{14.18}$$

Although this is of the same form as a Weibull hazard rate function, the underlying failure process is not Weibull. Integrating the intensity function provides the cumulative expected number of failures, $m(t)$:

$$m(t) = \int_0^t abx^{b-1}\,dx = at^b \tag{14.19}$$

Then with $n(T)$ the observed cumulative number of failures:

$$n(t) = at^b$$

and

$$\ln n(t) = \ln a + b \ln t \tag{14.20}$$

Observe that $b < 1$ is necessary for reliability growth. If no further design changes are made after time t_0, then future failure times are assumed to be exponential with an instantaneous MTTF found from

$$\text{MTTF}_i = \left[abt_0^{b-1}\right]^{-1}$$

14.4.1 Parameter Estimation for the Power Law Intensity Function

For the intensity function $\rho(t) = abt^{b-1}$, the parameters a and b may be estimated using a least-squares curve fitted to Eq. (14.20). However, the maximum likelihood estimates (MLEs) are preferred over the least-squares estimates. MLEs will be discussed in more detail in Chapter 15; however, the formulas for computing the MLEs are as follows[1]

Type I data

Given N successive failure times $t_1 < t_2 < \cdots < t_N$ that occur prior to the accumulated test time or observed system time, T,

$$\hat{b} = \frac{n}{n \ln T - \sum_{i=1}^{n} \ln t_i} \tag{14.21}$$

Then

$$\hat{a} = \frac{n}{T^{\hat{b}}} \tag{14.22}$$

$$\hat{\rho}(T) = \hat{a}\hat{b}T^{\hat{b}-1}$$

$$\text{MTTF}_i = \frac{1}{\hat{\rho}(T)} \tag{14.23}$$

Two-sided confidence intervals for the MTTF may be obtained from

$$\frac{L}{\hat{\rho}(T)} \le \text{MTTF} \le \frac{U}{\hat{\rho}(T)} \tag{14.24}$$

where L and U are confidence interval factors obtained from Table A.6 for Type I testing in the Appendix.

Type II data

Given N successive failure times $t_1 < t_2 < \cdots < t_N$ following accumulated test time or observed system time $T = t_N$,

$$\hat{b} = \frac{n}{(n-1)\ln t_n - \sum_{i=1}^{n-1} \ln t_i} \tag{14.25}$$

The parameter $\hat{a}$ would be estimated using Eq. (14.22), and Eq. (14.23) would then be used to estimate the MTTF at the conclusion of the current test cycle. Again, two-sided confidence intervals may be obtained using Eq. (14.24), with L and U found from Table A.6 for Type II testing.

EXAMPLE 14.3. Two prototype engines are tested concurrently with Type I testing for $T = 500$ hr. The first engine accumulates a total of 200 hr, and the second engine accu-

[1] These same formulas can be used to estimate the parameters of a nonhomogeneous Poisson process having an increasing power-law intensity function for a deteriorating system under minimal repair. In this case, the t_i are the failure times where $t_i = t_{i-1} + x_i$ and x_i is the time between failure $i - 1$ and failure i. T is the total time the system was observed.

mulates 300 hr. Times of failures (*) on each engine are identified below:

Engine 1, hr	Engine 2, hr	Cumulative, hr
5.6*	0	5.6
10.2	8.6*	18.8
20.4*	18.1	38.5
41.8*	36.0	77.8
72.3	61.5*	133.8
88.5*	75.0	163.5
120.0*	105.4	225.4
170.7	152.8*	323.5
190.2	181.3*	371.5
200.0	256.6*	456.6
200.0	300.0	500.0

Solution. Following Eqs. (14.21), (14.22), and (14.23),

Failure time	ln(failure time)
5.6	1.722767
18.8	2.933857
38.5	3.650658
77.8	4.354141
133.8	4.896346
163.5	5.096813
225.4	5.417876
323.5	5.779199
371.5	5.917549
456.6	6.123808
Total	45.89302

$$\hat{b} = \frac{10}{10\ln(500) - 45.89302} = 0.615268$$

$$\hat{a} = \frac{10}{500^{0.615268}} = 0.218479$$

and, therefore,

$$\rho(T) = 0.218479 \times 0.615268t^{0.615268-1} = 0.134423t^{-0.384732}.$$

The intensity at the end of the test is

$$\rho(500) = 0.134423(500)^{-0.384732} = 0.012305$$

The MTTF at the end of testing is then

$$\text{MTTF} = \frac{1}{\rho(500)} = 81.265 \text{ hr}$$

A 90 percent confidence interval for the MTTF is found using Table A.6 for Type I testing in the Appendix with $N = 10$: $(0.476 \times 81.26, 2.575 \times 81.26) = (38.68, 209.24)$.

EXAMPLE 14.4. Estimate the AMSAA parameters from the following failure times: 3, 15, 35, 58, 113, 187, 225, 465, 732, 1123, 1587, 2166, 5423, 8423, 12,035 (the test was terminated after 15 failures).

Solution. Using Eqs. (14.25), (14.22), and (14.23),

Failure time	ln(failure time)
3	1.098612
15	2.70805
35	3.555348
58	4.060443
113	4.727388
187	5.23109
225	5.4161
465	6.142038
732	6.595781
1123	7.023759
1587	7.369601
2166	7.680638
5423	8.598404
8423	9.038721
Total	79.24599

$$\hat{b} = \frac{15}{14\ln(12035) - 79.24599} = 0.28685$$

$$\hat{a} = \frac{15}{12035^{0.28685}} = 1.013$$

Then $\hat{\rho}(t) = 1.013 \times 0.28685t^{(0.28685-1)} = 0.29058t^{-0.71315}$

and, at the end of testing,

$$\text{MTTF} = \frac{1}{\hat{\rho}(12035)} = 2797$$

A 90 percent confidence interval for the MTTF is given by $(0.6299 \times 2797, 2.182 \times 2797) = (1762, 6103)$.

14.5 OTHER GROWTH MODELS

Numerous growth models have been proposed in the literature. *The Military Handbook: Reliability Growth Management* [1981] summarizes sixteen different growth models, including the Duane and AMSAA models. Healy [1987] provides an alternative to the Duane model that ignores early failures. Ascher and Feingold [1984] compare several growth models; here we briefly describe several of these alternative models. For example, Lloyd and Lipow [1962] present a model based on discrete trials and a single failure mode. If a failure occurs on a given trial, there is a constant probability of success in eliminating the failure. If the system does not fail on a particular trial, no action is taken. The probability of a failure on a given trial (if it has not been eliminated) is also constant. The resulting reliability on the nth trial is

$$R_n = 1 - ae^{-b(n-1)} \tag{14.26}$$

where a and b are constants to be estimated.

Barlow and Scheuer [1966] generalize on the Lloyd and Lipow model. For their model, a reliability growth program is conducted in k stages. The reliability in the ith stage is

$$r_i = 1 - q_0 - q_i \qquad i = 1, 2, \ldots, k \tag{14.27}$$

where q_0 is the probability of an inherent failure, which is constant and does not change for each stage, and q_i is the probability of an assignable-cause failure. Inherent failures reflect the state of the art, whereas an assignable-cause failure is one that can be corrected through equipment or operational modifications. Each trial results in either an inherent failure, an assignable-cause failure, or no failure. The q_i are assumed to be nonincreasing, indicating that the reliability cannot decrease during the test program. Reliability growth is achieved by decreasing q_i through engineering redesign. The number of trials in the ith stage may be fixed or random. The following maximum likelihood estimates are obtained for q_0 and q_i as a function of the number of inherent and assignable failures and successes observed at each stage:

$$\hat{q}_0 = \frac{\sum_{i=1}^{K} a_i}{\sum_{i=1}^{K}(a_i + b_i + c_i)} \tag{14.28}$$

$$\hat{q}_i = \frac{(1 - q_0)b_i}{b_i + c_i} \tag{14.29}$$

where a_i = the number of inherent failures at stage i
b_i = the number of assignable-cause failures at stage i
c_i = the number of successes at stage i

Then

$$\hat{r}_i = 1 - \hat{q}_0 - \hat{q}_i$$

If $\hat{q}_{i+1} > \hat{q}_i$, then, to ensure that the $\hat{q}_i$ are nonincreasing, the observations in stage i and stage $(i+1)$ are combined and $\hat{q}_i$ is recomputed using Eq. (14.29); this procedure may be repeated until a nonincreasing sequence is obtained.

Gompertz Curve. A growth model based on the Gompertz curve is given by

$$R = ab^{c^t} \tag{14.30}$$

where $0 < a, b, c \leq 1$ are constants to be determined and t is the development time. As $t \to \infty$, $c^t \to 0$, and therefore $R \to a$. As a result, the constant a is an upper bound on the reliability. A disadvantage of this model is the need to use nonlinear least squares to obtain estimates of the model parameters.

Exponential Model. The exponential model is simple, and like the Duane model, it can be estimated by using linear regression analysis. The model has the form

$$\text{MTTF}_c = ae^{bt} \tag{14.31}$$

where $a, b > 0$ are constants estimated from a least-squares analysis of the logarithm of Eq. (14.31) and t may be cumulative test time or development time.

Lloyd-Lipow Model. The Lloyd-Lipow model [1962] takes the following form:

$$\text{MTTF}_c = a - b/t$$

where $t \geq b/a$, and a and b are the parameters to be estimated. The parameter a in this model serves as an upper bound on the cumulative MTTF. Linear least squares can be used to estimate the parameters under the transformation $t' = 1/t$. The rate of growth for this model is inversely proportional to the square of the cumulative test time; that is, the cumulative MTTF increases at a decreasing rate—an attractive property.

Given these and many more models found in the literature, it is not obvious in most cases which model to use. The assumptions of each model and its applicability to the particular growth problem certainly must be carefully considered. A study conducted by the Hughes Aircraft Company for the Rome Air Development Center [1975] strongly supports the use of the AMSAA model. This study compared six continuous-growth models, including the Duane growth curve and the exponential model, against airborne equipment failure data. The AMSAA model consistently outperformed the others, having the smallest percentage error in comparing predicted versus actual values. Additional research comparing the performance of these various models is necessary.

EXERCISES

14.1 Using the idealized growth curve, if the growth parameter was 0.4 and initial testing at 1000 hours produced an average MTTF of 200, how many test hours will be required to achieve an MTTF of 800? What MTTF should be observed after 2000 cumulative test hours?

14.2 A test–fix–test reliability growth program conducted growth testing in 100-hr increments with the following results:

Hours of test	Number of failures
0–100	23
100–200	15
200–300	9
300–400	5
400–500	2
500–600	2
600–700	1
700–800	1

Fit a Duane growth curve and estimate the additional time necessary to achieve an MTTF of 50 hours.

14.3 As part of a reliability growth program, 5500 hr of Type I testing resulted in failures at the following times:

1, 3, 9, 19, 36, 60, 93, 135, 188, 252, 330, 421, 526, 647, 785, 941, 1115, 1309, 1523, 1758, 2015, 2295, 2600, 2929, 3283, 3664, 4073, 4509, 4975, 5470

Fit the AMSAA growth model, and compute a 90 percent confidence interval for the MTTF at the conclusion of the test program.

14.4 Will I. Fail, a reliability engineering student, has derived, after 745 hours of reliability growth testing, the following Duane growth curve for a new product:

$$\text{MTTF}_c = 0.07506\ T^{0.7526} \qquad \text{(in hours)}$$

If a final product MTTF = ~~100~~ hours is desired, how many additional hours of reliability growth testing will be required? What is the current MTTF?

14.5 A new automotive transmission is undergoing reliability growth testing. The following failure times (in thousands of miles) were observed during one test–fix–test growth cycle: 23, 45, 92, 210, 378, and 690. Fit the AMSAA growth model to this data and estimate the MTTF at the end of the test period (assume Type II testing). Construct a 90 percent confidence interval about the MTTF.

14.6 An initial 500 hours of testing resulted in an average MTTF of 150 hours. The reliability test plan calls for 3 additional test–fix–test cycles having the following cumulative hours: 500–1500; 1500–3000; 3000–4000. If a 30 percent growth slope is desired, determine the average MTTF which should be observed for each test period. What is the MTTF goal at the conclusion of the test program? Plot the idealized growth curve and identify the average MTTF at the proper points in time.

14.7 A reliability growth test plan calls for a growth parameter (α) of 0.5. The following test data have been obtained:

Cumulative hours	Cumulative failures
100	18
200	30
300	38
400	44
500	47

Track the actual progress to date of the reliability growth with the idealized growth curve. Using the Duane curve, estimate the reliability growth over the test period of 1000 hr of cumulative test time. What will be the MTTF at the conclusion of the program versus the MTTF goal as implied by the idealized curve?

14.8 A growth model has the following form:

$$\rho(t) = e^{a+bt}$$

where $\rho(t)$ is an intensity function and a and b are parameters to be estimated from the test data. Derive expressions for the instantaneous MTTF, the cumulative number of failures, and the cumulative MTTF. Suggest a procedure for estimating the parameters.

14.9 A system has a fixed (unknown) number of defects (failure modes), k_1. As a result of a test program, the rate of change of the number of defects after t test hours is proportional to the number of defects remaining at time t. That is, let $n(t)$ equal the number of defects remaining at test time t. Then,

$$\frac{dn(t)}{dt} = -k_2 n(t)$$

where k_2 is a proportionality constant and the initial conditions are $n(0) = k_1$. Assume that when a defect occurs during test, it is subsequently eliminated. Derive an expression for the cumulative number of defects (failures) found and eliminated, $N(t)$. Also, derive an expression for the amount of test time necessary to find and remove q percent of the defects and for the number remaining after test time t. What applications does this model suggest?

14.10 Failures have occurred at the following cumulative test times (Type II testing): 28, 146, 258, 426, 521, 1027, 1273 hours.

(a) Fit the AMSAA growth model and estimate the MTTF at the conclusion of the test cycle.

(b) On the basis of (a), how many more hours of test time will be necessary to achieve an MTTF of 3000 hours?

(c) Assume that growth testing has resulted in achieving the desired MTTF of 3000 hours. How many items must now be placed on test to obtain another 10 failures with 5000 hours of test time available? (Assume a constant failure rate.)

(d) If the testing in (c) continues for only 600 more hours, what is the expected number of failures?

CHAPTER 15

Identifying Failure and Repair Distributions

The objective of this and the following chapter is to fit a theoretical distribution to a random sample of failure or repair data. By *fit* we mean to perform a statistical test in order to accept or reject the hypothesis that the observed times come from a specified distribution. This chapter discusses the procedures for identifying and specifying candidate distributions for fitting the data. The following chapter outlines several goodness-of-fit tests available for conducting the hypothesis test. The particular probability distributions addressed include the exponential, Weibull, normal, and lognormal.

Chapter 12 developed methods for deriving empirical reliability distributions directly from failure data. These methods are called *nonparametric* or *distribution-free* methods since they do not require the specification of a theoretical distribution and the estimation of the distribution's parameters. An alternative approach is to identify an appropriate theoretical distribution, estimate the parameter(s), and perform a goodness-of-fit test. In general, fitting a theoretical distribution is preferred over empirically developing a model. First, empirical models do not provide information beyond the range of the sample data. In reliability engineering the tails of the distribution are of most interest. For singly censored data, extrapolation beyond the censored data is possible with a theoretical distribution. Second, we are interested in determining the probabilistic nature of the underlying failure process. A sample is only a small (random) subset of the population of failure times, and it is the distribution the sample came from and not the sample itself that we want to establish. Third, often the failure process is a result of some physical phenomena that can be associated with a particular distribution. For example, the central limit theorem provides justification for using the normal distribution when additive effects are present or the lognormal distribution when multiplicative effects are present. Chance, or random, processes may generate an exponential time to failure. Fourth, small sample sizes provide very little information concerning the failure process. However, if the sample is consistent with a theoretical distribution, then much stronger results based on the properties of the theoretical distribution are possible. Finally, use can be made of the theoretical reliability model in performing more complex analysis of the failure process. For example, the effect of changes on the MTTF can easily be accomplished. Without an analytical reliability model to use, it is difficult to derive more complicated relationships, such as the preventive maintenance reliability model discussed in Chapter 9.

15.1 IDENTIFYING CANDIDATE DISTRIBUTIONS

With the collection of failure or repair data, the fitting of a theoretical distribution may be viewed as a three-step process consisting of identifying candidate distributions, estimating parameters, and performing a goodness-of-fit test.

Identifying candidate distributions is both an art and a science. An understanding of the failure process, knowledge of the characteristics of the theoretical distributions, and a statistical analysis of the data will assist in selecting a failure or repair distribution. One suggested approach[1] is to

1. Construct a histogram of the failure or repair times.
2. Compute descriptive statistics.
3. Analyze the empirical failure rate.
4. Use prior knowledge of the failure process.
5. Use properties of the theoretical distribution.
6. Construct a probability plot.

A histogram is a graph resulting from grouping the failure times into classes and plotting the frequency or relative frequency of the number of observations within each class versus the interval times of each class. Determining the proper number of intervals is important if the histogram is to correctly reflect the shape of the underlying probability density function. Use of too many classes results in insufficient summarization of the data and the inability to discern the shape of the distribution. If too few classes are used, too much information is lost through summarization to properly identify the distribution. A good rule of thumb for the number of classes is given by Sturges' rule:

$$k = \lfloor 1 + 3.3 \log_{10} n \rfloor \tag{15.1}$$

where k = number of classes
n = sample size
$\lfloor x \rfloor$ = integer part of x

[1] The methodology outlined may be applied to individual failure modes rather than an entire set of failure data. However, the failure times of those failure modes not considered must be treated as censored data.

For example,

n	k
50	6
500	9
5000	13

EXAMPLE 15.1. The following 35 failure times (in operating hours) were obtained from field data over a 6-month period. Construct a histogram.

1476	300	98	221	157
182	499	552	1563	36
246	442	20	796	31
47	438	400	279	247
210	284	553	767	1297
214	428	597	2025	185
467	401	210	289	1024

Solution. From Sturges' rule: $k = \lfloor 1+3.3\log_{10} 35 \rfloor = \lfloor 1+3.3(1.544) \rfloor = \lfloor 6.0954 \rfloor = 6$. From the graphs of Fig. 15.1, it is apparent that too many or too few classes tend to hide the underlying distribution (exponential).

Descriptive statistics computed from the sample data may be useful in either identifying a candidate distribution or eliminating some distributions. For example, if the failure times came from a symmetrical or nearly symmetrical distribution, such as the normal or a Weibull with a shape parameter between 3 and 4, then the sample mean and median times to failure will be approximately equal. If the mean is considerably larger than the median, then the data are skewed to the right, and the exponential, lognormal, or Weibull will provide the better fit. If the failure process is exponential, we would expect the sample mean and the sample standard deviation to be approximately equal (which is the case for the exponential distribution).

EXAMPLE 15.2. From the failure data in Example 15.1, $\widehat{\text{MTTF}} = 485.2$. After the data have been rank-ordered, the sample median is defined to be the middle observation if the sample size is odd or the midpoint of the two middle observations if the sample size is even. Therefore, for this example, the data are rank-ordered as follows:

20	31	36	47	98
157	182	185	210	210
214	221	246	247	279
284	289	**300**	400	401
428	438	442	467	499
552	553	597	767	796
1024	1297	1476	1563	2025

Here $\hat{t}_{\text{med}} = t_{18} = 300$ hr. Since $\widehat{\text{MTTF}} > t_{\text{med}}$, it appears that the data are skewed to the right. Further, from Eq. (12.11), $s^2 = (20^2 + 31^2 + \cdots + 2025^2 - 35 \cdot \overline{485.2}^2)/34 = 220{,}712.3$, or $s = 469.8$. Because the sample mean and sample standard deviation are near one another, the exponential distribution appears to be a good candidate.

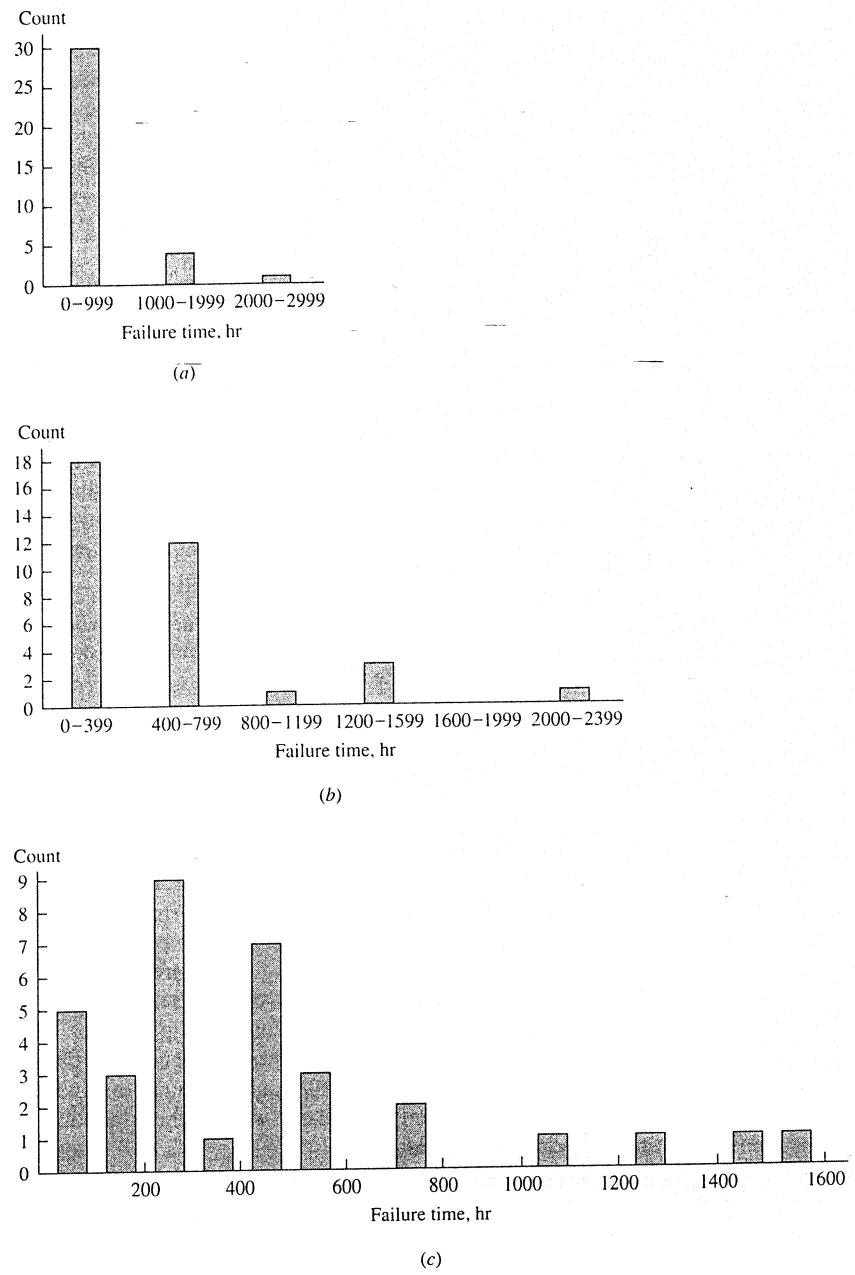


FIGURE 15.1
Histograms of failure times with differing numbers of classes: (*a*) 3 classes (too few); (*b*) 6 classes; (*c*) 17 classes (too many).

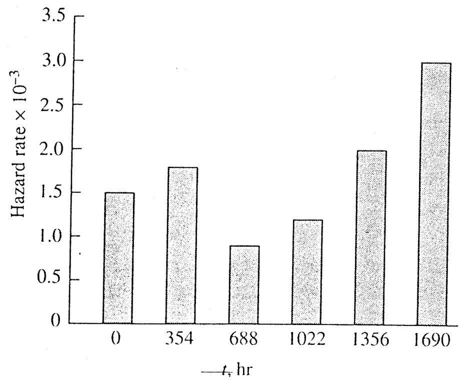


FIGURE 15.2
An empirical hazard rate curve.

As a next step in the analysis of the data, it may be useful to compute and graph the empirical hazard rate function. From the graph it may be possible to determine whether the hazard rate is decreasing, increasing, or constant. A constant failure rate will further support the use of the exponential distribution, and a decreasing failure rate will support the use of the Weibull distribution. An increasing failure rate may be modeled by either a Weibull, a normal, or a lognormal distribution.

EXAMPLE 15.3. For the data in Example 15.1, the hazard rate function shown in Fig. 15.2 is computed from Eq. (12.15). Since the empirical hazard rate function is not monotonic, we cannot rule out a constant failure rate at this point.

15.2 PROBABILITY PLOTS AND LEAST-SQUARES CURVE-FITTING

Probability plots provide an informal method of evaluating the fit of a set of data to a distribution. If we plot the points $(t_i, \hat{F}(t_i))$, $i = 1, 2, \ldots, n$, on appropriate graph paper, a proper fit to the distribution would graph as an approximate straight line. This is because the vertical scale and possibly the horizontal scale have been modified to linearize the cumulative distribution function. Since straight lines are easily identifiable, a probability plot provides a better visual test of a distribution than comparison of a histogram with a probability density function. In constructing the probability plot, the estimate of $F(t_i)$ may be any of the plotting positions discussed in Chapter 12. However, because of skewness, convenience of use, and the ability to plot multiply censored data using the rank adjustment method, we will use the approximation to the median plotting position given by Eq. (12.6).

From a probability plot it may also be possible to obtain initial estimates for the parameter(s) of the distribution being fitted. Probability plots may also be used when the sample size is too small to construct a meaningful histogram and may be used with incomplete data. If the data are singly censored on the right, the probability plot will graph the cumulative distribution up to the time of the last failure.

Our primary approach to probability plots is to fit a linear regression (least-squares) line of the form $y = a + bx$ to a set of transformed data. The nature of the transform will depend on the distribution as described below. However, if the failure or repair times fit the assumed distribution, the transformed data will graph as a straight line, and the fitted regression line will have a high index of fit, r. This approach is more accurate than manually plotting the data on special graph paper, although we will illustrate the manual plotting technique with a Weibull plot. The least-squares fits to the exponential, Weibull, normal, and lognormal distributions are discussed in turn.

15.2.1 Exponential Plots

The cumulative distribution function for the exponential distribution is $F(t) = 1 - e^{-\lambda t}$, or $1 - F(t) = e^{-\lambda t}$. Then taking the natural logarithm of both sides, $\ln[1 - F(t)] = -\lambda t$, and

$$-\ln[1 - F(t)] = \ln\left[\frac{1}{1 - F(t)}\right] = \lambda t \tag{15.2}$$

Given failure times $t_1, t_2, \ldots, t_n$, the points $(t_i, \hat{F}(t_i))$, where $\hat{F}(t_i)$ may be any of the plotting positions, may be plotted on exponential graph paper. The vertical scale is based on the transformation

$$\hat{F}(t) \leftrightarrow \ln\left[\frac{1}{1 - \hat{F}(t)}\right]$$

Since $F(\text{MTTF}) = 1 - e^{-1} = 0.632$, the MTTF of the distribution may be estimated directly from the graph by finding the value of t that corresponds to $F(t) = 0.632$.

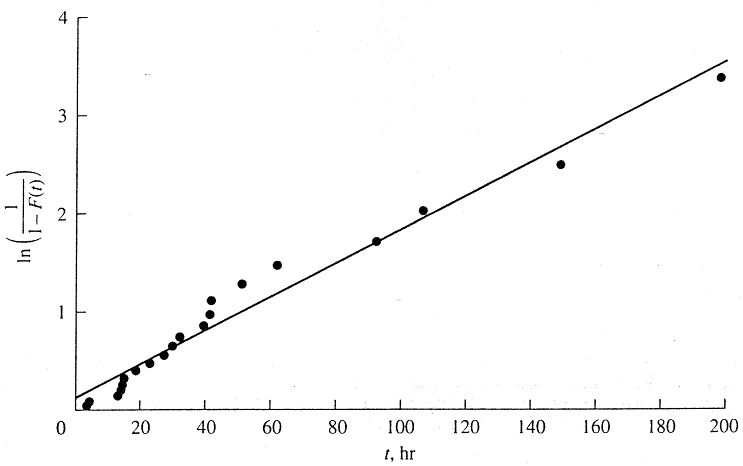


FIGURE 15.3
An exponential least-squares plot of failure data.

A more accurate fit to the data may be obtained by performing a least-squares fit of Eq. (15.2) using

$$\hat{\lambda} = b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \tag{15.3}$$

where $y_i = \ln\{1/[1 - F(t_i)]\}$ and $x_i = t_i$. This equation is used in place of the least-squares Eqs. (14.10) and (14.11) since the line passes through the origin with a slope of b. The estimate for the MTTF is then $1/b$.

EXAMPLE 15.4. From the following failure times, obtained from a reliability test program, a plot of $\ln(1/1 - \hat{F}(t_i))$ versus t_i is constructed and is shown in Fig. 15.3. In this example, $\hat{F}(t_i) = (i - 0.3)/(n + 0.4)$. The plot shows that an obvious linear relationship exists.

Failure time, $(x_i) = t_i$	$F(t_i)$	$y_i = \ln[1/(1 - F(t_i))]$
3.3	0.03431	0.03492
4.2	0.08333	0.08701
12.9	0.13235	0.14197
13.8	0.18137	0.20013
14.3	0.23039	0.26187
14.8	0.27941	0.32769
18.5	0.32843	0.39814
22.8	0.37745	0.47393
27.1	0.42647	0.55595
29.7	0.47549	0.64529
32	0.52451	0.74341
39.5	0.57353	0.85221
41.3	0.62255	0.97431
41.6	0.67157	1.11343
51.1	0.72059	1.27507
61.7	0.76961	1.46797
92.2	0.81863	1.70720
106.6	0.86765	2.02228
148.8	0.91667	2.48491
198.1	0.96569	3.37221

A least-squares fit is obtained from Eq. (15.3), yielding the slope $b = 0.01832$. The resulting index of fit, $r = 0.979$, indicates a strong linear fit to the data, thus supporting the hypothesis that the data came from an exponential distribution. The estimated MTTF is $1/b = 54.6$ hr. The sample MTTF, 48.7, is obtained from averaging the 20 failure times (sample mean).

15.2.2 Weibull Plots

From the Weibull cumulative distribution function,

$$F(t) = 1 - e^{-(t/\theta)^{\beta}}$$

and

$$\ln\left[\frac{1}{1 - F(t)}\right] = \left(\frac{t}{\theta}\right)^{\beta}$$

Taking logarithms again,

$$\ln\ln\left[\frac{1}{1 - F(t)}\right] = \beta \ln t - \beta \ln \theta \tag{15.4}$$

Therefore plot

$$\left(\ln t_i, \ln\ln\left[\frac{1}{1 - \hat{F}(t_i)}\right]\right)$$

or using Weibull probability paper, plot $(t_i, \hat{F}(t_i))$.

Weibull graphing

Generally, a least-squares fit to the data is recommended over a manual plot of the data on probability paper. It is more accurate and less subjective than fitting a straight line to the data by eye. In addition, measures of how well the curve fits the data, such as the index of fit, are available. With the use of the personal computer and one of the many statistical applications available with graphics capability, plotting data by hand is no longer necessary. However, because of the popularity of these plots and the fact that a practicing engineer is still likely to encounter them, an example of the use of Weibull probability paper is provided. Weibull graph paper is constructed so that data generated from a Weibull distribution will graph as a straight line. The abscissa is a logarithmic scale, and the ordinate, while labeled in terms of the cumulative percentage of failures, $F(t)$, is scaled on the basis of

$$\ln\ln\left(\frac{1}{1 - F(t)}\right)$$

Theta, θ, can be estimated from the point on the line that corresponds to 63.2 percent of the failures, since $F(\theta) = 0.632$. The 63.2 percent line is often identified on the graph paper. It can be seen from Eq. (15.4) that β can be estimated from the slope of the plotted line. However, care must be taken in using a ruler to find the slope since not all versions of Weibull paper are constructed where the unit length is the same on the abscissa and ordinate (for example, 1 unit equals 1 centimeter on both the abscissa and the ordinate). Some versions of Weibull paper will have a scale or legend that can be used to find the slope of the fitted line. The following example uses Weibull paper with a 1-to-2 (key) scale (that is, 1 unit on the abscissa equals 2 units on the ordinate). When the slope is being found, a scale adjustment must then be made. An alternative approach is to solve Eq. (15.4) for β, giving

$$\hat{\beta} = \frac{\ln\ln[1/1 - F(t)]}{\ln t - \ln\theta}$$

Once θ has been determined, β can be found for several pairs of values of t_i and $F(t_i)$ and an average can be computed. This approach is illustrated in the following example.

EXAMPLE 15.5. The following failure times were obtained after rank-ordering five units that were tested until failure (complete data). Construct a Weibull plot and estimate the distribution parameters from the plot.

i	Failure time, hr	$(i - 0.3)/(5 + 0.4)$
1	32	0.13
2	51	0.31
3	74	0.50
4	90	0.69
5	120	0.87

The paired values shown above were plotted on Weibull plotting paper to construct the graph of Fig. 15.4. The slope of the fitted line was estimated using a ruler with the vertical distance adjusted by a factor of 2 to account for the difference in scale between the two axes. The characteristic life of 85 hr was found where the fitted line intercepted the 63.2 cumulative percentage line. Using the alternate approach to estimate β:

$$\hat{\beta} = \frac{\ln\ln[1/(1-0.31)]}{\ln 51 - \ln 85} = 1.94 \quad \text{and} \quad \hat{\beta} = \frac{\ln\ln[1/(1-0.87)]}{\ln 120 - \ln 85} = 2.06$$

The average of the two values is 2.0, and from Fig. 15.4,

$$\beta = 2 \times \frac{1 \text{ inch}}{1 \text{ inch}} = 2.0.$$

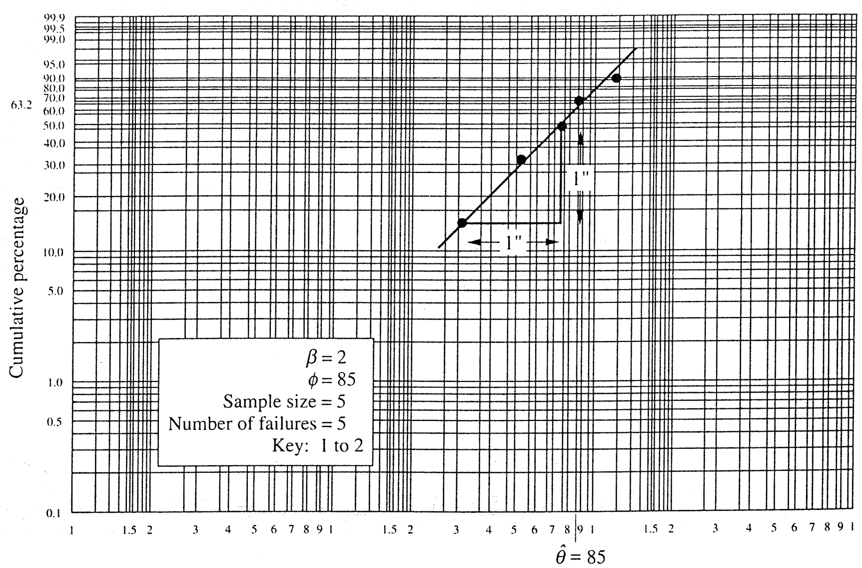


FIGURE 15.4
A Weibull plot on probability paper.

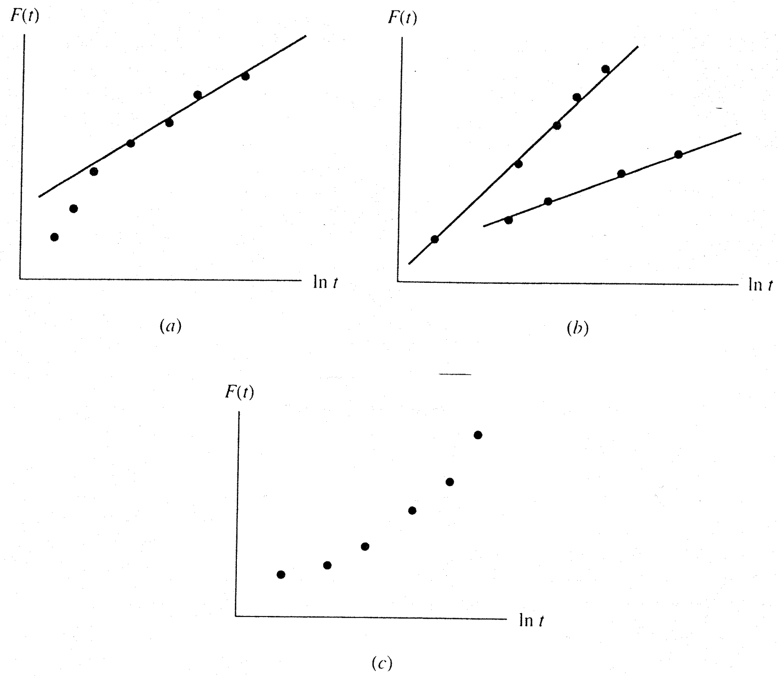


FIGURE 15.5
Nonlinear Weibull plots: (*a*) for a nonzero t_0; (*b*) with competing failure modes; (*c*) non-Weibull distribution.

When the Weibull plot is a curve rather than a straight line, either the data came from another distribution, they have a nonzero minimum lifetime, or two or more failure modes are present. Figure 15.5(a) shows a typical curvature resulting from a nonzero minimum time, and Fig. 15.5(b) is characteristic of competing failure modes being present. In the former case, the adjusted data, $t_i - t_0$, should be plotted. Estimates for t_0 are discussed in Section 15.3.6. In the latter case, a separate distribution should be fit to each failure mode. This will require treating the failure times of all but the times of the failure mode being fitted as censored times. The method discussed in Section 15.2.5, based on rank adjustments, can be used to accomplish this. Of course, if even after these considerations the Weibull distribution results in a poor fit, as is shown in Fig. 15.5(c), the data should be plotted against other distributions.

Least-squares approach

For a least-squares fit we use Eqs. (14.10) and (14.11) with

$$x_i = \ln t_i \quad \text{and} \quad y_i = \ln\ln\left[\frac{1}{1-\hat{F}(t_i)}\right]$$

From Eq. (15.4) it can be seen that β corresponds to the slope and $\beta \ln \theta$ is the intercept. Therefore, from the least-squares fit, $\hat{\beta} = b$ and an estimate of the shape parameter, θ, is obtained by setting $a = -\beta \ln \hat{\theta}$ and solving for $\hat{\theta}$, or $\hat{\theta} = e^{-a/\hat{\beta}}$.

EXAMPLE 15.5 (CONTINUED). A least-squares fit of the five failure times given in Example 15.5 yields the following results:

Failure time, hr	$F(t_i)$	$\ln \ln[1/(1 - F(t_i))]$
32	0.12963	−1.97446
51	0.31481	−0.97269
74	0.5	−0.36651
90	0.68519	0.14477
120	0.87037	0.71446

Intercept: $a = -8.95165$
Slope: $b = 2.01553$
Estimated β: $\hat{\beta} = 2.01553$
Estimated θ: $\hat{\theta} = 84.88845$
Index of fit: $r = 0.9986$

EXAMPLE 15.6. The following failure times were obtained from testing 15 units until each had failed:

112.2	139.8	156.8	113.6	75.5	88.5	73.9	95.5
218.0	25.1	403.1	150.3	164.5	138.5	151.9	

Solution. The data are rank-ordered in the following table. The plot of $\ln \ln[1/(1 - \hat{F}(t_i))]$ versus $\ln(t_i)$ shown in Fig. 15.6 has an approximately linear relationship.

i	t_i	$\hat{F}(t_i) = \dfrac{i - 0.3}{15 + 0.4}$
1	25.1	0.0455
2	73.9	0.1104
3	75.5	0.1753
4	88.5	0.2403
5	95.5	0.3052
6	112.2	0.3701
7	113.6	0.4351
8	138.5	0.5000
9	139.8	0.5649
10	150.3	0.6299
11	151.9	0.6948
12	156.8	0.7597
13	164.5	0.8247
14	218.0	0.8896
15	403.1	0.9545

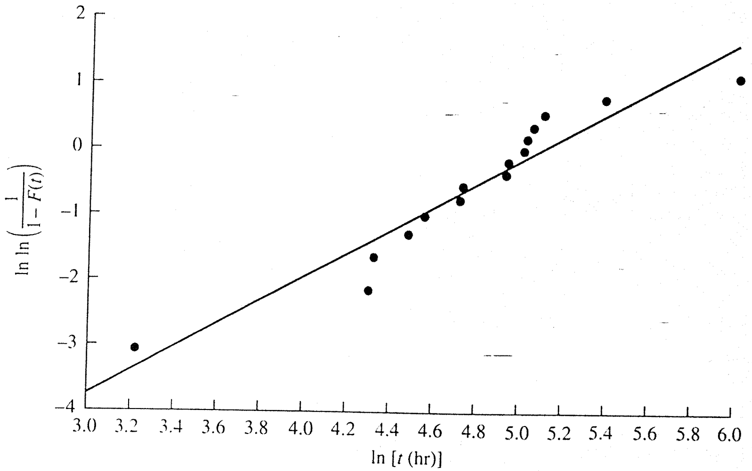


FIGURE 15.6
A Weibull least-squares plot of failure data.

The least-squares fit of the data in the following table was obtained.

Failure time, $x_i = t_i$	$F(t_i)$	$y_i = \ln \ln[1/(1 - F(t_i))]$
25.1	0.04545	−3.0679
73.9	0.11039	−2.1458
75.5	0.17532	−1.6463
88.5	0.24026	−1.2918
95.5	0.30519	−1.0103
112.2	0.37013	−0.77167
113.6	0.43506	−0.56029
138.5	0.5	−0.36651
139.8	0.56494	−0.18361
150.3	0.62987	−0.006120
151.9	0.69481	0.17126
156.8	0.75974	0.35490
164.5	0.82468	0.55453
218	0.88961	0.79016
403.1	0.95455	1.1285

Intercept: $a = -9.1649$
Slope: $b = 1.8027$
Estimated β: $\hat{\beta} = 1.8027$
Estimated θ: $\hat{\theta} = 161.41$
Index of fit: 0.9545

The estimated β is 1.80, and the estimated θ is 161. The index of fit, r, of 0.9545 indicates a good fit.

15.2.3 Normal Plots

For the normal distribution,

$$F(t) = \Phi\left(\frac{t-\mu}{\sigma}\right) = \Phi(z)$$

The inverse function can be written as

$$z_i = \Phi^{-1}[F(t)] = \frac{t_i - \mu}{\sigma} = \frac{t_i}{\sigma} - \frac{\mu}{\sigma} \tag{15.5}$$

which is linear in t. With the appropriate transformation of the vertical scale, the points $(t_i, \hat{F}(t_i))$ may be plotted. A least-squares fit is obtained by setting

$$x_i = t_i \qquad \text{and} \qquad y_i = z_i = \Phi^{-1}\left[F(t_i)\right]$$

The values of z_i may be obtained from Table A.1 in the Appendix on the basis of the corresponding value for $\hat{F}(t_i)$. From the least-squares fit and Eq. (15.5),

$$\hat{\sigma} = \frac{1}{b} \qquad \text{and} \qquad \hat{\mu} = -a\hat{\sigma} = -\frac{a}{b}$$

EXAMPLE 15.7. Data collected on the wearout of ball bearings (in hundreds of operating hours) resulted in the following failure times:

68.0	75.5	83.0	80.3	87.7	77.6	71.1	81.9	87.4	69.6
78.0	77.8	88.4	78.2	71.4	80.2	85.6	98.3	74.3	74.6

A least-squares fit consisted of fitting the above points in ascending order to the cumulative probabilities, as shown:

i	t_i	$\hat{F}(t_i) = \dfrac{i-0.3}{20+0.4}$	z_i
1	68.0	0.0343	−1.8211
2	69.6	0.0833	−1.3832
3	71.1	0.1324	−1.1151
4	71.4	0.1814	−0.9100
5	74.3	0.2304	−0.7375
6	74.6	0.2794	−0.5846
7	75.5	0.3284	−0.4443
8	77.6	0.3775	−0.3121
9	77.8	0.4265	−0.1853
10	78.0	0.4755	−0.0615
11	78.2	0.5245	0.0615
12	80.2	0.5735	0.1853
13	80.3	0.6225	0.3121
14	81.9	0.6716	0.4443
15	83.0	0.7206	0.5846
16	85.6	0.7696	0.7375
17	87.4	0.8186	0.9100
18	87.7	0.8676	1.1151
19	88.4	0.9167	1.3832
20	98.3	0.9657	1.8211

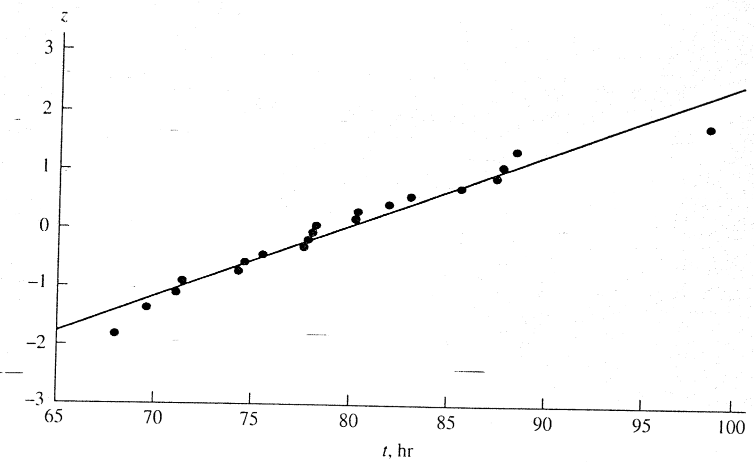


FIGURE 15.7
A normal least-squares plot of failure data.

The cumulative probabilities are transformed into the corresponding z_i values from the normal table, Table A.1, in the Appendix. A plot of z_i values versus the corresponding failure times is shown in Fig. 15.7. The least-squares fit yields the following:

Intercept: $a = -9.81565$
Slope: $b = 0.123553$
Estimated σ: $\hat{\sigma} = 1/b = 8.0937$
Estimated mean μ: $\hat{\mu} = -a/b = 79.445$
Index of fit: $r = 0.979$

The least-squares estimated mean and σ are approximately 79.4 and 8.1, respectively. The index of fit of 0.979 indicates a good fit.

15.2.4 Lognormal Plots

Lognormal probability plots are based on the relationship of the lognormal distribution to the normal distribution. Since

$$F(t) = \Phi\left(\frac{1}{s}\ln\frac{t}{t_{\text{med}}}\right) = \Phi(z)$$

then

$$z = \Phi^{-1}[F(t)] = \frac{1}{s}\ln t - \frac{1}{s}\ln t_{\text{med}} \tag{15.6}$$

and the points $(\ln t_i, z_i)$ are plotted or, on lognormal probability paper, the points $(t_i, \hat{F}(t_i))$ are plotted. For a least-squares fit, $x_i = \ln t_i$ and $y_i = z_i$. The shape parameter, s, is the reciprocal of the slope of the fitted line, and t_{med}, the median, is obtained from the y intercept of the fitted line. That is,

$$\hat{s} = \frac{1}{b} \qquad \text{and} \qquad \hat{t}_{\text{med}} = e^{-\hat{s}a}$$

EXAMPLE 15.8. Repair times of a mechanical pump are believed to follow a lognormal distribution. Twenty-five repair times (in minutes) were observed as part of a maintainability demonstration:

47.1	84.8	151.9	122.5	218.2	99.6	59.8	138.8	213.5	53.4
102.4	100.8	230.1	104.6	61.5	122.1	186.2	498.4	77.0	78.7
112.3	44.0	151.3	151.3	222.8					

The graph in Fig. 15.8 was obtained by plotting $\Phi^{-1}[\hat{F}(t_i)] = z_i$ versus $\ln t_i$:

i	t_i	$\hat{F}(t_i) = \dfrac{i - 0.3}{25 + 0.4}$	z_i
1	44.0	0.0276	−1.9173
2	47.1	0.0669	−1.4993
3	53.4	0.1063	−1.2465
4	59.8	0.1457	−1.0551
5	61.5	0.1850	−0.8965
6	77.0	0.2244	−0.7574
7	78.7	0.2638	−0.6317
8	84.8	0.3031	−0.5155
9	99.6	0.3425	−0.4057
10	100.8	0.3819	−0.3005
11	102.4	0.4213	−0.1986
12	104.6	0.4606	−0.0989
13	112.3	0.5000	0.0000
14	122.1	0.5394	0.0990
15	122.5	0.5787	0.1986
16	138.8	0.6181	0.3005
17	151.3	0.6575	0.4057
18	151.3	0.6969	0.5155
19	151.9	0.7362	0.6317
20	186.2	0.7756	0.7574
21	213.5	0.8150	0.8965
22	218.2	0.8543	1.0551
23	222.8	0.8937	1.2461
24	230.1	0.9331	1.4993
25	498.4	0.9724	1.9173

Intercept: $a = -7.7550$
Slope: $b = 1.631$
Estimated $s = 1/b = 0.613$
Estimated $t_{\text{med}} = e^{-sa} = 116.0$
Index of fit: $r = 0.986$

The estimated least-squares t_{med} and s are approximately 116 minutes and 0.61, respectively. The index of fit of 0.986 indicates a good fit.

15.2.5 Multiply Censored Time Plots

When the sample data include both failure times and multiply censored times, adjustments must be made to the cumulative probabilities on the basis of the discussion of the rank adjustment method in Section 12.2.3. The approach is illustrated with

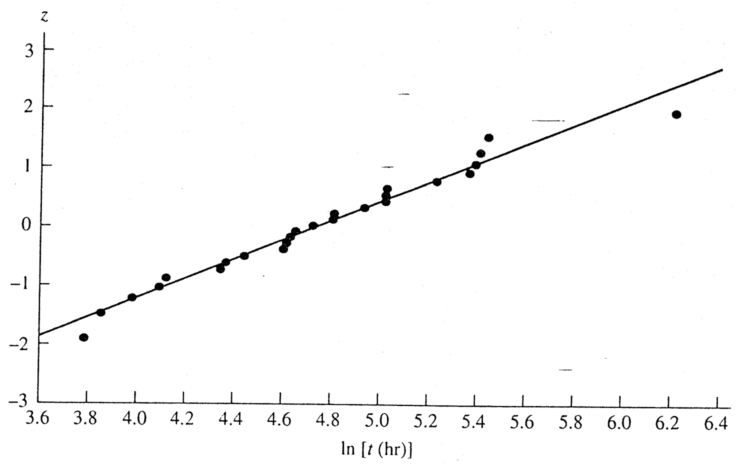


FIGURE 15.8
A lognormal least-squares plot of repair data.

the following example. Once the adjusted ranks (i_{t_i}) are determined and the corresponding $F(t_i) = (i_{t_i} - 0.3)/(n + 0.4)$ computed, the least-squares fit follows the previous discussion for the applicable distribution.

EXAMPLE 15.9. Thirty motors are placed on accelerated test with failures occuring at the following cycle times. A cycle consists of a motor starting up to its maximum number of revolutions per minute and then shutting down until it has come to a complete stop. Censored units resulted from motors being removed from test to satisfy other demands. With $n = 30$ and $F(t_i) = (i_{t_i} - 0.3)/(n + 0.4)$:

Time	Adjusted rank, i_{t_i}	$F(t_i)$	$y = \ln[1/(1 - F(t_i))]$	$\ln y$
141	1	0.023	0.0233	−3.735
391	2	0.056	0.0575	−2.855
399	3	0.089	0.0930	−2.375
410⁺				
463	4.04	0.123	0.1311	−2.031
465	5.07	0.157	0.1708	−1.767
497	6.11	0.191	0.2121	−1.550
501⁺				
559	7.19	0.227	0.2571	−1.358
563	8.27	0.262	0.3043	−1.190
579	9.36	0.298	0.3537	−1.039
580⁺				
586	10.50	0.336	0.4086	−0.8950
616	11.64	0.373	0.4666	−0.7622
683	12.77	0.410	0.5282	−0.6382
707	13.91	0.448	0.5939	−0.5211
713	15.05	0.485	0.6641	−0.4092
742⁺				
755⁺				
764	16.38	0.529	0.7529	−0.2838

Letting x be the failure time and using y as defined above, a least-squares curve fit to the exponential distribution was performed and yielded an index of fit of 0.79. Because the exponential fit was marginal, a least-squares fit to the Weibull distribution was computed regressing $\ln x$ against $\ln y$. The correlation was 0.94 with $b = 2.251$ and $a = -15.46$. Therefore, the time to failure of the motors can be described with a Weibull distribution having a shape parameter of 2.251 and a scale parameter of $e^{15.46/2.251} = 961$.

15.3 PARAMETER ESTIMATION

The previous discussion centered on the identification of candidate distributions for describing a failure or repair process. Once one or more distributions have been identified, the next step is to estimate the parameters of the distribution. Until the parameters are determined, the distribution is not completely specified. Although probability plots and least-squares fitting of the data provide a means of estimating the parameters of the distributions, they are not necessarily the preferred, or "best," estimates of the distribution parameters. This is especially true in certain goodness-of-fit tests that are based on the maximum likelihood estimator (MLE) for the distribution parameters. The MLE concept is first illustrated with an example and then defined in general. Finally, the MLEs for the basic reliability models are found.

EXAMPLE 15.10. Let X be the discrete random variable representing the number of trials necessary to obtain the first failure.[2] If we assume that the probability of a failure remains a constant p and each trial is independent, then

$$\Pr\{X = x\} = f(x) = (1-p)^{x-1}p \qquad x = 1, 2, \ldots \tag{15.7}$$

This is the probability of $x-1$ successes (probability $= (1-p)^{x-1}$) followed by a failure (probability $= p$).

If $x_1, x_2, \ldots, x_n$ represent a sample of size n from this distribution, then from Eq. (12.1) the joint distribution is

$$\begin{aligned} f_{x_1,\ldots,x_n}(x_1, x_2, \ldots, x_n) &= f(x_1)f(x_2)\cdots f(x_n) \\ &= (1-p)^{x_1-1}p(1-p)^{x_2-1}p\cdots(1-p)^{x_n-1}p \qquad (15.8) \\ &= p^n(1-p)^{\sum_{i=1}^{n}(x_i-1)} \end{aligned}$$

Equation (15.8) is called the *likelihood function* and represents the probability of obtaining the observed sample. Since Eq. (15.8) contains the unknown parameter, p, we would like to find a value of p consistent with the observed sample. If a value of p is found that maximizes the likelihood function, it also maximizes the probability of obtaining the observed sample. Therefore, we wish to solve the following problem:

$$\max_{0 \le p \le 1} g(p) = p^n(1-p)^{\sum_{i=1}^{n}(x_i-1)}$$

[2]This is the geometric distribution. It is the discrete version of the exponential distribution, in which the probability of a failure remains constant from trial to trial.

We can find the maximum of a function by finding the point at which the first derivative equals zero (horizontal tangent point). This is made easier if the logarithm of the likelihood function is used instead,[3] or

$$\max \ln g(p) = \ln\left\{\prod_{i=1}^{n} f(x_i)\right\} = n\ln p + \left[\sum_{i=1}^{n}(x_i - 1)\right]\ln(1-p)$$

Solving $d(\cdot)/dp = 0$

$$\frac{n}{p} + \frac{\sum_{i=1}^{n}(x_i-1)}{1-p}(-1) = 0$$

or

$$\hat{p} = \frac{n}{n + \sum_{i=1}^{n}(x_i - 1)} = \frac{n}{\sum_{i=1}^{n} x_i} \tag{15.9}$$

Therefore $\hat{p}$ as defined by Eq. (15.9) is the maximum likelihood estimator for this distribution.

EXAMPLE 15.11. The following data were collected on the number of production runs that resulted in a failure that stopped the production line: 5, 8, 2, 10, 7, 1, 2, 5. Therefore, if the random variable of interest, X, is the number of production runs necessary to obtain a failure, Eq. 15.7 is the probability distribution for X. The maximum likelihood estimate for the parameter p is

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{8}{40} = 0.2$$

and

$$\Pr\{X = x\} = f(x) = 0.8^{(x-1)}(0.2)$$

The mean of this distribution is $1/p$. Therefore, $\frac{40}{8} = 5$ is the mean number of production runs until a failure occurs. The probability that the line will fail during the third run is $\Pr\{x = 3\} = 0.8^2(0.2) = 0.128$.

15.3.1 Maximum Likelihood Estimator

In general, to find the MLE for any probability distribution with complete data, the maximum of the following likelihood function with respect to the unknown parameters $\theta_1, \ldots, \theta_k$ is found:

$$L(\theta_1, \ldots, \theta_k) = \prod_{i=1}^{n} f(t_i \mid \theta_1, \ldots, \theta_k)$$

The objective is to find the values of the estimators of $\theta_1, \ldots, \theta_k$ that render the likelihood function as large as possible for given values of $t_1, t_2, \ldots, t_n$. Because of the multiplicative form of the likelihood function, the maximum of the natural logarithm of the likelihood function is usually an easier problem to solve. In general,

[3]We can do this without changing the maximum point since the logarithm is a monotonically increasing function.

the necessary conditions for finding the MLEs are obtained by taking the first partial derivatives of the logarithm of the likelihood function with respect to $\theta_1, \ldots, \theta_k$ and setting these partials equal to zero. That is,

$$\frac{\partial \ln L(\theta_1, \ldots, \theta_k)}{\partial \theta_i} = 0 \qquad i = 1, 2, \ldots, k$$

With singly censored data on the right (Type I) present, the likelihood function is modified to be

$$L(\theta_1, \ldots, \theta_k) = \prod_{i=1}^{r} f(t_i \mid \theta_1, \ldots, \theta_k)[R(t_*)]^{n-r}$$

where r is the number of failures and n is the number at risk. The factor $[R(t_*)]^{n-r}$ is the probability that the $n - r$ censored units do not fail prior to the termination of the test. For Type II data, t_* is replaced with t_r.

15.3.2 Exponential MLE

For both complete and censored data, the MLE estimator for the parameter λ is given by:

$$\hat{\lambda} = \frac{r}{T} \tag{15.10}$$

where r is the number of failures and T is defined by Table 13.1.

To derive the MLE for Type II censoring, let n be the number on test and $r \leq n$ be the number of failures. Then with t_i the *ordered* sample of failure times,

$$f(t_i) = \lambda_i e^{-\lambda t_i} \qquad i = 1, 2, \ldots, r$$

is the probability density function for the random variable, T_i, representing the lifetime of the ith unit on test. For the $n - r$ units such that $t_i > t_r$,

$$\Pr\{T_i > t_r \text{ for all } i > r\} = (e^{-\lambda t_r})^{n-r}$$

Therefore the likelihood function is

$$L(t_1, \ldots, t_r) = \prod_{i=1}^{r} \lambda e^{-\lambda t_i}(e^{-\lambda t_r})^{n-r}$$

$$= \lambda^r \exp\left\{-\lambda \sum_{i=1}^{r} t_i - \lambda(n-r)t_r\right\}$$

and the natural logarithm of the likelihood function can be written as

$$\ln L = r \ln \lambda - \lambda \sum_{i=1}^{r} t_i - \lambda(n-r)t_r$$

Therefore
$$\frac{d \ln L}{d\lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} t_i - (n-r)t_r = 0$$

Then solving for λ,

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} t_i + (n-r)t_r} = \frac{r}{T}$$

For Type I data, t_r is replaced by t_*.

EXAMPLE 15.12. From the data in Example 15.1, $T = 16{,}981$ and $r = n = 35$. Therefore,

$$\hat{\lambda} = \frac{35}{16{,}981} = 0.00206 \qquad \text{or} \qquad \text{MTTF} = 485.2 \text{ hr}$$

15.3.3 Weibull MLE

The MLE for the two-parameter Weibull distribution must be computed numerically. For both complete and singly censored data, the estimate for the shape parameter, β, is found by solving the following equation (see Appendix 15A for its derivation):

$$g(\hat{\beta}) = \frac{\sum_{i=1}^{r} t_i^{\hat{\beta}} \ln t_i + (n-r)t_s^{\hat{\beta}} \ln t_s}{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)t_s^{\hat{\beta}}} - \frac{1}{\hat{\beta}} - \frac{1}{r}\sum_{i=1}^{r} \ln t_i = 0 \tag{15.11}$$

Then the MLE estimate of the characteristic life, θ, is found from

$$\hat{\theta} = \left\{\frac{1}{r}\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)t_s^{\hat{\beta}}\right]\right\}^{1/\hat{\beta}} \qquad \text{where } t_s = \begin{cases} 1 & \text{for complete data} \\ t_* & \text{for Type I data} \\ t_r & \text{for Type II data} \end{cases} \tag{15.12}$$

Eq. (15.11) must be solved numerically. The Newton-Raphson method for solving a nonlinear equation may be used. This requires solving for $\hat{\beta}$ iteratively using

$$\hat{\beta}_{j+1} = \hat{\beta}_j - \frac{g(\hat{\beta}_j)}{g'(\hat{\beta}_j)} \qquad \text{where } g'(x) = \frac{dg(x)}{dx} \tag{15.13}$$

An initial estimate for $\hat{\beta}$ may be obtained from a Weibull probability plot or a least-squares fit. Other numerical techniques, such as interval bisection and the method of golden sections, may also be used for finding the MLEs (see Reklaitis, Ravindran, and Ragsdell [1983] for a description of these techniques).

EXAMPLE 15.13. From the data of Example 15.6, the MLE for the shape parameter of the Weibull is obtained on the basis of Eq. 15.11. The calculations were done using the computer program accompanying this text. A starting value for β of 2.5 was selected, although the estimated β from a probability plot or a least-squares curve fit would provide an excellent starting value. In this example there are complete data with $r = n = 15$. Beginning with the initial β value and using the Newton-Raphson method, the following successive values were generated: 1.741574, 1.805401, 1.806654, 1.806655. The final value agrees closely with the least-square estimate of 1.8027. Using the MLE for β and Eq. 15.12,

$$\hat{\theta} = \left[\frac{1}{15}\sum_{i=1}^{15} t_i^{1.8}\right]^{1/1.8} = 158.655$$

15.3.4 Normal and Lognormal MLEs

The derivation of the maximum likelihood estimates for the normal distribution with complete data can be found in many statistics texts (such as Ross [1987]). The MLEs are based on the sample mean and sample variance:

$$\hat{\mu} = \bar{x} \tag{15.14}$$

$$\hat{\sigma}^2 = \frac{(n-1)s^2}{n} \tag{15.15}$$

where s is defined by Eq. (12.10) or Eq. (12.11).

For the lognormal distribution the relationship with the normal distribution is again utilized. Since the logarithm of the data would have a normal distribution, then the MLE estimates of the underlying normal parameters are used to compute the lognormal parameter estimates. Let

$$\hat{\mu} = \sum_{i=1}^{n} \frac{\ln t_i}{n}$$

Then

$$\hat{t}_{\text{med}} = e^{\hat{\mu}} \tag{15.16}$$

and

$$\hat{s} = \sqrt{\frac{\sum_{i=1}^{n}(\ln t_i - \hat{\mu})^2}{n}} \tag{15.17}$$

EXAMPLE 15.14. For the data in Example 15.8, the following MLEs are obtained for the lognormal distribution:

$$\hat{\mu} = \frac{\ln 44 + \ln 47.1 + \cdots + \ln 498.4}{25} = \frac{118.8}{25} = 4.752$$

$$\hat{t}_{\text{med}} = e^{4.753} = 115.932$$

$$\hat{s}^2 = \frac{(\ln 44 - 4.752)^2 + (\ln 47.1 - 4.752)^2 + \cdots + (\ln 498.4 - 4.752)^2}{25} = 0.31798$$

$$\hat{s} = 0.5639$$

These compare favorably to the least-squares estimates obtained in Example 15.8.

15.3.5 Maximum Likelihood Estimation with Multiply Censored Data

When multiply censored data are present, the likelihood function must be modified to reflect the fact that at the censored times no failure occurred. This can be accomplished by defining the likelihood function in the following manner:

$$L(\theta) = \prod_{i \in F} f(t_i;\theta) \prod_{i \in C} R(t_i^+;\theta) \tag{15.18}$$

where θ is the unknown parameter, F is the set of indices for the failure times, and C is the set of indices for the censored times. This general form of the likelihood function assumes that the probability distribution of the censored times, which are themselves random variables, is not dependent on the parameter θ. Even though t^+

is a censored time, it represents a survival (nonfailure) time in Eq. (15.18). Therefore the second product in Eq. (15.18) is the probability that none of the censored units will have failed prior to the time they were removed (censored). Equation (15.18) can also be used for singly censored data where t_i^+ becomes t_* (Type I) or t_r (Type II).

Exponential distribution

For the exponential distribution with r representing the number of failures, Eq. (15.18) becomes

$$L(\lambda) = \prod_{i \in F} \lambda^r \exp(-\lambda t_i) \prod_{i \in C} \exp(-\lambda t_i^+) = \lambda^r \exp\left(-\lambda \sum_{i \in F} t_i\right) \exp\left(-\lambda \sum_{i \in C} t_i^+\right) \tag{15.19}$$

Then taking logarithms,

$$\ln L(\lambda) = r \ln \lambda - \lambda \sum_{i \in F} t_i - \lambda \sum_{i \in C} t_i^+$$

$$\frac{d[\ln L(\lambda)]}{d\lambda} = \frac{r}{\lambda} - \sum_{i \in F} t_i - \sum_{i \in C} t_i^+ = 0$$

Finally, solving for λ,

$$\hat{\lambda} = \frac{r}{\sum_{i \in F} t_i + \sum_{i \in C} t_i^+} \tag{15.20}$$

This confirms a previous result that the MTTF for the exponential distribution can be estimated by taking the total time on test and dividing by the number of failures.

Weibull distribution

The derivation of the MLE with multiply censored data for the Weibull distribution is given in Appendix 15B. As is the case with singly censored data, the MLEs must be computed numerically. Following Nelson [1982], the following nonlinear equation must be solved for $\hat{\beta}$:

$$\sum_{i \in F} \frac{\ln t_i}{r} = \sum_{\text{all } i} t_i^{\hat{\beta}} \ln t_i \sum_{\text{all } i} (t_i^{\hat{\beta}})^{-1} - \frac{1}{\hat{\beta}} \tag{15.21}$$

Once the estimate for β has been found, then

$$\hat{\theta} = \left[\sum_{\text{all } i} \frac{t_i^{\hat{\beta}}}{r}\right]^{1/\hat{\beta}} \tag{15.22}$$

Equation (15.21) can be solved using the Newton-Raphson method (Eq. 15.13). However, the right-hand side is a monotonically increasing function of β and can therefore be solved by a direct-search technique as well. Singly censored failures may also be included with their censored times treated the same as multiply censored times.

EXAMPLE 15.15. Fifteen units were placed on test for 500 hr. The following failure times and censored times were observed prior to concluding the test:

34 136 145^+ 154 189 200^+ 286 287 334 353 380^+

Find the MLEs for both the exponential and Weibull distributions.

Solution. For the exponential distribution, the total time on test is given by

$$T = 34 + 136 + 145 + 154 + 189 + 200 + 286 + 287 + 334 + 353 + 380 + 4(500)$$
$$= 4498$$

and the MLE for the MTTF is $T/r = 4498/8 = 562.25$.

For the Weibull distribution, the four units that had not failed by the end of the test are each assigned a censored time of 500 hr. The left-hand side of Eq. (15.21) equals 5.21385. Beginning with $\beta = 0.1$ and increasing β in the right-hand side of Eq. (15.21) by 0.01 until it exceeds 5.21 results in $\beta = 1.43$. Then from Eq. (15.22), $\theta = 491$.[4]

Normal and lognormal distributions

Equation (15.18) provides the basis for determining the MLE for both the normal and lognormal distributions under both multiply censored and right-singly censored data. Because the reliability function of the normal distribution cannot be directly integrated, numerical procedures must be used. One of the simplest numerical techniques to implement is a recursive approximation based on the work of Sampford and Taylor [1959]. Since this algorithm is not very intuitive, its details are provided in Appendix 15C.

15.3.6 Location Parameter Estimation

When a probability plot displays a distinct bend in the curve rather than the expected straight line, it indicates that the data did not originate from the fitted distribution. The reason for the poor fit may be a nonzero minimum, or guarantee, life. In order for the data to be plotted on special graph paper, the minimum lifetime, or repair time, t_0, must be removed from the sample. This can be done by estimating t_0 and then transforming the data by letting $t_i' = t_i - t_0$. The plot is then based on the t_i' values, which reflect a shifting of the time axis so that the minimum time occurs at the origin. The MLE for the location (minimum lifetime) parameter of the Weibull and lognormal distributions is not well-defined. A simple but biased estimate for t_0 is $t_{(1)}$, the first failure time (minimum) from the ordered sample. Obviously, this is an upper bound on t_0. A trial-and-error procedure is to try different estimates for t_0 (obviously less than t_0), transforming the data, and performing a least-squares fit. The estimate that results in the best fit (perhaps the highest index of fit) would then be used.

[4]Data were generated from a Weibull distribution with $\beta = 1.5$ and $\theta = 400$ and arbitrary censored times.

Another estimate for t_0 is based on the work of Muralidhar, Huamurty, and Zanakis [1992] and applies to *complete data* samples from a Weibull (but not the exponential) or a log-normal distribution. Estimation techniques for the (two-parameter) exponential distribution are discussed in Section 15.4. Assuming that the failure times are in ascending order,

$$\hat{t}_0 = \frac{t_1 t_n - t_j^2}{t_1 + t_n - 2t_j} \tag{15.23}$$

where $j = \lceil np \rceil$, which is np rounded up to the closest integer, n is the sample size, and p represents an empirically determined percentile with $p = 0.50$ (median) for the lognormal distribution and

$$p = 0.8829n^{-0.3437} \tag{15.24}$$

for the Weibull distribution. Muralidhar, Huamurty, and Zanakis have shown that for the exponential distribution the best estimate for t_0 is given by

$$\hat{t}_0 = 2t_1 - t_2 \tag{15.25}$$

The "best" estimate is defined to be a minimum-bias percentile estimator for the location parameter. This estimate would not be the same as the MLE for the location parameter. The MLEs for the Weibull and lognormal location parameters are not available in closed form. It is possible for the estimate of t_0 to be negative. This may happen, for example, if a component experiences shelf life before entering service. On the other hand, quality control, or inspection that "weeds out" infant mortality failure modes, or engineering and material advances may generate a positive t_0.

EXAMPLE 15.16. The following 23 failure times were observed:

101	172	184	274	378	704	1423	2213
2965	5208	5879	6336	6428	6630	7563	10,435
30,138	30,580	38,265	47,413	81,607	158,007	182,958	

A Weibull least-squares fit provided a 0.517 estimate for β and 13,948 for the characteristic life. The index of fit was 0.97. With $\beta \ll 1$, earlier failures than those observed were expected. Therefore a minimum-life estimate was computed from $p = 0.8829(23)^{-0.3437} = 0.30$ and $j = \lceil 0.3(23) \rceil = \lceil 6.9 \rceil = 7$. Therefore $t_1 = 101$, $t_{23} = 182{,}958$, $t_7 = 1423$, and

$$\hat{t}_0 = \frac{(101)(182{,}958) - 1423^2}{101 + 182{,}958 - 2(1423)} = 91.3$$

Transforming the data by setting $t_i' = t_i - 91.3$ and performing a least-squares fit, we find the estimated beta to be 0.45 and the estimated characteristic life to be 13,079. The index of fit increased to 0.99.[5]

Alternative methods for estimating t_0 for the Weibull distribution may be found in Mitchell [1967] and Mann, Schafer, and Singpurwalla [1974].

[5]The data were generated from a Weibull distribution with $\beta = 0.4$, $\theta = 10{,}000$, and $t_0 = 100$.

15.4 CONFIDENCE INTERVALS

In order to determine the precision with which the MLE estimates the distribution parameter, confidence intervals may be constructed. We do not expect that the MLE will exactly equal the parameter. However, a confidence interval provides a range of values among which we have a high degree of confidence that the distribution parameter is included. Confidence intervals for the exponential distribution are easily obtained as shown below.

15.4.1 Confidence Intervals for the Constant Failure Rate Model

Confidence intervals pertaining to the exponential failure distribution are relatively easy to derive compared with those of other failure distributions (see, for example, Ross [1987] for the derivation). A $100(1-\alpha)$ percent confidence interval for the MTTF is given by

$$\text{MTTF}_\text{L} = \frac{2T}{\chi^2_{\alpha/2,k}}$$
$$\text{MTTF}_\text{U} = \frac{2T}{\chi^2_{1-\alpha/2,2r}} \tag{15.26}$$

where
$$k = \begin{cases} 2(r+1) & \text{for Type I testing} \\ 2r & \text{for Type II testing and complete data,} \end{cases}$$

T is the total test time, r is the number of failures, and χ^2 is the chi-square table value based on the degree of confidence desired and k degrees of freedom: $\Pr\{\chi^2 > \chi^2_{\alpha/2,k}\} = \alpha/2$ and $\Pr\{\chi^2 > \chi^2_{1-\alpha/2,2r}\} = 1-\alpha/2$. For Type I testing without replacement of failed units, the above confidence interval is an approximation; otherwise it is exact. A confidence interval for the reliability at a particular point in time, t_0, may then be computed from

$$e^{-t_0/\text{MTTF}_\text{L}} \le R(t_0) \le e^{-t_0/\text{MTTF}_\text{U}} \tag{15.27}$$

A design life, t_R, confidence interval to achieve a reliability of R is found from

$$-\text{MTTF}_\text{L}\ln(R) \le t_R \le -\text{MTTF}_\text{U}\ln(R) \tag{15.28}$$

A one-sided confidence interval may be found by replacing $x^2_{\alpha/2}$ with x^2_α (for a lower bound) in Eq. (15.26). From Eq. (15.26) it is apparent that the accuracy of the estimate of the MTTF depends on the number of failures (that is, degrees of freedom), r, and not on the number under test, n. Recall, however, that n affects the time required to obtain the r failures. It is also possible to compute a one-sided lower-bound confidence interval for the MTTF when no failures are observed. With $r = 0$:

$$\frac{2T}{x^2_{\alpha,2}} \le \text{MTTF} \tag{15.29}$$

EXAMPLE 15.17. Thirty units were placed on test until 20 failures were observed. The following failure times were obtained:

50.1	20.9	31.1	96.5	36.3	99.1	42.6	84.9	6.2	32.0
30.4	87.7	14.2	4.6	2.5	1.8	11.5	84.6	88.6	10.7

A 90 percent confidence interval for the MTTF of the exponential distribution is found from a Type II test with $n = 30$ and $r = 20$:

$$T = 50.1 + 20.9 + \cdots + 10.7 + (30-20)(99.1) = 1827.3$$
$$x^2_{0.975,40} = 24.433 \qquad x^2_{0.025,40} = 59.342$$
$$\text{MTTF}_\text{L} = \frac{2(1827.3)}{59.342} = 61.58$$
$$\text{MTTF}_\text{U} = \frac{2(1827.3)}{24.433} = 149.58$$

For a mission of $t_0 = 10$ hr,

$$0.850 = e^{-10/61.58} \le R(10) \le e^{-10/149.58} = 0.935$$

For a reliability of $R = 0.95$,

$$3.16 = -61.58\ln(0.95) \le t_R \le -149.58\ln(0.95) = 7.67$$

The two-parameter exponential distribution

The reliability function for the exponential distribution having a threshold, or guaranteed lifetime, is

$$R(t) = e^{-(t-t_0)/\text{MTTF}} \qquad t \ge t_0 \tag{15.30}$$

If t_0 is known, one may analyze the distribution as a one-parameter exponential by making the transformation $t_i' = t_i - t_0$ on all failure times. If t_0 is not known, the maximum likelihood estimates are obtained from

$$\hat{t}_0 = t_1$$
$$\widehat{\text{MTTF}} = \frac{\sum_{i=1}^r t_i + (n-r)t_s - nt_1}{r} \tag{15.31}$$

where
$$t_s = \begin{cases} t_* & \text{for Type I testing} \\ t_r & \text{for Type II testing} \end{cases}$$

An unbiased estimator for t_0 is given by

$$\tilde{t}_0 = t_1 - \frac{\widehat{\text{MTTF}}}{n} \tag{15.32}$$

Two-sided confidence intervals may then be found from

$$\text{MTTF}_\text{L} = \frac{2(r-1)\widehat{\text{MTTF}}}{x^2_{\alpha/2,2r-2}} \le \text{MTTF} \le \frac{2(r-1)\widehat{\text{MTTF}}}{x^2_{1-\alpha/2,2r-2}} = \text{MTTF}_\text{U}$$
$$t_{0L} = t_1 - \frac{\widehat{\text{MTTF}}}{n}F_{\alpha,2,2r-2} \le t_0 \le t_1 \tag{15.33}$$

and
$$e^{-(t-t_{0L})/\mathrm{MTTF_L}} \le R(t) \le e^{-(t-t_1)/\mathrm{MTTF_U}}$$
where
$$F_{\alpha,2,2r-2} = (r-1)[\alpha^{-1/(r-1)} - 1]$$

EXAMPLE 15.18. Given the following failure times as a result of 50 items on test with 30 failures generated, compute point and interval estimates for the parameters of the two-parameter exponential distribution.

27.4	34.2	34.6	39.7	40.3	45.0
45.2	47.3	47.8	50.8	53.1	53.4
54.7	54.9	55.6	58.3	58.4	61.8
65.3	67.8	68.5	74.2	75.1	75.2
77.9	78.0	82.4	86.0	99.3	105.4

Solution

$$\hat{t}_0 = 27.4$$

$$\widehat{\mathrm{MTTF}} = \frac{27.4 + 34.2 + \cdots + 105.4 + (50-30)(105.4) - 50(27.4)}{30}$$
$$= \frac{1817.7 + 2108 - 1370}{30} = 85.19$$

Then an unbiased estimate for the minimum life is

$$\tilde{t}_0 = 27.4 - \frac{85.19}{50} = 25.7$$

and 95 percent confidence intervals are given by

$$61.1 = \frac{2(29)(85.19)}{80.9} \le \mathrm{MTTF} \le \frac{2(29)(85.19)}{38.9} = 127.0$$

where
$$\chi^2_{0.025,58} \approx 80.9 \qquad \chi^2_{0.975,58} \approx 38.9$$
and
$$22.0 = 27.4 - \frac{85.19}{50}(3.16) \le t_0 \le 27.4$$
where
$$F_{0.05,2,58} \approx 3.16$$
and
$$e^{-(t-22)/61.6} \le R(t) \le e^{-(t-27.4)/127}$$

15.4.2 Confidence Intervals for Other Distributions

Confidence intervals for the Weibull distribution parameters are mathematically or computationally difficult to obtain and usually require the use of numerical techniques or specialized tables. Methods for constructing these intervals may be found in Lawless [1982], Nelson [1982], or Kececioglu [1993]. Kapur and Lamberson [1977] describe the procedure for constructing nonparametric confidence intervals for the Weibull probability plot and provide 5, 50, and 95 percent rank tables for use in generating the confidence bands. This method requires plotting, for example, the 5 percent and 95 percent ranks along with the median rank (50th percentile) for each failure or repair time. Points not falling on the median plot are projected horizontally to the plotted line, and then the 5th and 95th ranks are plotted below and

above the median line. A nonparametric confidence interval is not dependent on the form of the distribution. Therefore, in principle, this technique can be used with any distribution, although it is usually presented within the context of fitting a Weibull distribution. Both the slope of the fitted line and the sample size will affect the width of the resulting confidence band.

For large sample sizes with complete data, the following approximate confidence intervals [Abernethy et al., 1983] may be computed:

$$\hat{\beta}\exp\left(\frac{-0.78Z_{\alpha/2}}{\sqrt{n}}\right) \le \beta \le \hat{\beta}\exp\left(\frac{0.78z_{\alpha/2}}{\sqrt{n}}\right) \tag{15.34}$$

$$\hat{\theta}\exp\left(\frac{-1.05z_{\alpha/2}}{\hat{\beta}\sqrt{n}}\right) \le \theta \le \hat{\theta}\exp\left(\frac{1.05z_{\alpha/2}}{\hat{\beta}\sqrt{n}}\right) \tag{15.35}$$

where z is the standardized normal deviate, n is the sample size, and $\hat{\beta}$ and $\hat{\theta}$ are maximum likelihood estimates.

EXAMPLE 15.19. From the 15 failure times in Example 15.6 and the MLEs obtained from this data in Example 15.13, the following 90 percent confidence intervals are constructed:

$$1.297 \le \beta \le 2.52 \qquad \text{and} \qquad 124.1 \le \theta \le 203.3$$

where $n = 15$, $z_{0.05} = 1.645$, $\hat{\beta} = 1.807$, and $\hat{\theta} = 158.66$.

Confidence intervals for estimating the parameters of the normal distribution (and therefore the lognormal) under complete data may be found in any introductory statistics text, such as Ross [1987], and will not be discussed here. However, as discussed in Chapter 12, Eq. (12.12) under complete data provides an exact confidence interval for the mean of a normal distribution and an approximate confidence interval for the mean (MTTF or MTTR) of any distribution for large sample sizes (such as 30 or more failure or repair times). Confidence intervals with censored data present are discussed in Lawless [1982] and Nelson [1982]. Engelhardt and Bain [1978] derive confidence intervals for the power-law process.

15.5 PARAMETER ESTIMATION FOR COVARIATE MODELS

Covariate models containing one or more explanatory variables as part of the reliability model were introduced in Chapter 7. In order to apply these models, methods must be available to obtain estimates for the unknown parameters. In general, finding maximum likelihood estimates for these parameters requires solving a system of nonlinear equations. We will illustrate the procedure with the exponential covariate model, in which the failure rate is given by

$$\lambda = \exp\left[\sum_{i=0}^{k} a_i x_i\right]$$

where the x_i are explanatory variables with $x_0 = 1$, and the a_i are unknown parameters to be estimated. Assuming complete data, the likelihood function can be written as

$$L(a_1, \ldots, a_k) = \prod_{j=1}^{n} \exp\left[\sum_{i=0}^{k} a_i x_{ij}\right] \exp\left\{-t_j \exp\left[\sum_{i=0}^{k} a_i x_{ij}\right]\right\}$$

where x_{ij} is the value of the ith covariate associated with the jth failure. Then the log likelihood function is

$$\ln L(a_1, \ldots, a_k) = \sum_{j=1}^{n}\left\{\sum_{i=0}^{k} a_i x_{ij} - t_j \exp\left[-\sum_{i=0}^{k} a_i x_{ij}\right]\right\}$$

The resulting system of nonlinear equations to be solved is as follows:

$$\frac{\partial \ln L}{\partial a_i} = \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} t_j x_{ij} \exp\left[\sum_{i=0}^{k} a_i x_{ij}\right] = 0 \qquad i = 0, 1, \ldots, k$$

Lawless [1982] recommends using the Newton-Raphson method for solving the above equations.

A less efficient but computationally simpler estimator is obtained using least squares. To develop this estimate, the reliability function for the exponential can be written as

$$R(t) = \exp\left\{-t \exp\left[\sum_{i=o}^{k} a_i x_i\right]\right\}$$

or

$$-\ln R(t) = t \exp\left[\sum_{i=o}^{k} a_i x_i\right]$$

Then taking logarithms again,

$$\ln[-\ln R(t)] = \ln t + \sum_{i=o}^{k} a_i x_i$$

Therefore if $y_j = \ln\ln(1/[1 - F(t_j)]) - \ln t_j$ and $(1, x_{1j}, \ldots, x_{kj})$ are paired observations, then multiple regression can be used to find least-square estimates for the a_i's. If there is a single covariate, Eqs. (14.10) and (14.11) may be used.

A similar approach can be used with the Weibull distribution if the logarithm of the characteristic life is a linear function of one or more covariates. The reliability function can be written as

$$R(t) = \exp\left\{-\left(\frac{t}{\exp\left[\sum_{i=0}^{k} a_i x_i\right]}\right)^{\beta}\right\}$$

Then

$$\ln\left[\ln \frac{1}{1 - F(t)}\right] = \beta \ln t - \beta \sum_{i=0}^{k} a_i x_{ij}$$

and the least-squares estimates are obtained by regressing $y_j = \ln\ln(1/[1 - F(t_j)])$ with the corresponding independent variables $(1, t_j, x_{1j}, \ldots, x_{kj})$. The resulting least-squares estimates will be β and βa_i, $i = 0, 1, \ldots, k$.

EXAMPLE 15.20. The following data were obtained from a planned reliability test in which the covariate was a measured load, in volts, placed on an electronic system. Assume that failures are Weibull with the characteristic life a function of the applied load.

Failure time	Covariate	Computed y_j
4.7	160	−3.07
7.5	160	−2.15
10.3	160	−1.64
20.5	160	−1.29
141.6	120	−1.01
166.0	120	−0.772
209.1	120	−0.560
324.1	120	−0.367
551.3	100	−0.184
3124.5	90	0.171
4671.1	100	0.355
5048.7	100	0.555
5220.0	90	0.79
9657.6	90	1.13

From the multiple regression analysis,

$$\hat{\beta} = 0.50 \qquad -\hat{\beta} a_0 = -4.44 \qquad -\hat{\beta} a_1 = 0.00825$$

The index of fit was 0.96. Therefore

$$\hat{\theta}(x) = 8.88 - 0.0165x \qquad R(t) = \exp\left[-\left(\frac{t}{\hat{\theta}(x)}\right)^{0.5}\right]$$

Further discussions on the parameter estimation of covariate models may be found in Gertsbakh [1989] and Leemis [1995]. Parameter estimation in general is addressed in detail in Mann, Schafer, and Singpurwalla [1974], Nelson [1982], and Lawless [1982].

APPENDIX 15A
WEIBULL MAXIMUM LIKELIHOOD ESTIMATOR

Assume Type II censoring with n the number at risk and r the number of failures. Then

$$L(\theta, \beta) = \prod_{i=1}^{r} f(t)R(t_r)^{n-r} = \left[\prod_{i=1}^{r} \frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta-1} e^{-(t_i/\theta)^{\beta}}\right]\left[e^{-(t_r/\theta)^{\beta}}\right]^{n-r}$$

and

$$\ln L = r\ln\beta - \beta r\ln\theta + \sum_{i=1}^{r}(\beta-1)\ln t_i - \sum_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{\beta} - (n-r)\left(\frac{t_r}{\theta}\right)^{\beta}$$

Taking partial derivatives and setting them equal to zero:

$$\frac{\partial \ln L}{\partial\theta} = -\beta r + \frac{\beta}{\theta^{\beta}}\sum_{i=1}^{r} t_i^{\beta} + \frac{(n-r)\beta}{\theta^{\beta}}t_r^{\beta} = 0$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{r}{\beta} + \sum_{i=1}^{r}\ln t_i - r\ln\theta + r\ln\theta - \frac{r\sum_{i=1}^{r} t_i^{\beta} + (n-r)t_r^{\beta}\ln t_r}{\sum_{i=1}^{r} t_i^{\beta} + (n-r)t_r^{\beta}} = 0$$

Rearranging terms in the last equation and eliminating θ by solving the first equation for θ, results in Eq. (15.11). Solving $\partial L/\partial\theta = 0$ for θ results in Eq. (15.12).

APPENDIX 15B
WEIBULL MLE WITH MULTIPLY CENSORED DATA

Let F be the set of failure indices and C be the set of censored indices. Then

$$L(\theta,\beta) = \prod_F f(t_i)\prod_C R(t_i) = \left[\prod_F \frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta-1} e^{-(t_i/\theta)^{\beta}}\right]\prod_C\left[e^{-(t_i/\theta)^{\beta}}\right]$$

and

$$\ln L = \sum_F\left[\ln\beta - \beta\ln\theta + (\beta-1)\ln t_i - \left(\frac{t_i}{\theta}\right)^{\beta}\right] - \sum_C\left(\frac{t_i}{\theta}\right)^{\beta}$$

Taking partial derivatives and setting them equal to zero:

$$\frac{\partial \ln L}{\partial\theta} = \sum_F\left[\frac{-\beta}{\theta} + \frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta}\right] + \sum_C\left[\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta}\right] = 0$$

$$\frac{\partial \ln L}{\partial\beta} = \sum_F \frac{1}{\beta} + \ln\left(\frac{t_i}{\theta}\right) - \left(\frac{t_i}{\theta}\right)^{\beta}\ln\left(\frac{t_i}{\theta}\right) - \sum_C\left(\frac{t_i}{\theta}\right)^{\beta}\ln\left(\frac{t_i}{\theta}\right) = 0$$

Solving the above two equations results in Eq. (15.21) and (15.22).

APPENDIX 15C
MLE FOR NORMAL AND LOGNORMAL DISTRIBUTIONS WITH CENSORED DATA

Derivation of the following algorithm may be found in Lawless [1982] and is based on the work of Sampford and Taylor [1959]. Let t_i be the ith failure or censored time

for the normal distribution and the natural logarithm of the failure or censored time for the lognormal distribution. Let F be the set of indices of the failure times and C be the set of indices of the censored times. Define

$$w_i = \begin{cases} t_i & i \in F \\ \mu + \sigma V(z_i) & i \in C \end{cases}$$

where $z_i = (t_i - \mu)/\sigma$, $V(z) = \phi(z)/Q(z)$, $\phi(z)$ is the standardized normal density function (Eq. (4.22)), and $Q(z)$ is an approximation to the complementary cumulative normal distribution given by

$$Q(z) = \phi(z)(a_1 y + a_2 y^2 + a_3 y^3)$$

with $y = (1 + pz)^{-1}$, $a_1 = 0.4361836$, $a_2 = -0.1201676$, $a_3 = 0.937298$, and $p = 0.33267$.

Given initial values for the mean, μ, and the standard deviation, σ, recursively solve for w_i and

$$\tilde{\mu} = \frac{\sum_{i=1}^{n} w_i}{n} \qquad \text{and} \qquad \tilde{\sigma}^2 = \frac{\sum_{i=1}^{n}(w_i - \tilde{\mu})^2}{r + \sum_{i\in C}\lambda(z_i)}$$

where $\lambda(z) = V(z)[V(z) - z]$. The algorithm may terminate when the differences between successive $\tilde{\mu}$ and $\tilde{\sigma}$ values are less than a specified value. In implementing this algorithm, the author found that $Q(z)$ is more accurate for positive z values than negative values. Therefore when $z < 0$, $Q(|z|)$ is found and $Q(z) = 1 - Q(z)$, taking advantage of the symmetry of the normal distribution.

EXERCISES

Enough problems should be worked without the use of the computer to understand the technique. Computer software can then be used to check results.

For each of the data sets in Exercises 15.1 through 15.4,

(a) Compute descriptive statistics.
(b) Perform a least-squares fit (optionally graph).
(c) Determine the MLEs for the best fit from (b).

15.1 Complete data—failure times in days:

18.7	27.2	33.1	19.1	9.5	14.2	57.9
1.5	164.6	30.8	35.9	26.8	31.3	

15.2 Complete data—failure times in hours:

50.6	58.8	63.6	51.0	38.7	43.1	45.4	79.5	18.3

15.3 Repair times in minutes:

93.3	97.5	101.7	100.1	104.3	98.7
95.0	101.0	104.1	94.2	98.9	98.8

15.4 Complete data—failure times in hours:

68	134	262	205	396	162	90
236	386	79	167	421	171	93
204	330	1018	120	123		

15.5 Determine whether the following complete data came from an exponential distribution:

396	391	401	389	346	343	390	425	360	
380	413	437	481	408	413	414	392	374	433
402	435	362	376	354	434				

15.6 (a) Show that the MLE under Type II testing for the rectangular distribution (Exercise 2.8) is given by

$$\hat{b} = \frac{nt_r}{r}$$

(b) Compute the MLE if the following failure times (in hours) were recorded with 20 units on test: 8, 10, 43, 47, 62, 73, 80, 85, 87, 90.

15.7 The following failure times, in days, were obtained after placing 30 units on a 100-day test bed: 8, 12, 22, 51, 73, 85.
(a) Determine the empirical reliability at 100 days.
(b) Assuming a constant failure rate, find the MLE for the MTTF, and then find $R(100)$.
(c) Using (b), derive a 90 percent confidence interval for the MTTF.
(d) Find $R(100)$ from a least-square estimator for the MTTF.
(e) Find the MLEs for the two-parameter exponential, and then estimate $R(100)$ on the basis of the two-parameter model.

15.8 In an attempt to establish a repair-time probability distribution, an industrial engineer obtained the following times, in hours, to restore a failed machine:

3.0 3.8 0.5 10.0 2.4 1.5 0.6 4.3 3.0 1.2 1.8 2.8

(a) Compute the sample mean and determine a 90 percent confidence interval for the MTTR.
(b) Using least squares (probability plots), determine the best distribution.
(c) On the basis of this distribution, is the maintainability objective of 90 percent of the repairs completed in 3 hr being met?

15.9 Twenty-five units were tested until 15 failures were generated. The following failure times, in hours, were observed:

52 65 66 70 71 73 78 79 80 81 83 93 94 99 100

Compute a two-sided 90 percent confidence interval for the MTTF and the guarantee lifetime for the two-parameter exponential distribution. Estimate the design life if specifications call for a 95 percent reliability.

15.10 Compute an 80 percent confidence interval for the MTTF of a product having a constant failure rate, such that in a Type II test of 100 units, 40 units failed. The estimated MTTF was found to be 120 operating hours. Find an 80 percent confidence interval for the design life on the basis of a 95 percent reliability.

15.11 Derive the MLE for complete data for a Weibull distribution whose shape parameter is known to be 2 (that is, a Rayleigh distribution). Compare your results with Eq. (15.12).

15.12 Analyze the following multiply censored data to determine the best distribution. Twenty-five units were placed on test until the 10th failure was observed. Censored units were removed to meet product demands.

6 8 10^+ 12 18 36 40^+ 45^+ 37 49 55 60^+ 67 78

15.13 For the following (complete) data, compare a two-parameter Weibull distribution with a three-parameter Weibull distribution using least squares. For the three-parameter Weibull, follow the procedure in Example 15.16.

51	60	61	66	67	72	86	94	108	127
142	143	182	193	200	201	204	217	254	374

15.14 Derive the MLE for the discrete Poisson distribution.

15.15 Twelve units were placed on test. The test was terminated after eight failures had been obtained (Type II). In addition, two units had to be removed prior to their failure or the termination of the test. The following failure and censor times were recorded:

3 50^+ 65 157 244 245 341 382 710^+ 945

(a) Find the MLE for the MTTF for the exponential distribution.
(b) Find the MLEs for the Weibull distribution.
(c) Find the MLEs for the lognormal distribution.

15.16 A least-square estimate for the failure time of a hardened drill bit was obtained from fitting 20 failure times (in operating hours) to a (transformed) lognormal distribution. The fitted slope equals 0.8333, and the fitted intercept equals −5.46. Determine the reliability of the drill bit at 50 operating hours.

15.17 Derive the MLEs for q_0 and q_i of Eqs. (14.28) and (14.29) for the Borlaw and Scheuer reliability growth model discussed in Section 14.5. The likelihood function is based on the multinomial probability distribution:

$$L = \prod_{i=1}^{K} \frac{(a_i + b_i + c_i)!}{a_i!b_i!c_i!} q_0^{a_i} q_i^{b_i} (1 - q_0 - q_i)^{c_i}$$

15.18 Given the following failure times, in hours, derive an approximate 95 percent confidence interval for the parameters of the Weibull distribution.

352	334	204	70	107	113	228	129	302	68
202	67	271	120	234	170	100	73	81	146
310	200	203	204	266	239	130	32	396	92

CHAPTER 16

Goodness-of-Fit Tests

The final step in the selection of a theoretical distribution is to perform a statistical test for goodness of fit. Such a test compares a null hypothesis (H_0) with an alternative hypothesis (H_1) having the following form:

H_0: The failure times came from the specified distribution.

H_1: The failure times did not come from the specified distribution.

The test consists of computing a statistic based on the sample of failure times. This statistic is then compared with a critical value obtained from a table of such values. Generally, if the test statistic is less than the critical value, the null hypothesis (H_0) is accepted; otherwise, the alternative hypothesis (H_1) is accepted.

The critical value depends on the level of significance of the test and the sample size. The level of significance is the probability of erroneously rejecting the null hypothesis in favor of the alternative hypothesis. Table 16.1 shows the four possible cases that may occur. Because of the randomness inherent in the sampling process, the test statistic has a probability of exceeding the critical value even though H_0 is true. This results in a Type I error having a probability of occurring equal to the level of significance (α). It is also possible for the test statistic to be less than the critical value even though H_1 is true. This results in a Type II error. It occurs with a probability that is usually controlled indirectly by the specification of the level of significance and the sample size.

There are two types of goodness-of-fit tests: general tests and specific tests. A general test is applicable to fitting more than one theoretical distribution, and a specific test is tailored to a single distribution. When available, specific tests will be more powerful (have a higher probability of correctly rejecting a distribution) than general tests. What follows is a discussion of a general test, the chi-square goodness-of-fit test, and specific tests for the exponential, Weibull, normal, and lognormal

TABLE 16.1
Hypothesis tests

	H_0 true	H_1 true
Accept H_0	Correct decision	Type II error
Accept H_1	Type I error	Correct decision

failure distributions. Tests for identifying and fitting trend data to the nonhomogeneous Poisson process and the U.S. Army Material Systems Analysis Activity (AMSAA) model are then discussed.

16.1 CHI-SQUARE GOODNESS-OF-FIT TEST

This test is applicable to both continuous and discrete distributions and may be used when the parameters of the distribution are estimated from the maximum likelihood estimators (MLEs). The test is valid for large sample sizes only; however, it will accommodate singly censored data. The data must be grouped into classes (cells). The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \tag{16.1}$$

where
k = number of classes
O_i = observed number of failures (or repairs) in the ith class
$E_i = n\,p_i$ = expected number of failures (or repairs) in the ith class
n = total number at risk (sample size)
p_i = probability of a failure occurring in the ith class if H_0 is true

$$= F(a_i) - F(a_{i-1}) = \begin{cases} R(a_{i-1}) - R(a_i) & \text{for fitting failure data} \\ H(a_i) - H(a_{i-1}) & \text{for fitting repair data} \end{cases}$$

and the ith class is defined by $[a_{i-1}, a_i)$ with $a_0 = 0$. The probabilities are based on the distribution stated in the null hypothesis. To ensure an approximate chi-square distribution, $n\,P_i$ should be at least 5 for all i. Therefore the sample size should be large enough to achieve this result, or the classes can be combined as illustrated in Example 16.1. All singly censored times (on the right) should be in the last interval, which includes t_* (if Type I) or t_r (if Type II) and has ∞ as an upper bound. The test statistic, χ^2, has a chi-square distribution with degrees of freedom equal to $k - 1 -$ number of estimated parameters. Critical values are then obtained from Table A.3 in the Appendix on the basis of the number of degrees of freedom and the desired level of significance. Examples using the exponential, Weibull, normal, and lognormal distributions follow.

EXAMPLE 16.1. EXPONENTIAL DISTRIBUTION. The 35 failure times from Example 15.1 are grouped into the six cells as shown:

Cell	Upper bound	Count
1	354	18
2	688	10
3	1022	2
4	1356	2
5	1690	2
6	2026	1

(Cells 3–6 combined: 7)

The MLE for the parameter λ is $\hat{\lambda} = 1/\widehat{MTTF} = 1/485.4 = 0.00206$. Cells 3 through 6 are combined to ensure that the expected cell count will be at least 5. Then the expected cell counts are obtained as follows:

$$E_1 = 35\,P_1 = 35\,[1 - e^{-354/485.4}] = 18.120$$

$$E_2 = 35\,P_2 = 35\,[1 - e^{-688/485.4} - P_1] = 8.396$$

$$E_3 = 35\,P_3 = 35\,[1 - P_1 - P_2] = 8.483$$

The hypotheses are

H_0: Failure times are exponential with $\lambda = 0.00206$.

H_1: Failure times are not exponential with $\lambda = 0.00206$.

The level of significance is $\alpha = 0.10$.

Upper bound	Number observed	Probability	Expected value	$(O - E)^2/E$
354	18	0.5177	18.1204	0.0008
688	10	0.2399	8.3966	0.3062
∞	7	0.2424	8.4830	0.2593
				$\chi^2 = 0.5663$

The number of degrees of freedom is $3 - 1 - 1 = 1$. Since $\chi^2 = 0.5662 < \chi^2_{\text{crit},0.10,1} = 2.71$, we accept H_0.

Alternative approach

A frequently preferred method for defining classes is to set $p_i = 1/k$, in which case $E_i = n/k$. With this approach, the class endpoints must be solved for, and the resulting interval widths will vary. However, the expected number of observations in each class will be constant. For the exponential distribution:

$$F(a_i) = 1 - e^{-\lambda a_i} = \frac{i}{k} \qquad i = 1, 2, \ldots, k-1$$

or

$$a_i = \frac{-\ln(1 - i/k)}{\lambda} \qquad i = 1, 2, \ldots, k-1$$

EXAMPLE 16.1 (CONTINUED). Let $k = 5$. Then $p_i = 0.2$ and $E_i = 7$, and

$$a_i = \frac{-\ln(1 - i/5)}{0.00206}$$

Cell	Lower bound	Upper bound	Number observed	Expected value	$(O - E)^2/E$
1	0.00	108.3	5.00	7.00	0.57
2	108.36	247.8	9.00	7.00	0.57
3	247.8	444.6	9.00	7.00	0.57
4	444.6	780.8	6.00	7.00	0.14
5	780.8	∞	6.00	7.00	0.14
					$\chi^2 = 1.99$

Since $\chi^2_{\text{crit},0.10,3} = 6.25$, we accept H_0.

EXAMPLE 16.2. WEIBULL DISTRIBUTION. The following 35 failure times were observed from among 50 units placed on test. The test was terminated at the 35th failure (Type II censoring). The failures are believed to follow a Weibull distribution.

1.3	7.3	7.8	13.3	13.9
19.4	19.7	22.3	22.8	26.7
29.7	30.2	31.9	32.2	33
36.8	37	41.7	46.7	50.4
51.4	60	61.3	61.4	65.6
65.8	72.6	78.4	100.4	110.6
111.4	118.2	119.4	132.1	139.7

The MLEs were computed using Eqs. (15.11) and (15.12) with $\hat{\beta} = 1.032$ and $\hat{\theta} = 112.9$.

The failure times are then grouped into five classes of width 28 [(139.7 − 1.3)/5 = 27.68]. Therefore $a_1 = 28$, $a_2 = 56$, $a_3 = 84$, $a_4 = 112$, and $a_5 = 140$. The remaining failure times, those after failure time 140, are placed in the sixth class. The expected cell counts are computed in the following manner:

$$E_i = 50\,P_i = 50\left\{\exp\left[-\left(\frac{a_{i-1}}{112.9}\right)^{1.032}\right] - \exp\left[-\left(\frac{a_i}{112.9}\right)^{1.032}\right]\right\} \qquad \text{for } i = 1, 2, 3, 4, 5$$

$$E_6 = 50\,[1 - P_1 - P_2 - P_3 - P_4 - P_5]$$

The hypotheses are

H_0: Failure times are Weibull with $\beta = 1.03$, $\theta = 112.9$.

H_1: Failure times are not Weibull with $\beta = 1.03$, $\theta = 112.9$.

Letting $\alpha = 0.10$ for the level of significance of the test:

Upper bound	Observed	Probability	Expected	$(O - E)^2/E$
28	10	0.2117	10.5838	0.0322
56	11	0.1731	8.6525	0.6369
84	7	0.1369	6.8470	0.0034
112	3	0.1074	5.3710	1.0467
140	4	0.0839	4.1926	0.0088
∞	15	0.2871	14.3530	0.0292
				$\chi^2 = 1.7572$

The number of degrees of freedom is $6 - 1 - 2 = 3$.

Cells 4 and 5 were combined in order to generate expected counts of 5 or greater.

Upper bound	Observed	Probability	Expected	$(O-E)^2/E$
28	10	0.2117	10.5838	0.0322
56	11	0.1731	8.6525	0.6369
84	7	0.1369	6.8470	0.0034
140	7	0.1913	9.5636	0.6872
∞	15	0.2871	14.3530	0.0292
				$\chi^2 = 1.3889$

The number of degrees of freedom is $5 - 1 - 2 = 2$. Since $\chi^2 = 1.3889 < \chi^2_{\text{crit},0.10,2} = 4.61$, H_0 is accepted. Failure times have a Weibull distribution.[1]

EXAMPLE 16.3. NORMAL DISTRIBUTION. Fifty bearings were placed on an accelerated stress test until wearout failure was observed (complete data). It is believed that wearout is normally distributed. Failure times are in (accelerated) operating hours.

278.2	320.2	361.8	346.5	387.7
331.7	295.3	355.4	386.1	287.1
333.7	332.5	391.5	335.2	297.3
346.2	376.4	446.7	313.3	314.8
340.3	273.3	361.6	361.5	389.2
391.2	372.8	336.8	357.6	331.7
342.6	305.7	272.6	359.1	399.9
443.1	375.2	364.7	300.5	359.4
298.8	276.0	339.3	447.5	350.6
397.0	301.8	282.5	357.2	346.5

The sample mean (MLE) is found to be 345.5, and the sample standard deviation (MLE) (Eq. (15.15)) is 43.6. Therefore,

H_0: Failures are normal with $\mu = 345.5$ and $\sigma = 43.6$.

H_1: Failures are not normal with $\mu = 345.5$ and $\sigma = 43.6$.

Expected cell counts may be obtained from

$$E_i = 50\,P_i = 50\left[\Phi\left(\frac{a_i - 345.5}{43.6}\right) - \Phi\left(\frac{a_{i-1} - 345.5}{43.6}\right)\right]$$

Let $\alpha = 0.10$.

The data were placed in ascending order and grouped into seven classes, with the following results.

[1] Data were generated from a Weibull distribution with $\beta = 0.8$ and $\theta = 110$.

272.6	272.8	273.3	276.0	278.2
282.5	287.1	295.3	297.3	298.8
300.5	301.8	305.7	313.3	314.8
320.2	331.7	331.7	332.5	333.7
335.2	336.8	339.3	340.3	342.6
346.2	346.6	346.5	350.6	355.4
357.2	357.6	359.1	359.4	361.5
361.6	361.8	364.7	375.2	376.4
386.1	387.7	389.2	391.2	391.5
397.0	399.9	443.1	446.7	447.5

Upper bound	Observed	Probability	Expected	$(O-E)^2/E$
298	9	0.1378	6.8915	0.6451
322	7	0.1567	7.8370	0.0894
347	12	0.2174	10.8685	0.1178
372	10	0.2171	10.855	0.0673
397	8	0.1519	7.5965	0.0214
422	1	0.0789	3.9470	2.2004
448	3	0.0307	1.5335	1.4024
∞	0	0.0094	0.4700	0.4700
				$\chi^2 = 5.335079$

Combining the last three cells (422, 448, and ∞) resulted in the following:

Upper bound	Observed	Probability	Expected	$(O-E)^2/E$
298	9	0.1378	6.8915	0.6451
322	7	0.1567	7.8370	0.0894
347	12	0.2174	10.8685	0.1178
372	10	0.2171	10.855	0.0673
397	8	0.1519	7.5965	0.0214
∞	4	0.1190	5.9500	0.6391
				$\chi^2 = 1.58$

The number of degrees of freedom is $6-1-2 = 3$. Since $\chi^2 = 1.58 < \chi^2_{\text{crit},0.10,3} = 6.25$, H_0 is accepted. Failure times of the bearings are normally distributed.[2]

EXAMPLE 16.4. LOGNORMAL DISTRIBUTION. Seventy-five repair times (in minutes) were observed for removing and replacing a failed component. Repair times are believed to have a lognormal distribution.

[2] Data were generated from a normal distribution having a mean of 345 and a standard deviation of 50.

78.5	141.3	253.2	204.1	363.6
166.0	99.7	231.4	355.8	89.0
170.6	168.0	383.5	174.4	102.5
203.5	310.4	830.7	128.3	131.1
187.2	73.3	252.2	252.1	371.4
382.1	295.1	178.4	238.6	166.1
193.4	115.3	72.6	243.7	431.0
789.3	305.2	263.6	107.3	244.7
104.8	76.1	184.5	840.2	216.4
414.0	109.2	83.4	237.3	204.4
420.5	231.0	106.8	457.4	189.7
80.6	643.8	122.7	84.6	222.6
92.2	340.7	426.5	215.8	462.9
273.3	103.7	56.2	105.0	559.1
385.0	349.4	215.3	50.4	96.1

Maximum likelihood estimators for t_{med} and s were found to be 199.36 and 0.654, respectively. Therefore expected cell counts are obtained from:

$$E_i = 75\,P_i = 75\left\{\Phi\left[\frac{1}{0.654}\ln\frac{a_i}{199.36}\right] - \Phi\left[\frac{1}{0.654}\ln\frac{a_{i-1}}{199.36}\right]\right\}$$

for $i = 2, 3, \ldots$. For $i = 1$, the last term is set equal to zero. The last expected cell count (k) is computed from $E_k = 75[1 - P_1 - P_2 - \cdots - P_{k-1}]$. Therefore,

H_0: Repair times are lognormal with $t_{\text{med}} = 199.36$ and $s = 0.654$.

H_1: Repair times are not lognormal with $t_{\text{med}} = 199.36$ and $s = 0.654$.

The data were then rank-ordered and placed into four classes each having a width of 100 minutes with the remaining observations in the right-hand tail of the distribution.

50.4	56.2	72.6	73.3	76.1
78.5	80.6	83.4	84.6	89.0
92.2	96.1	99.7	102.5	103.7
104.8	105.0	106.8	107.3	109.2
115.3	122.7	128.3	131.1	141.3
166.0	166.1	168.0	170.6	174.4
178.4	184.5	187.2	189.7	193.4
203.5	204.1	204.4	215.3	215.8
216.4	222.6	231.0	231.4	237.3
238.6	243.7	244.7	252.1	252.2
253.2	263.6	273.3	295.1	305.2
310.4	340.7	349.4	355.8	363.6
371.4	382.1	383.5	385.0	414.0
420.5	426.5	431.0	457.4	462.9
559.1	643.8	789.3	830.7	840.2

Upper bound	Observed	Probability	Expected	$(O - E)^2/E$
100	13	0.1469	11.0123	0.3588
200	22	0.3731	27.9810	1.2785
300	19	0.2125	15.9323	0.5907
400	10	0.1231	9.22950	0.0643
∞	11	0.1446	10.8427	0.0023
				$\chi^2 = 2.29$

The number of degrees of freedom is $5-1-2 = 2$. Since $\chi^2 = 2.29 < \chi^2_{\text{crit},0.10,2} = 4.61$, the null hypothesis is accepted.[3]

16.2 BARTLETT'S TEST FOR THE EXPONENTIAL DISTRIBUTION

A specific test for fitting an exponential distribution is Bartlett's test. The hypotheses are

H_0: Failure times are exponential.

H_1: Failure times are not exponential.

The test statistic is

$$B = \frac{2r[\ln((1/r)\sum_{i=1}^{r} t_i) - (1/r)\sum_{i=1}^{r}\ln t_i]}{1 + (r+1)/(6r)} \tag{16.2}$$

where t_i = time of failure of the ith unit

r = number of failures

The test statistic, B, under the null hypothesis, has a chi-square distribution with $r - 1$ degrees of freedom. In this test, if

$$\chi^2_{1-\alpha/2,r-1} < B < \chi^2_{\alpha/2,r-1}$$

where

$$\Pr\{\chi^2 < \chi^2_{1-\alpha/2,r-1}\} = \Pr\{\chi^2 > \chi^2_{\alpha/2,r-1}\} = \frac{\alpha}{2}$$

then the null hypothesis is accepted; otherwise, the alternative hypothesis is accepted. Observe that the failure rate of the exponential need not be estimated to apply this test. A sample size of 20 or more failures is recommended in using this test in order to achieve an adequate power for the test to reject the exponential when appropriate.

EXAMPLE 16.5. BARTLETT'S TEST. Thirty units were placed on test until 20 failures were observed. The following failure times were obtained in accelerated test hours:[4]

50.1	20.9	31.1	96.5	36.3	99.1	42.6	84.9	6.2	32.0
30.4	87.7	14.2	4.6	2.5	1.8	11.5	84.6	88.6	10.7

[3]Data were generated from a lognormal distribution with $t_{\text{med}} = 200$ and $s = 0.70$.

[4]Data were generated from an exponential distribution having a mean of 75 hr.

A constant failure rate is assumed. Therefore,

H_0: Failures are exponential.

H_1: Failures are not exponential.

Let $\alpha = 0.10$. With $r = 20$,

$$\sum_{i=1}^{20} t_i = 836.3 \qquad \sum_{i=1}^{20} \ln t_i = 63.93848$$

Then
$$B = \frac{2(20)[\ln(836.3/20) - 63.93848/20]}{1 + (20+1)/[6(20)]} = 18.258$$

Since $\chi^2_{0.95,19} = 10.1 < B = 18.258 < \chi^2_{0.05,19} = 30.1$, the null hypothesis is accepted. The MLE for λ is found from $\hat{\lambda} = 20/T = 0.01094$ (or MTTF $= 91.4$), where

$$T = 50.1 + 20.9 + \cdots + 10.7 + (30 - 20)96.5 = 1827.4$$

EXAMPLE 16.6. Bartlett's test was applied to the normally distributed data in Example 16.3 with the following results:

$$B = \frac{100[\ln(17273.6/50) - 291.8577/50]}{1 + (50+1)/[6(50)]} = 0.663$$

where the sum of the failure times is 17273.6 and the sum of the logarithm of the failure times is 291.8577.

We test the following hypotheses with $\alpha = 0.10$:

H_0: Failures are exponential.

H_1: Failures are not exponential.

Since $B = 0.6630 < \chi^2_{0.95,49} = 34.7$, H_0 is rejected and H_1 is accepted. The failures are *not* generated by a constant failure rate process.

16.3 MANN'S TEST FOR THE WEIBULL DISTRIBUTION

A specific test for the Weibull failure distribution is a test developed by Mann, Schafer, and Singpurwalla [1974]. The hypotheses are

H_0: The failure times are Weibull.

H_1: The failure times are not Weibull.

The test statistic is

$$M = \frac{k_1 \sum_{i=k_1+1}^{r-1} [(\ln t_{i+1} - \ln t_i)/M_i]}{k_2 \sum_{i=1}^{k_1} [(\ln t_{i+1} - \ln t_i)/M_i]} \qquad (16.3)$$

where $k_1 = \lfloor \frac{r}{2} \rfloor \qquad k_2 = \lfloor \frac{r-1}{2} \rfloor$

$$M_i = Z_{i+1} - Z_i$$

$$Z_i = \ln\left[-\ln\left(1 - \frac{i - 0.5}{n + 0.25}\right)\right]$$

and $\lfloor x \rfloor$ is the integer portion of the number x. M_i is an approximation. If $M > F_{\text{crit}}$, then H_1 is accepted. Values for F_{crit} may be obtained from tables of the F-distribution if one lets the number of degrees of freedom for the numerator be $2k_2$ and the number of degrees of freedom for the denominator be $2k_1$.

This test is for the two-parameter Weibull distribution. Therefore, if the alternative hypothesis is accepted, the three-parameter Weibull as well as other distributions should be considered. Observe that the data must be rank-ordered for the test statistic to be computed.

EXAMPLE 16.7. MANN'S TEST. We use the failure times in Example 16.2 and we let $\alpha = 0.05$.

t_i	$\ln t_i$	M_i	$\ln t_{i+1} - \ln t_i$	$(\ln t_{i+1} - \ln t_i)/M_i$
1.3	0.2624	1.1087	1.7255	1.5563
7.3	1.9879	0.5211	0.0662	0.1271
7.8	2.0541	0.3469	0.5337	1.5384
13.3	2.5878	0.2620	0.0441	0.1683
13.9	2.6319	0.2115	0.3334	1.5762
19.4	2.9653	0.1781	0.0153	0.0860
19.7	2.9806	0.1544	0.1240	0.8030
22.3	3.1046	0.1367	0.0222	0.1621
22.8	3.1268	0.1229	0.1579	1.2845
26.7	3.2847	0.1120	0.1064	0.9504
29.7	3.3911	0.1032	0.0167	0.1622
30.2	3.4078	0.0959	0.0548	0.5715
31.9	3.4626	0.0897	0.0094	0.1044
32.2	3.4720	0.0845	0.0245	0.2900
33.0	3.4965	0.0800	0.1090	1.3625
36.8	3.6055	0.0762	0.0054	0.0711
37.0	3.6109	0.0728	0.1196	1.6429
41.7	3.7305	0.0699	0.1132	1.6201
46.7	3.8437	0.0674	0.0763	1.1319
50.4	3.9200	0.0651	0.0196	0.3017
51.4	3.9396	0.0632	0.1547	2.4485
60.0	4.0943	0.0615	0.0215	0.3493
61.3	4.1158	0.0600	0.0016	0.0268
61.4	4.1174	0.0587	0.0662	1.1274
65.6	4.1836	0.0576	0.0030	0.0524
65.8	4.1866	0.0566	0.0984	1.7379
72.6	4.2850	0.0558	0.0768	1.3768
78.4	4.3618	0.0552	0.2474	4.4812
100.4	4.6092	0.0547	0.0967	1.7682
110.6	4.7059	0.0543	0.0072	0.1331
111.4	4.7131	0.0541	0.0593	1.0957
118.2	4.7724	0.0541	0.0101	0.1863
119.4	4.7825	0.0542	0.1011	1.8646
132.1	4.8836	0.0545	0.0559	1.0256
139.7	4.9395			

Therefore $n = 50$, $r = 35$, $k_1 = k_2 = 17$, numerator $= 352.3682$, and denominator $= 211.7246$. $M = 1.664$ with 34 degrees of freedom for both the numerator and denominator. Since $M = 1.664 < F_{\text{crit},0.05,34,34}$, H_0 is accepted.

16.4 KOLMOGOROV-SMIRNOV TEST FOR NORMAL AND LOGNORMAL DISTRIBUTIONS

A goodness-of-fit test for use with the normal distribution when the parameters are estimated is a version of the Kolmogorov-Smirnov test developed by H. W. Lilliefors [1967]. It compares the empirical cumulative distribution function with the normal cumulative distribution function. The hypotheses are

H_0: The failure times are normal.

H_1: The failure times are not normal.

The test statistic is $D_n = \max\{D_1, D_2\}$, where

$$D_1 = \max_{1 \le i \le n} \left\{ \Phi\left(\frac{t_i - \bar{t}}{s}\right) - \frac{i-1}{n} \right\} \qquad D_2 = \max_{1 \le i \le n} \left\{ \frac{i}{n} - \Phi\left(\frac{t_i - \bar{t}}{s}\right) \right\}$$

$$\bar{t} = \sum_{i=1}^{n} \frac{t_i}{n} \qquad s^2 = \frac{\sum_{i=1}^{n}(t_i - \bar{t})^2}{n-1} \tag{16.4}$$

If $D_n < D_{\text{crit}}$, then accept H_0; if $D_n \ge D_{\text{crit}}$, then accept H_1. The values for D_{crit} may be found in Table A.7 in the Appendix. This test is appropriate for complete samples only.

EXAMPLE 16.8. KOLMOGOROV-SMIRNOV TEST FOR NORMAL DISTRIBUTION. The following 15 observations represent a sample of the repair times, in hours, of a complex piece of machinery. Test the hypothesis that the repair time is normal.[5]

61.6	70.0	78.4	75.3	83.5
72.3	65.1	77.1	83.2	63.4
72.7	72.5	84.3	73.0	65.5

Rank-ordering the data and computing the MLEs for μ and σ:

61.6	63.4	65.1	65.5	70.0
72.3	72.5	72.7	73.0	75.3
77.1	78.4	83.2	83.5	84.3

Sample mean is $\hat{\mu} = 73.20346$; MLE for the standard deviation is $\hat{\sigma} = 7.041$. We test the following hypotheses:

H_0: Repair time is normal with $\mu = 73.2$ and $\sigma = 7.041$.

H_1: Repair time is not normal with $\mu = 73.2$ and $\sigma = 7.041$.

Set $\alpha = 0.10$.

[5]Data were generated from a normal distribution having $\mu = 75$ and $\sigma = 10$.

$(i-1)/n$	i/n	Cumulative probability	$D_1(i)$	$D_2(i)$
0	0.0667	0.0495	0.0495	0.0172
0.0667	0.1333	0.0823	0.0156	0.0511
0.1333	0.2000	0.1251	−0.0083	0.0749
0.2000	0.2667	0.1379	−0.0621	0.1288
0.2667	0.3333	0.3264	0.0597	0.0070
0.3333	0.4000	0.4483	0.1149	−0.0483
0.4000	0.4667	0.4602	0.0602	0.0065
0.4667	0.5333	0.4721	0.0054	0.0612
0.5333	0.6000	0.4880	−0.0453	0.1120
0.6000	0.6667	0.6179	0.0179	0.0488
0.6667	0.7333	0.7088	0.0422	0.0245
0.7333	0.8000	0.7704	0.0370	0.0297
0.8000	0.8667	0.9222	0.1222	−0.0555
0.8667	0.9333	0.9279	0.0612	0.0055
0.9333	1.0000	0.9430	0.0096	0.0570

$$\max D_1 = 0.1222$$
$$\max D_2 = 0.1288$$
$$\text{Kolmogorov-Smirnov test stat} = 0.1288$$
$$\text{Sample size} = 15$$

Since $D_{15} = 0.1288 < D_{\text{crit},0.10} = 0.201$, H_0 is accepted.

This same test may be used to fit a lognormal distribution. Given failure times $t'_1, t'_2, \ldots, t'_n$, set $t_i = \ln t'_i$ and use Eq. (16.4) with $\bar{t}$ and s the sample mean and sample standard deviation of $t_1, t_2, \ldots, t_n$, respectively.

EXAMPLE 16.9. KOLMOGOROV-SMIRNOV TEST FOR LOGNORMAL DISTRIBUTION. The time to failure of hose assemblies, due to structural fatigue and chemical breakdown, is believed to have a lognormal distribution.The following 25 failure times were obtained from environmental stress testing (complete data).[6]

240.5	511.8	1083.4	821.3	1725.4
629.4	326.9	964.8	1677.8	282.3
652.3	639.2	1847.8	670.8	338.8
818.1	1407.5	4991.0	452.0	464.9
734.9	220.2	1078.1	1077.3	1773.0

Maximum likelihood estimators were found to be

$$\hat{t}_{\text{med}} = 765.426 \qquad \hat{s} = 0.725$$

We test the following hypotheses:

H_0: Failure times are lognormal with $t_{\text{med}} = 765.426$ and $s = 0.725$.

H_1: Failure times are not lognormal with $t_{\text{med}} = 765.426$ and $s = 0.725$.

Let $\alpha = 0.10$.

[6]Data were generated from a lognormal distribution having $t_{\text{MED}} = 800$ and $s = 0.90$.

$(i-1)/n$	i/n	Cumulative probability	$D_1(i)$	$D_2(i)$
0	0.04	0.0427	0.0427	−0.0027
0.04	0.08	0.0548	−0.0148	0.0252
0.08	0.12	0.0838	0.0038	0.0362
0.12	0.16	0.1210	0.0010	0.0390
0.16	0.20	0.1314	0.0286	0.0686
0.20	0.24	0.2327	0.0327	0.0073
0.24	0.28	0.2451	0.0051	0.0349
0.28	0.32	0.2877	0.0077	0.0323
0.32	0.36	0.3936	0.0736	−0.0336
0.36	0.40	0.4013	0.0413	−0.0013
0.40	0.44	0.4129	0.0129	0.0271
0.44	0.48	0.4286	−0.0114	0.0514
0.48	0.52	0.4761	−0.0039	0.0439
0.52	0.56	0.5359	0.0159	0.0241
0.56	0.60	0.5398	−0.0202	0.0602
0.60	0.64	0.6255	0.0255	0.0145
0.64	0.68	0.6808	0.0408	−0.0008
0.68	0.72	0.6808	0.0008	0.0392
0.72	0.76	0.6844	−0.0356	0.0756
0.76	0.80	0.7996	0.0395	0.0005
0.80	0.84	0.8599	0.0599	−0.0199
0.84	0.88	0.8686	0.0286	−0.0114
0.88	0.92	0.8770	0.0030	0.0430
0.92	0.96	0.8869	+0.0331	0.0731
0.96	1	0.9952	0.0352	0.0048

$$\max D_1 = 0.0736$$
$$\max D_2 = 0.0756$$
$$\text{Kolmogorov-Smirnov test statistic} = 0.0756$$
$$\text{Sample size} = 25$$

Since $D_{25} = 0.0756 < D_{\text{crit},0.10} = 0.165$, H_0 is accepted. Failure times are lognormal.

16.5 TESTS FOR THE POWER-LAW PROCESS MODEL

Both the minimal repair process, discussed in Chapter 9, and the AMSAA model, discussed in Chapter 14, assume a nonhomogeneous Poisson process based on the power-law formula. This model allows for the application of statistical techniques such as the calculation of the maximum likelihood estimators (MLE), confidence intervals, and the Cramer–von Mises test statistic for goodness of fit. To determine whether the nonhomogeneous Poisson process is a more appropriate model than the constant failure rate model (homogeneous Poisson process), a trend test on the failure times may be performed. For the intensity function, $\rho(t) = abt^{b-1}$, the hypotheses tested are

H_0: The intensity function is constant ($b = 1$).

H_1: The intensity function is not constant ($b \neq 1$).

If the intensity function is not constant, then either an increasing trend (deteriorating), $b > 1$, or decreasing trend (growth), $b < 1$, is present and the nonhomogeneous Poisson process should be considered. The test statistic is computed from

$$\chi^2 = \frac{2n}{\hat{b}} \tag{16.5}$$

where n is the number of failures and the denominator is the MLE for the growth or deterioration rate, b. The MLE for b is given by Eq. (14.21) for time-terminated data (for example, Type I testing) and by Eq. (14.25) for failure-terminated data (for example, Type II testing). Under the null hypothesis, the test statistic, χ^2, has a chi-square distribution with $2n$ degrees of freedom (df) for Type I testing and $2(n-1)$ df for Type II testing. Therefore the null hypothesis is rejected if $\chi^2 < \chi^2_{\text{crit},1-\alpha/2}$ or $\chi^2 > \chi^2_{\text{crit},\alpha/2}$. If the MLE of the slope parameter is less than 1, growth is present in the data, and if the MLE of the slope parameter is greater than 1, system deterioration is observed.

EXAMPLE 16.10. For the growth data of Example 14.4 (Type II testing), $n = 15$ and $b = 0.28685$. Therefore, $\chi^2 = 30/0.28685 = 104.58$ has a chi-square distribution with 28 degrees of freedom. The critical values are $\chi^2_{\text{crit},0.95} = 16.9$ and $\chi^2_{\text{crit},0.05} = 41.3$. Since $\chi^2 = 104.58 > 41.3$ a significant trend is present. Since the estimate for b is less than 1, there is significant growth.

Goodness-of-fit test

If a trend is present, the power-law intensity function may be a suitable model. To perform the goodness-of-fit test for this model, the following hypotheses are stated:

H_0: A nonhomogeneous Poisson process with intensity abt^{b-1} describes the data.

H_1: The above process does not describe the data.

First, an unbiased estimator for the parameter, b, is obtained from

$$\bar{b} = \begin{cases} \dfrac{n-1}{n}\hat{b} & \text{for Type I testing} \\[2ex] \dfrac{n-2}{n}\hat{b} & \text{for Type II testing or complete data} \end{cases} \tag{16.6}$$

where n is the number of failures and $\hat{b}$ is computed by Eq. (14.21) for Type I testing (time-terminated) and Eq. (14.25) for Type II testing (failure-terminated). The Cramer–von Mises goodness-of-fit test statistic is then computed from

$$C_M = \frac{1}{12M} + \sum_{i=1}^{M}\left[\left(\frac{t_i}{t_k}\right)^{\bar{b}} - \frac{2i-1}{2M}\right]^2 \tag{16.7}$$

$$\text{where } M = \begin{cases} n & \text{for time-terminated data} \\ n-1 & \text{for failure-terminated data} \end{cases}$$

$$t_k = \begin{cases} T & \text{for time-terminated data} \\ t_n & \text{for failure-terminated data} \end{cases}$$

and T is the total cumulative test time or total system operating time in the case of a reparable system. Critical values are found in Table A.8 in the Appendix.

EXAMPLE 16.11. For the growth data provided in Example 14.3, the statistic, C_M, is computed (Type I testing) as follows:

$$M = 10$$

$$\bar{b} = \tfrac{9}{10}(0.615268) = 0.5537$$

$$C_M = \frac{1}{12(10)} + \left[\left(\frac{5.6}{500}\right)^{0.5537} - \frac{2-1}{2(10)}\right]^2 + \left[\left(\frac{18.8}{500}\right)^{0.5537} - \frac{4-1}{2(10)}\right]^2$$

$$+ \cdots + \left[\left(\frac{456.6}{500}\right)^{0.5537} - \frac{20-1}{2(10)}\right]^2 = 0.01218$$

For $\alpha = 0.10$, the critical value is given in Table A.8 as 0.167. Since $C_M < 0.167$, H_0 is accepted.

EXAMPLE 16.12. For the failure times of Example 14.4, the test statistic is computed (Type II test) as follows:

$$M = 14$$

$$\bar{b} = \tfrac{13}{15}(0.28685) = 0.2486$$

$$C_M = \frac{1}{12(14)} + \left[\left(\frac{3}{12035}\right)^{0.2486} - \frac{2-1}{2(14)}\right]^2$$

$$+ \cdots + \left[\left(\frac{8423}{12035}\right)^{0.2486} - \frac{28-1}{2(14)}\right]^2 = 0.12714$$

For a level of significance $\alpha = 0.10$, the critical value is 0.169. Since $C_M < 0.169$, accept H_0.

EXAMPLE 16.13. The following failure times, in working days, were recorded on a numerical control machine that has been operating for 916 days: 211, 287, 345, 456, 567, 631, 705, 784, 817, 856, 893, 916. It is felt that failures are increasing as the machine ages. Assuming failure-terminated data, the maximum likelihood estimators were computed for the power-law process (nonhomogeneous Poisson process) with the following results:

$$\hat{a} = 8.51 \times 10^{-6} \qquad \hat{b} = 2.076 \qquad \rho(t) = 1.767 \times 10^{-5} t^{1.076}$$

The chi-square statistic for the trend test is $\chi^2 = 24/2.076 = 11.56 < \chi^2_{\mathrm{crit},0.95} = 12.3$ on the basis of 22 degrees of freedom. Therefore the hypothesis of a significant trend is accepted. Since the estimate of b is greater than 1, the machine is deteriorating. The goodness-of-fit test provided $C_M = 0.0239 < 0.214$, the critical value at the 5 percent level of significance. As a result, the computed intensity function was accepted. After four years of use (approximately 1000 working days), the MTBF of the machine is estimated to be

$$\mathrm{MTBF} = \frac{1}{\rho(1000)} = \frac{1}{1.767 \times 10^{-5}(1000)^{1.076}} = 33.5 \text{ days}$$

16.6 ON FITTING DISTRIBUTIONS

The goodness-of-fit tests described are not the only ones available, but they are among the more popular tests. The chi-square test, because it is an approximate and general test having less power to correctly reject a distribution, will require a larger sample size than the specific tests to maintain the same Type I and Type II error rates. It also suffers from a somewhat arbitrary specification of the classes regardless of whether the class boundaries are stated or the number of classes having equal probability is specified. It is therefore possible to reject a distribution under one specification and accept it under another. It is for these reasons that a specific test is included for each of the distributions. The specific tests are preferred over the chi-square test when the sample size is small. For very small sample sizes the power of the specific tests will also be small. If this is the case, a probability plot or least-square curve fit will serve as well.

Philosophically, one should hypothesize a distribution, collect the data, apply a goodness-of-fit test, and either reject or not reject the null hypothesis. A distinction is made between accepting the null hypothesis and not rejecting the null hypothesis. In the latter case we are saying that the evidence is insufficient to reject the hypothesized distribution, and as a result, we may continue to utilize the distribution until such time as the evidence is to the contrary. We are unable to prove that the data came from the hypothesized distribution. In fact, since the reliability models themselves are approximations to failure processes, it is very likely that the data did not come from the *exact* distribution hypothesized and, for large sample sizes, even slight differences will result in rejection. As a result, these tests allow us to reject unacceptable models while maintaining those that are reasonably correct.

If a second distribution is hypothesized, a second sample should be obtained and tested. It is only in this way that independence among tests and therefore control of the Type I or Type II error rates can be properly maintained. In practice, however, this is not done, primarily because of the time, cost, or difficulty in obtaining additional sample data. Instead, for a single sample, several distributions will be hypothesized and tested. It is often the case that more than one distribution will have an acceptable least-squares fit or will pass a goodness-of-fit test. For example, both the Weibull and lognormal distributions can take on similar shapes, and often data that fit one will also fit the other. If the exponential is a good fit, then a Weibull having a β value close to 1 may also pass a goodness-of-fit test. The engineer must select the best distribution from among the acceptable distributions. Several criteria have been used to identify the best fit. For example, the distribution having the best index of fit from a least-squares analysis or the distribution having the smallest chi-square value in a chi-square goodness-of-fit test may be selected. A generalized Kolmogorov-Smirnov statistic (that is, the maximum deviation between the empirical cumulative distribution and the hypothesized cumulative distribution) has also been used. Some commercial software packages available for fitting distributions will rank-order them on the basis of these computed statistics.

In those cases in which the data do not adequately fit any of the distributions tested, an empirical reliability distribution can be used. The reader is also advised to consider other theoretical distributions, such as the gamma, beta, and extreme value distributions, the Pearson family of distributions, and possibly a bimodal distribution.

Further readings

Many introductory statistic texts include discussions on goodness-of-fit tests. For example, Ross [1987] introduces the chi-square goodness-of-fit test. A text on simulation by Law and Kelton [1991] provides a comprehensive discussion on fitting distributions that includes probability plots, parameter estimation, the chi-square test, and the Kolmogorov-Smirnov test. Unfortunately, these references address only the case in which complete data are present. Kapur and Lamberson [1977] provide detailed discussions concerning fitting the exponential and Weibull distributions, including probability plots and the Bartlett and Mann tests. Lawless [1982] includes a chapter on goodness-of-fit testing that addresses censored data and the nonhomogeneous Poisson process. Leemis [1995] discusses the use of the Kolmogorov-Smirnov test, including the case in which right-censored data are present. Kececioglu [1993] has detailed chapters on the chi-square, Kolmogorov-Smirnov and Cramer–von Mises tests.

EXERCISES

For exercises 16.1 through 16.4,

(a) Compute the MLE parameters.
(b) Determine the goodness of fit (chi-square and other appropriate tests) for the indicated distributions.

16.1 **Normal distribution.** The following are complete data representing the number of miles driven before the oil of an automotive engine showed measurable contamination.

1033	1035	1056	1097	1106	1117	1126
1145	1152	1161	1164	1177	1182	1193
1195	1199	1202	1204	1206	1241	1222
1223	1225	1256	1263	1265	1273	1273
1280	1282	1303	1308	1322	1359	1365

16.2 **Lognormal distribution.** The following are complete data representing the number of operating hours to failure of a small appliance motor.

1174.9	1372.7	2003.0	1212.9	2031.7
2085.0	1887.2	2446.0	1593.2	1717.4
2084.4	2295.7	1285.2	2088.6	1842.9
2495.3	3626.0	1739.0	1889.8	1449.9
2528.3	1523.2	1726.3	2470.9	1169.6
2242.7	1538.7	2532.7	1716.7	2051.9
1775.2	1815.5	1756.6	1355.0	2673.9
3540.9				

16.3 **Weibull distribution.** The following are complete data representing the number of days until the first failure of the Gord Motor Company's new automobile, the Stallion.

200.6	341.7	119.7	118.9	370.6
693.7	837.5	174.1	544.0	9.4
521.8	133.7	211.7	38.4	1053.3
387.5	297.4	122.3	1060.9	196.9
212.1	418.9	61.1	323.3	229.5
161.2	112.4	80.0	321.5	171.3
555.2	400.4	58.9	244.1	200.5
89.4	113.7	91.0	133.2	80.8

16.4 **Exponential distribution.** The following are complete data representing the number of minutes required to perform corrective maintenance on a jet engine.

200.6	341.7	119.7	118.9	370.6
693.7	837.5	174.1	544.0	9.4
521.8	133.7	211.7	38.4	1053.3
387.5	297.4	122.3	1060.9	196.9
212.1	418.9	61.1	323.3	229.5
161.2	112.4	80.0	321.5	171.3
555.2	400.4	58.9	244.1	200.5
89.4	113.7	91.0	133.2	80.8

16.5 Seventy temperature sensors were placed on test for a period of 300 hr, and the following failure times (in hours) were recorded:

59	68	83	119	126	127
132	152	162	197	221	225
248	249	280	289	290	291

Determine an appropriate reliability model.

16.6 Forty-eight flashlamps used in laser pumping have been operational for some time, and the following failure times, measured in 10^5 shots, or cycles, have been observed. Determine an appropriate reliability model. Assume that the usage rate is the same for all the flashlamps.

92	106	110	116	128
138	138	140	150	155
156	156	166	166	168
171	176	177	182	187

16.7 Fifty compressors were tested for 20 days, and the following failure times (grouped) were recorded:

Interval	Number of failures
$0 \le x < 50$	9
$50 \le x < 75$	11
$75 \le x < 100$	9
$100 \le x < 150$	12

Fit a Weibull distribution using the chi-square goodness-of-fit test in which the MLEs are $\beta = 2$ and $\theta = 100$. Test at the 10 percent level.

16.8 Ten units were tested with the following five failures (time in days) recorded after 20 days of testing (Type I testing): 3.8, 12.9, 13.3, 15.9, 17.3. Fit a Weibull distribution by conducting Mann's goodness-of-fit test at the 10 percent level.

16.9 Derive a general formula for finding the ith class $[a_i - 1, a_i)$ in a chi-square goodness-of-fit test for the Weibull distribution using the alternative approach, presented in the second part of Example 16.1, in which k intervals are desired. What are the class boundaries for testing the hypothesis H_0: Times are from a Weibull distribution having a shape parameter of 2 and a characteristic life of 1000? The data are to be fitted into 5 classes.

16.10 Perform a chi-square goodness-of-fit test at the 10 percent significance level on the following grouped data to fit a rectangular failure distribution with $b = 1000$ hr.

Interval	Number of failures
0–200	16
200–400	18
400–600	25
600–800	23
800–1000	24

16.11 Perform Bartlett's goodness-of-fit test at the 10 percent level on the following failure data: 8, 12, 22, 51, 73, 85.

16.12 Apply the AMSAA goodness-of-fit test to the data found in Exercise 14.3.

16.13 Apply an appropriate goodness-of-fit test to the repair data given in problem 15.8.

16.14 The Major Motors Corporation is concerned with early wearout of brake linings on its new sports car. A new type of material is being tested that is believed to increase the design life to 40,000 miles for at least 95 percent of the linings. Ten brake linings constructed of the new material undergo an accelerated durability test that simulates customer usage. The following wearout times (in miles) were recorded:

50,395	35,326	48,211	39,015	42,202
51,845	36,403	39,611	47,075	45,913

Determine a best fit reliability model. On the basis of the fitted model, decide whether the desired design life is being met.

PART III

Application

CHAPTER 17

Reliability Estimation and Application

Through the following examples the analysis of failure or repair data and the application of appropriate reliability and maintainability models are illustrated. The process begins with the collection of failure or repair data as a result of a reliability testing program or a field reporting system. Data collection is addressed in the following chapter. Initial analysis may include histograms, sample statistics, empirical reliability functions, and probability plots (or least-squares curve fitting) to identify candidate reliability models. The determination of the distribution parameters followed by goodness-of-fit tests will establish the best reliability or maintainability model. From the selected model, various design alternatives, such as redundancy, preventive maintenance, screening or burn-in testing, reliability or maintainability allocation, or economic analysis, can be explored. The final analysis should determine the suitability of the end item reliability or maintainability and should suggest options for improvement, if necessary. It is in this chapter that we attempt to tie together in a meaningful way the mathematical models established in Part I with the statistical analysis of failure and repair data described in Part II. Practical considerations for implementing a reliability and maintainability program that will support the analysis illustrated here are discussed in the remaining chapter.

17.1 CASE 1: REDUNDANCY

In a large electronics company that manufactures small electrical parts such as resistors, capacitors, transistors, and inductors, a new component is experiencing a high failure rate. In order to meet government contract specifications, the function being performed by this component must have a 90 percent or better reliability over a 4-hour mission in a high-stress environment. Because of the cost and time required to redesign the component, design engineers are considering adding redundancy to achieve the desired reliability.

In order to determine the reliability, 75 units were placed on test in a high-stress environment. The test was terminated at the 50th failure. The following failure times (in hours) were recorded:

0.4	0.8	0.8	1.9	2.0
2.2	2.4	2.7	3.1	3.2
3.6	3.9	4.0	4.0	4.3
5.7	6.0	6.3	6.5	6.8
8.3	10.1	11.1	11.4	11.5
11.7	11.8	12.4	12.7	13.1
15.0	15.4	17.6	17.8	18.3
18.7	18.9	19.4	19.6	19.8
21.0	21.5	21.6	22.2	22.8
24.1	25.1	25.6	25.8	26.0

A least-squares fit of the exponential distribution ($F_i = (i - 0.3)/75.4$) was performed, yielding the following:

Slope: = 0.0375
Estimated MTTF: 26.7
Index of fit: 0.992

The index of fit (0.992) indicates a strong linear fit, thus supporting further analysis to fit an exponential distribution.[1]

In order to perform a chi-square goodness-of-fit test, the maximum likelihood estimator (MLE) was computed from the sample of failure times:

$$T = 0.4 + 0.8 + 0.8 + \cdots + 25.8 + 26.0 + (75 - 50)26.0 = 1250.9$$

$$\hat{\lambda} = \frac{50}{1250.9} = 0.04 \qquad \text{or} \qquad \widehat{\text{MTTF}} = 25$$

The chi-square goodness-of-fit test was then conducted with $\alpha = 0.10$. The hypotheses were

H_0: Failure times are exponential with MTTF = 25 hr.
H_1: Failure times are not exponential with MTTF = 25 hr.

For the chi-square test the data were grouped (somewhat arbitrarily) into the following intervals:

Interval	Frequency
0–5	15
5–10	6
10–15	9
15–20	10
20–26	10

[1] Data were generated from an exponential population with a mean of 25 hr.

The chi-square statistic for fitting the exponential with a failure rate of 0.04 or an MTTF of 25 was computed as follows:

Upper bound	Observed	Probability	Expected	$(O-E)^2/E$
5	15	0.1813	13.5952	0.1452
10	6	0.1484	11.1308	2.3651
15	9	0.1215	9.1131	0.0014
20	10	0.0995	7.4612	0.8639
26	10	0.0959	7.1906	1.0976
∞	25	0.3535	26.5091	0.0859
				4.5591

Since $4.559 < \chi^2_{0.10,4} = 7.779$, the null hypothesis is accepted. Other distributions could also be explored at this point to see whether a better fit (chi-square statistic) could be obtained. However, on the basis of the sample data, there is insufficient evidence for us to reject the exponential distribution. Therefore, we will proceed on the basis of the results of this test.

Once the distribution has been accepted as exponential with a mean of 25 hr, confidence intervals may be constructed in order to determine the accuracy by which the mean time to failure has been estimated.

a. Ninety percent confidence interval for the mean:

$$20.1 \text{ hr} = \frac{2(1250.9)}{124.3} < \text{MTTF} < \frac{2(1250.9)}{77.9} = 32.1 \text{ hr}$$

where $\chi^2_{0.95,100} = 77.9$ and $\chi^2_{0.05,100} = 124.3$.

b. Ninety percent confidence for the reliability:

$$e^{-t/20.1} < R(t) < e^{-t/32.1}$$

$$0.82 = e^{-4/20.1} < R(4) < e^{-4/32.1} = 0.88$$

On the basis of the computed confidence interval, the 90 percent reliability for a 4-hr mission has not been achieved, so a design modification must be considered in order to meet the objective. As discussed earlier, redundancy was considered to be the best design option. Two identical redundant units provide the following reliability, which will allow the contract specifications to be met:

$$R_s(t) = 2e^{-0.04t} - e^{-0.08t}$$

$$R_s(4) = 0.978$$

17.2 CASE 2: BURN-IN TESTING

A new fuel pump is experiencing manufacturing problems resulting in high infant mortality. The company has a 25-operating-hour warranty program that because of the high infant mortality rate is costing the company a considerable amount of money. Each failure under the warranty program costs the company approximately \$1200 in replacement costs, administrative costs, and an estimated cost for the loss of goodwill. The company would like to implement a burn-in program in order to eliminate marginal pumps before they are sold under warranty. However, it is not certain whether a burn-in program will be effective and, if so, how many hours of testing would be necessary to implement the program. To answer these questions, the company first needed to establish an appropriate reliability model.

Twenty-five fuel pumps selected randomly from the production line were placed on test. The test ran for 500 operating hours (Type I test). The following failure times were obtained:

0.005	0.1	0.2	1.0
1.0	2.3	9.3	10.1
14.2	16.4	29.7	155.2
172.6	393.1	442.8	445.0

A decreasing failure rate was assumed, and the Weibull distribution was selected for analysis. A least-squares fit to the Weibull was performed:

Failure time, t_i	$F(t_i)$	$\ln \ln[1/(1 - F(t_i))]$
0.005	0.0276	−3.5775
0.1	0.0669	−2.6697
0.2	0.1063	−2.1858
1.0	0.1457	−1.8487
1.0	0.1850	−1.5866
2.3	0.2244	−1.3699
9.3	0.2638	−1.1834
10.1	0.3031	−1.0184
14.2	0.3425	−0.8691
16.4	0.3819	−0.7317
29.7	0.4213	−0.6035
155.2	0.4606	−0.4823
172.6	0.5000	−0.3665
393.1	0.5394	−0.2547
442.8	0.5787	−0.1456
445.0	0.6181	−0.0381

Intercept: $a = -1.7948$
Slope: $b = 0.2933$
Estimated β: $\hat{\beta} = 0.2933$
Estimated θ: $\hat{\theta} = 454.94$
Index of fit: 0.986

The high index of fit (0.9860) supports the use of the Weibull.

The Mann goodness-of-fit test was run to statistically test the hypotheses

H_0: Failure times are Weibull.

H_1: Failure times are not Weibull.

The results are as follows:

t_i	$\ln t_i$	M_i	$\ln(t_i + 1) - \ln(t_i)$	$[\ln(t_i + 1) - \ln(t_i)]/M_i$
0.005	−5.2983	1.1191	2.9957	2.6769
0.1	−2.3026	0.5320	0.6932	1.3029
0.2	−1.6094	0.3585	1.6094	4.4893
1	0	0.2742	0	0
1	0	0.2245	0.8329	3.7101
2.3	0.8329	0.1919	1.3971	7.2804
9.3	2.2300	0.169	0.0825	0.4884
10.1	2.3125	0.1523	0.3407	2.2373
14.2	2.6532	0.1397	0.1441	1.0314
16.4	2.7973	0.1301	0.5938	4.5645
29.7	3.3911	0.1227	1.6536	13.4769
155.2	5.0447	0.1171	0.1063	0.9076
172.6	5.1510	0.1129	0.8231	7.2902
393.1	5.9741	0.11	0.1190	1.0820
442.8	6.0931	0.1083	0.0050	0.0459
445.0	6.0981			

Number on test: 25
Number of failures: 16
Value of k: 9
Numerator of test statistic: 227.1293
Denominator of test statistic: 155.3101
Mann's test statistic: $M = 1.462$
Numerator degrees of freedom for F_{crit}: 14
Denominator degrees of freedom for F_{crit}: 16

Since $M = 1.462 < F_{0.05,14,16} = 2.37$, the null hypothesis could not be rejected.

The parameters for the fitted Weibull were then determined by use of the maximum likelihood estimators (Type I data):

Sample size: 25
Number of failures: 16
$\hat{\beta} = 0.31$
$\hat{\theta} = 461$

Therefore

$$R(t) = e^{-(t/461)^{0.31}}$$

The probability of a failure over the 25 hr warranty period is found from the following:

$$1 - R(25) = 1 - e^{-(t/461)^{0.31}} = 0.333$$

With a 5-hr burn-in period,

$$R(25 \mid 5) = \frac{R(30)}{R(5)} = \frac{e^{(-30/461)^{0.31}}}{e^{-(5/461)^{0.31}}} = 0.833$$

or the probability of failure is reduced by half to $1 - 0.833 = 0.167$. Therefore it appears that a burn-in program could significantly reduce the number of failures observed under the warranty program.

To determine the optimal length of time for burn-in testing, an economic analysis was performed. The company determined that the cost of operating the program was $50 per unit tested per hour and that a fuel pump that failed during testing would incur its replacement cost of $350. The expected cost model over the warranty life of the fuel pump is given by Eq. (13.8):

$$E(C) = 50T + 350[1 - e^{-(T/461)^{0.31}}] + 1200[e^{-(T/461)^{0.31}} - e^{-[(T+25)/461]^{0.31}}]$$

A direct search of the above equation at 0.1-hr intervals resulted in the following:

T	0	0.1	0.2	0.3	0.4	**0.5**	0.6	0.7	0.8	0.9	1.0
$E(C)$	399.7	345.2	336.9	332.9	331.0	**330.3**	330.4	331.1	332.3	333.9	335.8

Therefore the minimum-cost solution occurs at 0.5 indicating that a 30-minute burn-in test would be sufficient. Observe that $1 - R(25 \mid T = 0.5) = 0.25$, which is not as good an improvement as the 5-hr burn-in period. However, given the cost data, it is the least-cost solution—a savings of almost $70 per unit over the alternative of not conducting a burn-in program.

17.3 CASE 3: PREVENTIVE MAINTENANCE ANALYSIS

A new turbine steam generator requires a 0.98 reliability over a 1000-cycle period. A cycle occurs whenever the superheated steam reaches a temperature in excess of 700°C. The first 40 generators had failures occurring at the following times (cycles):[2]

347	396	433	513	624
673	1008	1035	1055	1066
1162	1266	1298	1361	1367
1549	1561	1576	1708	1834
1840	2497	2554	2656	2666
2686	3261	3278	3281	3338
3421	4238	4242	4481	4845
5022	5744	6013	6238	13446

A preventive maintenance program is being considered, if necessary, to achieve the reliability objective. Preventive maintenance will replace the turbine, thus restoring the generator to as good as new condition.

To determine whether the reliability specifications are being met, the failure data were analyzed. Since failures occur from increased heat stress due to boiler expansion and contraction, it was felt that an increasing hazard rate would be observed. Therefore, a Weibull distribution was tested using the Mann test with the hypotheses

H_0: Failure times are Weibull.

H_1: Failure times are not Weibull.

The level of significance was chosen to be $\alpha = 0.05$. The results of Mann's test were as follows:

[2]Failure times are from a lognormal distribution with a median value of 2000 and $s = 0.836$.

t_i	$\ln t_i$	M_i	$\ln(t_i + 1) - \ln t_i$	$\ln(t_i + 1) - \ln t_i/M_i$
347	5.8493	1.1113	0.1321	0.1189
396	5.9814	0.5238	0.0893	0.1706
433	6.0707	0.3497	0.1696	0.4849
513	6.2403	0.2649	0.1959	0.7393
624	6.4362	0.2145	0.0755	0.3522
673	6.5117	0.1813	0.4040	2.2285
1008	6.9157	0.1577	0.0265	0.1678
1035	6.9422	0.1401	0.0191	0.1363
1055	6.9613	0.1266	0.0104	0.0819
1066	6.9717	0.1158	0.0862	0.7444
1162	7.0579	0.1072	0.0857	0.7996
1266	7.1436	0.1000	0.0250	0.2498
1298	7.1686	0.0941	0.0474	0.5035
1361	7.216	0.0892	0.0044	0.0490
1367	7.2204	0.0850	0.1250	1.4702
1549	7.3454	0.0814	0.0077	0.0944
1561	7.3531	0.0784	0.0095	0.1218
1576	7.3626	0.0758	0.0805	1.0617
1708	7.4431	0.0737	0.0712	0.9655
1834	7.5143	0.0719	0.0032	0.0448
1840	7.5175	0.0704	0.3053	4.3373
2497	7.8228	0.0693	0.0226	0.3263
2554	7.8454	0.0684	0.0392	0.5728
2656	7.8846	0.0679	0.0037	0.0550
2666	7.8883	0.0676	0.0075	0.1111
2686	7.8958	0.0676	0.1940	2.8697
3261	8.0898	0.0679	0.0052	0.0764
3278	8.0950	0.0686	0.0009	0.0132
3281	8.0959	0.0698	0.0172	0.2468
3338	8.1131	0.0714	0.0246	0.3444
3421	8.1377	0.0736	0.2141	2.9096
4238	8.3518	0.0765	0.0010	0.0129
4242	8.3528	0.0805	0.0548	0.6808
4481	8.4076	0.0860	0.0781	0.9082
4845	8.4857	0.0937	0.0359	0.3830
5022	8.5216	0.1049	0.1343	1.2804
5744	8.6559	0.1228	0.0458	0.3728
6013	8.7017	0.1557	0.0367	0.2358
6238	8.7384	0.2392	0.7680	3.2109
13446	9.5064			

Number on test: 40
Number of failures: 40
Value of k: 21
Numerator of test statistic: 378.900
Denominator of test statistic: 201.166
Mann's test statistic: $M = 1.884$
Numerator degrees of freedom for F_{crit}: 38
Denominator degrees of freedom for F_{crit}: 40

Since $1.884 = M > F_{0.05,38,40} = 1.68$, the Weibull (two-parameter) distribution was rejected.

A lognormal distribution was then hypothesized, and the Lilliefors (Kolmogorov-Smirnov) goodness-of-fit test was conducted.

H_0: Failure times are lognormal.

H_1: Failure times are not lognormal.

The MLEs were determined from the failure data: $\hat{t}_{med} = 1939.8$ and $\hat{s} = 0.83$. The level of significance was chosen to be $\alpha = 0.10$.

$(i-1)/n$	i/n	Cumulative probability	$D_1(i)$	$D_2(i)$
0.000	0.025	0.0192	0.0192	0.0058
0.025	0.050	0.0281	0.0031	0.0219
0.050	0.075	0.0352	−0.0149	0.0399
0.075	0.100	0.0548	−0.0202	0.0452
0.100	0.125	0.0853	−0.0147	0.0397
0.125	0.150	0.102	−0.0230	0.0480
0.150	0.175	0.2148	0.0648	−0.0398
0.175	0.200	0.2236	0.0486	−0.0236
0.200	0.225	0.2327	0.0327	−0.0077
0.225	0.250	0.2358	0.0108	0.0142
0.250	0.275	0.2676	0.0176	0.0074
0.275	0.300	0.3050	0.0300	−0.0050
0.300	0.325	0.3156	0.0156	0.0094
0.325	0.350	0.3336	0.0086	0.0164
0.350	0.375	0.3372	−0.0128	0.0378
0.375	0.400	0.3936	0.0186	0.0064
0.400	0.425	0.3974	−0.0026	0.0276
0.425	0.450	0.4013	−0.0237	0.0487
0.450	0.475	0.4404	−0.0096	0.0346
0.475	0.500	0.4721	−0.0029	0.0279
0.500	0.525	0.4761	−0.0239	0.0489
0.525	0.550	0.6179	0.0929	−0.0679
0.550	0.575	0.6293	0.0793	−0.0543
0.575	0.600	0.6480	0.0730	−0.0480
0.600	0.625	0.6480	0.0480	−0.0230
0.625	0.650	0.6517	0.0267	−0.0017
0.650	0.675	0.7357	0.0857	−0.0606
0.675	0.700	0.7357	0.0607	−0.0357
0.700	0.725	0.7357	0.0357	−0.0107
0.725	0.750	0.7422	0.0172	0.0078
0.750	0.775	0.7518	0.0017	0.0233
0.775	0.800	0.8264	0.0514	−0.0264
0.800	0.825	0.8264	0.0264	−0.0014
0.825	0.850	0.8413	0.0163	0.0087
0.850	0.875	0.8643	0.0143	0.0107
0.875	0.900	0.8729	−0.0021	0.0271
0.900	0.925	0.9032	0.0032	0.0218
0.925	0.950	0.9131	−0.0119	0.0369
0.950	0.975	0.9207	−0.0293	0.0543
0.975	1.000	0.9901	0.0151	0.0099

Maximum D_1: 0.0929
Maximum D_2: 0.0543
Kolmogorov-Smirnov test statistic: 0.0929
Sample size: 40

Since $D_{40} = 0.0929 < 0.805/\sqrt{40} = 0.1273$, the lognormal distribution is accepted. (The critical value is obtained from Table A.7.)

The reliability function (Eq. 4.31) is

$$R(t) = 1 - \Phi\left(\frac{\ln(t/1939.8)}{0.83}\right)$$

and the MTTF (Eq. 4.28) is

$$\text{MTTF} = 1939.8e^{(0.83^2/2)} = 2737 \text{ cycles}$$

Therefore

$$R(1000) = 1 - \Phi\left(\frac{\ln(1000/1939.8)}{0.83}\right) = 1 - \Phi(-0.80) = 0.78814$$

and the specification is not being met.

To determine a preventive maintenance program designed to meet the specification, the following table is constructed using the model

$$R_m(t) = R(T)^n R(t - nT) \qquad nT \le t \le (n+1)T$$

and

$$R_m(nT) = R(T)^n$$

where T is the maintenance interval

n	T	$z = [\ln(T/1939.8)]/0.83$	$R(T)$	$R(T)^n$
1	1000	−0.79	0.78881	0.78881
2	500	−1.63	0.9485	0.8997
4	250	−2.47	0.9932	0.9730
5	200	−2.74	0.9969	0.9860
8	125	−3.30	0.9995	0.9960

Therefore, a 200-cycle preventive maintenance program is suggested (initiated).

17.4 CASE 4: RELIABILITY ALLOCATION

A high-frequency radio receiver consists of the serially related components shown in Fig. 17.1. A reliability testing program conducted during initial development tested each component independently, with the following results. The test was concluded after 8000 operating hours (accelerated life testing). From the test data the engineering design team must determine the number of redundant components in order to achieve a 95 percent reliability at 1000 operating hours.

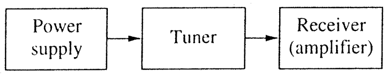


FIGURE 17.1
High-frequency radio.

	Power supply	Tuner	Receiver
Unit cost ($)	175	250	525
Number on test	30	20	15
Failure times	4779	588.0	82.8
	5051	3682.6	820.2
	5633	3819.0	858.3
	6317	4999.5	1201.9
	7491	5048.3	1412.3
	7573	5688.1	1989.1
	7637	7480.7	2042.7
	7953	7641.6	3113.9
			3176.6
			3706.7
			3790.3
			4057.0
			4488.0

Because the data were incomplete (Type I censored), a least-squares fit using the Weibull probability plot was accomplished with $F(t_i) = i - 0.3/(n + 0.4)$ where n is the total number on test. Summarizing the results:

	Power supply	Tuner	Receiver
Estimated β	3.98	0.986	0.928
Estimated θ	10,981	22,172	3893.5
Index of fit	0.95	0.93	0.95

Therefore the system reliability function, without redundancy, is expressed as

$$R_s(t) = R_1(t)R_2(t)R_3(t) = e^{-(t/10,981)^{3.98}} e^{-(t/22,172)^{0.986}} e^{-(t/3893.5)^{0.928}}$$

Allowing for redundancy with any of the three components, the system reliability may be written as

$$R_s(t) = \prod_{i=1}^{3} \{1 - [1 - R_i(t)]^{n_i}\}$$

where n_i is the number of parallel units of component i. $R_s(1000)$ was then evaluated for several different configurations in order to find the least-cost configuration that will meet the reliability specification. From the following calculations, it can be seen that the configuration having the minimum acceptable cost is the one having one power supply, two tuners, and three receivers, with a resulting cost of $2250.

n_1	n_2	n_3	R_1	R_2	R_3	$R_s = R_1R_2R_3$	Cost
1	1	2	0.9999279	0.9539904	0.9391559	**0.8958812**	1475
1	1	3	0.9999279	0.9539904	0.9849919	**0.939605**	2000
1	2	2	0.9999279	0.9978831	0.9391559	**0.9371002**	1725
1	**2**	**3**	**0.9999279**	**0.9978831**	**0.9849919**	**0.9828358**	**2250**
1	3	2	0.9999279	0.9999026	0.9391559	**0.9389968**	1975
1	3	3	0.9999279	0.9999026	0.9849919	**0.9848249**	2500
2	1	2	1	0.9539904	0.9391559	**0.8959458**	1650
2	1	3	1	0.9539904	0.9849919	**0.9396728**	2175
2	2	2	1	0.9978831	0.9391559	**0.9371678**	1900
2	2	3	1	0.9978831	0.9849919	**0.9829067**	2425
2	3	2	1	0.9999026	0.9391559	**0.9390644**	2150
2	3	3	1	0.9999026	0.9849919	**0.9848959**	2675

17.5
CASE 5: RELIABILITY GROWTH TESTING

A high-priced automobile will have a new road-sensing suspension system with four linear vertical-displacement sensors. The sensors provide a controller with information on the dynamic state of the vehicle. Prior to production, prototype sensors are to undergo reliability growth testing in order to reach a target MTTF of 8000 hr. Testing consists of measuring electrical performance, such as operating current, output impedance, and voltage; determining the effects of various (accelerated) environmental stresses, including temperature, temperature cycling, humidity, water spray, corrosion, salt spray, and dust; and subjecting the sensors to (accelerated) mechanical vibration tests.

Initial tests were performed on 43 sensors. Six failures occurred at individual test times of 24, 72, 102, 168, 216, and 250 hr. Each failure was analyzed, and the design team recommended and implemented design corrections. The test plan required 300 hr of testing for each unit that survived all the tests. The cumulative test hours that correspond to each failure time must be computed in order to fit the AMSAA growth model to the data. Therefore:

Failure time	Cumulative test time
24	$43 \times 24 = 1032$
72	$1032 + 42 \times (72 - 24) = 3048$
102	$3048 + 41 \times (102 - 72) = 4278$
168	$4278 + 40 \times (168 - 102) = 6918$
216	$6918 + 39 \times (216 - 168) = 8790$
250	$8790 + 38 \times (250 - 216) = 10,082$

The remaining units continued to be tested for 300 hr each. Therefore the total cumulative test time (Type I testing) was $10082 + 37 \times (300 - 250) = 11,932$ hr. The maximum likelihood estimates were found to be $\hat{b} = 1.024$ and $\hat{a} = 4.0 \times 10^{-4}$. This resulted in an intensity function of $\rho(t) = 0.00041t^{0.024}$ with an instantaneous

MTTF of 1949 hr after 11,932 hr of testing. Because $b > 1$, it was apparent that no increase in reliability was accomplished during this initial test cycle. In order to establish growth targets, the idealized growth curve was used. The growth parameter, $\alpha = 0.58$, was computed on the basis of Eq. (14.3) with $M_I = 11{,}932/6 = 1988$, $M_F = 8000$, $t_1 = 11{,}932 \approx 12{,}000$, and $T = 36{,}000$ hr. The total cumulative test time of 36,000 hr (which includes the initial 12,000 hr) was based on the resources available and the time remaining to complete development of the sensor. Since Eq. (14.3) is an approximation, a growth rate of 0.58 resulted in $M(36{,}000) = 8951$. Therefore the growth parameter was adjusted downward to $\alpha = 0.55$, resulting in $M(36{,}000) = 8083$ hr. Using the idealized growth curve, milestone targets were computed at 6000 hr increments, giving the following:

Cumulative test time, hr	$M(t)$	Cumulative failures	Average (interval) MTTF
18,000	5521	7.24	4966
24,000	6468	8.24	5992
30,000	7312	9.11	6889
36,000	8083	9.9	7697

In order to meet the reliability target, no more than four additional failures could be generated over the next 24,000 hr of testing. Because of the ambitious growth rate required, a complete redesign of the sensor from a reliability perspective was begun.

17.6 CASE 6: REPAIRABLE SYSTEM ANALYSIS

The Notso Reliable Manufacturing Company has been experiencing a high number of failures with its five-year-old industrial robot used for arc welding. Once the unit fails, it is often down for what is considered to be an excessive length of time for repair. Downtime costs the company \$750 per failure in lost production and repair costs. A replacement unit will cost \$21,000. The company wishes to determine whether it is economical to replace the unit. The unit was advertised as having a 10-year design life. The unit operates 8 hr per day for 240 days out of the year (1920 total operating hours per year).

From the time the robot was first installed, the company has recorded the following failure times, in operating hours, of the unit:

1339	1857	2307	3329	3792	3891
5541	5646	5726	5806	6530	6736
6771	6826	7056	7065	7097	7771
7779	7942	8045	8088	8558	8642
8764	8958	9034	9104	9318	9523

For the last 16 failures it has also maintained a log of the following repair times, in hours:

36.2	47.7	81	7.2	36.1	2.7	4.5	27.6
5.2	7.2	16.7	26.8	7.7	12.6	8.4	10.8

Because of the apparent decrease in time between failures over the five years, a nonhomogeneous Poisson process was hypothesized with a power-law intensity function. Maximum likelihood estimates were obtained from the above failure data:

$$\hat{a} = 5.2 \times 10^{-8} \qquad \hat{b} = 2.2 \qquad \rho(t) = 5.2 \times 10^{-8}(2.2)t^{1.2}$$

Since the estimate for b is greater than 1, the system is deteriorating. The test statistic for the presence of trend is

$$\chi^2 = \frac{2n}{\hat{b}} = \frac{2(30)}{2.2} = 27.27$$

with 58 degrees of freedom. The critical chi-square values at the 10 percent level of significance are approximately 43.2 and 79.1. Therefore the null hypothesis of no trend in the data is rejected. The Cramer–von Mises goodness-of-fit statistic was computed and found to be $C_M = 0.0616$ with $M = 29$ (failure-terminated data). Since the critical value is 0.128 at a 20 percent level of significance, the power-law intensity function was accepted as a good fit to the data.

Least-squares fits of the repair times to the exponential, Weibull, normal, and lognormal distributions resulted in the lognormal having the highest R^2 value, 0.976 (the normal was worst, with $R^2 = 0.79$). The maximum likelihood estimates for the lognormal distribution were then computed to be

$$\hat{t}_{\text{med}} = 13.8 \qquad \hat{s} = 0.93$$

providing a computed mean time to repair (MTTR) of 21.3 hr and a computed standard deviation of the repair times of 25.0 hr. The Kolmogorov-Smirnov goodness-of-fit statistic for the lognormal distribution is $D_{16} = 0.14$, which is less than the 10 percent critical value of 0.195. Therefore the null hypothesis that the repair times came from a lognormal distribution is accepted. At the five-year point (9600 operating hours), the unit will have an instantaneous MTBF of $1/[5.2 \times 10^{-8} \times 2.2 \times 9600^{1.2}] = 145.5$ hr. If no further deterioration is observed, a steady-state availability of $145.5/[145.5 + 21.3] = 87$ percent will be obtained. However, there is no reason to assume that the intensity function will remain constant. For the sixth year of operation,

$$m(5\text{ yr}, 6\text{ yr}) = \int_{5(1920)}^{6(1920)} 5.2 \times 10^{-8}(2.2)t^{1.2}\,dt = 14.8$$

is the expected number of failures, which has a total expected cost of \$750(14.8) = \$11,100 and an expected downtime time of 14.8(21.3) = 315 hr or almost 8 work weeks.

Therefore, with the failure process established, the replacement model discussed in Section 10.2.4 was applied, yielding the following:

$$t^* = \left[\frac{C_u}{C_f a(b-1)}\right]^{1/b} = \left[\frac{21{,}000}{750(5.2 \times 10^{-8})(1.2)}\right]^{1/2.2} = 8564 \text{ hr}$$

Since the unit has already accumulated over 9500 hr, it appears that it is operating beyond its economic life and should be replaced.

17.7
CASE 7: MULTIPLY CENSORED DATA

A company recorded the following failure or censored times (in hours) for 20 machine dies from a single manufacturer. Censored times resulted from dies that were no longer used because of design or manufacturing changes.

16^+	117	261^+	323^+	518
531^+	643^+	758	824	881
1323	1582	1795	1854^+	2556
2829	5002	6298	$13{,}991^+$	18,789

These dies were advertised by the manufacturer as having a 500-hr design life (with a 90 percent reliability). The company wants to determine whether the advertised design life is being met before it decides to order additional dies from this manufacturer. The rank adjustment method is used initially to derive an empirical reliability distribution. This method is selected because the adjusted ranks are used in the least-squares plots to account for the censored times.

i	Time	Rank increment	Adjusted rank	Reliability
1	16^+			
2	117	1.05	1.05	0.9632
3	261^+			
4	323^+			
5	518	1.173529	2.223529	0.9057
6	531^+			
7	643^+			
8	758	1.341177	3.564706	0.8400
9	824	1.341177	4.905882	0.7742
10	881	1.341177	6.247059	0.7085
11	1323	1.341177	7.588235	0.6427
12	1582	1.341177	8.929412	0.5770
13	1795	1.341177	10.27059	0.5112
14	1854^+			
15	2556	1.532773	11.80336	0.4361
16	2829	1.532773	13.33613	0.3610
17	5002	1.532773	14.86891	0.2858
18	6298	1.532773	16.40168	0.2107
19	$13{,}991^+$			
20	18,789	2.29916	18.70084	0.0980

From the empirical distribution it would appear that the reliability at 500 hr is approximately 90 percent. However, a more accurate estimate of the design life can be obtained by fitting a theoretical distribution. To find the best fit, a least-squares analysis is conducted using the exponential, Weibull, normal, and lognormal distributions. The results are as follows:

Exponential distribution

Slope—failure rate/repair rate: 1.54×10^{-4}
Least-squares estimate of mean: 6492.8
Index of fit: 0.81

Weibull distribution

Intercept: $a = -7.358218$
Slope: $b = 0.8889475$
Estimated β: $\hat{\beta} = 0.8889475$
Estimated θ: $\hat{\theta} = 3934.144$
Index of fit: 0.968

Normal distribution

Intercept: $a = -0.6514274$
Slope: $b = 1.307285 \times 10^{-4}$
Estimated σ: $\hat{\sigma} = 7649.442$
Estimated mean: $\hat{\mu} = 4983.057$
Index of fit: 0.750

Lognormal distribution

Intercept: $a = -5.190266$
Slope: $b = 0.6729616$
Estimated s: 1.485969
Estimated t_{med}: 2236.291
Index of fit: 0.983

Comparison of the indexes of fit shows that the lognormal distribution is preferred and is closely followed by the Weibull. Neither the exponential nor the normal distribution would be acceptable. To determine the best reliability model, the MLEs for the lognormal are found to be $t_{\text{med}} = 2228.5$ and $s = 1.33$. Therefore

$$R(t) = 1 - \Phi\left(\frac{1}{1.33} \ln \frac{t}{2228.5}\right)$$

and a 90 percent design life is given by

$$t_{0.90} = 2228.5e^{1.33(-1.28)} = 406.13 \text{ hr}$$

On the basis of the best fit reliability model, the stated design life is not being met, and the company should perhaps look elsewhere for its new dies.

EXERCISE

Forty small generators were placed on test for 300 hr. The following failure times in hours were recorded:

117	156	182	119	70	86	68	95
279	17	172	194	155	175	87	263
128	236	112	297	113	174	290	183
148	94	117	49	252	250	203	127

(a) Identify candidate distributions.
(b) Of the candidate distributions in part (a), perform a least-squares fit (probability plot).
(c) Determine the MLE(s) of the best fit in (b).
(d) Perform an appropriate goodness-of-fit test on the distribution selected in (c).
(e) From the fitted distribution, determine the following:
  (i) MTTF
  (ii) Median time to failure
  (iii) Mode
  (iv) Standard deviation
  (v) Design life for a reliability of 0.90
  (vi) Type of failure rate
  (vii) Reliability over the first 75 operating hours
(f) Find the reliability over 75 operating hours if an identical redundant unit is included.
(g) Find the design life for a reliability of 0.90 of the redundant system in (f).
(h) Find the design life for a reliability of 0.90 if a preventive maintenance program is considered (for a single unit). Assume that the preventive maintenance interval is 25 hr and that preventive maintenance restores the unit to its original condition.

CHAPTER 18

Implementation

This chapter is concerned with putting into practice the models and methods of reliability and maintainability (R&M) engineering. At this point the reader should be able to collect and analyze failure and repair data, determine the proper R&M models from this analysis, and manipulate these models as part of the engineering design process. Numerous examples, exercises, and applications have been presented to illustrate these procedures. Our objective now is to develop a formalized structure that defines the R&M processes and the supporting organizational and informational systems necessary to maintain a viable R&M program.

18.1 OBJECTIVES, FUNCTIONS, AND PROCESSES

Reliability and maintainability engineering, like other engineering and scientific disciplines, should adhere to the scientific method. In this context the steps required in following the scientific method include problem definition, observation and the collection of data, the analysis of the data and formulation of one or more mathematical models, verification of the model, solution of the model, and implementation of the solution. Hypotheses may include assertions concerning probability distributions (models) or parameter values with verification consisting of goodness-of-fit tests or other statistical hypothesis tests and confidence intervals. The procedures for analyzing data, constructing and solving R&M models, and performing hypothesis tests have been discussed in detail in the previous chapters.

The overall objective of reliability engineering is to ensure that the final product will be both economically reliable and reliably safe. Economically reliable means that the product's observed reliability has been established with consideration of the life-cycle costs involved. These costs include both the acquisition costs and the costs of failures (including, perhaps, customer goodwill costs). The economics of reliability is discussed further in the next section. Reliably safe requires designing sufficient reliability into the product to ensure that the probability of accidents, injury, or death resulting from a product failure is within an acceptable limit. The functions performed by the reliability engineer in meeting this objective include the following:

1. Quantifying reliability requirements as design goals or specifications
2. Allocating or apportioning reliability requirements to system components and parts
3. Applying reliability design methods during product development (such as parts selection and derating)
4. Performing R&M analysis (such as block diagrams, stress-strength analysis, redundancy, and R&M trade-offs)
5. Conducting a failure mode, effect, and criticality analysis (FMECA) program
6. Participating in design reviews
7. Establishing test procedures and conducting reliability testing
8. Performing a reliability prediction or demonstration
9. Developing a reliability plan

The objective of maintainability engineering is to ensure that restoration of a failed system or product is accomplished in a timely and safe manner. The economics of repair considers the trade-offs among product reliability, repair time, maintenance resources, and spare parts requirements. The functions performed by the maintainability engineer in meeting this objective are similar to those of the reliability engineer and include the following:

1. Quantifying maintainability requirements as design goals or specifications
2. Allocating or apportioning maintainability to system assemblies and components
3. Applying maintainability design methods during product development (fault isolation, modularization, and so on)
4. Performing maintainability and availability analysis (such as determining R&M trade-offs, maintenance resourcing, preventive maintenance intervals, and spares provisioning)
5. Participating in design reviews
6. Performing maintainability prediction and demonstration
7. Developing a maintainability plan

Additional discussions on implementing and managing reliability or maintainability programs may be found in Blanchard and Lowery [1969], Dhillow [1983], Dhillow and Reiche [1985], Guthrie [1990], Ireson and Coombs [1988], Lloyd and Lipow [1962], and O'Conner [1985].

18.2 THE ECONOMICS OF RELIABILITY AND MAINTAINABILITY AND SYSTEM DESIGN

The reliability and maintainability program must pay for itself. How much reliability and maintainability should be designed into a product depends to a large degree on the costs (or profits) to be realized from the operational use of the product. In Chapters 8, 10, and 11, cost trade-off models were presented in order to relate R&M

parameters to product life-cycle costs. The revenue and life-cycle cost model presented here is more comprehensive than the earlier models. Nevertheless, it is only an example of the many forms that such models may take. Our focus, not surprisingly, is on those costs affected by the system reliability and maintainability.

Figure 18.1 shows the total cost curve as the sum of the acquisition costs curve and the cost-of-failures curve. Acquisition cost includes the cost of implementing and operating a reliability program in addition to the overall development and production costs associated with the product. Acquisition cost consists of direct material and labor costs as well as indirect costs such as taxes, insurance, energy, production facilities and equipment, and overhead costs such as administrative, marketing, and product development costs. It is the product development that generally involves the engineering staff. Acquisition costs are increasing functions of reliability, not only because more organizational resources must be committed to achieve a higher reliability, but also because the material and production costs of the product must increase as well. This may be a result of more costly parts selection, added redundancy, stricter tolerances, excess strength, and increased quality control and inspection sampling during manufacture. The cost of failures may include warranty costs, liability costs, replacement or repair costs, the cost of the infrastructure necessary to support operational failures, and the loss of future profit (market share) as a result of loss of customer goodwill. These costs obviously decrease as reliability improves. The sum of the acquisition costs curve and the cost-of-failures curve generates the total cost curve. The minimum-cost point on the total cost curve, shown in Fig. 18.1, represents the desired reliability level. If the minimum-cost point exceeds a minimum required reliability necessary, for example, to meet a safety or contractual requirement, then

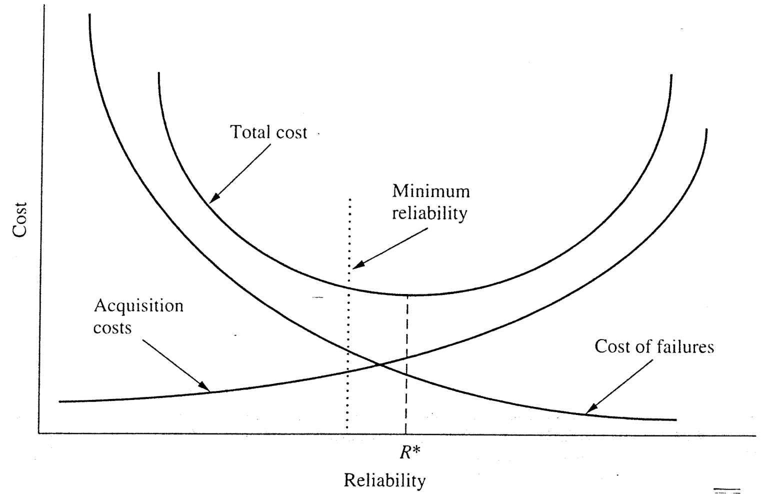


FIGURE 18.1
The total cost versus reliability curve.

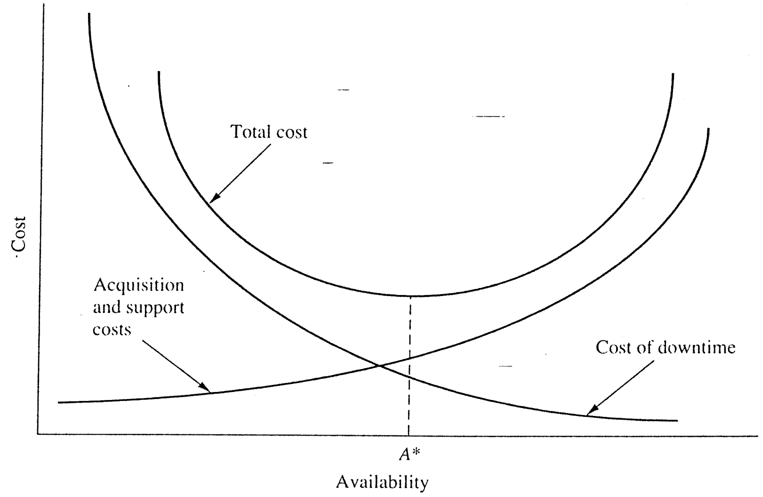


FIGURE 18.2
The cost versus availability curve.

it is the desired level of reliability. Otherwise, the minimum reliability becomes the desired level. If a safety or liability cost associated with injury or loss of life can be quantified, it also can be included as a failure cost. Often, however, we are unable or unwilling to assign a cost to an injury or death, and we must be content to establish a lower bound on safety-related reliability parameters.

A similar cost curve exists as a function of the maintainability of a repairable product. However, it is more useful to consider the economics of a repairable system in terms of availability, as shown in Fig. 18.2. With reliability fixed, as maintainability improves and restoration time decreases, system (operational) availability will increase. Therefore there will be less downtime and the costs associated with downtime will decrease. These costs include the repair costs, consisting of labor, facilities, equipment, and spares, and any loss of revenue associated with the operation of the system. Those acquisition and support costs that increase as the maintainability increases include the infrastructure necessary to implement the maintainability program; design and development costs associated with increased fault isolation, modularization, accessibility, interchangeability, and other design methods; higher salaries for increased maintenance skill levels; maintenance training; added repair capability; and increased availability of spare parts. To the extent that it increases availability, the cost of a preventive maintenance program would be included in the category of increasing costs.

18.2.1 Life-Cycle Cost Model

A generalization of the life-cycle costs given by Eq. (11.37) that takes the above cost elements into account is based on the following assumptions:

1. Failures result in a renewal process (unit replacement or repair to as good as new condition).
2. All units are identical and are acquired at the same time ($t = 0$).
3. Annual operating requirements are constant.
4. The system is in a steady state (equilibrium).
5. There is no preventive maintenance.
6. No failures occur in standby, and perfect switching occurs with negligible downtime.

Mathematically, the life-cycle cost, LCC, can be expressed as

$$\begin{aligned} \mathrm{LCC}(m, s, k, \mathrm{MTBF}, \mathrm{MTTR}, s_i, k_i) = {} & C_u(\mathrm{MTBF}, \mathrm{MTTR})(m + s) + F_o \\ & + A_{sys} P_A(r, t_d) C_o m \\ & + P_A(r, t_d) \frac{t_0}{\mathrm{MTBF}} A_{sys} m (C_f + L\,\mathrm{MTTR}) \\ & + F_{rep} k + P_A(r, t_d) C_{rep} k \\ & + \sum [C_i s_i + P_A(r, t_d) C_{rep,i} k_i] \\ & - P_F(r, t_d) S_a (m + s) \end{aligned} \tag{18.1}$$

where $C_u(\mathrm{MTBF}, \mathrm{MTTR})$ = unit acquisition cost
MTBF = the MTBF of the system failure distribution in operating hours
MTTR = repair or replacement time in hours
m = programmed number of operating units
s = number of spare units (standby redundancy)
k = number of repair channels
s_i = number of spares of component i
k_i = number of repair channels for component i
A_{sys} = effective system availability (average percentage of the m units operating)
F_o = fixed cost of operating
C_o = annual operating costs per unit
F_{rep} = initial acquisition cost per repair channel
C_{rep} = annual (support) cost per repair channel
C_f = fixed cost per failure
C_i = unit cost of component i
$C_{rep,i}$ = annual cost per repair channel for component i
L = labor rate ($ per hour)
t_0 = number of operating hours per year per unit
t_d = design life (in years)
S_a = unit salvage value (a negative value is a disposal cost)
r = discount rate
$P_F(r, t_d)$ = $1/(1 + r)^{t_d}$ is a present-value factor of a future amount at time t_d years at a discount rate of r.
$P_A(r, t_d)$ = $[(1 + r)^{t_d} - 1]/[r(1 + r)^{t_d}]$ is the present-value factor of an annuity over t_d years at a discount rate of r.

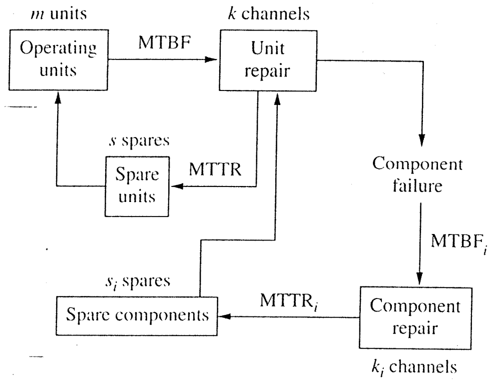


FIGURE 18.3
The system repair cycle.

The above cost model treats m, s, k, s_i, k_i, MTBF, and MTTR as design variables for the repair-cycle system shown in Fig. 18.3. The component mean time between failures (MTBF_i) and the component mean repair time (MTTR_i) can be obtained from a reliability and maintainability allocation based on the computed values of the system MTBF and MTTR. Through the use of this model, trade-offs among these decision variables can be made. The unit acquisition cost, C_u, is assumed to be a function of the inherent reliability (MTBF) and maintainability (MTTR). This cost relationship is not written explicitly since it is very problem-specific and may take on many different functional forms. For many applications, some of the cost terms in Eq. (18.1) may be zero. For example, if repair is accomplished by a fixed number of repair (labor) resources, k, the labor cost per failure, L, may be zero. On the other hand, if only a replacement cost, C_f, is incurred when a failure occurs, there may be no repair channels necessary, or $k = 0$. For those components that are not repairable, $k_i = 0$, and an annual spares replenishment cost is incurred for those units that are discarded.

The effective system availability, A_{sys}, is a function of m, s, k, s_i, k_i, MTBF, and MTTR and is based on the concept of an operational availability (for example, Eq. (11.8) or Eq. (11.38)) when $s = 0$. If $s \geq 1$, then A_{sys} must also take into account the effect that the spare units have on the expected number of units operating (from among the m that are programmed). If constant failure rates and repair rates are assumed, A_{sys} can be computed by use of the queuing model presented in Section 10.4. That is, $A_{\text{sys}} = L_o/m$ where L_o is the expected number of units operating (Eq. (10.20)). If the failure times or repair times are not exponential, computer simulation can be used to find the steady-state system availability as a function of the number operating (m), number of standby spares (s), repair capability (k), number of component spares (s_i), and the inherent unit MTBF and MTTR.

In some applications it may be more desirable to compare alternative designs with regard to expected life-cycle profits rather than costs. Since profit = revenue − cost,

$$E[\text{profit}] = P_A(r, t_d)A_{\text{sys}}Rm - \text{LCC} \tag{18.2}$$

where R = revenue generated per operating unit per year

Although it is desirable to minimize Eq. (18.1) or maximize Eq. (18.2), this can be difficult since m, s, k, and s_i must be integers and the relationships are nonlinear. In most cases it is not possible to express A_{sys} in a simple closed form. On the other hand, given values for the design variables and the cost and revenue coefficients, the numerical evaluation of either equation should be straightforward, as shown in the following example.

EXAMPLE 18.1. A factory must select one of two alternative machines to be used in its production process. It will purchase 11 machines, and one machine will serve as a (standby) spare. Engineering has obtained the following data on two different machines from their manufacturer.

Machine	Unit cost	MTBF	MTTR
ABC	$4000	120 hr	8 hr
DEF	$4700	150 hr	6 hr

Both machines have a 15-year economic life with no salvage value. Machine ABC would be maintained with a full-time maintenance technician at an annual cost of $18,000. Machine DEF would be maintained with a service contract at a labor cost of $40 per repair hour (MTTR) per failure. Under the service agreement, a repair person will be dispatched immediately with the travel time included as part of the specified MTTR. The higher unit cost of DEF is a result of the increased reliability and maintainability. It has an inherent availability of 0.9615, and machine ABC has an inherent availability of 0.9375. Each operating machine will generate $8000 in revenue per year. Both machines will incur a $30 fixed cost per failure for spare parts. The cost analysis addresses two issues. The first issue is whether the higher reliability and maintainability of the second machine justify its higher unit cost. The second issue deals with the relative costs of two different maintenance concepts: dedicated in-house repair capability versus an external maintenance contract. Either machine will have an energy (operating) cost of $1000 a year and will be operated 8 hr a day over a 240-day work year. All other costs are identical for both machines and therefore can be assumed to be zero for this analysis. Assume that component spares are identical in cost and are in ample supply.

The effective system availability of both maintenance concepts must first be determined. For machine ABC, we use the steady-state queuing formulas (Section 10.4) with $\lambda = 1/120 = 0.00833$, $\mu = 1/8 = 0.125$, $k = 1$, and $s = 1$, obtaining $A_{\text{sys}} = 0.9285$. For machine DEF, with $\lambda = 1/150 = 0.00667$, $\mu = 1/6 = 0.1667$, $k = \infty$, and $s = 1$, we obtain $A_{\text{sys}} = 0.993$. In the latter case, under the infinite server assumption, no queuing for repair takes place. That is, $\mu_i = i\mu$ for all i.

Assuming a discount rate of 7 percent, $\text{P}_\text{A}(0.07, 15) = 9.1079$. Therefore

$$\begin{aligned}\text{LCC}_{\text{ABC}} &= 4000(11) + 0.9285(9.1079)(10)[1000 + (8)(240)(30)/120] \\ &\quad + (9.1079)(18,000) = \$333,101 \\ \text{LCC}_{\text{DEF}} &= 4700(11) + 0.993(9.1079)(10)[1000 + ((8)(240)/150)(30 + 40(6))] \\ &= \$454,707\end{aligned}$$

Revenues and profits generated by either machine will also depend upon the effective system availability:

$$P_{ABC} = 0.9285(9.1079)(8000)(10) - LCC_{ABC} = 676{,}534 - 333{,}101 = \$343{,}433$$

$$P_{DEF} = 0.993(9.1079)(8000)(10) - LCC_{DEF} = 723{,}531 - 454{,}707 = \$268{,}824$$

The large difference in expected profits generated by the two machines illustrates the importance of the maintenance concept. The more reliable machine with the higher availability generates a lower profit primarily because of higher repair costs. Even if its unit cost were identical to that of machine ABC, its profit would be considerably less.

18.2.2 Minimal Repair

The alternative repair concept to the renewal process assumed above is that of minimal repair. Under this concept, the effective system availability decreases and the number of failures increases as the units age. On the basis of a nonhomogeneous Poisson process with intensity function $\rho(t)$, t in operating hours, the expected number of failures per unit in year j can be found from

$$f_j = \int_{(j-1)t_0}^{jt_0} \rho(t)\,dt$$

Therefore the cost per failure in Eq. (18.1), $P_A(r, t_d)(t_0/\text{MTBF})m(C_f + L\,\text{MTTR})$, is replaced by

$$\sum_{j=1}^{t_d} P_F(r, j) f_j m(C_f + L\,\text{MTTR})$$

Since the unit intensity rate is increasing, the system availability must be recomputed each year. The average failure rate for year j is f_j/t_0, and therefore an approximate system availability for year j is computed from[1]

$$A_{sys,j} = f(m, s, k, \lambda_j = f_j/t_d, \mu) \qquad \text{where } \mu = \frac{1}{\text{MTTR} + \text{SDT}(s_i, k_i)}$$

and $\text{SDT}(s_i, k_i)$ is the mean system supply delay time. Annual operating costs are now represented by

$$\sum_{j=1}^{t_d} A_{sys,j} P_F(r, j) C_o m$$

where $P_F(r, j)$ is the present-value factor for year j at a discount rate of r. The annual revenue expression in Eq. (18.2) becomes

$$\sum_{j=1}^{t_d} A_{sys,j} P_F(r, j) R m$$

[1]This is an approximation since the failure rate is treated as constant for the year by the queuing model.

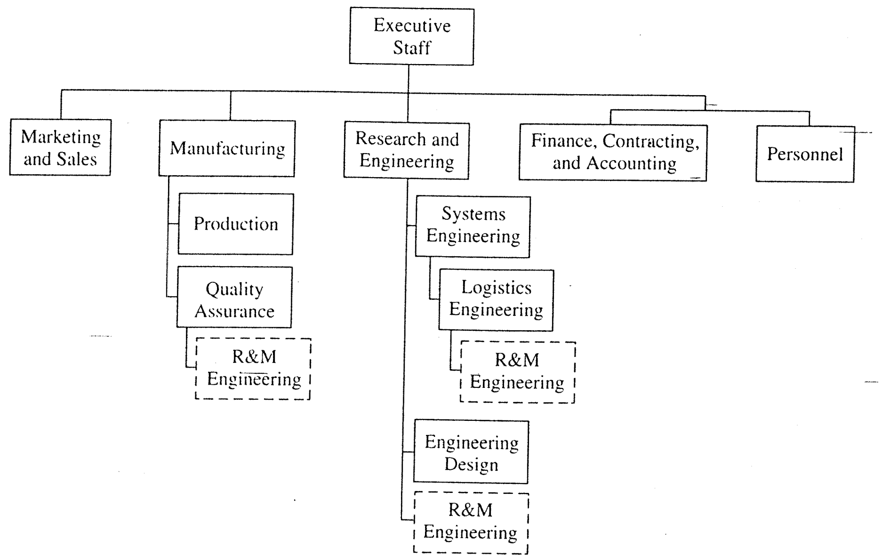


FIGURE 18.4
A functional organizational chart.

18.3
ORGANIZATIONAL CONSIDERATIONS

The reliability and maintainability function may be found in one of several divisions within an organization. Typically, however, it is found within an engineering division, a logistics or support division, or a quality assurance staff. Its size may vary from one or two individuals (or perhaps none if the company contracts out or hires a consultant to advise on R&M matters) to an entire department or division within the company. Rather than attempt to depict the many different organizational settings the R&M function may find itself in, we will outline the three primary organizational philosophies and where the R&M function usually appears in each. The three organizational forms are functional, project oriented, and composite or matrix.

As shown in Fig. 18.4, in the functional, or line organization, the reliability and maintainability engineering staff report vertically to the manager of the functional division to which they are assigned. This may be in quality assurance or in the engineering division. For this organizational structure the R&M function is highly centralized, leading to consistent and standardized R&M design techniques and testing procedures across all products. There is a single focal point within the company for all issues related to R&M. With the R&M resources pooled, some economy may be achieved. Information may flow vertically, with little or no direct communication between the R&M staff and other lateral organizations in marketing, production, or engineering. There may be no close identification with individual product types. Obtaining production and field data may be more difficult since it requires support from other divisions.

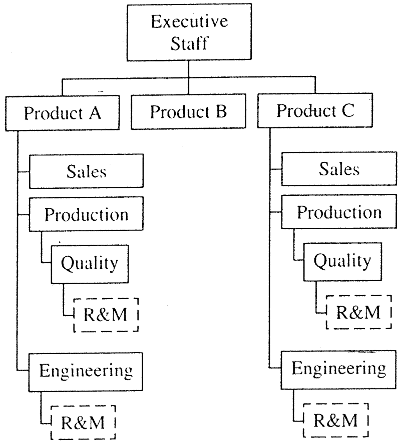


FIGURE 18.5
A product-oriented organizational chart.

In the project-oriented structure, the R&M staff is assigned within a product or development project, as shown in Fig. 18.5. Within each product line there are separate and dedicated production and support staff. The R&M function is decentralized, with specified individuals assigned to a particular project. Therefore there is stronger project allegiance than is found in the functional organization structure. This results in increased support for the product with unique problems receiving dedicated attention. With decentralization comes the added cost of having duplicate functions and processes within each product line. There may be little or no standardization of R&M processes such as FMECA and reliability test procedures. However, there may be separate data collection systems and informational flows by product type, thereby improving the quality and quantity of available failure and repair data. On the other hand, there may be no single R&M "champion" within the company functioning at a high level, so less emphasis might be placed on reliability and maintainability.

In the composite, or matrix, organization (Fig. 18.6), the R&M staff belong to a functional group such as engineering but are temporarily *matrixed* to particular products or projects as needed. This structure attempts to combine some of the advantages of both the functional and product-oriented organizations. Standardization of R&M processes can be achieved with the functional organization tasked to develop common procedures for R&M design, testing, FMECA, and data collection. There is a single champion and point of contact for R&M matters within the company. Product allegiance is obtained by assigning individuals to development teams. The structure is flexible and dynamic, with individuals returning to their parent division to be reassigned when no longer needed with a particular product. There is still a cost associated with decentralization, which requires staffing each product team and the functional group with R&M engineers. Conflicts may also arise because staff members are tasked by and report to both the product team and the functional division.

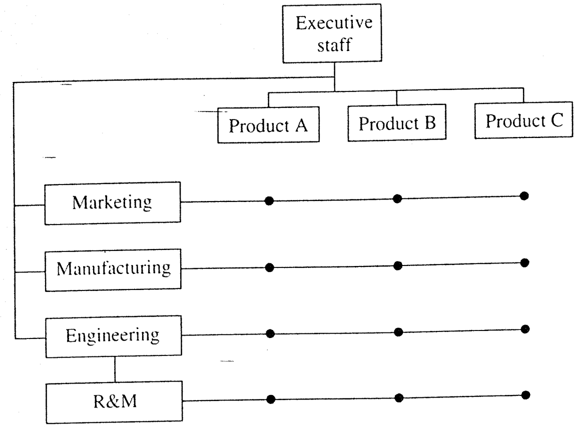


FIGURE 18.6
A matrix organizational chart.

18.4 DATA SOURCES AND DATA COLLECTION METHODS

The need for failure and repair data generally continues long after the system has been developed. R&M data are necessary, for example, for life-cycle costing, evaluating warranty programs, computing spare parts inventory levels, determining the proper amount of maintenance resources, analyzing trends, identifying areas for engineering redesign and modifications, establishing the proper level of preventive maintenance and overhaul programs, performing system effectiveness studies, and establishing operational capability. In designing a management information system, one must complete the basic tasks of establishing the system's objectives, identifying what data to collect, specifying how they are to be collected (for example, sampling versus census, manual versus automated), and determining what analysis is to be performed.

R&M data may be obtained from sources within the organization and, in many instances, from external sources as well. Table 18.1 identifies primary sources and applications of R&M data. Internal data sources include product and reliability testing, acceptance sampling, manufacturing rejects and quality control results, and operational or field service data. External sources include vendor part specifications, the Government-Industry Data Exchange Program (GIDEP), the Reliability Analysis Center (RAC), and other databases that are commercially available. Reliability testing has been discussed in Chapters 13 and 14. Field data and external sources of R&M data can provide useful information as part of the R&M effort.

TABLE 18.1
Product life cycle and R&M data sources

Product phase	Primary sources of R&M data	R&M tasks
Conceptual (definitional) and initial design	Databases (e.g., MIL-HDBK, GIDEP) Vendor specifications Comparability analysis	R&M goals and specifications Allocation and apportionment Life-cycle costing
Detailed design and prototyping	Reliability testing Product testing Maintainability demonstration	R&M predictions Trade-off analysis Development of R&M plans Design methods
Production	Environmental and stress testing Quality assurance program Burn-in testing Acceptance testing Inspection/sampling	R&M demonstrations Redesign and modifications
Operations	Field data	Establishment of preventive maintenance programs and overhaul programs Determination of spares and parts levels and maintenance resource levels

18.4.1 Field Data

It is generally more difficult to obtain accurate information from operational failures than it is to obtain data under controlled test conditions. However, operational failures occur under actual conditions of use and environment and therefore provide valuable information. For example, maintenance-induced failures, maintenance actions in which no problem was found, and operator errors are not normally observed during reliability testing. With the increased computing power and reduced costs of personal computers along with new technologies in computer networking and communications, there is no reason not to automate the collection and analysis of reliability and maintainability data. A transition is taking place from a data-poor environment to a data-rich environment. Older data systems, which were designed to process only aggregate numbers such as total failures per month and total maintenance worker hours per month, are being replaced with systems that can receive and analyze complete failure and repair histories. Until recently, because data collection was limited to the number of failures per time period, the exponential distribution was the only choice available for the analyst. In order to apply the techniques discussed in Part II, individual times to failure must be captured and recorded. The same is true for repair times. R&M data collection systems are in place now in which a failure, for example on a production line, is picked up automatically by a sensor and a record of the time and cause of the failure is written to a personal-computer database. A request is then made to the operator to enter more specific information on the failure along with the repair action. Codes for specific failure modes, failure causes, and repair actions taken can be selected from pull-down menus. When the system returns to operation, the time is automatically recorded. Commercial statistical software is then used to perform the analysis, including least-squares curve fitting and maximum likelihood estimation.

The type of data to be collected depends on the objectives to be met. The following R&M data elements should be collected for every failure. The manner and format in which the data are recorded are system-specific, and as a result, no attempt is made to design a data collection instrument.

In addition to the data displayed in Table 18.2, secondary data may also be recorded. For example, the location of the failure, any parts needed during repair,

TABLE 18.2
R&M data elements

Data element	Definition
Failure number	A sequential number identifying a failure record (date and time may be used instead).
Date and time	Date and time when the failure is recorded (not necessarily the actual failure time).
Part ID	The specific component or part that has failed (at the lowest identifiable level).
Failure time	The age of the part (in a reliability sense) at the time of failure (not when the failure was discovered) measured in failure time units (operating hours, cycles, miles). For repairable components, the time should be measured from a specific reference point, such as the beginning of the current operating cycle or the start of the mission.
Failure mode	The exact nature of the failure (short, overload, impact fracture, power failure, break).
Failure cause	An event or situation causing the failure, such as excessive vibration, fatigue, corrosion, power surge, human error, maintenance-induced error, or normal wear.
Start repair	The date and time when hands-on corrective maintenace began (does not include time waiting for spares or maintenance resources)
Stop repair	The date and time when all restoration has been completed, including verification, validation, and the return of the system to an operational (or standby) status.
Action taken	The type of maintenance performed to correct the failure, such as removal and replacement, minor adjustment, calibration, and rebuilding.
Crew size	The number of individuals performing corrective maintenance (if the number varies during restoration, use a weighted average).

delay times waiting for parts or repair resources, and the level of repair may be needed depending on the type of analysis to be performed. To ensure consistency and facilitate the analysis, standard codes should be defined representing common failure modes, failure causes, and actions taken. As an example, if a competing failure mode analysis is to be performed, the analyst can easily identify those failures associated with the particular failure mode while treating the remaining failures as censored times. A repair-time distribution can be developed for each maintenance subtask ("action taken" code).

In the collection and analysis of field data, there are several issues that may have to be resolved. These include (1) differing ages of the product and its components, (2) the unit of time in which the time between failures is to be measured, and (3) the reference point for measuring the time between failures. It is normally the case that products are manufactured and enter service at different times. Therefore, simply recording the calendar date and time of a failure will not reflect the product time to failure. The date and time the product entered service must also be retained as a baseline. It is also important to maintain the chronology of the failures in order to perform trend analysis. For example, different production lots may reflect changes in the production process or design changes in the product itself. As a result, the failure times may not be from a stationary process and therefore would not represent a sample from a single probability distribution. In the case of repairable systems, sequential failure times may not be independent if restoration is not a renewal process (repair to as good as new condition). Unless failure times are exponentially distributed and independent, a different statistical analysis from that used under the normal assumptions of independent, identically distributed random variables is necessary. However, if each of n repairable systems or age cohorts results in a homogeneous Poisson renewal process, that is, in an independent and identically distributed exponential time between failures, then the superposition (the union of all such renewal processes) is itself a Poisson renewal process. It is also true that the steady-state (equilibrium) superposition of an infinite number of independent renewal processes is a homogeneous Poisson process. When the superposition is a Poisson process, then times between failures can be treated as being identically distributed. In practice, many of the components may never fail, and they may therefore never reach a steady state. If data collected on components and their failure times follow different probability distributions, simulation may be necessary to determine the system failure distribution.

In the definition of reliability, the unit of measurement for time must be specified. If the unit of time is operating hours or cycles, it may be necessary to use calendar time as a surrogate if data on operating hours or cycles are not maintained. If the number of operating hours or cycles per unit of calendar time is constant, the conversion to the proper time unit is straightforward. However, if this is not the case, it may not be possible to measure failure times properly without also measuring the number of elapsed operating hours or cycles as well. For example, if aircraft failures are recorded by date and time of failure but time to failure is measured in flying hours, then conversion to flying hours between failures can only be accomplished if it is known that there are x flying hours per unit of calendar time.

18.4.2 Process Reliability and Operational Failures

The primary focus in this book has been on product reliability. However, important applications of reliability are also found in processes. Process reliability is concerned with the operation and failure of processes such as production, distribution, or transportation systems. For example, a production line may consist of several machines, a conveyor system, auxiliary feeds into the line, and several processes performed in the manufacture of the product. Process failures in general can be divided into two groups: operational failures and equipment failures. Equipment failures are no different from product failures. Operational failures, however, may include jams on the production line, failure in the splicing of material, improper feeds, depletion or contamination of input material, operator error, and inadvertent spills. Operational failures will shut down the system as effectively as equipment failures. For example, in observing the production and packaging of coffee, the system feeding the coffee into cans may gradually build up residue because of moisture present in the coffee. The buildup eventually prevents the proper amount of coffee from being processed into the cans, resulting in the cans being rejected for insufficient weight. This in turn will cause the production line to be stopped in order for the residue to be removed. Operational failures in a transportation system, such as that of a commercial airline, may include the nonavailability of an aircrew due to sickness or a strike, weather delays and cancellations, and airport congestion delaying arrivals or departures. It is interesting to note that airlines typically define a failure to be a delay in the scheduled departure time of 15 minutes or more regardless of the failure mode.

In the analysis of operational failures, the reference point for measuring failure times must be selected with care. For example, consider a stoppage on an automotive assembly line caused by improper feed speeds or spacing of work-in-process material with respect to the speed of the line. The time between successive failures may not be the relevant measurement. Instead, the time to failure from the start-up of the assembly line may be a more accurate measurement since the initial conditions (feeds, speeds, and spacing) of the line on start-up will determine whether a problem will occur or not. This is certainly the case if the line is readjusted for each production run or model type. In fact, one would expect to see decreasing failure rates over a production run. The longer the line has operated, the less likely it will fail as a result of an operational failure. By using the start-up time as the point of reference for failure times, competing failure modes are by necessity introduced. When a line fails as a result of another failure mode, such as an equipment failure, the length of time the line operated before failure becomes a censored time with respect to the particular failure mode being investigated. Therefore, reliability models must be developed using independent samples of failure and censor times obtained for each failure mode.

18.4.3 External Data Sources

During the conceptual and initial design phases of product development, R&M data must be obtained from sources external to the product itself. If a comparable product currently exists, it may provide a source of R&M data if adjustments are made to account for known differences between the two products. Other sources of data include the Reliability Analysis Center (RAC). The RAC is a Department of Defense Information Analysis Center managed by Rome Laboratory in Rome, New York [Rome Laboratory, 1993]. The RAC collects, analyzes, and disseminates reliability information pertaining primarily to electronic equipment and components. The RAC is also responsible for *Military Handbook: Reliability Prediction of Electronic Equipment* [1991] (MIL-HDBK-217F), which contains failure rates for various electronic parts and devices.

Military Handbook 217F [1991] provides two methods for reliability prediction of electronic components. The first of these, the parts-count reliability prediction, can be used when detailed design data and stress levels have not as yet been established. The technique assumes "average stress levels" and requires an estimation of the part types and quantities. A constant failure rate is then computed from

$$\lambda = \sum_{i=1}^{n} q_i \lambda_{e,i} \pi_{q,i}$$

where q_i = quantity of the ith generic part
$\lambda_{e,i}$ = base failure rate for the ith generic part operating in environment e
$\pi_{q,i}$ = quality factor for the ith generic part
n = number of different part types

Eleven different operating environments are included, such as ground benign, ground fixed, ground mobile, naval sheltered, naval unsheltered, airborne inhabited cargo, and space flight. Quality factors may be based on the degree of compliance with military standards. Electronic components addressed under this methodology include microcircuits, semiconductors, resistors, capacitors, inductors, and electromechanical devices.

The other method is the parts-stress technique, which also computes a constant failure rate but requires estimates of the stresses, such as temperature and the voltages to be applied to the parts.

Alternative prediction methods comparable to *Military Handbook* 217F include Bellcore Reliability Prediction Procedure for Electronic Equipment, Nippon Telegraph and Telephone Corporation Standard Reliability Table for Semiconductor Devices, methods found in British Telecom Handbook of Reliability Data for Components Used in Telecommunications Systems, methods developed by the French National Center for Telecommunications Studies, and methods found in Siemens Reliability and Quality Specification Failure Rates of Components. Bowles [1992] compared the failure rate estimates obtained from all of these (including *Military Handbook* 217F) for a dynamic random access memory chip under similar operating and environmental conditions. He found that they varied considerably from one another (from 8 to 1950 failures per 10^9 operating hours). All of the prediction methods are based on the constant failure rate model.

Military Handbook 472, *Maintainability Prediction of Electronic Equipment,* identifies five procedures for estimating maintenance downtimes such as MTTR and maximum corrective downtime (95th percentile). Many of the techniques are designed for airborne and shipboard electronic systems. Information needed generally includes the number of replaceable components, failure rates, locations, packaging, and maintenance procedures.

The Government Industry Data Exchange Program (GIDEP) is a cooperative exchange of technical data among government and industry participants. In order to obtain R&M data and reports from GIDEP, participants must be willing to share their data with the other members. R&M data available include part failure modes and rates and replacement rates.

Similar to *Military Handbook* 217F is the *Handbook of Reliability Prediction Procedures for Mechanical Equipment* [Ploe and Skewis 1990]. This handbook provides techniques for estimating the failure rates of many mechanical devices including seals, gaskets, springs, solenoids, valves, bearings, gears, splines, actuators, pumps, filters, brakes, clutches, compressors, and motors. Adjustments to basic failure rates are based on material properties and operating environments. Raze [1987] also provides estimating techniques for selected mechanical equipment.

18.5 PRODUCT LIABILITY, WARRANTIES, AND RELATED MATTERS

Liability shifts the cost of reliability from the consumer to the manufacturer. Liability implies the obligation to rectify or compensate for any injury or damage. From the consumer's point of view, poor reliability results in cost, inconvenience, and loss of safety. Although reliability and safety are closely related, they are not synonymous. For example, the failure of a gas-driven chain saw to start will probably not cause a safety hazard although the chain saw itself when operating may have several potential safety hazards. Nevertheless, certain failure modes may cause injury or death. Under court rulings concerning strict liability, the manufacturer of a product is liable for injuries due to product defects without a plaintiff necessarily showing negligence or fault. It is assumed the manufacturer will understand the failure process sufficiently to design reliable products. In general, a manufacturer is expected to exercise due care to ensure the safety of others in the use of the product. Failure to exercise a reasonable amount of care may be considered negligence. Product safety deficiencies may be a result of inadequate or improper design, unanticipated use or unplanned environmental conditions, human error, or an unlikely malfunction (failure). The manufacturer must demonstrate that a reasonable effort was expended in eliminating, mitigating, or warning of such deficiencies. Products that are not foolproof should contain a warning concerning the dangers of their misuse. Generally for a manufacturer to be held liable for an unsafe product, it must be demonstrated that (1) the product contained a dangerous defect, (2) the defect existed when the product was sold, and (3) the defect caused an accident or injury. In many states liability lawsuits may be filed against manufacturers for products that are outdated, have been misused, or have been improperly maintained over an indefinite period of time. Often, negligent actions of a plaintiff are not taken into consideration when the economic damages to be awarded are being determined. When there are two or more defendants in a product liability case, each defendant may be held 100 percent responsible for all damages regardless of their individual roles in causing the plaintiff's harm. Further discussion on liability may be found in Hammer [1989] and Witherell [1994].

Warranties have been discussed in previous chapters primarily in the context of conditional reliabilities. A warranty is a contractual guarantee to the buyer concerning product performance. Failure and repair costs are allocated between the manufacturer and the buyer. In one respect, a warranty limits the manufacturer's liability by specifying consumer responsibilities and operating and servicing conditions. Reliability Improvement Warranty (RIW) is an incentive-type contract in which the manufacturer, for a fixed fee, provides all the repairs and spares needed to maintain the product over a fixed time period, usually several years in duration. Options available under warranty programs include replacing the entire unit, repairing or paying to repair the unit, replacing defective components, and reimbursing the buyer a predetermined dollar amount.

Service contracts

A service contract is an extended warranty. It differs from the original warranty in that the consumer purchases the extension at an additional cost to the purchase price of the product. Most service contracts are for a one-year period, although multiyear contracts are available. They generally cover the cost of most repairs and may cover the cost of some replacement parts. There can be a deductible as well.

EXAMPLE 18.2. Statistics show that 7 percent of all VCRs sold need servicing during their first year and 43 percent need servicing within five years. The average cost of a repair is \$95.00. If Weibull failure times are assumed, the parameters can be estimated by solving the following two nonlinear equations:

$$1 - e^{-(1/\theta)^\beta} = 0.07$$

$$1 - e^{-(5/\theta)^\beta} = 0.43$$

Rearranging terms and taking the natural logarithm of both sides of each equation results in

$$\frac{1}{\theta} = [-\ln 0.93]^{1/\beta}$$

$$\frac{5}{\theta} = [-\ln 0.57]^{1/\beta}$$

By multiplying the first equation by -5 and adding it to the second equation, one equation in one unknown is obtained:

$$0 = -5[-\ln 0.93]^{1/\beta} + [-\ln 0.57]^{1/\beta}$$

This equation can be solved directly, resulting in $\beta = 1.272$. Substituting this value into one of the original equations and solving for θ gives $\theta = 7.86$. Assuming that a failure occurring during the first year is covered under warranty, the expected cost of a failure the second year is

$$95\left[e^{-(1/7.86)^{1.272}} - e^{-(2/7.86)^{1.272}}\right] = 95(0.091) = \$8.63$$

Therefore, \$8.63 is the expected amount that a second-year service contract will cost the manufacturer. Since presumably the manufacturer will sell a large number of such contracts, the expected cost is a reasonable estimate of its average cost. The amount by which the service contract price exceeds \$8.63 is the manufacturer's profit.

EXAMPLE 18.3. A well-known clothing and appliance store offers a 3-year maintenance agreement costing \$270 on a 2-year-old self-propelled lawnmower. A recent popular magazine quotes the typical life (MTTF) of a self-propelled lawnmower as 7 years. If we assume Weibull failures and a high degree of wearout with $\beta = 2$, then $\theta = 7/\Gamma(1.5) = 7.9$ years, and the probability of a failure occurring during the next 3 years given that the lawnmower is currently operating is

$$1 - R(5 \mid 2) = \left[1 - \frac{e^{-(5/7.9)^2}}{e^{-(2/7.9)^2}}\right] = 1 - 0.714 = 0.286$$

Assuming that only one failure will occur over the 3 years, a repair cost of 270/0.286 = 944 would justify the cost of the maintenance agreement. Since the repair cost would be considerably less than \$944, the cost of the maintenance agreement appears to be excessive. The possibility of more than one failure occurring complicates the analysis. However, treating the failure-repair process as a stochastic point process with an intensity function of $\rho(t) = (2/\theta^2)t$ (power law), the expected number of failures over the three-year period is

$$m(2, 5) = \int_2^5 \frac{2}{7.9^2} t\, dt = 0.336$$

These last two examples illustrate how insight into a decision problem can be obtained even though very limited information is available. The analysis can be repeated for several different assumed values of β to determine the sensitivity of the expected costs.

18.6 SOFTWARE RELIABILITY

Software reliability has inherently different characteristics from hardware and process reliability. A computer program may have a fixed number of faults ("bugs"). Software will either fail ("abend") or perform as intended. If a failure occurs, it will always occur. Therefore, failures are not random. However, in the execution of complex software, the time until a path is taken that will generate a failure may be treated as random. As a result, probabilistic models may be useful in describing software reliability. Normally, a decreasing failure rate is observed if software failures are fixed as they occur and the fix does not generate any new failures. Software testing can be likened to reliability growth testing in that the software is executed in an attempt to discover bugs, analyze the causes (failure mechanisms), and initiate correction (redesign and recode). We can define software reliability as the probability that a given program module or modules will operate for a specified time without a software error (failure) when executed within its intended design specifications.

There are three general classifications of software failure modes. *Specification errors* occur when the software is coded according to the specifications but the specifications themselves are incorrectly documented or stated. *Design errors* may result from incorrect interpretation of the specifications, from incomplete specifications, or from design tools, such as flow charts, structure charts, and pseudocode, that contain incorrect or incomplete logic. *Coding errors* result from typographical errors, incorrect numerical values, and improper or omitted symbols or syntax. Through use of good software engineering techniques, failures can be reduced or eliminated. These techniques include top-down structured programming, use of program modules, formal walkthroughs, fault-tolerant programming, program redundancy, use of fourth-generation computer languages, and unit and system testing. Program redundancy, for example, can be achieved through fault-tolerant software. Fault-tolerant software will detect a fault in the software within a module, recover the input conditions to the module, and execute another module that performs the same function. Several independently designed and coded modules performing the same function must be developed. Modern programming techniques also call for the use of data flow diagrams, process descriptions, entity-relationship diagrams, data dictionaries, and structure charts.

There have been numerous software reliability growth models proposed but very little published concerning their actual use. Therefore, it is difficult to evaluate and compare these models. Typically, software reliability models estimate the time to the next failure or the expected number of remaining failures. For time-to-failure models, the common assumption is that the failure time is related to the number of remaining errors in the software. "Time" may be measured in test time, CPU execution time, system operating (calendar) time, lines of code tested, and so on. Failure intensity functions will be either time-dependent or error-dependent. Most models assume instantaneous and perfect removal of detected errors.

The Jelinski-Moranda model assumes that the time to the ith failure, t_i, is exponential with the failure rate proportional to the number of failures remaining in the software, and it assumes that a failure is perfectly removed once detected. Therefore $\lambda_i = (N - i + 1)k$ where N is the (unknown) number of errors in the software at the start of the test phase and k is a proportionality constant. Therefore the number of failures per unit time is a nonhomogeneous Poisson process.

Shooman's model also assumes that the time to the next failure is exponential but with an intensity function given by $\rho(t) = k[N/I - n_c(t)]$ where N and k are the same as above, t is the operating time of the system measured from its activation time, I is the total number of instructions in the software, and $n_c(t)$ is the total number of failures corrected during time t and is normalized with respect to I.

An example of an error-counting model is the Goel-Okumoto nonhomogeneous Poisson process, which assumes that the probability of detecting an error is proportional to the number of errors remaining. The expected number of failures observed in time t has the form $m(t) = a(1 - e^{-bt})$ with a failure intensity of $\rho(t) = abe^{-(b-1)t}$.

Additional discussions and models of software reliability may be found in Neufelder [1993], Xie [1991], and Rook [1990].

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APPENDIX

Statistical and Numerical Tables

TABLE A.1

Standardized normal probabilities: $\Phi(z) = \int_{-\infty}^{z} (1/\sqrt{2\pi})e^{-y^2/2}\,dy$

z	Φ(z)	1 − Φ(z)	z	Φ(z)	1 − Φ(z)	z	Φ(z)	1 − Φ(z)
−4.00000	0.00003	0.99997	−3.51000	0.00022	0.99978	−3.02000	0.00126	0.99874
−3.99000	0.00003	0.99997	−3.50000	0.00023	0.99977	−3.01000	0.00131	0.99869
−3.98000	0.00003	0.99997	−3.49000	0.00024	0.99976	−3.00000	0.00135	0.99865
−3.97000	0.00004	0.99996	−3.48000	0.00025	0.99975	−2.99000	0.00139	0.99861
−3.96000	0.00004	0.99996	−3.47000	0.00026	0.99974	−2.98000	0.00144	0.99856
−3.95000	0.00004	0.99996	−3.46000	0.00027	0.99973	−2.97000	0.00149	0.99851
−3.94000	0.00004	0.99996	−3.45000	0.00028	0.99972	−2.96000	0.00154	0.99846
−3.93000	0.00004	0.99996	−3.44000	0.00029	0.99971	−2.95000	0.00159	0.99841
−3.92000	0.00004	0.99996	−3.43000	0.00030	0.99970	−2.94000	0.00164	0.99836
−3.91000	0.00005	0.99995	−3.42000	0.00031	0.99969	−2.93000	0.00169	0.99831
−3.90000	0.00005	0.99995	−3.41000	0.00032	0.99968	−2.92000	0.00175	0.99825
−3.89000	0.00005	0.99995	−3.40000	0.00034	0.99966	−2.91000	0.00181	0.99819
−3.88000	0.00005	0.99995	−3.39000	0.00035	0.99965	−2.90000	0.00187	0.99813
−3.87000	0.00005	0.99995	−3.38000	0.00036	0.99964	−2.89000	0.00193	0.99807
−3.86000	0.00006	0.99994	−3.37000	0.00038	0.99962	−2.88000	0.00199	0.99801
−3.85000	0.00006	0.99994	−3.36000	0.00039	0.99961	−2.87000	0.00205	0.99795
−3.84000	0.00006	0.99994	−3.35000	0.00040	0.99960	−2.86000	0.00212	0.99788
−3.83000	0.00006	0.99994	−3.34000	0.00042	0.99958	−2.85000	0.00219	0.99781
−3.82000	0.00007	0.99993	−3.33000	0.00043	0.99957	−2.84000	0.00226	0.99774
−3.81000	0.00007	0.99993	−3.32000	0.00045	0.99955	−2.83000	0.00233	0.99767
−3.80000	0.00007	0.99993	−3.31000	0.00047	0.99953	−2.82000	0.00240	0.99760
−3.79000	0.00008	0.99992	−3.30000	0.00048	0.99952	−2.81000	0.00248	0.99752
−3.78000	0.00008	0.99992	−3.29000	0.00050	0.99950	−2.80000	0.00255	0.99745
−3.77000	0.00008	0.99992	−3.28000	0.00052	0.99948	−2.79000	0.00264	0.99736
−3.76000	0.00008	0.99992	−3.27000	0.00054	0.99946	−2.78000	0.00272	0.99728
−3.75000	0.00009	0.99991	−3.26000	0.00056	0.99944	−2.77000	0.00280	0.99720
−3.74000	0.00009	0.99991	−3.25000	0.00058	0.99942	−2.76000	0.00289	0.99711
−3.73000	0.00009	0.99991	−3.24000	0.00060	0.99940	−2.75000	0.00298	0.99702
−3.72000	0.00010	0.99990	−3.23000	0.00062	0.99938	−2.74000	0.00307	0.99693
−3.71000	0.00010	0.99990	−3.22000	0.00064	0.99936	−2.73000	0.00317	0.99683
−3.70000	0.00011	0.99989	−3.21000	0.00066	0.99934	−2.72000	0.00326	0.99674
−3.69000	0.00011	0.99989	−3.20000	0.00069	0.99931	−2.71000	0.00336	0.99664
−3.68000	0.00012	0.99988	−3.19000	0.00071	0.99929	−2.70000	0.00347	0.99653
−3.67000	0.00012	0.99988	−3.18000	0.00074	0.99926	−2.69000	0.00357	0.99643
−3.66000	0.00013	0.99987	−3.17000	0.00076	0.99924	−2.68000	0.00368	0.99632
−3.65000	0.00013	0.99987	−3.16000	0.00079	0.99921	−2.67000	0.00379	0.99621
−3.64000	0.00014	0.99986	−3.15000	0.00082	0.99918	−2.66000	0.00391	0.99609
−3.63000	0.00014	0.99986	−3.14000	0.00084	0.99916	−2.65000	0.00402	0.99598
−3.62000	0.00015	0.99985	−3.13000	0.00087	0.99913	−2.64000	0.00415	0.99585
−3.61000	0.00015	0.99985	−3.12000	0.00090	0.99910	−2.63000	0.00427	0.99573
−3.60000	0.00016	0.99984	−3.11000	0.00094	0.99906	−2.62000	0.00440	0.99560
−3.59000	0.00016	0.99984	−3.10000	0.00097	0.99903	−2.61000	0.00453	0.99547
−3.58000	0.00017	0.99983	−3.09000	0.00100	0.99900	−2.60000	0.00466	0.99534
−3.57000	0.00018	0.99982	−3.08000	0.00103	0.99897	−2.59000	0.00480	0.99520
−3.56000	0.00019	0.99981	−3.07000	0.00107	0.99893	−2.58000	0.00494	0.99506
−3.55000	0.00019	0.99981	−3.06000	0.00111	0.99889	−2.57000	0.00508	0.99492
−3.54000	0.00020	0.99980	−3.05000	0.00114	0.99886	−2.56000	0.00523	0.99477
−3.53000	0.00021	0.99979	−3.04000	0.00118	0.99882	−2.55000	0.00539	0.99461
−3.52000	0.00022	0.99978	−3.03000	0.00122	0.99878	−2.54000	0.00554	0.99446

(continued)

TABLE A.1 (CONTINUED)

z	Φ(z)	1 − Φ(z)	z	Φ(z)	1 − Φ(z)	z	Φ(z)	1 − Φ(z)
−2.53000	0.00570	0.99430	−2.03000	0.02118	0.97882	−1.53000	0.06301	0.93699
−2.52000	0.00587	0.99413	−2.02000	0.02169	0.97831	−1.52000	0.06426	0.93574
−2.51000	0.00604	0.99396	−2.01000	0.02222	0.97778	−1.51000	0.06552	0.93448
−2.50000	0.00621	0.99379	−2.00000	0.02275	0.97725	−1.50000	0.06681	0.93319
−2.49000	0.00639	0.99361	−1.99000	0.02330	0.97670	−1.49000	0.06811	0.93189
−2.48000	0.00657	0.99343	−1.98000	0.02385	0.97615	−1.48000	0.06944	0.93056
−2.47000	0.00676	0.99324	−1.97000	0.02442	0.97558	−1.47000	0.07078	0.92922
−2.46000	0.00695	0.99305	−1.96000	0.02500	0.97500	−1.46000	0.07214	0.92786
−2.45000	0.00714	0.99286	−1.95000	0.02559	0.97441	−1.45000	0.07353	0.92647
−2.44000	0.00734	0.99266	−1.94000	0.02619	0.97381	−1.44000	0.07493	0.92507
−2.43000	0.00755	0.99245	−1.93000	0.02680	0.97320	−1.43000	0.07636	0.92364
−2.42000	0.00776	0.99224	−1.92000	0.02743	0.97257	−1.42000	0.07780	0.92220
−2.41000	0.00798	0.99202	−1.91000	0.02807	0.97193	−1.41000	0.07927	0.92073
−2.40000	0.00820	0.99180	−1.90000	0.02872	0.97128	−1.40000	0.08076	0.91924
−2.39000	0.00842	0.99158	−1.89000	0.02938	0.97062	−1.39000	0.08226	0.91774
−2.38000	0.00866	0.99134	−1.88000	0.03005	0.96995	−1.38000	0.08379	0.91621
−2.37000	0.00889	0.99111	−1.87000	0.03074	0.96926	−1.37000	0.08534	0.91466
−2.36000	0.00914	0.99086	−1.86000	0.03144	0.96856	−1.36000	0.08691	0.91309
−2.35000	0.00939	0.99061	−1.85000	0.03216	0.96784	−1.35000	0.08851	0.91149
−2.34000	0.00964	0.99036	−1.84000	0.03288	0.96712	−1.34000	0.09012	0.90988
−2.33000	0.00990	0.99010	−1.83000	0.03362	0.96638	−1.33000	0.09176	0.90824
−2.32000	0.01017	0.98983	−1.82000	0.03438	0.96562	−1.32000	0.09342	0.90658
−2.31000	0.01044	0.98956	−1.81000	0.03515	0.96485	−1.31000	0.09510	0.90490
−2.30000	0.01072	0.98928	−1.80000	0.03593	0.96407	−1.30000	0.09680	0.90320
−2.29000	0.01101	0.98899	−1.79000	0.03673	0.96327	−1.29000	0.09853	0.90147
−2.28000	0.01130	0.98870	−1.78000	0.03754	0.96246	−1.28000	0.10027	0.89973
−2.27000	0.01160	0.98840	−1.77000	0.03836	0.96164	−1.27000	0.10204	0.89796
−2.26000	0.01191	0.98809	−1.76000	0.03920	0.96080	−1.26000	0.10383	0.89617
−2.25000	0.01222	0.98778	−1.75000	0.04006	0.95994	−1.25000	0.10565	0.89435
−2.24000	0.01255	0.98745	−1.74000	0.04093	0.95907	−1.24000	0.10749	0.89251
−2.23000	0.01287	0.98713	−1.73000	0.04182	0.95818	−1.23000	0.10935	0.89065
−2.22000	0.01321	0.98679	−1.72000	0.04272	0.95728	−1.22000	0.11123	0.88877
−2.21000	0.01355	0.98645	−1.71000	0.04363	0.95637	−1.21000	0.11314	0.88686
−2.20000	0.01390	0.98610	−1.70000	0.04457	0.95543	−1.20000	0.11507	0.88493
−2.19000	0.01426	0.98574	−1.69000	0.04551	0.95449	−1.19000	0.11702	0.88298
−2.18000	0.01463	0.98537	−1.68000	0.04648	0.95352	−1.18000	0.11900	0.88100
−2.17000	0.01500	0.98500	−1.67000	0.04746	0.95254	−1.17000	0.12100	0.87900
−2.16000	0.01539	0.98461	−1.66000	0.04846	0.95154	−1.16000	0.12302	0.87698
−2.15000	0.01578	0.98422	−1.65000	0.04947	0.95053	−1.15000	0.12507	0.87493
−2.14000	0.01618	0.98382	−1.64000	0.05050	0.94950	−1.14000	0.12714	0.87286
−2.13000	0.01659	0.98341	−1.63000	0.05155	0.94845	−1.13000	0.12924	0.87076
−2.12000	0.01700	0.98300	−1.62000	0.05262	0.94738	−1.12000	0.13136	0.86864
−2.11000	0.01743	0.98257	−1.61000	0.05370	0.94630	−1.11000	0.13350	0.86650
−2.10000	0.01786	0.98214	−1.60000	0.05480	0.94520	−1.10000	0.13567	0.86433
−2.09000	0.01831	0.98169	−1.59000	0.05592	0.94408	−1.09000	0.13786	0.86214
−2.08000	0.01876	0.98124	−1.58000	0.05705	0.94295	−1.08000	0.14007	0.85993
−2.07000	0.01923	0.98077	−1.57000	0.05821	0.94179	−1.07000	0.14231	0.85769
−2.06000	0.01970	0.98030	−1.56000	0.05938	0.94062	−1.06000	0.14457	0.85543
−2.05000	0.02018	0.97982	−1.55000	0.06057	0.93943	−1.05000	0.14686	0.85314
−2.04000	0.02067	0.97933	−1.54000	0.06178	0.93822	−1.04000	0.14917	0.85083

(continued)

TABLE A.1 (CONTINUED)

z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$
−1.03000	0.15150	0.84850	−0.53000	0.29806	0.70194	−0.03000	0.48803	0.51197
−1.02000	0.15386	0.84614	−0.52000	0.30153	0.69847	−0.02000	0.49202	0.50798
−1.01000	0.15625	0.84375	−0.51000	0.30503	0.69497	−0.01000	0.49601	0.50399
−1.00000	0.15866	0.84134	−0.50000	0.30854	0.69146	0.00000	0.50000	0.50000
−0.99000	0.16109	0.83891	−0.49000	0.31207	0.68793	0.01000	0.50399	0.49601
−0.98000	0.16354	0.83646	−0.48000	0.31561	0.68439	0.02000	0.50798	0.49202
−0.97000	0.16602	0.83398	−0.47000	0.31918	0.68082	0.03000	0.51197	0.48803
−0.96000	0.16853	0.83147	−0.46000	0.32276	0.67724	0.04000	0.51595	0.48405
−0.95000	0.17106	0.82894	−0.45000	0.32636	0.67364	0.05000	0.51994	0.48006
−0.94000	0.17361	0.82639	−0.44000	0.32997	0.67003	0.06000	0.52392	0.47608
−0.93000	0.17619	0.82381	−0.43000	0.33360	0.66640	0.07000	0.52790	0.47210
−0.92000	0.17879	0.82121	−0.42000	0.33724	0.66276	0.08000	0.53188	0.46812
−0.91000	0.18141	0.81859	−0.41000	0.34090	0.65910	0.09000	0.53586	0.46414
−0.90000	0.18406	0.81594	−0.40000	0.34458	0.65542	0.10000	0.53983	0.46017
−0.89000	0.18673	0.81327	−0.39000	0.34827	0.65173	0.11000	0.54380	0.45620
−0.88000	0.18943	0.81057	−0.38000	0.35197	0.64803	0.12000	0.54776	0.45224
−0.87000	0.19215	0.80785	−0.37000	0.35569	0.64431	0.13000	0.55172	0.44828
−0.86000	0.19489	0.80511	−0.36000	0.35942	0.64058	0.14000	0.55567	0.44433
−0.85000	0.19766	0.80234	−0.35000	0.36317	0.63683	0.15000	0.55962	0.44038
−0.84000	0.20045	0.79955	−0.34000	0.36693	0.63307	0.16000	0.56356	0.43644
−0.83000	0.20327	0.79673	−0.33000	0.37070	0.62930	0.17000	0.56749	0.43251
−0.82000	0.20611	0.79389	−0.32000	0.37448	0.62552	0.18000	0.57142	0.42858
−0.81000	0.20897	0.79103	−0.31000	0.37828	0.62172	0.19000	0.57535	0.42465
−0.80000	0.21186	0.78814	−0.30000	0.38209	0.61791	0.20000	0.57926	0.42074
−0.79000	0.21476	0.78524	−0.29000	0.38591	0.61409	0.21000	0.58317	0.41683
−0.78000	0.21770	0.78230	−0.28000	0.38974	0.61026	0.22000	0.58706	0.41294
−0.77000	0.22065	0.77935	−0.27000	0.39358	0.60642	0.23000	0.59095	0.40905
−0.76000	0.22363	0.77637	−0.26000	0.39743	0.60257	0.24000	0.59483	0.40517
−0.75000	0.22663	0.77337	−0.25000	0.40129	0.59871	0.25000	0.59871	0.40129
−0.74000	0.22965	0.77035	−0.24000	0.40517	0.59483	0.26000	0.60257	0.39743
−0.73000	0.23269	0.76731	−0.23000	0.40905	0.59095	0.27000	0.60642	0.39358
−0.72000	0.23576	0.76424	−0.22000	0.41294	0.58706	0.28000	0.61026	0.38974
−0.71000	0.23885	0.76115	−0.21000	0.41683	0.58317	0.29000	0.61409	0.38591
−0.70000	0.24196	0.75804	−0.20000	0.42074	0.57926	0.30000	0.61791	0.38209
−0.69000	0.24510	0.75490	−0.19000	0.42465	0.57535	0.31000	0.62172	0.37828
−0.68000	0.24825	0.75175	−0.18000	0.42858	0.57142	0.32000	0.62552	0.37448
−0.67000	0.25143	0.74857	−0.17000	0.43251	0.56750	0.33000	0.62930	0.37070
−0.66000	0.25463	0.74537	−0.16000	0.43644	0.56356	0.34000	0.63307	0.36693
−0.65000	0.25785	0.74215	−0.15000	0.44038	0.55962	0.35000	0.63683	0.36317
−0.64000	0.26109	0.73891	−0.14000	0.44433	0.55567	0.36000	0.64058	0.35942
−0.63000	0.26435	0.73565	−0.13000	0.44828	0.55172	0.37000	0.64431	0.35569
−0.62000	0.26763	0.73237	−0.12000	0.45224	0.54776	0.38000	0.64803	0.35197
−0.61000	0.27093	0.72907	−0.11000	0.45620	0.54380	0.39000	0.65173	0.34827
−0.60000	0.27425	0.72575	−0.10000	0.46017	0.53983	0.40000	0.65542	0.34458
−0.59000	0.27760	0.72240	−0.09000	0.46414	0.53586	0.41000	0.65910	0.34090
−0.58000	0.28096	0.71904	−0.08000	0.46812	0.53188	0.42000	0.66276	0.33724
−0.57000	0.28434	0.71566	−0.07000	0.47210	0.52790	0.43000	0.66640	0.33360
−0.56000	0.28774	0.71226	−0.06000	0.47608	0.52392	0.44000	0.67003	0.32997
−0.55000	0.29116	0.70884	−0.05000	0.48006	0.51994	0.45000	0.67364	0.32636
−0.54000	0.29460	0.70540	−0.04000	0.48405	0.51595	0.46000	0.67724	0.32276

(continued)

TABLE A.1 (CONTINUED)

z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$
0.47000	0.68082	0.31918	0.97000	0.83398	0.16602	1.47000	0.92922	0.07078
0.48000	0.68439	0.31561	0.98000	0.83646	0.16354	1.48000	0.93056	0.06944
0.49000	0.68793	0.31207	0.99000	0.83891	0.16109	1.49000	0.93189	0.06811
0.50000	0.69146	0.30854	1.00000	0.84134	0.15866	1.50000	0.93319	0.06681
0.51000	0.69497	0.30503	1.01000	0.84375	0.15625	1.51000	0.93448	0.06552
0.52000	0.69847	0.30153	1.02000	0.84614	0.15386	1.52000	0.93574	0.06426
0.53000	0.70194	0.29806	1.03000	0.84850	0.15150	1.53000	0.93699	0.06301
0.54000	0.70540	0.29460	1.04000	0.85083	0.14917	1.54000	0.93822	0.06178
0.55000	0.70884	0.29116	1.05000	0.85314	0.14686	1.55000	0.93943	0.06057
0.56000	0.71226	0.28774	1.06000	0.85543	0.14457	1.56000	0.94062	0.05938
0.57000	0.71566	0.28434	1.07000	0.85769	0.14231	1.57000	0.94179	0.05821
0.58000	0.71904	0.28096	1.08000	0.85993	0.14007	1.58000	0.94295	0.05705
0.59000	0.72240	0.27760	1.09000	0.86214	0.13786	1.59000	0.94408	0.05592
0.60000	0.72575	0.27425	1.10000	0.86433	0.13567	1.60000	0.94520	0.05480
0.61000	0.72907	0.27093	1.11000	0.86650	0.13350	1.61000	0.94630	0.05370
0.62000	0.73237	0.26763	1.12000	0.86864	0.13136	1.62000	0.94738	0.05262
0.63000	0.73565	0.26435	1.13000	0.87076	0.12924	1.63000	0.94845	0.05155
0.64000	0.73891	0.26109	1.14000	0.87286	0.12714	1.64000	0.94950	0.05050
0.65000	0.74215	0.25785	1.15000	0.87493	0.12507	1.65000	0.95053	0.04947
0.66000	0.74537	0.25463	1.16000	0.87698	0.12302	1.66000	0.95154	0.04846
0.67000	0.74857	0.25143	1.17000	0.87900	0.12100	1.67000	0.95254	0.04746
0.68000	0.75175	0.24825	1.18000	0.88100	0.11900	1.68000	0.95352	0.04648
0.69000	0.75490	0.24510	1.19000	0.88298	0.11702	1.69000	0.95449	0.04551
0.70000	0.75804	0.24196	1.20000	0.88493	0.11507	1.70000	0.95543	0.04457
0.71000	0.76115	0.23885	1.21000	0.88686	0.11314	1.71000	0.95637	0.04363
0.72000	0.76424	0.23576	1.22000	0.88877	0.11123	1.72000	0.95728	0.04272
0.73000	0.76731	0.23270	1.23000	0.89065	0.10935	1.73000	0.95818	0.04182
0.74000	0.77035	0.22965	1.24000	0.89251	0.10749	1.74000	0.95907	0.04093
0.75000	0.77337	0.22663	1.25000	0.89435	0.10565	1.75000	0.95994	0.04006
0.76000	0.77637	0.22363	1.26000	0.89617	0.10383	1.76000	0.96080	0.03920
0.77000	0.77935	0.22065	1.27000	0.89796	0.10204	1.77000	0.96164	0.03836
0.78000	0.78230	0.21770	1.28000	0.89973	0.10027	1.78000	0.96246	0.03754
0.79000	0.78524	0.21476	1.29000	0.90147	0.09853	1.79000	0.96327	0.03673
0.80000	0.78814	0.21186	1.30000	0.90320	0.09680	1.80000	0.96407	0.03593
0.81000	0.79103	0.20897	1.31000	0.90490	0.09510	1.81000	0.96485	0.03515
0.82000	0.79389	0.20611	1.32000	0.90658	0.09342	1.82000	0.96562	0.03438
0.83000	0.79673	0.20327	1.33000	0.90824	0.09176	1.83000	0.96638	0.03362
0.84000	0.79955	0.20045	1.34000	0.90988	0.09012	1.84000	0.96712	0.03288
0.85000	0.80234	0.19766	1.35000	0.91149	0.08851	1.85000	0.96784	0.03216
0.86000	0.80511	0.19489	1.36000	0.91309	0.08691	1.86000	0.96856	0.03144
0.87000	0.80785	0.19215	1.37000	0.91466	0.08534	1.87000	0.96926	0.03074
0.88000	0.81057	0.18943	1.38000	0.91621	0.08379	1.88000	0.96995	0.03005
0.89000	0.81327	0.18673	1.39000	0.91774	0.08226	1.89000	0.97062	0.02938
0.90000	0.81594	0.18406	1.40000	0.91924	0.08076	1.90000	0.97128	0.02872
0.91000	0.81859	0.18141	1.41000	0.92073	0.07927	1.91000	0.97193	0.02807
0.92000	0.82121	0.17879	1.42000	0.92220	0.07780	1.92000	0.97257	0.02743
0.93000	0.82381	0.17619	1.43000	0.92364	0.07636	1.93000	0.97320	0.02680
0.94000	0.82639	0.17361	1.44000	0.92507	0.07493	1.94000	0.97381	0.02619
0.95000	0.82894	0.17106	1.45000	0.92647	0.07353	1.95000	0.97441	0.02559
0.96000	0.83147	0.16853	1.46000	0.92786	0.07214	1.96000	0.97500	0.02500

(continued)

TABLE A.1 (CONTINUED)

z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$
1.97000	0.97558	0.02442	2.47000	0.99324	0.00676	2.97000	0.99851	0.00149
1.98000	0.97615	0.02385	2.48000	0.99343	0.00657	2.98000	0.99856	0.00144
1.99000	0.97670	0.02330	2.49000	0.99361	0.00639	2.99000	0.99861	0.00139
2.00000	0.97725	0.02275	2.50000	0.99379	0.00621	3.00000	0.99865	0.00135
2.01000	0.97778	0.02222	2.51000	0.99396	0.00604	3.01000	0.99869	0.00131
2.02000	0.97831	0.02169	2.52000	0.99413	0.00587	3.02000	0.99874	0.00126
2.03000	0.97882	0.02118	2.53000	0.99430	0.00570	3.03000	0.99878	0.00122
2.04000	0.97933	0.02067	2.54000	0.99446	0.00554	3.04000	0.99882	0.00118
2.05000	0.97982	0.02018	2.55000	0.99461	0.00539	3.05000	0.99886	0.00114
2.06000	0.98030	0.01970	2.56000	0.99477	0.00523	3.06000	0.99889	0.00111
2.07000	0.98077	0.01923	2.57000	0.99492	0.00508	3.07000	0.99893	0.00107
2.08000	0.98124	0.01876	2.58000	0.99506	0.00494	3.08000	0.99897	0.00103
2.09000	0.98169	0.01831	2.59000	0.99520	0.00480	3.09000	0.99900	0.00100
2.10000	0.98214	0.01786	2.60000	0.99534	0.00466	3.10000	0.99903	0.00097
2.11000	0.98257	0.01743	2.61000	0.99547	0.00453	3.11000	0.99906	0.00094
2.12000	0.98300	0.01700	2.62000	0.99560	0.00440	3.12000	0.99910	0.00090
2.13000	0.98341	0.01659	2.63000	0.99573	0.00427	3.13000	0.99913	0.00087
2.14000	0.98382	0.01618	2.64000	0.99585	0.00415	3.14000	0.99916	0.00084
2.15000	0.98422	0.01578	2.65000	0.99598	0.00402	3.15000	0.99918	0.00082
2.16000	0.98461	0.01539	2.66000	0.99609	0.00391	3.16000	0.99921	0.00079
2.17000	0.98500	0.01500	2.67000	0.99621	0.00379	3.17000	0.99924	0.00076
2.18000	0.98537	0.01463	2.68000	0.99632	0.00368	3.18000	0.99926	0.00074
2.19000	0.98574	0.01426	2.69000	0.99643	0.00357	3.19000	0.99929	0.00071
2.20000	0.98610	0.01390	2.70000	0.99653	0.00347	3.20000	0.99931	0.00069
2.21000	0.98645	0.01355	2.71000	0.99664	0.00336	3.21000	0.99934	0.00066
2.22000	0.98679	0.01321	2.72000	0.99674	0.00326	3.22000	0.99936	0.00064
2.23000	0.98713	0.01287	2.73000	0.99683	0.00317	3.23000	0.99938	0.00062
2.24000	0.98745	0.01255	2.74000	0.99693	0.00307	3.24000	0.99940	0.00060
2.25000	0.98778	0.01222	2.75000	0.99702	0.00298	3.25000	0.99942	0.00058
2.26000	0.98809	0.01191	2.76000	0.99711	0.00289	3.26000	0.99944	0.00056
2.27000	0.98840	0.01160	2.77000	0.99720	0.00280	3.27000	0.99946	0.00054
2.28000	0.98870	0.01130	2.78000	0.99728	0.00272	3.28000	0.99948	0.00052
2.29000	0.98899	0.01101	2.79000	0.99736	0.00264	3.29000	0.99950	0.00050
2.30000	0.98928	0.01072	2.80000	0.99745	0.00255	3.30000	0.99952	0.00048
2.31000	0.98956	0.01044	2.81000	0.99752	0.00248	3.31000	0.99953	0.00047
2.32000	0.98983	0.01017	2.82000	0.99760	0.00240	3.32000	0.99955	0.00045
2.33000	0.99010	0.00990	2.83000	0.99767	0.00233	3.33000	0.99957	0.00043
2.34000	0.99036	0.00964	2.84000	0.99774	0.00226	3.34000	0.99958	0.00042
2.35000	0.99061	0.00939	2.85000	0.99781	0.00219	3.35000	0.99960	0.00040
2.36000	0.99086	0.00914	2.86000	0.99788	0.00212	3.36000	0.99961	0.00039
2.37000	0.99111	0.00889	2.87000	0.99795	0.00205	3.37000	0.99962	0.00038
2.38000	0.99134	0.00866	2.88000	0.99801	0.00199	3.38000	0.99964	0.00036
2.39000	0.99158	0.00842	2.89000	0.99807	0.00193	3.39000	0.99965	0.00035
2.40000	0.99180	0.00820	2.90000	0.99813	0.00187	3.40000	0.99966	0.00034
2.41000	0.99202	0.00798	2.91000	0.99819	0.00181	3.41000	0.99968	0.00032
2.42000	0.99224	0.00776	2.92000	0.99825	0.00175	3.42000	0.99969	0.00031
2.43000	0.99245	0.00755	2.93000	0.99831	0.00169	3.43000	0.99970	0.00030
2.44000	0.99266	0.00734	2.94000	0.99836	0.00164	3.44000	0.99971	0.00029
2.45000	0.99286	0.00714	2.95000	0.99841	0.00159	3.45000	0.99972	0.00028
2.46000	0.99305	0.00695	2.96000	0.99846	0.00154	3.46000	0.99973	0.00027

(continued)

TABLE A.1 (CONTINUED)

z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$	z	$\Phi(z)$	$1-\Phi(z)$
3.47000	0.99974	0.00026	3.65000	0.99987	0.00013	3.83000	0.99994	0.00006
3.48000	0.99975	0.00025	3.66000	0.99987	0.00013	3.84000	0.99994	0.00006
3.49000	0.99976	0.00024	3.67000	0.99988	0.00012	3.85000	0.99994	0.00006
3.50000	0.99977	0.00023	3.68000	0.99988	0.00012	3.86000	0.99994	0.00006
3.51000	0.99978	0.00022	3.69000	0.99989	0.00011	3.87000	0.99995	0.00005
3.52000	0.99978	0.00022	3.70000	0.99989	0.00011	3.88000	0.99995	0.00005
3.53000	0.99979	0.00021	3.71000	0.99990	0.00010	3.89000	0.99995	0.00005
3.54000	0.99980	0.00020	3.72000	0.99990	0.00010	3.90000	0.99995	0.00005
3.55000	0.99981	0.00019	3.73000	0.99990	0.00010	3.91000	0.99995	0.00005
3.56000	0.99981	0.00019	3.74000	0.99991	0.00009	3.92000	0.99995	0.00005
3.57000	0.99982	0.00018	3.75000	0.99991	0.00009	3.93000	0.99996	0.00004
3.58000	0.99983	0.00017	3.76000	0.99992	0.00008	3.94000	0.99996	0.00004
3.59000	0.99983	0.00017	3.77000	0.99992	0.00008	3.95000	0.99996	0.00004
3.60000	0.99984	0.00016	3.78000	0.99992	0.00008	3.96000	0.99996	0.00004
3.61000	0.99985	0.00015	3.79000	0.99993	0.00007	3.97000	0.99996	0.00004
3.62000	0.99985	0.00015	3.80000	0.99993	0.00007	3.98000	0.99996	0.00004
3.63000	0.99986	0.00014	3.81000	0.99993	0.00007	3.99000	0.99997	0.00003
3.64000	0.99986	0.00014	3.82000	0.99993	0.00007	4.00000	0.99997	0.00003

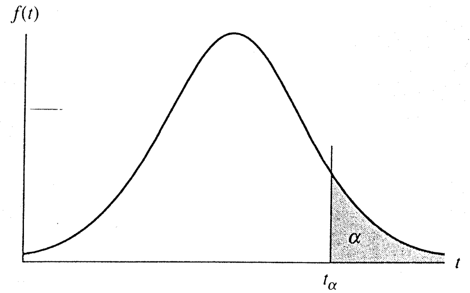

TABLE A.2
Critical t values with ν degrees of freedom

ν	α = 0.100	0.050	0.025	0.010	0.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.695	9.925
3	1.639	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.799
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
∞	1.282	1.645	1.960	2.326	2.576

Table Courtesy of Dr. Ronald Deep.

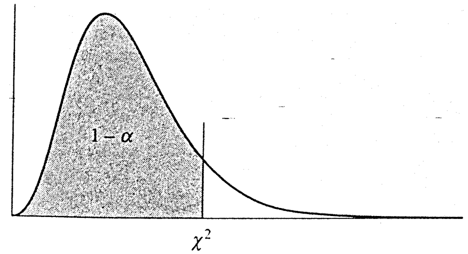

TABLE A.3
Chi-square distribution with ν degrees of freedom

ν	$1-\alpha$ = 0.005	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99	0.995	0.999
1	0.0000	0.0002	0.0010	0.0039	0.0158	2.71	3.84	5.02	6.63	7.88	10.8
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.6	13.8
3	0.0717	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.3	12.8	16.3
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.1	13.3	14.9	18.5
5	0.412	0.554	0.831	1.15	1.61	9.24	11.1	12.8	15.1	16.7	20.5
6	0.676	0.872	1.24	1.64	2.20	10.6	12.6	14.4	16.8	18.5	22.5
7	0.989	1.24	1.69	2.17	2.83	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	1.65	2.18	2.73	3.49	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.09	2.70	3.33	4.17	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.05	3.82	4.57	5.58	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	3.57	4.40	5.23	6.30	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	4.11	5.01	5.89	7.04	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	4.66	5.63	6.57	7.79	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	5.23	6.26	7.26	8.55	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	5.81	6.91	7.96	9.31	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	6.41	7.56	8.67	10.1	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	7.01	8.23	9.39	10.9	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	7.63	8.91	10.1	11.7	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	8.26	9.59	10.9	12.4	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	8.90	10.3	11.6	13.2	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	9.54	11.0	12.3	14.0	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	10.2	11.7	13.1	14.8	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	10.9	12.4	13.8	15.7	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	11.5	13.1	14.6	16.5	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	12.2	13.8	15.4	17.3	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	12.9	14.6	16.2	18.1	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	13.6	15.3	16.9	18.9	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	14.3	16.0	17.7	19.8	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	15.0	16.8	18.5	20.6	40.3	43.8	47.0	50.9	53.7	59.7

Table courtesy of Dr. Ronald Deep.

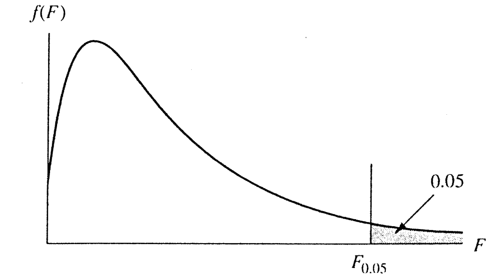

TABLE A.4
Critical values for the F-distribution for $\alpha = 0.05$.

Denominator degrees of freedom, ν_2	Numerator degrees of freedom, ν_1 1	2	3	4	5	6	7	8	9	10	11
1	161	199	216	225	230	234	237	239	241	242	243
2	18.51	19.0	19.16	19.25	19.30	19.33	19.36	19.37	19.38	19.39	19.40
3	10.13	9.55	9.28	9.12	9.01	8.94	8.88	8.84	8.81	8.78	8.76
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.93
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.78	4.74	4.70
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.63	3.60
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.34	3.31
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.13	3.10
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.97	2.94
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.86	2.82
12	4.75	3.89	3.49	3.26	3.11	3.00	2.92	2.85	2.80	2.76	2.72
13	4.67	3.80	3.41	3.18	3.02	2.92	2.84	2.77	2.72	2.67	2.63
14	4.60	3.74	3.34	3.11	2.96	2.85	2.77	2.70	2.65	2.60	2.56
15	4.54	3.68	3.29	3.06	2.90	2.79	2.70	2.64	2.59	2.55	2.51
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.45
17	4.45	3.59	3.20	2.96	2.81	2.70	2.62	2.55	2.50	2.45	2.41
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37
19	4.38	3.52	3.13	2.90	2.74	2.63	2.55	2.48	2.43	2.38	2.34
20	4.35	3.49	3.10	2.87	2.71	2.60	2.52	2.45	2.40	2.35	2.31
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28
22	4.30	3.44	3.05	2.82	2.66	2.55	2.47	2.40	2.35	2.30	2.26
23	4.28	3.42	3.03	2.80	2.64	2.53	2.45	2.38	2.32	2.28	2.24
24	4.26	3.40	3.01	2.78	2.62	2.51	2.43	2.36	2.30	2.26	2.22
25	4.24	3.38	2.99	2.76	2.60	2.49	2.41	2.34	2.28	2.24	2.20
26	4.22	3.37	2.89	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.30	2.25	2.20	2.16
28	4.20	3.34	2.95	2.71	2.56	2.44	2.36	2.29	2.24	2.19	2.15
29	4.18	3.33	2.93	2.70	2.54	2.43	2.35	2.28	2.22	2.18	2.14
30	4.17	3.32	2.92	2.69	2.53	2.43	2.34	2.27	2.21	2.16	2.12

Table courtesy of Dr. Ronald Deep.

TABLE A.5
Median ranks

Rank order	Sample size 1	2	3	4	5	6	7	8	9	10
1	50.0	29.2	20.6	15.9	12.9	10.9	9.4	8.3	7.4	6.6
2		70.7	50.0	38.5	31.3	26.4	22.8	20.1	17.9	16.2
3			79.3	61.4	50.0	42.1	36.4	32.0	28.6	25.8
4				84.0	68.6	57.8	50.0	44.0	39.3	35.5
5					87.0	73.5	63.5	55.9	50.0	45.1
6						89.0	77.1	67.9	60.6	54.8
7							90.5	79.8	71.3	64.4
8								91.7	82.0	74.1
9									92.5	83.7
10										93.3

Rank order	Sample size 11	12	13	14	15	16	17	18	19	20
1	6.1	5.6	5.1	4.8	4.5	4.2	3.9	3.7	3.5	3.4
2	14.7	13.5	12.5	11.7	10.9	10.2	9.6	9.1	8.6	8.2
3	23.5	21.6	20.0	18.6	17.4	16.3	15.4	14.5	13.8	13.1
4	32.3	29.7	27.5	25.6	23.9	22.4	21.1	20.0	18.9	18.0
5	41.1	37.8	35.0	32.5	30.4	28.5	26.9	25.4	24.1	22.9
6	50.0	45.9	42.5	39.5	36.9	34.7	32.7	30.9	29.3	27.8
7	58.3	54.0	50.0	46.5	43.4	40.8	38.4	36.3	34.4	32.7
8	67.6	62.1	57.4	53.4	50.0	46.9	44.2	41.8	39.6	37.7
9	76.4	70.2	64.9	60.4	56.5	53.0	50.0	47.2	44.8	42.6
10	85.2	78.3	72.4	67.4	63.0	59.1	55.7	52.7	50.0	47.5
11	93.8	86.4	79.9	74.3	69.5	65.2	61.5	58.1	55.1	52.4
12		94.3	87.4	81.3	76.0	71.4	67.2	63.6	60.3	57.3
13			94.8	88.2	82.5	77.5	73.0	69.0	65.5	62.2
14				95.1	89.0	83.6	78.8	74.5	70.6	67.2
15					95.4	89.7	84.5	79.9	75.8	72.1
16						95.7	90.3	85.4	81.0	77.0
17							96.0	90.8	86.1	81.9
18								96.2	91.3	86.8
19									96.4	91.7
20										96.5

Source: *Weibull Analysis Handbook*, AFWAL-TR-83-2079, Wright-Patterson AFB, Ohio, 1983. *(continued)*

TABLE A.5 (CONTINUED)

Rank order	Sample size 21	22	23	24	25	26	27	28	29	30
1	3.2	3.1	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2
2	7.8	7.5	7.1	6.8	6.6	6.3	6.1	5.9	5.7	5.5
3	12.5	11.9	11.4	10.9	10.5	10.1	9.7	9.4	9.1	8.8
4	17.2	16.4	15.7	15.0	14.4	13.9	13.4	12.9	12.5	12.1
5	21.8	20.9	20.0	19.1	18.4	17.7	17.0	16.4	15.9	15.3
6	26.5	25.3	24.2	23.2	22.3	21.5	20.7	20.0	19.3	18.6
7	30.2	29.8	28.5	27.4	26.3	25.3	24.3	23.5	22.7	21.9
8	35.9	34.3	32.8	31.5	30.2	29.1	28.0	27.0	26.1	25.2
9	40.8	38.8	37.1	35.8	34.2	32.9	31.7	30.5	29.5	28.5
10	45.3	43.2	41.4	39.7	38.1	36.7	35.3	34.1	32.9	31.8
11	50.0	47.7	45.7	43.8	42.1	40.5	39.0	37.6	36.3	35.1
12	54.6	52.2	50.0	47.9	46.0	44.2	42.6	41.1	39.7	38.4
13	58.3	56.7	54.2	52.0	50.0	48.1	46.3	44.7	43.1	41.7
14	64.0	61.1	58.5	56.1	53.9	51.8	50.0	48.2	46.5	45.0
15	68.7	65.8	62.8	60.2	57.8	55.6	53.6	51.7	50.0	48.3
16	73.4	70.1	67.1	64.2	61.8	59.4	57.3	55.2	53.4	51.6
17	78.1	74.6	71.4	68.4	65.7	63.2	60.9	58.8	56.8	54.9
18	82.7	79.0	75.7	72.5	69.7	67.0	64.6	62.3	60.2	58.2
19	87.4	83.5	79.9	76.7	73.6	70.8	68.2	65.8	63.6	61.5
20	92.1	88.0	84.2	80.8	77.6	74.6	71.9	69.4	67.0	64.8
21	96.7	92.4	88.5	84.9	81.5	78.4	75.6	72.9	70.4	68.1
22		96.8	92.8	89.0	85.5	82.2	79.2	76.4	73.8	71.4
23			97.0	93.1	89.4	86.0	82.9	79.9	77.2	74.7
24				97.1	93.3	89.8	86.5	83.5	80.6	78.0
25					97.2	93.6	90.2	87.0	84.0	81.3
26						97.2	93.8	90.5	87.4	84.6
27							97.4	94.0	90.8	87.8
28								97.5	94.2	91.1
29									97.6	94.4
30										97.7

(continued)

TABLE A.5 (CONTINUED)

Rank order	Sample size 31	32	33	34	35	36	37	38	39	40
1	2.2	2.1	2.0	2.0	1.9	1.9	1.8	1.8	1.7	1.7
2	5.3	5.1	5.0	4.8	4.7	4.6	4.4	4.3	4.2	4.1
3	8.5	8.2	8.0	7.7	7.5	7.3	7.1	6.9	6.7	6.6
4	11.7	11.3	11.0	10.6	10.3	10.1	9.8	9.5	9.3	9.1
5	14.9	14.4	14.0	13.6	13.2	12.8	12.5	12.1	11.8	11.5
6	18.0	17.5	17.0	16.5	16.0	15.6	15.1	14.7	14.4	14.0
7	21.2	20.6	20.0	19.4	18.8	18.3	17.8	17.3	16.9	16.5
8	24.4	23.7	23.0	22.3	21.7	21.1	20.5	20.0	19.4	19.0
9	27.6	26.8	26.0	25.2	24.5	23.8	23.2	22.6	22.0	21.4
10	30.8	29.9	29.0	28.1	27.3	26.6	25.8	25.2	24.5	23.9
11	34.0	32.9	32.0	31.0	30.1	29.3	28.5	27.8	27.1	26.4
12	37.2	36.0	35.0	33.9	33.0	32.1	31.2	30.4	29.6	28.9
13	40.4	39.1	38.0	36.8	35.8	34.8	33.9	33.0	32.2	31.4
14	43.6	42.2	41.0	39.8	38.6	37.6	36.6	35.6	34.7	33.8
15	46.8	45.3	44.0	42.7	41.5	40.3	39.2	38.2	37.2	36.3
16	50.0	48.4	47.0	45.6	44.3	43.1	41.9	40.8	39.8	38.8
17	53.1	51.5	50.0	48.5	47.1	45.8	44.6	43.4	42.3	41.3
18	56.3	54.6	52.9	51.4	50.0	48.6	47.3	46.0	44.9	43.8
19	59.5	57.7	55.9	54.3	52.8	51.3	50.0	48.6	47.4	46.2
20	62.7	60.8	58.9	57.2	55.6	54.1	52.6	51.3	50.0	48.7
21	65.9	63.9	61.9	60.1	58.4	56.8	55.3	53.9	52.5	51.2
22	69.1	67.0	64.9	63.1	61.3	59.6	58.0	56.5	55.0	53.7
23	72.3	70.0	67.9	66.0	64.1	62.3	60.7	59.1	57.6	56.1
24	75.5	73.1	70.9	68.9	66.9	65.1	63.3	61.7	60.1	58.6
25	78.7	76.2	73.9	71.8	69.8	67.8	66.0	64.3	62.7	61.1
26	81.9	79.3	76.9	74.7	72.6	70.6	68.7	66.9	65.2	63.6
27	85.0	82.4	79.9	77.6	75.4	73.3	71.4	69.5	67.7	66.1
28	88.2	85.5	82.9	80.5	78.2	76.1	74.1	72.1	70.3	68.5
29	91.4	88.6	85.9	83.4	81.1	78.8	76.7	74.7	72.8	71.0
30	94.6	91.7	88.9	86.3	83.9	81.6	79.4	77.3	75.4	73.5
31	97.7	94.8	91.9	89.3	86.7	84.3	82.1	79.9	77.9	76.0
32		97.8	94.9	92.2	89.6	87.1	84.8	82.6	80.5	78.5
33			97.9	95.1	92.4	89.8	87.4	85.2	83.0	80.9
34				97.9	95.2	92.6	90.1	87.8	85.5	83.4
35					98.0	95.3	92.8	90.4	88.1	85.9
36						98.0	95.5	93.0	90.8	88.4
37							98.1	95.6	93.2	90.8
38								98.1	95.7	93.3
39									98.2	95.8
40										98.2

(continued)

TABLE A.5 (CONTINUED)

Rank order	Sample size 41	42	43	44	45	46	47	48	49	50
1	1.6	1.6	1.5	1.5	1.5	1.4	1.4	1.4	1.4	1.3
2	4.0	3.9	3.8	3.7	3.7	3.6	3.5	3.4	3.4	3.3
3	6.4	6.3	6.1	6.0	5.8	5.7	5.6	5.5	5.4	5.3
4	8.8	8.6	8.4	8.2	8.0	7.9	7.7	7.5	7.4	7.2
5	11.3	11.0	10.7	10.5	10.3	10.0	9.8	9.6	9.4	9.2
6	13.7	13.3	13.0	12.7	12.5	12.2	11.9	11.7	11.4	11.2
7	16.1	15.7	15.3	15.0	14.7	14.3	14.0	13.7	13.5	13.2
8	18.5	18.1	17.6	17.2	16.9	16.5	16.2	15.8	15.5	15.2
9	20.9	20.4	20.0	19.5	19.1	18.7	18.3	17.9	17.5	17.2
10	23.3	22.8	22.3	21.8	21.3	20.8	20.4	20.0	19.5	19.2
11	25.8	25.2	24.6	24.0	23.5	23.0	22.5	22.0	21.6	21.1
12	28.2	27.5	26.9	26.3	25.7	25.1	24.6	24.1	23.6	23.1
13	30.6	29.9	29.2	28.5	27.9	27.3	26.7	26.2	25.6	25.1
14	33.0	32.2	31.5	30.8	30.1	29.4	28.8	28.2	27.7	27.1
15	35.4	34.6	33.8	33.0	32.3	31.6	30.9	30.3	29.7	29.1
16	37.9	37.0	36.1	35.3	34.5	33.8	33.1	32.4	31.7	31.1
17	40.3	39.3	38.4	37.5	36.7	35.9	35.2	34.4	33.7	33.1
18	42.7	41.7	40.7	39.8	38.9	38.1	37.3	36.5	35.8	35.1
19	45.1	44.0	43.0	42.1	41.1	40.2	39.4	38.6	37.8	37.0
20	47.5	46.4	45.3	44.3	43.3	42.4	41.5	40.6	39.8	39.0
21	50.0	48.8	47.6	46.6	45.5	44.6	43.6	42.7	41.8	41.0
22	52.4	51.1	50.0	48.8	47.7	46.7	45.7	44.8	43.9	43.0
23	54.8	53.5	52.3	51.1	50.0	48.9	47.8	46.8	45.9	45.0
24	57.2	55.9	54.6	53.3	52.2	51.0	50.0	48.9	47.9	47.0
25	59.6	58.2	56.9	55.6	54.4	53.2	52.1	51.0	50.0	49.0
26	62.0	60.6	59.2	57.8	56.6	55.3	54.2	53.1	52.0	50.9
27	64.5	62.9	61.5	60.1	58.8	57.5	56.3	55.1	54.0	52.9
28	66.9	65.2	63.8	62.4	61.0	59.7	58.4	57.2	56.0	54.9
29	69.3	67.7	66.1	64.6	63.2	61.8	60.5	59.3	58.1	56.9
30	71.7	70.0	68.4	66.9	65.4	64.0	62.6	61.3	60.1	58.9
31	74.1	72.4	70.7	69.1	67.6	66.1	64.7	63.4	62.1	60.9
32	76.6	74.8	73.0	71.4	69.8	68.3	66.9	65.5	64.1	62.9
33	79.0	77.1	75.3	73.6	72.0	70.5	69.0	67.5	66.2	64.8
34	81.4	79.5	77.6	75.9	74.2	72.6	71.1	69.6	68.2	66.8
35	83.8	81.8	79.9	78.1	76.4	74.8	73.2	71.7	70.2	68.8
36	86.2	84.2	82.3	80.4	78.6	76.9	75.3	73.7	72.2	70.8
37	88.7	86.6	84.6	82.7	80.8	79.1	77.4	75.8	74.3	72.8
38	91.1	88.9	86.9	84.9	83.0	81.2	79.5	77.9	76.3	74.8
39	93.5	91.3	89.2	87.2	85.2	83.4	81.6	79.9	78.3	76.8
40	95.9	93.6	91.5	89.4	87.4	85.6	83.7	82.0	80.4	78.8
41	98.3	96.0	93.8	91.7	89.6	87.7	85.9	84.1	82.4	80.7
42		98.3	96.1	93.9	91.9	89.9	88.0	86.2	84.4	82.7
43			98.4	96.2	94.1	92.0	90.1	88.2	86.4	84.7
44				98.4	96.2	94.2	92.2	90.3	88.5	86.7
45					98.4	96.3	94.3	92.4	90.5	88.7
46						98.5	96.4	94.4	92.5	90.7
47							98.5	96.5	94.5	92.7
48								98.5	96.5	94.6
49									98.5	96.6
50										98.6

TABLE A.6(A)
Confidence interval factors for the power-law intensity model (AMSAA): Type I testing

	γ 0.80		0.90		0.95		0.98	
N	L	U	L	U	L	U	L	U
2	0.261	18.66	0.200	38.66	0.159	78.66	0.124	198.7
3	0.333	6.326	0.263	9.736	0.217	14.55	0.174	24.10
4	0.385	4.243	0.312	5.947	0.262	8.093	0.215	11.81
5	0.426	3.386	0.352	4.517	0.300	5.862	0.250	8.043
6	0.459	2.915	0.385	3.764	0.331	4.738	0.280	6.254
7	0.487	2.616	0.412	3.298	0.358	4.061	0.305	5.216
8	0.511	2.407	0.436	2.981	0.382	3.609	0.328	4.539
9	0.531	2.254	0.457	2.750	0.403	3.285	0.349	4.064
10	0.549	2.136	0.476	2.575	0.421	3.042	0.367	3.712
11	0.565	2.041	0.492	2.436	0.438	2.852	0.384	3.441
12	0.579	1.965	0.507	2.324	0.453	2.699	0.399	3.226
13	0.592	1.901	0.521	2.232	0.467	2.574	0.413	3.050
14	0.604	1.846	0.533	2.153	0.480	2.469	0.426	2.904
15	0.614	1.800	0.545	2.087	0.492	2.379	0.438	2.781
16	0.624	1.759	0.556	2.029	0.503	2.302	0.449	2.675
17	0.633	1.723	0.565	1.978	0.513	2.235	0.460	2.584
18	0.642	1.692	0.575	1.933	0.523	2.176	0.470	2.503
19	0.650	1.663	0.583	1.893	0.532	2.123	0.479	2.432
20	0.657	1.638	0.591	1.858	0.540	2.076	0.488	2.369
21	0.664	1.615	0.599	1.825	0.548	2.034	0.496	2.313
22	0.670	1.594	0.606	1.796	0.556	1.996	0.504	2.261
23	0.676	1.574	0.613	1.769	0.563	1.961	0.511	2.215
24	0.682	1.557	0.619	1.745	0.570	1.929	0.518	2.173
25	0.687	1.540	0.625	1.722	0.576	1.900	0.525	2.134
26	0.692	1.525	0.631	1.701	0.582	1.873	0.531	2.098
27	0.697	1.511	0.636	1.682	0.588	1.848	0.537	2.068
28	0.702	1.498	0.641	1.664	0.594	1.825	0.543	2.035
29	0.706	1.486	0.646	1.647	0.599	1.803	0.549	2.006
30	0.711	1.475	0.651	1.631	0.604	1.783	0.554	1.980
35	0.729	1.427	0.672	1.565	0.627	1.699	0.579	1.870
40	0.745	1.390	0.690	1.515	0.646	1.635	0.599	1.788
45	0.758	1.361	0.705	1.476	0.662	1.585	0.617	1.723
50	0.769	1.337	0.718	1.443	0.676	1.544	0.632	1.671
60	0.787	1.300	0.739	1.393	0.700	1.481	0.657	1.591
70	0.801	1.272	0.756	1.356	0.718	1.435	0.678	1.533
80	0.813	1.251	0.769	1.328	0.734	1.399	0.695	1.488
100	0.831	1.219	0.791	1.286	0.758	1.347	0.722	1.423

Source: *Military Handbook: Reliability Growth Management* [1981].

TABLE A.6(B)

Confidence interval factors for the power-law intensity model (AMSAA): Type II testing

	γ							
	0.80		0.90		0.95		0.98	
N	L	U	L	U	L	U	L	U
2	0.8065	33.76	0.5552	72.67	0.4099	151.5	0.2944	389.9
3	0.6840	8.927	0.5137	14.24	0.4054	21.96	0.3119	37.60
4	0.6601	5.328	0.5174	7.651	0.4225	10.65	0.3368	15.96
5	0.6568	4.000	0.5290	5.424	0.4415	7.147	0.3603	9.995
6	0.6600	3.321	0.5421	4.339	0.4595	5.521	0.3815	7.388
7	0.6656	2.910	0.5548	3.702	0.4760	4.595	0.4003	5.963
8	0.6720	2.634	0.5668	3.284	0.4910	4.002	0.4173	5.074
9	0.6787	2.436	0.5780	2.989	0.5046	3.589	0.4327	4.469
10	0.6852	2.287	0.5883	2.770	0.5171	3.286	0.4467	4.032
11	0.6915	2.170	0.5979	2.600	0.5285	3.054	0.4595	3.702
12	0.6975	2.076	0.6067	2.464	0.5391	2.870	0.4712	3.443
13	0.7033	1.998	0.6150	2.353	0.5488	2.721	0.4821	3.235
14	0.7087	1.933	0.6227	2.260	0.5579	2.597	0.4923	3.064
15	0.7139	1.877	0.6299	2.182	0.5664	2.493	0.5017	2.921
16	0.7188	1.829	0.6367	2.144	0.5743	2.404	0.5106	2.800
17	0.7234	1.788	0.6431	2.056	0.5818	2.327	0.5189	2.695
18	0.7278	1.751	0.6491	2.004	0.5888	2.259	0.5267	2.604
19	0.7320	1.718	0.6547	1.959	0.5954	2.200	0.5341	2.524
20	0.7360	1.688	0.6601	1.918	0.6016	2.147	0.5411	2.453
21	0.7398	1.662	0.6652	1.881	0.6076	2.099	0.5478	2.390
22	0.7434	1.638	0.6701	1.848	0.6132	2.056	0.5541	2.333
23	0.7469	1.616	0.6747	1.818	0.6186	2.017	0.5601	2.281
24	0.7502	1.596	0.6791	1.790	0.6237	1.982	0.5659	2.235
25	0.7534	1.578	0.6833	1.765	0.6286	1.949	0.5714	2.192
26	0.7565	1.561	0.6873	1.742	0.6333	1.919	0.5766	2.153
27	0.7594	1.545	0.6912	1.720	0.6378	1.892	0.5817	2.116
28	0.7622	1.530	0.6949	1.700	0.6421	1.866	0.5865	2.083
29	0.7649	1.516	0.6985	1.682	0.6462	1.842	0.5912	2.052
30	0.7676	1.504	0.7019	1.664	0.6502	1.820	0.5957	2.083
35	0.7794	1.450	0.7173	1.592	0.6681	1.729	0.6158	1.905
40	0.7894	1.410	0.7303	1.538	0.6832	1.660	0.6328	1.816
45	0.7981	1.378	0.7415	1.495	0.6962	1.606	0.6476	1.747
50	0.8057	1.352	0.7513	1.460	0.7076	1.562	0.6605	1.692
60	0.8184	1.312	0.7678	1.407	0.7267	1.496	0.6823	1.607
70	0.8288	1.282	0.7811	1.367	0.7423	1.447	0.7000	1.546
80	0.8375	1.259	0.7922	1.337	0.7553	1.409	0.7148	1.499
100	0.8514	1.225	0.8100	1.293	0.7759	1.355	0.7384	1.431

Source: *Military Handbook: Reliability Growth Management* [1981].

TABLE A.7

Critical values for the Kolmogorov-Smirnov test for normality (Lilliefors test)

	α				
Sample size, n	0.20	0.15	0.10	0.05	0.01
4	0.300	0.319	0.352	0.381	0.417
5	0.285	0.299	0.315	0.337	0.405
6	0.265	0.277	0.294	0.319	0.364
7	0.247	0.258	0.276	0.300	0.348
8	0.233	0.244	0.261	0.285	0.331
9	0.223	0.233	0.249	0.271	0.311
10	0.215	0.224	0.239	0.258	0.294
11	0.206	0.217	0.230	0.249	0.284
12	0.199	0.212	0.223	0.242	0.275
13	0.190	0.202	0.214	0.234	0.268
14	0.183	0.194	0.207	0.227	0.261
15	0.177	0.187	0.201	0.220	0.257
16	0.173	0.182	0.195	0.213	0.250
17	0.169	0.177	0.189	0.206	0.245
18	0.166	0.173	0.184	0.200	0.239
19	0.163	0.169	0.179	0.195	0.235
20	0.160	0.166	0.174	0.190	0.231
25	0.149	0.153	0.165	0.180	0.203
30	0.131	0.136	0.144	0.161	0.187
$n > 30$	$\frac{0.736}{\sqrt{n}}$	$\frac{0.768}{\sqrt{n}}$	$\frac{0.805}{\sqrt{n}}$	$\frac{0.886}{\sqrt{n}}$	$\frac{1.031}{\sqrt{n}}$

TABLE A.8
Critical values for the Cramer–von Mises goodness-of-fit test

	α				
M	**0.20**	**0.15**	**0.10**	**0.05**	**0.01**
2	0.138	0.149	0.162	0.175	0.186
3	0.121	0.135	0.154	0.184	0.230
4	0.121	0.134	0.155	0.191	0.280
5	0.121	0.137	0.160	0.199	0.300
6	0.123	0.139	0.162	0.204	0.310
7	0.124	0.140	0.165	0.208	0.320
8	0.124	0.141	0.165	0.210	0.320
9	0.125	0.142	0.167	0.212	0.320
10	0.125	0.142	0.167	0.212	0.320
11	0.126	0.143	0.169	0.214	0.320
12	0.126	0.144	0.169	0.214	0.320
13	0.126	0.144	0.169	0.214	0.330
14	0.126	0.144	0.169	0.214	0.330
15	0.126	0.144	0.169	0.215	0.330
16	0.127	0.145	0.171	0.216	0.330
17	0.127	0.145	0.171	0.217	0.330
18	0.127	0.146	0.171	0.217	0.330
19	0.127	0.146	0.171	0.217	0.330
20	0.128	0.146	0.172	0.217	0.330
30	0.128	0.146	0.172	0.218	0.330
60	0.128	0.147	0.173	0.220	0.330
100	0.129	0.147	0.173	0.220	0.340

For $M > 100$, use values for $M = 100$.

Source: *Military Handbook: Reliability Growth Management* [1981].

TABLE A.9
Gamma function

x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$
1.01	.99433	1.51	.88659	2.01	1.00427	2.51	1.33875
1.02	.98884	1.52	.88704	2.02	1.00862	2.52	1.34830
1.03	.98355	1.53	.88757	2.03	1.01306	2.53	1.35798
1.04	.97844	1.54	.88818	2.04	1.01758	2.54	1.36779
1.05	.97350	1.55	.88887	2.05	1.02218	2.55	1.37775
1.06	.96874	1.56	.88964	2.06	1.02687	2.56	1.38784
1.07	.96415	1.57	.89049	2.07	1.03164	2.57	1.39807
1.08	.95973	1.58	.89142	2.08	1.03650	2.58	1.40844
1.09	.95546	1.59	.89243	2.09	1.04145	2.59	1.41896
1.10	.95135	1.60	.89352	2.10	1.04649	2.60	1.42962
1.11	.94740	1.61	.89468	2.11	1.05161	2.61	1.44044
1.12	.94359	1.62	.89592	2.12	1.05682	2.62	1.45140
1.13	.93993	1.63	.89724	2.13	1.06212	2.63	1.46251
1.14	.93642	1.64	.89864	2.14	1.06751	2.64	1.47377
1.15	.93304	1.65	.90012	2.15	1.07300	2.65	1.48519
1.16	.92980	1.66	.90167	2.16	1.07857	2.66	1.49677
1.17	.92670	1.67	.90330	2.17	1.08424	2.67	1.50851
1.18	.92373	1.68	.90500	2.18	1.09000	2.68	1.52040
1.19	.92089	1.69	.90678	2.19	1.09585	2.69	1.53246
1.20	.91817	1.70	.90864	2.20	1.10180	2.70	1.54469
1.21	.91558	1.71	.91057	2.21	1.10785	2.71	1.55708
1.22	.91311	1.72	.91258	2.22	1.11399	2.72	1.56964
1.23	.91075	1.73	.91467	2.23	1.12023	2.73	1.58237
1.24	.90852	1.74	.91683	2.24	1.12657	2.74	1.59528
1.25	.90640	1.75	.91906	2.25	1.13300	2.75	1.60836
1.26	.90440	1.76	.92137	2.26	1.13954	2.76	1.62162
1.27	.90250	1.77	.92376	2.27	1.14618	2.77	1.63506
1.28	.90072	1.78	.92623	2.28	1.15292	2.78	1.64868
1.29	.89904	1.79	.92877	2.29	1.15976	2.79	1.66249
1.30	.89747	1.80	.93138	2.30	1.16671	2.80	1.67649
1.31	.89600	1.81	.93408	2.31	1.17377	2.81	1.69068
1.32	.89464	1.82	.93685	2.32	1.18093	2.82	1.70506
1.33	.89338	1.83	.93969	2.33	1.18819	2.83	1.71963
1.34	.89222	1.84	.94261	2.34	1.19557	2.84	1.73441
1.35	.89115	1.85	.94561	2.35	1.20305	2.85	1.74938
1.36	.89018	1.86	.94869	2.36	1.21065	2.86	1.76456
1.37	.88931	1.87	.95184	2.37	1.21836	2.87	1.77994
1.38	.88854	1.88	.95507	2.38	1.22618	2.88	1.79553
1.39	.88785	1.89	.95838	2.39	1.23412	2.89	1.81134
1.40	.88726	1.90	.96177	2.40	1.24217	2.90	1.82736
1.41	.88676	1.91	.96523	2.41	1.25034	2.91	1.84359
1.42	.88636	1.92	.96877	2.42	1.25863	2.92	1.86005
1.43	.88604	1.93	.97240	2.43	1.26703	2.93	1.87673
1.44	.88581	1.94	.97610	2.44	1.27556	2.94	1.89363
1.45	.88566	1.95	.97988	2.45	1.28421	2.95	1.91077
1.46	.88560	1.96	.98374	2.46	1.29298	2.96	1.92814
1.47	.88563	1.97	.98769	2.47	1.30188	2.97	1.94574
1.48	.88575	1.98	.99171	2.48	1.31091	2.98	1.96358
1.49	.88595	1.99	.99581	2.49	1.32006	2.99	1.98167
1.50	.88623	2.00	1	2.50	1.32934	3.00	2

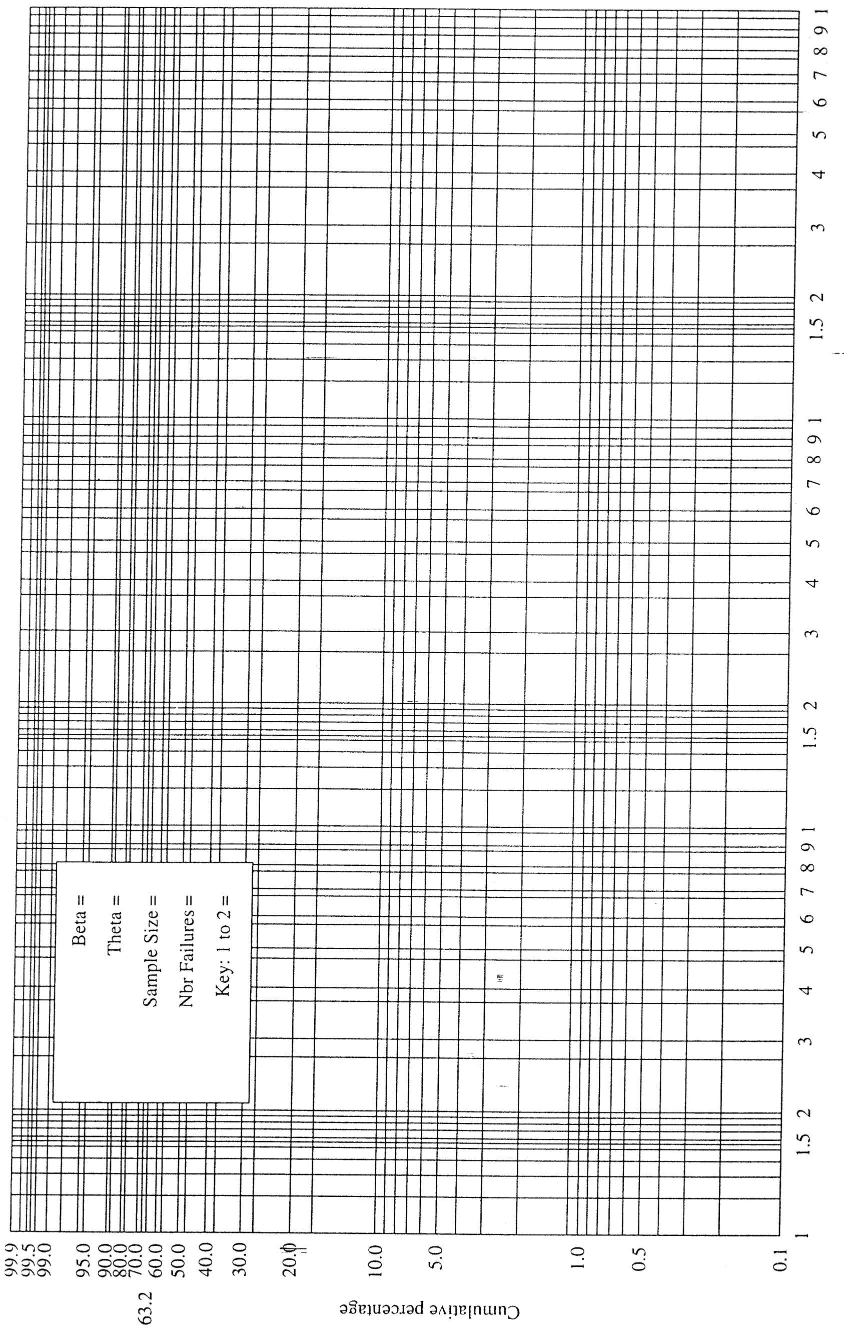

FIGURE A.1
Weibull distribution probability paper.

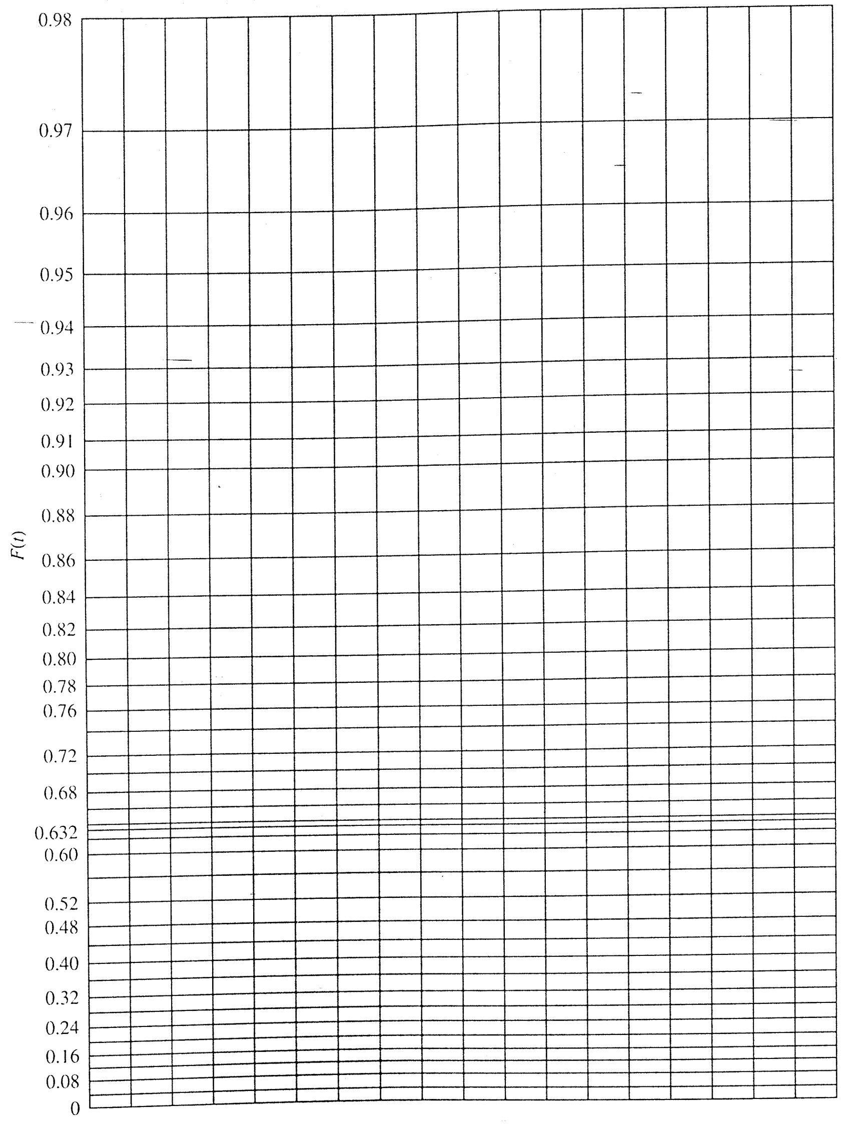

FIGURE A.2
Exponential distribution probability paper.

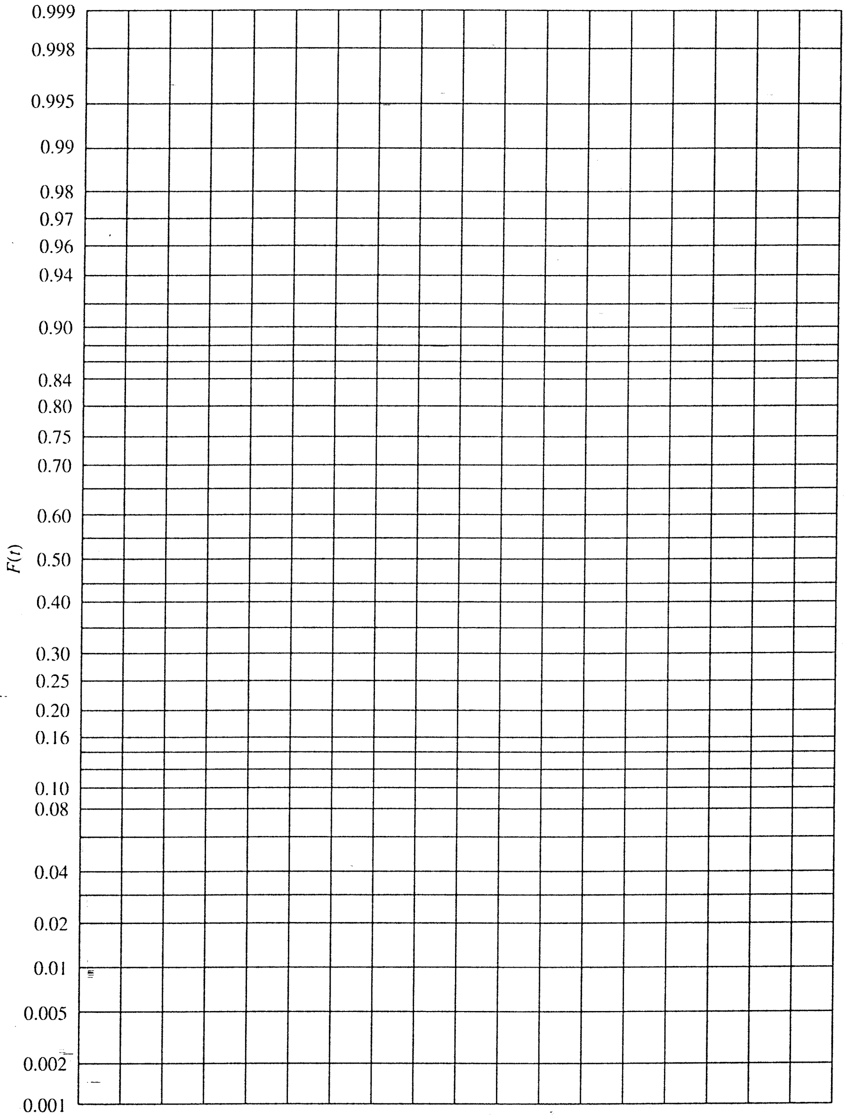

FIGURE A.3
Normal distribution probability paper.

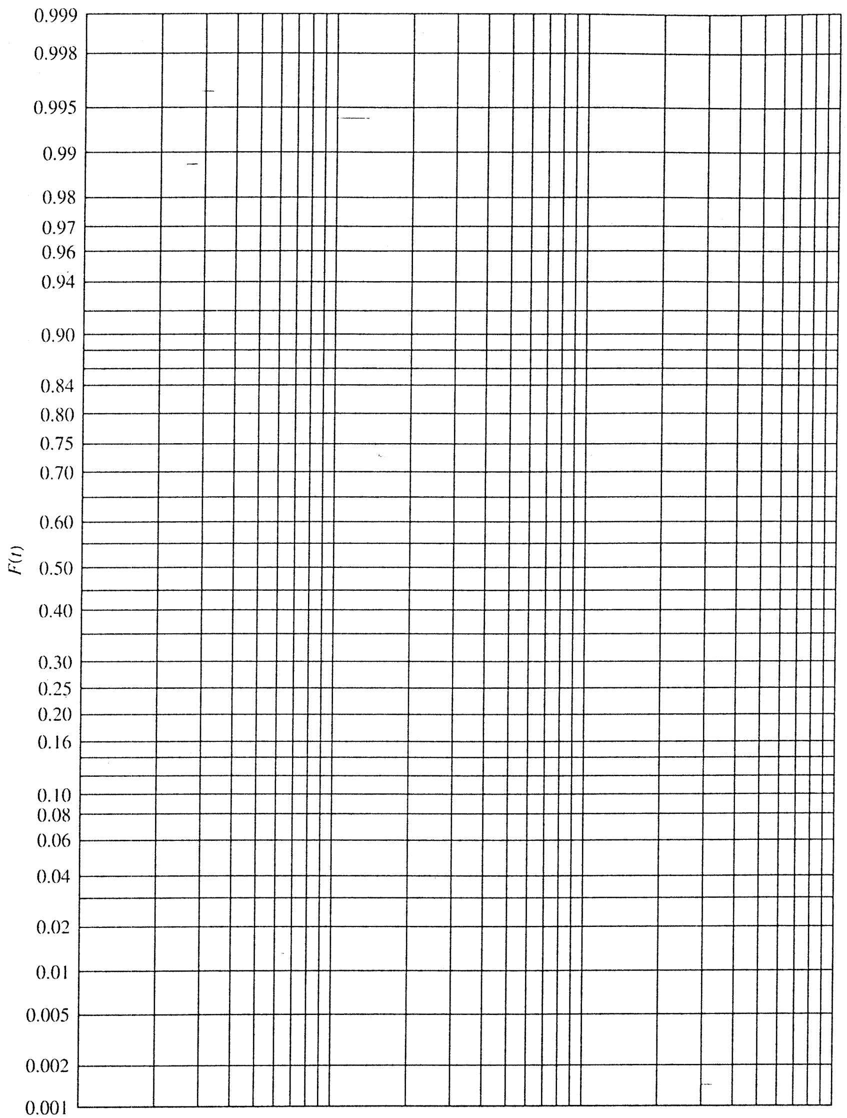

FIGURE A.4
Lognormal distribution probability paper.

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