Signal-Plus-Noise Models vs. Stationary Models

Deterministic Signal-Plus-Noise Models

$$\boxed{X_t = s_t + Z_t}$$

s_t is a deterministic signal

Z_t is a zero-mean, stationary process

Example signals: $s_t = a + bt$

$s_t = a + bt + ct^2$

$s_t = A\cos(2\pi ft + C)$ C constant

X_t is non-stationary because of <u>non-constant mean.</u>

Deterministic Signal-Plus-Noise Models

$$\boxed{X_t = s_t + Z_t}$$

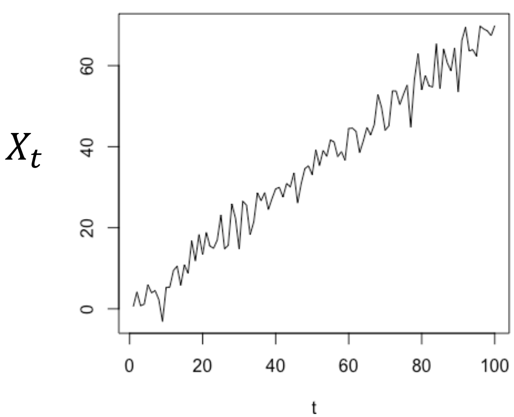
s_t is a deterministic signal

Z_t is a zero-mean, stationary process

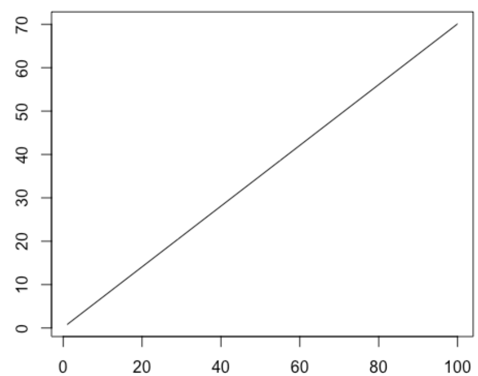
Example signal: $s_t = a + bt$

$$X_t = .1 + .7t + Z_t \qquad Z_t \sim \mathrm{N}(0,10)$$

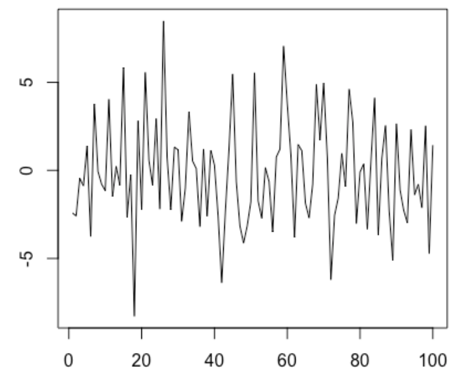
Realization



Signal



Noise



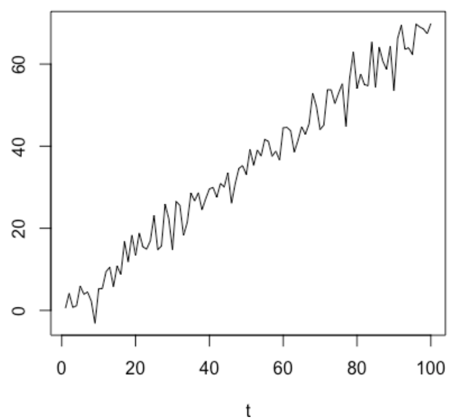
X_t is non-stationary because of non-constant mean.

Deterministic Signal-Plus-Noise Models

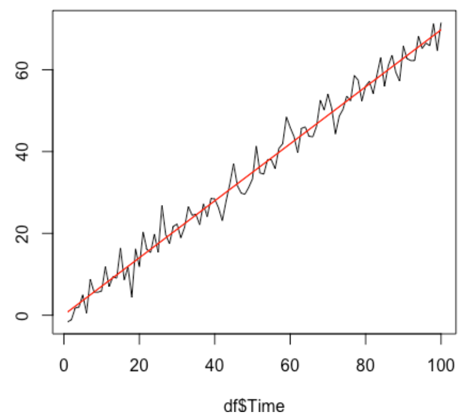
Example signal: $X_t = .1 + .7t + Z_t \qquad Z_t \sim$ N(0,10)

In practice, we only have the realization and we don't know that beta_0 = .1 and beta_1 = .7. In practice, we will estimate fit a line and estimate them.

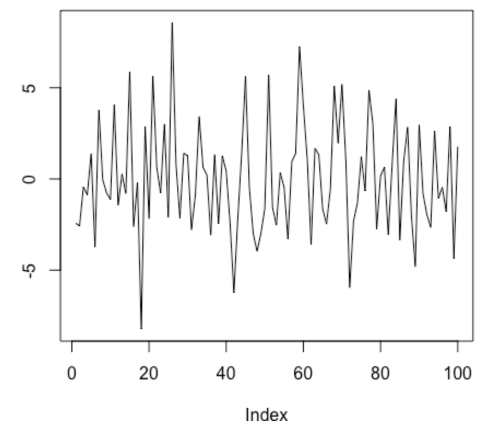
Realization



Estimated signal



Residuals (noise)




```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.10661    0.61375   0.174    0.862    
Time         0.69652    0.01055  66.012   <2e-16 ***
---
```

```
Five Smallest Values of  aic 
     p   q        aic
16   5   0   2.215608
18   5   2   2.225088
1    0   0   2.227298
10   3   0   2.227450
17   5   1   2.232907
```

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Deterministic Signal-Plus-Noise Models

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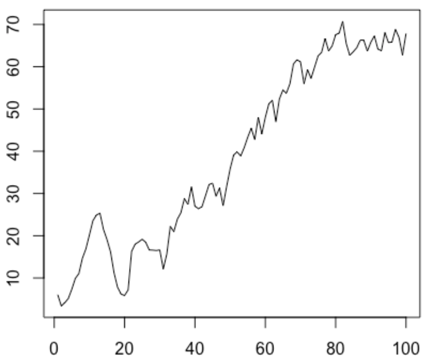
s_t is a deterministic signal

Z_t is a zero-mean, stationary process

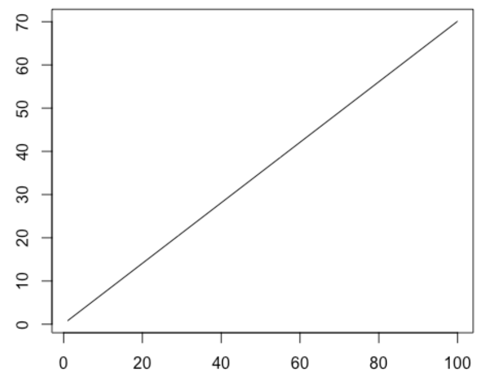
Example signals: $s_t = a + bt$

$X_t = .1 + .7t + Z_t$ $\quad\quad$ Z_t are AR(1)

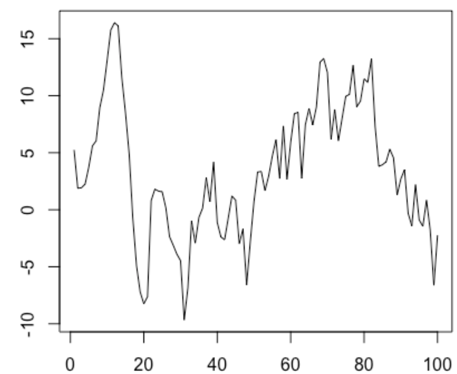
Realization



Signal



Noise



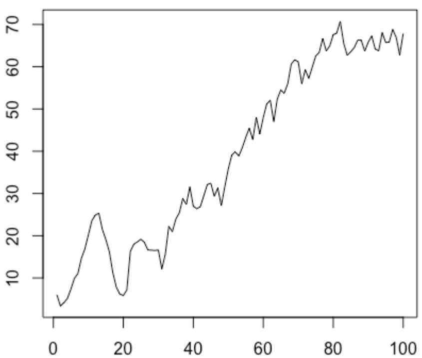
X_t is non-stationary because of non-constant mean.

Deterministic Signal-Plus-Noise Models

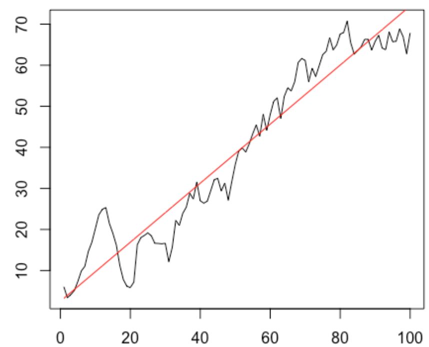
Example signals: $X_t = .1 + .7t + Z_t$ $\quad$ Z_t are AR(1)

In practice, we only have the realization and we don't know that beta_0 = .1 and beta_1 = .7. In practice, we will estimate fit a line and estimate them.

Realization

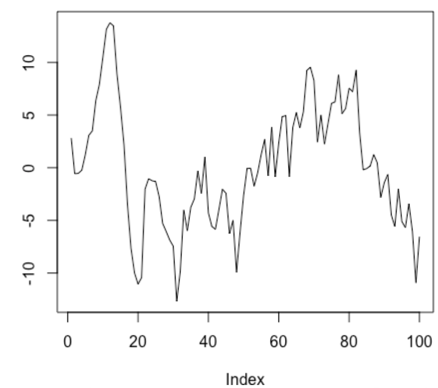


Estimated signal




```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.5118     1.1811   2.127    0.036 *
Time          0.7190     0.0203  35.411   <2e-16 ***
---
```

Residuals (noise)




```
Five Smallest Values of  aic
   p  q      aic
15 4  2 2.067117
4  1  0 2.106889
7  2  0 2.125041
5  1  1 2.125327
8  2  1 2.138012
```

We know that the noise is AR(1), and that is one of the models suggested by the AIC. We will wait to show how to identify the model and estimate the coefficients in future Units.

Deterministic Signal-Plus-Noise Models

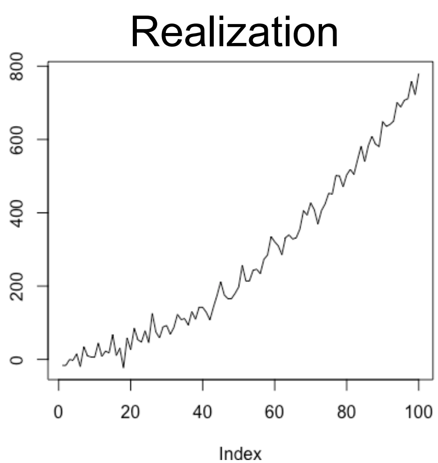
$$\boxed{X_t = s_t + Z_t}$$

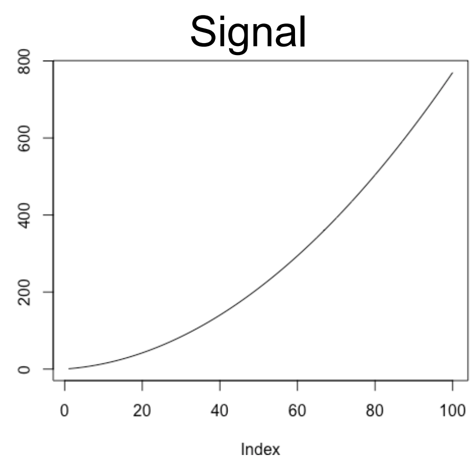
s_t is a deterministic signal

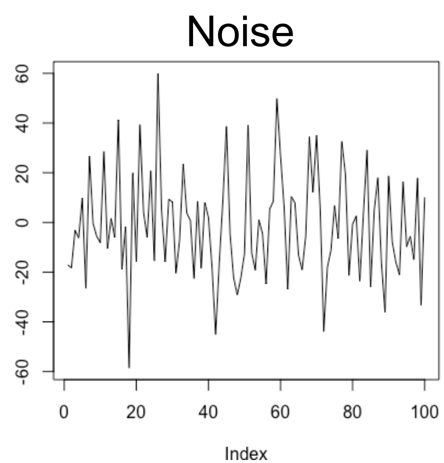
Z_t is a zero-mean, stationary process

Example signals: $s_t = a + bt + ct^2$

$$X_t = .1 + .7t + .07t^2 + Z_t \qquad Z_t \sim \text{N}(0{,}500)$$







X_t is nonstationary because of non-constant mean.

Deterministic Signal-Plus-Noise Models

$$\boxed{X_t = s_t + Z_t}$$

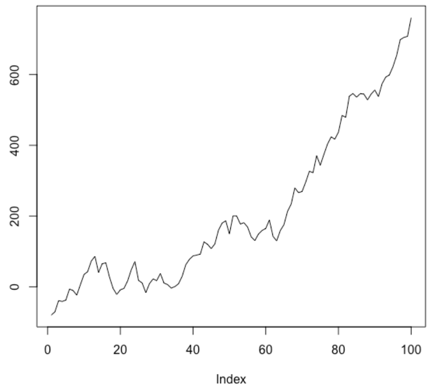
s_t is a deterministic signal

Z_t is a zero-mean, stationary process

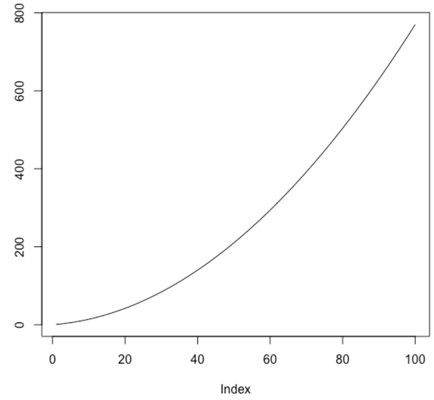
Example signals: $s_t = a + bt + ct^2$

$$X_t = 2 + 4t + .07t^2 + Z_t \qquad Z_t \text{ are AR(1)}$$

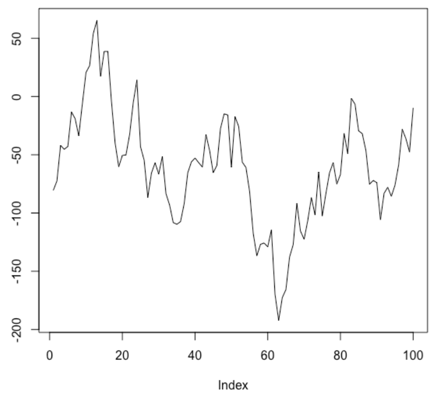
Realization



Signal



Noise



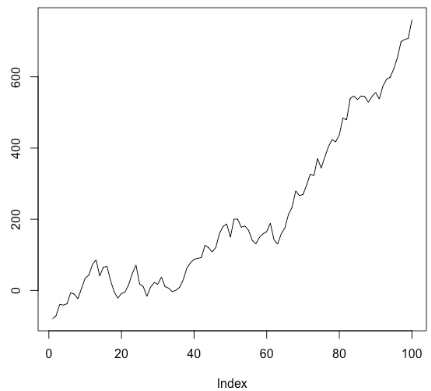
X_t is nonstationary because of non-constant mean.

Deterministic Signal-Plus-Noise Models

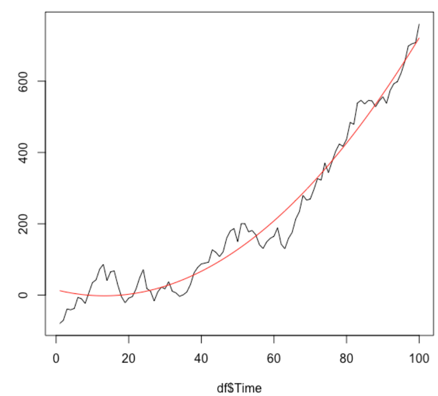
Example signals: $X_t = 2 + 4t + .07t^2 + Z_t$ $\quad$ Z_t are AR(1)

In practice, we only have the realization and we don't know that beta_0 = 2 and beta_1 = 4 and beta_2 = .07. In practice, we will estimate fit a model and estimate them.

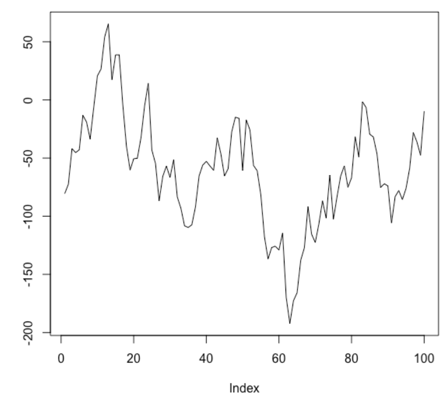
Realization



Estimated signal



Residuals (noise)




```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.755414  12.618202   1.169    0.245
Time        -2.543920   0.576686  -4.411 2.66e-05 ***
Time2        0.096116   0.005532  17.375  < 2e-16 ***
```

```
Five Smallest Values of  aic
     p    q        aic
4    1    0   6.199118
7    2    0   6.215963
5    1    1   6.216040
13   4    0   6.225594
6    1    2   6.233495
```

We know that the noise is AR(1), and that is one of the models suggested by the AIC. We will wait to show how to identify the model and estimate the coefficients in future Units.

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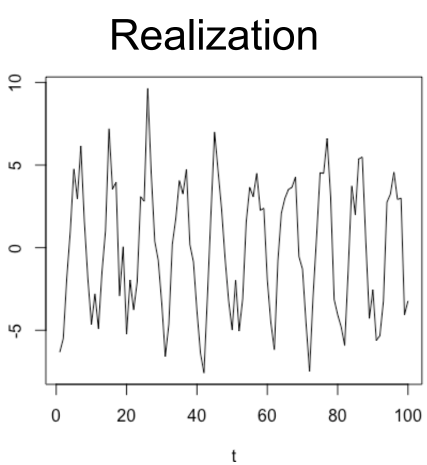
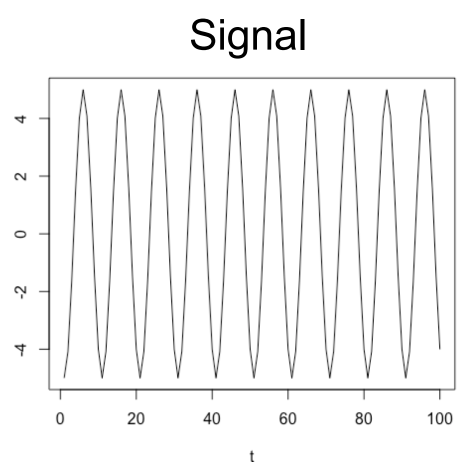
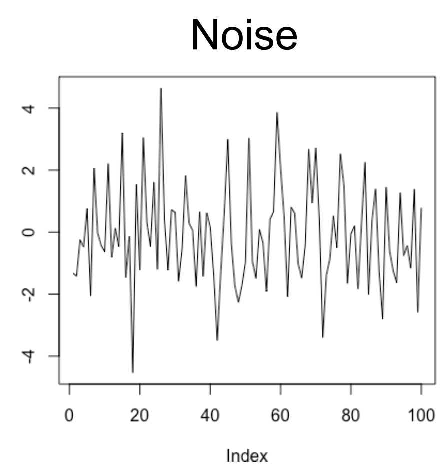
Example signals: $s_t = A\cos(2\pi ft + C)$ C constant

$$X_t = 5\cos(2\pi(.1)t + 2.5) + Z_t \qquad Z_t \sim \mathrm{N}(0,3)$$







X_t is nonstationary because of non-constant mean.

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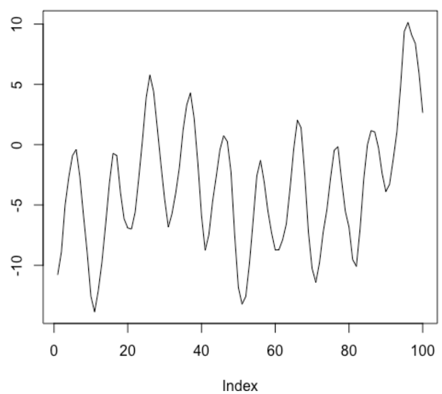
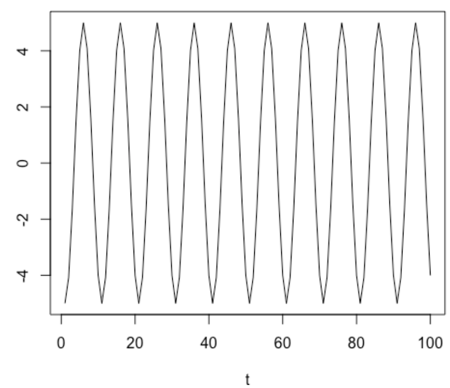
Example signals: $s_t = A\cos(2\pi f t + C)$ C constant

$$X_t = 5\cos(2\pi(.1)t + 2.5) + Z_t \qquad Z_t \sim \text{ARMA}(1,1)$$

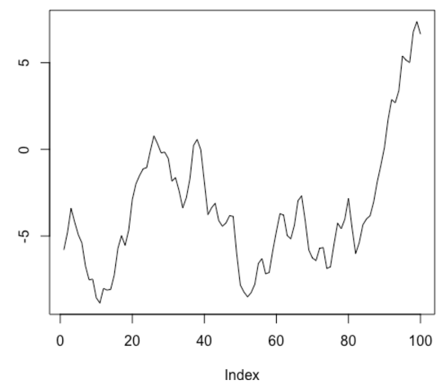
Realization



Signal



Noise



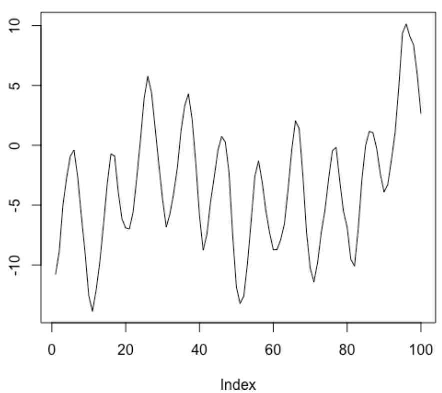
X_t is nonstationary because of non-constant mean.

Deterministic Signal-Plus-Noise Models

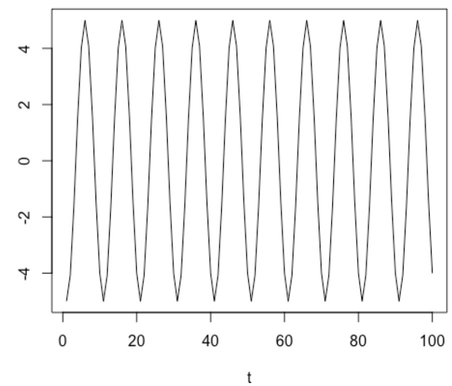
Example signals: $s_t = A\cos(2\pi f t + C)$ C constant

$$X_t = 5\cos(2\pi(.1)t + 2.5) + Z_t \qquad Z_t \sim \text{ARMA}(1,1)$$

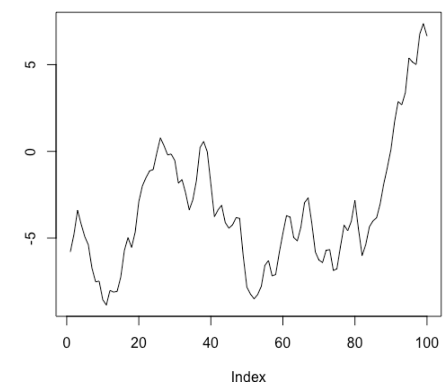
Realization



Signal



Noise



We know that the noise is AR(1,1), and that is one of the models suggested by the AIC. We will wait to show how to identify the model and estimate the coefficients in future Units.

Five Smallest Values of aic

	p	q	aic
16	5	0	2.541818
13	4	0	2.544026
10	3	0	2.653253
7	2	0	2.744524
5	1	1	2.886489

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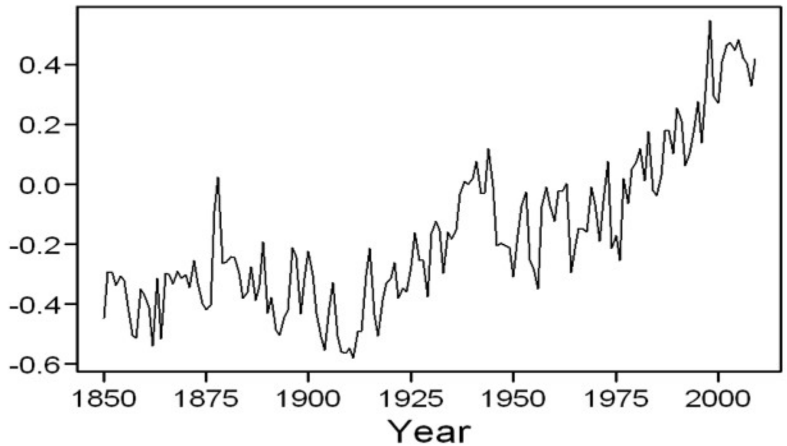
Signal Present?

Sometimes it's not easy to tell whether a deterministic signal is present in the data

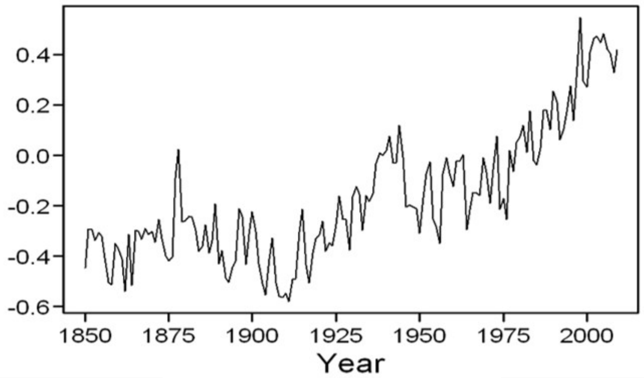


(a) Global Temperature Data

Is there a deterministic signal?

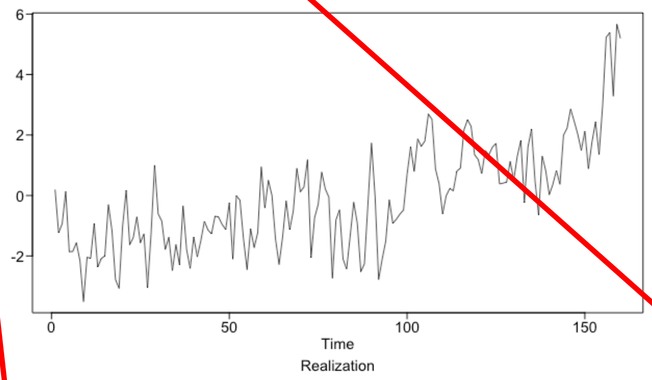
Realizations

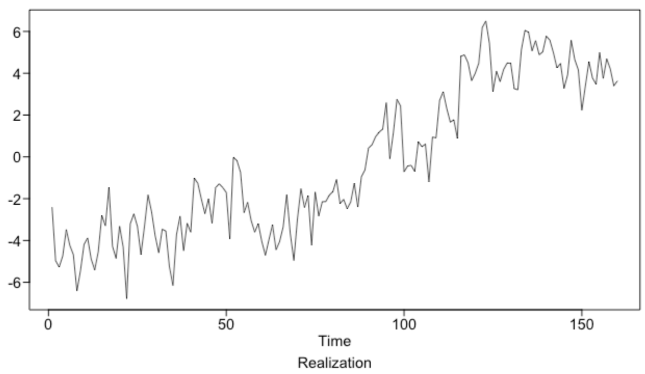
Sometimes it's not easy to tell whether a deterministic signal is present in the data



Global warming?

We know these were generated from a stationary (AR(4)) model!





parms = mult.wge(c(.975),c(.2,-.45),c(-.53))

gen.arma.wge(160,phi = parms$model.coef,vara = 1, sn = 23)

parms = mult.wge(c(.975),c(.2,-.45),c(-.53))

gen.arma.wge(160,phi = parms$model.coef,vara = 1, sn = 13)

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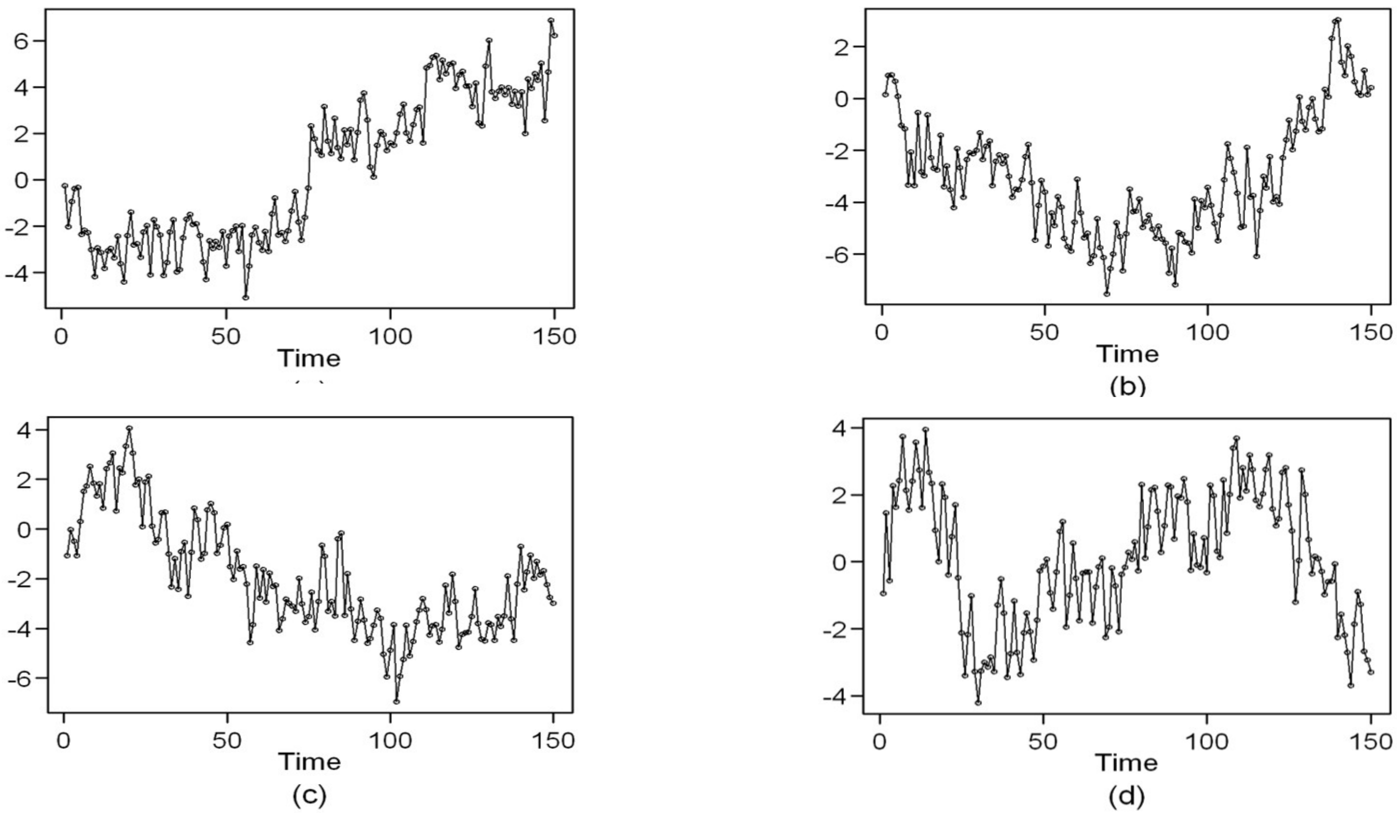
$$X_t = 5\cos(2\pi(.1)t + 2.5) + Z_t \quad Z_t \sim \text{AR(1)}$$

Four Realizations from the Stationary AR(4) Model

$$(1-.975B)(1-.2B+.45B^2)(1+.53B)X_t = a_t$$

$$(1-.645B+.02225B^2-.0969B^3-.2325B^4)Xt = at$$



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Screencast for Signal-Plus-Noise Models

Screencast: Just run these a few times to show that sometimes you just randomly get something that looks like it has trend, even when it does not.

```
gen.sigplusnoise.wge(100,2,4,vara = 100)

gen.sigplusnoise.wge(100,0,0,vara = 10)
```

```
parms = mult.wge(c(.975),c(.2,-.45),c(-.53))
gen.arma.wge(160,phi = parms$model.coef,vara = 1)
```

Also, be able to generate realizations from a model in factored form

$$(1-.975B)(1-.2B+.45B^2)(1+.53B)X_t = a_t$$

```
parms = mult.wge(c(.975),c(.2,-.45),c(-.53))
parms$model.coef

# Output:
# [1]  0.6450000 -0.0222500  0.0969000  0.2325375



#Now that we have the coefficients, () we can generate a realization.
gen.arma.wge(160,phi = parms$model.coef,vara = 1)
```

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ARIMA Properties and Characteristics

ARIMA(*p*,*d*,*q*) Models

The *autoregressive integrated moving average process* of orders p, d, and q (denoted ARIMA(p,d,q)) is a process, X_t, whose differences $(1-B)^d X_t$ satisfy a (stationary) ARMA(p,q) model, where d is a non-negative integer.

We use the notation:

$$(B)(1 \quad B)^d X_t = (B)a_t$$

d = 1

$$(1 - .67B)(1 - 1.6B + .8B^2)(1 - B)X_t = (1 - .84B)a_t$$

ARIMA(3,1,1)

You can generate realizations from the ARIMA(*p*,*d*,*q*) model using **`gen.arima.wge`**

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ARIMA(*p*,*d*,*q*) Models

$$(B)(1 \quad B)^d X_t = \quad (B)a_t$$

(B) and   (B) are stationary (and invertible) parts of the ARIMA(p,d,q) model. That is, all the roots of

$$\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p = 0$$

and

$$\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q = 0$$

lie outside the unit circle.

And

$$(1 - B)^d$$

clearly has roots on the unit circle.

The non-stationary part!

In fact… they are *d* roots of 1!

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Extended Autocorrelations

Question: What is ρ_k for an ARIMA(p,d,q) process?

Consider AR(1) $(1 \quad {}_1 B)X_t = a_t$ $\qquad {}_k = \frac{{}_k}{{}_0}$

if $\varphi_1 = .8$ $\qquad {}^2 = {}_0 = \frac{{}^2_a}{1 \quad {}^2_1}$

Light board?

$$= \frac{{}^2_a}{1 \quad .8^2} = \frac{{}^2_a}{.36} = 2.78 \; {}^2_a \quad \text{if} \quad {}_1 = .8$$

$$= 5.26 \; {}^2_a, \qquad {}_1 = .9$$

$$= 100 \; {}^2_a, \qquad {}_1 = .99$$

$$= 10{,}000 \; {}^2_a, \qquad {}_1 = .9999$$

as ${}_1 \quad 1$

What is ${}_k$ when ${}_1 = 1$?

An ARIMA(p,d,q) model is non-stationary because σ^2 $(= \gamma_0)$ **is not finite**.

For an AR(1) $(1 - \varphi_1 B)X_t = a_t$, we know that $\gamma_k = \gamma_0 \varphi_1^k$.

So, $\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\gamma_0 \varphi_1^k}{\gamma_0}$ and for the case $\varphi_1 = 1$,
$\gamma_0 = \infty$, so it follows that $\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\infty}{\infty} = ?$

Light board ?

Clearly, we need a new definition of autocorrelation for the case $(1-B)X_t = a_t$.

Note that if $|\;_1| < 1$, then $\;_0 = \;^2$ is finite.

So, $\rho_k = \dfrac{\gamma_0 \varphi_1^k}{\gamma_0} = \varphi_1^k$ for each $|\varphi_1| < 1$ as $\varphi_1 \to 1$.

Light board?

It seems reasonable to define ρ_k for the limiting case, i.e. $\varphi_1 = 1$, as the limit of $\rho_k = \varphi_1^k$ as φ_1 goes to 1. That is, the limiting autocorrelation is $\rho_k^* = 1^k = 1$ for each k.

In general, for ARIMA (and ARUMA) models, we define the "autocorrelation" as a *limit*.

“Woodward and colleagues use the term “extended autocorrelations.” We will continue to call them “autocorrelations” for simplicity.

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ARIMA and tswge

For $(1 \quad B)X_t = a_t$ use

```
x1=gen.arima.wge(n=200, d=1)
plotts.sample.wge(x1)
```

For $(1 \quad B)(1 \quad 1.5B + .8B^2)X_t = a_t$ use

```
x2=gen.arima.wge(n=200, phi=c(1.5,-.8),d=1)
plotts.sample.wge(x2)
```

For $(1 \quad B)^2(1 \quad 1.5B + .8B^2)X_t = (1 + .8B)a_t$ use

```
x3=gen.arima.wge(n=200, phi=c(1.5,-.8),d=2,theta=-.8)
plotts.sample.wge(x3)
```

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ARIMA

ARIMA(0,1,0)

ARIMA(0,1,0)

$$(1 \quad B)X_t = a_t$$

Notes:

- Let $Y_t = (1 - B)X_t$, so $Y_t = a_t$ (i.e. white noise)
- $X_t - X_{t-1} = a_t$ or $X_t = X_{t-1} + a_t$

The value of X_t at time t is equal to the value at time $t-1$ plus a random, zero-mean noise component.

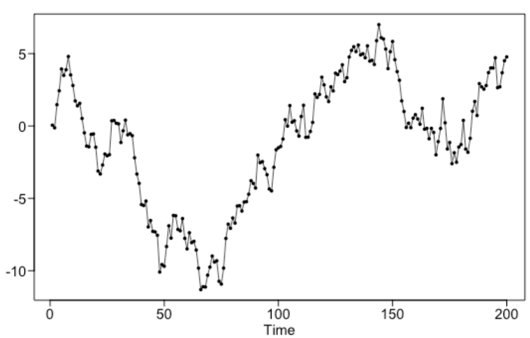
- At each time, *t*, the process is equally likely to go up or down from time $t-1$.
- This is wandering (random walk) behavior.

ARIMA(0,1,0)

$$(1 \quad B)X_t = a_t$$

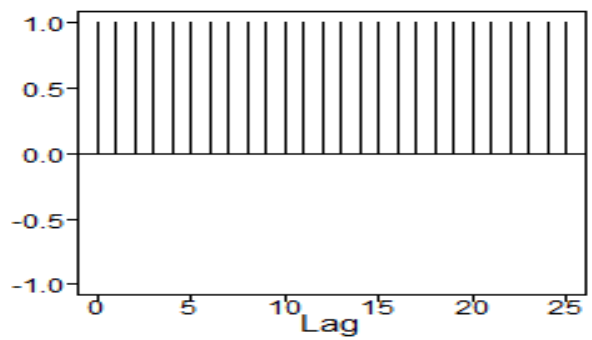
```
#R Code:
x = gen.arima.wge(200,phi = 0, var = 1,d = 1,sn = 31)
acf(x)
```

Realization



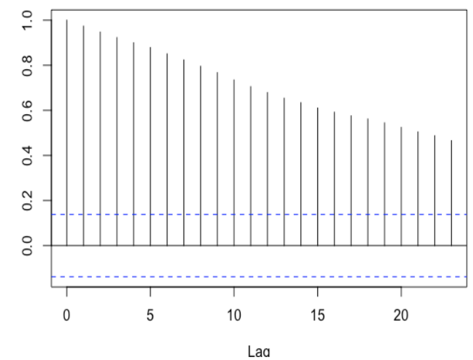
Strictly wandering since there is one factor with a root of 1

True autocorrelations



Identically equal to $\rho_k = 1$

Sample autocorrelations



Slowly damping, even though true values don't damp

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Differencing Data

“Difference” the data using tswge.

tswge transformation function…. *run this code!*

```
# the following command differences the data in x
y=artrans.wge(x,phi.tr=1)
# This simply means that y(i) = x(i) - x(i-1)
# y has length n-1 because x(1) has no x(0) before it.
# Example
x = c(1,3,6,10,25)
y = artrans.wge(x,phi.tr = 1)
y # shows the 4 differences
```

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ARIMA

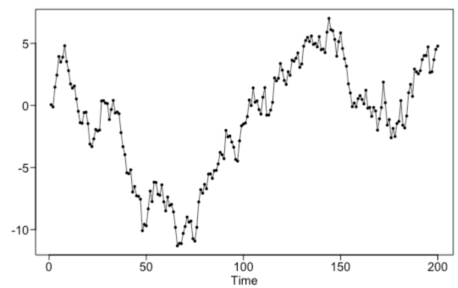
Examples

Stationarize the ARIMA(0,1,0): Taking out the (1 – *B*)

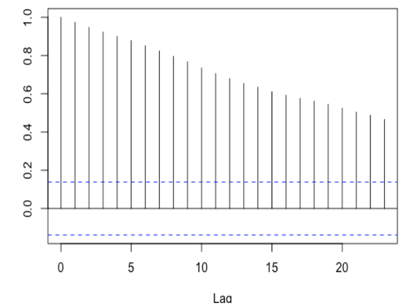
$$(1 \quad B)X_t = a_t \;\rightarrow\; X_t - X_{t-1} = a_t$$

```
#R Code:
x = gen.arima.wge(200,phi = 0, var = 1,d = 1,sn = 31)
acf(x)
```

Realization



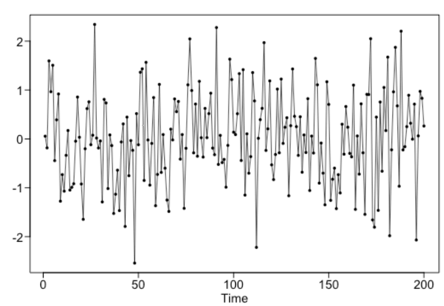
Sample autocorrelations



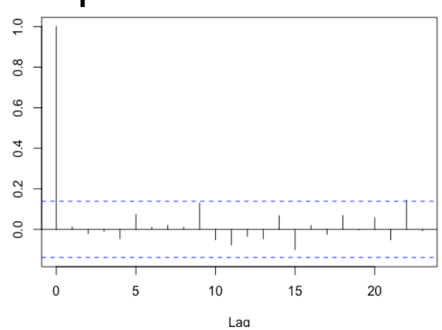
To make the ARIMA(0,1,0) model stationary, we simply need to “difference” the data. As we saw in the filtering unit, the first difference is ($\tilde{X}_t = X_t - X_{t-1}$) and, by the definition of the model, is equal to white noise (a_t). Since white noise is a stationary process, we can simply take the first difference to “stationarize” these data. In other words, differencing the data will remove the (1 – *B*) so that: $\tilde{X}_t = a_t$

```
#R Code:
Xtilde = artrans.wge(x,1)
plotts.wge(Xtilde)
acf(Xtilde)
```

Realization



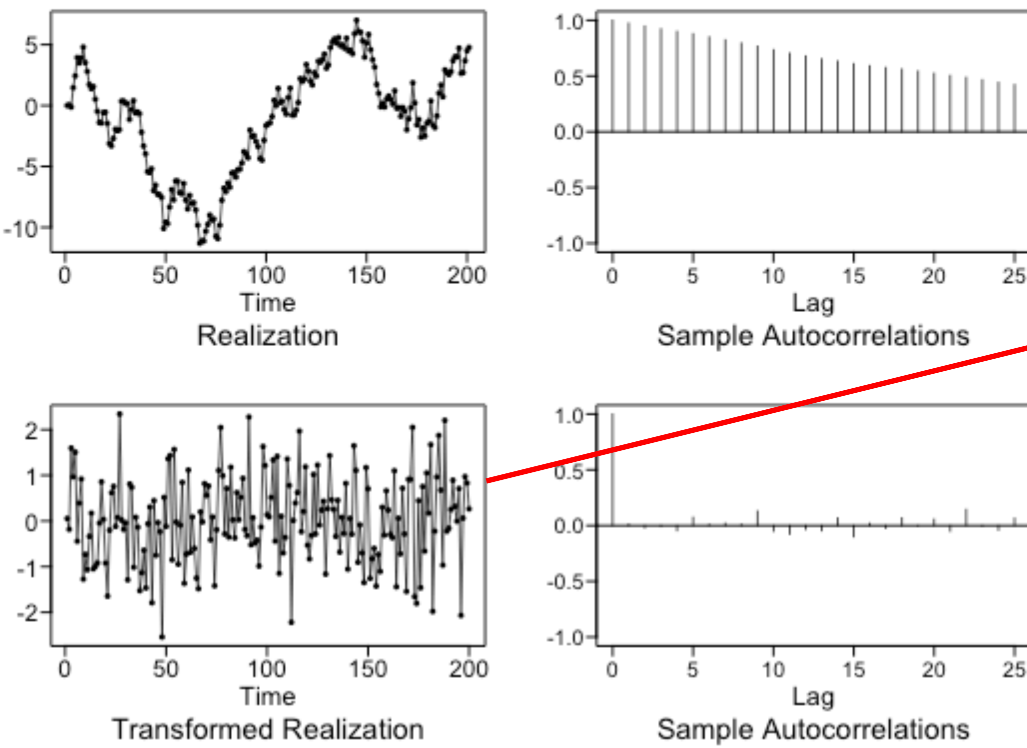
Sample autocorrelations



Stationarize the ARIMA(0,1,0): Taking out the (1 – *B*)

$$(1 \quad B)X_t = a_t$$

```
x = gen.arima.wge(200,phi = 0, var = 1,d = 1,sn = 31)
artrans.wge(x,1)
aic5.wge(artans.wge(x,1))
```




```
Five Smallest Values of  aic
      p     q          aic
1     0     0 -0.08725030   *
2     0     1 -0.07740763
4     1     0 -0.07740123
5     1     1 -0.07016072
3     0     2 -0.06785942
```


* We will estimate the coefficients of these models in a later unit.

ARIMA(0,1,0): Stock Example

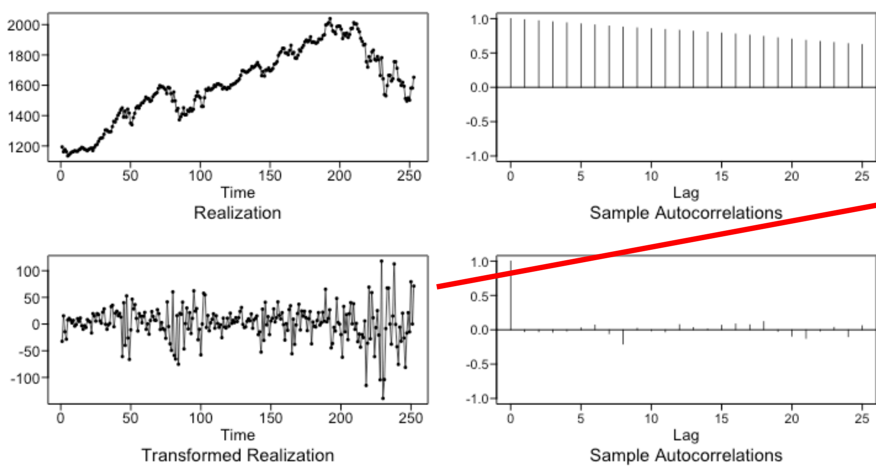
$$(1 \quad B)X_t = a_t$$

Amazon Stock

Plot care of yahoo finance.

ARIMA(0,1,0): Stock Example

Amazon stock



Five Smallest Values of aic

	p	q	aic
1	0	0	6.989672 *
2	0	1	6.996810
4	1	0	6.996860
5	1	1	7.001863
3	0	2	7.003753

```
amzn = read.csv(file.choose(),header = TRUE)
x = artrans.wge(amzn$Adj.Close,1)
```

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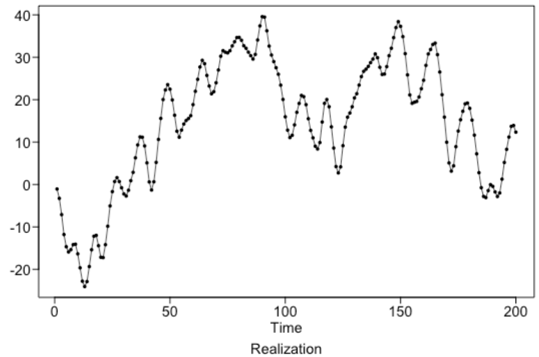
* We will estimate the coefficients of these models in a later unit.

ARIMA(2,1,0)

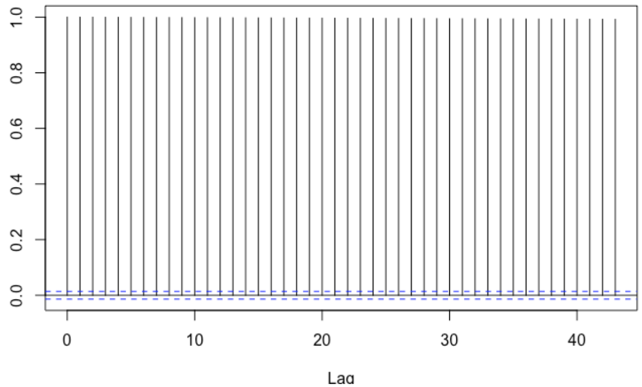
$$(1 - 1.5B + .8B^2)(1 - B)X_t = a_t$$

```
a = gen.arima.wge(200,phi = c(1.5,-.8), var = 1,d = 1,sn = 31)
acf(a)
```

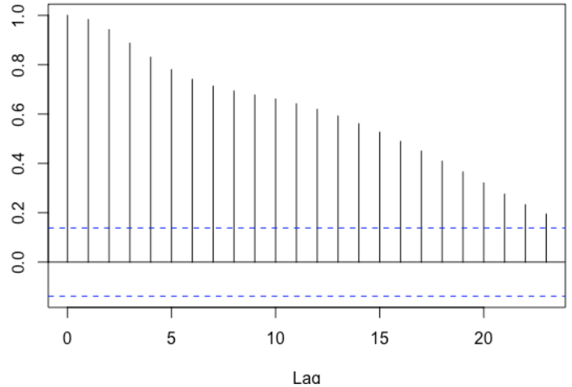
Realization



True autocorrelations



Sample autocorrelations



Wandering with possibly a hint of the cyclic behavior

Identically equal to $\rho_k = 1$ (even though there is second order cyclic stationary factor)

Slowly damping with very little indication of cyclic behavior

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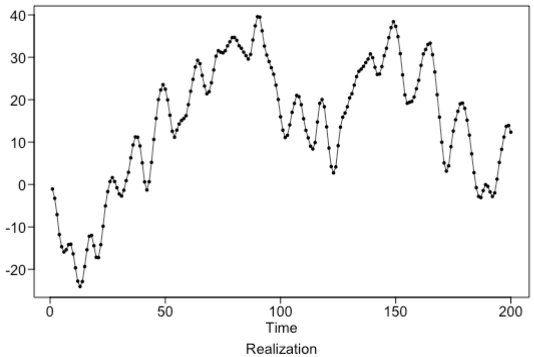
ARIMA(2,1,0)

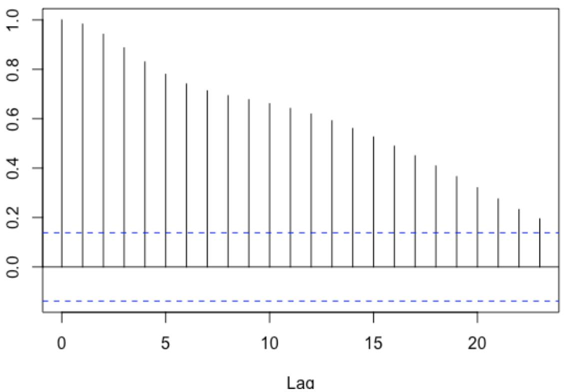
$$(1 - 1.5B + .8B^2)(1 - B)X_t = a_t$$

```
> model = mult.wge(fac1 = c(1.5,-.8), fac2 = 1)
> factor.wge(model$model.coef)

Coefficients of Original polynomial:
2.5000 -2.3000 0.8000

Factor                 Roots              Abs Recip    System Freq
1-1.0000B              1.0000             1.0000       0.0000
1-1.5000B+0.8000B^2    0.9375+-0.6092i    0.8944       0.0917
```

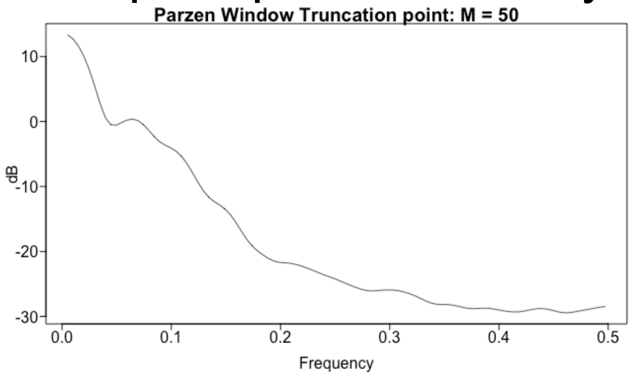
Realization



Sample autocorrelations



Sample Spectral Density



Wandering with possibly a hint of the cyclic behavior

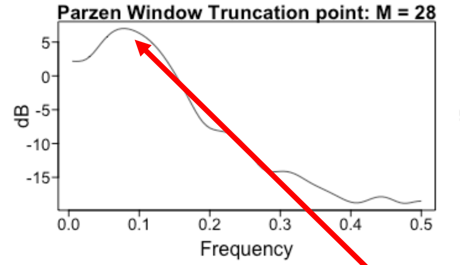
Slowly damping with very little indication of cyclic behavior

There is a slight peak at .09, but the dominating behavior is at 0

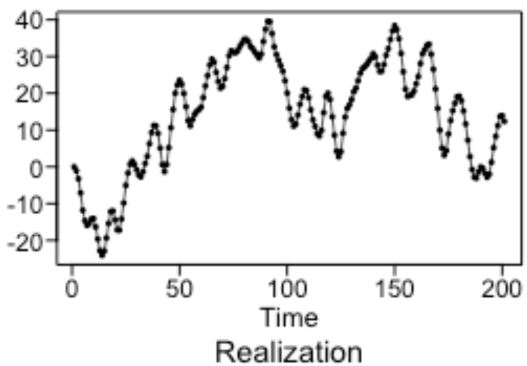
Stationarize the ARIMA(2,1,0): Taking out the (1–*B*)

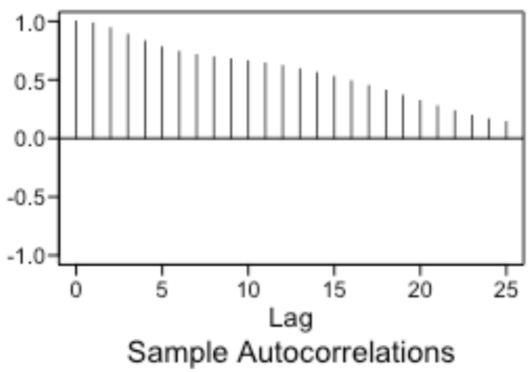
$$(1 - 1.5B + .8B^2)(1 - B)X_t = a_t$$

```
x = gen.arima.wge(200,phi = c(1.5,-.8), var = 1,d = 1,sn = 31)
FirstDif = artrans.wge(x,1) #Take out the (1-B)
parzen.wge(FirstDif)
aic5.wge(FirstDif) #Check the structure of the noise
```



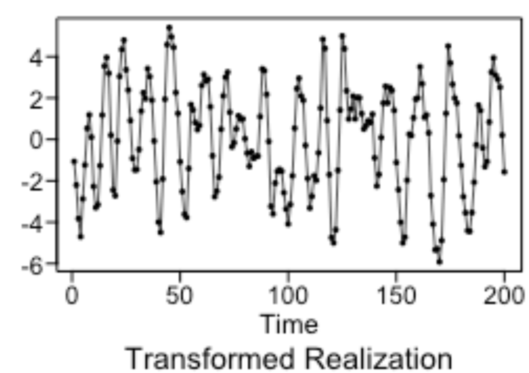
Factor	Roots	Abs Recip	System Freq
1-1.0000B	1.0000	1.0000	0.0000
1-1.5000B+0.8000B^2	0.9375+-0.6092i	0.8944	0.0917

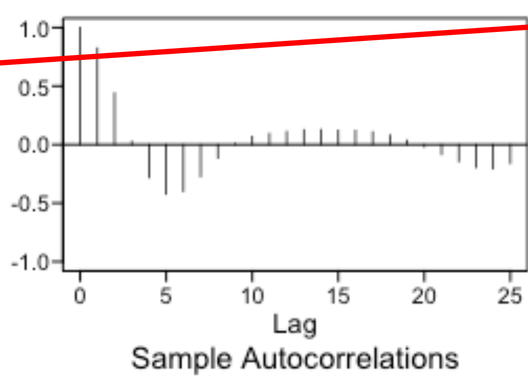




Five Smallest Values of aic

	p	q	aic
7	2	0	-0.04218408 *
8	2	1	-0.03402526
10	3	0	-0.03396977
14	4	1	-0.02959502
13	4	0	-0.02416244





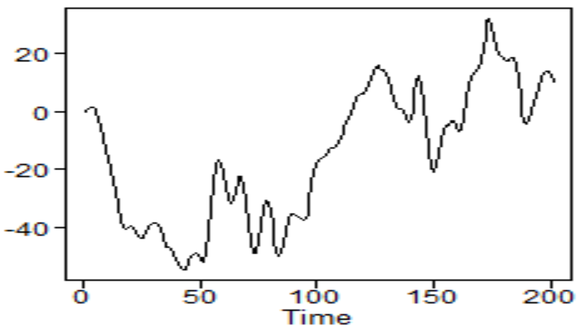
* We will estimate the coefficients of these models in a later unit.

ARIMA(2,1,0)

$$(1 - 1.5B + .8B^2)(1 - B)X_t = a_t$$

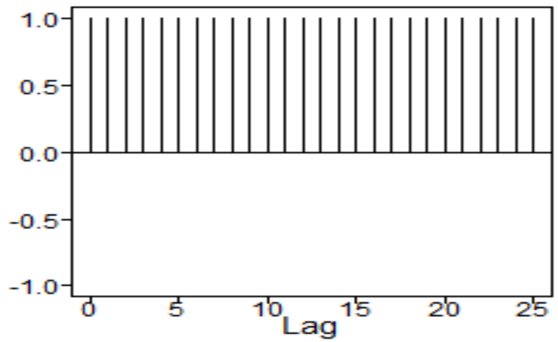
```
> model = mult.wge(fac1 = c(1.5,-.8), fac2 = 1)
> factor.wge(model$model.coef)

Coefficients of Original polynomial:
2.5000 -2.3000 0.8000

Factor                Roots              Abs Recip    System Freq
1-1.0000B             1.0000             1.0000       0.0000
1-1.5000B+0.8000B^2   0.9375+-0.6092i    0.8944       0.0917
```

Realization



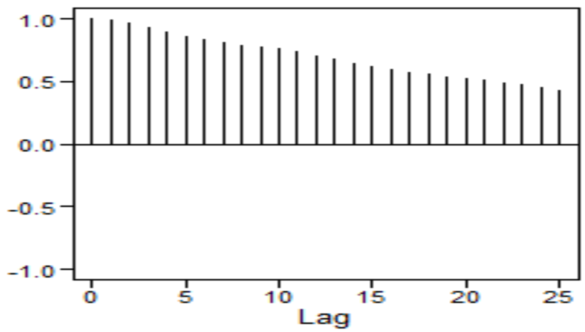
Wandering with possibly a hint of the cyclic behavior

True autocorrelations



Identically equal to $\rho_k = 1$ (even though there is second order cyclic stationary factor)

Sample autocorrelations



Slowly damping with very little indication of cyclic behavior

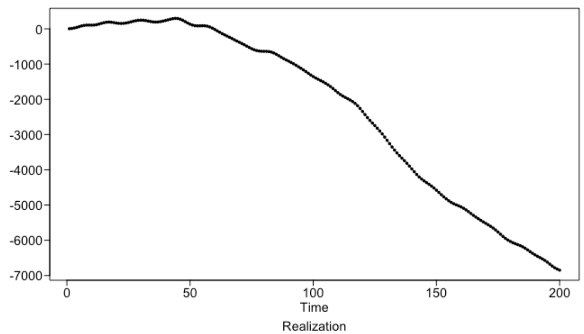
ARIMA(2,2,1)

$$(1-1.5B+.8B^2)(1-B)^2 X_t = (1+.8B)a_\tau$$

```
> a = gen.arima.wge(200, phi = c(1.5,-.8), theta = -.8, d = 2, vara = 1, sn = 21)
> acf(a)
> p = parzen.wge(a,trunc = 40)
> ar = mult.wge(fac1 = c(1.5,-.8), fac2 = 1, fac3 = 1)
> factor.wge(ar$model.coef)

Coefficients of Original polynomial:
3.5000 -4.8000 3.1000 -0.8000

Factor                 Roots                Abs Recip    System Freq
1-1.0000B              1.0000               1.0000       0.0000
1-1.0000B              1.0000               1.0000       0.0000
1-1.5000B+0.8000B^2    0.9375+-0.6092i      0.8944       0.0917
```

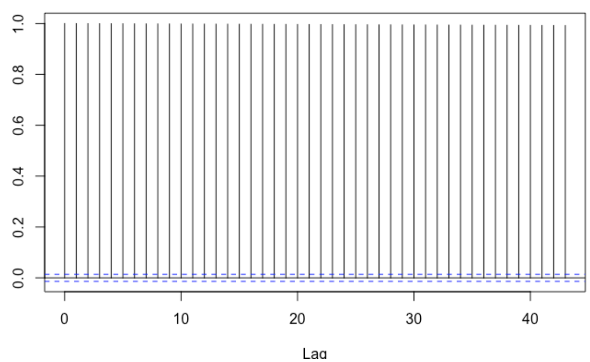
```
> factor.wge(-.8)

Coefficients of Original polynomial:
-0.8000

Factor                 Roots                Abs Recip    System Freq
1+0.8000B              -1.2500              0.8000       0.5000
```

Realization



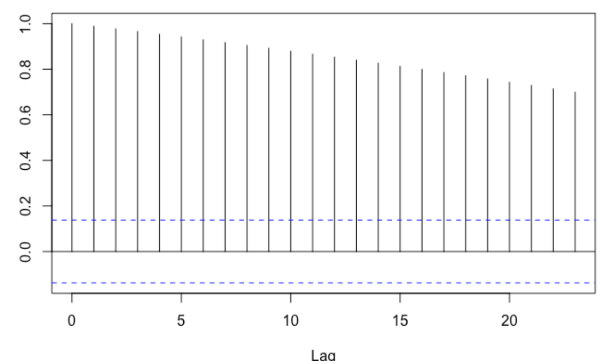
Very highly correlated wandering behavior with no hint of the cyclic factor (dominated by the two roots of 1)

True autocorrelations



Again, identically equal to $\rho_k = 1$

Sample autocorrelations



Very slowly damping with no indication of cyclic or MA behavior… mostly because of the way they are calculated

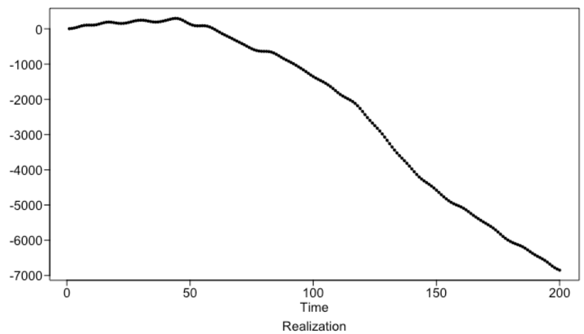
ARIMA(2,2,1)

$$(1 - 1.5B + .8B^2)(1 - B)^2 X_t = (1 + .8B)a_\tau$$

```
> a = gen.arima.wge(200, phi = c(1.5,-.8), theta = -.8, d = 2, vara = 1, sn = 21)
> acf(a)
> p = parzen.wge(a,trunc = 40)
> ar = mult.wge(fac1 = c(1.5,-.8), fac2 = 1, fac3 = 1)
> factor.wge(ar$model.coef)

Coefficients of Original polynomial:
3.5000 -4.8000 3.1000 -0.8000

Factor                 Roots              Abs Recip    System Freq
1-1.0000B              1.0000             1.0000       0.0000
1-1.0000B              1.0000             1.0000       0.0000
1-1.5000B+0.8000B^2    0.9375+-0.6092i    0.8944       0.0917
```

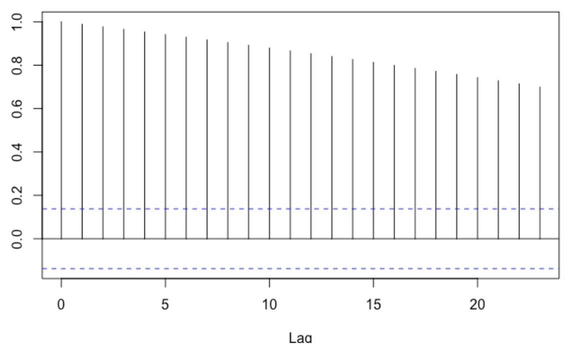
```
> factor.wge(-.8)

Coefficients of Original polynomial:
-0.8000

Factor              Roots             Abs Recip    System Freq
1+0.8000B           -1.2500           0.8000       0.5000
```

Realization



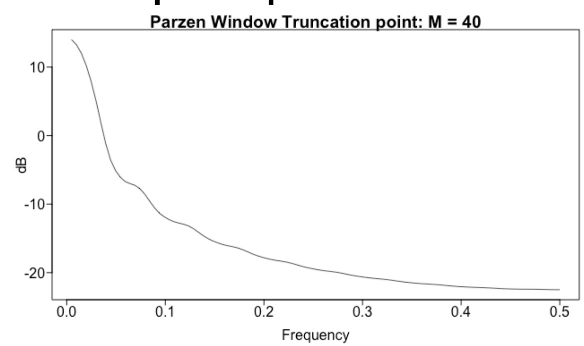
Very highly correlated wandering behavior with no hint of the cyclic factor (dominated by the two roots of 1)

Sample autocorrelations



Very slowly damping with no indication of cyclic or MA behavior

Sample spectral density

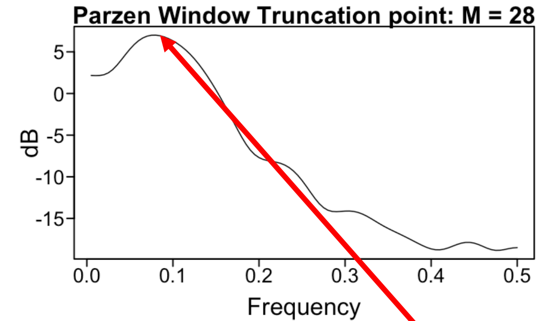


There is almost no evidence of the frequency at .0917

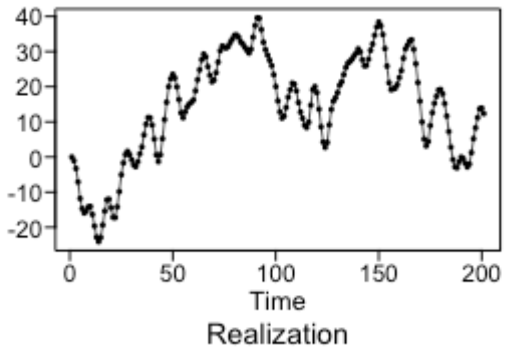
Stationarize the ARIMA(2,2,1): Taking out the $(1 - B)^2$

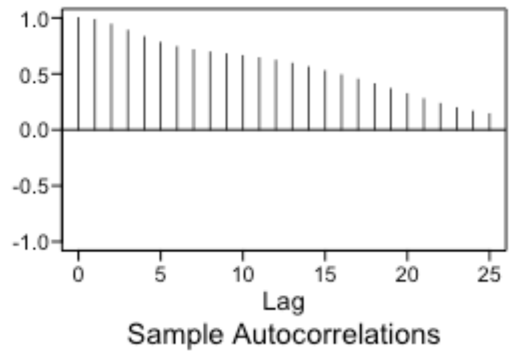
$$(1 - 1.5B + .8B^2)(1 - B)^2 X_t = (1 + .8B)a_\tau$$

```
x = gen.arima.wge(200,phi = c(1.5,-.8), var = 1,d = 2,sn = 31)
FirstDif = artrans.wge(x,1) #Take out the (1-B)
SecondDif = artrans.wge(FirstDif,1) # Take out the other (1-B)
parzen.wge(SecondDif)
aic5.wge(SecondDif) #Check the structure of the noise
```



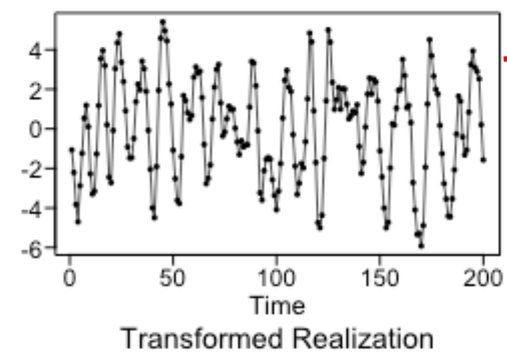
Factor	Roots	Abs Recip	System Freq
1-1.0000B	1.0000	1.0000	0.0000
1-1.0000B	1.0000	1.0000	0.0000
1-1.5000B+0.8000B^2	0.9375+-0.6092i	0.8944	0.0917

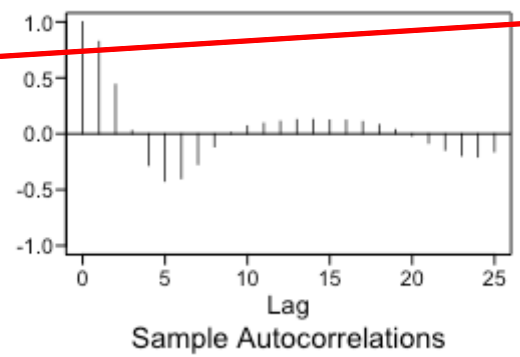




Five Smallest Values of aic

	p	q	aic
7	2	0	-0.04218408
8	2	1	-0.03402526 *
10	3	0	-0.03396977
14	4	1	-0.02959499
13	4	0	-0.02416244





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* We will estimate the coefficients of these models in a later unit.

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ARIMA

General Comments

General Comments about ARIMA Models

- The $(1-B)^d$ factor dominates the stationary components.
  - In the realizations
  - Autocorrelations
  - Spectral densities all have peaks at $f = 0$
- For $d > 1$, this domination is even more striking.
- The true autocorrelations are equal to 1 for all k.
- The sample autocorrelations will always damp (in part because of the way they are calculated).
- **Slowly damping sample autocorrelations is an indication of ARIMA data.**

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Seasonal Models | Properties and Characteristics

Seasonal Models

$$\boxed{(B)(1 \quad B^s)X_t = \quad (B)a_t}$$

- Contain factor $(1 - B^{s})$
- Monthly data $(1 - B^{12})$, quarterly data $(1 - B^{4})$,…

$\varphi(B)$ =1 $\theta(B)$ =1

Example seasonal models

(a) $(1 \quad B^4)X_t = a_t$

(b) $(1 \quad B + .6B^2)(1 \quad B^4)X_t = (1 + .5B)a_t$

(c) $(1 \quad B + .6B^2)(1 \quad B^{12})X_t = (1 + .5B)a_t$

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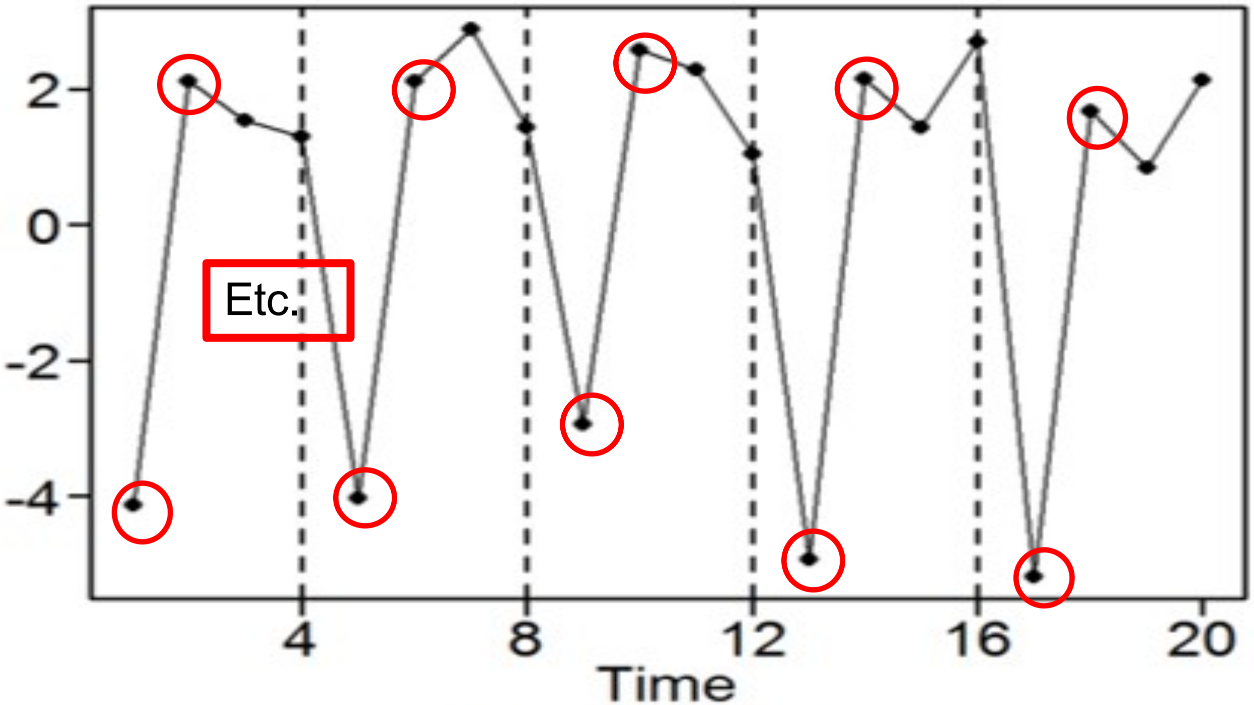
Seasonal Models | Examples

$$(1 \quad B^4)X_t = a_t$$

i.e.

$$X_t = X_{t \quad 4} + a_t$$



For quarterly sales data, this model says that sales in the current quarter are equal to the sales for this quarter last year plus a random-noise term.

First: Note the difference between (a) $(1 \quad B)^4$ and (b) $(1 \quad B^4)$

(a) $(1 \quad B)^4 X_t = a_t$ is an ARIMA(0,4,0) model.

$$(1 \quad B)^4 = (1 \quad B)(1 \quad B)(1 \quad B)(1 \quad B)$$

(b) $(1 \quad B^4) = (1 \quad B^2)(1 + B^2) = (1 \quad B)(1 + B)(1 + B^2)$ (simple factoring)

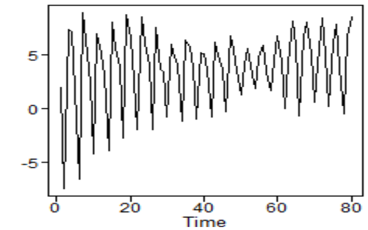
More intuitively, $(1 \quad B^4)X_t = a_t$ can be written

$X_t \quad B^4 X_t = a_t$ or $X_t \quad X_{t \ 4} = a_t$ or $\boxed{X_t = X_{t \ 4} + a_t}$

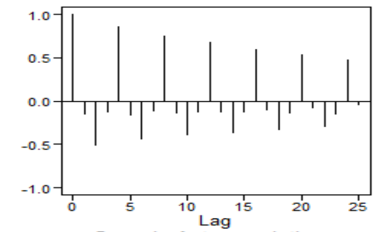
$$(1\quad B^4)X_t = a_t$$

Notes about the models:

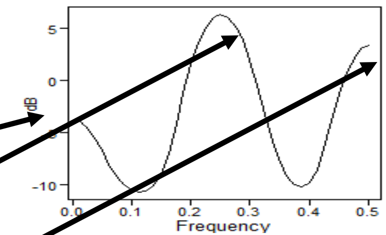
- "Quarterly" behavior is present in realization (but not as clear as it was in the initial realization of length $n = 20$).



- Sample autocorrelations at lags 4, 8,… are "large," which is consistent with the model.
  - That is, $X_t, X_{t+4}, X_{t+8},\ldots$ would be expected to be "similar."



- The spectral estimate has peaks at $f = 0$, $.25$, and $.5$ (we'll come back to this).

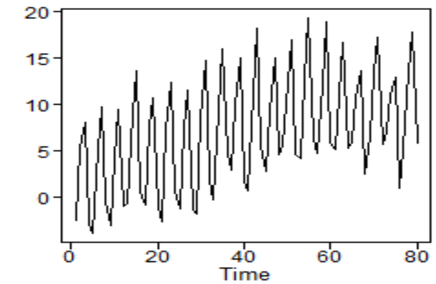


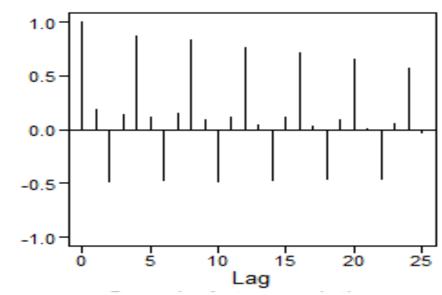
$$(1 - B^4) = (1 - B)(1 + B^2)(1 + B)$$

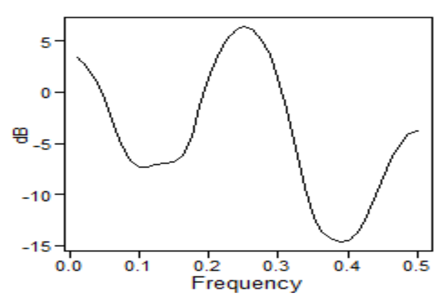
$$(1 \quad B + .6B^2)(1 \quad B^4)X_t = (1 + .5B)a_t$$

Notes about the models:

- "Quarterly" behavior is present in realization.
- Sample autocorrelations at lags 4, 8,… are "large."
- The spectral estimate has peaks at $f = 0$, $.25$, and $.5$).



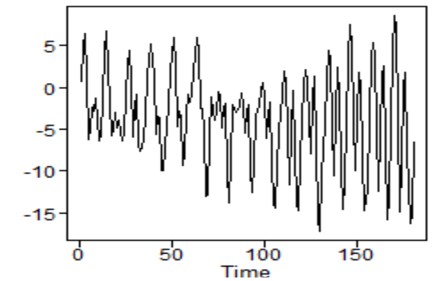


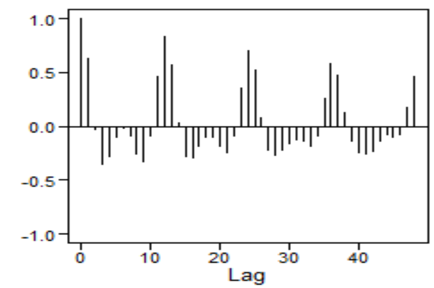


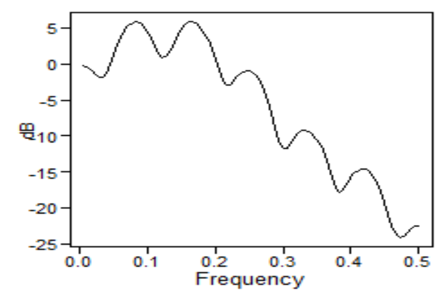
$$(1 \quad B + .6B^2)(1 \quad B^{12})X_t = (1 + .5B)a_t$$

Notes about the models:

- "Monthly" (seasonal behavior at lag 12) behavior is present in realization.
- Sample autocorrelations at lags 12, 24,… are "large."
- The spectral estimate has mild peaks at $f = 0$ and .5 along with 5 fairly equally spaced peaks between 0 and $.5$ (we'll come back to this shortly).







Notes

Before proceeding, we note that Woodward, Gray, and Elliott (2017) also define an ARUMA model as a generalization of the ARIMA model.

Although we will not discuss these in detail, we do mention that seasonal models (which will be discussed shortly) are special cases of ARUMA models.

We will use the tswge function **`gen.aruma.wge`** to generate realizations from seasonal models.

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Seasonal Models, ARUMA and tswge

tswge demo

Below, we use the tswge commands.

- gen.aruma.wge (to generate realizations from seasonal models)
- plotts.sample.wge to plot the realizations, sample autocorrelations, and spectral estimates.
- We recommend running these commands several times to get a better idea of the typical behavior

For $(1 \quad B^4)X_t = a_t$ use

x1=gen.aruma.wge(n=80, s=4, sn = 5)

plotts.sample.wge(x1)

For $(1 \quad B+.6B^2)(1 \quad B^4)X_t = (1+.5B)a_t$ use

x2=gen.aruma.wge(n=80, phi=c(1,-.6),s=4,theta=-.5)

plotts.sample.wge(x2)

For $(1 \quad B+.6B^2)(1 \quad B^{12})X_t = (1+.5B)a_t$ use

x3=gen.aruma.wge(n=180, phi=c(1,-.6),s=12,theta=-.5)

plotts.sample.wge(x3,lag.max=48)

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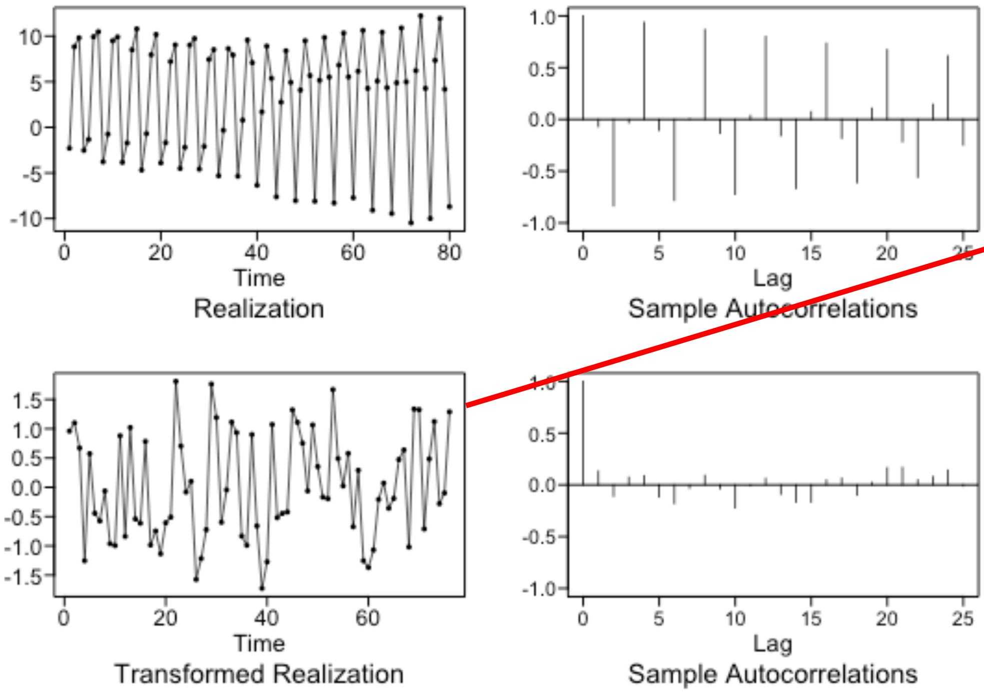
Stationarize an ARUMA

Stationarize Quarterly Data: Taking out the $(1 - B^4)$

$$\boxed{(1 - B^4)X_t = a_t}$$

```
x=gen.aruma.wge(n=80, s=4, sn = 81) #tswge function to generate ARIMA and Seasonal Models
Dif = artrans.wge(x,c(0,0,0,1)) #Take out the (1-B^4)
aic5.wge(Dif) #Check the structure of the noise
```



Five Smallest Values of aic

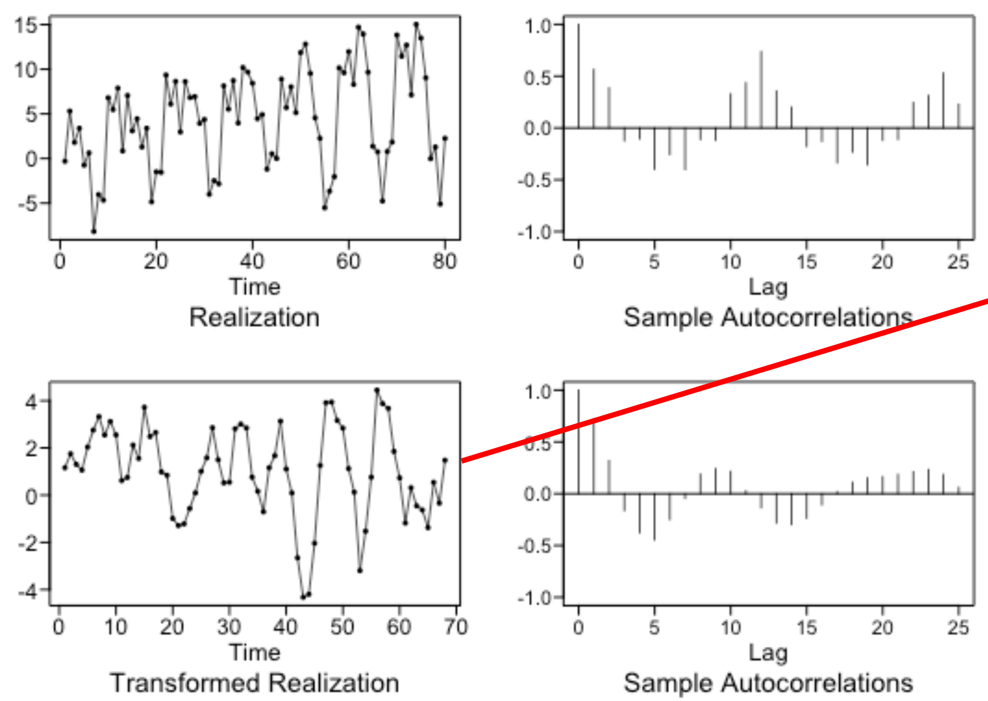
	p	q	aic
1	0	0	-0.1887305 *
2	0	1	-0.1885748
17	5	1	-0.1849357
4	1	0	-0.1814065
3	0	2	-0.1768610

* We will estimate the coefficients of these models in a later unit.

Stationarize Monthly Seasonal Data: Taking out the $(1 - B^{12})$

$$(1 - .4B - .6B^2 + .74B^3)(1 - B^{12})\, X_t = (1 + .7B)a_t$$

```
x=gen.aruma.wge(n=80, phi = c(.4,.6,-.74), theta = c(-.7), s=12, sn = 31)
Dif = artrans.wge(x,c(rep(0,11),1)) #Take out the (1-B^12)
aic5.wge(Dif) #Check the structure of the noise
```



Five Smallest Values of aic

	p	q	aic	
15	4	2	0.1844447	
11	3	1	$\boxed{0.1992944}$	*
12	3	2	0.2166751	
17	5	1	0.2195760	
7	2	0	0.2224280	

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* We will estimate the coefficients of these models in a later unit.

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Seasonal Models | Factor Tables

Return again to the simple quarterly model

$$(1 - B^4)X_t = a_t$$

Note that $(1 - B^4)$ is a 4th order polynomial operator associated with characteristic equation $1 - z^4 = 0$.

Previously, we noted that, using simple factoring, we obtain

$$1 - z^4 = (1+z^2)(1 - z^2) = (1+z^2)(1+z)(1 - z) = 0$$

Recall, we can factor a polynomial using the tswge function factor.wge. In this case, the command is

factor.wge(phi=c(0,0,0,1)) or factor.wge(phi=c(rep(0,3),1)).

To factor $1-z^{12}$, we use

factor.wge(phi=c(0,0,0,0,0,0,0,0,0,0,0,1)) or factor.wge(phi=c(rep(0,11),1)).

Factor Tables

$$1 - B^4$$

Factor	Roots	Abs Recip	*f*
$1 - B$	1	1	0
$1 + B^2$	$\pm i$	1	.25
$1 + B$	1	1	.5

$$1 - B^{12}$$

Factor	Root(s)	Abs Recip	*f*
$1 - B$	1	1	0
$1 - \sqrt{3}B + B^2$	$.866 \pm .5i$	1	.083
$1 - B + B^2$	$.5 \pm .866i$	1	.167
$1 + B^2$	$+i$	1	.25
$1 + B + B^2$	$-.5 \pm .866i$	1	.333
$1 + \sqrt{3}B + B^2$	$-.866 \pm .5i$	1	.417
$1 + B$	-1	1	.5

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Seasonal Models | tswge and Factor Tables

tswge Screen Share

- Screen Share: factor.wge

```
factor.wge(c(0,0,0,1))
factor.wge(c(0,0,0,0,0,0,0,0,0,0,0,1))
factor(c(rep(0,3),1))
factor(c(rep(0,11),1))
factor(c(rep(0,4),1))
```

- Factor the model below and show that it is a monthly seasonal model:

$$(.2B - .4B^2 - .49B^3 - 1B^{12} - .2B^{13} + .4B^{14} + .4B^{14} + .49B^{15})\, X_t = (1 + .92B)a_\tau$$

$$(1 + .2B - .4B^2 - .49B^3)(1 - B^{12})\, X_t = (1 + .29B)a_\tau$$

- Factor the model below and show that it is a 5-month seasonal model:

$$(.3B - .8B^2 + 1B^5 - .3B^6 + .8B^7)\, X_t = (1 + .29B)a_\tau$$

$$(1 + .3B - .8B^2)(1 - B^5)\, X_t = (1 + .29B)a_\tau$$

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A More General Seasonal Model

A More General Seasonal Model

$$\boxed{(B)(1 \quad B)^d(1 \quad B^s)X_t = \quad (B)a_t}$$

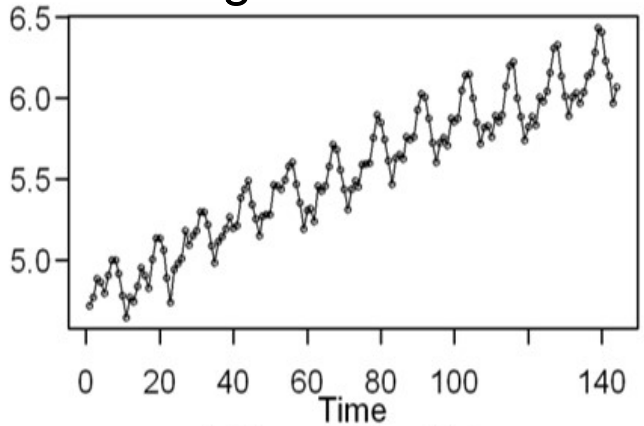
Note that this model has:

- Stationary factors: $\varphi(B)$ and $\theta(B)$
- An ARIMA-type factor: $(1 - B)^d$
- A seasonal factor: $1 - B^s$

Airline Data

Recall the classical airline data (log of the monthly number of airline passengers for a 12-year period).

Log airline data



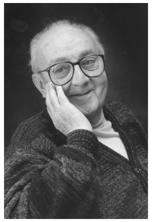
Some analysts have fit models of the following form to the log airline data)

$$(B)(1 \quad B)(1 \quad B^{12})X_t = \quad (B)a_t$$

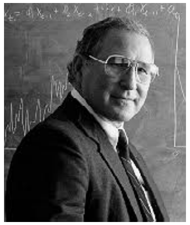
In fact, models containing $(1-B)(1-B^{12})$ are sometimes referred to as "airline models."

Airline Competition Among the Greats!

$$(1 - .74B)(1 + .38B^{12})(1 - B^{12})(X_t - \mu) = a_t$$



$$(1 - B)(1 - B^{12})(X_t - \mu) = (1 - .4B)\ (1 - .6B^{12})a_t$$



$$(1 + .36B + .05B^2 + .14B^3 + .11B^4 - .04B^5 - .09B^6 + .02B^7 - .02B^8 - .17B^9 - .03B^{10} + .10B^{11} + .38B^{12})(1 - B)(1 - B^{12})(X_t - \mu) = a_t$$

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Screen Share: Airline Data

Woodward/Gray, Box and Parzen

tswge Screen Share

- Screen Share page 258 in book

Screen Cast Showing Gray and Woodward Model (1981)

Versus Parzen Model (1980)

Show PDFs!

Show that the AIC is better for Box than Parzen or Woodward and Gray.

```
data(airlog)
SA1 = artrans.wge(airlog,1) # take first differences of the data
SA1_12 = artrans.wge(SA1,c(rep(0,11),1)) # take the 12th difference of the first difference (1-B)(1-B^12)
SA12 = artrans.wge(airlog,c(rep(0,11),1)) # take the 12th difference of the data (1-B^12)
Parzen = aic.wge(SA12, p = 13) #Phi(B)(1-B12)(Xt-mu) = at
Box = aic.wge(SA1_12, q = 13) #(1-B)(1-B^12) (Xt-mu) = Theta(B)at
WoodwardAndGray = aic.wge(SA1_12, p = 12) # #Phi(B)(1-B)(1-B^12) (Xt-mu) = at
Parzen$value
Box$value
WoodwardAndGray$value
# We will return to this competition when we use the model to forecast!
```

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