Table of Contents Proposal

For Part III Essay. Topic 97. Modular forms and representation theory

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1 Abstract

The goal of the essay will be to give an introduction to the spectral adelic generalization of classic modular forms. To justify this generalization, we aim to give 2 applications of the new framework. First, we would like to describe how this setting completes the theory of Hecke operators. Second, we will see a reinterpretation of the theory of newforms.

The primary references for this essay (so far) have been [Gel75] and [Del73]. We have also consulted [KL06] and [Bum97] and a set of online notes on Atkin-Lehmer theory by Andreea Mocanu [Moc].

2 Table of Contents

1. **Introduction** **~ 1 page**
   - Historical introduction
   - Relation to the Langlands' program
2. **Modular Forms Revisited** **~ 6 pages**
   - 2.1 Functions on lattices ~ 1 page
     - Following [Del73, p.8-20]. For arbitrary level, work on a universal cover.
     - Comment on how this generalizes for Hilbert modular forms
   - 2.2 The decomposition of $L^2(\Gamma \setminus \mathrm{SL}_2(\mathbb{R}))$ ~ 2 pages
     - Discrete and continuous spectrum
     - Casimir operator and K-action
     - Wave forms
     - Multiplicites are finite
     - [Gel75, Chapter 2] and [Del73, Section 2.1]
   - 2.3 Hecke Theory for $\Gamma_0(N)$ ~ 2 pages
     - Hecke operators as a double coset action or as an averaging operator over sublattices
     - Detailed computation of the proof of $T(p)$ being self-adjoint, to see what breaks when $(p, N) \neq 1$.

- [KL06, p.20, p.39] and [Gel75, p.17, p.20]

2.4 Atkin–Lehner Theory of Newforms ∼ 1 page

- Show the full computation of $\Delta(z)$ and $\Delta(2z)$ have the same eigenvalues at $p \neq 2$ to motivate theory of new-forms.
- No classical proofs, only statements. [Moc]
- [Gel75, Section 1.D]

3. **Automorphic Forms of** $\mathrm{GL}_2(\mathbb{A})$ **∼ 8 pages**

3.1 Functions on Adelic Lattices ∼ 1 page

- Strong Approximation Theorem for $\mathrm{GL}_2(\mathbb{A})$ [KL06, p. 6.3]
- Relation between classical lattices and adelic lattices [Del73, Section 1.2 and 2]

3.2 The Spectrum of $\mathrm{GL}_2(\mathbb{A})$ ∼ 3 pages

- [Del73, Section 2.2-2.4] and [Gel75, Section 3.A]

3.3 Jacquet-Langlands' theory ∼ 4 pages

- Kirillov and Whittaker models [Del73, page 24-30]
- [Gel75, Section 6]

4. **Applications** **∼ 3 pages**

4.1 Hecke Operators Revisited ∼ 2 pages

- I want to give the detailed computations for $\Gamma_0(N)$, which is now $K_0(N)$.
- [Gel75, Section 5, page 86]

4.2 Newforms revisited ∼ 1 page

- You can use Jacquet-Langlands' main theorem to prove the multiplicity one theorem of Atkin Lehner
- [Gel75, Section 6. D]

5. **Further reading** **∼ 1 page**

- Jacquet Langland's Theory, L-functions and meromorphic continuation
- Arthur-Selberg Trace formula

References

[Bum97] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[Del73] P. Deligne. "Formes Modulaires et Representations De GL(2)". In: *Modular Functions of One Variable II*. Ed. by Pierre Deligne and Willem Kuijk. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 55–105. ISBN: 978-3-540-37855-6.

[Gel75] Stephen S. Gelbart. *Automorphic Forms on Adele Groups. (AM-83)*. Princeton University Press, 1975. ISBN: 9780691081564. URL: http://www.jstor.org/stable/j.ctt1b7x82z (visited on 03/07/2024).

[KL06] Andrew Knightly and Charles Li. "Traces of Hecke Operators". In: (Dec. 2006). DOI: 10.1090/surv/133/01.

[Moc] Andreea Mocanu. "Atkin Lehner Theory". In: (). URL: https://andreeamocanu.github.io/atkin-lehner-theory.pdf.