Table of Contents Proposal

For Part III Essay. Topic 97. Modular forms and representation theory

Javier López-Contreras

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1 Abstract

The goal of the essay will be to give an introduction to the spectral adelic generalization of classic modular forms. To justify this generalization, we aim to give 2 applications of the new framework. First, we would like to describe how this setting completes the theory of Hecke operators. Second, we will see a reinterpretation of the theory of newforms.

The primary references for this essay (so far) have been [Gel75] and [Del73]. We have also consulted [KL06], [Bum97], a set of online notes on Atkin-Lehmer theory by Andreea Mocanu [Moc] and a set of online notes on Hecke Operators by Samanda Hu [Hu].

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- Show the full computation of $\Delta(z)$ and $\Delta(2z)$ have the same eigenvalues at $p \neq 2$ to motivate theory of new-forms.
- [Moc] and [Gel75, Section 1.D]

3. **Automorphic Forms of** $\mathrm{GL}_2(\mathbb{A})$ **~ 8 pages**

   3.1 Functions on Adelic Lattices ~ 1 page
   - Strong Approximation Theorem for $\mathrm{GL}_2(\mathbb{A})$ [KL06, p. 6.3]
   - Relation between classical lattices and adelic lattices [Del73, Section 1.2 and 2]

   3.2 The Spectrum of $\mathrm{GL}_2(\mathbb{A})$ ~ 3 pages
   - [Del73, Section 2.2-2.4] and [Gel75, Section 3.A]

   3.3 Jacquet-Langlands' theory ~ 4 pages
   - Kirillov and Whittaker models [Del73, page 24-30]
   - [Gel75, Section 6]

4. **Applications** **~ 7 pages**

   4.1 Hecke Operators Revisited ~ 5 pages
   - Detailed computations for $\Gamma_0(N)$, which is now $K_0(N)$.
   - [Gel75, Section 5, page 86]

   4.2 Newforms revisited ~ 2 page
   - Jacquet-Langlands' main theorem to prove the multiplicity one theorem of Atkin-Lehner
   - [Gel75, Section 6. D]

5. **Further reading** **~ 1 page**
   - Jacquet Langland's Theory, L-functions and meromorphic continuation
   - Arthur-Selberg Trace formula

6. **Extra question**. What are the next things to read canonically if I am interested in Langlands?
   - Modularity?
   - Papers by Langlands?
   - Trace Formula? I think I am not too far to understand the adelic version of the trace formula

3 Questions

3.1 Direction Questions

1. What **hard** proof should I include?
   - Uniqueness of Whitaker models (Every reference avoids it)
   - Multiplicity one theorem

  - Two newforms have the same eigenvalues for good Hecke operators iff they are the same.
  - This is true even if you initially choose them from diferent levels and find and N where they have same eigenvalues to good hecke operators on N.
- Strong multiplicity one (m_π are 0 or 1)
- Classification of admissible irreduble reps of $\mathrm{GL}_2(F)$ for F local field.
  - I have seen the full proof for archimedean local fields [Bet]
  - May be too long for the essay?
- Analytic properties of Jacquet-Langlands L-function associated to a cuspidal automorphic representation?

2. Smaller Lemmas to include
   - Strong Aproximation Theorem for $\mathrm{GL}_2(\mathbb{A})$
   - Hecke Operators for general level and their conjugates
   - Explanation on how the map from cusp forms to cuspidal automorphic representations is not one to one and this comes form theory of newforms.
3. Should I include motivation from $\mathrm{GL}_1(A)$?

3.2 Mathematical Questions

1. Why do we always concentrate on $\Gamma_i(N)$
   - For a general Γ, can you have a theory of newforms?
   - The Hecke Theory of $\Gamma(N)$ is reduced to the one of $\Gamma_1(N^2)$
2. Can we repeat what happens to the bad Hecke operators
   - I have seen the computation that they are not normal but don't understand the bigger picture
   - I recall that we discussed that they "miss-send" the cusp forms of a given level to another one? But this is incorrect recalling [Gel75, Ch 6].

$$S_k(N) = S_k^{\mathrm{new}}(N) \oplus \bigoplus_{p|N} i_p S_k(N/p)$$

References

[Bet] L. Alexander Betts. *Notes on Classification of irreducible admissible* $\mathrm{GL}(2,\mathbb{R})$*-modules.* URL: `https://pub.math.leidenuniv.nl/~vonkjb/other/automorphic/GL2R.pdf`.

[Bum97] Daniel Bump. *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[Del73] P. Deligne. "Formes Modulaires et Representations De GL(2)". In: *Modular Functions of One Variable II.* Ed. by Pierre Deligne and Willem Kuijk. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 55–105. ISBN: 978-3-540-37855-6.

[DS06] F. Diamond and J. Shurman. *A First Course in Modular Forms*. Graduate Texts in Mathematics. Springer New York, 2006. ISBN: 9780387272269. URL: https://books.google.co.uk/books?id=EXZCAAAAQBAJ.

[Gel75] Stephen S. Gelbart. *Automorphic Forms on Adele Groups. (AM-83)*. Princeton University Press, 1975. ISBN: 9780691081564. URL: http://www.jstor.org/stable/j.ctt1b7x82z (visited on 03/07/2024).

[Hu] Samanda (Yuting) Hu. "Modular Forms and Hecke Operators". In: (). URL: https://math.uchicago.edu/~may/REU2020/REUPapers/Hu,Samanda.pdf.

[KL06] Andrew Knightly and Charles Li. "Traces of Hecke Operators". In: (Dec. 2006). DOI: 10.1090/surv/133/01.

[Moc] Andreea Mocanu. "Atkin Lehner Theory". In: (). URL: https://andreeamocanu.github.io/atkin-lehner-theory.pdf.