Undergraduate Thesis
in Mathematics and Computer Science

Artin's Conjecture on primes with prescribed primitive roots

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Abstract

TODO: Write Abstract
TODO: Keywords + I also need to add the AMS classification number

English version

Catalan version

Spanish version

Acknowledgements

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1. Introduction

As it often happens in Mathematics, the history of Artin's Conjecture can be traced back to the writtings of Carl Friedrich Gauss. In the articles 314-317 of his 1801 *Disquisitiones Arithmeticae* [GWC86], Gauss asks the following elementary question. Why does the decimal expression of $\frac{3}{7}$ have a period of length 6, while the expression of $\frac{1}{11}$ has a shorter period, of only 2 digits?

$$\frac{3}{7} = 0.428571\ 428571\ 428571\ldots \qquad \frac{1}{11} = 0.09\ 09\ 09\ldots \tag{1.1}$$

When p is a prime $\notin \{2,5\}$ and $a \in \mathbb{Z} \cap [1, p-1]$, it turns out that the length of the period of $\frac{a}{p}$ is exactly $\mathrm{ord}_{(\mathbb{Z}/p\mathbb{Z})^\times}(10)$. To see this, note that

$$\frac{a}{p} = \left(\frac{a_1}{10} + \cdots + \frac{a_s}{10^s}\right)\left(1 + \frac{1}{10^s} + \cdots\right) = \left(10^{s-1}a_1 + \cdots + a_s\right)\frac{1}{10^s - 1} \tag{1.2}$$

This in turn implies that $10^s = 1 \mod p$. But for any $s' < s$ with $10^{s'} = 1 \mod p$, let $M \in \mathbb{Z}$ such that $a(10^s - 1) = pM$. Choosing the a_i to be the base 10 digits of M, we could give a shorter periodic expression of $\frac{a}{p}$.

The article continues with the following remark. If one had another $b \in \mathbb{Z} \cap [1, p-1]$ such that $b = 10^\lambda a \mod p$ for some λ, then period of $\frac{b}{p}$ would just be the period of $\frac{a}{p}$ translated λ decimal places to the right.

$$b_i = \left\lfloor \frac{10^i b}{p} \right\rfloor \mod 10 = \left\lfloor \frac{10^i(10^\lambda a + Np)}{p} \right\rfloor \mod 10 = \left\lfloor \frac{10^{i+\lambda} a}{p} \right\rfloor \mod 10 = a_{i+\lambda} \tag{1.3}$$

Therefore, if 10 was a primitive root modulo p, the periods of the $\frac{a}{p}$ would be in bijection with the possible translations of the period of $\frac{1}{p}$. The question of when is 10 a primitive root as we iterate p over the primes was not addressed by Gauss but will be the central topic of this Undergraduate Thesis.

In September of 1927, in a private conversation with Helmut Hasse, Emil Artin gave a precise conjecture about the density of primes with a prescribed primitive root [LT65, Pages vii-x]. Namely, he stated that given a non-square integer $a \in \mathbb{Z}_{>1} \setminus \mathbb{Z}^2$, the density of primes where a is a primitive root should be

$$A(a) = \prod_{l \text{ prime}} \left(1 - \frac{1}{l(l-1)}\right) \approx 0.3739558 \tag{1.4}$$

which has since become known as Artin's constant. To obtain such a precise conjecture, he used

the Chebotarev's Density Theorem [Che26] over a certain family of Kummer Fields. This theorem had been proven in 1922 on the doctoral thesis of Nikolai Chebotarev and had just been published in 1926. With this powerful tool and assuming a certain *passing to the limit* argument, one reaches the conjectured density. Nonetheless, this limit argument is where the incredible difficulty of Artin's Conjecture lies. Since its conception in 1927, there have been many attempts at solving Artin's problem yet, so far, no mathematician has been able to give an unconditional proof.

In 1934, Hasse would propose Artin's problem to his doctoral student, Herbert Bilharz. After one year of work, they heard from Harold Davenport that Paul Erdős believed to have a proof. In April 5th 1935, Hasse wrote a letter to Erdős

"... My friend Davenport has told me that you believe to have solved a problem which is close to my heart: the problem of the density of those primes that have a given number as a primitive root... In case you have already dealt with this problem, I obviously have to find as quickly as possible a new PhD subject for Mr. Bilharz, who is working on this topic for already a year"

Because of this, Bilharz was forced to develop his thesis about the equivalent conjecture over the function field $\mathbb{F}_q(x)$. In the end, Erdős' attempt ended up depending both on the Riemann Hypothesis and on an argument about the distribution of primes that he was unable to justify and was never published. To this day, Erdős argument remains a mystery from which we only know the few details that he wrote in a letter to Hasse [Coj02, Appendix II]. Nonetheless, the sudden change on Bilharz' thesis had a happy ending. In 1937, he would publish a proof of Artin's Conjecture over $\mathbb{F}_q(x)$ [Bil37] that depended on the Riemann Hypothesis over Function Fields of Curves over $\mathbb{F}_q$. This version of the Riemann Hypothesis was settled by André Weil 1940 [Wei40].

In 1957, Emma and Derrick Lehmer computed some numerical estimates of Artin's constant with the aid of a computer. They realized that density of the set of primes with a prescribed primitive root at a didn't seem to be independent of a, as Artin had conjectured. After some correspondence, Artin realized that for certain values of a, his conjectured density formula missed a correction factor that could be given explicitly. This mistake came from a miss-calculaton of the degree of a certain Kummer extension. As Artin's sums it up in [Leh90]

"... So I was careless but the machine caught up with me.
Cordially, E. Artin"

The first major advancement towards a proof of the conjecture came in 1967, by Christopher Hooley [Hoo67]. He showed that the Generalized Riemann Hypothesis on a certain family of Kummer Fields would imply Artin's Conjecture. His proof is heavily inspired by the development of Sieve Theory in recent years. Hooley was able to reduce Artin's Conjecture to a problem about counting primes. Under the assumption of the Riemann Hypothesis, we proved a statement about the vertical distribution of the Riemann zeroes. With this, he gave a sufficiently fine estimation of the prime counting function which was enough to settle the conjecture.

Removing the Riemann Hypothesis condition has proven to be a hard problem on its own. In 1983,

an important step in this direction was given by Rajiv Gupta and Ram Murty [Gup84]. They were able to give a set of 13 integers and proved unconditionally that at least one of these must follow Artin's Conjecture. In 1985, Heath-Brown [Hea86] refined their argument showing that at least one of $a \in \{2, 3, 5\}$ follows Artin Conjecture.

Over the last century, Artin's Conjecture has remained one of the few elusive problems originated in Elementary Number Theory. As such, it has sparked interest about a number of related problems, from which we give two notable examples. First, in 1976, J. P. Serre [Ser03] used a version of Hooley's argument to count the number of primes $p \leq x$ where the modulo p reduction of given Elliptic Curve is cyclic. Second, in 1977, H. W. Lenstra [W77] showed that Hooley's argument can be extended to settle a more general conjecture which has implication in the discovery of Euclidean Algorithms for certain rings of integers.

Note from the author: The historical introduction that you have just read has been pieced together from a number of sources that I would like to exhaustively list in the interest of full acknowledgement and proper book-keeping.

- The prologue of [LT65], written by two of Artin's doctoral students, John T. Tate and Serge Lang, gives a detailed accounting of the birth of the conjecture.
- The correspondence between Erdős, Davenport and Hasse seems to have been compiled and publicly published for the first time in 2002, in A. Cojocaru's PhD thesis [Coj02, Appendix II].
- The correspondence between the Lehmers and Artin was re-discovered in 2001 and is currently available in the Lehmer Archives [Leh90] of the Bancroft Library at U. C. Berkeley.
- A general overview of the history Artin's Conjecture can be found in Ram Murty's Survey [Mur88].
- A delightful historical exposition about the correction factor in Artin's constant is available in Stevenhagen's article [Ste03].

2. Notation and background

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the integer, rational, real and complex numbers respectively. To avoid confusion, we will use either $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{\geq 0}$ instead of $\mathbb{N}$.
- The functions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ give the real and imaginary part of a $z \in \mathbb{C}$, respectively.
- Unless otherwise stated, p will an arbitrary prime, $q = p^r$ and l will be a prime $l \neq p$.
- $\mathbb{F}_q$ denotes the finite field of q elements
- Given a group G and $a \in G$, $\mathrm{ord}_G(a)$ denotes the multiplicative order of a and $\mathrm{ind}_G(a) \coloneqq \frac{|G|}{\mathrm{ord}_G(a)}$ is the index.
- For $p \in \mathbb{Z}$ a prime and $a \in \mathbb{Q}$, $\mathrm{ord}_p(a) = \max\{k \in \mathbb{Z} \mid p^k \mid a\}$. *Warning!* Note that $\mathrm{ord}_{\mathbb{F}_p^*}$ and ord_p are different functions. This distinction could be a source of confusion.
- For $a, b \in \mathbb{Z}$, we denote its greatest common divisor with (a, b) or $\gcd(a, b)$
- A function $f : \mathbb{Z} \to \mathbb{C}$ is weakly multiplicative if $f(ab) = f(a)f(b)\ \forall a, b \in \mathbb{Z}$ with $(a, b) = 1$. The following functions are weakly multiplicative
  - $\mu(n)$ denotes the Möebius Inversion function, namely

$$\begin{aligned} \mu : \mathbb{Z} &\longrightarrow \{-1, 0, 1\} \\ n &\mapsto \begin{cases} 0 & k^2 \mid n \\ (-1)^r & n = p_1 \dots p_r \end{cases} \end{aligned} \tag{2.1}$$

  - $\phi(n)$ denotes Euler Totient function, namely

$$\phi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \tag{2.2}$$

  - $w(n)$ denotes the number of distinct prime divisors of n
- If L/K is a finite extension of algebraic fields with Dedekind Domains B/A, we denote
  - ΔL the discriminant over K
  - $\mathrm{Tr}, \mathcal{N} : L \to K$ the trace and norm respectively
  - $\mathrm{Spl}(L/K)$ is the set of primes in L that split completely over K. When the base field is clear by context, we will write $\mathrm{Spl}(L)$.

2.1 Facts from Algebraic Number Theory

2.2 Facts from Function Fields

Constant extensions, L-series of cyclotomic fields

2.3 Facts from Analytic Number Theory

2.3.1 Dirichlet Density

Let K be a global field with ring of integers $\mathcal{O}_K$.

Definition 2.1 (Dirichlet's Density)**.** Let $S \subseteq \operatorname{Spec} \mathcal{O}_K$, define

$$\delta(S, s) = \frac{\sum_{\mathfrak{p} \in S} \frac{1}{(\mathcal{N}\mathfrak{p})^s}}{\sum_{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K} \frac{1}{(\mathcal{N}\mathfrak{p})^s}} \tag{2.3}$$

We say S has Dirichlet Density $\delta(S)$ if $\lim_{s \to 1} \delta(S, s)$ exists and is equal to $\delta(S)$

TODO: Generalization to global fields and saying that in those, it doesnt match the usual denisty, even though in $\mathbb{Q}$ it does

Theorem 2.2 (Chebotarev's Density Theorem)**.**

2.3.2 Sieving methods

Selberg Sieve

3. Artin's Conjecture

In 1927, Emil Artin famously asked the following question pertaining to Elementary Number Theory.

Question 3.1. For a given $a \in \mathbb{Z}$, are there infinitely many primes $p \in \mathbb{Z}$ such that $a \mod p$ is a primitive root in $\mathbb{Z}/p\mathbb{Z}$?

Definition 3.2. We will denote $P(a) = \{p \in \mathbb{Z} \mid a \text{ is a primitive root } \mod p\}$.

We are interested in whether the cardinal of $P(a)$ is infinite or not. For certain values of $a \in \mathbb{Z}$, the question is answer in the negative.

Lemma 3.3 (Necessary condition in A.C.)**.** If $a \in \mathbb{Z}$ is $\in \{-1, 0, 1\}$ or a perfect square, then there are only finitely many primes for which it is a primitive root. Namely, $P_{-1} = \{2, 3\}$ and, for $k \geq 0$,

$$P_{k^2} = \begin{cases} \emptyset & 2 \mid k \\ \{2\} & \text{otherwise} \end{cases} \tag{3.1}$$

Proof. If $a = 0$, then $a \mod p = 0$ is not invertible $\forall p$. If $a = -1$, then $a \mod p$ has order $\in \{1, 2\}$ as $(-1)^2 = 1 \mod p$. Hence, -1 can be, at most, a primitive root for primes $p \in \{2, 3\}$.

On the other hand, suppose $a = k^2$ with $k \geq 1$ has $\operatorname{ord}_{\mathbb{F}_p^*}(a) = p - 1$. Denote $r = \operatorname{ord}_{\mathbb{F}_p^*}(k)$. Then, $r \mid p - 1$ and $k^{2r} = 1 = a^r \mod p \implies p - 1 \mid r$ so $r = p - 1$. But if $p > 2$, then $r = p - 1$ is even and $a^{r/2} = k^r = 1$, which contradicts $\operatorname{ord}_{\mathbb{F}_p^*}(a) = p - 1$ ∎

Remark 3.4. The previous lemma does not have an analogue for l-th powers, with $l > 2$. This is because $p - 1 \neq 0 \mod 2$ only happens at $p = 2$, yet $p - 1 \neq 0 \mod l$ happens for infinitely many primes.

Remark 3.5. Note that $a \in \{-1, 0, 1\}$ do not follow the conjecture. We can exclude them from all our future attempts to prove that these conditions are sufficient. This resolves irrelevant corner cases in future lemmas.

Conjecture 3.6 (Artin's primitive root conjecture)**.** If $a \in \mathbb{Z}$ is not $\in \{-1, 0, 1\}$ or a square, the set $P(a)$ has positive density over the set of primes.

There are no values of a for which the conjecture has been proven to hold.

3.1 Studied generalizations

This long-lasting conjecture has raised interest in a number of related problems. This section describes some of these generalizations, which will be studied in more detail in the rest of the document. These generalizations were explored and resolved, often conditionally to some version of the Riemann Hypothesis, in [W77]. Section 6.1 gives an exposition of this paper.

3.1.1 Prescribed root at $a \in \mathbb{Q}$

One could be interested in asking A.C. about $a \in \mathbb{Q}$ instead of restricting to only $a \in \mathbb{Z}$, which creates the following problem.

Problem 3.7. Let $a \in \mathbb{Q}^*$ and P_a the set of primes in $\mathbb{Z}$ following

$$(1)\ \mathrm{ord}_p(a) = 0 \qquad \text{and} \qquad (2)\ \mathrm{ord}_{\mathbb{F}_p^*}(a) = p-1$$

Is P_a infinite?

Remark 3.8. Note that condition (1) is placed so that $a \mod p$ is well-defined and non-zero, which makes $\mathrm{ord}_{\mathbb{F}_p^*}(a)$ well-defined.

3.1.2 AC over Global Fields

The original conjecture studies the set of $p \in \mathbb{Z}$ for which $a \mod p$ generates the multiplicative group of the residue field $(\mathbb{Z}/p\mathbb{Z})^*$. The same question can be naturally extended to more general rings. We will be specially interested in the rings of integers of field extensions of $\mathbb{Q}$ and $\mathbb{F}_q(x)$, also known as Global Fields. Global Fields are examples of Dedekind Domains with infinitely many primes and finite residue fields. Without both of these conditions the conjecture is trivially false. Note that this excludes Local Fields and extensions of $K(t)$ for any non-finite field K.

Problem 3.9 (A.C. over Global Fields)**.** Let K be a Global Field, $\mathcal{O}_K$ its ring of integers and $a \in K^*$. Are there infinitely many prime ideals in $\mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_K$ such that

$$(1)\ \mathrm{ord}_{\mathfrak{p}}(a) = 0 \qquad \text{and} \qquad (2)\ a \mod \mathfrak{p} \text{ generates } (D/\mathfrak{p})^*?$$

For instance, writing Problem 3.9 for $\mathbb{F}_q(x)$ we obtain the following question.

Question 3.10 (A.C. over $\mathbb{F}_q(x)$)**.** Given an $a(x) \in \mathbb{F}_q[x]$ monic, are there infinitely many $v(x) \in \mathbb{F}_q[x]$ monic and irreducible such that $\overline{a}(x)$ is a primitive root of $\mathbb{F}_q[x]/(v) \simeq \mathbb{F}_{q^{\deg v}}$?

Section 4.1 focuses on Artin's conjecture over Function Fields. In this case, the necessary and sufficient conditions were found by Bilharz in 1937 [Bil37] conditional to the Riemann Hypothesis over Function Fields of Curves, which was settled shortly after by André Weil [Wei40]. Note that Bilharz's result came three decades before significant progress was made on the original conjecture over $\mathbb{Q}$ by Hooley [Hoo67]. On reason for this is that certain L-functions related to Artin's Conjecture can only be explicitly computed over Function Fields.

Remark 3.11. By the same rationale exposed in Remark 3.5, the values $a \in \lambda(K) \cup \{0\}$ will never follow the conjecture, where $\lambda(K)$ are the roots of unity of the Global Field K. We will ignore these values in all further considerations.

3.1.3 Restricting $\mathrm{Frob}_{T/\mathbb{Q}}(p)$

One may be interested in imposing congruence conditions for the primes being counted. For example, one can show that there are no primes $p = \pm 1 \mod 8$ where 2 is a primitive root, as for those p, $\left(\frac{2}{p}\right) = 1$. A natural question would be to ask if there are infinitely many primes $p = 3 \mod 8$ such that 2 is a primitive root. For the general conjecture over the Global Field K, these modular restrictions are expressed as restrictions on the Frobenius element over an arbitrary Abelian extension T/K.

Problem 3.12. Let K be a Global Field, $a \in K^*$, T/K an Abelian field extension and $C \subseteq \mathrm{Gal}(T/K)$ a subset formed of conjugacy classes. Are there infinitely many prime ideals $\mathfrak{p} \in \operatorname{Spec} K$ such that

(1) $\mathrm{ord}_{\mathfrak{p}}(a) = 0$, (2) $\mathrm{ord}_{(K/\mathfrak{p})^*}(a) = \mathcal{N}\mathfrak{p} - 1$, (3) $\mathrm{Frob}_{\mathfrak{p}}(T/K) \in C$?

Remark 3.13. Note that $T = K$ and $C = \{1\}$ recovers the original problem.

3.1.4 Arbitrary set of generators

One more way Artin's Conjecture can be generalized is by choosing a more general set W to take the role of the prescribed primitive root a.

Problem 3.14. Let $W \subseteq \mathbb{Q}^*$ and let $\Gamma = \langle W \rangle$ be the multiplicative group $\Gamma \subseteq \mathbb{Q}$ generated by W. Are there infinitely many primes $p \in \mathbb{Z}$ such that the quotient $\Gamma \to F_p^*$ is well-defined and surjective?

Note that this is equivalent to $\mathrm{ord}_p(w) = 0 \ \forall w \in W$ and $\Gamma_p = \{\gamma \mod p \mid \gamma \in \Gamma\} = \mathbb{F}_p^*$

Remark 3.15. Note that $W = \{a\}$ recovers the original conjecture.

This generalization comes up in applications of Artin's Conjecture in finding Euclidean Algorithms on Global Fields [CW75].

3.1.5 Primes with $\mathrm{ind}_{F_p^*}(a) \mid m, m \in \mathbb{Z}$

One can weaken the surjectivity condition of the quotient map $\langle a \rangle \to \mathbb{F}_p^*$. This results in the following problem.

Problem 3.16. Given a $m \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Q}$, are there infinitely many primes such that $\mathrm{ord}_p(a) = 0$ and $\mathrm{ind}_{\mathbb{F}_p^*}(a) \mid m$.

Remark 3.17. $m = 1$ recovers de original conjecture.

3.2 Artin's observation

In the letter that proposed the conjecture, Artin gave a relevant observation that links the set $P(a)$ with the set of completely split rational primes over an explicit family of Kummer fields. This link with Algebraic Number Theory is a central piece in the attempts at solving the conjecture. It begins to explain why the Generalized Riemann Hypothesis will play an important role.

The work presented in this section can be generalized to the related conjectures described in Section 3.1. We have chosen to expose the classical setting first, as the general setting doesn't introduce any new ideas but complicates the notation. We will discuss a general version of Artin's Observation in Section 6.1.

Let $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $p > 2$ a prime with $p \nmid a$.

Remark 3.18. The prime $p = 2$ is a corner case in some of the following Lemmas. We explicitly exclude it from consideration as, in Artin's conjecture, we are only interested in density problems unaffected by finite exceptions.

Lemma 3.19. a is a primitive root mod p if and only if there isn't any $l \in \mathbb{Z}$ prime such that

$$(1)\ l \mid p-1 \qquad \text{and} \qquad (2)\ a^{\frac{p-1}{l}} = 1 \mod p$$

Proof. If the $\mathrm{ord}_{\mathbb{F}_p^*}(a) = r \neq p-1$, it must $r \mid p-1$. Take l any non-trivial prime factor of $\frac{p-1}{r} \neq 1$ and b such that $bl = \frac{p-1}{r}$. Then $l \mid \frac{p-1}{r} \mid p-1$ and $a^{\frac{p-1}{l}} = a^{rb} = 1 \mod p$.

For the reciprocal, note that $\mathrm{ord}_{\mathbb{F}_p^*}(a) \leq \frac{p-1}{l} < p-1$. ∎

Lemma 3.20. Let l be a prime $l \mid p-1$. Then $a^{\frac{p-1}{l}} = 1 \mod p$ is equivalent to $x^l = a \mod p$ having a solution in $\mathbb{F}_p^*$.

Proof. Recall that $\mathbb{F}_p^*$ is a cyclic group, with some primitive root ζ. Let $a = \zeta^i$, so $\zeta^{i\frac{p-1}{l}} = 1 \mod p$. Hence, $p-1 \mid i\frac{p-1}{l}$. There is a $b \in \mathbb{Z}$ such that $b(p-1) = i\frac{p-1}{l} \implies bl = i \implies l \mid i$. Then $u = \zeta^{\frac{i}{l}}$ is a solution of $x^l = a \mod p$.

For the reciprocal, if $u \in \mathbb{F}_p^*$ is the solution to $u^l = a$, then $a^{\frac{p-1}{l}} = u^{p-1} = 1$. ■

Remark 3.21. Note that $x^l = a \mod p$ might have solutions when $l \nmid p-1$. In that case, all the elements in $\mathbb{F}_p^*$ are l-residues as the group endomorphism $x \mapsto x^l$ must have trivial kernel and, hence, full image.

Definition 3.22 (Kummer Fields relevant to Artin's Conjecture)**.** For l prime $l \nmid a$ and k square-free integer coprime with a, let $L_l = \mathbb{Q}(\zeta_q, \sqrt[l]{a})$ and $L_k = \prod_{\substack{l \mid k \\ \text{prime}}} L_l$ the compositum. Denote $C_k = \mathbb{Q}(\zeta_k)$.

Lemma 3.23. Let l be a prime. A prime $p \in \mathbb{Z}_{>2}$ splits completely in $C_l/\mathbb{Q}$ if and only if $l \mid p-1$.

Proof. For $l = 2$, $C_2 = \mathbb{Q}$ and the result is trivial. Otherwise, recall that the ring of integers of a cyclotomic field is $\mathbb{Z}[\zeta_l]$ [Lan94, Th. 4 Page 75], which is generated by the primitive element. By the classical theorem in Ramification Theory [Neu99, Ch 1 Prop. 8.3], the splitting behavior of p is equivalent to the splitting of the minimal polynomial of ζ_l, namely $\Phi_l(x) = \frac{x^l-1}{x-1}$, modulo p.

If $\Phi_l(x) \mod p$ splits completely, in particular it has one root $u \neq 1 \mod p$ which $u^l = 1 \underset{l \text{ prime}}{\implies} l \mid p-1$. For the reciprocal, let ζ be a primitive root of $\mathbb{F}_p^*$. Then, if $l \mid p-1$, $x^l = 1 \mod p$ has solutions $\{\zeta^{\frac{p-1}{l}}, \zeta^{2\frac{p-1}{l}}, \ldots, \zeta^{l\frac{p-1}{l}} = 1\}$ which are all unique. Hence, $\Phi_l(x)$ splits completely.

■

Second proof (using Frobenius substitution). For $l = 2$, $C_2 = \mathbb{Q}$ and the result is trivial. Otherwise, recall that the discriminant of a prime cyclotomic field is $(-1)^{\frac{l-1}{2}} l^{l-2}$. Hence, p ramifies at $p = l$ which does not follow $l \mid p-1$. For p unramified, p is completely split if and only if $\text{Frob}_p(C_l/\mathbb{Q}) = 1$. Now, $\zeta_l^p = \zeta_l \mod p \implies \zeta_l^{p-1} = 1 \mod p \implies l \mid p-1$ or $l = p = 2$.

For the other direction, let $\text{Frob}_p(C_l/\mathbb{Q}) = a \in \text{Gal}(C_l/\mathbb{Q}) = (\mathbb{Z}/l\mathbb{Z})^*$ such that $\zeta_l \mapsto \zeta_l^a$. By the property of the Frobenius element on the residue field $\zeta_l^a = \zeta_l^p \mod p \implies \zeta_l^{p-a} = 1 \mod p \implies l \mid p-a$. As $l \mid p-1$ and $1 \leq a \leq l-1$, the only possibility is $a = 1$. ■

The proofs of the following Lemmas 3.25 and 3.26 are taken from M. Rosen book *Number Theory in Function Fields*, where they are given for Function Fields [Ros02, Propositions 10.3-4]. A version of these Lemmas is true for general Dedekind Domains.

Remark 3.24. Recall, for l prime, $x^l - a$ is irreducible over K if and only if a is not an l-th power over K. [Lan05, Th. 9.1 Page 297]

Lemma 3.25. Let l be a prime. Let $\mathfrak{p}$ be a prime ideal of C_l with $(p) = \mathfrak{p} \cap \mathbb{Z}$, such that $p > 2$ and $l \mid p - 1$. Then, $\mathfrak{p}$ ramifies over L_l/C_l if and only if $l \mid \operatorname{ord}_{\mathfrak{p}}(a)$

Proof. Let $O = \mathbb{Z}[\zeta_l]$ be the ring of integers of C_l and $O_{\mathfrak{p}}$ its localization ring at P and π a uniformizer element of $O_{\mathfrak{p}}$. Let $R_{\mathfrak{p}}$ be the integral closure of $O_{\mathfrak{p}}$ over L_l.

If $l \mid \operatorname{ord}_{\mathfrak{p}}(a)$, then $a = \pi^{lh}u$ with u a unit of $O_{\mathfrak{p}}$. Then $\mu := \frac{\sqrt[l]{a}}{\pi^h} \in L_l$. Clearly, $L_l = C_l(\mu)$. Now, $O_{\mathfrak{p}}[\mu]$ is a full rank free $O_{\mathfrak{p}}$-module under $R_{\mathfrak{p}}$. By a classical theorem in Algebraic Number Theory, if the discriminant of $O_{\mathfrak{p}}[\mu]$ is a unit in $O_{\mathfrak{p}}$ we must have $R_{\mathfrak{p}} = O_{\mathfrak{p}}[\mu]$.

Hence, let's compute $\operatorname{Disc}_{O_{\mathfrak{p}}[\mu]/O_{\mathfrak{p}}} = \operatorname{Det}((\operatorname{Tr}(\mu^i\mu^j))_{ij})$. If $l \nmid k$, then u^k cannot be an l-th power as u is not one and $l \nmid k$. Hence, the minimal polynomial of μ^k is $x^l - u^k$. On the other hand, if $l \mid k$, we must have $l = 0$ or $l = k$. In the first case, $\operatorname{Tr}(1) = l$ and in the second, $\operatorname{Tr}(\mu^l) = \operatorname{Tr}(u) = lu$. We conclude that

$$\operatorname{Tr}_{L_l/C_l}(\mu^k) = \begin{cases} l & k = 0 \\ lu & k = l \\ 0 & 0 \leq k \leq 2l-1, k \notin \{0, l\} \end{cases}$$

From this, we can compute $\operatorname{Disc}_{O_{\mathfrak{p}}[\mu]/O_{\mathfrak{p}}} = \pm l^l u^{l-1}$. Indeed, this is a unit in $O_{\mathfrak{p}}$ as u is one by definition and $l \neq p$, hence $R_{\mathfrak{p}} = O_{\mathfrak{p}}[\mu]$. Furthermore, $\mathfrak{p} \nmid \operatorname{Disc}$ so $\mathfrak{p}$ is unramified.

For the other direction, suppose $l \nmid \operatorname{ord}_{\mathfrak{p}}(a)$. Let $\mathfrak{P}$ be a prime over $\mathfrak{p}$ in L_l/C_l. Since $(\sqrt[l]{a})^l = a$, we have

$$l \operatorname{ord}_{\mathfrak{P}}(\sqrt[l]{a}) = \operatorname{ord}_{\mathfrak{P}}(a) = e(\mathfrak{P}/\mathfrak{p}) \operatorname{ord}_{\mathfrak{p}}(a) \tag{3.2}$$

Hence, $l \mid e(\mathfrak{P}/\mathfrak{p})$. Because the extension has degree l, we know $e \leq l$ so $e = l$. This means that $\mathfrak{p}$ is totally ramified. ■

Lemma 3.26 (Key Lemma)**.** Let l be a prime. Let $\mathfrak{p}$ be a prime in C_l and $(p) = \mathfrak{p} \cap \mathbb{Z}$, such that $\operatorname{ord}_{\mathfrak{p}}(a) = \operatorname{ord}_p(a) = 0$, $p > 2$ and $l \mid p - 1$. $\mathfrak{p}$ splits completely over L_l/C_l if and only if $x^l = a \mod \mathfrak{p}$ has a solution.

Proof. Let $O_{\mathfrak{p}}$ be the localization of the ring of integers of C_l away from $\mathfrak{p}$ and let $R_{\mathfrak{p}}$ be its integral closure over L_l. The hypothesis $\operatorname{ord}_{\mathfrak{p}}(a) = 0$ implies that a is a unit over $O_{\mathfrak{p}}$ and, as shown in the

proof of Lemma 3.25, $R_\mathfrak{p} = O_\mathfrak{p}[\sqrt[l]{a}]$. Note that, by Lemma 3.25, $\mathfrak{p}$ does not ramify over L_l/C_l as $l \mid 0 = \mathrm{ord}_\mathfrak{p}(a)$. Also note that $l \mid p-1 \implies l \mid \mathcal{N}\mathfrak{p} - 1 = |O_\mathfrak{p}/\mathfrak{p}|$. Hence, the residue field contains some primitive l-root, $\zeta_l = \zeta^{\frac{\mathcal{N}\mathfrak{p}-1}{l}}$, where ζ is the generator of $(O_\mathfrak{p}/\mathfrak{p})^*$.

The case where a is an l-th power over C_l is trivial. Discard that case, which implies $x^l - a$ is irreducible over C_l. Now, the extension $R_\mathfrak{p}/O_\mathfrak{p}$ is generated by a power basis with minimal polynomial $x^l - a$. Hence, the ramification properties of $\mathfrak{p}$ are equal to the ramification of $x^l - a \mod \mathfrak{p}$. If $\mathfrak{p}$ is totally split, $x^l - a$ splits $\mod \mathfrak{p}$, so there is at least one solution. If $x^l = a \mod \mathfrak{p}$ has one solution, as $\zeta_l \in C_l$, all the solutions are $\{\zeta_l\sqrt[l]{a}, \zeta_l^2\sqrt[l]{a}, \ldots, \zeta_l^l\sqrt[l]{a} = \sqrt[l]{a}\}$ which are all distinct $\mod \mathfrak{p}$. Hence, $\mathfrak{p}$ totally splits. ■

Second proof (using Frobenius Substitution). See [Ros02, Proposition 10.4] ■

Lemma 3.27. A prime l follows the conditions of Lemma 3.19 for $p > 2$ if and only if p is completely split over $L_l/\mathbb{Q}$.

Proof. Application of Lemmas 3.23 and 3.26. Recall that $x^l = a \mod \mathfrak{P}$ has a solution if and only if $a^{\frac{\mathcal{N}\mathfrak{P}-1}{l}} = 1 \mod \mathfrak{P}$. As $\mathfrak{p}$ splits completely, $\mathcal{N}\mathfrak{P} = \mathcal{N}\mathfrak{p}$. Also, because both sides of the identity are in $O_\mathfrak{p}/\mathfrak{p} \subseteq R_\mathfrak{p}/\mathfrak{P}$, we can lower the modulo $a^{\frac{\mathcal{N}\mathfrak{p}-1}{l}} = 1 \mod \mathfrak{p} \iff x^l = a \mod \mathfrak{p}$ is solvable. ■

Lemma 3.28. For k square free, all the primes $l_i \mid k$ follow conditions of Lemma 3.19 if and only if p is completely split over $L_k/\mathbb{Q}$. By Chebotarev's theorem, these primes p have density $\frac{1}{[L_k:\mathbb{Q}]}$.

Proof. A prime splits completely in the compositum if and only if it splits completely in each factor. Using the previous Lemmas, we obtain the desired result. ■

Theorem 3.29 (Artin's observation)**.** Let $a \in \mathbb{Z}$ not -1 nor a square and k a square free integer coprime to a. The density of primes for which there is no $l \mid k$ following the conditions of Lemma 3.19 is

$$A_k(a) = \sum_{\substack{k' \mid k \\ k' \geq 1}} \frac{\mu(k')}{[L_{k'} : \mathbb{Q}]} \tag{3.3}$$

where μ is the Moebius Inversion function.

Proof. By Lemma 3.28, we know the density of primes such that all $l \mid k$ follow conditions of Lemma 3.19. The Inclusion-Exclusion Principle yields the desired result. ■

Remark 3.30. Note that taking $k \to \infty$ over the primorials coprime to a, the density $A_k(a)$ counts primes where a is "close" to being a primitive root, in the sense that an l following the conditions of Lemma 3.19 would need to be very large. Hence, one might expect the limit of $A_k(a)$ to be the density of primes with a prescribed primitive root at a. This is precisely what Artin conjectured. Nonetheless, the step of taking the limit is where the difficulty in Artin's conjecture lies.

Hence, Artin arrived at the following specific conjecture.

Definition 3.31 (Artin's constant)**.** For $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$, we define Artin's constant as

$$A(a) = \sum_{k \geq 1} \frac{\mu(k)}{[L_k : \mathbb{Q}]} \tag{3.4}$$

Conjecture 3.32 (Artin primitive root Conjecture II)**.** Given $a \in \mathbb{Z} \setminus \{-1, 0, 1\}$, the set of P_a has Dirichlet density $A(a)$. Furthermore, $A(a) > 0$ if and only if a is not a perfect square.

Assuming this conjecture was true, one can compute the $[L_k : \mathbb{Q}]$ and show positivity without using any version of the Riemann Hypothesis. We do so in the following sections.

3.2.1 Computation of the degree

Definition 3.33 (Constants relevant in Artin's observation)**.** Let $h = \max\{h' \mid a \text{ is an } h'\text{-perfect power in } \mathbb{Z}\}$, which is well-defined as $a \notin \{-1, 0, 1\}$. Let $k = l_1 \dots l_r$ be square-free integer coprime to a and $k_a = \frac{k}{(k,h)}$. Note that k_a is the product of the prime divisors l of k such that a is not a l-th power.

Definition 3.34 (Fields relevant to Artin's Conjecture II)**.** Denote $R_k = \mathbb{Q}(\sqrt[k]{a})$ and $I_k = C_k \cap R_k$.

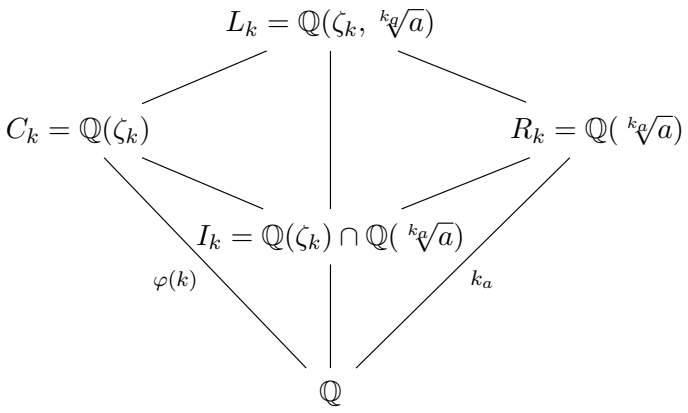
Lemma 3.35. The field $L_k = \prod_{\substack{l|k \\ \text{prime}}} \mathbb{Q}(\zeta_q, \sqrt[l]{a})$ is precisely $\mathbb{Q}(\zeta_k, \sqrt[k_a]{a})$. It is also $\mathbb{Q}(\zeta_k, \sqrt[k]{a})$.

Proof. First we prove $\mathbb{Q}(\zeta_k, \sqrt[k_a]{a}) \subseteq L_k$. Let $x_i = \frac{k}{l_i} \in \mathbb{Z}$. The $\gcd(x_1, \dots, x_r) = 1$ and Bézout's identity gives $a_i \in \mathbb{Z}$ such that $\sum a_i x_i = 1$. Now, $\prod_{\substack{l|k \\ \text{prime}}} (\zeta_q)^{a_i} = e^{2\pi i \cdot \sum \frac{a_i}{l}} = e^{2\pi i \frac{1}{k}} = \zeta_k$. By the same method that $\sqrt[k]{a} \in L_k$. The other inclusion holds because $\zeta_q = \zeta_k^{k/l}$ and

$$\sqrt[l]{a} = \begin{cases} \in \mathbb{Q} & \text{if } l|h \\ (\sqrt[k_a]{a})^{k_a/l} & \text{otherwise} \end{cases} \tag{3.5}$$

An analogous argument proves the second expression. ■

Remark 3.36. Even though the second expression might seem more canonical, in the computation of the degree, the first expression will be more useful. This is because the extension $\mathbb{Q}(\zeta_k, \sqrt[k]{a})/\mathbb{Q}(\zeta_k)$ could be trivial if, for example, a was a k-th power in $\mathbb{Z}$. This is accounted by substituting k by k_a.



Following the identity $[L_k : \mathbb{Q}] = [L_k : C_k][C_k : \mathbb{Q}] = [L_k : C_k]\varphi(k)$, we aim to compute $[L_k : C_k]$. When Artin proposed the conjecture, he claimed $[L_k : C_k] = k_a$. This was found to be incorrect by D. H. and E. Lehmer and corrected in a private correspondence with Artin. Independently, Hooley [Hoo67] attributes this correction to Heilbronn. The full history of this correction is delightfully exposed in the first part of [Ste03], including the original letters from the Berkeley archives.

Lemma 3.37 (Degree correction, Heilbronn)**.** Let $a = a_1 a_2^2$ be the square free decomposition of a. Then, the degree $[L_k : C_k]$ is

$$[L_k : C_k] = \begin{cases} \frac{k_a}{2} & \text{if } 2a_1 | k \text{ and } a_1 = 1 \mod 4 \\ k_a & \text{otherwise} \end{cases} \tag{3.6}$$

Proof. $C_k/\mathbb{Q}$ is Galois. A classical proposition of Galois Theory [Mil22, Proposition 3.19] concerning the Galois group of a compositum states

$$[C_k : \mathbb{Q}][R_k : \mathbb{Q}] = [L_k : \mathbb{Q}][I_k : \mathbb{Q}] \implies k_a = [L_k : C_k][I_k : \mathbb{Q}] \tag{3.7}$$

If q is a prime factor of $[I_k : \mathbb{Q}]$, then $[C_k(\sqrt[q]{a}) : C_k]$ is either 1 or q and $[C_k(\sqrt[q]{a}) : C_k] \mid [L_k : C_k] = \frac{k_a}{[I_k : Q]}$. But q does not divide $\frac{k_a}{[I_k:\mathbb{Q}]}$ as k_a is square-free and $q \mid [I_k : \mathbb{Q}]$. Hence, $[C_k(\sqrt[q]{a}) : C_k] = 1 \implies \sqrt[q]{a} \in C_k$. Lastly, because $\mathbb{Q}(\zeta_q, \sqrt[q]{a}) \subseteq C_k$, the extension $\mathbb{Q}(\zeta_q, \sqrt[q]{a})/\mathbb{Q}$ must be an Abelian extension. Hence, q can only be an even prime and $[I_k : \mathbb{Q}]$ can only be either 1 or 2. It will be 2 precisely when k is even and $\sqrt{a} \in C_k \iff \sqrt{a_1} \in C_k$.

A classical application of Gauss Sums [Neu99, Ex. 4 Chapter 1.10] proves that the only quadratic subfields in the k-th cyclotomic field are of the form

$$\mathbb{Q}\left(\sqrt{\left(\frac{-1}{D}\right)D}\right) \subseteq \mathbb{Q}(\zeta_k) \tag{3.8}$$

where $D > 1$ is a square-free odd divisor of k. Hence, we need a_1 to be an odd divisor of k and $a_1 = 1 \mod 4 \iff \left(\frac{-1}{a_1}\right) = 1$. ■

Warning 3.38. This corner case is an inconvenience in further computations. Artin's conjecture is already an interesting and open problem for any particular value of $a \in \mathbb{Z}$. For the duration of this document, we will ignore these exceptional a and refer the reader to the precise bookkeeping in other references.

3.2.2 Positivity of Artin's constant

In Artin's conjecture over $\mathbb{Q}$, we end up having a conjectured density

$$A(a) = \sum_{k\geq 1} \frac{\mu(k)e(k)}{\phi(k)k_a}, \qquad \text{where } e(k) = \begin{cases} 2 & 2a_1|k \text{ and } a_1 = 1 \mod 4 \\ 1 & \text{otherwise} \end{cases} \tag{3.9}$$

Lemma 3.39 (Euler product of $A(a)$)**.** Let $a = a_1 a_2^2$ be the square-free decomposition of a, and let h be the largest integer such that a is an h-power in $\mathbb{Z}$. The following identity is true.

$$A(a) = \delta_{a_1} \prod_{q|h \text{ prime}} \left(1 - \frac{1}{q-1}\right) \prod_{q\nmid h \text{ prime}} \left(1 - \frac{1}{q(q-1)}\right) \tag{3.10}$$

where, $\delta_{a_1} = 1$ if $a_1 \neq 1 \mod 4$ and

$$\delta_{a_1} = 1 - \mu(a_1) \prod_{\substack{q|a_1 \\ q|h \\ \text{prime}}} \frac{1}{q-2} \prod_{\substack{q|a_1 \\ q\nmid h \\ \text{prime}}} \frac{1}{q(q-1)-1} \tag{3.11}$$

otherwise.

Proof. When $a_1 \neq 1 \mod 4$, note that $e(k) = 1\ \forall k$ and the function $\psi(k) = \frac{\mu(k)}{\phi(k)k_a}$ is weakly multiplicative. Hence, it has a representation as an Euler Product.

$$A(a) = \sum_{k\geq 1} \frac{\mu(k)1}{\phi(k)k_a} = \prod_{q \text{ prime}} \left(1 - \frac{1}{q_a(q-1)}\right) \tag{3.12}$$

Now, q_a is either q or 1 precisely when a is a q-th power or not, respectively. Or equivalently, precisely when $q|h$ or not, respectively.

When $a_1 = 1 \mod 4$, this computation is more cumbersome. See [Hoo67, Eq. 31-32]. ■

Lemma 3.40 (Positivity of Artin's constant)**.** Let $a \notin \{-1, 0, 1\}$. Then, $A(a) > 0$ if and only if a is not perfect square.

Proof. If a is perfect square, h would be even and the term $1 - \frac{1}{2-1} = 0$. Hence, $A(a) = 0$. For the other direction, if $A(a) = 0$, either it has a 0 factor in its product expression or it tends to 0 in the limit. Yet the infinite product is

$$\prod_{q \text{ prime}} \left(1 - \frac{1}{q(q-1)}\right) > \prod_{q \text{ prime}} \left(1 - \frac{1}{q^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} > 0 \tag{3.13}$$

Hence, if $A(a) = 0$, we must have a 0 term. The only possibility is $2 \mid h \iff a$ is a perfect square. ■

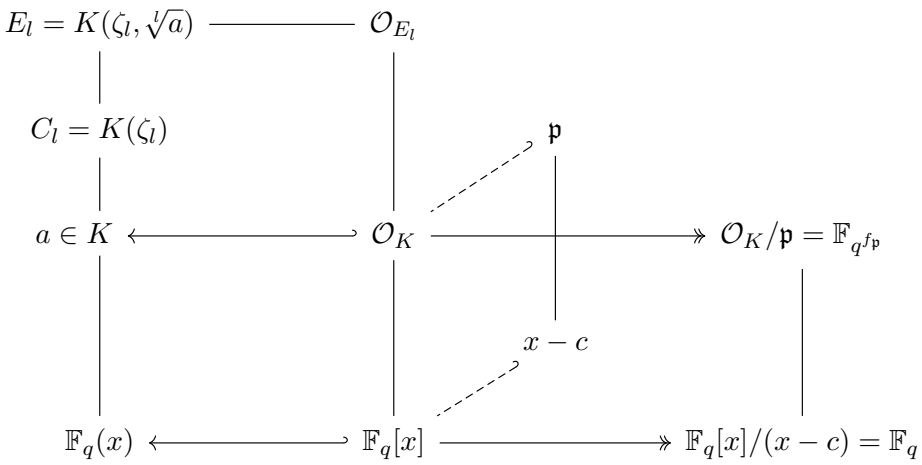
4. Function Field setting

This chapter focuses in A.C in the Function Field setting. First, we give an exposition of the original proof of A.C. over Function Fields by Bilharz. The original paper [Bil37] is in German, so the main source for our exposition has been the translation of Bilharz's result found in the book *Number Theory in Function Fields* by M. Rosen [Ros02, Chapter 10]. On the other hand, we present a second independent proof of the result found in 2020 by Kim-Murty [KR20; KM22] developing on ideas of Davenport [Dav39].

The abstract of [KR20] announces that their proof is independent of the Riemann Hypothesis on Function Fields. Yet we have found a small technical error in their paper that invalidates this claim. The proof can be fixed assuming a weaker version of R. H. The author of the present document has been unable to find a condition-less fix.

Notation 4.1 (Relevant constants and fields in Artin's Conjecture over Function Fields)**.** For the remaining of this section, $q = p^r$ is an arbitrary prime power, K is a Function Field with field of constants $\mathbb{F}_q$ and let $a \in K^*$. To study Problem 3.9 over K, we will need to study the ramification properties of primes $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ over extensions L_l/K with l a rational prime and $L_l = K(\zeta_l, \sqrt[l]{a})$. Also, for $k \in \mathbb{Z}^+$ denote $C_k = K(\zeta_k)$.

Remark 4.2. If $a \in \mathbb{F}_q^* \subseteq K^*$, then $\operatorname{ord}_{K^*}(a) \mid q$, hence a can only be a primitive root for finitely many primes. We may assume $a \in K^* \setminus \mathbb{F}_q^*$.



Remark 4.3. Note that we define the rings of integers of a Function Field as the integral closure of $\mathbb{F}_q[x]$. As discussed in [Neu99, Chapter 1.14], this decision is somewhat arbitrary and, for example, we could choose to center over $\mathbb{F}_q[1/x]$. Nonetheless, for Artin's Conjecture this makes no difference, as it only changes the behavior of the finitely many primes at infinity.

4.1 Original proof by Bilharz

An analogue of Artin's observation, presented in Section 3.2, can be given for general Global Fields, as will be discussed in Section 6.1. From this starting point, formalized by Theorem 4.7, Bilharz [Bil37] gave an argument to justify the *step to the limit* in the Function Field setting. This final step in his argument is remarkably ad-hoc and only valid for Function Fields. For the advancement of the conjecture over Number Fields, Hooley [Hoo67] was able to replace this ad-hoc argument by a more general reduction to the problem of counting primes.

As discussed in Section 3.2.1, Artin's original conjecture had a small flaw in the density formula that came from a miss-computation of the degree $L_l/\mathbb{Q}$ for some values of a. Bilharz original proof contains a similar error. When $a \in K$ is a Geometric Element at $l = 2$, Bilharz's proof is correct as is. By the same rationale exposed in the Warning 3.38, this document will limit the exposition to that case.

Definition 4.4 (Geometric Element)**.** Let K be a function field with constant field $\mathbb{F}_q$. An element $a \in K$ is said to be geometric at a prime $l \in \mathbb{Z}$ if and only if the integral closure of $\mathbb{F}_q$ over $K(\sqrt[l]{a})$ is $\mathbb{F}_q$, or, in order words, if $K(\sqrt[l]{a})/K$ is a geometric extension.

Remark 4.5. If a is not geometric at $l = 2$, then the extension $K(\sqrt{a})/K$ must be exactly $K \cdot \mathbb{F}_{q^2}/K$ as it is of degree 2 and must extend the constants. Nonetheless, if $a \in K^* \setminus (K^*)^2$, the extension $K(\sqrt{a})/K$ ramifies at $\mathfrak{p} \mid a$ but $K \cdot \mathbb{F}_{q^2}/K$ doesn't ramify anywhere.

Wow! This must be wrong, if so, Bilharz Proof would work in general

Lemma 4.6. $a \in K$ is a primitive root modulo $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ if and only if there is no $l \in \mathbb{Z}$ prime that follows both

$$\text{(1)} \quad l \mid \mathcal{N}\mathfrak{p} - 1 \qquad \text{and} \qquad \text{(2)} \quad a^{\frac{\mathcal{N}\mathfrak{p}-1}{l}} = 1 \mod \mathfrak{p}$$

We can assume $l \neq p = \operatorname{char} K$ as condition 1 is never true for $l = p$.

Theorem 4.7 (Artin's observation for Function Fields)**.** Let $a \in K$, and k square-free and $p = \mathrm{char}(K) \nmid k$. The density of primes such that there is no $l \mid k$ that follows conditions of Lemma 4.6 is

$$A_k(a) = \sum_{k' \mid k} \frac{\mu(k')}{[L_{k'} : \mathbb{Q}]} \tag{4.1}$$

4.1.1 Computation of the degree

Definition 4.8. Given $k \in \mathbb{Z}$ square free $p \nmid k$, let $f(k) = \mathrm{ord}_{(\mathbb{Z}/k\mathbb{Z})^\times}(q)$, where $\mathbb{F}_q$ is the field of constants of K. This is well-defined as $(q, k) = (p^r, k) = 1$. Analogous to Definition 3.33, we denote k_a the product of all $l \mid k$ primes such that a is not an l-th power in K.

Lemma 4.9. The extension $K(\zeta_k)/K$ is Galois and has degree $[K(\zeta_k) : K] = f(k)$.

Proof. Notice that $K(\zeta_k) = K \cdot \mathbb{F}_q(\zeta_k)$. On one hand, $\mathbb{F}_q(\zeta_k)/\mathbb{F}_q$ is a finite field extension, so it is Galois and has a Galois group generated by $\phi_q : x \mapsto x^q$. Hence it has degree $[\mathbb{F}_q(\zeta_k) : \mathbb{F}_q] = f(k)$. On the other hand $\mathbb{F}_q(\zeta_k) \cap K = \mathbb{F}_q$ as we have chosen q such that $\mathbb{F}_q$ is the field of constants of K. A classical proposition of Galois Theory [Mil22, Proposition 3.19] concerning the Galois group of a compositum states that, with the given conditions, $K(\zeta_k)/K$ is Galois and its Galois group is isomorphic to $\mathrm{Gal}(\mathbb{F}_q(\zeta_k)/\mathbb{F}_q)$. This concludes $[K(\zeta_k) : K] = f(l)$. ∎

Lemma 4.10. Let $a \in K$ be a Geometric Element at $l = 2$. Then, the degree $[L_k : K(\zeta_k)] = k_a$, where $L_k = K(\zeta_k, \sqrt[k]{a}) \underset{\text{Lemma 3.35}}{=} K(\zeta_k, \sqrt[k_a]{a})$.

Proof. Following the argument of Lemma 3.37, it is sufficient to see that $I_k := K(\sqrt[k_a]{a}) \cap K(\zeta_k)$ is $I_k = K$. Suppose not, then for some $l \mid k$, $K(\sqrt[l]{a}) \subseteq K(\zeta_k)$. If $l \neq 2$, we find a non-abelian extension $K(\zeta_l, \sqrt[l]{a})/K$ embedded in an abelian one $K(\zeta_k)/K$, which is a contradiction.

For $l = 2$, we use the condition of $a \in K$ being a Geometric Element at $l = 2$. To begin with, a subextension of a constant extension must also be constant extension. This would imply that the field of constants of $K(\sqrt{a})$ is $\mathbb{F}_{q^2}$. But by a Geometric at $l = 2$, the field of constants must be $\mathbb{F}_q$, which is a contradiction. ∎

4.1.2 Bilharz's contribution

Notation 4.11. Let $\mathbb{P} = \{p_1 = 2, p_2 = 3, \ldots\}$ be the usual enumeration of the rational primes. Let $\mathrm{Pr}_n = \prod_{i \le n} p_i$ be the n-th primorial.

To match our notation with the source [Ros02, Ch.10] we define $\mathcal{M}_k(a) \coloneqq P_{\mathrm{Pr}_k}(a)$ and $\mathcal{M}(a) = P(a)$. The value a will remain constant throughout the section, so we drop the parenthesis and use $\mathcal{M}_k$ and $\mathcal{M}$.

The remaining step in Artin's conjecture is to relate the family $\mathcal{M}_k$ with the set $\mathcal{M}$.

$$\mathcal{M}_k = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K \mid \nexists\, l \le k \text{ prime following the conditions of Lemma 4.6}\}$$

$$\begin{aligned}\mathcal{M} &= \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K \mid a \text{ is a primitive root } \mod \mathfrak{p}\} = \\ &= \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K \mid \nexists\, l \text{ prime following the conditions of Lemma 4.6}\}\end{aligned}$$

From Artin's observation, we compute the density $\delta(\mathcal{M}_k)$ as the finite sum found in Theorem 4.7. We aim to prove that the density of $\mathcal{M}$ is $\delta(\mathcal{M}) \coloneqq \lim_k \delta(\mathcal{M}_k)$.

Remark 4.12. This is not trivial as the Dirichlet measure is not well-behaved with respect to infinite intersection of sets. For example, note that for $S_n = \{p \text{ prime} \mid p \ge n\}$, we have $0 = \delta(\cap_n S_n) \neq \lim_n \delta(S_n) = 1$. Weinberger [Wei72] found an example close to Artin's conjecture where this equality also fails.

We begin by introducing two preliminary theorems without proof.

Theorem 4.13 (Romanoff)**.** Let $q \in \mathbb{Z}_{>1}$ be a prime power, $m \in Z$ with $(q, m) = 1$ and $f(m) = \operatorname{ord}_{(\mathbb{Z}/m\mathbb{Z})^\times}(q)$ which is well-defined as $(q, m) = 1$. Then the following sum converges.

$$\sum_{\substack{m \in \mathbb{Z}_{>0} \\ m \text{ square-free} \\ (m,q)=1}} \frac{1}{m \cdot f(m)} \tag{4.2}$$

Proof. See [Ros02, Theorem 10.8] ∎

Lemma 4.14 (Upper bound on genus of L_l)**.** Let g_l be the genus of the field L_l. There exist constants $A, B \in \mathbb{R}$, $A > 0$ such that $\forall l$ prime $\; g_{L_l} = Al + B$. This implies there are $A_1, A_2 \in \mathbb{R}^+$ such that $\forall l$ prime, $\; A_1 l < g_l < A_2 l$.

Proof. Application of Riemann-Hurwitz Identity. See [Ros02, Proposition 10.4] ∎

The next step of the proof uses a finer version of Chebotarev Theorem to upper bound the function $\delta(\mathcal{M}_n, s) - \delta(\mathcal{M}, s)$. Following Artin's observation, we can give the following properties of the sets $\mathcal{M}_n$ and $\mathcal{M}$.

Observation 4.15. The sets $\mathcal{M}_n$ and $\mathcal{M}$ follow

1. $\mathcal{M} \subseteq \mathcal{M}_m \subseteq \mathcal{M}_n$ for all $m > n$
2. $\cap_{n\geq 1}\mathcal{M}_n = \mathcal{M}$
3. $\mathcal{M}_n \setminus \mathcal{M} \subseteq \cup_{i\geq n+1} \mathrm{Spl}(L_{l_i})$

For $s \in \mathbb{R}$, these properties translate to Dirichlet Densities as

1. $\delta(\mathcal{M}, s) \leq \delta(\mathcal{M}_m, s) \leq \delta(\mathcal{M}_n, s)$ for all $m > n$
2. $\lim_n \delta(\mathcal{M}_n, s)$ exists and is $\geq \delta(\mathcal{M}, s)$.
3. $\delta(\mathcal{M}_n, s) - \delta(\mathcal{M}, s) \leq \sum_{i\geq n+1} \delta(\mathrm{Spl}(L_{l_i}), s)$

Lemma 4.16 (Fine version of Chebotarev's Theorem)**.** If L/K is Galois, and $s \in R$

$$\delta(\mathrm{Spl}(L), s) < \frac{1}{[L:K]} \frac{\log \zeta_L(s)}{\log \zeta_K(s)} \tag{4.3}$$

Proof. The classical proof of Chebotarev's Theorem 2.2 shows this finer result, before taking the limit $s \to 1$. ∎

Lemma 4.17 (Main Lemma for Theorem 4.20)**.** There exists a real number $s_1 > 1$ such that

$$\sum_{i\geq 1} \frac{1}{[L_{l_i}:K]} \frac{\log \zeta_{L_{l_i}}(s)}{\log \zeta_K(s)} \tag{4.4}$$

converges uniformly on the interval $(1, s_1)$.

Proof. For a geometric, $[L_l : K] = lf(l)$ for all but a finite amount of l. Hence, it suffices to prove

$$\sum_{\substack{l \text{ prime} \\ l\neq p \\ l\nmid h}} \frac{1}{lf(l)} \frac{\log \zeta_{L_l}(s)}{\log \zeta_K(s)} \tag{4.5}$$

is uniformly convergent in an interval $(1, s_1)$.

A classical theorem of Function Field extensions [Ros02, Theorem 3.5] states

$$\zeta_{L_l}(s) = \zeta_{R_l}(s) P_{L_l}(s) \tag{4.6}$$

where $P_{L_l}(s)$ is a polynomial in $q^{-f(l)s}$ of degree $2g_l$, where g_l is the genus of L_l. Substituting back, the sum in Equation 4.5 splits in two parts. It is sufficient to see that these two terms uniformly

converge.

First we bound the ζ_{R_l} term. Note that the zeta function of a cyclotomic field has a closed formula

$$\zeta_{R_l}(s) = \frac{1}{(1-q^{-f(l)s})(1-q^{f(l)(1-s)})} \leq \frac{1}{(1-q^{-s})(1-q^{1-s})} = \zeta_R(s) \tag{4.7}$$

Hence, the term is bounded as follows. Note that order 1 pole in each ζ cancels out and the sum converges by Romanoff result 4.13.

$$\sum_{\substack{l \text{ prime} \\ l \neq p \\ l \nmid h}} \frac{1}{lf(l)} \frac{\log \zeta_{R_l}(s)}{\log \zeta_K(s)} \leq \frac{\log \zeta_R(s)}{\log \zeta_K(s)} \sum_{\substack{l \text{ prime} \\ l \neq p \\ l \nmid h}} \frac{1}{lf(l)} \tag{4.8}$$

Now we turn to the P_{L_l} term. If one writes the monomial factorization of P as

$$P_{L_l}(s) = \prod_{j=1}^{2g_l} \left(1 - \pi_j q^{-f(l)s}\right) \tag{4.9}$$

the Riemann Hypothesis on the Function Field L_l states that the π_j have absolute value $q^{f(l)/2}$. This, together with Lemma 4.14, gives the following bounds.

$$2A_1 l \log\left(1 - q^{-\frac{f(l)}{2}}\right) < \log P_{L_l}(s) < 2A_2 l \log\left(1 + q^{-\frac{f(l)}{2}}\right) \tag{4.10}$$

Using that for $x > 0$, $\log(1+x) < x$ and $-\log(1-x) = \sum_{k\geq 1} \frac{x^k}{k} < \sum_{k\geq 1} x^k = \frac{x}{1+x}$ and letting $r = \max(A_1, A_2)$, we conclude

$$\left|\log(P_{L_l}(s))\right| < rl \frac{\sqrt{q}}{\sqrt{q}-1} q^{-\frac{f(l)}{2}} \tag{4.11}$$

Because $|\log \zeta_K(s)|$ has a pole at 1, $\frac{1}{|\log \zeta_K(s)|} < C$ for s close to 1. Hence,

$$\sum_{l\neq p} \frac{|\log \zeta_{P_{L_l}}(s)|}{|\log \zeta_K(s)|} < rlC \frac{\sqrt{q}}{\sqrt{q}-1} \sum_{l\neq p} \frac{1}{f(l) q^{f(l)/2}} \tag{4.12}$$

■

Lemma 4.18. The sum $\sum_{l\neq p} \frac{1}{f(l)q^{f(l)/2}}$ converges

Proof. Separate the sum in two parts, regarding if $l \leq q^{f(l)/2}$ or not. The first term is now conver-

gent by Romanoff's result 4.13.

$$\sum_{\substack{l\neq p\\ l\leq q^{f(l)/2}}} \frac{1}{f(l)q^{f(l)/2}} < \sum_{\substack{l\neq p\\ l\leq q^{f(l)/2}}} \frac{1}{f(l)l} \tag{4.13}$$

For the second term, we may use a sieving method. For a given $f \in \mathbb{Z}^+$, we can estimate how many primes $l > q^{f/2}$ will have $f(l) = f$. All such l will be prime divisors of $q^f - 1$, and since $l_1 l_2 > (q^{f/2})^2 > q^f - 1$, there can be at most 1. Hence

$$\sum_{\substack{l\neq p\\ l> q^{f(l)/2}}} \frac{1}{f(l)q^{f(l)/2}} < \sum_{f=1}^{\infty} \frac{1}{fq^{f/2}} = -\log(1-q^{1/2}) < \infty \tag{4.14}$$

■

Remark 4.19. Note that the full R.H. is not needed for this result. A $\mathrm{Re}(z) > 1-\epsilon$ zerofree region is enough as

$$\sum_{f=1}^{\infty} \frac{1}{fq^{(1-\epsilon)f}} = -\log(1-q^{1-\epsilon}) < \infty \tag{4.15}$$

Theorem 4.20 (Bilharz)**.** Let K be a function field with field of constants $\mathbb{F}_q$. Let $a \in K$ an arbitrary element Geometric at $l = 2$. The Dirichlet density of the set $\mathcal{M}_a$ is

$$\delta(\mathcal{M}_a) = \sum_{\substack{k\geq 1\\ p\nmid k}} \frac{\mu(k)}{[L_k : K]} = \sum_{\substack{k\geq 1\\ p\nmid k}} \frac{\mu(k)}{k_a f(k)} \tag{4.16}$$

This sum converges by Theorem 4.13.

Proof. By Property 3 of Observation 4.15 and Lemma 4.16

$$0 \leq \delta(\mathcal{M}_n, s) - \delta(\mathcal{M}, s) \underset{4.15}{\leq} \sum_{i\geq n+1} \delta(\{L_{l_i}\}, s) \underset{4.16}{<} \sum_{i\geq n+1} \frac{1}{[L_{l_i} : K]} \frac{\log \zeta_{L_{l_i}}(s)}{\log \zeta_K(s)} \tag{4.17}$$

Fixing $s < s_1$, by Lemma 4.17, the right-hand side converges to 0 as $n \to \infty$. By a classical Lemma of Uniform Convergence [Ros02, Lemma 10.2], the limits $n \to \infty$ and $s \to 1$ can be swapped, which concludes

$$\delta(\mathcal{M}) = \lim_{n\to\infty} \delta(\mathcal{M}_n) \tag{4.18}$$

as desired. ■

4.1.3 Positivity conditions

In the previous section, we have concluded that when $a \in K$ is a Geometric Element at $l = 2$, then Artin's constant for the function field K is

$$\delta(\mathcal{M}) = \sum_{\substack{k \geq 1 \\ p \nmid k}} \frac{\mu(k)}{f(k) k_a} \tag{4.19}$$

Note that $f(k)$ is weakly multiplicative. If a, b are coprime integers, $f(ab) = \mathrm{ord}_{\mathbb{Z}/ab\mathbb{Z}}(q) = \mathrm{ord}_{\mathbb{Z}/a\mathbb{Z}}(q) \cdot \mathrm{ord}_{\mathbb{Z}/b\mathbb{Z}}(q) = f(a)f(b)$ by the Chinese Reminder Theorem. Hence, this series has an Euler Product.

Lemma 4.21 (Euler product)**.** Let K be a function field with field of constants $\mathbb{F}_q$ and $a \in K$ a Geometric Element at $l = 2$. Let S_a the finite set of primes l such that a is an l-th power. Then, Artin's Constant can be expressed as the following Euler Product.

$$\delta(\mathcal{M}) = \prod_{\substack{l \text{ prime} \\ l \in S_a}} \left(1 - \frac{1}{f(l)}\right) \prod_{\substack{l \text{ prime} \\ l \notin S_a}} \left(1 - \frac{1}{l f(l)}\right) \tag{4.20}$$

Proof. Perform the product expansion formally. Romanoff's Theorem 4.13 shows convergence. ■

Theorem 4.22 (Positivity conditions)**.** Let K be a function field with field of constants $\mathbb{F}_q$ and $a \in K$ a Geometric Element at $l = 2$. Then a follows Artin's Conjecture on K if and only if a is not an l-th power for a prime l with $f(l) = 1$. In order words, for a prime l such that $l \mid q - 1$.

Proof. If a is an l-th power with $l \mid q - 1$ then there is a 0-factor, so $\delta(\mathcal{M}) = 0$. For the other direction, if $\delta(\mathcal{M}) = 0$ it can either have a 0-factor or it can tend to 0 in the limit. If it has a 0-factor, it must be for some $l \in S_a$ with

$$1 - \frac{1}{f(l)} = 0 \implies f(l) = 1 \tag{4.21}$$

On the other hand, the infinite part of the product can be lower bounded using that $\log(1 - x) > -x$ for $0 < x \leq \frac{1}{2}$.

$$\prod_{l \text{ prime}} \left(1 - \frac{1}{l f(l)}\right) = \exp\left(\sum_{l \text{ prime}} \log\left(1 - \frac{1}{l f(l)}\right)\right) > \exp\left(\sum_{l \text{ prime}} \frac{1}{l f(l)}\right) \tag{4.22}$$

The exponent converges by Romanoff Theorem 4.13 which in turn implies that its exponential converges to a non-zero value. ■

4.2 Modern proof by Kim-Murty

The article [KR20] (and its corrigendum [KM22]) present a new proof of Theorem 4.20 only for the case of $K = \mathbb{F}_q(x)$. The paper's abstract claims that their proof doesn't depend on the Riemann Hypothesis over Function Fields, unlike the original [Bil37]. We believe that there is a small flaw in their argument that invalidates this claim.

We first give an exposition of the strategy followed by this paper. After this, we describe the technical error in their argument and how a reduced Riemann Hypothesis patches it. This proof with a reduced R.H. was already observed by Davenport [Dav39] without details.

To this day, the author of the present document has not found a way to patch this proof without blackboxing the Riemann Hypothesis in Function Fields.

4.2.1 Proof Strategy

The paper aims to prove the conjecture by proving a series of bounds of polynomial character sums, following the next Lemma.

Lemma 4.23 (Sufficient condition)**.** Given $a(x) \in \mathbb{F}_q[x]$ monic. If there is some $c > \log_q(2)$ such that for all $n \in \mathbb{Z}_{\geq 1}$ and all non-trivial characters $\chi : \mathbb{F}_{q^n} \to \mathbb{C}$, we have

$$\left| \sum_{\theta \in \mathbb{F}_{q^n}} \chi(a(\theta)) \right| = o\left(\frac{q^{n\left(1-\frac{c}{\log n}\right)}}{\log n} \right)$$

then, Artin's Conjecture holds for $a(x)$.

Remark 4.24. Here is where the necessary condition is needed. If a was a d-th power for some $d \mid q^i - 1$ for some i, there would be a character in $\mathbb{F}_{q^i}$ for which the character sum was trivial, hence it would sum to q^n.

The condition in Rosen and in Kim Murty differs!!

In the rest of the section, we aim to give a sketch of the proof of Lemma 4.23.

Definition 4.25 (Sifting function)**.** Given a cyclic group G, define

$$S_G : G \to \mathbb{C}$$
$$g \mapsto \frac{\varphi(m)}{m}\left(1 + \sum_{\substack{d|m \\ d>1}} \frac{\mu(d)}{\varphi(d)} \sum_{\operatorname{ord}\chi = d} \chi(g)\right)$$

where φ is Euler's totient function and where the last sum runs over all group characters of order exactly d.

Remark 4.26. Note that the first term comes from the trivial character and $d = 1$. We only separate the first term as a presentation convenience, as it will be the asymptotically significant term.

Lemma 4.27. With the definition above, we have

$$S_G(g) = \begin{cases} 1, & g \text{ is a primitive root of } G \\ 0, & \text{otherwise} \end{cases}$$

Proof. To-do ■

Definition 4.28. Given an $a(x) \in \mathbb{F}_q[x]$ monic, define $W_a : \mathbb{F}_q[x]^{\text{irr}} \to \mathbb{Z}$,

$$W_a(v) = \begin{cases} \deg v, & a \text{ is a primitive root modulo } v \\ 0, & \text{otherwise} \end{cases}$$

We aim to count irreducible v where a is a primitive root modulo v, but we will find it easier to count them if we weight them with a multiplicity $\deg v$. This is analogous to the role that the Von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \\ 0, & \text{otherwise} \end{cases}$$

takes in the original proof of the prime number theorem, by Hadamart and de la Vallée Poussin.

Lemma 4.29. For all $n \in \mathbb{Z}_{>0}$, the following equality holds.

$$\sum_{\substack{v \in \mathbb{F}_q[x]^{\text{irr}} \\ \deg v | n}} W_a(v) = \sum_{\theta \in \mathbb{F}_{q^n}^*} S_{\mathbb{F}_{q^n}^*}(a(\theta))$$

Proof. To-do ■

Lemma 4.30. The set of upper bounds described in Lemma 4.23 imply that $\sum_{\theta\in\mathbb{F}_{q^n}^*} S_{\mathbb{F}_{q^n}^*}(a(\theta))$ diverges as $n\to\infty$.

Proof. Use Definition 4.25 to fully expand the sum. Then, applying a triangular inequality and using the set of upper bounds in Lemma 4.23, the leading term is absolutely asymptotically bigger than all the other combined. Hence, the sum diverges. Probably make more clear

$$\begin{aligned}\sum_{\theta\in\mathbb{F}_q^*} S_{\mathbb{F}_{q^n}^*}(a(\theta)) &= \sum_{\theta\in\mathbb{F}_q^*} \frac{\varphi(q^n-1)}{q^n-1}\left(1+\sum_{\substack{d\mid q^n-1\\ d>1}}\frac{\mu(d)}{\varphi(d)}\sum_{\substack{\chi:\mathbb{F}_{q^n}^*\to\mathbb{C}\\ \operatorname{ord}\chi=d}}\chi\left(a(\theta)\right)\right) = \\ &= \varphi(q^n-1)+\sum_{\substack{d\mid q^n-1\\ d>1}}\frac{\mu(d)}{\varphi(d)}\sum_{\substack{\chi:\mathbb{F}_{q^n}^*\to\mathbb{C}\\ \operatorname{ord}\chi=d}}\sum_{\theta\in\mathbb{F}_q^*}\chi\left(a(\theta)\right)\end{aligned} \tag{4.23}$$

Using the upper bounds of Lemma 4.23, we find that the first term is the asymptotically relevant one.

$$\begin{aligned}\left|\sum_{\substack{d\mid q^n-1\\ d>1}}\frac{\mu(d)}{\varphi(d)}\sum_{\substack{\chi:\mathbb{F}_{q^n}^*\to\mathbb{C}\\ \operatorname{ord}\chi=d}}\sum_{\theta\in\mathbb{F}_q^*}\chi\left(a(\theta)\right)\right| &\le \sum_{\substack{d\mid q^n-1\\ d>1\\ d\text{ sq-free}}}\frac{1}{\varphi(d)}\sum_{\substack{\chi:\mathbb{F}_{q^n}^*\to\mathbb{C}\\ \operatorname{ord}\chi=d}}\left|\sum_{\theta\in\mathbb{F}_q^*}\chi\left(a(\theta)\right)\right| = \\ &\underset{\text{Lemma 4.23}}{=} o\left(2^{w(q^n-1)}\frac{q^{n\left(1-\frac{c}{\log n}\right)}}{\log n}\right) = o\left(\varphi(q^n-1)\right)\end{aligned} \tag{4.24}$$

On the last step we have used that $w(N)<\frac{\log N}{\log\log N}$ and $\varphi(q^n-1)>\frac{q^n}{\log\log q^n}$ ■

4.2.2 Bound of the Polynomial Character Sums

Remark 4.31. Bounding for each n independently is not enough, as we need the B to be independent on n. That's why proving the case $n=1$ and then base changing from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$ doesn't work.

Here is where the paper makes its initial mistake, which is, a priori, fixed in the corrigendum. Their method only works for characters of $\mathbb{F}_{q^n}$ that are lifts of characters of $\mathbb{F}_q$. By "lifts" we mean that $\chi:\mathbb{F}_{q^n}\to\mathbb{C}$ decomposes as $\chi=\chi'\circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}:\mathbb{F}_{q^n}\to\mathbb{F}_q\to\mathbb{C}$, where $N_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ is the norm of the field extension and χ' is a character of $\mathbb{F}_q$

Apart from this error, which is supposedly fixed in the corrigendum, I have found another flaw that I think invalidates the proof. The details are described in the next section.

4.2.3 Potential error in the corrigendum

These are the details of a potential error in the corrigendum that would invalidate the proof of Artin's conjecture.

The second page of the corrigendum [KM22] introduces the following L-function.

Definition 4.32. Given a fix $a \in \mathbb{F}_q[x]$ monic of degree K and an arbitrary character of the algebraic closure $\chi : \overline{\mathbb{F}_q} \to \mathbb{C}$, define

$$L(s,\chi) \coloneqq \exp\left(\sum_{n\geq 1} N_n(\chi)\frac{q^{-sn}}{n}\right)$$

with

$$N_n(\chi) \coloneqq \sum_{\theta\in\mathbb{F}_{q^n}} \chi(a(\theta))$$

The next paragraph states that this L-function is another form of the L-function given in the original paper [KR20]. I believe the error is in this equality of L-functions.

The L-function of the original paper is defined as follows.

Definition 4.33. Given an r-tuple of characters $\chi_i' : \mathbb{F}_q \to \mathbb{C}$ and an r-tuple of monic irreducible polynomials $f_i \in \mathbb{F}_q[x]$, define

$$\begin{aligned} \widehat{\chi} : \mathbb{F}_q[x] &\to \mathbb{C} \\ g &\mapsto \prod_{i=1}^{r} \chi_i'(\ (f_i, g)\ ) \end{aligned}$$

where (f_i, g) indicates the resultant. Then, define

$$\mathcal{L}'(s,\widehat{\chi}) = \sum_{\substack{g\in\mathbb{F}_q[x]\\ \text{monic}}} \frac{\widehat{\chi}(g)}{(q^{\deg g})^s}$$

To equalize Definition 4.33 with Definition 4.32, I understand that the natural choice is to take $r =$ #irreducible factors of a, $(f_1, \ldots, f_r)$ the irreducible components of a.

Setting the $\chi_i' = \chi$ doesn't work as, to start, the χ_i should be characters of $\mathbb{F}_q$ and χ is a character of $\overline{\mathbb{F}}_q$. Even if we stretch the Definition 4.33 to include characters of $\overline{F}_q$, this choice of χ_i will still not work, as I will show in a moment. For now, let's just set them all equal to each other $\chi_i' = \chi'$, letting

χ' be an arbitrary character of $\mathbb{F}_q$ (possibly a character of $\overline{F}_q$, if we need to stretch the definition).

Note that we have $\widehat{\chi}(g) = \chi'(\ (a,g)\ )$ as $a = \prod f_i$. We have split a into irreducible components just to match the conditions of the Definition 4.33.

Question 4.34. Is $\mathcal{L} = \mathcal{L}'$?

Taking the logarithm of the Euler product of second L-function, we get

$$\begin{aligned}
\log \mathcal{L}'(s,\widehat{\chi}) &= \sum_{\substack{v\in\mathbb{F}_q[x]\\ \text{monic irreducible}}} -\log\left(1-\frac{\widehat{\chi}(v)}{q^{\deg vs}}\right)\\
&= \sum_{\substack{v\in\mathbb{F}_q[x]\\ \text{monic irreducible}}} \sum_{k\geq 1}\frac{1}{k}\cdot\left(\frac{\widehat{\chi}(v)}{q^{\deg vs}}\right)^k\\
&= \sum_{m\geq 1}\sum_{\substack{v\in\mathbb{F}_q[x]\\ \text{monic irreducible}\\ \deg v=m}}\sum_{k\geq 1}\frac{1}{k}\cdot\widehat{\chi}(v)^k q^{-mk\cdot s}\\
&= \sum_{n\geq 1}\left(\sum_{m|n}\sum_{\substack{v\in\mathbb{F}_q[x]\\ \text{monic irreducible}\\ \deg v=m}} m\cdot\widehat{\chi}(v)^{n/m}\right)\frac{q^{-sn}}{n}
\end{aligned}$$

where, in the last equality, we have set $n = mk$

For this to be equal to Definition 4.32, we would need the equality of all the coefficients. Namely, $\forall n \geq 1$

$$N_n(\chi) = \sum_{\theta\in\mathbb{F}_{q^n}}\chi(a(\theta)) \stackrel{?}{=} \sum_{m|n}\sum_{\substack{v\in\mathbb{F}_q[x]\\ \text{monic irreducible}\\ \deg v=m}} m\cdot\chi'(\ (a,v)\ )^{n/m}$$

If $\chi = \chi'\circ N_{\mathbb{F}_q^n/\mathbb{F}_q}$, this is true. For any $v\in\mathbb{F}_q[x]$ irreducible polynomial of degree m, let $\theta_1,\dots,\theta_m$ be its roots. Now

$$\begin{aligned}
\chi(a(\theta_1))+\cdots+\chi(a(\theta_m)) &= \chi'(N(a(\theta_1)))+\cdots+\chi'(N(a(\theta_m)))\\
&= \sum_i \chi'\left(\left(\prod_j a(\theta_j)\right)^{n/m}\right)\\
&= m\cdot\chi'\left(\prod_i a(\theta_i)\right)^{n/m}\\
&= m\cdot\chi'(\ (a,v)\ )^{n/m}
\end{aligned}$$

Adding over all conjugation classes, we get the desired identity.

But, given an arbitrary $\chi : \overline{F}_q \to \mathbb{C}$ which is not the lift of any character on the base field, there doesn't seem to be a natural choice of χ' that makes the identity true.

4.2.4 Flaw in the proof of Artin's conjecture

The equality of the two L-functions is not merely a presentation problem. It is logically used in the proof of Artin's conjecture.

Davenport [Dav39] proves that the L-function on Definition 4.33 is a polynomial. Only in the case $\chi = \chi' \circ N$ he uses this to find an equality of the character sum with a sum over the zeroes of the L-function. Because there are only finitely many characters on the base field, one can take the $B = \max |s_i|$ of all the finitely many zeroes (as Davenport has seen $\mathcal{L}$ is a polynomial) of all the finitely many L-series. This will be a uniform bound on all the infinitely many lifts and $B < 1$ by the result analogous to the classical argument by Hadamart and de la Vallée Poussin.

For $\chi \neq \chi' \circ N$, the character sum that one needs to bound doesn't even come up as a coefficient in the L-series of Definition 4.33. It only comes up as a coefficient in the Definition 4.32, which, a priori, is not a polynomial nor does it follow an equality similar to the one found by Davenport.

4.2.5 Conditional fix

The character sum you want to bound would also come up in an L-series like the one in Definition 4.33 via base change from $\mathbb{F}_q$ to $\mathbb{F}_{q'}$ with $q' = q^n$. But in this case, the zeroes of this L series are not linked in any way to the family of L-series considered when defining B. Hence, the zeros of this L-function are not necessarily $\leq B$. So one would have to take $B = \sup |s_i|$ which, a priori, can be 1.

This would be solved if you knew that the zeroes of all the L series are in the region $Re(s) < 1 - \epsilon$ for some ϵ independent of n and χ. This looks similar to Theorem 4 in [KR20] but the bound given in the paper isn't enough. Under base change, it seems to be

$$1 - \frac{c}{(K-1)\log(q^n)} = 1 - \frac{c}{n(K-1)\log q}$$

which is not enough as, when $n \to \infty$ it goes to 1.

5. Number Field Setting

This chapter focuses on Artin's conjecture over Number Fields, were, even after a century, it remains an open problem. In Section 5.1, we give an exposition of Hooley's conditional proof of Artin's Conjecture [Hoo67] over $\mathbb{Q}$. Following that, in Section 5.2, we propose a new Conjecture 5.18. This self-contained conjecture would reduce the strength of the Riemann Hypothesis assumed in Hooley's result. There is strong numerical evidence that the conjecture holds, as shown in Figure 5.1.

5.1 Hooley's conditional result

In his 1967 paper [Hoo67], Hooley published a conditional proof of Artin's conjecture 3.32 under the assumption that the Generalized Riemann Hypothesis holds for an explicit family of Kummer fields. This section gives a sketch of this proof.

One of the relevant takeaways of the following proof is that it reduces Artin's Conjecture to the problem of counting primes over certain Kummer Fields. The assumption of the Riemann Hypothesis is used to give an extremely fine estimate of the prime counting function.

5.1.1 Preparation

For this chapter, we return to the notations of Definition 3.33. To match the notations used in Hooley's paper [Hoo67], we introduce the following functions.

Definition 5.1 (Prime counting functions in [Hoo67])**.**

1. $R_a(q,p) = \begin{cases} 1 & q \text{ follows Lemma 3.19} \\ 0 & \text{otherwise} \end{cases}$
2. $N_a(x) = \#\{p < x \mid a \text{ is a primitive root } \mod p\}$
3. $N_a(x,\xi) = \#\{p < x \mid \nexists\, q \text{ following Lemma 3.19 in the range } q < \xi\}$
4. $M_a(x,\xi_1,\xi_2) = \#\{p < x \mid \exists\, q \text{ following Lemma 3.19 in the range } \xi_1 < q \leq \xi\}$
5. $P_a(x,k) = \#\{p < x \mid \forall q | k, q \text{ follows Lemma 3.19}\}$

Lemma 5.2 (Basic observations of the newly defined functions)**.**

1. $N_a(x) = N_a(x, x-1)$
2. $N_a(x) \leq N_a(x, \xi)$
3. $N_a(x) \geq N_a(x, \xi) - M_a(x, \xi, x-1)$
4. $M_a(x, \xi_1, \xi_2) \leq \sum_{\xi_1 < q \leq \xi_2} P_a(x, q)$

Lemma 5.3. $N_a(x, \xi) = \sum_{l'} \mu(l') P_a(x, l')$, where the sum is over all l' square free with factors $\leq \xi$. Note that

$$l' \leq \prod_{q \leq \xi} q = e^{\sum_{q \leq \xi} \log q} \leq e^{2\xi}$$

where in the last inequality we have used the prime number theorem.

Lemma 5.4. Let $\xi_1 = \frac{1}{6}\log x, \xi_2 = x^{1/2}\log^{-2} x, \xi_3 = x^{1/2}\log x$. From the previous observations, we get

$$\begin{aligned} N_a(x) = N_a(x, \xi_1) + O(M_a(x, \xi_1, \xi_2)) + \\ + O(M_a(x, \xi_2, \xi_3)) + O(M_a(x, \xi_3, x-1)) \end{aligned} \tag{5.1}$$

Hooley proves that the first is the leading term, being $\sim A(a)\frac{x}{\log x}$ for an explicit constant $A(a)$. Moreover, he proves that, the other 3 terms will be asymptotically smaller, upper bounded by $O\left(\frac{x}{\log^2 x}\right)$. This concludes that $N_a(x) \sim A(a)\frac{x}{\log x}$, which is precisely Artin's conjecture. The choice of ξ_i is taken carefully to fulfill the estimates.

Remark 5.5. The bounds of terms 3 and 4 use elementary techniques. For terms 1 and 2, the R.H. is needed. As we will detail in the following section, the estimation of term 1 only needs the $2/3$-zero free region but the upper bounding of term 2 will need the full $1/2$ R.H.

The conjecture that we propose gives an equally good bound for term 2 using less strength of the R.H. We do so by improving the bound on term 4, which makes it possible to choose a lower ξ_3, which at its turn makes it possible to choose lower ξ_2 without disrupting the bound of term 3. Having a lower ξ_2 gives the possibility of conserving the bound of the second term but using less strength of the R.H.

The estimation of the first term still needs the $2/3$ R.H., so the best this possible improvement can hope to do is lower the conditions, but not give a condition-less proof.

5.1.2 Bounds on the 3rd and 4th term

Lemma 5.6 (Bound of the 4th term)**.** Let $\xi_3 = x^{1/2}\log x$, then

$$M_a(x,\xi_3,x-1) = O\left(\frac{x}{\log^2 x}\right)$$

Proof. If q follows Lemma 3.19, in particular $a^{\frac{p-1}{q}} = 1 \mod p$. Hence, if there is a $q > \xi_3$ that follows the Lemma, there will be an $m < \frac{x}{\xi_3}$ such that $p|a^m - 1$. All the primes counted on $M_a(x,\xi_3,x-1)$ need to be divisors of

$$S_a(x/\xi_3) \coloneqq \prod_{m<x/\xi_3}(a^m-1)$$

Hence, $2^{M_a(x,\xi_3,x-1)} < S_a(x/\xi_3)$ which implies $M_a(x,\xi_3,x-1) < \log S_a(x/\xi_3) < \log a \sum_{m<x/\xi_3} m = O\left((x/\xi_3)^2\right) = O\left(\frac{x}{\log^2 x}\right)$. ∎

Remark 5.7. One is forced to choose $\xi_3 = x^{1/2}\log x$ for the last equality to be true. Yet, in this document we conjecture a refined upper bound for the number of primes diving $S_a(n) = \prod_{m<n}(a^m-1)$. Using our conjecture, one will be able to choose a lower ξ_3.

Lemma 5.8 (Bound of the 3rd term)**.** Let $\xi_2 = x^{1/2}\log^{-2} x$ and $\xi_3 = x^{1/2}\log x$. Then $M_a(x,\xi_2,\xi_3) = O\left(\frac{x}{\log^2 x}\right)$.

Proof. By Lemma 5.2, we may express $M_a(x,\xi_2,\xi_3) \le \sum_{\xi_2<q\le\xi_3} P_a(x,q)$.

Now, if q follows Lemma 3.19, then in particular $p \equiv 1 \mod q$. By Brun's method, which is an inequality related to Dirichlet's Theorem, we have

$$P_a(x,q) \le \sum_{\substack{p\le x \\ p\equiv 1 \mod q}} 1 \le \frac{A_1 x}{(q-1)\log(x/q)}$$

From this we obtain the bound

$$\begin{aligned} M_a(x,\xi_2,\xi_3) &= O\left(\frac{x}{\log x}\sum_{\xi_2<q\le\xi_3}\frac{1}{q}\right) = \\ &= O\left(\frac{x}{\log^2 x}\left(\log\frac{\xi_3}{\xi_2} + O(1)\right)\right) = O\left(\frac{x\log\log x}{\log^2 x}\right) \end{aligned} \tag{5.2}$$

∎

Remark 5.9. This lemma forces to choose the polynomial degree of ξ_2 to be the same as ξ_3, a priori $1/2$. Yet a key takeaway from this lemma is that the bound only depends on the ratio ξ_3/ξ_2. If we manage to lower ξ_3, we can automatically lower ξ_2 without disturbing this bound.

5.1.3 Reduction to counting primes

The point of view found by Artin's observation gives a clearer line of attack to the conjecture. This is exemplified by the following lemmas, linking the prime counting function to the sums we are interested in computing.

Definition 5.10 (Prime counting function)**.**

$$\pi(x,k) := \#\{\mathfrak{p} \text{ prime ideal of } L_k | N\mathfrak{p} \leq x\}$$

Lemma 5.11.

$$n(k)P_a(x,k) = \pi(x,k) + O(n(k)w(k)) + O(n(k)x^{1/2}) \tag{5.3}$$

Proof. This is an implication of elementary ramification theory applied to L_k, check the article [Hoo67] for the details. Maybe add? ■

5.1.4 Prime counting theorem

By Lemma 5.11, an estimate of $\pi(x,k)$ will give an estimate of $P_a(x,k)$ and which in turn will give an estimate of the first and second term in Equation 5.1, by Lemmas 5.2 and 5.3. The final part of Hooley's article deduces a good enough prime counting theorem.

Theorem 5.12. Assuming the GRH for ζ_{L_k}, we have the estimate

$$\pi(x,k) = \frac{x}{\log x} + O(n(k)x^{1/2}\log kx) \tag{5.4}$$

Proof. Hooley starts from the classical idea that π can be expressed in terms of the zeroes of ζ_{L_k}. He deduces a theorem about the vertical distribution of zeroes and, together with the assumption that the zeroes are in the $1/2$ line, he is able to deduce the desired bound. ■

Remark 5.13. If you follow Hooley's proof only assuming the zero-free region $Re(s) > f$, you get the estimate

$$\pi(x,k) = \frac{x}{\log x} + O(n(k)x^{f}\log kx) \tag{5.5}$$

From the rest of the document, f will note the value up to which the RH is assumed.

5.1.5 Bounds for the 1st and 2nd term

By Lemma 5.11, one gets an estimate of P_a and unrolling Lemmas 5.2 and 5.3 one gets estimates of the first and second term in Equation 5.1. They are explained in the following lemmas.

Lemma 5.14 (Estimation of the 1st term)**.**

$$\begin{aligned} N_a(x,\xi_1) &= \sum_{l'} \mu(l') \left(\frac{x}{\log x \cdot n(l')} + O(x^f \log x) \right) = \\ &\underset{l'<e^{2\xi_1} \text{ by Prop. 5.3}}{=} \frac{x}{\log x} \sum_{l'} \frac{\mu(l')}{n(l')} + O\left(\sum_{l<e^{2\xi_1}} x^f \log x \right) = \\ &= A(a)\frac{x}{\log x} + O(e^{2\xi_1} x^f \log x) = \\ &= A(a)\frac{x}{\log x} + O(x^{f+1/3} \log x) \end{aligned} \tag{5.6}$$

Remark 5.15. Very significantly, note that for the extra term to be irrelevant, we only need f to be $f < 2/3$. For this, it is sufficient to assume an $R(s) \geq 2/3$ zero-free region.

Lemma 5.16 (Bound of the 2nd term)**.**

$$\begin{aligned} M_a(x,\xi_2,\xi_3) &\leq \sum_{\xi_1<q\leq\xi_2} \left(\frac{x}{\log x \cdot q(q-1)} + O(x^f \log x) \right) = \\ &= O\left(\frac{x}{\log x} \sum_{q>\xi_2} \frac{1}{q^2} \right) + O\left(x^f \log x \sum_{q\leq\xi_2} 1 \right) = \\ &= O\left(\frac{x}{\xi_1 \log x} \right) + O\left(\frac{x^f \xi_2 \log x}{\log \xi_2} \right) = O\left(\frac{x}{\log^2 x} \right) \end{aligned} \tag{5.7}$$

Remark 5.17. Note that in the last equality we did need $f = 1/2$ because $\xi_2 = x^{1/2} \log^{-2} x$. If we manage to lower the polynomial degree of ξ_2, we would be able to conserve the bound using a higher f, hence reducing the conditions in Hooley's proof.

5.2 Proposed improvement

We propose the following self-contained conjecture.

Conjecture 5.18. Let $S_a(n) := \prod_{m<n} (a^m - 1)$. Let $w(N) = \#\{\text{distinct primes } p|N\}$. Is it true that $w(S_a(n)) = O(n \cdot \text{poly-log})$?

We state that this would reduce the conditions on Hooley's conditional proof from the full R. H. to

an $R(s) \geq 2/3$ zero free region. The weaker conjecture $w(S_a(n)) = O(n^{2-\epsilon} \cdot \text{poly-log})$ for $\epsilon > 0$ would already improve the conditions to an $R(s) \geq 1/2 + \epsilon/3$ zero-free region.

The conjecture can be reformulated as follows. Note that it is asking a similar question to the original AC but instead of asking for primes with high $\text{ord}_p(a) = p-1$ it asks for primes with low $\text{ord}_p(a)$.

Conjecture 5.19. Let $P(n) = \#\{p \text{ prime} \mid \text{ord}_p(a) < n\}$, is $P(n) = O(n \cdot \text{ poly-log})$?

Seems like the conjecture is as hard as Artin's conjecture

Remark 5.20. For the application on AC, the value of a can be asked to be a non-square. Yet, numerical evidence in Figure 5.1 seems to imply that the conjecture is true regardless. This doesn't contradict the necessary condition in AC as a being a non-square is still used in Artin's observation.

Remark 5.21. The polylogarithmic part will take no paper in the application to AC, can be taken as large as one wants.

Remark 5.22. Note that, following the factorization $a^m - 1 = \prod_{d|m} \Phi_d(a)$, the conjecture is very related to the values of $w(\Phi_d(a))$, where Φ_d is the d-th cyclotomic polynomial. There seems to be a conjecture by Erdós [MS19] on $P(\Phi(a))$, the largest prime divisor which has a very similar flavor.

5.2.1 Upper bound $w(S_a(n)) = O(n^2)$

It is not hard to prove $w(S_a(n)) = O(n^2)$. For example, $2^{w(S_a(n))} < S_a(n)$, from which the desired bound follows. This bound can be improved by logarithmic factors in a number of ways. For instance using the well-known bound $w(N) = O\left(\frac{\log N}{\log \log N}\right)$, which can be proven by looking at $N = \prod_{p<n} p$ the primorials.

5.2.2 Lower bound $w(S_a(n)) = \Omega(n)$

A trivial application of Zsigmondy's theorem[Zsi92] shows $w(S_a(n)) = \Omega(n)$.

5.2.3 Numerical evidence

We believe that the strong conjecture is true. Numerical evidence is shown in Figure 5.1, for $a = 2$.

The limitation of these numerical computations is the number of primes can be saved in a computer in practice. The current program, found in the Appendix, checks for primes up to $L = 10^8$ through an Eratosthenes Sieve. Yet $S_2(n)$ grows very quickly so, a priori, it could start having prime factors

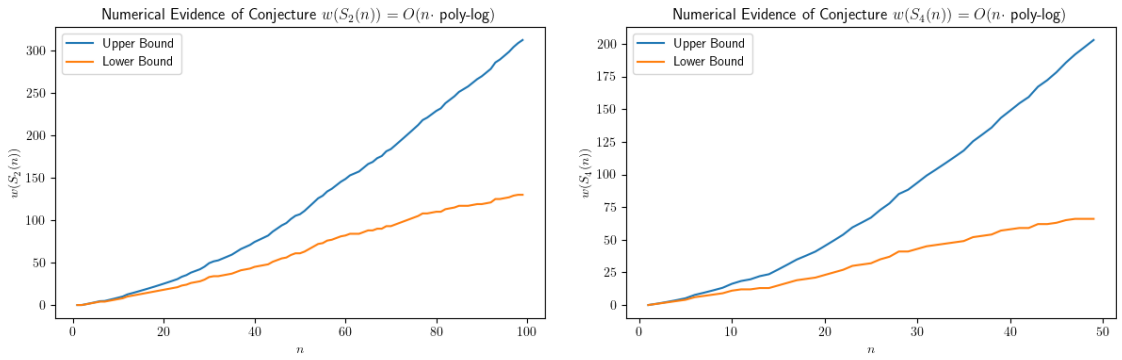


Figure 5.1: Numerical Evidence of Conjecture 5.18. The lower bound w' is the number of distinct primes in $S_2(n)$ in the range $< 10^8$. The upper bound has an extra correction term of $\frac{n(n-1)}{2}\log_{10^8}(2)$ which over counts the number of primes that $S_2(n)$ can have on the range $\geq 10^8$.

larger than our range. We can only give an exact value of $w(S_a(n))$ for n relatively small (~ 10). For higher values, we compute a lower and higher bound for $w(S_2(n))$.

The lower bound $w'(S_2(n))$ is just the number of distinct primes dividing $S_2(n)$ that are in the range $p < L$ which we compute by counting. The upper bound is $w' + \frac{n(n-1)}{2}\log_L(2)$. This is an upper bound because any extra prime of $S_2(n)$ not in our range is at least $\geq L$, hence there can only be, at most, $\log_L(S_2(n)) \leq \log_L(2^{\sum_{m<n} m}) = \frac{n(n-1)}{2}\log_L(2)$.

5.2.4 Improvement on Artin's conjecture

Conjecture 5.18 gives a finer upper bound for the 4th term in Equation 5.1. This will let us choose a smaller ξ_3'. For this section, we assume Conjecture 5.18 and, to simplify the computations, we let the polylogarithmic part be trivial $L(n) = 1$. Hence, suppose $w(S_a(n)) \leq C_a \cdot n$

Lemma 5.23 (New Bound of the 4th Term)**.** Let $\xi_3' = \log^2 x$, then

$$M_a(x, \xi_3', x-1) = O\left(\frac{x}{\log^2 x}\right)$$

Proof. As seen in the original proof $M_a(x, \xi_3, x-1) \leq w(S_a(x/\xi_3))$. Now Hooley uses the trivial bound $w(S_a(n)) = O(n^2)$ and concludes that $M_a(x, \xi_3, x-1) = O\left((x^2/\xi_3^2)\right) = O\left(\frac{x}{\log^2 x}\right)$. In the new case, $M_a(x, \xi_3', x-1) = O(w(S_a(x/\xi_3'))) = O(x/\xi_3') = O\left(\frac{x}{\log^2 x}\right)$. ∎

Now let $\xi_2' = \log^{-3} x$, which makes the ratio $\xi_3'/\xi_2' = \log^5 x$. Lemma 5.8 still holds with these new brackets. But now, having $\xi_2' = \log^{-3} x$ makes the bound of the 2 term condition-free. This can be seen in the last equality of Lemma 5.16.

Hence, the only condition that remains is the $R(s) \geq 2/3$ zero-free region used for estimation the

first term.

6. Common Factor

Write almost from scratch again

6.1 Lenstra's paper

6.1.1 Artin's observation revisited

Bibliography

[Zsi92] K. Zsigmondy. "Zur Theorie der Potenzreste". In: *Monatshefte für Mathematik und Physik* 3 (1892), pp. 265–284. doi: https://doi.org/10.1007/BF01692444. url: https://link.springer.com/article/10.1007/BF01692444#citeas.

[Che26] N. Chebotarev. "Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören". In: *Mathematische Annalen* 95 (1926).

[Bil37] Herbert Bilharz. "Primdivisoren mit vorgegebener Primitivwurzel". In: *Mathematische Annalen* 114.1 (1937). Cited by: 20, pp. 476–492. doi: 10.1007/BF01594189.

[Dav39] H Davenport. "On character sums in finite fields". In: *Acta Math.* 71 (1939), pp. 99–121.

[Wei40] André Weil. "Sur les fonctions algebriques á corps de constantes fini." In: *C. R. Acad. Sci. Paris* 210 (1940), pp. 592–594.

[LT65] Artin Emil Serge Lang and John Torrence Tate. *The Collected Papers of Emil Artin*. Springer-Verlang, 1965.

[Hoo67] Christopher Hooley. "On Artin's conjecture." In: *Journal für die reine und angewandte Mathematik* 225 (1967), pp. 209–220. url: http://eudml.org/doc/150785.

[Wei72] Peter J. Weinberger. "A counterexample to an analogue of Artin's conjecture". In: 1972.

[CW75] George Cooke and Peter J. Weinberger. "On the construction of division chains in algebraic number rings,with applications to SL2". In: *Communications in Algebra* 3.6 (1975), pp. 481–524. doi: 10.1080/00927877508822057. eprint: https://doi.org/10.1080/00927877508822057. url: https://doi.org/10.1080/00927877508822057.

[W77] Lenstra H. W. "On Artin's conjecture and Euclid's algorithm in global fields". In: *Inventiones mathematicae* (1977). doi: 10.1007/BF01389788.

[Gup84] Murty M. R. Gupta R. "A remark on Artin's conjecture". In: *Inventiones mathematicae* 78 (1984).

[GWC86] C.F. Gauss, W.C. Waterhouse, and A.A. Clarke. *Disquisitiones Arithmeticae*. Springer-Verlag, 1986. isbn: 9783540962540. url: https://books.google.com/books?id=Y-49PgAACAAJ.

[Hea86] D. R. Heath-Brown. "Artin's conjecture for primitive roots". In: *The Quarterly Journal of Mathematics* 37 (1986).

[Mur88] M. Ram Murty. "Artin's conjecture for primitive roots". In: *The Mathematical Intelligencer* 10 (1988).

[Lan94] S. Lang. *Algebraic Number Theory*. Graduate Texts in Mathematics. Springer, 1994. isbn: 9780387942254. url: https://books.google.es/books?id=u5eGtA0YalgC.

[Neu99] Jürgen Neukirch. *Algebraic number theory*. Vol. Grundlehren der mathematischen Wissenschaften. Berlin: Springer, 1999.

[Coj02] Alina Carmen Cojocaru. "Cyclicity of Elliptic Curves Modulo p". PhD thesis. Queen's University, 2002.

[Ros02] M. Rosen. *Number Theory in Function Fields*. Graduate Texts in Mathematics. Springer New York, 2002. isbn: 9780387953359. url: https://books.google.com/books?id=vDpa%5C_C5DIbkC.

[Ser03] Jean-Pierre Serre. "Résumé des cours de 1977–1978". In: 2003.

[Ste03] Peter Stevenhagen. "The correction factor in Artin's primitive root conjecture". en. In: *Journal de théorie des nombres de Bordeaux* 15.1 (2003), pp. 383–391. url: http://www.numdam.org/item/JTNB_2003__15_1_383_0/.

[Lan05] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2005. isbn: 9780387953854. url: https://books.google.es/books?id=Fge-BwqhqIYC.

[MS19] M. Ram Murty and François Séguin. "Prime divisors of sparse values of cyclotomic polynomials and Wieferich primes". In: *Journal of Number Theory* 201 (2019), pp. 1–22. issn: 0022-314X. doi: https://doi.org/10.1016/j.jnt.2019.02.016. url: https://www.sciencedirect.com/science/article/pii/S0022314X19300927.

[KR20] Seoyoung Kim and M. Ram Murty. "Artin's primitive root conjecture for function fields revisited". In: *Finite Fields and Their Applications* 67 (2020), p. 101713. issn: 1071-5797. doi: https://doi.org/10.1016/j.ffa.2020.101713. url: https://www.sciencedirect.com/science/article/pii/S1071579720300824.

[KM22] Seoyoung Kim and M. Ram Murty. "Corrigendum to "Artin's primitive root conjecture for function fields revisited" [Finite Fields Appl. 67 (2020) 101713]". In: *Finite Fields and Their Applications* 78 (2022), p. 101963. issn: 1071-5797. doi: https://doi.org/10.1016/j.ffa.2021.101963. url: https://www.sciencedirect.com/science/article/pii/S107157972100157X.

[Mil22] J. S. Milne. *Fields and Galois Theory*. Ann Arbor, MI: Kea Books, 2022.

[Leh90] Derrick Lehmer. *Lehmer Papers, Approximately 1926-1990*. University Archives, The Bancroft Library, University of California, Berkeley, 1926-1990.