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For Part III Essay. Topic 97. Modular forms and representation theory

Javier López-Contreras

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1 Abstract

The goal of the essay will be to give an introduction to the spectral adelic generalization of classic modular forms. To justify this generalization, we aim to give 2 applications of the new framework. First, we would like to describe how this setting completes the theory of Hecke operators. Second, we will see a reinterpretation of the theory of newforms.

The primary references for this essay (so far) have been [Gel75] and [Del73]. We have also consulted [KL06] and [Bum97] and a set of online notes on Atkin-Lehmer theory by Andreea Mocanu [Moc].

2 Table of Contents

1. Introduction	~ 1 page
• Historical introduction	
• Relation to the Langlands' program	
2. Modular Forms Revisited	~ 6 pages
2.1 Functions on lattices	~ 1 page
• [Del73, p.8-20].	
2.2 The decomposition of $L^2(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))$	~ 2 pages
• Discrete and continuous spectrum	
• Casimir operator and K -action	
• Wave forms	
• Multiplicities are finite	
• [Gel75, Chapter 2] and [Del73, Section 2.1]	
2.3 Hecke Theory for $\Gamma_0(N)$	~ 2 pages
• Detailed computation of the proof of $T(p)$ being self-adjoint, to see what breaks when $(p, N) \neq 1$.	
• [KL06, p.20, p.39] and [Gel75, p.17, p.20]	
2.4 Atkin-Lehner Theory of Newforms	~ 1 page

<ul style="list-style-type: none"> • Show the full computation of $\Delta(z)$ and $\Delta(2z)$ have the same eigenvalues at $p \neq 2$ to motivate theory of new-forms. • [Moc] and [Gel75, Section 1.D] 	
3. Automorphic Forms of $GL_2(\mathbb{A})$	~ 8 pages
3.1 Functions on Adelic Lattices	~ 1 page
<ul style="list-style-type: none"> • Strong Approximation Theorem for $GL_2(\mathbb{A})$ [KL06, p. 6.3] • Relation between classical lattices and adelic lattices [Del73, Section 1.2 and 2] 	
3.2 The Spectrum of $GL_2(\mathbb{A})$	~ 3 pages
<ul style="list-style-type: none"> • [Del73, Section 2.2-2.4] and [Gel75, Section 3.A] 	
3.3 Jacquet-Langlands' theory	~ 4 pages
<ul style="list-style-type: none"> • Kirillov and Whittaker models [Del73, page 24-30] • [Gel75, Section 6] 	
4. Applications	~ 3 pages
4.1 Hecke Operators Revisited	~ 2 pages
<ul style="list-style-type: none"> • Detailed computations for $\Gamma_0(N)$, which is now $K_0(N)$. • [Gel75, Section 5, page 86] 	
4.2 Newforms revisited	~ 1 page
<ul style="list-style-type: none"> • Jacquet-Langlands' main theorem to prove the multiplicity one theorem of Atkin-Lehner • [Gel75, Section 6. D] 	
5. Further reading	~ 1 page
<ul style="list-style-type: none"> • Jacquet Langland's Theory, L-functions and meromorphic continuation • Arthur-Selberg Trace formula 	

References

- [Bum97] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.
- [Del73] P. Deligne. "Formes Modulaires et Représentations De $GL(2)$ ". In: *Modular Functions of One Variable II*. Ed. by Pierre Deligne and Willem Kuyk. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 55–105. ISBN: 978-3-540-37855-6.
- [Gel75] Stephen S. Gelbart. *Automorphic Forms on Adele Groups. (AM-83)*. Princeton University Press, 1975. ISBN: 9780691081564. URL: <http://www.jstor.org/stable/j.ctt1b7x82z> (visited on 03/07/2024).
- [KL06] Andrew Knightly and Charles Li. "Traces of Hecke Operators". In: (Dec. 2006). DOI: 10.1090/surv/133/01.
- [Moc] Andreea Mocanu. "Atkin Lehner Theory". In: (). URL: <https://andreeamocanu.github.io/atkin-lehner-theory.pdf>.