Hurwitz Automorphism Theorem

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1 Summary

The aim of this document is to give a proof of Hurwitz Automorphism Theorem, that states that the number of conformal automorphisms of a compact Riemann Surface M of genus $g \geq 2$ is upper-bounded by

$$|\mathrm{Aut}(M)| \leq 84(g-1)$$

.

The proof is divided in two parts

1. Given $H \in \mathrm{Aut}(M)$ a finite subgroup, define the topological and complex structure of M/H and the properties of the ramificated covering given by the quotient map $\pi : M \mapsto M/H$.

2. Apply the Riemann-Hurwitz identity to the aforementioned ramified covering to prove the theorem.

Comment 1. *The main reference for this work is Farkas-Kra, Riemann Surfaces, Chapters III.7, V.1.*

2 Preliminaries

We will need the following theorems.

Theorem 1 (Riemman-Hurwitz Identity)**.** *Let R and T be compact Riemann Surfaces of genus g and γ respectively. Let $f : R \mapsto T$ a non constant holomorphism. Let N be the degree of the ramified covering defined by f and*

$$B = \sum_{P \in R} b_f(P),$$

where $b_f(P)$ is the branching number of P by f. Then,

$$g - 2 = N(\gamma - 2) + 1 + \frac{B}{2}$$

PROOF. Proven in class.

Theorem 2 (Weiestrass gap theorem)**.** *Let M be a compact Riemann Surface of positive genus g and $P \in M$ an arbitrary point. Then, there exist exactly g integers*

$$1 \leq n_1 < n_2 < \cdots < n_g < 2g$$

such that there does not exist a holomorphic function in M $\{p\}$ with a pole of order n_i at P.

Definition 1 (Weierstrass points)**.** *A point is called a Weierstrass point if it doesn't have gaps precisely at $\{1, 2, \cdots, g\}$.*

Corollary 1. *The set of Weierstrass points is discrete. Hence, by M compact, finite.*

PROOF. These results are a consequence of Riemann-Roch's theorem. The proof is outside of the scope of this document.

3 Structure of M/H

Definition 2. *Let $\mathrm{Aut}(M)$ be the set of conformal (holomorphic, bijective and inverse holomorphic) maps $f : M \mapsto M$.*

Proposition 1 (Farkas-Kra III.7.7)**.** *Let M be an arbitrary R.S, $H \subseteq \mathrm{Aut}(M)$ a finite subgroup, $P \in M$ and $H_P = \{h | h(P) = P, h \in H\}$. Then, H_P is cyclic.*

Using the same notations,

Definition 3 (Farkas-Kra III.7.8)**.** *We can provide M/H with a Riemann Surface structure compatible with $\pi : M \mapsto M/H$.*

PROOF. For a given $P \in M$, we know that $H_P = \langle h \rangle$.

If $h = 1$, $H_P = \{1\}$, then $\pi^{-1}(\pi(P)) = \{P\}$, so the local coordinate of P will also be of $\pi(P)$ and, in such local coordinates, $\pi(z) = z$, so the branching number $b_\pi(P) = 0$.

Otherwise, $H_P = \langle h \rangle$ with $h : M \mapsto M$ a non constant conformal automorphism. Let $k = |H_P|$. Choose a neighbourhood U of P such that $(U) \in U$. Then, $h(z) = e^{2\pi i/k} z$ for some local coordinate at U, then (U, z^k) will be a local coordinate of $[P] \in M/H$ and $b_\pi(P) = k - 1$.

4 Hurwitz theorem

Comment 2. *In this section, we asume M is a compact Riemann Surface of genus $g \geq 2$.*

Proposition 2. *Let $1 \neq T \in \mathrm{Aut}(M)$, the set of fixed points of T is discrete. Hence, as M compact, it is finite.*

Given $T : M \mapsto M$ and $P \in M$ such that $T(P) = P$. If P is a branching point, it is clearly the only fixed point in a small neighbourhood. If P is not a branching point, $f(z) = z$ for some local coordinate M.

Proposition 3. *(Farkas-Kra V.1.1) Let $1 \neq T \in \mathrm{Aut}(M)$, then T has at most $2g+2$ fixed points.*

Proposition 4. *(Farkas-Kra V.1.2) Let $W(M)$ the the finite set of Weierstrass points of M and $T \in \mathrm{Aut}(M)$. Then $T(W(M)) = W(M)$. Hence, there is a group homomorphism $\lambda : \mathrm{Aut}(M) \mapsto \mathrm{Perm}(W(M))$.*

Proposition 5. *If M is not hyperelliptic, λ is injective. If M is hyperelliptic, $\lambda = \langle J \rangle$, the hyperelliptic involution.*

Corollary 2. $\mathrm{Aut}(M)$ *is finite.*

`PROOF.` If M is not hyperelliptic, we have injected $\mathrm{Aut}(M)$ into a finite group, so $\mathrm{Aut}(M)$ mush be finite.

If M is hyperelliptic, by the isomorphism theorem, $\lambda(\mathrm{Aut}(M)) \simeq \mathrm{Aut}(M)/\langle J \rangle$. Since both $\langle J \rangle$ and $\lambda(\mathrm{Aut}(M)) \subseteq \mathrm{Perm}(W(M))$ are finite, $\mathrm{Aut}(M)$ must be finite.

Theorem 3 (Hurwitz)**.** *Let M be a compact Riemann Surface of genus $g \geq 2$, then* $\mathrm{Aut}(M) \leq 84(g-1)$*.*

`PROOF.` We abbreviate $\mathrm{Aut}(M) = G$. We study the ramified covering $\pi : M \mapsto M/G$.

The degree of the ramified covering is $N = |G|$ as if P is not a branching point of π (which exists as branching points are finite), then none of the $Q = h(P)$ are branching points ($\pi(Q) = [P]$) and, as we have seen in Defintion **3**, $|G_P| = 1$

$$N = \sum_{Q \in \pi^{-1}([P])} (b_\pi(Q) + 1) = \sum_{Q \in G \cdot P} (b_\pi(Q) + 1) = |G \cdot P| = |G|/|G_P| = |G|$$

Otherwise $h(P) = g(P) \implies g^{-1}h(P) = P$, which would be contradictory.

From Group Theory, we know that $P \in M$, $|G_P| \cdot |G \cdot P| = |G|$ and we have seen that $b_\pi(P) = |G_P| - 1 = \frac{N}{|G \cdot P|} - 1$.