

1 PART I

Consider the Neo-Classical growth model. Time is discrete and goes on forever. There is a representative agent that derives utility only from consumption and discounts future utility at a rate β . The agent owns k_0 units of capital and has an endowment of time that can be used for labor or leisure every period. The time endowment is normalized to 1. There is a representative firm that hires labor and rents capital to produce using a constant returns to scale technology. Capital rental rate is r and the wage is w . Capital depreciates fully after use (i.e. $\rho = 1$).

The utility function is:

$$u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta}$$

1. Define a competitive equilibrium for this economy.

A competitive equilibrium is a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and allocations for the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that

- Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm solves:

$$\max_{\{y_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t (y_t - r_t k_t - w_t l_t)$$

subject to:

$$y_t = F(k_t, l_t) \forall t \geq 0$$

- Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household solves:

$$\max_{\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t (c_t + i_t) \leq \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi$$

$$x_{t+1} = (1 - \delta)x_t + i_t, \quad \forall t \geq 0$$

$$0 \leq l_t \leq 1, \quad 0 \leq k_t \leq x_t, \quad \forall t \geq 0$$

$$c_t, x_{t+1} \geq 0, \quad \forall t \geq 0$$

x_0 given

- Markets clear

$$y_t = c_t + i_t \quad \text{Goods market}$$

$$k_t^d = k_t^s \quad \text{Capital market}$$

$$l_t^d = l_t^s \quad \text{Labor market}$$

2. Find the steady state value for $\{c, l, k, y, r, w\}$

For this problem we set up the social planners problem as:

$$\max_{c_t, l_t, k_{t+1}} \sigma_{t=0}^{\infty} \beta^t \left[\frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta} \right]$$

subject to

$$c_t + k_{t+1} - (1 - \delta)k_t = z k_t^\alpha l_t^{1-\alpha}$$

and the non negativity constraints

$$y_t, k_t, l_t \geq 0, l_t \leq 1, k_0 = \bar{k}_0 \text{ given}$$

The procedure requires obtaining the first order conditions for c_t, l_t and k_{t+1} and rearranging them. After all the algebra we obtain the following:

$$\frac{k}{l} = \left(\frac{1 - \beta(1 - \delta)}{\alpha z \beta} \right)^{\frac{1}{\alpha-1}} = \Psi$$

$$\frac{c}{l} = z \left(\frac{k}{l} \right)^\alpha - \delta \left(\frac{k}{l} \right) = \Lambda = z \Psi^\alpha - \delta \Psi$$

Combining the f.o.c of consumption and labor we obtain:

$$l = \left(\frac{z(1 - \alpha)\Psi^\alpha}{\chi \Lambda^\sigma} \right)^{\frac{1}{\sigma+\eta}}$$

Finally,

$$\begin{aligned} k^{ss} &= \Psi l^{ss} \\ c^{ss} &= \Lambda l^{ss} \\ y^{ss} &= z(k^{ss})^\alpha (l^{ss})^{1-\alpha} = z \Psi^\alpha l^{ss} \\ r^{ss} &= \alpha z (k^{ss})^{\alpha-1} (l^{ss})^{1-\alpha} = \alpha z M^{\alpha-1} \\ w^{ss} &= (1 - \alpha) z (k^{ss})^\alpha (l^{ss})^{-\alpha} = (1 - \alpha) z \Psi^\alpha \end{aligned}$$

3. Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

$$V(k) = \max_{k', l} \left\{ \frac{(z k^\alpha l^{1-\alpha} + (1 - \delta)k - k')^{1-\sigma}}{1 - \sigma} - \chi \frac{l^{1+\eta}}{1 + \eta} + \beta V(k') \right\}$$

Subject to

$$\begin{aligned} 0 &\leq k' \leq z k^\alpha l^{1-\alpha} + (1 - \delta)k \\ 0 &\leq l \leq 1 \end{aligned}$$

For the following exercises, assume that $\alpha = 1, z = 1, \sigma = 2, \eta = 1$

4. Find χ such that $l_{ss} = 0.4$
5. Solve the planner's problem numerically using value function iteration.
 - (a) Plain VFI.
 - (b) Modified Howard's Policy Iteration (you must choose the number of policy iterations).
 - (c) MacQueen-Porteus Bounds.

Use an equally space grid for capital between $[10^{-5}, 2k_{ss}]$, vary the number of grid points until you get a maximum error of 1% in your Euler Equation. For each method report the time and number of iterations.

6. Use the solution to the planner's problem to obtain the path of $\{c, k, r, w, y\}$ starting from the steady state after the following changes:

I. Capital decreases to 80%

- Consumption

AAA

Type	Number of Calls	Time	%tot	alloc
Plain VFI $n_k = 20$	1	7.213361 seconds	2.35	2.405 GiB
Plain VFI $n_k = 50$	1	44.228189 seconds	2.18	14.860 GiB
Plain VFI $n_k = 200$	1	705.513128 seconds	2.2	237.686 GiB
Plain VFI $n_k = 1000$	1			
VFI-HPI $n_k = 20$	1	1.620049 seconds	2.68	184.976 MiB
VFI-HPI $n_k = 50$	1	0.155176 seconds	7.74	61.352 MiB
VFI-HPI $n_k = 200$	1	1.366485 seconds	3.90	696.695 MiB
VFI-HPI $n_k = 1000$	1	27.642289 seconds	4.14	15.158 GiB
VFI-MBP $n_k = 20$	1	0.566068 seconds	1.73	65.938 MiB
VFI-MBP $n_k = 50$	1	0.089633 seconds	8.67	24.117 MiB
VFI-MBP $n_k = 200$	1	1.274607 seconds	2.60	481.320 MiB
VFI-MBP $n_k = 1000$	1	45.568152 seconds	4.56	26.500 GiB

- Capital AAA
- Interest Rate AAA
- Wage AAA

• Production	AAA
II. Productivity increases permanently by 5%	
• Consumption	AAA
• Capital	AAA

- Interest Rate AAA
- Wage AAA
- Production AAA

7. Prove that the mapping used in Howard's policy iteration algorithm is a contraction.