1 PART I

Consider the Neo-Classical growth model. Time is discrete and goes on forever. There is a representative agent that derives utility only from consumption and discounts future utility at a rate β. The agent owns k_0 units of capital and has an endowment of time that can be used for labor or leisure every period. The time endowment is normalized to 1. There is a representative firm that hires labor and rents capital to produce using a constant returns to scale technology. Capital rental rate is r and the wage is w. Capital depreciates fully after use (i.e. $\rho = 1$).

The utility function is:

$$u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta}$$

1. Define a competitive equilibrium for this economy.

   A competitive equilibrium is a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and allocations for the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that

   - Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm solves:

     $$\max_{\{y_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t)$$

     subject to:

     $$y_t = F(k_t, l_t) \forall t \geq 0$$

   - Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household solves:

     $$\begin{aligned} \max_{\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \\ subject\ to \\ \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi \\ x_{t+1} = (1-\delta)x_t + i_t, \quad \forall t \geq 0 \\ 0 \leq l_t \leq 1,\ 0 \leq k_t \leq x_t,\ \forall t \geq 0 \\ c_t, x_{t+1} \geq 0,\ \forall t \geq 0 \\ x_0\ given \end{aligned}$$

   - Markets clear

     $$\begin{aligned} y_t = c_t + i_t \quad Goods\ market \\ k_t^d = k_t^s \quad Capital\ market \\ l_t^d = l_t^s \quad Labor\ market \end{aligned}$$

2. Find the steady state value for $\{c, l, k, y, r, w\}$

   For this problem we set up the social planners problem as:

   $$\max_{c_t, l_t, k_{t+1}} \sigma_{t=0}^{\infty} \beta^t \left[\frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta} \right]$$

subject to

$$c_t + k_{t+1} - (1-\delta)k_t = zk_t^{\alpha} l_t^{1-\alpha}$$

and the non negativity constraints

$$y_t, k_t, l_t \geq 0, l_t \leq 1, k_0 = \bar{k}_0 \; given$$

The procedure requires obtaining the first order conditions for $c_t, l_t and k_{t+1}$ and rearranging them. After all the algebra we obtain the following:

$$\frac{k}{l} = \left(\frac{1-\beta(1-\delta)}{\alpha z \beta}\right)^{\frac{1}{\alpha-1}} = \Psi$$

$$\frac{c}{l} = z\left(\frac{k}{l}\right)^{\alpha} - \delta\left(\frac{k}{l}\right) = \Lambda = z\Psi^{\alpha} - \delta\Psi$$

Combining the f.o.c of consumption and labor we obtain:

$$l = \left(\frac{z(1-\alpha)\Psi^{\alpha}}{\chi\Lambda^{\sigma}}\right)^{\frac{1}{\sigma+\eta}}$$

Finally,

$$\begin{aligned} k^{ss} &= \Psi l^{ss} \\ c^{ss} &= \Lambda l^{ss} \\ y^{ss} &= z(k^{ss})^{\alpha}(l^{ss})^{\alpha} = z\Psi^{\alpha} l^{ss} \\ r^{ss} &= \alpha z(k^{ss})^{\alpha-1}(l^{ss})^{1-\alpha} = \alpha z M^{\alpha-1} \\ w^{ss} &= (1-\alpha)z(k^{ss})^{\alpha}(l^{ss})^{-\alpha} = (1-\alpha)z\Psi^{\alpha} \end{aligned}$$

3. Pose the planner's dynamic programming problem. Write down the appropriate Bellman equation.

$$V(k) = \max_{k',l}\left\{\frac{(zk^{\alpha}l^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi\frac{l^{1+\eta}}{1+\eta} + \beta V(k')\right\}$$

Subject to

$$\begin{aligned} 0 \leq k' \leq zk^{\alpha}l^{1-\alpha} + (1-\delta)k \\ 0 \leq l \leq 1 \end{aligned}$$

For the following exercises, assume that $\alpha = 1, z = 1, \sigma = 2, \eta = 1$

4. Find χ such that $l_{ss} = 0.4$
5. Solve the planner's problem numerically using value function iteration.
   (a) Plain VFI.
   (b) Modified Howard's Policy Iteration (you must choose the number of policy iterations).
   (c) MacQueen-Porteus Bounds.

   Use an equally space grid for capital between $[10^{-5}, 2k_{ss}]$, vary the number of grid points until you get a maximum error of 1% in your Euler Equation. For each method report the time and number of iterations.

6. Use the solution to the planner's problem to obtain the path of $\{c, k, r, w, y\}$ starting from the steady state after the following changes:

   I. Capital decreases to 80%
      - Consumption

AAA

Type	Number of Calls	Time	%tot	alloc
Plain VFI $n_k = 20$	1	7.213361 s	2.35	2.405
Plain VFI $n_k = 50$	1			
Plain VFI $n_k = 100$	1			
Plain VFI $n_k = 500$	1			
VFI-HPI $n_k = 20$	1			
VFI-HPI $n_k = 50$	1			
VFI-HPI $n_k = 100$	1			
VFI-HPI $n_k = 500$	1			
VFI-MBP $n_k = 20$	1			
VFI-MBP $n_k = 50$	1			
VFI-MBP $n_k = 100$	1			
VFI-MBP $n_k = 500$	1			

- Capital

  AAA

- Interest Rate

  AAA

- Wage

  AAA

- Production

AAA

II. Productivity increases permanently by 5%

- Consumption

AAA

- Capital

AAA

- Interest Rate

AAA

- Wage

AAA

- Production

AAA

7. Prove that the mapping used in Howard's policy iteration algorithm is a contraction.