b. *The Indirect methods (Iterative Methods)*

Indirect methods (Iterative techniques) are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

*An iterative technique to solve the n * n linear system A**x** = **b** starts with an initial approximation* $\boldsymbol{x}^{(0)}$ *to the solution* $\boldsymbol{x}$ *and generates a sequence of vectors* $\{\boldsymbol{x}(k)\}_{k=0}^{\infty}$ *that converges to* $\boldsymbol{x}$.

1- *Jacobi's Method*

Two assumptions made on Jacobi Method:

1. The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation,* $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3,\dots$

The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(k)}$ from $\boldsymbol{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}x_j^{(k-1)}) + b_i\right], \qquad \text{for } i = 1,2,,\dots n$$

Example. *The linear system A**x** = **b** given by*

$$\begin{aligned}
10x_1 - x_2 + 2x_3 \quad &= 6\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25\\
2x_1 - x_2 + 10x_3 - x_4 &= -11\\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

has the unique solution $\boldsymbol{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\boldsymbol{x}^{(k)}$ *to* $\boldsymbol{x}$ starting with $\boldsymbol{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|x^{(k)} - x^{(k-1)}\|_\infty}{\|x^{(k)}\|_\infty} < 10^{-3}$$

Solution We first solve equation xi, for each i = 1, 2, 3, 4, to obtain

$$\begin{aligned}
x_1 &= \qquad\quad \frac{1}{10}x_2 - \frac{1}{5}x_3 \qquad\qquad + \frac{3}{5},\\
x_2 &= \frac{1}{11}x_1 \qquad\quad + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11},\\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 \qquad\quad + \frac{1}{10}x_4 - \frac{11}{10},\\
x_4 &= \qquad - \frac{3}{8}x_2 + \frac{1}{8}x_3 \qquad\qquad + \frac{15}{8}.
\end{aligned}$$

From the initial approximation $\boldsymbol{x}(0) = (0, 0, 0, 0)^t$ we have $\boldsymbol{x}^{(1)}$ given by:

$$\begin{aligned}
x_1^{(1)} &= \qquad\quad \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} \qquad\qquad + \frac{3}{5} = 0.6000\\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} \qquad\quad + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727\\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} \qquad\quad + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000\\
x_4^{(1)} &= \qquad - \frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} \qquad\qquad + \frac{15}{8} = 1.8750
\end{aligned}$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, *are generated in a similar manner and are presented in next table:*

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$relative\ error = \frac{\|x_2^{10} - x_2^{9}\|_\infty}{\|x_2^{10}\|_\infty} = \frac{8.0 * 10^{-4}}{1.9998} < 10^{-3}$$

The Jacobi's Method in Matrix Form :

Consider to solve an $n \times n$ size system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ -a_{21} & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = D - L - U$$

$A\boldsymbol{x} = \boldsymbol{b}$ is transformed into $(D - L - U)\boldsymbol{x} = \boldsymbol{b}$

$$D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b}$$

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\boldsymbol{x} = D^{-1}(L + U)\boldsymbol{x} + D^{-1}\boldsymbol{b}$$

The matrix form of Jacobi iterative method is

$$\boldsymbol{x}^{(k)} = D^{-1}(L + U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} \qquad k = 1,2,3, \ldots$$

Define $T = D^{-1}(L + U)$ and $\boldsymbol{c} = D^{-1}\boldsymbol{b}$, Jacobi iteration method can also be written as

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad k = 1,2,3, \ldots$$

The Gauss-Seidel Method

With the Jacobi method, the values of $x_i^{(k)}$ obtained in the th iteration remain unchanged until the entire *(k+1)* th iteration has been calculated. With **the Gauss-Seidel method**, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$ and so on.

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right]$$

for each $i = 1, 2, \ldots, n$.

Namely,

$$\begin{aligned} a_{11}x_1^{(k)} &= -a_{12}x_2^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1 \\ a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2 \\ &\vdots \\ a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots a_{nn}x_n^{(k)} &= b_n \end{aligned}$$

Example *Use the Gauss-Seidel iterative technique to find approximate solutions to*

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ *until*

$$\frac{\|x^{(k)} - x^{(k-1)}\|_\infty}{\|x^{(k)}\|_\infty} < 10^{-3}$$

Solution For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$
\begin{aligned}
x_1^{(k)} &= \qquad\qquad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= \qquad\qquad - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad + \frac{15}{8}.
\end{aligned}
$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in next table :

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

We stopped after five iterations because

$$
relative\ error = \frac{\left\|x_2^5 - x_2^4\right\|_\infty}{\left\|x_2^5\right\|_\infty} = \frac{3.0 * 10^{-4}}{2} < 10^{-3}
$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in last example required twice as many iterations for the same accuracy.