Microeconomics
Homework 2: Utility & the Utility Maximization Problem

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1. The consumption set is $X = \mathbb{R}^2_+$.

   (a) Solve the following utility maximization problem: $\max_{x_1,x_2} \ln(x_1) + x_2$ s.t. $x_1 + 6x_2 = 12$. At the optimal bundle, is the marginal rate of substitution equal to the price ratio? Explain why.

   (b) Now solve $\max_{x_1,x_2} \ln(x_1) + x_2$ s.t. $x_1 + 6x_2 = 3$. At the optimal bundle, is the marginal rate of substitution equal to the price ratio? Explain why.

2. Consider the following utility functions:

   - Cobb-Douglas: $u(x_1, x_2) = x_1^2 x_2$
   - Perfect substitutes: $u(x_1, x_2) = x_1 + 2x_2$
   - Perfect complements: $u(x_1, x_2) = \min\{x_1, 2x_2\}$
   - Quasi-linear: $u(x_1, x_2) = \sqrt{x_1} + x_2$

   For each of them:

   (a) Plot an indifference curve.

   (b) Whenever possible, calculate the marginal rate of substitution (MRS).

   (c) Are the preferences strongly monotone? Are they weakly monotone?

   (d) Are the preferences strictly convex? Are they convex?

3. Consider the utility function $u(x_1, x_2) = \sqrt{x_1} + x_2$. In the same graph, plot the marginal utility of good 1 and marginal utility of good 2. The horizontal axis should be x_1 (or x_2) and the vertical axis the marginal utility. Using your graph, explain why we may have corner solutions with these preferences.

4. Consider the utility functions $u_1(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2}$ and $u_2(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ with $\gamma_1, \gamma_2 > 0$ and $0 < \alpha < 1$.

   (a) Find the relation between α, γ_1, and γ_2 such that they represent the same preferences.

   (b) Solve the utility maximization problem.

5. The constant elasticity of substitution (CES) utility function. Consider the utlity function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$. Here $\rho \leq 1$ is a parameter.

   (a) Find the marginal rate of substitution (MRS).

   (b) Set MRS equal to p_1/p_2 and solve for the ratio $x \stackrel{\text{Def}}{=} x_2/x_1$. This is the relative demand of good x_2 (relative to good x_1) as a function of the relative prices. How does the relative demand change with an increase of the relative price $p \stackrel{\text{Def}}{=} p_2/p_1$?

(c) Define $x \overset{\text{Def}}{=} x_2/x_1$ and $p \overset{\text{Def}}{=} p_2/p_1$. Find the elasticity of substitution: $\frac{dx}{dp} \cdot \frac{p}{x}$ and show it is constant.

(d) Comment on the value of the elasticity of substitution when $\rho = 1$ and $\rho \to -\infty$.

6. Consider the CES utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$.

   (a) Plot an indifference curve when $\rho = 1$.

   (b) Plot an indifference curve when $\rho = -20$.

7. Consider another version of the CES utility function:

$$u(x_1, x_2) = \frac{1}{\rho} \ln \left(\frac{x_1^\rho}{2} + \frac{x_2^\rho}{2} \right)$$

   (a) Why is this version equivalent to $(x_1^\rho + x_2^\rho)^{1/\rho}$? Explain your answer.

   (b) Take the limit of $u(x_1, x_2)$ when $\rho \to 0$ and use L'Hopital's rule to show this utility function converges to Cobb-Douglas.

8. Prove the following statement. If $u(x)$ represents $\succeq$, then $\succeq$ is complete and transitive.

9. Given the following utility functions:

   - Cobb-Douglas: $u(x_1, x_2) = x_1 x_2^3$
   - Perfect Substitutes: $u(x_1, x_2) = x_1 + 3x_2$
   - Perfect Complements: $u(x_1, x_2) = \min\{x_1, 3x_2\}$
   - Quasi-Linear: $u(x_1, x_2) = \ln(x_1) + x_2$

   (a) Solve the utility maximization problem. Your answer should include $x_1(p_1, p_2, m)$, $x_2(p_1, p_2, m)$, and $v(p_1, p_2, m)$.

   (b) Verify Roy's identity.

10. Suppose the utility function is $u(x_1, x_2) = \max\{x_1, x_2\}$.

    (a) Plot an indifference curve.

    (b) Are the preferences monotone?

    (c) Are the preferences strictly convex?

    (d) Find the Marshallian demands and the indirect utility function. Hint: there will be corner solutions.

11. Consider the CES utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$. Prove that it converges to $\min\{x_1, x_2\}$ when $\rho \to -\infty$.

12. Isoelastic demand. Consider the following utility function where $\varepsilon > 0$ is a parameter.

$$u(x_1, x_2) = \frac{x_1^{1-\frac{1}{\varepsilon}} - 1}{1 - \frac{1}{\varepsilon}} + x_2$$

    (a) Plot the marginal utilities of goods 1 and 2 in the same graph for $\varepsilon = 0.5, 1, 2$. Argue intuitively that there will be corner solutions.

    (b) Focus on the first term $f(\varepsilon) = \frac{x_1^{1-\frac{1}{\varepsilon}} - 1}{1-\frac{1}{\varepsilon}}$. Use L'Hopital's rule to show that $f(\varepsilon) \to \ln(x_1)$ as $\varepsilon \to 1$.

(c) Assume the solution is interior. Find the Marshallian demand for good 1 $x_1(p_1, p_2, m)$. The budget constraint is the usual $p_1x_1 + p_2x_2 = m$.

(d) Calculate the elasticity of demand: $\left|\frac{\partial x_1(p_1,p_2,m)}{\partial p_1} \cdot \frac{p_1}{x_1(p_1,p_2,m)}\right|$. Show it is a constant.

Answers

1. (a) $x_1^* = 6$, $x_2^* = 1$. MRS equals the price ratio because we are in an interior solution.
   (b) $x_1^* = 3$, $x_2^* = 0$. MRS is not equal to the price ratio because this is a corner solution.

2. In order:
   (a) See figure.
   (b) $\frac{1}{2}$, $\frac{2x_2}{x_1}$, undefined, $\frac{1}{2\sqrt{x_1}}$.
   (c) SM (whenever consumption is positive), SM, M, SM.
   (d) SC (whenever consumption is positive), C, C, SC.

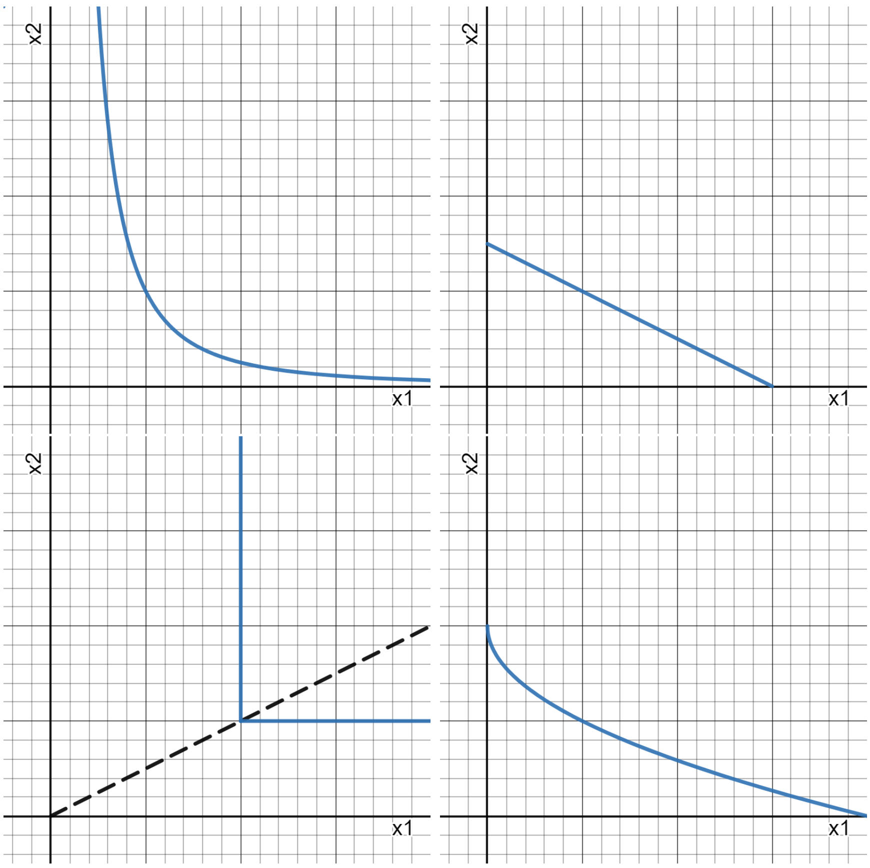

Figure 1: Exercise 2

3. The marginal utility of good 2 is constant, while the marginal utility of good 1 is decreasing. For small values of x_1, its marginal utility is higher. If the consumer has very little money, she will choose to buy only x_1 because it gives her higher marginal utility. This is the corner solution we see in quasi-linear preferences.

4. (a) $\alpha = \frac{\gamma_1}{\gamma_1+\gamma_2}$.
   (b) $x_1(p_1, p_2, m) = \alpha \frac{m}{p_1}$ and $x_2(\cdot) = (1-\alpha)\frac{m}{p_2}$. The indirect utility is $v(p_1, p_2, m) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} m$

5. (a) $\text{MRS} = \left(\frac{x_2}{x_1}\right)^{1-\rho}$
   (b) $x = \left(\frac{1}{p}\right)^{\frac{1}{1-\rho}}$ or $\frac{x_2}{x_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{1-\rho}}$
   (c) $\varepsilon = \frac{dx}{dp} \cdot \frac{p}{x} = -\frac{1}{1-\rho}$.
   (d) When $\rho \to 1$, $\varepsilon \to -\infty$, goods are very substitutes with each other. When $\rho \to -\infty$, $\varepsilon \to 0$, goods are not substitutes with each other.

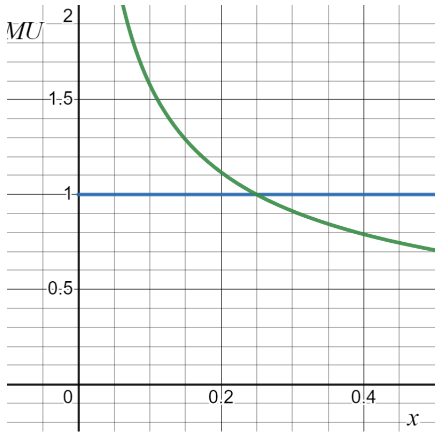

Figure 2: Exercise 3

6. (a) This should be perfect substitutes.
   (b) This should be close to perfect complements.

7. (a) Because it's a increasing transformation of $(x_1^\rho + x_2^\rho)^{1/\rho}$. First divide by $2^{1/\rho}$ and then apply $\ln(\cdot)$.
   (b) Consider x_1 and x_2 constant.

$$\begin{aligned}\lim_{\rho\to 0} \quad \frac{1}{\rho}\ln\left(\frac{x_1^\rho}{2}+\frac{x_2^\rho}{2}\right) &= \lim_{\rho\to 0} \quad \frac{x_1^\rho \ln(x_1) + x_2^\rho \ln(x_2)}{x_1^\rho + x_2^\rho} \\ &= \frac{\ln(x_1)+\ln(x_2)}{2}\end{aligned}$$

   The last step is an increasing transformation of $x_1 x_2$.

8. Use the definition of utility function $x \succeq y \iff u(x) \geq u(y)$.
   (a) Completeness. Take two real numbers $u(x)$ and $u(y)$. You can always tell $u(x) \geq u(y)$ or $u(x) \leq u(y)$ (or both). Using the definition of utility function we have $x \succeq y$ or $y \succeq x$ (or both).
   (b) Transitivity. Take three real numbers $u(x)$, $u(y)$, and $u(z)$ such that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Because these are numbers, it must be that $u(x) \geq u(z)$. We have shown that if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

9. You can use Roy's identity to find the other demand. If there are corner solutions, you should apply Roy's identity by cases.
   - Cobb-Douglas: $v(p_1,p_2,m) = \frac{27}{256}\frac{m^4}{p_1 p_2^3}$. $x_1(\cdot) = \frac{m}{4p_1}$.
   - Perfect Substitutes: $v(p_1,p_2,m) = \frac{m}{\min\{p_1,p_2\}}$. $x_1(\cdot) = \begin{cases} m/p_1 & \text{if } \ 3p_1 < p_2 \\ [0, m/p_1] & \text{if } \ 3p_1 = p_2 \\ 0 & \text{if } \ 3p_1 > p_2 \end{cases}$
   - Perfect Complements: $v(p_1,p_2,m) = \frac{3m}{3p_1+p_2}$. $x_2(\cdot) = \frac{m}{3p_1+p_2}$
   - Quasi-Linear: $v(p_1,p_2,m) = \ln\left(\min\left\{\frac{p_2}{p_1}, \frac{m}{p_1}\right\}\right) + \max\left\{\frac{m-p_2}{p_2}, 0\right\}$. $x_1(\cdot) = \min\left\{\frac{p_2}{p_1}, \frac{m}{p_1}\right\}$ and $x_2(\cdot) = \max\left\{\frac{m-p_2}{p_2}, 0\right\}$

10. (a) They are inverted L shaped. See figure.
    (b) Yes, they are strongly monotone.

(c) No, they are not strictly convex convex. A linear combination of bundles can be strictly worse.

(d) $v(p_1, p_2, m) = \frac{m}{\min\{p_1,p_2\}}$, the Marshallian demand are similar to perfect substitutes:

$$x_1(p_1, p_2, m) = \begin{cases} \frac{m}{p_1} & \text{if} \quad p_1 < p_2 \\ \frac{m}{p_1} \text{ or } 0 & \text{if} \quad p_1 = p_2 \\ 0 & \text{if} \quad p_1 > p_2 \end{cases}$$

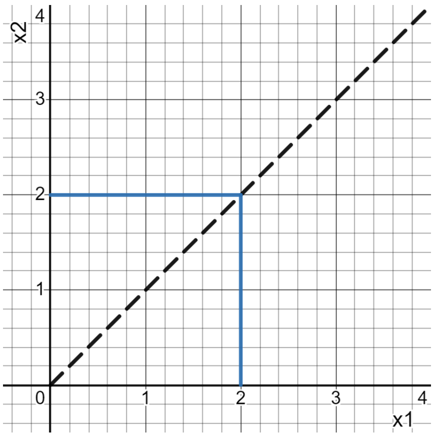

Figure 3: Exercise 10

11. Suppose $x_1 \leq x_2$. We want to show that $u(x_1, x_2) \to x_1$ as $\rho \to -\infty$.

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$$
$$u(x_1, x_2) = [x_1^\rho \cdot (1 + (\frac{x_2}{x_1})^\rho)]^{1/\rho}$$
$$u(x_1, x_2) = x_1 \cdot [1 + \left(x_2/x_1\right)^\rho]^{1/\rho}$$

Now make $\rho \to -\infty$. Note that $\left(x_2/x_1\right)^\rho \to 0$ because x_1/x_2 is less than 1. Then we see that $u(x_1, x_2) = x_1$.

Next assume $x_1 \geq x_2$ and repeat the same steps to show $u(x_1, x_2) = x_2$. This completes the proof: when $\rho \to -\infty$ the utility function is either x_1 or x_2, whichever is smaller.

12. (a) See figure. These are quasi-linear preferences. If the income is low enough, the consumer will buy only good 1 so $x_1 > 0$ while $x_2 = 0$, a corner solution. $\text{MU}_2 = 1$ and $\text{MU}_1 = x_1^{-\frac{1}{\varepsilon}}$.

(b) The first limit is undetermined. Differentiate the numerator and the denominator both with respect to ε and take the limit again.

$$\lim_{\varepsilon \to 1} \frac{x_1^{1-\frac{1}{\varepsilon}} - 1}{1 - \frac{1}{\varepsilon}} = \lim_{\varepsilon \to 1} \frac{x_1^{1-\frac{1}{\varepsilon}} \ln(x_1) \frac{1}{\varepsilon^2}}{\frac{1}{\varepsilon^2}} = \ln(x_1)$$

(c) $x_1(p_1, p_2, m) = \left(\frac{p_2}{p_1}\right)^\varepsilon$.

(d) The elasticity of demand is ε.

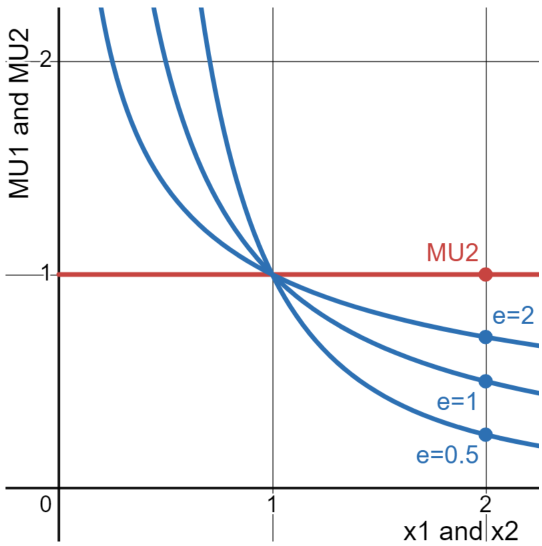

Figure 4: Exercise 12