Microeconomics
Homework 9: Intertemporal Choice

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1. Let $u(c) = \ln(c)$ be the period utility function. There are two periods. Define $m \stackrel{\text{Def}}{=} m_0 + \frac{m_1}{1+r}$.

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1) \quad \text{s.t.} \quad c_0 + \frac{c_1}{1+r} = m$$

   (a) Find the solutions $c_0(\beta, r, m)$, $c_1(\beta, r, m)$.
   (b) How does the solution change with changes to β, r, and m?
   (c) Suppose $r = 0.25$, $\beta = 0.6$, $m_0 = 240$ and $m_1 = 0$ comment on the saving/borrowing decision.
   (d) Now $m_0 = 0$ and $m_1 = 240$. The other parameters remain the same. Comment on the saving/borrowing decision.

2. Let $u(c) = \ln(c)$ be the period utility function. There are three periods. Define $m \stackrel{\text{Def}}{=} m_0 + \frac{m_1}{1+r} + \frac{m_2}{(1+r)^2}$.

   (a) Define $z_0 \stackrel{\text{Def}}{=} m$, z_1, and z_2 as the available total resources the agent has at the beginning of each period. Find expressions for each z_i. You can call c_i the consumption in period i.
   (b) Focus on $t = 2$. Given z_2, what will be the consumption in that period?
   (c) Focus on $t = 1$. Given z_1, what will be the consumption in that period?
   (d) Focus on $t = 0$. Given z_0, what will be the consumption in that period?
   (e) Use your previous answers to find the optimal consumption $c_i(\beta, r, m)$ each period.
   (f) Consider the following two situations of income over time. Comment on the saving/borrowing decision. Assume $\beta = 0.8$ and $r = 0.5$.
      - $m_0 = 100$ and $m_1 = m_2 = 0$. This gives a present value of $m = 100$.
      - $m_0 = 0$, $m_1 = 120$, and $m_2 = 45$. This also gives a present value of $m = 100$.
   (g) Assume $\beta = 0.8$, $r = 0.5$, and $m = 100$. Create a situation in which he borrows money in both $t = 0$ and $t = 1$.

3. Consider the following infinite problem.

$$\max_{\{c\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = m$$

   (a) Write the problem in recursive form.
   (b) Find the first order condition and solve for c_{t+1} as a function of c_t.
   (c) Show that given c_0, one can express $c_t = [\beta(1+r)]^t c_0$.

(d) Use the budget constraint and find $c_0(\beta, r, m)$.

(e) Assume $m = 100$, $\beta = 0.8$, and $r = 0.25$. What is the stream of consumption? Explain.

4. Let $u(c) = \frac{c^{1-\rho}}{1-\rho}$ (with $\rho > 0$ be the period utility function. There are two periods. Define $m \stackrel{\text{Def}}{=} m_0 + \frac{m_1}{1+r}$.

$$\max_{c_0, c_1} \quad u(c_0) + \beta u(c_1) \quad \text{s.t.} \quad c_0 + \frac{c_1}{1+r} = m$$

(a) Calculate $c_0(\beta, r, m)$ and $c_1(\beta, r, m)$. Your answers will depend on the parameter ρ.

(b) Set $\rho = 1$ and verify that your previous answers are equal to the case of $u(c) = \ln(c)$.

(c) Let $\rho \to \infty$ and verify that the consumer chooses $c_1 = c_2$ no matter the value of β. Can you explain why?

5. Quasi-hyperbolic discounting produces time inconsistent results. Let $\beta \leq 1$ and $\delta \leq 1$. The consumer down-weights the future by δ as described below.

$$\max_{c_0, c_1, c_2} \quad u(c_0) + \delta \cdot \left[\beta u(c_1) + \beta^2 u(c_2)\right] \quad \text{s.t.} \quad c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} = m \tag{1}$$

(a) Calculate the MRS between c_1 and c_2.

Now consider the sequential formulation when $t = 1$.

$$v(z_1) = \max_{c_1, c_2} \quad u(c_1) + \delta \cdot \left[\beta u(c_2)\right] \quad \text{with} \quad c_2 = (1+r)(z_1 - c_1) \tag{2}$$

(b) Calculate the MRS between c_1 and c_2. Comment how δ makes your answer different from a).

In what follows you'll show there will be time inconsistent results. For simplicity assume $r = 0$, $\beta = 1$. Let $\delta = 0.5$ and $(m_0, m_1, m_2) = (4200, 0, 0)$. The period utility function is $u(c) = \ln(c)$

(c) Solve problem (1). How much money does the agent have at the beginning of $t = 1$? Let z_1 be that amount.

(d) After having consumed c_0 you obtained before, solve the following problem. Argue that the consumer cannot commit to the choice he made in part c).

$$\max_{c_1, c_2} \quad u(c_1) + \delta \cdot [u(c_2)] \quad \text{where} \quad c_1 + c_2 = z_1$$

Answers

1. (a) $c_0(\cdot) = \frac{1}{1+\beta} \cdot m$, $c_1(\cdot) = \frac{\beta}{1+\beta} \cdot (1+r)m$.
   (b) Take derivatives to make comparative statics. Careful: m also depends on r.
   (c) When $m_0 = 240$, the agent chooses to consume $c_0 = 150$ in the first period and to save 90. Next period he has $90 \cdot (1+0.25) = 112.5$ and he consumes everything.
   (d) When $m_1 = 240$ he borrows and consume $c_0 = 120$. Next period he has to pay back $120 \cdot (1+0.25) = 150$ so he is left with $240 - 150 = 90$. He consume all of that.

2. (a) $z_0 = m$, $z_1 = (1+r)(z_0 - c_0)$, and $z_2 = (1+r)(z_1 - c_1)$.
   (b) $c_2(z_2) = z_2$.
   (c) $c_1(z_1) = \frac{z_1}{1+\beta}$.
   (d) $c_0(z_0) = \frac{z_0}{1+\beta+\beta^2}$.
   (e) $c_0(\beta, r, m) = \frac{1}{1+\beta+\beta^2} \cdot m$, $c_1(\beta, r, m) = \frac{\beta}{1+\beta+\beta^2} \cdot (1+r)m$, and $c_2(\beta, r, m) = \frac{\beta^2}{1+\beta+\beta^2} \cdot (1+r)^2 m$
   (f) In both situations the consumption is $c_0 \simeq 41$, $c_1 \simeq 49$, and $c_2 \simeq 59$. In the first situation, the person always saves. In the second situation, in $t = 0$ he borrows money and in $t = 1$ repays and saves a little for the future.
   (g) The easiest one is $m_0 = m_1 = 0$ and $m_2 = 225$.

3. (a) In rescursive form:

$$v(z_t) = \max_{c_t} \left\{ \ln(c_t) + \beta v(z_{t+1}) \right\} \quad \text{with} \quad z_{t+1} = (1+r)(z_t - c_t)$$

   (b) The first order condition is:

$$\frac{1}{c_t} = \frac{\beta(1+r)}{c_{t+1}}$$

   Solving for c_{t+1} gives you $c_{t+1} = \beta(1+r) \cdot c_t$
   (c) Start with c_1, then c_2 and substitute one into the other.
   (d) Rewrite the budget constraint and sum the geometric series.

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = m$$
$$\sum_{t=0}^{\infty} \frac{[\beta(1+r)]^t c_0}{(1+r)^t} = m$$
$$\sum_{t=0}^{\infty} \beta^t c_0 = m$$
$$c_0 \sum_{t=0}^{\infty} \beta^t = m$$
$$c_0 \cdot \frac{1}{1-\beta} = m$$
$$c_0 = (1-\beta) \cdot m$$

   (e) Consumption is constant over time $c_t^* = 20$.
   - In the first period he consumes $c_0 = 20$ and saves the remaining 80.
   - His savings become $(1+0.25)80 = 100$ next period. He consumes $c_1 = 20$ and saves the rest.
   - ...

4. (a) Optimal consumption is given below.

$$c_0(\cdot) = \frac{1}{1 + \beta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}} \cdot m \quad \text{and} \quad c_1(\cdot) = \frac{\beta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}}{1 + \beta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}} \cdot (1+r)m$$

   (b) Set $\rho = 1$. The answers matches the ones in exercise 1) a).

   (c) Let $\rho \to \infty$. You'll see $\beta^{1/\rho} \to 1$ and:

$$c_1 = c_2 = \frac{1}{1 + \frac{1}{1+r}} \cdot m$$

   The reason is that with $\rho \to \infty$ preferences are perfect complements and the consumer would want to smooth consumption as much as possible.

5. (a) MRS $= \frac{u'(c_1)}{\beta u'(c_2)}$

   (b) MRS $= \frac{u'(c_1)}{\delta \beta u'(c_2)}$. Only if $\delta = 1$ the answers are the same. When $\delta < 1$ the MRS here is higher than the previous one.

   (c) $c_0 = 2100$ and $c_1 = c_2 = 1050$. At the beginning of $t = 1$ the agent has $4200 - 2100 = 2100$.

   (d) Now $c_2 = 700$ and $c_1 = 1400$. Because she has this extra inclination for the present (captured by $\delta < 1$), when $t = 1$ arrives, her previous choice is not optimal now.