Producer Theory

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Production Function

Production Function

$$y = f(k, l)$$

- Summarizes the productive process.
- Summarizes the best way of combining capital and labor (and more) to produce y units of the output.
- Theoretical tool so we can describe production without actually getting into the details of each productive process.

Marginal Product

$$\mathrm{MP_K} = \frac{\partial f(k,l)}{\partial k} \qquad \mathrm{MP_L} = \frac{\partial f(k,l)}{\partial l}$$

- Positive quantities.
- Decreasing. Law of diminishing marginal product.

Average Product

$$\mathrm{AP_L} = \frac{f(k, l)}{l}$$

- Average vs Marginal.
- At the amount of labor where the AP is maximum, AP=MP.

Isoquant & MRTS

$$q = f(k, l)$$

- For some fixed quantity q, plot the isoquant in (l, k) plane.
- Its slope (if defined) is the Marginal Rate of Technical Substitution MRTS.

$$\text{MRTS} = -\frac{dk}{dl} = \frac{\text{MP}_\text{L}}{\text{MP}_\text{K}}$$

Returns to Scale

$$f(tk, tl) \lesseqqgtr t \cdot f(k, l)$$

- All the inputs are multiplied by the same constant $t > 1$. This is meant to represent that we increase the scale of production.
- We classify the production function as having:
    - Decreasing returns to scale (DRS).
    - Constant returns to scale (CRS).
    - Increasing returns to scale (IRS).

CRS Production Function

$$f(tk, tl) = t \cdot f(k, l)$$

- Output per worker depends only on the ratio of capital per worker: $\frac{y}{l} = f\left(\frac{k}{l}\right)$.
- Marginal products also depend only on the ratio of capital per worker.
- $f(k, l) = \mathrm{MP_K} \cdot k + \mathrm{MP_L} \cdot l$, a consequence of Euler's theorem.

Elasticity of Substitution

$$f(k,l) = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}, \quad \text{with} \quad \rho \leq 1$$

- CES production function.
- $\sigma = \frac{1}{1-\rho} > 0$ is the elasticity of substitution.
- $\gamma > 0$ controls the returns to scale.

Three Cases

- Perfect Substitutes ($\sigma \to \infty$): $f(k, l) = (k + l)^{\gamma}$.
- Perfect Complements ($\sigma = 0$): $f(k, l) = \big(\min\{k, l\}\big)^{\gamma}$.
- Cobb-Douglas ($\sigma = 1$): $f(k, l) = k^{\alpha} \cdot l^{\beta}$

Cost Minimization Problem

Total Cost Definition

$$wl + rk$$

- Economic cost. Includes the opportunity cost.
- Assume prices (w, r) are taken as given. This is a competitive firm.
- In economics if a firm makes zero profits it means it makes just enough revenue to cover the opportunity cost.

Cost Minimization Problem

$$\min_{k,l} \quad wl + rk \quad \text{s.t.} \quad q = f(k, l)$$

- For a target production level q, find how to allocate capital and labor to minimize total cost.
- This problem is identical to the expenditure minimization problem in consumer theory.

Cost Minimization Problem II

$$\min_{k,l} \quad wl + rk \quad \text{s.t.} \quad q = f(k, l)$$

- The solution are the conditional factor demands $l^c(w, r, q)$ and $k^c(w, r, q)$.
- Minimum cost attained is the cost function $c(w, r, q)$.

Cost Minimization Problem III

- If the solution is interior and the MRTS is defined, then:

$$\text{MRTS} = \frac{w}{r}$$

- Everything we studied in the expenditure minimization problem applies here.

Cost Function

$$c(w, r, q)$$

- Homogeneous of degree 1 in prices.
- (Weakly) Increasing in prices and q.
- Concave in prices.
- Marginal Cost: $\frac{\partial c(w,r,q)}{\partial q}$
- Average Cost: $\frac{c(w,r,q)}{q}$
- Shephard's lemma: $\frac{\partial c(w,r,q)}{\partial w} = l^C(w, r, q)$

Profit Maximization Problem

Profit Definition

$$pf(k, l) - wl - rk$$

- A competitive firm takes (p, w, r) as given and chooses the amount of capital and labor to maximize profits.
- Total revenue $pf(k, l)$.
- Total cost $wl + rk$, which includes the opportunity cost.

Profit Maximization Problem

$$\max_{k,l} \quad pf(k,l) - wl - rk$$

- Not identical to utility maximization problem. Why?
- May not have a solution.
- Typically need DRS to have a solution.

Profit Maximization Problem: Characterization

$$\mathrm{MP_L} = \frac{w}{p}$$
$$\mathrm{MP_K} = \frac{r}{p}$$

- In an interior solution marginal product must be equal to the real price of the input.

Profit Maximization Problem: Characterization II

$$\text{MRTS} = \frac{w}{r}$$

- Usual condition for an interior solution.
- Profit maximization implies cost minimization.

Profit Maximization Problem

$$\max_{k,l} \quad pf(k,l) - wl - rk$$

- Input demands $k(p,w,r)$ and $l(p,w,r)$.
- Profit function $\pi(p,w,r)$.
- Supply of the firm $y(p,w,r)$.

Profit Function

- Homogeneous of degree 1 in (p, w, r).
- (Weakly) Increasing in p.
- (Weakly) Decreasing in (w, r).
- Convex in p.
- Envelope results:

$$\frac{\partial \pi(\cdot)}{\partial p} = y(p, w, r), \quad \frac{\partial \pi(\cdot)}{\partial w} = -l(p, w, r), \quad \text{and} \quad \ldots$$

Profit Maximization Revisited

- First solve the cost minimization problem and store the cost function.
- Then choose the quantity that maximizes profit.
- This is equivalent to what we did before, but it does not say anything about input demands.

Profit Maximization Revisited II

$$\max_{q} \quad pq - c(q, w, r)$$

- FOC gives us the usual $p = mc$.
- With DRS, marginal cost will be increasing.
- The solution is the supply curve $q(p, w, r)$ and it matches our previous supply.

Constant Returns to Scale

CRS & Competitive Markets

- Recall that under CRS: $f(k, l) = MP_K \cdot k + MP_L \cdot l$.
- If markets are competitive, then:

$$p \cdot f(k, l) = r \cdot k + w \cdot l$$

- $TR = TC$ so profits are 0.
- CRS are attractive in competitive models because one does not need to keep track of profits.

CRS & Competitive Markets

Typically:

- DRS & competitive markets → Positive profits.
- CRS & competitive markets → Zero profits.
- IRS → Not consistent with competitive markets. Firms have a strong incentive to increase their scale of production.