# Uncertainty

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# St. Petersburg Paradox

# Gamble

- A coin is flipped until a heads appears.
- The game pays you as follows. The more tails you get, the more money.

$$L = \begin{cases} \$2 & \text{with prob.} \quad \frac{1}{2} \\ \$4 & \text{with prob.} \quad \frac{1}{4} \\ \$8 & \text{with prob.} \quad \frac{1}{8} \\ \dots \end{cases}$$

- How much money would you pay to play this game?
- $L$ for lottery.

# St. Petersburg Paradox

- It's very likely you would't pay much to play this gamble.
- Let's calculate the expected value of the gamble or lottery $L$.

$$\mathbb{E}[L] = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots \to \infty$$

- How come people pay so little money to play a gamble that has an infinite expected value?
- In some sense this gamble is not worth its infinite expected dollar value.

# Bernoulli's Solution

- People do not respond directly to the dollar prizes.
- Instead, they care about the utility these dollars give them.
- Assume the utility of $w$ dollars is given by $u(w) = \ln(w)$.
- Even if the expected monetary value of the lottery (what we call $\mathbb{E}[L]$) is infinite, the expected utility people derive (called $\mathrm{EU}[L]$) may be finite.
- This solves the paradox.

# Bernoulli's Solution II

$$\begin{aligned} \mathrm{EU}[L] &= \frac{1}{2} \cdot \ln(2) + \frac{1}{4} \cdot \ln(4) + \frac{1}{8} \cdot \ln(8) + \dots \\ &= 2 \cdot \ln(2) \simeq 1.39 \end{aligned}$$

- This lottery gives the consumer an expected utility of 1.39.
- \$4 correspond to a satisfaction of 1.39.
- \$4 is closer to what people would pay for this game.

# The von Neumann-Morgenstern Utility Index

# Some Notation

- $L$ is a **lottery** that specifies some monetary prizes as well as the probability of getting those.
- $\mathbb{E}[L]$ is the **expected value of the lottery**. Note this is an objective number. It involves the probabilities and the monetary prizes.
- $u(w)$ is the **von Neumann-Morgenstern utility function**, defined over monetary prizes. Give it a monetary prize and it returns the satisfaction associated with it. I will also call this the **Bernoulli's utility function**.
- $\mathrm{EU}(L)$ is the **expected utility of the lottery**. This is not an objective number. It depends on consumer's preferences.

# Construct the vN-M Utility Index

- Suppose there are $n$ prizes. We sort them such that $w_1 < w_2 < \cdots < w_n$.
- Assign $u(w_1) = 0$ and $u(w_n) = 1$, though any other pair of numbers work.
- Now consider $w_2$ and ask the consumer to state the probability $\pi$ such that she is indifferent between:
  - Receiving $w_2$ for sure.
  - Getting $w_n$ with probability $\pi$ and $w_1$ with probability $(1 - \pi)$.
- Which probability would make you indifferent?
- $w_2$ is the certainty equivalent.

# Construct the vN-M Utility Index II

$$\begin{aligned} u(w_2) &= \pi \cdot u(w_n) + (1 - \pi) \cdot u(w_1) \\ &= \pi \cdot 1 + (1 - \pi) \cdot 0 \\ &= \pi \end{aligned}$$

- Repeat for $w_3, w_4, \ldots$ and we've found the vN-M utility function for the consumer.
- Example. Let $w_1 = 0$, $w_2 = 50$, and $w_3 = 100$. Set $u(0) = 0$, $u(100) = 1$, and let's find $u(50)$.
- vN-M utility is invariant to linear transformations.

# Expected Utility Maximization

- Assume consumers have a vN-M utility index.
- Assume also that consumers know the true probabilities.
- They decide before the uncertainty is realized. In this sense their decision is risky.
- They will maximize the expected utility.

# Certainty Equivalent - CE

- Given a lottery $L$, the consumer has some expected utility $\mathrm{EU}[L]$ about it. We call certainty equivalent the amount of money that gives her the same expected utility.

$$u(w_{CE}) = \mathrm{EU}[L]$$

- Participating in the gamble $L$ or receiving $w_{CE}$ for sure. Both options give her the same satisfaction.

# Risk Aversion

# Risk Aversion

- Would you bet \$1000 on the outcome of a coin flip?
- Most people won't. Even though this is a fair bet. $\mathbb{E}[L] = 0$.

$$L = \begin{cases} 1000 & \text{with prob.} \quad 0.5 \\ -1000 & \text{with prob.} \quad 0.5 \end{cases}$$

- In this sense people are typically risk averse.

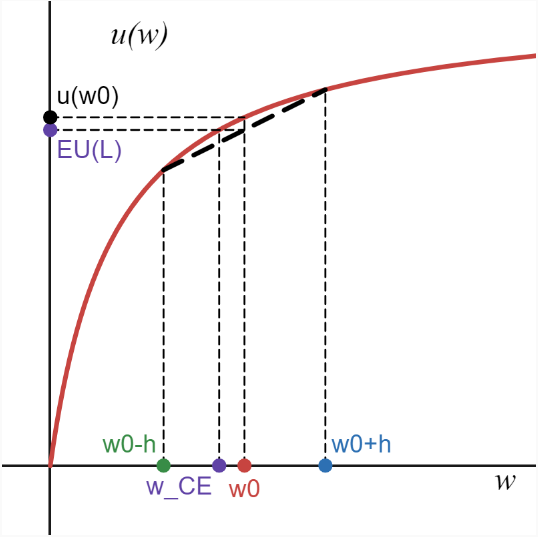
# What explains risk aversion?

- Decreasing marginal utility of money.
- More money is always good. But it is specially better when I have little of it.
- In terms of the vN-M utility, we have that $u'(w) > 0$, but $u''(w) < 0$.
- In words $u(w)$ is concave (facing down, sad, or n shapped).

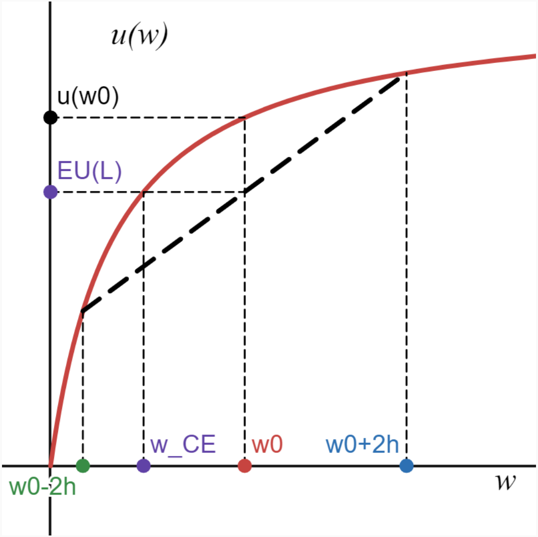
# Risk aversion II

- Consider the following fair bets.
  - $L_0$. Get $w_0$ for sure.
  - $L_1$. Get $w_0 + h$ with probability 0.5 and $w_0 - h$ with probability 0.5.
  - $L_2$. Get $w_0 + 2h$ with probability 0.5 and $w_0 - 2h$ with probability 0.5.
- The three lotteries have the same expected value $\mathbb{E}[L_0] = \mathbb{E}[L_1] = \mathbb{E}[L_2] = w_0$.
- But $L_2$ is riskier than $L_1$ which is also riskier than $L_0$.
- Risk averse consumers will rank $L_0 \succ L_1 \succ L_2$.

# Graphically



# Graphically II



# Risk neutrality and loving

- If $u(w)$ is linear, then the consumer is risk neutral.
- If $u(w)$ is convex (upwards, u shaped or happy), then the consumer is a risk lover.
- For a convex $u(w)$ the expected utility of a fair bet is always higher than receiving $w_0$ for sure.
- Check graph.

# Arrow Pratt Measure

$$r(w) = -\frac{u''(w)}{u'(w)}$$

- Invariant to linear transformations.
- Positive for risk averse individuals.
- This measures absolute risk aversion.
- CARA utility $u(w) = -e^{-\rho w}$, with $\rho > 0$.

# Relative Arrow Pratt Measure

$$rr(w) = -\frac{u''(w)}{u'(w)} \cdot w$$

- CRRA utility $u(w) = \frac{w^{1-\rho}-1}{1-\rho}$, with $\rho > 0$.

# Risk aversion III

Given a lottery $L$ with expected value $\mathbb{E}[L]$. A consumer is risk averse if:

- Arrow-Pratt measures are positive.
- $u(w)$ is concave.
- She'd pay money not to take risk.
- $u(\mathbb{E}[L]) > \text{EU}[L]$
- $w_{CE} < \mathbb{E}[L]$

# Insurance

# Insurance Problem

- A consumer has some wealth $w$.
- With probability $p$ something bad happens and she loses some of her wealth.

$$L = \begin{cases} w & \text{with prob.} \quad 1-p \\ w-D & \text{with prob.} \quad p \end{cases}$$

- Assume this consumer is risk averse.

# Insurance Problem II

- A risk neutral insurance company offers her the following deal.
- You can choose to insurance $x$ dollars paying me $qx$ with $q < 1$ (why? What is $q$ doing?).
- If the bad thing happens, you'll receive $x$ dollars back.

$$L' = \begin{cases} w - qx & \text{with prob.} \quad 1 - p \\ w - qx - D + x & \text{with prob.} \quad p \end{cases}$$

# Consumer's Choice

$$
\begin{aligned}
\max_{x} \quad & \mathrm{EU}[L'] = (1-p)\cdot u[w-qx] + p\cdot u[w-D+(1-q)x] \\
\text{(FOC)} \quad & (1-p)q\cdot u'[w-qx] = p(1-q)\cdot u'[w-D+(1-q)x]
\end{aligned}
$$

- If you know $u(w)$, you can solve for the insurance demand $x(\cdot)$.
- The insurance demand will be a function of $(p, q, w)$.

# Insurance Company's Profit

- Insurance company is risk neutral.
- Always, no matter what, the company receives $qx$ from the consumer. This is the total revenue.
- Only if the bad thing happens (with probability $p$), the insurance company returns $x$. This is the expected cost.

$$\text{E. Prof} = qx - px = (q - p)x$$

# Actuarially Fair Premium

- A competitive insurance company will have zero expected profits.
- It will set a premium $q = p$.
- We call this premium, an Actuarially Fair Premium.
- It's fair because it matches the probability of the bad thing happening.
- A non-competitive insurance company will charge $q > p$.

# Actuarially Fair Premium II

- Actuarially fair premiums are good for consumers.
- If consumers face an actuarially fair premium, they will fully insure themselves.
- In other words, they will buy the maximum amount of insurance. And face no risk at all.

# Full Insurance

- Set $q = p$ in the FOC and solve for x.

$$
\begin{aligned}
(1-p)q \cdot u'[w - qx] &= p(1-q) \cdot u'[w - D + (1-q)x] \\
(1-p)p \cdot u'[w - px] &= p(1-p) \cdot u'[w - D + (1-p)x] \\
u'[w - px] &= u'[w - D + (1-p)x] \\
w - px &= w - D + (1-p)x \\
x &= D
\end{aligned}
$$

# Full Insurance II

- Compare $L$, $L'$ and $L^{\mathrm{F}}$.

$$L^{\mathrm{F}} = \begin{cases} w - pD & \text{with prob.} \quad 1 - p \\ w - pD & \text{with prob.} \quad p \end{cases}$$

- $L^{\mathrm{F}}$ has no risk at all.
- In general the amount chosen will be $x \in [0, D]$ depending on how expensive the premium $q$ is.

# State-Preference Approach

# Revisiting Insurance

$$L = \begin{cases} w & \text{with prob.} \quad 1-p \\ w-D & \text{with prob.} \quad p \end{cases}$$

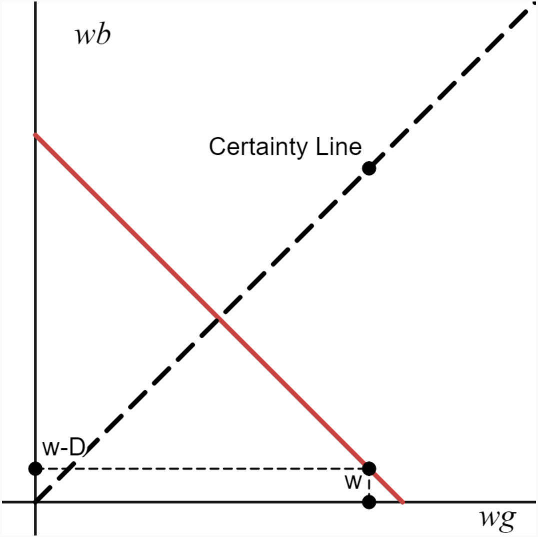
- $w_g$ is my income in good times. With no insurance I know that $w_g = w$, but I could change that with insurance.
- $w_b$ is my income in bad times. With no insurance $w_b = w - D$, but I could change this with insurance.

# Budget Constraint

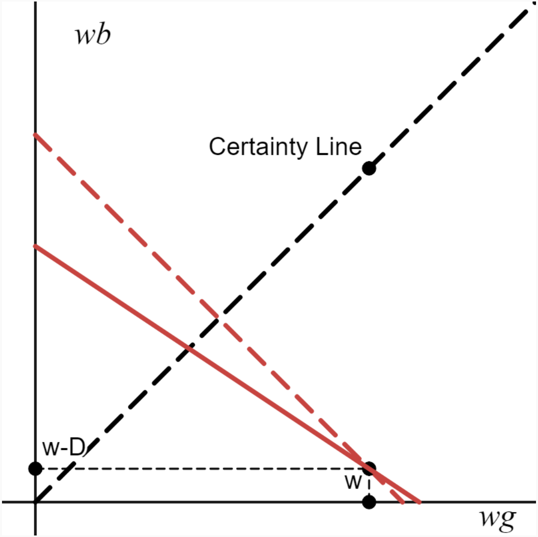
- Let $(r_g, r_b)$ be the prices. $r_g/r_b$ measures the price of an unit of income in good times relative to bad times.
- I can get a valuation of my contingent wealth using those prices. $\overline{w} = r_g \cdot w + r_b \cdot (w - D)$.
- Now write down the budget constraint.
- $\frac{r_g}{r_b} = \frac{1-q}{q}$

$$r_g \cdot w_g + r_b \cdot w_b = \overline{w}$$

# Budget Constraint II



# Budget Constraint III: $\uparrow r_b$



# Preferences

- Use the expected utility theory to write down an utility function over $w_g$ and $w_b$.

$$v(w_g, w_b) = (1-p) \cdot u(w_g) + p \cdot u(w_b)$$

- Now plot indifference curves.
- Example: use $u(w) = \ln(w)$. What is this?

# Utility Maximization Problem

$$\max_{w_g, w_b} \quad v(w_g, w_b) \quad \text{s.t.} \quad r_g \cdot w_g + r_b \cdot w_b = \overline{w}$$

- The solutions depend on $(r_g, r_b, w, D)$.
- In an interior solution we'll have MRS $= \frac{r_g}{r_b}$.

$$\frac{1-p}{p} \cdot \frac{u'(w_g)}{u'(w_b)} = \frac{r_g}{r_b}$$

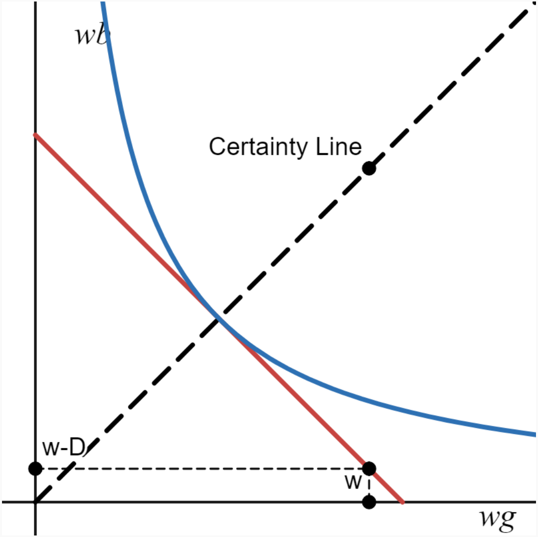
# Actuarially Fair Premium

- In general we cannot continue without a specific utility function.
- If $r_g = 1 - p$ and $r_b = p$, then:

$$w_g = w_b$$

- Same result as before. If the insurance company charges the actuarially fair premium, the consumer buys full insurance.

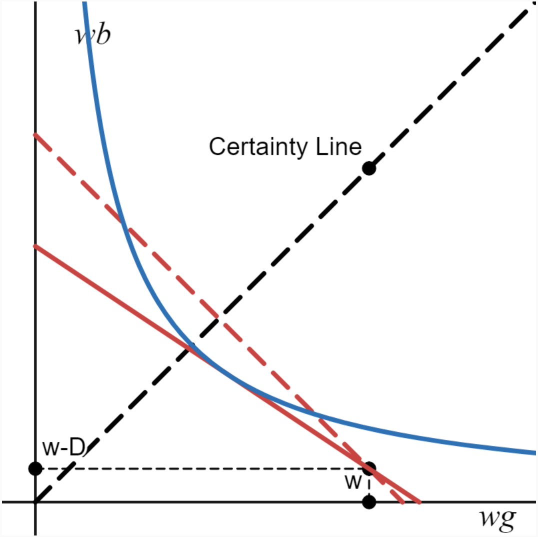
# Graph - Full Insurance



## General Case

- Without full insurance, we won't have this result.
- In general $w_g \geq w_b$, only with equality if prices are actuarially fair.

# Graph - General Case



# Risk Aversion

- The shape of the indifference curve is related to risk aversion.
- What would be the shape of the indifference curve for a risk neutral consumer?
- And for an extremely risk averse consumer?
- Recall CES utility.