Microeconomics
Homework 8: Uncertainty

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1. Consider the Bernoulli utility function $u(w) = -e^{-\rho w}$ where $\rho > 0$ is a parameter. Find the Arrow-Pratt measure of absolute risk aversion.

2. Consider the Bernoulli utility function $u(w) = \frac{w^{1-\rho}-1}{1-\rho}$ where $\rho \geq 0$ is a parameter. Find the Arrow-Pratt measure of relative risk aversion.

3. Consider the Bernoulli utility function $u(w) = \frac{w^{1-\rho}-1}{1-\rho}$ where $\rho \geq 0$ is a parameter. Using L'Hopital's rule to show that $u(w) \to \ln(w)$ when $\rho \to 1$.

4. Consiser Aang and Bumi who have the following Bernoulli utility functions $u_A(w) = \sqrt{w}$ and $u_B(w) = w^{1/4}$. Who is more risk averse?

5. Aang and Bumi from the previous exercise choose between the following loterries:

$$L_1 = \begin{cases} 81 & \text{with} \quad p = \frac{1}{4} \\ 16 & \text{with} \quad 1-p = \frac{3}{4} \end{cases}$$

$$L_2 = 26 \quad \text{with} \quad p = 1$$

   (a) Calculate the expected value of each lottery.

   (b) What do Aang and Bumi choose?

   (c) Calculate the certainty equivalent of L_1 for both agents.

6. You have the following Bernoulli utility function $u(w) = 2\sqrt{w}$ and face a lottery that gives you 4 or 16 with equal probabilities.

   (a) Calculate the expected value of the lottery. What would be your utility if you received this amount for sure?

   (b) Calculate the expected utility of the lottery.

   (c) Calculate the certainty equivalent of the lottery.

   (d) Plot all your answers in a graph with w in the horizontal axis.

7. A farmer believes there is a $50-50$ chance that the next growing season will be abnormally rainy. His Bernoulli utility is $u(w) = \ln(w)$.

   (a) Suppose the farmer must choose between wheat and corn, where these give the following income prospects. Which of the crops will he plant?

      Wheat: 28000 if it does not rain, 10000 if it rains.

      Corn: 19000 if it does not rain, 15000 if it rains.

   (b) Suppose he can plant half of his field with each crop. Would he choose to do so? Explain.

(c) What mix of wheat and corn would provided maximum expected utility to this farmer? To answer, solve the following problem. Find his maximum expected utility.

$$\max_{0\leq\alpha\leq 1} \quad \frac{1}{2}\ln\left[28000\alpha + 19000(1-\alpha)\right] + \frac{1}{2}\ln\left[10000\alpha + 15000(1-\alpha)\right]$$

(d) There is a possibility of insurance. Farmers that grow only wheat can pay the insurance company 4000. In a rainy season, the insurance company will pay farmers 8000. Would this possibility cause the farmer to change his plans?

8. A consumer has the following Bernoulli utility function defined over her wealth, in this case, her house $u(w) = \sqrt{w}$. Her house is worth $w = 36$ with probability $p = 1/2$, but with probability $1 - p = 1/2$ a flooding happens and completely destroys it, meaning $w = 0$.

   (a) Calculate her expected utility.

   There is an insurance company that offers her the following deal. Pay the insurance company $\frac{3}{4}x$ where x is the amount she chooses to be covered, and receive x back if the flooding happens.

   (b) Find the amount she chooses to cover. And the expected utility she gets.

   (c) Find the expected profits of the insurance company.

9. Refer to the previous exercise. Define the expected utility as follows, where w_g is the wealth when the flooding does not happen and w_b when the flooding happens.

$$v(w_g, w_b) = \frac{1}{2}\sqrt{w_g} + \frac{1}{2}\sqrt{w_b}$$

   (a) Plot the indifference curve of $v(\cdot,\cdot)$ at the expected utility with no insurance. The horizontal axis should be w_g and the vertical axis w_b. Also plot the indifference curve at the expected utility level with insurance you found in the previous question.

   (b) Plot the following budget constraint $\frac{1}{4}w_g + \frac{3}{4}w_b = \frac{1}{4}36 + \frac{3}{4}0$ and show the optimal solution.

   (c) Plot the certainty line. Is the premium actuarially fair?

   (d) Interpret the budget constraint.

   (e) Argue graphically what would happen if the budget constraint was $\frac{1}{2}w_g + \frac{1}{2}w_b = \frac{1}{2}36 + \frac{1}{2}0$. You can follow these steps:

      i. Argue this is the actuarially fair premium.

      ii. Argue she will fully insure herself. So $w_b = w_g = w^*$. What's the value for the wealth?

      iii. Even if you do not maximize the expected utility, you know that the tangency will happen in the point (w^*, w^*). Use this idea to make your plot.

10. An agent has the following Bernoulli utility function $u(w) = \ln(w)$. Her income this year was 100, which are subject to an income tax of 10%. She chooses how much of her income to report to the IRS to maximize her expected utility according to the following lottery. Let x be the amount of money she reports.

    With $p = 0.95$ she doesn't get caught and enjoys $100 - 0.1x$.

    With $(1-p) = 0.05$ she gets caught and the IRS confiscates everything she did not file in the tax return. Leaving her with $100 - 0.1x - (100 - x)$.

    Find the value of x she'll report to the IRS.

11. You have $w = 100$ today and are considering investing part of it on an asset. The asset has a random return given by 1.2 with $p = 1/2$ and 0.85 with $1 - p = 1/2$. Let α the the proportion of your wealth that you invest (and $1 - \alpha$ the proportion that you keep cash), the lottery you face is:

$$L = \begin{cases} (1-\alpha)100 + \alpha 120 & \text{with} \quad p = \frac{1}{2} \\ (1-\alpha)100 + \alpha 85 & \text{with} \quad 1 - p = \frac{1}{2} \end{cases}$$

Your Bernoulli utility function is $u(w) = \ln(w)$.

(a) Calculate the expected return of the asset.

(b) Why would you choose to hold cash even when the expected return is greater than 1?

(c) Calculate the portfolio weights that maximize your expected utility.

12. (Extra practice) Consider the following Bernoulli utility function $u(w) = -e^{-\rho w}$ with $\rho > 0$. A consumer has a total wealth of w, but with probability p she loses everything. There's a possibility of insurance described below.

$$L = \begin{cases} w & \text{with prob.} \quad 1 - p \\ 0 & \text{with prob.} \quad p \end{cases}$$

$$L_I = \begin{cases} w - qx & \text{with prob.} \quad 1 - p \\ (1-q)x & \text{with prob.} \quad p \end{cases}$$

$q \geq p$ is what the insurance company charges for each dollar insured. Find the demand for insurance. Denote it $x(w, q, p, \rho)$.

(a) Show that if $q = p$ there is full insurance.

(b) Show that an increase in ρ increases the demand for insurance.

(c) Show that an increase in q decreases the demand for insurance.

(d) Show that an increase in p increases the demand for insurance.

Answers

1. $-\frac{u''(w)}{u'(w)} = \rho$
2. $-\frac{u''(w)}{u'(w)} \cdot w = \rho$
3. Consider w to be a constant. First rewrite the utility function, second apply L'Hopital's rule.

$$\begin{aligned}\lim_{\rho \to 1} \frac{w^{1-\rho} - 1}{1-\rho} &= \lim_{\rho \to 1} \frac{e^{(1-\rho)\ln(w)} - 1}{1-\rho} \\ &= \lim_{\rho \to 1} \frac{e^{(1-\rho)\ln(w)} \cdot \ln(w) \cdot (-1)}{-1} \\ &= \ln(w)\end{aligned}$$

4. Bumi is more risk averse. To see it, calculate the Arrow-Pratt measure.
5. (a) $\mathbb{E}[L_1] = 32.25$ and $\mathbb{E}[L_2] = 26$.
   (b) For Aang $U^{\mathrm{E}}(L_1) = 5.25 > u(26) = 5.09$ so he chooses the lottery. For Bumi $U^{\mathrm{E}}(L_1) = 2.25 > u(26) = 2.2581$ so he chooses the 26 dollars for sure.
   (c) For Bumi $w = 25.6289$ and for Aang $w = 27.5625$. Bumi is more risk averse, so less money is needed to convince him not to play the lottery.
6. (a) $\mathbb{E}[L] = 10$ and $u(10) = 2\sqrt{10} \simeq 6.32$.
   (b) $U^{\mathrm{E}}(L) = 6$.
   (c) $w = 9$.
   (d) See figure.
7. (a) He will choose to plant corn which gives him an expected utility of 9.734. Planting wheat gives him an expected utility of 9.725.
   (b) He will choose to do so. Planting half and half gives him an expected utility of 9.7491 which is higher than only doing corn. The intuition is that he is distributing the risk between the crops.
   (c) $\alpha^* = 4/9 \simeq 0.44$ the maximum expected utility is 9.7493 which is higher than before.
   (d) Planting wheat and buying the insurance, he gets an expected utility of 9.8163. This is higher than the optimal mix without insurance. The farmer will take the insurance and plant wheat.
8. (a) 3
   (b) She chooses $x^* = 12$, and she pays $\frac{3}{4} \cdot 12 = 9$ to the insurance company. Her expected utility is 3.46.
   (c) The expected profits of the insurance company are $9 - \frac{1}{2} \cdot 12 = 3$. Note the insurance company receives 9 from the consumer no matter what, but in the case of a flooding (which happens with probability 1/2) it must pay 12 to the consumer.
9. (a) See plot.
   (b) See plot.
   (c) See plot. It is not, if the premium was actuarially fair, the agent will choose to be in the 45 degree line.
   (d) $p_1/p_2 = 1/3$ meaning the agent has to resign 3 units of wealth in the good state in order to obtain 1 unit of wealth in the bad state. Also $\frac{1-q}{q} = \frac{1/4}{3/4} = \frac{1}{3}$. This is the deal proposed by the insurance company.

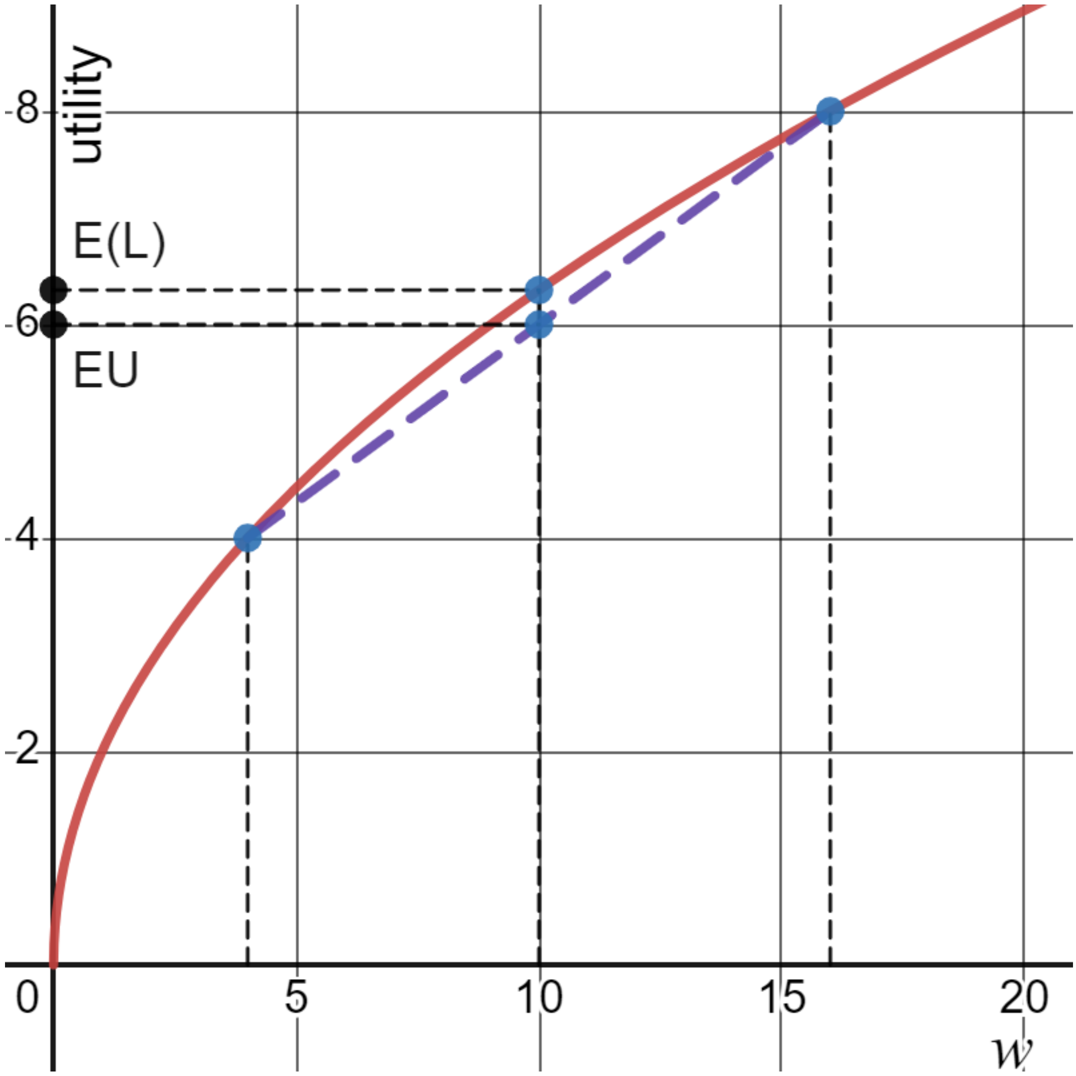

Figure 1: Exercise 6

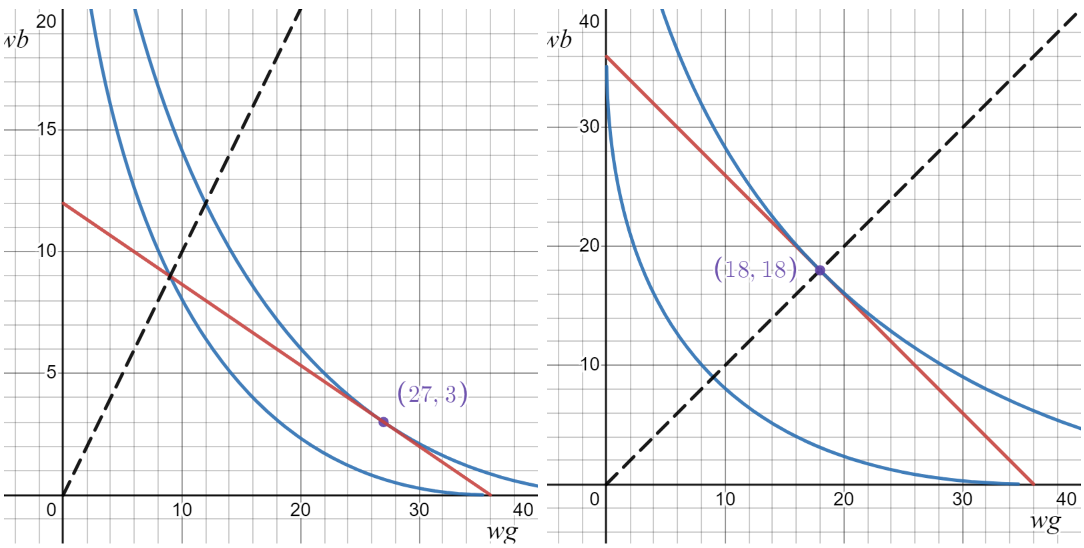

Figure 2: Exercise 9

(e) See graph. In this case she would choose to fully insure herself. So the optimal solution is on the 45 degree line, and no matter if the flooding happens or not. She always gets a wealth level of 18. Either because she pays the insurance and nothing happens or the flooding happens and she files a claim for a net gain of 18.

10. $x^* = 50$.

11. (a) 1.025.
    (b) Because of the risk.
    (c) $\alpha^* = 5/6$.

12. Finding the demand for insurance requires some algebra. You can verify $(a) - (d)$ by taking partial derivatives. Let $\tilde{q} \stackrel{\text{Def}}{=} \frac{q}{1-q}$ and $\tilde{p} \stackrel{\text{Def}}{=} \frac{p}{1-p}$ and note that $\tilde{q} \geq \tilde{p}$.

$$x(w, q, p, \rho) = w - \frac{1}{\rho} \cdot \left[\ln(\tilde{q}) - \ln(\tilde{p}) \right]$$