Derivation of the Normal Form for a Three-Alternative Decision Model (Based on Roxin, 2019)

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1 Model and Expansion

We consider the following system for three alternatives:

$$\tau\,\dot{r}_L = -r_L + \phi\Big(s_L\,r_L - c\,r_I + I_L + U\Big) + \xi_L(t), \tag{1}$$

$$\tau\,\dot{r}_C = -r_C + \phi\Big(s_C\,r_C - c\,r_I + I_C + U\Big) + \xi_C(t), \tag{2}$$

$$\tau\,\dot{r}_R = -r_R + \phi\Big(s_R\,r_R - c\,r_I + I_R + U\Big) + \xi_R(t), \tag{3}$$

$$\tau_I\,\dot{r}_I = -r_I + \phi_I\Big(\frac{g}{3}(r_L + r_R + r_C) + I_I\Big) + \xi_I(t). \tag{4}$$

In a summarized form:

$$\tau\,\dot{r}_i = -r_i + \phi\Big(s_i\,r_i - c\,r_I + I_i + U\Big) + \xi_i(t), \quad i = L, C, R, \tag{5}$$

$$\tau_I\,\dot{r}_I = -r_I + \phi_I\Big(\frac{g}{3}(R_L + R_C + R_R) + I_I\Big) + \xi_I(t), \tag{6}$$

where:

- r_i is the firing rate of excitatory population i.
- r_I is the firing rate of the inhibitory population.
- s_i is the self-excitation coefficient (which may differ among populations).
- c is the inhibition coefficient.
- I_i are each population external input.
- U is a ramping urgency signal. We assume it varies slowly:

  $$U = U_0 + \varepsilon^2\,\Delta U.$$

  As U is a ramping signal, we have that $U_0 = 0$
- τ and τ_I are the time constants for the excitatory and inhibitory populations, respectively.

Our goal is to derive a reduced (normal form) model that captures the critical dynamics in a two-dimensional competitive subspace.

An schematic representation of the model is shown in figure 1.

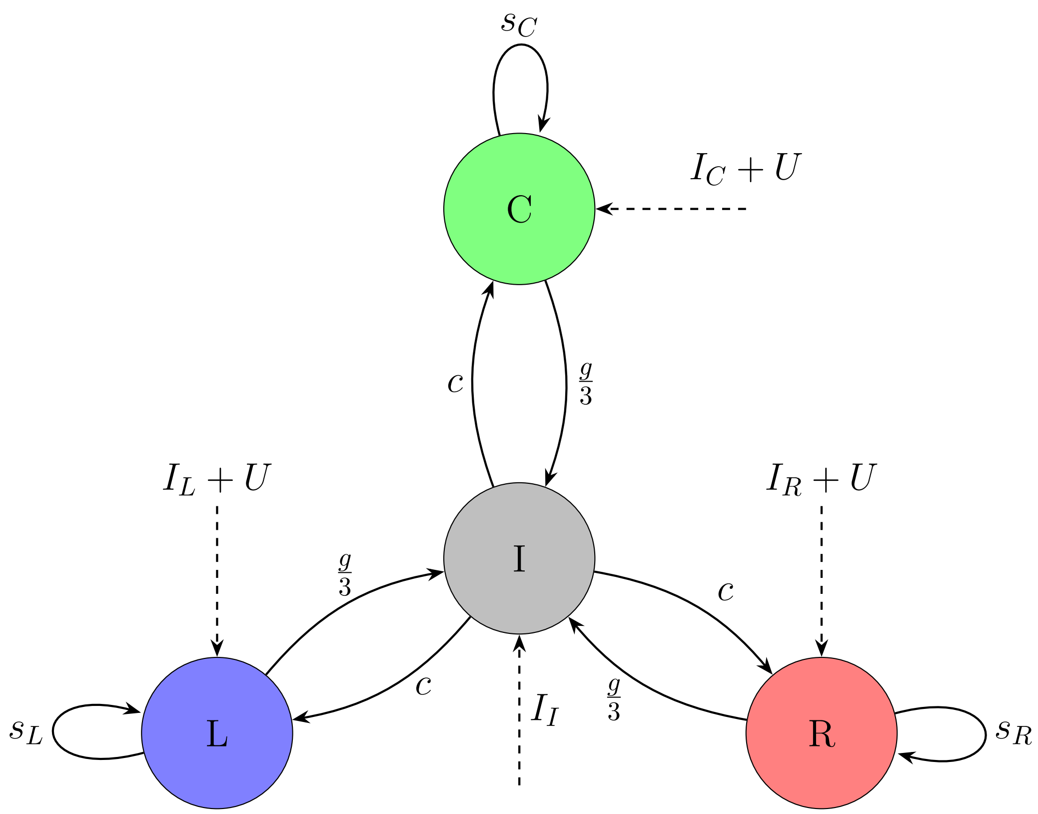

Figure 1: Schematic representation of the rate model for 3-choice tasks. The choices are represented by three excitatory populations labeled L (Left), C (Center), and R (Right), which receive external inputs (I_L, I_C, and I_R, respectively) and exhibit self-excitation with strengths s_L, s_C, and s_R. There's also an external ramping input U common to all the excitatory populations. Each excitatory population affects to an inhibitory population (I) with weight $g/3$, and the inhibitory neuron provides feedback with strength c to all excitatory populations. An external input I_I is also applied to the inhibitory neuron.

1.1 Derivation of the normal form

We assume a series expansion for the solution:

$$r_i(t) = R_i + \varepsilon\, r_{1,i}(t) + \varepsilon^2\, r_{2,i}(t) + O(\varepsilon^3), \quad i = L, C, R, \tag{7}$$

$$r_I(t) = R_I + \varepsilon\, r_{1,I}(t) + \varepsilon^2\, r_{2,I}(t) + O(\varepsilon^3). \tag{8}$$

The fixed point $R = (R_1, R_2, R_3, R_I)$ is determined at $O(1)$:

$$R_i = \phi\Big(s_i\, R_i - c\, R_I + I_i\Big), \quad i = L, C, R, \tag{9}$$

$$R_I = \phi_I\Big(\frac{g}{3}(R_L + R_C + R_R) + I_I\Big). \tag{10}$$

Additionally, we introduce a slow time scale:

$$T = \varepsilon\, t,$$

so that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\, \frac{\partial}{\partial T}.$$

We define

$$X_i = s_i\, r_i - c\, r_I + I_i + U. \tag{11}$$

Substituting the expansions and using $U = \varepsilon^2\, \Delta U$, we have

$$X_i = s_i\, (R_i + \varepsilon\, r_{1,i} + \cdots) - c\, (R_I + \varepsilon\, r_{1,I} + \cdots) + I_i + \varepsilon^2\, \Delta U.$$

We then group:

- Order $O(1)$:
$$X_{i,0} = s_i\, R_i - c\, R_I + I_i.$$
- Order $O(\varepsilon)$:
$$\delta X_i^{(1)} = \varepsilon\Big[s_i\, r_{1,i} - c\, r_{1,I}\Big].$$
- Order $O(\varepsilon^2)$: includes the term $\varepsilon^2\, \Delta U$.

The Taylor expansion of ϕ about $X_{i,0}$ is

$$\phi(X_i) = \phi(X_{i,0}) + \phi'(X_{i,0})\, \delta X_i + \frac{1}{2}\, \phi''(X_{i,0})\, (\delta X_i)^2 + O(\varepsilon^3), \tag{12}$$

with

$$\delta X_i = \varepsilon\Big[s_i\, r_{1,i} - c\, r_{1,I}\Big] + \varepsilon^2\, \Delta U.$$

Thus,

$$\phi\Big(s_i\, r_i - c\, r_I + I_i + U\Big) = \phi(X_{i,0}) + \varepsilon\, \phi'(X_{i,0})\Big(s_i\, r_{1,i} - c\, r_{1,I}\Big) + \varepsilon^2\, \left(\phi'(X_{i,0})\, \Delta U + \frac{1}{2}\, \phi''(X_{i,0})\Big(s_i\, r_{1,i} - c\, r_{1,I}\Big)^2\right) + O(\varepsilon^3).$$

For the equation (5), substituting the expansion we have:

At $O(1)$:

$$R_i = \phi(X_{i,0}), \quad i = L, C, R.$$

At $O(\varepsilon)$:

$$\begin{aligned} \tau\, \frac{d}{dt}\Big(R_i + \varepsilon\, r_{1,i}\Big) &= -\Big(R_i + \varepsilon\, r_{1,i}\Big) \\ &\quad + \phi(X_{i,0}) + \varepsilon\, \phi'(X_{i,0})\Big[s_i\, r_{1,i} - c\, r_{1,I}\Big] + O(\varepsilon^2). \end{aligned} \tag{13}$$

Since $R_i = \phi(X_{i,0})$, the $O(1)$ terms cancel. Thus, dividing by ε we obtain:

$$\tau\,\dot r_{1,i} = \Big[-1 + s_i\,\phi'(X_{i,0})\Big] r_{1,i} - c\,\phi'(X_{i,0})\,r_{1,I}, \quad i = L, C, R. \tag{14}$$

The **bifurcation condition** we impose is:

$$s_i\,\phi'(X_{i,0}) = 1, \quad i = L, C, R. \tag{15}$$

Under this condition, (14) reduces to:

$$\tau\,\dot r_{1,i} = -c\,\phi'(X_{i,0})\,r_{1,I}, \quad i = L, C, R. \tag{16}$$

Similarly, linearizing the inhibitory equation (6) gives:

$$\tau_I\,\dot r_{1,I} = -r_{1,I} + \phi_I'\Big(\frac{g}{3}(R_L + R_C + R_R) + I_I\Big)\frac{g}{3}\sum_{i=L,C,R} r_{1,i}. \tag{17}$$

Then the $O(\epsilon)$ equations can be written as

$$L_0\,r_1 = 0,$$

with .

$$L_0 = \begin{pmatrix} 0 & 0 & 0 & c\,\phi'(X_{L,0}) \\ 0 & 0 & 0 & c\,\phi'(X_{C,0}) \\ 0 & 0 & 0 & c\,\phi'(X_{R,0}) \\ -\frac{g}{3}\,\phi_I'\Big(\frac{g}{3}(R_L{+}R_C{+}R_R){+}I_I\Big) & -\frac{g}{3}\,\phi_I'\Big(\frac{g}{3}(R_L{+}R_C{+}R_R){+}I_I\Big) & -\frac{g}{3}\,\phi_I'\Big(\frac{g}{3}(R_L{+}R_C{+}R_R){+}I_I\Big) & 1 \end{pmatrix}.$$

Here, $X_{i,0} = s_i\,R_i - c\,R_I + I_i$.

Due to the slow time scale $T = \varepsilon\,t$, the time derivative expands as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\,\frac{\partial}{\partial T}.$$

The contribution from the slow scale (when acting on r_1) appears at order ε^2 and is grouped into the operator L_1:

$$L_1 = \begin{pmatrix} \tau\,\partial_T & 0 & 0 & 0 \\ 0 & \tau\,\partial_T & 0 & 0 \\ 0 & 0 & \tau\,\partial_T & 0 \\ 0 & 0 & 0 & \tau_I\,\partial_T \end{pmatrix}. \tag{18}$$

At order ε^2, we collect terms from the Taylor expansion (12). Recall that

$$\delta X_i = \varepsilon\Big[s_i\,r_{1,i} - c\,r_{1,I}\Big] + \varepsilon^2\,\Delta U.$$

Thus, from (12) we have:

$$\phi(X_i) = \phi(X_{i,0}) + \varepsilon\,\phi'(X_{i,0})\Big(s_i\,r_{1,i} - c\,r_{1,I}\Big) + \varepsilon^2\left(\phi'(X_{i,0})\,\Delta U + \frac{1}{2}\,\phi''(X_{i,0})\Big(s_i\,r_{1,i} - c\,r_{1,I}\Big)^2\right) + O(\varepsilon^3). \tag{19}$$

If the inputs have the form

$$I_i = I_0 + \varepsilon^2\,\bar I_i,$$

then the order ε^2 term for the ith excitatory equation is:

$$N_{2,i} = \phi'(X_{i,0})\Big(\bar{I}_i + \Delta U\Big) + \frac{1}{2}\,\phi''(X_{i,0})\Big[s_i\,r_{1,i} - c\,r_{1,I}\Big]^2. \tag{20}$$

An analogous term $N_{2,I}$ is obtained for the inhibitory equation from the expansion of ϕ_I.

Thus, the order ε^2 equation is written as

$$L_0\,r_2 + L_1\,r_1 = N_2, \tag{21}$$

Since L_0 has a nontrivial null space (by the bifurcation condition), a solution for r_2 exists only if the forcing term $N_2 - L_1\,r_1$ is orthogonal to the null space of the adjoint of L_0. Let $\{e_1, e_2\}$ be a basis for this null space, for example, we choose:

$$e_1 = (1, -1, 0, 0), \quad e_1 = (1, 1, -2, 0).$$

The **solvability condition** is given by:

$$\langle e_j,\ L_1\,r_1 - N_2\rangle = 0, \quad j = 1, 2. \tag{22}$$

Expressing the first-order solution in the form

$$r_1 = e_1\,X_1(T) + e_2\,X_2(T),$$

Where $X_1 = r_L - r_C, X_2 = r_L + r_C - 2r_R,$

The normal form equations are:

$$\tau\,\dot{X}_1 = A_1\left(\bar{I}_L - \bar{I}_C\right) + A_1\,\Delta U + C_1\,X_1\,X_2 + A_1(\xi_L(t) - \xi_C(t)) + O(\epsilon^3), \tag{23}$$

$$\tau\,\dot{X}_2 = A_2\left(\bar{I}_L + \bar{I}_C - 2\bar{I}_R\right) + A_2\,\Delta U + C_2\left(X_1^2 - 3X_2^2\right) + A_2(\xi_L(t) + \xi_C(t) - 2\xi_R(t)) + O(\epsilon^3). \tag{24}$$

Here, the coefficients are given in terms of the initial parameters as follows:

$$A_1 = \frac{1}{2}\Big[\phi'(X_{L,0}) - \phi'(X_{C,0})\Big], \qquad A_2 = \frac{1}{6}\Big[\phi'(X_{L,0}) + \phi'(X_{C,0}) - 2\,\phi'(X_{R,0})\Big],$$

$$C_1 = \frac{1}{2}\Big[s_L^2\,\phi''(X_{L,0}) - s_C^2\,\phi''(X_{C,0})\Big], \qquad C_2 = \frac{1}{6}\Big[s_L^2\,\phi''(X_{L,0}) + s_C^2\,\phi''(X_{C,0}) - 2\,s_R^2\,\phi''(X_{R,0})\Big].$$