Derivation of the Normal Form for a Three-Alternative Decision Model (Based on Roxin, 2019)

F. Javier Rodríguez Martínez

1 Model and Expansion

We consider the following system for three alternatives:

$$\tau\,\dot{r}_L = -r_L + \phi\Big(s_L\,r_L - c\,r_I + I_L\Big) + \xi_L(t), \tag{1}$$

$$\tau\,\dot{r}_C = -r_C + \phi\Big(s_C\,r_C - c\,r_I + I_C\Big) + \xi_C(t), \tag{2}$$

$$\tau\,\dot{r}_R = -r_R + \phi\Big(s_R\,r_R - c\,r_I + I_R\Big) + \xi_R(t), \tag{3}$$

$$\tau_I\,\dot{r}_I = -r_I + \phi_I\Big(\frac{g}{3}\big(r_L + r_R + r_C\big) + I_I\Big) + \xi_I(t). \tag{4}$$

Where $I_i = I_0 + \bar{I}_i$ for $i = L, C, R$.

In a summarized form:

$$\tau\,\dot{r}_i = -r_i + \phi\Big(s_i\,r_i - c\,r_I + I_i) + \xi_i(t), \quad i = L, C, R, \tag{5}$$

$$\tau_I\,\dot{r}_I = -r_I + \phi_I\Big(\frac{g}{3}(R_L + R_C + R_R) + I_I\Big) + \xi_I(t), \tag{6}$$

where:

- r_i is the firing rate of excitatory population i.
- r_I is the firing rate of the inhibitory population.
- s_i is the self-excitation coefficient (which may differ among populations).
- c is the inhibition coefficient.
- $\bar{I}_i$ are each population external input.
- I_0 is a ramping urgency signal.
- τ and τ_I are the time constants for the excitatory and inhibitory populations, respectively.

Our goal is to derive a reduced (normal form) model that captures the critical dynamics in a two-dimensional competitive subspace.

An schematic representation of the model is shown in figure 1.

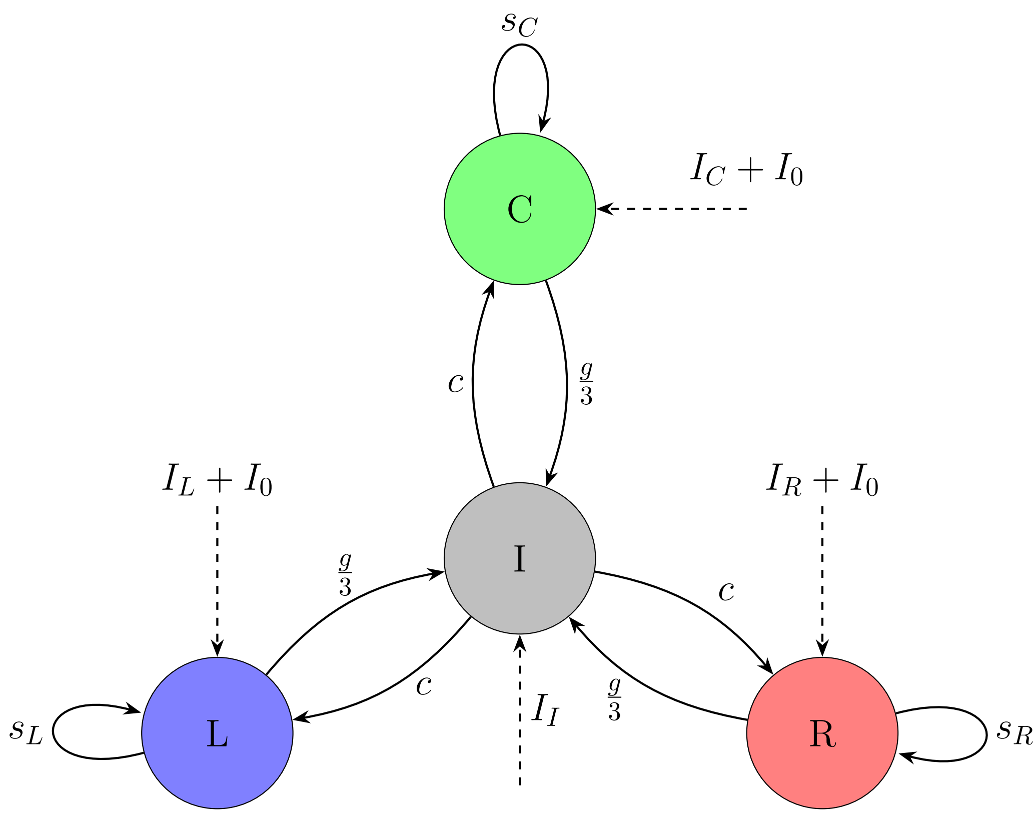

Figure 1: Schematic representation of the rate model for 3-choice tasks. The choices are represented by three excitatory populations labeled L (Left), C (Center), and R (Right), which receive external inputs ($\bar{I}_L$, $\bar{I}_C$, and $\bar{I}_R$, respectively) and exhibit self-excitation with strengths s_L, s_C, and s_R. There's also an external ramping input I_0 common to all the excitatory populations. Each excitatory population affects to an inhibitory population (I) with weight $g/3$, and the inhibitory neuron provides feedback with strength c to all excitatory populations. An external input I_I is also applied to the inhibitory neuron.

1.1 Derivation of the normal form of order 2

1.1.1 Expansion of the system

We assume a series expansion for the solution:

$$r_i(t) = R_i + \varepsilon\, r_{1,i}(t) + \varepsilon^2\, r_{2,i}(t) + O(\varepsilon^3), \quad i = L, C, R, \tag{7}$$

$$r_I(t) = R_I + \varepsilon\, r_{1,I}(t) + \varepsilon^2\, r_{2,I}(t) + O(\varepsilon^3). \tag{8}$$

We will also consider variations of s, $s = s_0 + \varepsilon^2 \bar{s}_i + O(\varepsilon^3)$ and $I_i = I_0 + \varepsilon^2 \bar{I}_i$. Thus, the fixed point $R = (R, R, R, R_I)$ is determined at $O(1)$:

$$R = \phi\Big(s_0\, R - c\, R_I + I_0\Big), \quad i = L, C, R, \tag{9}$$

$$R_I = \phi_I\Big(gR + I_I\Big). \tag{10}$$

Additionally, we introduce a slow time scale:

$$T = \varepsilon\, t,$$

so that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\, \frac{\partial}{\partial T}.$$

We define

$$X_i = s_i\, r_i - c\, r_I + I_i. \tag{11}$$

Substituting the expansions:

$$X_i = (s_0 + \varepsilon^2 \bar{s}_i)\,(R + \varepsilon\, r_{1,i} + \varepsilon^2 r_{2,i} \cdots) - c\,(R_I + \varepsilon\, r_{1,I} + \varepsilon^2 r_{2,i} + \cdots) + I_0 + \varepsilon^2 \bar{I}_i.$$

We then group:

- Order $O(1)$:
$$X_0 = s_0 R - c R_I + I_0$$
- Order $O(\varepsilon)$:
$$X_{i,1} = \varepsilon[s_0 r_{1,i} - c r_{1,I}]$$
- Order $O(\varepsilon^2)$:
$$X_{i,2} = \varepsilon^2[s_0 r_{2,i} + R\bar{s}_i + \bar{I}_i]$$
- Order $O(\varepsilon^3)$:
$$X_{i,3} = \varepsilon^3[s_0 r_{3,i} + r_{1,i}\bar{s}_i]$$

The Taylor expansion of ϕ about $X_{i,0}$ is

$$\phi(X_i) = \phi(X_0) + \phi'(X_0)(\Delta X_i) + \frac{1}{2}\,\phi''(X_0)(\Delta X_i)^2 + \frac{1}{6}\,\phi'''(X_0)(\Delta X_i)^3 + O(\varepsilon^4) \tag{12}$$

with

$$\Delta X_i = \varepsilon[s_0 r_{1,i} - c r_{1,I}] + \varepsilon^2[R\bar{s}_i + \bar{I}] + O(\varepsilon^3)$$

From this point on, we will exclude the dependence of $\phi(X)$ and it's derivatives and write them as ϕ, ϕ'...

For the equation (5), substituting the expansion we have:

At $O(1)$:

$$R = \phi(X_0), \quad i = L, C, R.$$

The **bifurcation condition** we impose is:

$$s_0\,\phi'(X_0) = 1, \quad i = L, C, R. \tag{13}$$

Thus, at $O(\varepsilon)$:

$$\tau \dot{r}_{1,i} = \underbrace{(-1 + s_0\phi'(X_0))}_{0}\, r_{1,i} - c\phi'(X_0) r_{1,I}$$

Similarly, linearizing the inhibitory equation (6) gives:

$$\tau_I\,\dot{r}_{1,I} = -r_{1,I} + \phi_I'\Big(\frac{g}{3}(r_{L,1} + r_{C,1} + r_{R,1}) + I_I\Big) \sum_{i=L,C,R} r_{i,1}. \tag{14}$$

The eigenvectors for the underlying DM process can be found solving:

$$L_0\, r_1 = 0,$$

with

$$L_0 = \begin{pmatrix} 0 & 0 & 0 & c\,\phi' \\ 0 & 0 & 0 & c\,\phi' \\ 0 & 0 & 0 & c\,\phi' \\ -\frac{g}{3}\,\phi_I' & -\frac{g}{3}\,\phi_I' & -\frac{g}{3}\,\phi_I' & 1 \end{pmatrix}.$$

Due to the slow time scale $T = \varepsilon\, t$, the time derivative expands as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\,\frac{\partial}{\partial T}.$$

The order ε^2 equation is written as

$$L_0\, r_2 + L_1\, r_1 = N_2, \tag{15}$$

Where,

$$L_1 = \begin{pmatrix} \tau\,\partial_T & 0 & 0 & 0 \\ 0 & \tau\,\partial_T & 0 & 0 \\ 0 & 0 & \tau\,\partial_T & 0 \\ 0 & 0 & 0 & \tau_I\,\partial_T \end{pmatrix}. \tag{16}$$

To compute N_2 we make use of the Taylor formula (12):

$$\phi(X_i) = \phi(X_0) + \varepsilon\phi'(X_0)(s_0 r_{1,i} - c r_{1,I}) + \varepsilon^2\left(\phi'(X_0)(R\bar{s}_i + \bar{I}_i) + \frac{1}{2}\,\phi''(X_0)(s_0 r_{1,i} - c r_{1,I})^2\right) + O(\varepsilon^3) \tag{17}$$

Then,

$$N_2 = \phi' \begin{pmatrix} \bar{I}_L + R\bar{s}_L \\ \bar{I}_C + R\bar{s}_C \\ \bar{I}_R + R\bar{s}_R \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \phi''(s_0 r_{L,1} - c r_{1,I})^2 \\ \phi''(s_0 r_{C,1} - c r_{1,I})^2 \\ \phi''(s_0 r_{R,1} - c r_{1,I})^2 \\ \phi_I'' \frac{g^2}{9}(r_{L,1} + r_{C,1} + r_{R,1}) \end{pmatrix} \tag{18}$$

Since L_0 has a nontrivial null space (by the bifurcation condition), we can write the solution of $L_0 r_1 = 0$ in terms of a basis of this null space, for example, we choose:

$$e_1 = (1, -1, 0, 0), \quad e_2 = (1, 1, -2, 0).$$

Thus, the $O(\varepsilon)$ solution can be written as:

$$r_1 = e_1\, X_1(T) + e_2\, X_2(T),$$

Where $X_1 = r_L - r_C, X_2 = r_L + r_C - 2r_R$,

In consequence, we have that:

$$r_1 = \begin{pmatrix} X_1 + X_2 \\ -X_1 + X_2 \\ -2X_2 \\ 0 \end{pmatrix}$$

We can now rewrite N_2 in terms of X_1, X_2 as:

$$N_2 = \phi' \begin{pmatrix} \bar{I}_L + R\bar{s}_L \\ \bar{I}_C + R\bar{s}_C \\ \bar{I}_R + R\bar{s}_R \\ 0 \end{pmatrix} + \frac{s_0^2\phi''}{2} \begin{pmatrix} (X_1 + X_2)^2 \\ (-X_1 + X_2)^2 \\ 4X_2^2 \\ 0 \end{pmatrix}$$

1.1.2 Solvability conditions

If we multiply by a vector $v^T \in ker(L_0)$, we obtain:

$$v^T N_2 = \underbrace{v^T L_0 r_1}_{0} + v^T L_1 r_0 = v^T L_1 r_0$$

Thus,

$$v^T N_2 - v^T L_1 r_0 = 0 \implies \langle v^T, N_2 - L_1 r_0 \rangle = 0 \implies \langle v^T, N_2 \rangle = 0$$

In consequence, our solvability conditions will be:

$$\begin{aligned} \langle e_1, L_1 r_1 \rangle &= \langle e_1, N_2 \rangle \\ \langle e_2, L_1 r_1 \rangle &= \langle e_2, N_2 \rangle \end{aligned} \tag{19}$$

This results in the **second order normal form** equations:

$$\begin{aligned} \tau \dot{X}_1 &= \frac{\phi'}{2}(\bar{I}_L - \bar{I}_C + R(\bar{s}_L - \bar{s}_C)) + s_0^2\phi''(X_1X_2) + \frac{1}{2}(\xi_L(t) - \xi_C(t)) \\ \tau \dot{X}_2 &= \frac{\phi'}{6}(\bar{I}_L + \bar{I}_C - 2\bar{I}_R + R(\bar{s}_L + \bar{s}_C - 2\bar{s}_R)) + \frac{s_0^2\phi''}{6}(X_1^2 - 3X_2^2) + \frac{1}{6}(\xi_L(t) + \xi_C(t) - 2\xi_R(t)) \end{aligned} \tag{20}$$

1.1.3 Potential derivation

We can think of this equations as the field in $\mathbb{R}^2$, $F(X_1, X_2) = (F_1, F_2) = (\bar{F}_1, 3\bar{F}_2)$, where

$$\begin{aligned} \bar{F}_1 &= \frac{\phi'}{2}(\bar{I}_L - \bar{I}_C + R(\bar{s}_L - \bar{s}_C)) + s_0^2\phi''(X_1X_2) \\ \bar{F}_2 &= \frac{\phi'}{6}(\bar{I}_L + \bar{I}_C - 2\bar{I}_R + R(\bar{s}_L + \bar{s}_C - 2\bar{s}_R)) + \frac{s_0^2\phi''}{6}(X_1^2 - 3X_2^2) \end{aligned}$$

The field defined by this equations is irrotational as:

$$
\begin{aligned}
\frac{\partial \bar{F}_1}{\partial y} &= X_1 \phi'' s_0^2 \\
\frac{\partial \bar{F}_2}{\partial x} &= \frac{X_1 \phi'' s_0^2}{3}
\end{aligned}
\tag{21}
$$

In consequence, it exists a potential ψ such that $\frac{\partial \psi}{\partial x} = F_1$, $\frac{\partial \psi}{\partial y} = F_2$:

$$\psi(X_1, X_2) = \frac{1}{2}(X_1^2 X_2 \phi'' s_0^2 - X_1 \phi' \left(I_C - I_L + R(s_C - s_L)\right) - X_2^3 \phi'' s_0^2 + X_2 \phi' \left(I_C + I_L - 2I_R + R(s_C + s_L - 2s_R)\right))$$

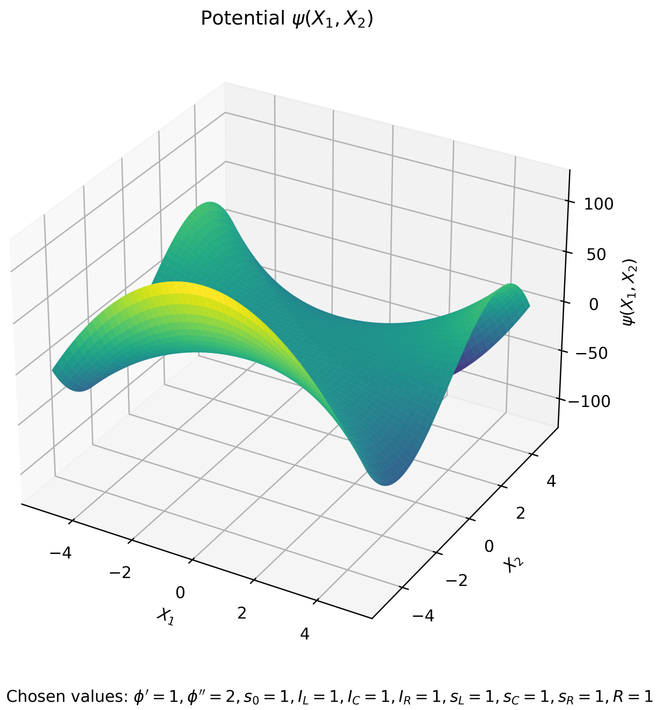

Figure 2: Potential obtained from the order 2 normal equations.

1.2 Derivation of the normal form of order 3

We can go up to third order to obtain more terms in this potential:

Now, r_2 has components in the directions $e_c = \big(1, 1, 1, 0\big)$ and $e_I = \big(0, 0, 0, 1\big)$.

Taking $r_2 = e_C R_{2C} + e_I R_{2I}$:

$$r_2 = \begin{pmatrix} R_{2C} \\ R_{2C} \\ R_{2C} \\ R_{2I} \end{pmatrix}$$

Projecting the equations:

$$
\begin{aligned}
\langle e_c, L_0 r_2 \rangle &= \langle e_c, N_2 \rangle \\
\langle e_I, L_0 r_2 \rangle &= \langle e_I, N_2 \rangle
\end{aligned}
\tag{22}
$$

Then as,

$$L_0 r_2 = \left(c\phi' R_{2I}, c\phi' R_{2I}, c\phi' R_{2I}, -g\phi'_I R_{2C} + R_{2I}\right)$$

We obtain the system of equations:

$$\begin{aligned} 3c\phi' R_{2I} &= \phi'(\bar{I}_L + \bar{I}_C + \bar{I}_R + R(\bar{s}_L + \bar{s}_C + \bar{s}_R)) + \frac{s_0^2\phi''}{2}\left(2X_1^2 + 6X_2^2\right) \\ -g\phi' R_{2C} + R_{2I} &= 0 \end{aligned} \tag{23}$$

Isolating, we find:

$$R_{2C} = \frac{1}{g\phi'_I} R_{2I} \qquad R_{2I} = \frac{1}{3c}(\bar{I}_L + \bar{I}_C + \bar{I}_R + R(\bar{s}_L + \bar{s}_C + \bar{s}_R)) + \frac{s_0^2\phi''}{3c\phi'}(X_1^2 + 3X_2^2)$$

Including r_2 and the terms $\mathrm{O}(\varepsilon^3)$ in ΔX_i:

$$\Delta X_i = \varepsilon[s_0 r_{1,i} - c r_{1,I}] + \varepsilon^2[R\bar{s}_i + s_0 r_{2,i} - c r_{2,I} + \bar{I}] + \varepsilon^3[r_{1,i}\bar{s}_i] \tag{24}$$

The $O(\varepsilon^3)$ term in the Taylor expansion of $\phi(X_i)$ is:

$$\varepsilon^3\left(\phi'(r_{1,i}\bar{s}_i) + \phi''(s_o r_{1,i} - c r_{1,I})(R\bar{s}_i + s_0 r_{2,i} - c r_{2,I} + \bar{I}_i) + \frac{1}{6}\phi'''(s_o r_{1,i} - c r_{1,I})^3\right) \tag{25}$$

Now, the equations up to $O(\varepsilon^3)$ are:

$$L_0 r_3 + L_1 r_2 = N_3$$

Where,

$$N_3 = \phi' \begin{pmatrix} \bar{s}_L(X_1 + X_2) \\ \bar{s}_C(-X_1 + X_2) \\ \bar{s}_R(-2X_2) \\ 0 \end{pmatrix} + \phi'' s_0 \begin{pmatrix} (X_1 + X_2)(R\bar{s}_L + \underbrace{s_0 R_{2C} - c R_{2I}}_{(s_0 - cg\phi'_I)R_{2C}} + \bar{I}_L) \\ (-X_1 + X_2)(R\bar{s}_C + (s_0 - cg\phi'_I)R_{2C} + \bar{I}_C) \\ (-2X_2)(R\bar{s}_R + (s_0 - cg\phi'_I)R_{2C} + \bar{I}_R) \\ 0 \end{pmatrix} + \frac{\phi''' s_0^3}{6} \begin{pmatrix} (X_1 + X_2)^3 \\ (-X_1 + X_2)^3 \\ -8X_2^3 \\ 0 \end{pmatrix}$$

Applying the solvability conditions, we obtain the equations:

$$0 = \langle e_1, N_3 \rangle \quad 0 = \langle e_2, N_3 \rangle \tag{26}$$

Solving, we can add this terms to the second order normal form and obtain the **third order normal form** equations:

$$\begin{aligned} 0 &= \phi'(X_1(\bar{s}_L + \bar{s}_C) + X_2(\bar{s}_L - \bar{s}_C)) \\ &\quad + s_0\phi''\Big(X_1(R(\bar{s}_L + \bar{s}_C) + 2(s_0 - cg\phi'_I)R_{2C} + \bar{I}_L + \bar{I}_C) + X_2\big(R(\bar{s}_L - \bar{s}_C) + \bar{I}_L - \bar{I}_C\big)\Big) \\ &\quad + \frac{\phi''' s_0^3}{3} X_1\left(X_1^2 + 3X_2^2\right) \end{aligned} \tag{27}$$

$$\begin{aligned} 0 &= \phi'(X_1(\bar{s}_L - \bar{s}_C) + X_2(\bar{s}_L + \bar{s}_C + 4\bar{s}_R)) \\ &\quad + s_0\phi''\Big(X_1(R(\bar{s}_L - \bar{s}_C) + \bar{I}_L - \bar{I}_C) + X_2(R(\bar{s}_L + \bar{s}_C + 4\bar{s}_R) + 6(s_0 - cg\phi'_I)R_{2C} + (\bar{I}_L + \bar{I}_C + 4\bar{I}_R))\Big) \\ &\quad + \phi''' s_0^3 X_2\left(X_1^2 + 3X_2^2\right) \end{aligned} \tag{28}$$

Put in terms of X_1, X_2 and adding to the equations that we found at $O(\varepsilon^2)$:

$$\begin{aligned}\tau\dot{X}_1 =& \frac{\phi'}{2}(\bar{I}_L - \bar{I}_C + X_1(\bar{s}_L + \bar{s}_C) + (R + X_2)(\bar{s}_L - \bar{s}_C)) \\ &+ \frac{s_0\phi''}{2}\Big(R(\bar{s}_L + \bar{s}_C) + \bar{I}_L + \bar{I}_C + \frac{2(s_0 - cg\phi'_I)}{3cg\phi'_I}\big(\bar{I}_L + \bar{I}_C + \bar{I}_R + R(\bar{s}_L + \bar{s}_C + \bar{s}_R)\big)\Big)X_1 \\ &+ \frac{s_0\phi''}{2}\big(R(\bar{s}_L - \bar{s}_C) + \bar{I}_L - \bar{I}_C\big)X_2 + s_0^2\phi''(X_1X_2) + \frac{(s_0 - cg\phi'_I)}{3cg\phi'_I}s_0^3(\phi'')^2X_1(X_1^2 + 3X_2^2) \\ &+ \frac{\phi'''s_0^3}{6}X_1\big(X_1^2 + 3X_2^2\big) + \frac{1}{2}(\xi_L(t) - \xi_C(t))\end{aligned} \tag{29}$$

$$\begin{aligned}\tau\dot{X}_2 =& \frac{\phi'}{6}(X_1(\bar{s}_L - \bar{s}_C) + X_2(\bar{s}_L + \bar{s}_C + 4\bar{s}_R) + \bar{I}_L + \bar{I}_C - 2\bar{I}_R + R(\bar{s}_L + \bar{s}_C - 2\bar{s}_R)) \\ &+ \frac{s_0\phi''}{6}\Big(R(\bar{s}_L + \bar{s}_C + 4\bar{s}_R) + (\bar{I}_L + \bar{I}_C + 4\bar{I}_R) + \frac{(s_0 - cg\phi'_I)}{2cg\phi'_I}(\bar{I}_L + \bar{I}_C + \bar{I}_R + R(\bar{s}_L + \bar{s}_C + \bar{s}_R))\Big)X_2 \\ &+ \frac{s_0\phi''}{6}(R(\bar{s}_L - \bar{s}_C) + \bar{I}_L - \bar{I}_C)X_1 + \frac{s_0^2\phi''}{6}(X_1^2 - 3X_2^2) + \frac{(s_0 - cg\phi'_I)}{3cg\phi'_I}s_0^3(\phi'')^2X_2(X_1^2 + 3X_2^2) \\ &+ \frac{\phi'''s_0^3}{6}X_2\big(X_1^2 + 3X_2^2\big) + \frac{1}{6}(\xi_L(t) + \xi_C(t) - 2\xi_R(t))\end{aligned} \tag{30}$$

This system of equations is irrotational, as in the $O(\varepsilon^2)$ case. Thus we can obtain the potential:

$$\begin{aligned}\psi(x,y) =& \frac{1}{24cg\phi'_I}X_1^4s_0^3(-2cg\phi'_I\phi''^2 + cg\phi'_I\phi''' + 2\phi''^2s_0) + 2X_1^2(I_Ccg\phi'_I\phi''s_0 \\ &+ 2I_C\phi''s_0^2 + I_Lcg\phi'_I\phi''s_0 + 2I_L\phi''s_0^2 - 2I_Rcg\phi'_I\phi''s_0 + 2I_R\phi''s_0^2 + Rcg\phi'_I\phi''s_0s_C \\ &+ Rcg\phi'_I\phi''s_0s_L - 2Rcg\phi'_I\phi''s_0s_R + 2R\phi''s_0^2s_C + 2R\phi''s_0^2s_L + 2R\phi''s_0^2s_R \\ &- 6X_2^2cg\phi'_I\phi''^2s_0^3 + 3X_2^2cg\phi'_I\phi'''s_0^3 + 6X_2^2\phi''^2s_0^4 + 6X_2cg\phi'_I\phi''s_0^2 \\ &+ 3cg\phi'_I\phi's_C + 3cg\phi'_I\phi's_L) + 9X_2^4s_0^3(-2cg\phi'_I\phi''^2 + cg\phi'_I\phi''' + 2\phi''^2s_0) \\ &+ 3X_2^2(I_Ccg\phi'_I\phi''s_0 + I_C\phi''s_0^2 + I_Lcg\phi'_I\phi''s_0 + I_L\phi''s_0^2 + 7I_Rcg\phi'_I\phi''s_0 \\ &+ I_R\phi''s_0^2 + Rcg\phi'_I\phi''s_0s_C + Rcg\phi'_I\phi''s_0s_L + 7Rcg\phi'_I\phi''s_0s_R + R\phi''s_0^2s_C \\ &+ R\phi''s_0^2s_L + R\phi''s_0^2s_R + 2cg\phi'_I\phi's_C + 2cg\phi'_I\phi's_L + 8cg\phi'_I\phi's_R) + 12cg\phi'_I(-X_1(I_CX_2\phi''s_0 \\ &+ I_C\phi' - I_LX_2\phi''s_0 - I_L\phi' + RX_2\phi''s_0s_C - RX_2\phi''s_0s_L + R\phi's_C - R\phi's_L + X_2\phi's_C \\ &- X_2\phi's_L) - X_2^3\phi''s_0^2 + X_2\phi'(I_C + I_L - 2I_R + Rs_C + Rs_L - 2Rs_R))\end{aligned} \tag{31}$$