A Generalized Theory of Gravitation

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IN the following we shall give a new presentation of the generalized theory of gravitation, which constitutes a certain progress in clarity as compared to the previous presentations.* It is our aim to achieve a theory of the total field by a generalization of the concepts and methods of the relativistic theory of gravitation.

1. THE FIELD STRUCTURE

The theory of gravitation represents the field by a symmetric tensor g_{ik}, i.e., $g_{ik}=g_{ki}(i, k=1, \cdots, 4)$, where the g_{ik} are real functions of $x_1, \cdots, x_4$.

In the generalized theory the total field is represented by a Hermitian tensor. The symmetry property of the (complex) g_{ik} is

$$g_{ik}=\overline{g_{ki}}.$$

If we decompose g_{ik} into its real and imaginary components, then the former is a symmetric tensor ($g_{\underline{ik}}$), the latter an antisymmetric tensor ($g_{\underset{\vee}{ik}}$). The g_{ik} are still functions of the real variables $x_1, \cdots, x_4$.

The formally natural character of this generalization of the symmetric tensor becomes particularly noticeable by the following consideration: From the covariant vector A_i one can form through multiplication the particular symmetric covariant tensor A_iA_k. From such tensors every symmetric tensor of rank 2 can be obtained through summation with real coefficients:

$$g_{ik}=\sum_{\alpha} \underset{\alpha}{c}\underset{\alpha}{A_i}\underset{\alpha}{A_k}.$$

In an analogous manner we form from a complex vector A_i the special Hermitian tensor $A_i\overline{A_k}$ (remains fixed if we interchange i and k and take the complex conjugate). We then get the representation of a general Hermitian tensor of rank 2,

$$g_{ik}=\sum_{\alpha} \underset{\alpha}{c}\underset{\alpha}{A_i}\overline{\underset{\alpha}{A_k}},$$

where the $\underset{\alpha}{c}$ are again real constants.

The determinant $g=|g_{ik}|(\neq 0)$ is real.

Proof:

$$|g_{ik}|=|g_{ki}|=|\overline{g_{ik}}|=\overline{|g_{ik}|}.$$

We can associate a contravariant g^{ik} to the covariant g_{ik} just as in the case of real fields by setting

$$g_{is}g^{ls}=\delta_i{}^l \quad (\text{or} \quad g_{si}g^{sl}=\delta_i{}^l),$$

where $\delta_i{}^l$ is the Kronecker tensor. Here the order of indices is important and, for example, $g_{is}g^{sl}$ does *not* equal $\delta_i{}^l$. In the following the tensor density $\mathfrak{g}^{ik}=g^{ik}(g)^{\frac{1}{2}}$ plays an important role.

From a group theoretical point of view the introduction of a Hermitian tensor is somewhat arbitrary, since both individual additive components $g_{\underline{ik}}$ and $g_{\underset{\vee}{ik}}$ have tensor character. However, this flaw is somewhat ameliorated by the fact that, just as in the case of real fields, there is a natural way of associating parallel translations to the Hermitian g_{ik}; this is the main basis for the claim that the introduction of a Hermitian g_{ik} is natural.

2. INFINITESIMAL PARALLEL TRANSLATIONS, ABSOLUTE DIFFERENTIATION AND CURVATURE

In the theory of real fields we give the infinitesimal parallel translation of a vector A^i or A_i by

$$\left.\begin{aligned}\delta A^i &= -\Gamma^i{}_{st}A^s dx^t\\ \delta A_i &= \quad\Gamma^s{}_{it}A_s dx_t\end{aligned}\right\} \tag{1}$$

with a corresponding introduction of infinitesimal parallel translations for tensors of higher rank.

The second equation of (1) is connected with the first by the demand that

$$0=\delta(\delta^k{}_i)=(\delta^s{}_i\Gamma^k{}_{sl}-\delta^k{}_s\Gamma^s{}_{il})dx^l.$$

* A. Einstein, "A generalization of the relativistic theory of gravitation," Ann. Math. **46** (1945); A. Einstein and E. G. Straus, "A generalization of the relativistic theory of gravitation II," Ann. Math. **47** (1946).

From (1) we get in the well-known manner the tensor character of

$$dA^i - \delta A^i = \left(\frac{\partial A^i}{\partial x_t} + A^s \Gamma^i{}_{st}\right) dx^t,$$

which yields the concept of covariant differentiation

$$\left.\begin{aligned} A^i{}_{;t} &= \frac{\partial A^i}{\partial x_t} + A^s \Gamma^i{}_{st} \\ A_{i;t} &= \frac{\partial A_i}{\partial x_t} - A_s \Gamma^s{}_{it} \end{aligned}\right\}. \tag{2}$$

In order to obtain the covariant derivative of g_{ik} we write

$$A_{i;l} = \frac{\partial A_i}{\partial x_l} - A_s \Gamma^s{}_{il},$$

$$A_{k;l} = \frac{\partial A_k}{\partial x_l} - A_s \Gamma^s{}_{kl},$$

multiplying the first equation by A_k, the second by A_i and adding we get

$$\begin{aligned} A_i A_{k;l} + A_k A_{i;l} &= (A_i A_k)_{;l} \\ &= (A_i A_k)_{,l} - (A_s A_k)\Gamma^s{}_{il} - (A_i A_s)\Gamma^s{}_{kl}, \end{aligned}$$

and since g_{ik} can be constructed as the sum of such special tensors we get

$$g_{ik;l} = g_{ik,l} - g_{sk}\Gamma^s{}_{il} - g_{is}\Gamma^s{}_{kl}.$$

The Γ are now determined from the g and their first derivatives by the demand that the absolute derivative of the g_{ik} vanish

$$0 = g_{ik;l} - g_{sk}\Gamma^s{}_{il} - g_{is}\Gamma^s{}_{kl}. \tag{3}$$

However, since the g_{ik} are symmetric, these are only 40 equations for the 64 Γ. In order to complete the determination of the Γ one uses the only possible invariant algebraic condition, namely, the condition of symmetry

$$\Gamma^l{}_{ik} = \Gamma^l{}_{ki}. \tag{4}$$

We now transfer this development to the complex case by defining parallel translation as in (1). However, this gives rise to a certain complication, since if we start from the translation of a complex vector,

$$\delta A^i = \Gamma^s{}_{it} A_s dx^t,$$

where the Γ will, in general, also be complex, and pass to the complex conjugate of this equation

$$\overline{\delta A_i} = \overline{\Gamma^s{}_{it}}\,\overline{A_s} dx^t,$$

then we see that we have there an equation which also defines a parallel translation, but this parallel translation may differ from the first. We define then two kinds of parallel translation

$$\left.\begin{aligned} \delta A^{i}_{+} &= -\Gamma^i{}_{st} A^s dx^t \\ \delta A_{\underset{+}{i}} &= \Gamma^s{}_{it} A_s dx^t \end{aligned}\right\} \tag{1a}$$

and

$$\left.\begin{aligned} \delta A^{i}_{-} &= -\overline{\Gamma^i{}_{st}} A^s dx^t \\ \delta A_{\underset{-}{i}} &= \overline{\Gamma^s{}_{it}} A_s dx^t \end{aligned}\right\} \tag{1b}$$

and, correspondingly, two kinds of covariant differentiation $A^{i}_{+}{}_{;t}$, $A_{\underset{+}{i};t}$, and $A^{i}_{-}{}_{;t}$, $A_{\underset{-}{i},t}$ as in (2). From (1a) and (1b) we get

$$\delta\overline{A^{i}_{-}} = \overline{\delta A^{i}_{+}} \quad \text{and} \quad \delta\overline{A^{i}_{+}} = \overline{\delta A^{i}_{-}}$$

In order that conjugate vectors have conjugate translations and derivatives it is necessary upon passage to the conjugate to change the character of translation or of differentiation, i.e., *to pass to the conjugate* Γ. In order to obtain the covariant derivative of a Hermitian tensor we write in analogy to the real case:

$$A_{\underset{+}{i};l} = \frac{\partial A_i}{\partial x_l} - A_s \Gamma^s{}_{il},$$

$$\overline{A_{\underset{-}{k};l}} = \frac{\partial \overline{A_k}}{\partial x_l} - \overline{A_s}\,\overline{\Gamma^s{}_{kl}}.$$

From this we get as before

$$\begin{aligned} A_i \overline{A_{\underset{-}{k}}}{}_{;l} + \overline{A_k} A_{\underset{+}{i};l} &= (A_{\underset{+}{i}} A_{\underset{-}{k}})_{;l} \\ &= (A_i A_k)_{,l} - (A_s \overline{A_k})\Gamma^s{}_{il} - (A_i \overline{A_s})\overline{\Gamma^s{}_{kl}}, \end{aligned}$$

and since g_{ik} can be constructed as the sum of such special tensors we get

$$g_{\underset{+}{i}\underset{-}{k};l} = g_{ik,l} - g_{sk}\Gamma^s{}_{il} - g_{is}\overline{\Gamma^s{}_{kl}}.$$

The analog to (3) is the requirement that this absolute derivative vanish

$$0 = g_{\underset{+}{i}\underset{-}{k};l} = g_{ik,l} - g_{sk}\Gamma^s{}_{il} - g_{is}\overline{\Gamma^s{}_{kl}}. \tag{3a}$$

These equations are Hermitian in the indices i, k (go into themselves if we interchange i, k and pass to the conjugate complex) and therefore again do not suffice to determine the complex Γ. In analogy to (4) we have as the only possible invariant algebraic determination the condition of Hermiticity

$$\Gamma^l{}_{ik}=\overline{\Gamma^l{}_{ki}}. \tag{4a}$$

Instead of (3a) we can then write

$$0=g_{\underset{+}{i}\underset{-}{k};l}=g_{ik,\,l}-g_{sk}\Gamma^s{}_{il}-g_{is}\Gamma^s{}_{lk}, \tag{3b}$$

which implies both (3a) and (4a).

Absolute differentiation of vector densities. If we multiply (3b) by $\frac{1}{2}g^{ik}$ and sum over i and k, then we get the vector equation

$$\frac{1}{(g)^{\frac{1}{2}}}\frac{\partial(g)^{\frac{1}{2}}}{\partial x_l}-\tfrac{1}{2}(\Gamma^a{}_{al}+\Gamma^a{}_{la})=0,$$

or shorter

$$\frac{\partial(g)^{\frac{1}{2}}}{\partial x_l}-(g)^{\frac{1}{2}}\Gamma^a{}_{\underline{la}}=0. \tag{3c}$$

$(g)^{\frac{1}{2}}$ is a scalar density, the left side of (3c) is a vector density. The latter will also hold if $(g)^{\frac{1}{2}}$ is replaced by an arbitrary scalar density ρ. We may therefore introduce as the absolute derivative of a scalar density ρ:

$$\rho_{;l}=\rho_{,\,l}-\rho\Gamma^a{}_{\underline{la}}. \tag{5}$$

This permits us to introduce absolute differentiation of tensor densities.

Example: If we multiply the right side of the equation

$$A^{\underset{+}{i}}{}_{;l}=A^i{}_{,\,l}+A^s\Gamma^i{}_{sl}$$

by a scalar density ρ, then we get the tensor density

$$(\rho A^i)_{,\,l}+(\rho A^s)\Gamma^i{}_{sl}-A^i\rho_{,\,l}$$

or, after introducing the vector density $\mathfrak{A}^i=\rho A^i$

$$\mathfrak{A}^i{}_{,\,l}+\mathfrak{A}^s\Gamma^i{}_{sl}-\mathfrak{A}^i\frac{\rho_{,\,l}}{\rho},$$

or according to (5)

$$(\mathfrak{A}^i{}_{,\,l}+\mathfrak{A}^s\Gamma^i{}_{sl}-\mathfrak{A}^i\Gamma^a{}_{\underline{la}})-\mathfrak{A}^i\rho_{;l}.$$

Since the last term is a tensor density, the term in brackets is also a tensor density which we may define as the absolute derivative $\mathfrak{A}^{\underset{+}{i}}{}_{;l}$ of a vector density $\mathfrak{A}^i$:

$$\mathfrak{A}^{\underset{+}{i}}{}_{;l}=\mathfrak{A}^i{}_{,\,l}+\mathfrak{A}^s\Gamma^i{}_{sl}-\mathfrak{A}^i\Gamma^a{}_{\underline{la}}. \tag{6}$$

In an analogous manner we may define the absolute derivatives of arbitrary tensor densities. They differ from the absolute derivative of the tensor by a last term like $-\mathfrak{A}^i\Gamma^a{}_{\underline{la}}$.

Just as in the case of real fields we can bring (3a) into a contravariant form; however, we have to be careful about the order of indices. We obtain the equivalent equations

$$0=g^{\underset{+}{i}\underset{-}{k}}{}_{;l}=g^{ik}{}_{,\,l}+g^{sk}\Gamma^i{}_{sl}+g^{is}\Gamma^k{}_{ls} \tag{3d}$$

or, after introducing the contravariant tensor density, $\mathfrak{g}^{ik}=g^{ik}(g)^{\frac{1}{2}}$

$$0=\mathfrak{g}^{\underset{+}{i}\underset{-}{k}}{}_{;l}=\mathfrak{g}^{ik}{}_{,\,l}+\mathfrak{g}^{sk}\Gamma^i{}_{sl}+\mathfrak{g}^{is}\Gamma^k{}_{ls}-\mathfrak{g}^{ik}\Gamma^s{}_{\underline{ls}}. \tag{3e}$$

The Eqs. (3a), (3d), and (3e) are equivalent.

Curvature: The change which a vector undergoes upon parallel translation around the boundary curve of an infinitesimal element of area has vector character. This leads to the formation of a curvature tensor also in the case of our generalized field. We have here the choice whether to use a "+" translation or a "−" translation; however, the results of the two translations are conjugate complex, so that it suffices to consider *one* form.

We obtain the tensor

$$R^i{}_{klm}=\Gamma^i{}_{kl,\,m}-\Gamma^i{}_{km,\,l}-\Gamma^i{}_{al}\Gamma^a{}_{km}+\Gamma^i{}_{am}\Gamma^a{}_{kl}, \tag{7}$$

and the corresponding contracted tensor (contraction with respect to i and m)

$$R^*{}_{kl}=\Gamma^a{}_{kl,\,a}-\Gamma^a{}_{ka,\,l}-\Gamma^a{}_{kb}\Gamma^b{}_{al}+\Gamma^a{}_{kl}\Gamma^b{}_{ab}. \tag{8}$$

There also exists a non-vanishing contraction with respect to i and k which yields the tensor

$$\Gamma^a{}_{al,\,m}-\Gamma^a{}_{am,\,l}. \tag{9}$$

However, we shall not use this tensor as we shall justify later. The tensor $R^*{}_{kl}$ is not Hermitian. We form the Hermitian tensor $R_{ik}=\frac{1}{2}(R^*{}_{ik}+\overline{R^*{}_{ki}})$. We thus get

$$R_{ik}=\Gamma^a{}_{ik,\,a}-\tfrac{1}{2}(\Gamma^a{}_{ia,\,k}+\Gamma^a{}_{ak,\,i})-\Gamma^a{}_{ib}\Gamma^b{}_{ak}+\Gamma^a{}_{ik}\Gamma^b{}_{\underline{ab}}. \tag{8a}$$

3. HAMILTONIAN PRINCIPLE. FIELD EQUATIONS

In the case of the real symmetric field one obtains the field equations most simply in the following manner. We use as Hamilton function the scalar density

$$\mathfrak{H}=\mathfrak{g}^{ik}R_{ik}. \tag{10}$$

If we vary the volume integral of $\mathfrak{H}$ independently with respect to Γ and $\mathfrak{g}$, then (in the case of real fields) variation with respect to Γ yields Eq. (3), and variation with respect to $\mathfrak{g}$ yields the equations $R_{ik}=0$. If we apply the same method to our case of a complex field (where $\mathfrak{H}$ is still real) then we see a complication, since the variation with respect to Γ does not immediately yield Eq. (3a), which we wish to keep in any case. The variation with respect to Γ yields

$$\begin{aligned}
-\{\mathfrak{g}^{ik}{}_{,a}+\mathfrak{g}^{sk}\Gamma^{i}{}_{sa}+\mathfrak{g}^{is}\Gamma^{k}{}_{as}-\mathfrak{g}^{ik}\Gamma^{b}{}_{\underline{ab}}\}&\\
+\tfrac{1}{2}\{\mathfrak{g}^{is}{}_{,s}+\mathfrak{g}^{st}\Gamma^{i}{}_{st}-\mathfrak{g}^{is}\Gamma^{a}{}_{\underset{\vee}{sa}}\}\delta_a{}^k&\\
+\tfrac{1}{2}\{\mathfrak{g}^{sk}{}_{,s}+\mathfrak{g}^{st}\Gamma^{k}{}_{st}+\mathfrak{g}^{sk}\Gamma^{a}{}_{\underset{\vee}{sa}}\}\delta_a{}^i&\\
+\tfrac{1}{2}\{\mathfrak{g}^{is}\Gamma^{a}{}_{\underset{\vee}{sa}}\delta_a{}^k-\mathfrak{g}^{sk}\Gamma^{a}{}_{\underset{\vee}{sa}}\delta_a{}^i\}.&
\end{aligned} \tag{11}$$

The first bracket is $\mathfrak{g}^{\,i\,k}_{+\,-;a}$; the second and third brackets are contractions of this quantity. If there were no fourth bracket then (11) would imply the vanishing of $\mathfrak{g}^{\,i\,k}_{+\,-;a}$, that is, (3a). However, this would require the vanishing of $\Gamma^{a}{}_{\underset{\vee}{sa}}$ to which demand we have no right for the time being.

We can resolve this difficulty in the following manner. We form the imaginary part of (11):

$$\begin{aligned}
-\mathfrak{g}^{\underset{\vee}{ik}}{}_{,a}-\mathfrak{g}^{\underline{sk}}\Gamma^{i}{}_{\underset{\vee}{sa}}-\mathfrak{g}^{\underset{\vee}{sk}}\Gamma^{i}{}_{\underline{sa}}&\\
-\mathfrak{g}^{\underline{is}}\Gamma^{k}{}_{\underset{\vee}{as}}-\mathfrak{g}^{\underset{\vee}{is}}\Gamma^{k}{}_{\underline{as}}+\mathfrak{g}^{\underset{\vee}{ik}}\Gamma^{b}{}_{\underline{ab}}&\\
+\tfrac{1}{2}\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}\,\delta_a{}^k+\tfrac{1}{2}\mathfrak{g}^{\underset{\vee}{sk}}{}_{,s}\,\delta_a{}^i=0.&
\end{aligned}$$

If we contract this equation with respect to k and a we get

$$\tfrac{1}{2}\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}+\mathfrak{g}^{\underline{is}}\Gamma^{a}{}_{\underset{\vee}{sa}}=0. \tag{11a}$$

From this we can deduce that the necessary and sufficient** condition for the vanishing of the $\Gamma^{s}{}_{\underset{\vee}{is}}$ is the vanishing of the $\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}$. In order to satisfy this *identically* it suffices to assume

$$\mathfrak{g}^{\underset{\vee}{is}}=\mathfrak{g}^{ist}{}_{,t}, \tag{12}$$

where $\mathfrak{g}^{ist}$ is a tensor density which is antisymmetric in all three indices. That is, we require that $\mathfrak{g}^{\underset{\vee}{is}}$ be derived from a "vector potential." We therefore substitute in the Hamilton function

$$\mathfrak{g}^{ik}=\mathfrak{g}^{\underline{ik}}+\mathfrak{g}^{ikl}{}_{,l} \tag{13}$$

and vary independently with respect to the Γ, $\mathfrak{g}^{\underline{ik}}$ and $\mathfrak{g}^{ikl}$. The variation with respect to the Γ then yields (3a), as we have shown. The variation with respect to the other quantities yields the equations

$$R_{\underline{ik}}=0, \tag{14}$$

$$R_{\underset{\vee}{ik},l}+R_{\underset{\vee}{kl},i}+R_{\underset{\vee}{li},k}=0. \tag{15}$$

In addition, we have the equations

$$\mathfrak{g}^{\,i\,k}_{+\,-;l}=0\quad\text{or}\quad g_{\underset{+\,-}{i\,k};l}=0, \tag{3a}$$

$$\Gamma^{s}{}_{\underset{\vee}{is}}=0, \tag{16}$$

$$\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}=0\quad\text{or}\quad\mathfrak{g}^{\underset{\vee}{is}}=\mathfrak{g}^{ist}{}_{,t}. \tag{17}$$

Considering (3a), each of the systems (16), (17) implies the other; this is proven by showing that (3a) implies the equation

$$\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}-\mathfrak{g}^{\underline{is}}\Gamma^{t}{}_{\underset{\vee}{st}}=0.$$

The system of field equations is therefore not weakened if we omit (17).

This is worth mentioning also for the following reason. While in the given derivation of the equations, special emphasis is given to the density $\mathfrak{g}^{ik}$ rather than to the tensor g_{ik} (or g^{ik}), the resulting system itself is free of such discrimination.

We now see that because of (16) the tensor (9) reduces to $\Gamma^{a}{}_{\underline{al},m}-\Gamma^{a}{}_{\underline{am},l}$, which vanishes because of Eq. (3c).

The derivation used here has the advantage, as compared to the previous one, that the Hamiltonian principle used is one without side conditions. This behavior is the same as that encountered in a (specially relativistic) derivation of Maxwell's equations from a variational principle. There (for imaginary time coordinate) the Hamiltonian function is $\mathfrak{H}=\varphi_{\underset{\vee}{ik}}\varphi_{\underset{\vee}{ik}}$. If we set here $\varphi_{\underset{\vee}{ik}}=\varphi_{i,k}-\varphi_{k,i}$ and vary with respect

** This holds for all points if we demand that the Γ be continuous and determined uniquely by the equations (3b); because then the determinant $|\mathfrak{g}^{\underline{is}}|$ can vanish nowhere.

to the φ_i, then we get the one system of equations ($\varphi_{\underset{\vee}{ik},k}=0$) directly, the other through elimination of the φ_i. This method corresponds to the one used above. One may, however, avoid the introduction of the potentials φ_i and instead adjoin the system of equations

$$\varphi_{\underset{\vee}{ik},l}+\varphi_{\underset{\vee}{kl},i}+\varphi_{\underset{\vee}{li},k}=0$$

as side conditions for the $\varphi_{\underset{\vee}{ik}}$ in the variation. This corresponds to the treatment of $\mathfrak{g}^{\underset{\vee}{is}}{}_{,s}=0$ as side condition for the variation in the previous paper. The side condition $\Gamma^s{}_{\underset{\vee}{is}}=0$ which was introduced there could have been omitted.

REMARKS

In order to preserve the special character of locally space-like and time-like directions it is essential that the index of inertia of $g_{\underline{ik}}dx^idx^k$ be the same everywhere, i.e., that the determinant $|g_{\underline{ik}}|$ vanish nowhere. This can indeed be deduced from the requirement that the Γ-field be finite and determined everywhere by Eq. (3a). My assistant has given the following simple proof of this:

If the determinant $|g_{\underline{ik}}|$ should vanish in a point P then there would exist a vector ξ^s different from 0, such that $g_{\underline{is}}\xi^s=0$. We now consider the real part of Eq. (3a):

$$g_{\underline{ik},l}-g_{\underline{sk}}\Gamma^s{}_{\underline{il}}-g_{\underline{is}}\Gamma^s{}_{\underline{lk}}-g_{\underset{\vee}{sk}}\Gamma^s{}_{\underset{\vee}{il}}-g_{\underset{\vee}{is}}\Gamma^s{}_{\underset{\vee}{lk}}=0.$$

If we multiply this equation (at the point P) by $\xi^i\xi^k\xi^l$ and sum over i, k, l, then the second and third terms vanish by definition of ξ, and the fourth and fifth because of the antisymmetry of the Γ. There exists, therefore, a linear combination of Eq. (3a) which does not contain the Γ. Hence at such a point the Γ either become infinite or not completely determined, in contradiction to our requirement.

Concerning the physical interpretation we remark that the antisymmetric density $\mathfrak{g}^{ikl}$ plays the role of an electromagnetic vector potential, the tensor $g_{\underset{\vee}{ik},l}+g_{\underset{\vee}{kl},i}+g_{\underset{\vee}{li},k}$ the role of current density. The latter quantity is the "complement" of a contravariant vector density with (identically) vanishing divergence.

Above we have used complex fields. However, there exists a theoretical possibility in which the g_{ik} and $\Gamma^l{}_{ik}$ are real though not symmetric. Thus one can obtain a theory which in its final formulas corresponds, except for certain signs, to the one developed above. E. Schrödinger, too, has based his affine theory (i.e., based on the Γ as fundamental field quantities) on real fields. I therefore wish to give here some formal reasons for the preferability of complex fields.

A Hermitian tensor g_{ik} can be constructed additively from vectors according to the scheme $g_{ik}=\sum_{\alpha}\underset{\alpha}{c}\,\underset{\alpha}{A_i}\,\underset{\alpha}{\overline{A_k}}$. The essential fact here is that with the use of *one* complex vector A_i one can construct the Hermitian tensor $A_i\overline{A_k}$ through multiplication, which is a close analogy to the case of symmetric real fields. A nonsymmetric real tensor cannot be constructed from vectors in such close analogy.

We now consider translation quantities $\Gamma^l{}_{ik}$ which are not symmetric in the lower indices. To them we have in both the real and the complex cases the adjoined ("conjugate") translation quantities $\tilde{\Gamma}^l{}_{ik}=\Gamma^l{}_{ki}$. In the complex case we have associated with the parallel translation of a vector

$$\delta A^i=-\Gamma^i{}_{st}A^sdx^t$$

the parallel translation of its conjugate complex vector

$$\delta\overline{A^i}=-\overline{\Gamma^i{}_{st}}\overline{A^s}dx^t.$$

Hence in the case of complex fields the adjoined translation corresponds to adjoined objects, while in the case of real fields there is no such adjoined object.