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Relativistic Cut-Off for Quantum Electrodynamics

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A relativistic cut-off of high frequency quanta, similar to that suggested by Bopp, is shown to produce a finite invariant self-energy for a free electron. The electromagnetic line shift for a bound electron comes out as given by Bethe and Weisskopf's wave packet prescription. The scattering of an electron in a potential, without radiation, is discussed. The cross section remains finite. The problem of polarization of the vacuum is not solved. Otherwise, the results will in general agree essentially with those calculated by the prescription of Schwinger. An alternative cut-off procedure analogous to one proposed by Wataghin, which eliminates high frequency intermediate states, is shown to do the same things but to offer to solve vacuum polarization problems as well.

THE main problems of quantum electrodynamics have been essentially solved by the observations of Bethe[1] and of Weisskopf[2] that the divergent terms in the line shift problem can be thought to be contained in a renormalization of the mass of a free electron. That this principle applies as well to other problems was demonstrated by Lewis[3] in analyzing the radiationless scattering of an electron in a potential. Ambiguities which remained in the subtraction procedures are removed by Schwinger.[2,4] He formulated, in a general way, which terms are to be identified in a future correct theory with rest mass, and hence should be omitted from a calculation which does not renormalize the mass. These results are remarkable because they solve the problem without the addition of any new fundamental lengths or dimensions.

The solution given by Schwinger does, however, assume that in some future theory the divergent self-energy terms will be finite. Therefore, it is of interest to point out that there is a model, a modification of ordinary electrodynamics, for which all quantities automatically do come out finite. With this model the ideas of Bethe, Oppenheimer, and Lewis and Schwinger can be directly confirmed.

The model results from the quantization of a classical theory described in a previous paper.[5] In this paper we describe only the results for processes in which only virtual quanta are emitted and absorbed. The problems of permanent emission and the position of positron theory must be more completely studied. It is hoped that a complete physical theory may be published in the near future. Lacking such a complete picture, the present paper may be looked upon merely as presenting an arbitrary rule to cut off at high frequencies in a relativistically invariant manner, the otherwise divergent integrals appearing in quantum field theories. For electrodynamics the rule is to consider the (positive) frequency ω and wave number $\mathbf{k}$ of the field oscillators as independent and to integrate them over the density function $g(\omega^2-k^2)d\omega d\mathbf{k}$ where

$$g(\omega^2-k^2)=\int_0^\infty [\delta(\omega^2-k^2) - \delta(\omega^2-k^2-\lambda^2)]G(\lambda)d\lambda. \quad (1)$$

Here $\delta(x)$ is Dirac's delta function and $G(\lambda)$ is some smooth function such that $\int_0^\infty G(\lambda)d\lambda=1$ and for which the mean values of λ which are important are of order of the frequency 137 $mc^2/\hbar$, or higher. Ordinary quantum electrodynamics replaces the function $g(\omega^2-k^2)$ by $\delta(\omega^2-k^2)$. According to (1), the density g is not everywhere positive.[5] Therefore, the model is essentially that due to Bopp.[6]

The model therefore contains an arbitrary function and the numerical results depend on the

[1] H. A. Bethe, Phys. Rev. **72**, 339 (1947); **73**, 1271A (1948).
[2] J. Schwinger and V. Weisskopf, Phys. Rev. **73**, 1272A (1948).
[3] H. W. Lewis, Phys. Rev. **73**, 173 (1948).
[4] J. Schwinger, Phys. Rev. **73**, 415A (1948).
[5] R. P. Feynman, Phys. Rev. **74**, 939 (1948).
[6] F. Bopp, Ann. d. Physik **42**, 573 (1942).

form of $G(\lambda)$. However, the only term that depends seriously (logarithmically) on the cut-off frequency is the self-energy, which can be used to renormalize the electron mass. After this is done, the remaining terms are nearly independent of the function $G(\lambda)$.

We shall illustrate these points by studying the particular examples of self-energy and radiationless scattering. We shall then discuss an alternative cut-off procedure in which the density of electron states is cut off rather than that of the quanta. This promises to solve problems of vacuum polarization which are not touched by the former procedure.

SELF-ENERGY

The transverse self-energy of a free electron, of mechanical mass μ, in state of momentum $\mathbf{P}_0$ energy $E_0=(\mu^2+P_0^2)^{\frac{1}{2}}$ is given to the first order in e^2 by the second-order perturbation theory, using the one-electron theory of Dirac, by

$$\Delta E=-\frac{e^2}{4\pi^2}\sum_i\int\frac{d\mathbf{k}}{k}\Big\{\sum_+\frac{(0|\alpha_i|f)(f|\alpha_i|0)}{E_f-E_0+k}+\sum_-\frac{(0|\alpha_i|f)(f|\alpha_i|0)}{-E_f-E_0+k}\Big\}. \quad (2)$$

Here the intermediate state f arises from the initial state through emission of a quantum of momentum $\mathbf{k}$ and of energy $k=|\mathbf{k}|$ (the velocity of light is taken as unity, as is Planck's constant). Thus in the intermediate state the electron has momentum $\mathbf{P}_f=\mathbf{P}_0-\mathbf{k}$ and an energy of magnitude $E_f=+(\mu^2+P_f^2)^{\frac{1}{2}}$ but which may be either plus or minus in sign. The sums indicate the sum over all such intermediate states (actually just two) for each sign of the energy. The terms for positive and negative energy have been separated and the sums are written $\sum_+$ and $\sum_-$ for these two cases. The $(f|\alpha_i|0)$ are the matrix elements of Dirac's α-matrices, the sum on i being over the two directions of polarization of the quanta. We shall henceforth write the integral $d\mathbf{k}/k$ over $\mathbf{k}$ space by its equivalent $2\int d\omega d\mathbf{k}\delta(\omega^2-k^2)$, the integral being over all *positive* ω, and all wave numbers $\mathbf{k}$. We shall also write ω for k in the energy denominators as we shall later wish to distinguish the energy of a quantum and the magnitude of the momentum change that its recoil represents. We may further simplify the expression by the use of the well-known projection operators:

$$\Lambda_f^{\pm}=(E_f\pm H_f)/2E_f=(E_f\pm\boldsymbol{\alpha}\cdot\mathbf{P}_f\pm\beta\mu)/2E_f.$$

According to the theory of holes, the last term, the transition to negative energy states, is to be left out; such transitions are prevented because the negative levels are already occupied. On the other hand, in the vacuum, electrons in state of energy $-E_f$ could make virtual transitions to positive energy state E_0. This is now prevented by the presence of an electron in the state E_0, so that, relative to the vacuum, the transverse self-energy is

$$\Delta E=-\frac{e^2}{2\pi^2}\sum_i\int d\omega d\mathbf{k}\delta(\omega^2-k^2)\times\Big\{\frac{(0|\alpha_i\Lambda_f{}^+\alpha_i|0)}{E_f-E_0+\omega}-\frac{(0|\alpha_i\Lambda_f{}^-\alpha_i|0)}{E_f+E_0+\omega}\Big\}. \quad (3)$$

The treatment of the longitudinal self-energy is usually different, for the longitudinal oscillators are first eliminated from the Hamiltonian, their effect being the term e^2/r_{00} where r_{00} is the meaningless distance of the electron from itself. These terms must be expressed as integrals over oscillators and combined with (3) before the change suggested by (1) is to be performed. An additional point of confusion is that the longitudinal elimination assumes the intermediate states to form a complete set as they do in (2), but the situation in (3) is not so clear. Fortunately, all these points may be most easily circumvented by simply not eliminating the longitudinal oscillators from the field Hamiltonian at all. One need simply to specify that the sum on i in (3) now be interpreted to mean the sum over each of three perpendicular space directions minus a term for the time direction. We may write $\sum_i\alpha_i\Lambda\alpha_i=\boldsymbol{\alpha}\cdot\Lambda\boldsymbol{\alpha}-\Lambda$, which is a relativistic combination since $\alpha_4=1$. One does not need to be concerned about the gauge condition in a problem in which all quanta are virtual, for the quanta are created by a charge which is conserved. This solution automatically insures the gauge condition just as the Lienard Wiechert classical solution of the Maxwell equations will automatically satisfy the gauge

condition if the charge which produces the potential is conserved.

With this convention for $\sum_i$, Eq. (3) represents the total self-energy. It is easily calculated. The numerator of first term may be written as $1/2E_f$ times $\sum_i(0|\alpha_i(H_f+E_f)\alpha_i|0)$ where H_f is $\boldsymbol{\alpha}\cdot\mathbf{P}_f+\beta\mu$. Now since $\sum_i \alpha_i\alpha_i=+2$, $\sum_i \alpha_i\beta\alpha_i=-4\beta$, and $\sum_i \alpha_i\boldsymbol{\alpha}\alpha_i=-2\boldsymbol{\alpha}$, this becomes

$$-2(0|-E_f+2\beta\mu+\boldsymbol{\alpha}\cdot\mathbf{P}_f|0).$$

The diagonal elements of β and $\boldsymbol{\alpha}$ for the state 0 are μ/E_0 and $\mathbf{P}_0/E_0$, respectively.

The change in energy ΔE_0 can, since the momentum is given, be represented as a change $\Delta\mu$ in rest mass of the electron. In virtue of the general relation $E^2=\mu^2+P^2$, the relation between these quantities is $\mu\Delta\mu=E_0\Delta E_0$. Thus we find, treating the sum of negative energies in a similar manner,

$$\Delta\mu_0=\frac{e^2}{2\pi^2\mu}\int d\omega d\mathbf{k}\,\delta(\omega^2-k^2)\left\{\frac{2\mu^2-E_0E_f+\mathbf{P}_0\cdot\mathbf{P}_f}{E_f(E_f-E_0+\omega)}+\frac{2\mu^2+E_0E_f+\mathbf{P}_0\cdot\mathbf{P}_f}{E_f(E_f+E_0+\omega)}\right\}. \quad (4)$$

The integral diverges logarithmically and $\Delta\mu_0$ defined here is meaningless. If the $\delta(\omega^2-k^2)$ is replaced by $g(\omega^2-k^2)$ defined in (1), the result is finite and invariant (i.e., does not depend on the momentum $\mathbf{P}_0$ of the electron).

How this comes about may be seen by calculating the integral in (4) for

$$g(\omega^2-k^2)=\delta(\omega^2-k^2)-\delta(\omega^2-k^2-\lambda^2)$$

and reserving an integration on λ until later. The integral (4) will converge with this $g(\omega^2-k^2)$, but it is convenient to divide it for purposes of calculation into the difference of two diverging ones.

This is legitimate providing the divergent integrals are first both computed over the same finite region of $\mathbf{k}$ space, the difference taken, and then the region allowed to pass to infinity. Therefore, we shall define $\Delta\mu_0$ by (4), in which we choose the region arbitrarily to be first over all (positive) ω and then over a sphere in $\mathbf{k}$ space of very large radius K. Likewise $\Delta\mu_\lambda$ is defined as expression (4) with $\delta(\omega^2-k^2-\lambda^2)$ replacing $\delta(\omega^2-k^2)$, and the integration taken over the same region.

The true self-mass is therefore

$$\Delta\mu=\int_0^\infty [\operatorname*{Lim}_{K\to\infty}(\Delta\mu_0-\Delta\mu_\lambda)]G(\lambda)d\lambda. \quad (5)$$

We may now calculate these integrals, starting with $\Delta\mu_\lambda$. Since $\mathbf{P}_0\cdot\mathbf{P}_f=\mathbf{P}_0\cdot(\mathbf{P}_0-\mathbf{k})=E_0^2-\mu^2-\mathbf{P}_0\cdot\mathbf{k}$ and $E_f^2=E_0^2+k^2-2\mathbf{P}_0\cdot\mathbf{P}_f$, the $\mathbf{P}_0\cdot\mathbf{P}_f$ term in the numerator of the first term may be eliminated, the numerator becoming

$$\tfrac{1}{2}(E_f^2+E_0^2-k^2)-E_0E_f+\mu^2=\mu^2+\tfrac{1}{2}(\omega^2-k^2)+\tfrac{1}{2}(E_f-E_0-\omega)(E_f-E_0+\omega).$$

Thus the first term in $\Delta\mu_\lambda$ becomes

$$\int\frac{\mu^2+\frac{1}{2}(\omega^2-k^2)}{E_f(E_f-E_0+\omega)}\delta(\omega^2-k^2-\lambda^2)d\omega d\mathbf{k}+\tfrac{1}{2}\int\delta(\omega^2-k^2-\lambda^2)d\omega d\mathbf{k}(E_f-E_0-\omega)/E_f. \quad (6)$$

Adding the corresponding second term which differs from the first only in the sign of E_0, and performing the integral on ω (which requires simply division by 2ω), we find

$$(2\pi^2/e^2)\mu\Delta\mu_\lambda=(\mu^2+\tfrac{1}{2}\lambda^2)\int\frac{1}{(E_f+\omega)^2-E_0^2}\cdot\frac{E_f+\omega}{E_f}\frac{d\mathbf{k}}{\omega}+\frac{1}{2}\int\frac{d\mathbf{k}}{\omega}-\frac{1}{2}\int\frac{d\mathbf{k}}{E_f}, \quad (7)$$

where $\omega=(k^2+\lambda^2)^{\frac{1}{2}}$ and the integration is to be taken over a sphere of radius K in $\mathbf{k}$ space. The first and, obviously, the second integrals turn out to be invariant; the third is not, but its contribution will cancel out on taking $\Delta\mu_0-\Delta\mu_\lambda$ as it does not depend on λ.[7] The result of the integrations[8] is, dropping terms of order $1/K$ and

[7] Pais has suggested that one subtract from $\Delta\mu_0$ the $-\Delta\mu_\lambda$ that one gets not from electrodynamics but from the scalar f field (for which $\beta\cdots\beta$ replaces $\Sigma_i\,\alpha_i\cdots\alpha_i$). Proceeding in this way the integrals $\int d\mathbf{k}/E_f$ do not appear with the same coefficient. Therefore, although this procedure leads to a finite rest mass it is not invariant in the sense here, that the limits of $\mathbf{k}$ space integration can be taken to be independent of the momentum of the electron. A. Pais, Kon. Ned. Akad. v. Wet. Verh. D1, 19, 1 (1947).

[8] The first integral may be performed in the following manner: First integrate over the directions in $\mathbf{k}$ space at constant magnitude k. Only E_f depends on the direction of k and one may therefore replace the solid angle integral by one on E_f. The limits of E_f are $E_+=(\mu^2+(P_0+k)^2)^{\frac{1}{2}}$ and $E_-=(\mu^2+(P_0-k)^2)^{\frac{1}{2}}$ but both terms may be considered together as one if the integral on k be extended from $-K$ to K instead of 0 to K. To integrate this on k, substitute the variable $x=E_++\omega-E_0$ and (the algebra is long) integrate by parts to reduce it to elementary integrals.

smaller:

$$(\pi/e^2)\mu\Delta\mu_\lambda=(\mu^2+\tfrac{1}{2}\lambda^2)[N_\lambda+\mu^2X_\lambda(\mu,\mu)]$$
$$+\tfrac{1}{2}[K^2-\lambda^2(\ln(2K/\lambda)-\tfrac{1}{2})]$$
$$-\tfrac{1}{2}[K^2-\tfrac{1}{3}P_0^2-\mu^2(\ln(2K/\mu)-\tfrac{1}{2})],$$

where

$$N_\lambda=N_0-[\lambda^2/(\lambda^2-\mu^2)]\ln(\lambda/\mu), \tag{7a}$$

with $N_0=\ln(2K/\mu)-\frac{1}{2}$, and the quantity $X_\lambda(\mu,\mu)$ is finite as $K\to\infty$. It is given by setting $\mu_0=\mu$ in the complicated expression

$$2\mu_0^4X_\lambda(\mu,\mu_0)=((\lambda^2-\mu^2-\mu_0^2)^2-4\mu^2\mu_0^2)^{\frac{1}{2}}$$
$$\times\ln\frac{\lambda^2+\mu^2-\mu_0^2+((\lambda^2-\mu^2-\mu_0^2)^2-4\mu^2\mu_0^2)^{\frac{1}{2}}}{2\lambda\mu}$$
$$+\left(\lambda^2-\mu^2+\mu_0^2-\frac{2\lambda^2\mu_0^2}{\lambda^2-\mu^2}\right)\ln\frac{\mu}{\lambda}+\mu_0^2. \tag{7b}$$

Thus $X_0(\mu,\mu)=1/2\mu^2$ and for λ large compared to μ, $X_\lambda(\mu,\mu)=1/4\lambda^2$. Hence

$$(\pi/e^2)(\Delta\mu_0-\Delta\mu_\lambda)=\frac{3\mu}{2}\cdot\frac{\lambda^2}{\lambda^2-\mu^2}\cdot\ln\frac{\lambda}{\mu}+\frac{\mu}{2}$$
$$-(\mu^2+\tfrac{1}{2}\lambda^2)\mu X_\lambda(\mu,\mu), \tag{8}$$

which is independent of K (in the limit $K\to\infty$). If the important values of λ are much greater than μ, we find approximately (to terms of order $(\mu/\lambda)^2$)

$$\Delta\mu=\mu(e^2/\pi)[\tfrac{3}{2}\ln(\lambda_0/\mu)+\tfrac{3}{8}], \tag{9}$$

where

$$\ln\lambda_0=\int_0^\infty\ln\lambda G(\lambda)d\lambda.$$

Judging from the classical case we would have expected to take λ_0 of order 137μ, for then all mass would be electromagnetic. But $\Delta\mu$ here is too small for this to represent a real possibility. The experimental electron mass m is of course $\mu+\Delta\mu$.

The value of λ would have to be of phenomenal size ($\sim e^{137}\mu$) before $\Delta\mu$ can represent a sizeable fraction of the experimental mass. However, to go to the limit of the conventional electrodynamics, λ_0 should be taken as infinite. Then the self-energy diverges logarithmically in the manner found by Weisskopf.[9]

[9] V. Weisskopf, Phys. Rev. **56**, 72 (1939).

The emission and subsequent absorption of a quantum acts similarly to the effect of a change in mass not only on the diagonal matrix element which we have just calculated, but on non-diagonal elements as well. Consider that the state appearing on the left of all the matrices in (3) were arbitrary, say x. Then the numerator of the first term can be expressed, as we have seen, by $(-1/E_f)(x|-E_f+2\beta\mu+\boldsymbol{\alpha}\cdot\mathbf{P}_f|0)$. The second term can be expressed similarly. The two terms can be combined so that the whole expression in brackets in (3) can be written

$$2\left\{\frac{(x|-E_fE_0+(E_f+\omega)(2\beta\mu+\boldsymbol{\alpha}\cdot\mathbf{P}_0-\boldsymbol{\alpha}\cdot\mathbf{k})|0)}{E_f((E_f+\omega)^2-E_0^2)}\right\}. \tag{10}$$

This expression may be multiplied by

$$\delta(\omega^2-k^2-\lambda^2)$$

and integrated with respect to ω and over a sphere of radius K in $\mathbf{k}$ space. We make use of the following integrals which can be directly verified:

$$\int\frac{1}{(E_f+\omega)^2-E_0^2}\cdot\frac{E_f+\omega}{E_f}\delta(\omega^2-k^2-\lambda^2)d\omega d\mathbf{k}/\pi$$
$$=N_\lambda+\mu_0^2X_\lambda(\mu,\mu_0),$$
$$\int\frac{1}{(E_f+\omega)^2-E_0^2}\delta(\omega^2-k^2-\lambda^2)d\omega d\mathbf{k}/\pi$$
$$=\tfrac{1}{2}N_\lambda+\tfrac{1}{2}(\mu^2+\mu_0^2-\lambda^2)X_\lambda(\mu,\mu)_0, \tag{11}$$
$$\int\frac{\mathbf{k}}{(E_f+\omega)^2-E_0^2}\cdot\frac{E_f+\omega}{E_f}\delta(\omega^2-k^2-\lambda^2)d\omega d\mathbf{k}/\pi$$
$$=\tfrac{1}{2}\mathbf{P}_0[\tfrac{1}{3}+N_\lambda+(\lambda^2+\mu_0^2-\mu^2)X_\lambda(u,\mu_0)].$$

The integrals have been calculated under the assumption that $E_0^2=\mu_0^2+P_0^2$. In our application we should take $\mu_0=\mu$. The quantities N_λ and $X_\lambda(\mu,\mu_0)$ are given by (7a), (7b). (The extra parameter μ_0 is helpful in obtaining other integrals, useful in the radiationless scattering problem, by differentiations with respect to the various parameters under the integral sign.)

The result of integration (10) with the density $\delta(\omega^2-k^2)-\delta(\omega^2-k^2-\lambda^2)$ is therefore

$$(e^2/\pi)(x|-E_0(\tfrac{1}{2}(N_0-N_\lambda)+\tfrac{1}{2}-(\mu^2-\tfrac{1}{2}\lambda^2)X_\lambda)$$
$$+(2\beta\mu+\boldsymbol{\alpha}\cdot\mathbf{P}_0)(N_0-N_\lambda+\tfrac{1}{2}-\mu^2X_\lambda)$$
$$-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\mathbf{P}_0(N_0-N_\lambda-\lambda^2X_\lambda)|0).$$

Now the energy of state 0 is E_0 so that $\boldsymbol{\alpha}\cdot\mathbf{P}_0=H_0-\beta\mu$ is equivalent to $E_0-\beta\mu$, since it operates on state 0 (no implication about state x is involved). Making this replacement, all the terms in E_0 are seen to cancel and the result is simply

$$(x|\beta|0)\cdot(\Delta\mu_0-\Delta\mu_\lambda), \tag{12}$$

where $\Delta\mu_0-\Delta\mu_\lambda$ is given in (8). On integrating over $G(\lambda)d\lambda$ then we find $(x|\beta\Delta\mu|0)$. But this is just the perturbation element which would result from a change of mass by $\Delta\mu$ in the Dirac equation.

We may use this result to show that the level shift for an electron in a bound state given in the present theory will be essentially that given by Weisskopf and Bethe according to their so-called wave-packet method. The change in energy of our electron in a bound state may be calculated in a straightforward manner according to the present formulation. One would simply start with Eq. (2) but with the wave functions and energies for states 0 and f being appropriate for the potential by which the electron is bound. Then one would integrate over $g(\omega^2-k^2)$ rather than $\delta(\omega^2-k^2)$ and obtain a definite finite result. The result would show a fairly large change in E_0 depending logarithmically on λ.

A good part of this change could be accounted for as simply due to the change in E_0 that would occur if the mass of the electron were altered from μ to $m=\mu+\Delta\mu$. We can define the true term shift, then, as the complete change in E_0, less $\Delta\mu(\partial E_0/\partial\mu)$, the change due to using μ instead of m in computing the energy with radiation absent. But $\partial E_0/\partial\mu$ is by perturbation theory the expected value $(\psi_0^*|\beta|\psi_0)$ of β for the state ψ_0 in question. From the result (12), however, this is also equivalent to computing the self-energy of a wave packet ψ_0, assuming the electron as free. But Bethe[1] and Weisskopf[2] compute their term shift by just this prescription: the total effect less the self-energy of the free packet. The only difference here is that we would compute the term shift integral on $g(\omega^2-k^2)$ rather than $\delta(\omega^2-k^2)$. But since the integral converges either way, the difference between the two results is very small, being of order of (μ^2/λ_0^2) times smaller than the result.

RADIATIONLESS SCATTERING

We can study the radiationless scattering problem in a similar manner. This problem is the correction to the scattering by a first-order potential due to the possibility of emission and absorption of a virtual quantum. For example, this emission and absorption can occur at any time previous to the scattering. (It would, in this case, be nearly equivalent to a change in mass in the wave function of the electron arriving at the scatterer.) There will be a large change in cross section, which would be expected as the result of a change in mass of the electron plus a smaller change caused essentially by emissions previous to and absorptions subsequent to the scattering. As in the case of the self-energy in a field and, in fact, in all such problems, we will really be interested in those effects of radiation over and above that resulting from the change in mass. It is, therefore, simpler to compute the difference between the desired quantity calculated with no radiation and electrons of mass m, and the same quantity computed with the possibility of a virtual quantum emission and absorption with an electron of mass μ. This difference, which we shall call the radiative correction, can be looked upon as the result of perturbation due to the addition to the Hamiltonian of both the radiative interaction terms and a term $-\beta\Delta\mu$. The latter term can, as we have shown, be represented by the integral over oscillators of

$$-\sum_i\left(\frac{\alpha_i\Lambda_f^+\alpha_i}{E_f-E_0+\omega}-\frac{\alpha_i\Lambda_f^-\alpha_i}{E_f+E_0+\omega}\right) \tag{13}$$

when acting on a free electron state of positive energy E_0 and momentum $\mathbf{P}_0$. When acting on a state of negative energy $-E_0$, the term can be shown in a similar manner to be the expression (13) with the sign of E_0 changed in the denominator.

Terms like these are just the ones that Schwinger[4] thought should be omitted from the Hamiltonian if one wishes to get meaningful results, so that the present model agrees with Schwinger's prescription.

When this proces is applied to the scattering problem to obtain the radiative correction to the matrix elements, we are left with several

residual terms. First, the emissions and absorptions previous to scattering are not exactly equivalent to a change in mass. If the emission occurs too close (in time) to the scattering, the absorption must occur in a restricted time, rather than at leisure as for a free electron forming $\beta\Delta\mu$. The correction to the matrix element (in the theory of holes) for this is proportional to

$$-\tfrac{1}{2}\sum_i \frac{(2|V\Lambda_1{}^+\alpha_i\Lambda_f{}^+\alpha_i|1)}{(E_f+\omega-E_1)^2} -\tfrac{1}{2}\sum_i \frac{(2|V\Lambda_1{}^+\alpha_i\Lambda_f{}^-\alpha_i|1)}{(E_f+\omega+E_1)^2}. \quad (14)$$

We assume the potential V (vector or scalar) depending on position like $e^{i\mathbf{q}\cdot\mathbf{R}}$ and time like e^{-iQt} induces transitions from a state 1 of momentum $\mathbf{P}_1$, energy E_1, to the state 2 of momentum $\mathbf{P}_2=\mathbf{P}_1+\mathbf{q}$, energy $E_2=E_1+Q=(\mu^2+P_2{}^2)^{\frac{1}{2}}$. The operator V is just 1 for scalar potential, α_x for vector potential in x direction, etc. The term (14) represents only that contribution due to a quantum of momentum $\mathbf{k}$, frequency ω. We expect later to integrate over ω and $\mathbf{k}$, times $g(\omega^2-k^2)$. We put $\mathbf{P}_f=\mathbf{P}_1-\mathbf{k}$, $E_f=(\mu^2+P_f{}^2)^{\frac{1}{2}}$. This term can also be regarded as due to the second-order normalization correction in the ordinary perturbation theory on the incoming wave function. There is a corresponding correction for the final wave function resulting from virtual quanta emitted and absorbed after the scattering: ($\mathbf{P}_g=\mathbf{P}_2-\mathbf{k}$, $E_g=(\mu^2+P_g{}^2)^{\frac{1}{2}}$).

$$-\tfrac{1}{2}\sum_i \frac{(2|\alpha_i\Lambda_g{}^+\alpha_i\Lambda_2{}^+V|1)}{(E_g+\omega-E_2)^2} -\tfrac{1}{2}\sum_i \frac{(2|\alpha_i\Lambda_g{}^-\alpha_i\Lambda_2{}^+V|1)}{(E_g+\omega+E_2)^2}. \quad (15)$$

All the effects of $\beta\Delta\mu$ are now included. The remaining terms are those for which the potential scattering occurs between the emission and absorption. They may be worked out as by Dancoff[10] (except that we include the longitudinal waves by summing i from 1 to 4). They are

$$+\sum_i \frac{(2|\alpha_i\Lambda_g{}^+V\Lambda_f{}^+\alpha_i|1)}{(E_f+\omega-E_1)(E_g+\omega-E_2)} +\sum_i \frac{(2|\alpha_i\Lambda_g{}^-V\Lambda_f{}^-\alpha_i|1)}{(E_f+\omega+E_1)(E_g+\omega+E_2)} \quad (16)$$

and

$$-\sum_i \frac{(2|\alpha_i\Lambda_g{}^+V\Lambda_f{}^-\alpha_i|1)}{(E_g+\omega-E_2)(E_f+\omega+E_1)} \times\left[1+\frac{2\omega}{E_f+E_g-E_2+E_1}\right] -\sum_i \frac{(2|\alpha_i\Lambda_g{}^-V\Lambda_f{}^+\alpha_i|1)}{(E_f+\omega-E_1)(E_g+\omega+E_2)} \times\left[1+\frac{2\omega}{E_f+E_g+E_2-E_1}\right]. \quad (17)$$

Although each separate term diverges, the sum of (14), (15), (16), (17) will lead to an integral convergent for large $\mathbf{k}$ even if integrated in the conventional manner on $\delta(\omega^2-k^2)$. This is the result of Lewis. Integration on $g(\omega^2-k^2)$ will make each term converge for large $\mathbf{k}$, but will then only make correction to the sum of order $(\mu/\lambda)^2$ smaller. These we shall neglect.

The integrals do, however, diverge logarithmically at the lower limit of small momentum transfer. This infra-red catastrophe has been completely cleared up by Bloch and Nordsieck.[11] They show that for very long wave-length quanta the amplitude for emission and reabsorption of more than one quantum is not negligible. Inclusion of these higher order terms, which is necessary only in the non-relativistic region, solves the problem. To keep the results given here in a simple form, we can imagine the integrals to be performed down to some minimum momentum $k_{\min}$, small compared to μ. What is effectively the same thing but which is easier (because relativistic invariance is maintained) for practical purposes, is to imagine that the quanta have a very small rest mass $\lambda_{\min}$. Thus we integrate the density

$$\delta(\omega^2-k^2-\lambda_{\min}{}^2)d\omega d\mathbf{k}$$

[10] S. M. Dancoff, Phys. Rev. **55**, 959 (1939).

[11] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

and assume $\lambda_{\min} \ll \mu$. The two methods are equivalent if one replaces $\ln\lambda_{\min}$ by $\ln(2k_{\min})-1$.

The integrals may be expanded in powers of $\mathbf{q}$ and Q, say up to the second.[12] The constant term vanishes on integration. The integrals appearing may all be expressed in terms of various parametric derivatives of the integrals already given in (11). The result may be expressed in terms of a general potential in a very simple way. A term linear in $\mathbf{q}$, such as proportional to q_x say, is equivalent to taking the matrix element $(2|q_x \exp(i\mathbf{q}\cdot\mathbf{R})|1)$ directly between the two states 2, 1. But this is also equivalent to the matrix element of $-i(\partial/\partial x)\exp(i\mathbf{q}\cdot\mathbf{R})$. Thus if the potential varied in any other manner in space, one sees by superposition that the matrix element is the same as that of $-i\partial V/\partial x$. Thus the terms up to second order can be represented by matrix elements of first and second space and time derivatives of the potential. That is, the radiative correction to the scattering in any potential is equivalent to the first order in e^2 and in the potential, to the scattering produced by a perturbation ΔH to the Dirac Hamiltonian. The perturbation up to terms of first and second derivatives of the vector potential $\mathbf{A}$ and the scalar potential φ is calculated in this manner to be

$$\Delta H=\frac{e^2}{2\pi\hbar c}\left\{-\frac{\hbar e}{2\mu c}(\beta(\boldsymbol{\sigma}\cdot\mathbf{B})-i\beta\boldsymbol{\alpha}\cdot\mathbf{E})\right.$$
$$\left.+\frac{2\hbar^2 e}{3\mu^2c^2}(\Box^2\varphi-\boldsymbol{\alpha}\cdot\Box^2\mathbf{A})\left(\ln\frac{\mu}{\lambda_{\min}}-\frac{3}{8}\right)\right\}. \quad (18)$$

The first term, where $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\varphi-(1/c)\partial\mathbf{A}/\partial t$, has the same effect as an alteration in the electron magnetic moment[13] by a fraction $e^2/2\pi\hbar c$. This effect was first discovered by Schwinger.[4]

LINE SHIFT

The perturbation to H given here is useful not only for scattering problems but also for the line-shift problem. The actual motion of an electron in a binding potential can be visualized as simply a continued sequence of scatterings in this potential. For each scattering we can calculate the effect of virtual quanta in the way outlined above. However, it is possible, if the potential is strong, that *two* scatterings occur between the emission and reabsorption of the quantum, in which case the above formula for ΔH is incorrect. In hydrogen the potential over most of the atom is sufficiently weak that this does not occur with effective probability. The very long wave-length quanta do have a tendency to exist in the virtual state for long periods, but they have been eliminated by the cut-off $\lambda_{\min}$ at low frequencies.

In hydrogen, then, the line shift due to quanta above minimum wave number $k_{\min}$ is the expected value, for the state in question, of

$$\Delta H=\frac{e^2}{2\pi\hbar c}\left\{-\frac{\hbar e i}{2\mu c}\beta\boldsymbol{\alpha}\cdot\nabla\varphi+\frac{2\hbar^2 e}{3\mu^2c^2}(\nabla^2\varphi)\right.$$
$$\left.\times\left(\ln\frac{\mu c}{2\hbar k_{\min}}+\frac{5}{8}\right)\right\}, \quad (19)$$

where $\varphi=e/r$, r being the distance to the proton, and we have used the relation

$$\ln\lambda_{\min}=\ln(2k_{\min})-1.$$

The first term insures that the fine structure separation correction will be that expected from the change in the electron's magnetic moment. The second may be combined with Bethe's nonrelativistic calculation for quanta below $k_{\min}$.[14]

APPLICATION TO OTHER PROCESSES

The important problem of verifying that the self-energy will not diverge in higher-order approximations has not been carried to completion. It appears unlikely that trouble will arise here. If that is true the model probably gives sensible answers to all problems of quantum electrodynamics other than those involving Uehling polarization effects, discussed below. It has been found to give finite self-mass if we have, instead of a vector field, a scalar field or a pseudoscalar field, coupled to the electron in the simplest way possible without gradient operators. If the field

[12] The integrals have also been worked out, by other methods, for arbitrarily large $\mathbf{q}$ and Q. These will appear in a future publication.

[13] W. Pauli, *Handbuch der Physik* (1933), Vol. 24/1, p. 233.

[14] Using Eq. (18), Professor Bethe finds 1050 megacycles for the separation between $2p_{3/2}$ and $2s_{1/2}$ in hydrogen. (Solvay Report.)

quanta have mass M, $g(\omega^2-k^2)$ is replaced by $g(\omega^2-k^2-M^2)$, and the values of λ of importance are chosen to be large compared to M.

The results for electrodynamics, then, after mass renormalization, depend only slightly on the form of $G(\lambda)$ and the size of λ_0. Since λ_0 may be taken to be extremely large without spoiling the smallness of $\Delta\mu$, there would appear to be good reason to drop the dependence on λ altogether. Thus the $G(\lambda)$ appears only as a complicated scaffold which is removed after the calculation is done.

On the other hand, electrodynamics probably does break down somewhere and it is interesting to keep the terms in λ for various phenomena to see if one might be selected which is particularly sensitive to λ. This phenomena would then be a promising one to study experimentally. The Møller interaction between two electrons is modified by the present theory. There is, of course, the radiative correction, but in addition to that there is simply a change due to the change in the density function for the quanta which can be exchanged. The Møller interaction ordinarily is proportional to $1/q^2$, where q is the magnitude of the momentum transferred from one electron to the other in the center of gravity system. The modification is only that this factor is changed to $\int_0^\infty(1/q^2-1/(q^2+\lambda^2))G(\lambda)d_\lambda$. This represents a decrease in cross section for hard collisions. If λ is of order 137 μc^2, we would need electrons in the center of gravity system of roughly 30 Mev to find a strong effect. This corresponds, however, to bombardment of stationary electrons by electrons of $3\frac{1}{2}$ Bev.[15]

It is interesting to note that the Møller interaction can be viewed as simply a correction to self-energy due to the exclusion principle. The self-energy of two electrons, 1 and 2, is not the sum of the self-energy of each, for one of the virtual states that 2 could ordinarily enter by emission of a quantum is now occupied by 1. The difference between the self-energy of two electrons and the sum of the self-energy of each separately comes out to be just their interaction energy.

[15] A more promising way to obtain processes with high momentum transfer would be wide-angle scattering of electrons from nuclei. But here deviations from expectations might be associated with uncertainties in the nuclear charge distributions rather than electrodynamics. Very wide angle pair production is a phenomena which does occur for high energy incident γ-rays with large momentum transfer in a region not too close to the nucleus.

VACUUM POLARIZATION. ALTERNATIVE CUT-OFF PROCEDURES

In the above calculation, terms of the type discussed by Uehling[16] have been omitted. These terms represent processes involving a pair production followed by annihilation of the *same* pair. For example, a pair produced by the potential may annihilate again emitting a quantum. This quantum is then absorbed by the electron in state 1 transferring it to state 2. These terms are infinite and are not made convergent by the present scheme. There is some point, nevertheless, to solving problems at first without taking them into account. This is because their net effect is only to alter the effective potential in which the electron finds itself, for it may be scattered either directly or by the quantum produced by the Uehling terms. That is, if this problem of polarization of the vacuum is solved it will mean, if there is any effect, simply that the potential A, φ appearing in the Dirac equation and (to high order) in such terms as (18) should be replaced by new "polarized" potentials A', φ'.

These polarization terms can be characterized in a relativistically invariant manner. All the terms which have been calculated above contain matrix elements of operators between states in a sequence such as 1 to f, f to g, g to 2. The omitted polarization terms contain transitions like f to g, g to f, 1 to 2. For higher order processes the polarization terms are those which do not contain a continued sequence of transitions from the initial to the final state.

The polarization terms are not affected in any helpful way by the changes in the density of quanta. It is likely that this problem will have its answer in a changed physical viewpoint. However, there is a simple alternative procedure to produce finite self-energies which also makes convergent the integrals appearing in Serber's[17] treatment of the polarization problem. (Since, however, this treatment of Serber already presupposes a partial subtraction procedure of Heisenberg and Dirac, the situation is not so clear here as in the self-energy problem.)

[16] E. A. Uehling, Phys. Rev. **48**, 55 (1935).
[17] R. Serber, Phys. Rev. **48**, 49 (1935).

From the point of view of coordinate space, the reason that the electronic self-energy diverges appears to be this. A virtual light quantum emitted at one point spreads out as $\delta(t^2-r^2)$ from the origin. The wave packet of the electron spreading out after the emission of the quantum has, as a consequence of Dirac's equation, a similar discontinuous value along the light cone. It is the continued coincidence of these singularities which makes the matrix element for the subsequent absorption of the quantum infinite. The method outlined above of changing $\delta(\omega^2-k^2)$ to $g(\omega^2-k^2)$ has the effect of changing $\delta(t^2-r^2)$ to $f(t^2-r^2)$ where $f(s^2)$ is everywhere finite and goes to zero rapidly for $|s^2|>1/\lambda_0^2$. The quanta have been moved away from the electrons so that overlap on the light cone is reduced.

An obvious alternative procedure is to move the electron wave function away from the quanta. This is easily done in a very similar manner. We assume the density of electron states of energy E, momentum $\mathbf{P}$ to be $g(E^2-P^2-\mu^2)$ rather than $\delta(E^2-P^2-\mu^2)$.[18] The quanta are conventional, $\omega=k$, density $d\mathbf{k}/k$. The self-energy integrals (2) can, of course, be expressed as an integral over the intermediate state momentum $\mathbf{P}_f$ rather than $\mathbf{k}$. Replacing $d\mathbf{P}_f/E_f$ by $g(E_f^2-P_f^2-\mu^2)dE_f d\mathbf{P}_f$, we find

$$\Delta E_0' = -\frac{e^2}{2\pi^2}\int g(E_f^2-P_f^2-\mu^2)dE_f d\mathbf{P}_f \cdot\frac{E_f}{k}\sum_i\left\{\frac{(0|\alpha_i\Lambda_f^+\alpha_i|0)}{E_f+k-E_0}-\frac{(0|\alpha_i\Lambda_f^-\alpha_i|0)}{E_f+k+E_0}\right\},$$

where $k=|\mathbf{P}_f-\mathbf{P}_0|$, $E_0=(\mu^2+P_0^2)^{\frac{1}{2}}$. The projection operators are unchanged since it is only the density of states which we wish to alter. They are still $\Lambda_f^{2\pm}=(E_f\pm\boldsymbol{\alpha}\cdot\mathbf{P}_f\pm\beta\mu)/2E_f$. The result of this calculation is to verify that $\Delta E_0'$ is finite, (depending logarithmically on λ_0). The other problems can be analyzed in the same way.

In the problem of polarization of the vacuum, the wave functions of both electron and positron ordinarily spread with a singularity on the light core. The matrix element for their subsequent annihilation is therefore infinite. With the modification here described these wave functions are made less singular and their overlap integral is finite. The polarization integrals in Serber's article[17] may now be integrated to yield finite results.

Other than terms which might be removed by a small renormalization of charge (depending logarithmically on λ_0), the net effect in (17) would be to change the $-(\frac{3}{8})$ in the last term of (17) to $-(\frac{3}{8})-(\frac{1}{5})$. However, the real existence of such polarization corrections is, in the author's view, uncertain. These matters will be discussed in much more detail in future publications. Also reserved for future publication is a more complete physical theory from which the results reported here may be directly deduced. It yields much more powerful techniques for setting up problems and performing the required integrations.

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[18] This is seen to be essentially the method proposed by Wataghin. G. Wataghin, Zeits. f. Physik **88**, 92 (1934).