The Works of Archimedes
On the Equilibrium of Planes - On Floating Bodies

Archimedes
212 BC

ON THE EQUILIBRIUM OF PLANES

OR

THE CENTRES OF GRAVITY OF PLANES.

BOOK I.

"I POSTULATE the following:

1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance.

2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they are not in equilibrium but incline towards the weight from which nothing was taken.

4. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide.

5. In figures which are unequal but similar the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.

7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure."

Proposition 1.

Weights which balance at equal distances are equal.

For, if they are unequal, take away from the greater the difference between the two. The remainders will then not balance [*Post.* 3]; which is absurd.

Therefore the weights cannot be unequal.

Proposition 2.

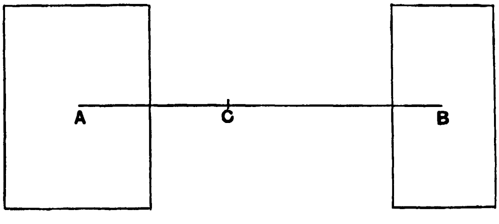
Unequal weights at equal distances will not balance but will incline towards the greater weight.

For take away from the greater the difference between the two. The equal remainders will therefore balance [*Post.* 1]. Hence, if we add the difference again, the weights will not balance but incline towards the greater [*Post.* 2].

Proposition 3.

Unequal weights will balance at unequal distances, the greater weight being at the lesser distance.

Let A, B be two unequal weights (of which A is the greater) balancing about C at distances AC, BC respectively.



Then shall AC be less than BC. For, if not, take away from A the weight $(A - B.)$ The remainders will then incline towards B [*Post.* 3]. But this is impossible, for (1) if $AC = CB$, the equal remainders will balance, or (2) if $AC > CB$, they will incline towards A at the greater distance [*Post.* 1].

Hence $AC < CB$.

Conversely, if the weights balance, and $AC < CB$, then $A > B$.

Proposition 4.

If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity.

[Proved from Prop. 3 by *reductio ad absurdum*. Archimedes assumes that the centre of gravity of both together is on the straight line joining the centres of gravity of each, saying that this had been proved before (προδέδεικται). The allusion is no doubt to the lost treatise *On levers* (περὶ ζυγῶν).]

Proposition 5.

If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude.

[This follows immediately from Prop. 4.]

COR 1. *The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal, while the distances between their centres of gravity are equal.*

COR. 2. *If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones.*

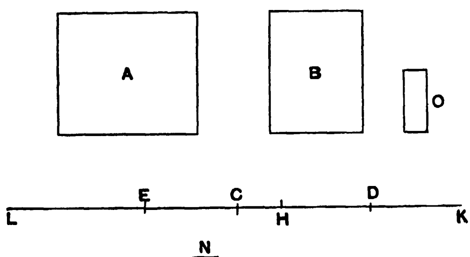
Propositions 6, 7.

Two magnitudes, whether commensurable [Prop. 6] *or incommensurable* [Prop. 7], *balance at distances reciprocally proportional to the magnitudes.*

I. Suppose the magnitudes A, B to be commensurable, and the points A, B to be their centres of gravity. Let DE be a straight line so divided at C that

$$A : B = DC : CE.$$

We have then to prove that, if A be placed at E and B at D, C is the centre of gravity of the two taken together.



Since A, B are commensurable, so are DC, CE. Let N be a common measure of DC, CE. Make DH, DK each equal to CE, and EL (on CE produced) equal to CD. Then $EH = CD$, since $DH = CE$. Therefore LH is bisected at E, as HK is bisected at D.

Thus LH, HK must each contain N an even number of times.

Take a magnitude O such that O is contained as many times in A as N is contained in LH, whence

$$A : O = LH : N.$$

But
$$B : A = CE : DC$$
$$= HK : LH.$$

Hence, *ex aequali*, $B : O = HK : N$, or O is contained in B as many times as N is contained in HK.

Thus O is a common measure of A, B.

Divide LH, HK into parts each equal to N, and A, B into parts each equal to O. The parts of A will therefore be equal in number to those of LH, and the parts of B equal in number to those of HK. Place one of the parts of A at the middle point of each of the parts N of LH, and one of the parts of B at the middle point of each of the parts N of HK.

Then the centre of gravity of the parts of A placed at equal distances on LH will be at E, the middle point of LH [Prop. 5, Cor. 2], and the centre of gravity of the parts of B placed at equal distances along HK will be at D, the middle point of HK.

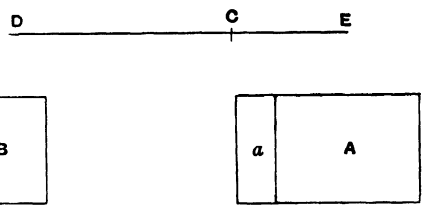
Thus we may suppose A itself applied at E, and B itself applied at D.

But the system formed by the parts O of A and B together is a system of equal magnitudes even in number and placed at equal distances along LK. And, since $LE = CD$, and $EC = DK$, $LC = CK$, so that C is the middle point of LK. Therefore C is the centre of gravity of the system ranged along LK.

Therefore A acting at E and B acting at D balance about the point C.

II. Suppose the magnitudes to be incommensurable, and let them be $(A + a)$ and B respectively. Let DE be a line divided at C so that

$$(A + a) : B = DC : CE.$$



Then, if $(A + a)$ placed at E and B placed at D do not balance about C, $(A + a)$ is either too great to balance B, or not great enough.

Suppose, if possible, that $(A + a)$ is too great to balance B. Take from $(A + a)$ a magnitude a smaller than the deduction which would make the remainder balance B, but such that the remainder A and the magnitude B are commensurable.

Then, since A, B are commensurable, and

$$A : B < DC : CE,$$

A and B will not balance [Prop. 6], but D will be depressed.

But this is impossible, since the deduction a was an insufficient deduction from $(A + a)$ to produce equilibrium, so that E was still depressed.

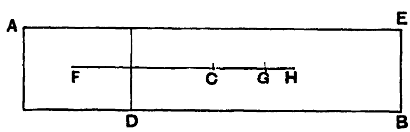
Therefore $(A + a)$ is not too great to balance B; and similarly it may be proved that B is not too great to balance $(A + a)$.

Hence $(A + a)$, B taken together have their centre of gravity at C.

Proposition 8.

If AB be a magnitude whose centre of gravity is C, and AD a part of it whose centre of gravity is F, then the centre of gravity of the remaining part will be a point G on FC produced such that

$$GC : CF = (AD) : (DE).$$



For, if the centre of gravity of the remainder (DE) be not G, let it be a point H. Then an absurdity follows at once from Props. 6, 7.

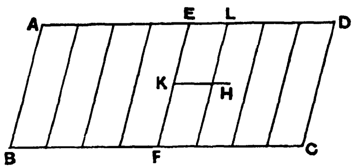
Proposition 9.

The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides.

Let $ABCD$ be a parallelogram, and let EF join the middle points of the opposite sides AD, BC.

If the centre of gravity does not lie on EF, suppose it to be H, and draw HK parallel to AD or BC meeting EF in K.

Then it is possible, by bisecting ED, then bisecting the halves, and so on continually, to arrive at a length EL less



than KH. Divide both AE and ED into parts each equal to EL, and through the points of division draw parallels to AB or CD.

We have then a number of equal and similar parallelograms, and, if any one be applied to any other, their centres of gravity coincide [*Post.* 4]. Thus we have an even number of equal magnitudes whose centres of gravity lie at equal distances along a straight line. Hence the centre of gravity of the whole parallelogram will lie on the line joining the centres of gravity of the two middle parallelograms [Prop. 5, Cor. 2].

But this is impossible, for H is outside the middle parallelograms.

Therefore the centre of gravity cannot but lie on EF.

Proposition 10.

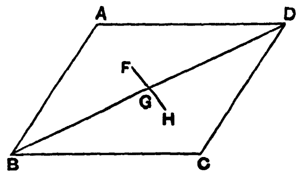
The centre of gravity of a parallelogram is the point of intersection of its diagonals.

For, by the last proposition, the centre of gravity lies on each of the lines which bisect opposite sides. Therefore it is at the point of their intersection; and this is also the point of intersection of the diagonals.

Alternative proof.

Let $ABCD$ be the given parallelogram, and BD a diagonal. Then the triangles ABD, CDB are equal and similar, so that [*Post.* 4], if one be applied to the other, their centres of gravity will fall one upon the other.

Suppose F to be the centre of gravity of the triangle ABD. Let G be the middle point of BD. Join FG and produce it to H, so that $FG = GH$.



If we then apply the triangle ABD to the triangle CDB so that AD falls on CB and AB on CD, the point F will fall on H.

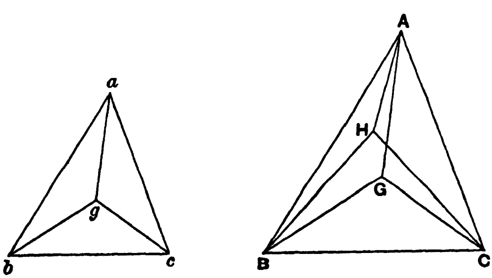
But [by *Post.* 4] F will fall on the centre of gravity of CDB. Therefore H is the centre of gravity of CDB.

Hence, since F, H are the centres of gravity of the two equal triangles, the centre of gravity of the whole parallelogram is at the middle point of FH, i.e. at the middle point of BD, which is the intersection of the two diagonals.

Proposition 11.

If abc, ABC be two similar triangles, and g, G two points in them similarly situated with respect to them respectively, then, if g be the centre of gravity of the triangle abc, G must be the centre of gravity of the triangle ABC.

Suppose $ab : bc : ca = AB : BC : CA$.



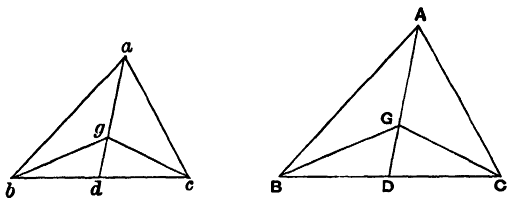
The proposition is proved by an obvious *reductio ad absurdum*. For, if G be not the centre of gravity of the triangle ABC, suppose H to be its centre of gravity.

Post. 5 requires that g, H shall be similarly situated with respect to the triangles respectively; and this leads at once to the absurdity that the angles HAB, GAB are equal.

Proposition 12.

Given two similar triangles abc, ABC, and d, D the middle points of bc, BC respectively, then, if the centre of gravity of abc lie on ad, that of ABC will lie on AD.

Let g be the point on ad which is the centre of gravity of abc.



Take G on AD such that

$$ad : ag = AD : AG,$$

and join gb, gc, GB, GC.

Then, since the triangles are similar, and bd, BD are the halves of bc, BC respectively,

$$ab : bd = AB : BD,$$

and the angles abd, ABD are equal.

Therefore the triangles abd, ABD are similar, and

$$\angle bad = \angle BAD.$$

Also $\qquad ba : ad = BA : AD,$

while, from above, $\qquad ad : ag = AD : AG.$

Therefore $ba : ag = BA : AG$, while the angles bag, BAG are equal.

Hence the triangles bag, BAG are similar, and

$$\angle abg = \angle ABG.$$

And, since the angles abd, ABD are equal, it follows that

$$\angle gbd = \angle GBD.$$

In exactly the same manner we prove that

$$\angle gac = \angle GAC,$$

$$\angle acg = \angle ACG,$$

$$\angle gcd = \angle GCD.$$

Therefore g, G are similarly situated with respect to the triangles respectively; whence [Prop. 11] G is the centre of gravity of ABC.

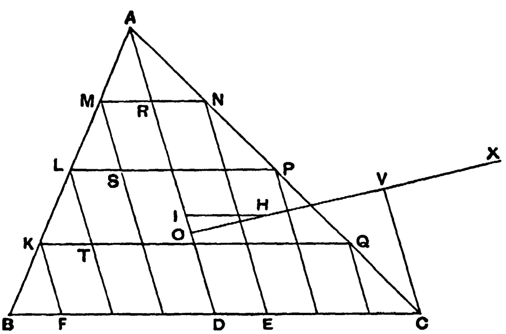
Proposition 13.

In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side.

Let ABC be a triangle and D the middle point of BC. Join AD. Then shall the centre of gravity lie on AD.

For, if possible, let this not be the case, and let H be the centre of gravity. Draw HI parallel to CB meeting AD in I.

Then, if we bisect DC, then bisect the halves, and so on, we shall at length arrive at a length, as DE, less than HI.



Divide both BD and DC into lengths each equal to DE, and through the points of division draw lines each parallel to DA meeting BA and AC in points as K, L, M and N, P, Q respectively.

Join MN, LP, KQ, which lines will then be each parallel to BC.

We have now a series of parallelograms as FQ, TP, SN, and AD bisects opposite sides in each. Thus the centre of gravity of each parallelogram lies on AD [Prop. 9], and therefore the centre of gravity of the figure made up of them all lies on AD.

Let the centre of gravity of all the parallelograms taken together be O. Join OH and produce it; also draw CV parallel to DA meeting OH produced in V.

Now, if n be the number of parts into which AC is divided,

$$\begin{aligned}\triangle ADC : (\text{sum of triangles on } AN, NP, \ldots) &= AC^2 : (AN^2 + NP^2 + \ldots)\\ &= n^2 : n\\ &= n : 1\\ &= AC : AN.\end{aligned}$$

Similarly

$$\triangle ABD : (\text{sum of triangles on } AM, ML, \ldots) = AB : AM.$$

And $$AC : AN = AB : AM.$$

It follows that

$$\begin{aligned}\triangle ABC : (\text{sum of all the small } \triangle\text{s}) &= CA : AN\\ &> VO : OH, \text{ by parallels.}\end{aligned}$$

Suppose OV produced to X so that

$$\triangle ABC : (\text{sum of small } \triangle\text{s}) = XO : OH,$$

whence, *dividendo*,

$$(\text{sum of parallelograms}) : (\text{sum of small } \triangle\text{s}) = XH : HO.$$

Since then the centre of gravity of the triangle ABC is at H, and the centre of gravity of the part of it made up of the parallelograms is at O, it follows from Prop. 8 that the centre of gravity of the remaining portion consisting of all the small triangles taken together is at X.

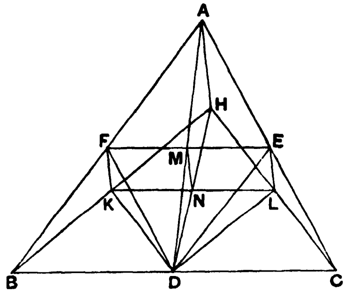
But this is impossible, since all the triangles are on one side of the line through X parallel to AD.

Therefore the centre of gravity of the triangle cannot but lie on AD.

Alternative proof.

Suppose, if possible, that H, not lying on AD, is the centre of gravity of the triangle ABC. Join AH, BH, CH. Let E, F be the middle points of CA, AB respectively, and join DE, EF, FD. Let EF meet AD in M.

Draw FK, EL parallel to AH meeting BH, CH in K, L respectively. Join KD, HD, LD, KL. Let KL meet DH in N, and join MN.



Since DE is parallel to AB, the triangles ABC, EDC are similar.

And, since $CE = EA$, and EL is parallel to AH, it follows that $CL = LH$. And $CD = DB$. Therefore BH is parallel to DL.

Thus in the similar and similarly situated triangles ABC, EDC the straight lines AH, BH are respectively parallel to EL, DL; and it follows that H, L are similarly situated with respect to the triangles respectively.

But H is, by hypothesis, the centre of gravity of ABC. Therefore L is the centre of gravity of EDC. [Prop. 11]

Similarly the point K is the centre of gravity of the triangle FBD.

And the triangles FBD, EDC are equal, so that the centre of gravity of both together is at the middle point of KL, i.e. at the point N.

The remainder of the triangle ABC, after the triangles FBD, EDC are deducted, is the parallelogram $AFDE$, and the centre of gravity of this parallelogram is at M, the intersection of its diagonals.

It follows that the centre of gravity of the whole triangle ABC must lie on MN; that is, MN must pass through H, which is impossible (since MN is parallel to AH).

Therefore the centre of gravity of the triangle ABC cannot but lie on AD.

Proposition 14.

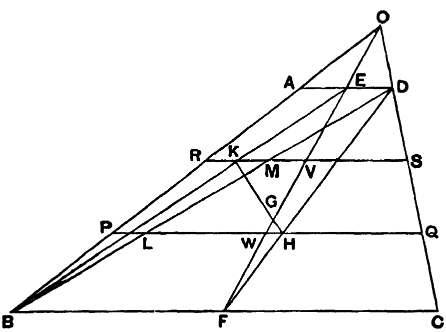
It follows at once from the last proposition that *the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.*

Proposition 15.

If AD, BC be the two parallel sides of a trapezium $ABCD$, AD being the smaller, and if AD, BC be bisected at E, F respectively, then the centre of gravity of the trapezium is at a point G on EF such that

$$GE : GF = (2BC + AD) : (2AD + BC).$$

Produce BA, CD to meet at O. Then FE produced will also pass through O, since $AE = ED$, and $BF = FC$.



Now the centre of gravity of the triangle OAD will lie on OE, and that of the triangle OBC will lie on OF. [Prop. 13]

It follows that the centre of gravity of the remainder, the trapezium $ABCD$, will also lie on OF. [Prop. 8]

Join BD, and divide it at L, M into three equal parts. Through L, M draw PQ, RS parallel to BC meeting BA in P, R, FE in W, V, and CD in Q, S respectively.

Join DF, BE meeting PQ in H and RS in K respectively.

Now, since $\qquad BL = \frac{1}{3} BD,$

$$FH = \tfrac{1}{3} FD.$$

Therefore H is the centre of gravity of the triangle DBC*.

Similarly, since $EK = \frac{1}{3} BE$, it follows that K is the centre of gravity of the triangle ADB.

Therefore the centre of gravity of the triangles DBC, ADB together, i.e. of the trapezium, lies on the line HK.

But it also lies on OF.

Therefore, if OF, HK meet in G, G is the centre of gravity of the trapezium.

Hence [Props. 6, 7]

$$\triangle DBC : \triangle ABD = KG : GH$$
$$= VG : GW.$$

But
$$\triangle DBC : \triangle ABD = BC : AD.$$

Therefore
$$BC : AD = VG : GW.$$

It follows that

$$(2BC + AD) : (2AD + BC) = (2VG + GW) : (2GW + VG)$$
$$= EG : GF.$$

Q. E. D.

* This easy deduction from Prop. 14 is assumed by Archimedes without proof.

ON THE EQUILIBRIUM OF PLANES.

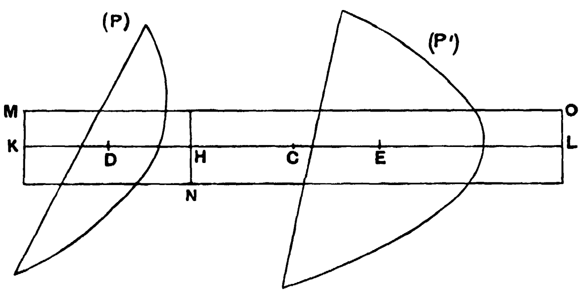
BOOK II.

Proposition 1.

If P, P' be two parabolic segments and D, E their centres of gravity respectively, the centre of gravity of the two segments taken together will be at a point C on DE determined by the relation

$$P : P' = CE : CD^*.$$

In the same straight line with DE measure EH, EL each equal to DC, and DK equal to DH; whence it follows at once that $DK = CE$, and also that $KC = CL$.



* This proposition is really a particular case of Props. 6, 7 of Book I. and is therefore hardly necessary. As, however, Book II. relates exclusively to parabolic segments, Archimedes' object was perhaps to emphasize the fact that the magnitudes in I. 6, 7 might be parabolic segments as well as rectilinear figures. His procedure is to substitute for the segments rectangles of equal area, a substitution which is rendered possible by the results obtained in his separate treatise on the *Quadrature of the Parabola*.

Apply a rectangle MN equal in area to the parabolic segment P to a base equal to KH, and place the rectangle so that KH bisects it, and is parallel to its base.

Then D is the centre of gravity of MN, since $KD = DH$.

Produce the sides of the rectangle which are parallel to KH, and complete the rectangle NO whose base is equal to HL. Then E is the centre of gravity of the rectangle NO.

Now
$$\begin{aligned}(MN) : (NO) &= KH : HL\\ &= DH : EH\\ &= CE : CD\\ &= P : P'.\end{aligned}$$

But
$$(MN) = P.$$

Therefore
$$(NO) = P'.$$

Also, since C is the middle point of KL, C is the centre of gravity of the whole parallelogram made up of the two parallelograms (MN), (NO), which are equal to, and have the same centres of gravity as, P, P' respectively.

Hence C is the centre of gravity of P, P' taken together.

Definition and lemmas preliminary to Proposition 2.

"If in a segment bounded by a straight line and a section of a right-angled cone [a parabola] a triangle be inscribed having the same base as the segment and equal height, if again triangles be inscribed in the remaining segments having the same bases as the segments and equal height, and if in the remaining segments triangles be inscribed in the same manner, let the resulting figure be said to be **inscribed in the recognised manner** (γνωρίμως ἐγγράφεσθαι) in the segment.

And it is plain

(1) that *the lines joining the two angles of the figure so inscribed which are nearest to the vertex of the segment, and the next*

pairs of angles in order, will be parallel to the base of the segment,

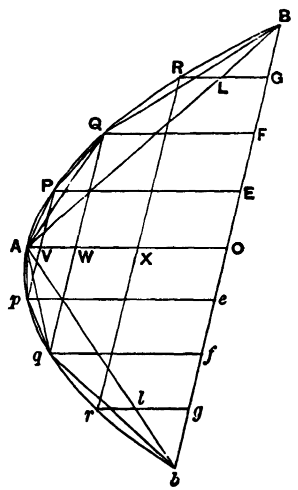
(2) that *the said lines will be bisected by the diameter of the segment,* and

(3) that *they will cut the diameter in the proportions of the successive odd numbers, the number one having reference to* [*the length adjacent to*] *the vertex of the segment.*

And these properties will have to be proved in their proper places (ἐν ταῖς τάξεσιν)."

[The last words indicate an intention to give these propositions in their proper connexion with systematic proofs; but the intention does not appear to have been carried out, or at least we know of no lost work of Archimedes in which they could have appeared. The results can however be easily derived from propositions given in the *Quadrature of the Parabola* as follows.

(1) Let $BRQPApqrb$ be a figure inscribed 'in the recognised manner' in the parabolic segment BAb of which Bb is the base, A the vertex and AO the diameter.



Bisect each of the lines BQ, BA, QA, Aq, Ab, qb, and through the middle points draw lines parallel to AO meeting Bb in G, F, E, e, f, g respectively.

These lines will then pass through the vertices R, Q, P, p, q, r of the respective parabolic segments [*Quadrature of the Parabola*, Prop. 18], i.e. through the angular points of the inscribed figure (since the triangles and segments are of equal height).

Also $BG = GF = FE = EO$, and $Oe = ef = fg = gb$. But $BO = Ob$, and therefore all the parts into which Bb is divided are equal.

If now AB, RG meet in L, and Ab, rg in l, we have

$$\begin{aligned} BG : GL &= BO : OA, \text{ by parallels,} \\ &= bO : OA \\ &= bg : gl, \end{aligned}$$

whence $GL = gl$.

Again [*ibid.*, Prop. 4]

$$\begin{aligned} GL : LR &= BO : OG \\ &= bO : Og \\ &= gl : lr; \end{aligned}$$

and, since $GL = gl$, $LR = lr$.

Therefore GR, gr are equal as well as parallel.

Hence $GRrg$ is a parallelogram, and Rr is parallel to Bb.

Similarly it may be shown that Pp, Qq are each parallel to Bb.

(2) Since $RGgr$ is a parallelogram, and RG, rg are parallel to AO, while $GO = Og$, it follows that Rr is bisected by AO.

And similarly for Pp, Qq.

(3) Lastly, if V, W, X be the points of bisection of Pp, Qq, Rr,

$$\begin{aligned} AV : AW : AX : AO &= PV^2 : QW^2 : RX^2 : BO^2 \\ &= 1 : 4 : 9 : 16, \end{aligned}$$

whence $\quad AV : VW : WX : XO = 1 : 3 : 5 : 7.$]

Proposition 2.

If a figure be 'inscribed in the recognised manner' in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment.

For, in the figure of the foregoing lemmas, the centre of gravity of the trapezium $BRrb$ must lie on XO, that of the trapezium $RQqr$ on WX, and so on, while the centre of gravity of the triangle PAp lies on AV.

Hence the centre of gravity of the whole figure lies on AO.

Proposition 3.

If BAB', bab' be two similar parabolic segments whose diameters are AO, ao respectively, and if a figure be inscribed in each segment 'in the recognised manner,' the number of sides in each figure being equal, the centres of gravity of the inscribed figures will divide AO, ao in the same ratio.

[Archimedes enunciates this proposition as true of *similar* segments, but it is equally true of segments which are not similar, as the course of the proof will show.]

Suppose $BRQPAP'Q'R'B'$, $brqpap'q'r'b'$ to be the two figures inscribed 'in the recognised manner.' Join PP', QQ', RR' meeting AO in L, M, N, and pp', qq', rr' meeting ao in l, m, n.

Then [Lemma (3)]

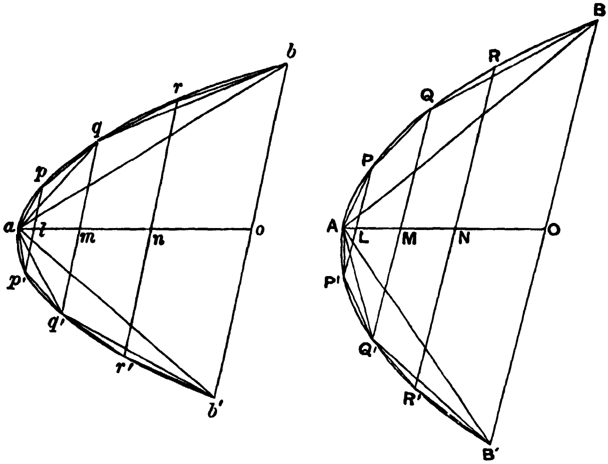
$$\begin{aligned} AL : LM : MN : NO &= 1 : 3 : 5 : 7 \\ &= al : lm : mn : no, \end{aligned}$$

so that AO, ao are divided in the same proportion.

Also, by reversing the proof of Lemma (3), we see that

$$PP' : pp' = QQ' : qq' = RR' : rr' = BB' : bb'.$$

Since then $RR' : BB' = rr' : bb'$, and these ratios respectively determine the proportion in which NO, no are divided by the centres of gravity of the trapezia $BRR'B'$, $brr'b'$ [I. 15], it follows that the centres of gravity of the trapezia divide NO, no in the same ratio.



Similarly the centres of gravity of the trapezia $RQQ'R'$, $rqq'r'$ divide MN, mn in the same ratio respectively, and so on.

Lastly, the centres of gravity of the triangles PAP', pap' divide AL, al respectively in the same ratio.

Moreover the corresponding trapezia and triangles are, each to each, in the same proportion (since their sides and heights are respectively proportional), while AO, ao are divided in the same proportion.

Therefore the centres of gravity of the complete inscribed figures divide AO, ao in the same proportion.

Proposition 4.

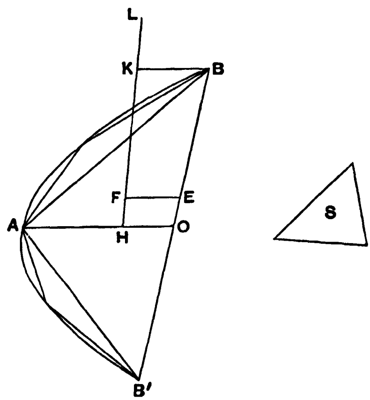
The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment.

Let BAB' be a parabolic segment, A its vertex and AO its diameter.

Then, if the centre of gravity of the segment does not lie on AO, suppose it to be, if possible, the point F. Draw FE parallel to AO meeting BB' in E.

Inscribe in the segment the triangle ABB' having the same vertex and height as the segment, and take an area S such that

$$\triangle ABB' : S = BE : EO.$$



We can then inscribe in the segment 'in the recognised manner' a figure such that the segments of the parabola left over are together less than S. [For Prop. 20 of the *Quadrature of the Parabola* proves that, if in any segment the triangle with the same base and height be inscribed, the triangle is greater than half the segment; whence it appears that, each time that we increase the number of the sides of the figure inscribed 'in the recognised manner,' we take away more than half of the remaining segments.]

Let the inscribed figure be drawn accordingly; its centre of gravity then lies on AO [Prop. 2]. Let it be the point H.

Join HF and produce it to meet in K the line through B parallel to AO.

Then we have

$$\begin{aligned}(\text{inscribed figure}) : (\text{remainder of segmt.}) &> \triangle ABB' : S \\ &> BE : EO \\ &> KF : FH.\end{aligned}$$

Suppose L taken on HK produced so that the former ratio is equal to the ratio $LF : FH$.

Then, since H is the centre of gravity of the inscribed figure, and F that of the segment, L must be the centre of gravity of all the segments taken together which form the remainder of the original segment. [I. 8]

But this is impossible, since all these segments lie on one side of the line drawn through L parallel to AO [Cf. *Post.* 7].

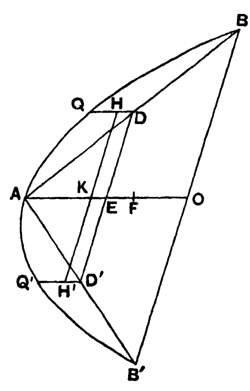
Hence the centre of gravity of the segment cannot but lie on AO.

Proposition 5.

If in a parabolic segment a figure be inscribed 'in the recognised manner,' the centre of gravity of the segment is nearer to the vertex of the segment than the centre of gravity of the inscribed figure is.

Let BAB' be the given segment, and AO its diameter.

First, let ABB' be the *triangle* inscribed 'in the recognised manner.'



Divide AO in F so that $AF = 2FO$; F is then the centre of gravity of the triangle ABB'.

Bisect AB, AB' in D, D' respectively, and join DD' meeting AO in E. Draw DQ, $D'Q'$ parallel to OA to meet the curve. QD, $Q'D'$ will then be the diameters of the segments whose bases are AB, AB', and the centres of gravity of those segments will lie respectively on QD, $Q'D'$ [Prop. 4]. Let them be H, H', and join HH' meeting AO in K.

Now QD, $Q'D'$ are equal*, and therefore the segments of which they are the diameters are equal [*On Conoids and Spheroids*, Prop. 3].

* This may either be inferred from Lemma (1) above (since QQ', DD' are both parallel to BB'), or from Prop. 19 of the *Quadrature of the Parabola*, which applies equally to Q or Q'.

Also, since QD, $Q'D'$ are parallel*, and $DE = ED'$, K is the middle point of HH'.

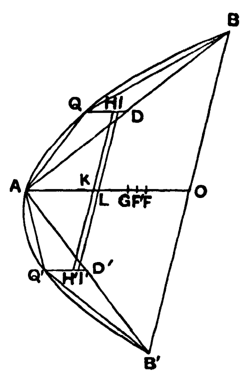
Hence the centre of gravity of the equal segments AQB, $AQ'B'$ taken together is K, where K lies between E and A. And the centre of gravity of the triangle ABB' is F.

It follows that the centre of gravity of the whole segment BAB' lies between K and F, and is therefore nearer to the vertex A than F is.

Secondly, take the *five-sided* figure $BQAQ'B'$ inscribed 'in the recognised manner,' QD, $Q'D'$ being, as before, the diameters of the segments AQB, $AQ'B'$.

Then, by the first part of this proposition, the centre of gravity of the segment AQB (lying of course on QD) is nearer to Q than the centre of gravity of the triangle AQB is. Let the centre of gravity of the segment be H, and that of the triangle I.

Similarly let H' be the centre of gravity of the segment $AQ'B'$, and I' that of the triangle $AQ'B'$.



It follows that the centre of gravity of the two segments AQB, $AQ'B'$ taken together is K, the middle point of HH', and that of the two triangles AQB, $AQ'B'$ is L, the middle point of II'.

If now the centre of gravity of the triangle ABB' be F, the centre of gravity of the whole segment BAB' (i.e. that of the triangle ABB' and the two segments AQB, $AQ'B'$ taken together) is a point G on KF determined by the proportion

(sum of segments AQB, $AQ'B'$) : $\triangle ABB' = FG : GK$. [I. 6, 7]

* There is clearly some interpolation in the text here, which has the words καὶ ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΘΖΗΙ. It is not yet proved that $H'D'DH$ is a *parallelogram*; this can only be inferred from the fact that H, H' divide QD, $Q'D'$ respectively in the same ratio. But this latter property does not appear till Prop. 7, and is then only enunciated of *similar* segments. The interpolation must have been made before Eutocius' time, because he has a note on the phrase, and explains it by gravely assuming that H, H' divide QD, $Q'D'$ respectively in the same ratio.

And the centre of gravity of the inscribed figure $BQAQ'B'$ is a point F' on LF determined by the proportion

$$(\triangle AQB + \triangle AQ'B') : \triangle ABB' = FF' : F'L. \quad [\text{I. } 6, 7]$$

[Hence $\qquad FG : GK > FF' : F'L,$

or $\qquad GK : FG < F'L : FF',$

and, *componendo*, $\qquad FK : FG < FL : FF'$, while $FK > FL$.]

Therefore $FG > FF'$, or G lies nearer than F' to the vertex A.

Using this last result, and proceeding in the same way, we can prove the proposition for *any* figure inscribed 'in the recognised manner.'

Proposition 6.

Given a segment of a parabola cut off by a straight line, it is possible to inscribe in it 'in the recognised manner' a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length.

Let BAB' be the segment, AO its diameter, G its centre of gravity, and ABB' the triangle inscribed 'in the recognised manner.'

Let D be the assigned length and S an area such that

$$AG : D = \triangle ABB' : S.$$

In the segment inscribe 'in the recognised manner' a figure such that the sum of the segments left over is less than S. Let F be the centre of gravity of the inscribed figure.

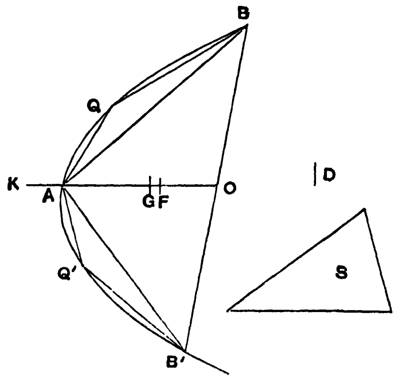
We shall prove that $FG < D$.

For, if not, FG must be either equal to, or greater than, D.

And clearly

$$\begin{aligned}(\text{inscribed fig.}) &: (\text{sum of remaining segmts.})\\ &> \triangle ABB' : S\\ &> AG : D\\ &> AG : FG, \text{ by hypothesis (since } FG \not< D).\end{aligned}$$

Let the first ratio be equal to the ratio $KG : FG$ (where K lies on GA produced); and it follows that K is the centre of gravity of the small segments taken together. [I. 8]



But this is impossible, since the segments are all on the same side of a line drawn through K parallel to BB'.

Hence FG cannot but be less than D.

Proposition 7.

If there be two similar parabolic segments, their centres of gravity divide their diameters in the same ratio.

[This proposition, though enunciated of *similar* segments only, like Prop. 3 on which it depends, is equally true of *any* segments. This fact did not escape Archimedes, who uses the proposition in its more general form for the proof of Prop. 8 immediately following.]

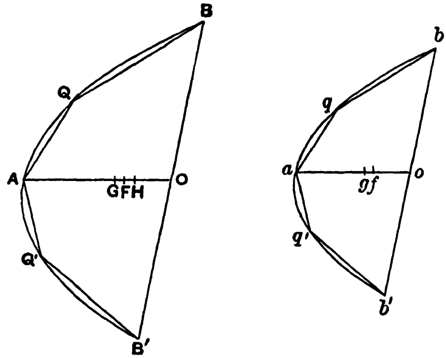
Let BAB', bab' be the two similar segments, AO, ao their diameters, and G, g their centres of gravity respectively.

Then, if G, g do not divide AO, ao respectively in the same ratio, suppose H to be such a point on AO that

$$AH : HO = ag : go;$$

and inscribe in the segment BAB' 'in the recognised manner' a figure such that, if F be its centre of gravity,

$$GF < GH. \qquad \text{[Prop. 6]}$$



Inscribe in the segment bab' 'in the recognised manner' a similar figure; then, if f be the centre of gravity of this figure,

$$ag < af. \qquad \text{[Prop. 5]}$$

And, by Prop. 3, $af : fo = AF : FO$.

But $AF : FO < AH : HO$

$< ag : go$, by hypothesis.

Therefore $af : fo < ag : go$; which is impossible.

It follows that G, g cannot but divide AO, ao in the same ratio.

Proposition 8.

If AO be the diameter of a parabolic segment, and G its centre of gravity, then

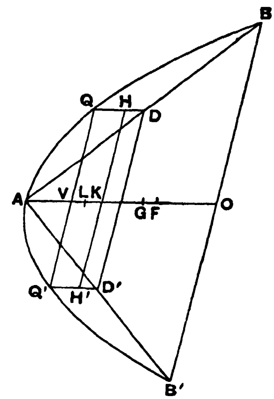
$$AG = \tfrac{3}{2}\, GO.$$

Let the segment be BAB'. Inscribe the triangle ABB' 'in the recognised manner,' and let F be its centre of gravity.

Bisect AB, AB' in D, D', and draw DQ, $D'Q'$ parallel to OA to meet the curve, so that QD, $Q'D'$ are the diameters of the segments AQB, $AQ'B'$ respectively.

Let H, H' be the centres of gravity of the segments AQB, $AQ'B'$ respectively. Join QQ', HH' meeting AO in V, K respectively.

K is then the centre of gravity of the two segments AQB, $AQ'B'$ taken together.



Now $AG : GO = QH : HD$, [Prop. 7]

whence $AO : OG = QD : HD$.

But $AO = 4QD$ [as is easily proved by means of Lemma (3), p. 206].

Therefore $OG = 4HD$;

and, by subtraction, $AG = 4QH$.

Also, by Lemma (2), QQ' is parallel to BB' and therefore to DD'. It follows from Prop. 7 that HH' is also parallel to QQ' or DD',

and hence $QH = VK$.

Therefore $AG = 4VK$,

and $AV + KG = 3VK$.

Measuring VL along VK so that $VL = \frac{1}{3}AV$, we have

$$KG = 3LK \quad\text{.........................}(1).$$

Again $AO = 4AV$ [Lemma (3)]

$= 3AL$, since $AV = 3VL$,

whence $AL = \frac{1}{3}AO = OF$ .... (2).

Now, by I. 6, 7,

$$\triangle ABB' : (\text{sum of segmts. } AQB,\ AQ'B') = KG : GF,$$

and $\triangle ABB' = 3$ (sum of segments AQB, $AQ'B'$)

[since the segment ABB' is equal to $\frac{4}{3}\triangle ABB'$ (*Quadrature of the Parabola*, Props. 17, 24)].

Hence $KG = 3GF$.

But $KG = 3LK$, from (1) above.

Therefore $LF = LK + KG + GF$

$= 5GF$.

And, from (2),

$$LF = (AO - AL - OF) = \tfrac{1}{3} AO = OF.$$

Therefore $$OF = 5GF,$$
and $$OG = 6GF.$$
But $$AO = 3OF = 15GF.$$

Therefore, by subtraction,

$$AG = 9GF$$
$$= \tfrac{3}{2} GO.$$

Proposition 9 (Lemma).

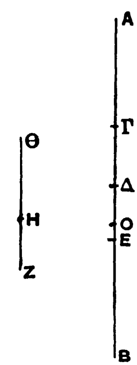
If a, b, c, d be four lines in continued proportion and in descending order of magnitude, and if

$$d : (a-d) = x : \tfrac{3}{5}(a-c),$$

and $(2a + 4b + 6c + 3d) : (5a + 10b + 10c + 5d) = y : (a - c)$,
it is required to prove that

$$x + y = \tfrac{2}{5} a.$$

[The following is the proof given by Archimedes, with the only difference that it is set out in algebraical instead of geometrical notation. This is done in the particular case simply in order to make the proof easier to follow. Archimedes exhibits his lines in the figure reproduced in the margin, but, now that it is possible to use algebraical notation, there is no advantage in using the figure and the more cumbrous notation which only obscures the course of the proof. The relation between Archimedes' figure and the letters used below is as follows;



AB $= a$, ΓB $= b$, ΔB $= c$, EB $= d$, ZH $= x$, HΘ $= y$, ΔO $= z$.]

We have $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} \qquad (1),$$

whence $$\frac{a-b}{b} = \frac{b-c}{c} = \frac{c-d}{d},$$

and therefore $$\frac{a-b}{b-c} = \frac{b-c}{c-d} = \frac{a}{b} = \frac{b}{c} = \frac{c}{d} \qquad (2).$$

Now $$\frac{2(a+b)}{2c} = \frac{a+b}{c} = \frac{a+b}{b} \cdot \frac{b}{c} = \frac{a-c}{b-c} \cdot \frac{b-c}{c-d} = \frac{a-c}{c-d}.$$

And, in like manner,

$$\frac{b+c}{d}=\frac{b+c}{c}\cdot\frac{c}{d}=\frac{a-c}{c-d}.$$

It follows from the last two relations that

$$\frac{a-c}{c-d}=\frac{2a+3b+c}{2c+d} \quad \text{.................. (3).}$$

Suppose z to be so taken that

$$\frac{2a+4b+4c+2d}{2c+d}=\frac{a-c}{z} \quad \text{............... (4),}$$

so that $z<(c-d)$.

Therefore $$\frac{a-c+z}{a-c}=\frac{2a+4b+6c+3d}{2(a+d)+4(b+c)}.$$

And, by hypothesis,

$$\frac{a-c}{y}=\frac{5(a+d)+10(b+c)}{2a+4b+6c+3d},$$

so that $$\frac{a-c+z}{y}=\frac{5(a+d)+10(b+c)}{2(a+d)+4(b+c)}=\frac{5}{2} \quad \text{........ (5).}$$

Again, dividing (3) by (4) crosswise, we obtain

$$\frac{z}{c-d}=\frac{2a+3b+c}{2(a+d)+4(b+c)},$$

whence $$\frac{c-d-z}{c-d}=\frac{b+3c+2d}{2(a+d)+4(b+c)} \quad \text{............ (6).}$$

But, by (2),

$$\frac{c-d}{d}=\frac{a-b}{b}=\frac{3(b-c)}{3c}=\frac{2(c-d)}{2d},$$

so that $$\frac{c-d}{d}=\frac{(a-b)+3(b-c)+2(c-d)}{b+3c+2d} \quad \text{......... (7).}$$

Combining (6) and (7), we have

$$\frac{c-d-z}{d}=\frac{(a-b)+3(b-c)+2(c-d)}{2(a+d)+4(b+c)},$$

whence $$\frac{c-z}{d}=\frac{3a+6b+3c}{2(a+d)+4(b+c)} \quad \text{............... (8).}$$

And, since [by (1)]

$$\frac{c-d}{c+d}=\frac{b-c}{b+c}=\frac{a-b}{a+b},$$

we have $$\frac{c-d}{a-c} = \frac{c+d}{b+c+a+b},$$

whence $$\frac{a-d}{a-c} = \frac{a+2b+2c+d}{a+2b+c} = \frac{2(a+d)+4(b+c)}{2(a+c)+4b} \quad\ldots\ldots(9).$$

Thus $$\frac{a-d}{\frac{3}{5}(a-c)} = \frac{2(a+d)+4(b+c)}{\frac{3}{5}\{2(a+c)+4b\}},$$

and therefore, by hypothesis,

$$\frac{d}{x} = \frac{2(a+d)+4(b+c)}{\frac{3}{5}\{2(a+c)+4b\}}.$$

But, by (8), $$\frac{c-z}{d} = \frac{3a+6b+3c}{2(a+d)+4(b+c)};$$

and it follows, *ex aequali*, that

$$\frac{c-z}{x} = \frac{3(a+c)+6b}{\frac{3}{5}\{2(a+c)+4b\}} = \frac{5}{3}\cdot\frac{3}{2} = \frac{5}{2}.$$

And, by (5), $$\frac{a-c+z}{y} = \frac{5}{2}.$$

Therefore $$\frac{5}{2} = \frac{a}{x+y},$$

or $$x+y = \tfrac{2}{5}a.$$

Proposition 10.

If $PP'B'B$ be the portion of a parabola intercepted between two parallel chords PP', BB' bisected respectively in N, O by the diameter ANO (N being nearer than O to A, the vertex of the segments), and if NO be divided into five equal parts of which LM is the middle one (L being nearer than M to N), then, if G be a point on LM such that

$$LG : GM = BO^2 . (2PN + BO) : PN^2 . (2BO + PN),$$

G will be the centre of gravity of the area $PP'B'B$.

Take a line ao equal to AO, and an on it equal to AN. Let p, q be points on the line ao such that

$$ao : aq = aq : an \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$ao : an = aq : ap \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

[whence $ao : aq = aq : an = an : ap$, or ao, aq, an, ap are lines in continued proportion and in descending order of magnitude].

Measure along GA a length GF such that

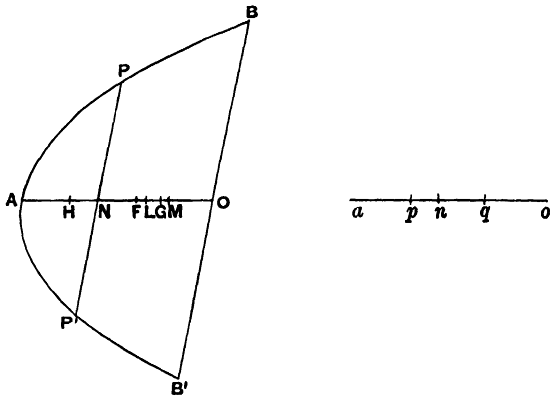
$$op : ap = OL : GF \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Then, since PN, BO are ordinates to ANO,

$$BO^2 : PN^2 = AO : AN$$
$$= ao : an$$
$$= ao^2 : aq^2, \text{ by (1)},$$

so that $$BO : PN = ao : aq \quad\ldots\ldots\ldots\ldots (4),$$

and $$BO^3 : PN^3 = ao^3 : aq^3$$
$$= (ao : aq) . (aq : an) . (an : ap)$$
$$= ao : ap \quad\ldots\ldots\ldots\ldots (5).$$



Thus (segment BAB') : (segment PAP')

$$= \triangle BAB' : \triangle PAP'$$
$$= BO^3 : PN^3$$
$$= ao : ap,$$

whence

$$(\text{area } PP'B'B) : (\text{segment } PAP') = op : ap$$
$$= OL : GF, \text{ by (3)},$$
$$= \tfrac{3}{5}ON : GF \quad\ldots\ldots\ldots (6).$$

Now $$BO^2 . (2PN + BO) : BO^3 = (2PN + BO) : BO$$
$$= (2aq + ao) : ao, \text{ by (4)},$$
$$BO^3 : PN^3 = ao : ap, \text{ by (5)},$$

and $$PN^3 : PN^2 . (2BO + PN) = PN : (2BO + PN)$$
$$= aq : (2ao + aq), \text{ by (4)},$$
$$= ap : (2an + ap), \text{ by (2)}.$$

Hence, *ex aequali*,

$$BO^2 . (2PN + BO) : PN^2 . (2BO + PN) = (2aq + ao) : (2an + ap),$$

so that, by hypothesis,

$$LG : GM = (2aq + ao) : (2an + ap).$$

Componendo, and multiplying the antecedents by 5,

$$ON : GM = \{5 (ao + ap) + 10 (aq + an)\} : (2an + ap).$$

But $ON : OM = 5 : 2$

$$= \{5 (ao + ap) + 10 (aq + an)\} : \{2 (ao + ap) + 4 (aq + an)\}.$$

It follows that

$$ON : OG = \{5 (ao + ap) + 10 (aq + an)\} : (2ao + 4aq + 6an + 3ap).$$

Therefore

$$(2ao + 4aq + 6an + 3ap) : \{5 (ao + ap) + 10 (aq + an)\} = OG : ON$$
$$= OG : on.$$

And $\qquad ap : (ao - ap) = ap : op$

$= GF : OL$, by hypothesis,

$= GF : \frac{3}{5} on$,

while ao, aq, an, ap are in continued proportion.

Therefore, by Prop. 9,

$$GF + OG = OF = \tfrac{3}{5} ao = \tfrac{3}{5} OA.$$

Thus F is the centre of gravity of the segment BAB'. [Prop. 8]

Let H be the centre of gravity of the segment PAP', so that $AH = \frac{3}{5} AN$.

And, since $\qquad AF = \frac{3}{5} AO$,

we have, by subtraction, $HF = \frac{3}{5} ON$.

But, by (6) above,

$$(\text{area } PP'B'B) : (\text{segment } PAP') = \tfrac{3}{5} ON : GF$$
$$= HF : FG.$$

Thus, since F, H are the centres of gravity of the segments BAB', PAP' respectively, it follows [by I. 6, 7] that G is the centre of gravity of the area $PP'B'B$.

ON FLOATING BODIES.

BOOK I.

Postulate 1.

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else."

Proposition 1.

If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere.

For, if not, there will be some two lines drawn from the point to the surface which are not equal.

Suppose O to be the fixed point, and A, B to be two points on the surface such that OA, OB are unequal. Let the surface be cut by a plane passing through OA, OB. Then the section is, by hypothesis, a circle whose centre is O.

Thus $OA = OB$; which is contrary to the assumption. Therefore the surface cannot but be a sphere.

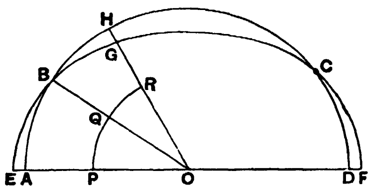
Proposition 2.

The surface of any fluid at rest is the surface of a sphere whose centre is the same as that of the earth.

Suppose the surface of the fluid cut by a plane through O, the centre of the earth, in the curve $ABCD$.

$ABCD$ shall be the circumference of a circle.

For, if not, some of the lines drawn from O to the curve will be unequal. Take one of them, OB, such that OB is greater than some of the lines from O to the curve and less than others. Draw a circle with OB as radius. Let it be EBF, which will therefore fall partly within and partly without the surface of the fluid.



Draw OGH making with OB an angle equal to the angle EOB, and meeting the surface in H and the circle in G. Draw also in the plane an arc of a circle PQR with centre O and within the fluid.

Then the parts of the fluid along PQR are uniform and continuous, and the part PQ is compressed by the part between it and AB, while the part QR is compressed by the part between QR and BH. Therefore the parts along PQ, QR will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compressed the more.

Therefore there will not be rest; which is contrary to the hypothesis.

Hence the section of the surface will be the circumference of a circle whose centre is O; and so will all other sections by planes through O.

Therefore the surface is that of a sphere with centre O.

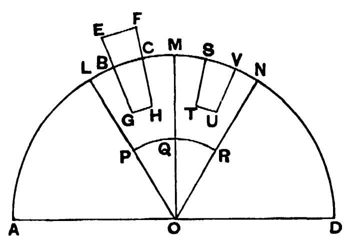
Proposition 3.

Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.

If possible, let a certain solid *EFHG* of equal weight, volume for volume, with the fluid remain immersed in it so that part of it, *EBCF*, projects above the surface.

Draw through *O*, the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle *ABCD*.

Conceive a pyramid with vertex *O* and base a parallelogram at the surface of the fluid, such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of *ABCD* in *OL*, *OM*. Also let a sphere within the fluid and below *GH* be described with centre *O*, and let the plane of *ABCD* cut this sphere in *PQR*.



Conceive also another pyramid in the fluid with vertex *O*, continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut in *OM*, *ON* by the plane of *ABCD*.

Lastly, let *STUV* be a part of the fluid within the second pyramid equal and similar to the part *BGHC* of the solid, and let *SV* be at the surface of the fluid.

Then the pressures on *PQ*, *QR* are unequal, that on *PQ* being the greater. Hence the part at *QR* will be set in motion by that at PQ, and the fluid will not be at rest; which is contrary to the hypothesis.

Therefore the solid will not stand out above the surface.

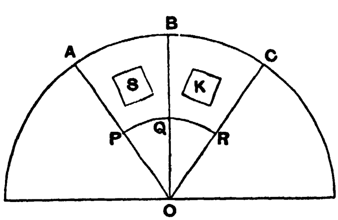
Nor will it sink further, because all the parts of the fluid will be under the same pressure.

Proposition 4.

A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.

In this case, after the manner of the previous proposition, we assume the solid, if possible, to be completely submerged and the fluid to be at rest in that position, and we conceive (1) a pyramid with its vertex at O, the centre of the earth, including the solid, (2) another pyramid continuous with the former and equal and similar to it, with the same vertex O, (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid, (4) a sphere with centre O whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre O cutting the surface of the fluid in the circle ABC, the solid in S, the first pyramid in OA, OB, the second pyramid in OB, OC, the portion of the fluid in the second pyramid in K, and the inner sphere in PQR.

Then the pressures on the parts of the fluid at PQ, QR are unequal, since S is lighter than K. Hence there will not be rest; which is contrary to the hypothesis.

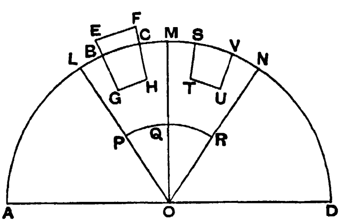


Therefore the solid S cannot, in a condition of rest, be completely submerged.

Proposition 5.

Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.

For let the solid be $EGHF$, and let $BGHC$ be the portion of it immersed when the fluid is at rest. As in Prop. 3, conceive a pyramid with vertex O including the solid, and another pyramid with the same vertex continuous with the former and equal and similar to it. Suppose a portion of the fluid $STUV$ at the base of the second pyramid to be equal and similar to the immersed portion of the solid; and let the construction be the same as in Prop. 3.



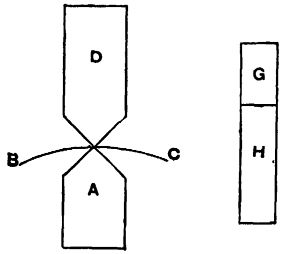
Then, since the pressure on the parts of the fluid at PQ, QR must be equal in order that the fluid may be at rest, it follows that the weight of the portion $STUV$ of the fluid must be equal to the weight of the solid $EGHF$. And the former is equal to the weight of the fluid displaced by the immersed portion of the solid $BGHC$.

Proposition 6.

If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.

For let A be completely immersed in the fluid, and let G represent the weight of A, and $(G + H)$ the weight of an equal volume of the fluid. Take a solid D, whose weight is H

and add it to A. Then the weight of $(A+D)$ is less than that of an equal volume of the fluid; and, if $(A+D)$ is immersed in the fluid, it will project so that its weight will be equal to the weight of the fluid displaced. But its weight is $(G+H)$.



Therefore the weight of the fluid displaced is $(G+H)$, and hence the volume of the fluid displaced is the volume of the solid A. There will accordingly be rest with A immersed and D projecting.

Thus the weight of D balances the upward force exerted by the fluid on A, and therefore the latter force is equal to H, which is the difference between the weight of A and the weight of the fluid which A displaces.

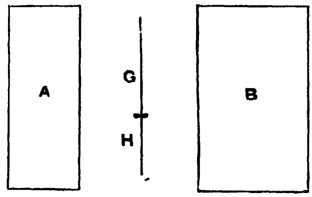
Proposition 7.

A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

(1) The first part of the proposition is obvious, since the part of the fluid under the solid will be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom.

(2) Let A be a solid heavier than the same volume of the fluid, and let $(G+H)$ represent its weight, while G represents the weight of the same volume of the fluid.

Take a solid B lighter than the same volume of the fluid, and such that the weight of B is G, while the weight of the same volume of the fluid is $(G+H)$.



Let A and B be now combined into one solid and immersed. Then, since $(A+B)$ will be of the same weight as the same volume of fluid, both weights being equal to $(G+H)+G$, it follows that $(A+B)$ will remain stationary in the fluid.

Therefore the force which causes A by itself to sink must be equal to the upward force exerted by the fluid on B by itself. This latter is equal to the difference between $(G+H)$ and G [Prop. 6]. Hence A is depressed by a force equal to H, i.e. its weight in the fluid is H, or the difference between $(G+H)$ and G.

[This proposition may, I think, safely be regarded as decisive of the question how Archimedes determined the proportions of gold and silver contained in the famous crown (cf. Introduction, Chapter I.). The proposition suggests in fact the following method.

Let W represent the weight of the crown, w_1 and w_2 the weights of the gold and silver in it respectively, so that $W=w_1+w_2$.

(1) Take a weight W of pure gold and weigh it in a fluid. The apparent loss of weight is then equal to the weight of the fluid displaced. If F_1 denote this weight, F_1 is thus known as the result of the operation of weighing.

It follows that the weight of fluid displaced by a weight w_1 of gold is $\frac{w_1}{W}\cdot F_1$.

(2) Take a weight W of pure silver and perform the same operation. If F_2 be the loss of weight when the silver is weighed in the fluid, we find in like manner that the weight of fluid displaced by w_2 is $\frac{w_2}{W} \cdot F_2$.

(3) Lastly, weigh the crown itself in the fluid, and let F be the loss of weight. Therefore the weight of fluid displaced by the crown is F.

It follows that $$\frac{w_1}{W} \cdot F_1 + \frac{w_2}{W} \cdot F_2 = F,$$

or $$w_1 F_1 + w_2 F_2 = (w_1 + w_2) F,$$

whence $$\frac{w_1}{w_2} = \frac{F_2 - F}{F - F_1}.$$

This procedure corresponds pretty closely to that described in the poem *de ponderibus et mensuris* (written probably about 500 A.D.)* purporting to explain Archimedes' method. According to the author of this poem, we first take two equal weights of pure gold and pure silver respectively and weigh them against each other when both immersed in water; this gives the relation between their weights in water and therefore between their loss of weight in water. Next we take the mixture of gold and silver and an equal weight of pure silver and weigh them against each other in water in the same manner.

The other version of the method used by Archimedes is that given by Vitruvius†, according to which he measured successively the *volumes* of fluid displaced by three equal weights, (1) the crown, (2) the same weight of gold, (3) the same weight of silver, respectively. Thus, if as before the weight of the crown is W, and it contains weights w_1 and w_2 of gold and silver respectively,

(1) the crown displaces a certain quantity of fluid, V say.

(2) the weight W of gold displaces a certain volume of

* Torelli's *Archimedes*, p. 364; Hultsch, *Metrol. Script.* II. 95 sq., and Prolegomena § 118.

† *De architect.* IX. 3.

fluid, V_1 say; therefore a weight w_1 of gold displaces a volume $\frac{w_1}{W} \cdot V_1$ of fluid.

(3) the weight W of silver displaces a certain volume of fluid, say V_2; therefore a weight w_2 of silver displaces a volume $\frac{w_2}{W} \cdot V_2$ of fluid.

It follows that $$V = \frac{w_1}{W} \cdot V_1 + \frac{w_2}{W} \cdot V_2,$$

whence, since $$W = w_1 + w_2,$$
$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1};$$

and this ratio is obviously equal to that before obtained, viz. $\frac{F_2 - F}{F - F_1}$.]

Postulate 2.

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity."

Proposition 8.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in such a position that its axis is perpendicular to the surface; and, if the solid be forced into such a position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position.

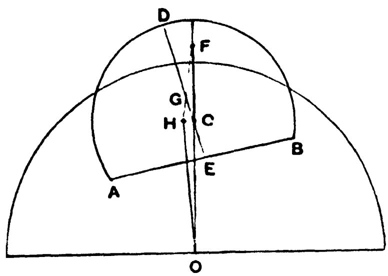
[The proof of this proposition is wanting in the Latin version of Tartaglia. Commandinus supplied a proof of his own in his edition.]

Proposition 9.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base is completely below the surface, the solid will rest in such a position that its axis is perpendicular to the surface.

[The proof of this proposition has only survived in a mutilated form. It deals moreover with only one case out of three which are distinguished at the beginning, viz. that in which the segment is greater than a hemisphere, while figures only are given for the cases where the segment is equal to, or less than, a hemisphere.]

Suppose, first, that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth; and, if possible, let it be at rest in the position shown in the figure, where AB is the intersection of the plane with the base of the segment, DE its axis, C the centre of the sphere of which the segment is a part, O the centre of the earth.



The centre of gravity of the portion of the segment outside the fluid, as F, lies on OC produced, its axis passing through C.

Let G be the centre of gravity of the segment. Join FG, and produce it to H so that

$FG : GH =$ (volume of immersed portion) : (rest of solid).

Join OH.

Then the weight of the portion of the solid outside the fluid acts along FO, and the pressure of the fluid on the immersed portion along OH, while the weight of the immersed portion acts along HO and is by hypothesis less than the pressure of the fluid acting along OH.

Hence there will not be equilibrium, but the part of the segment towards A will ascend and the part towards B descend, until DE assumes a position perpendicular to the surface of the fluid.