Theorem 3.52

(a) Finite dimensional normed vector spaces are separable.

(b) An infinite-dimensional Hilbert space $\mathcal{H}$ is separable if and only if it has an orthonormal basis.

Proof

(a) Let X be a finite-dimensional, real normed vector space, and let $\{e_1, \ldots, e_k\}$ be a basis for X. Then the set of vectors having the form $\sum_{n=1}^{k} \alpha_n e_n$, with α_n a rational is countable and dense (the proof of density is similar to the proof below in part (b)), so X is separable. In the complex case we use complex rational coefficients α_n.

(b) ($\Rightarrow$) Suppose that $\mathcal{H}$ is infinite-dimensional and separable, and let $\{x_n\}$ be a countable, dense sequence in $\mathcal{H}$. We construct a new sequence $\{y_n\}$ by omitting every member of the sequence $\{x_n\}$ which is a linear combination of the preceding members of the sequence. By this construction the sequence $\{y_n\}$ is linearly independent. Now, by inductively applying the Gram–Schmidt algorithm (see the proof of part (b) of Lemma 3.20) to the sequence $\{y_n\}$ we can construct an orthonormal sequence $\{e_n\}$ in $\mathcal{H}$ with the property that for each $k \geq 1$, $\mathrm{Sp}\,\{e_1, \ldots, e_k\} = \mathrm{Sp}\,\{y_1, \ldots, y_k\}$. Thus,

$$\mathrm{Sp}\,\{e_n : n \in \mathbb{N}\} = \mathrm{Sp}\,\{y_n : n \in \mathbb{N}\} = \mathrm{Sp}\,\{x_n : n \in \mathbb{N}\}.$$

Since the sequence $\{x_n\}$ is dense in $\mathcal{H}$ it follows that $\overline{\mathrm{Sp}}\,\{e_n : n \in \mathbb{N}\} = \mathcal{H}$, and so, by Theorem 3.47, $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. → (P15 3.49)

($\Leftarrow$) Now suppose that $\mathcal{H}$ has an orthonormal basis $\{e_n\}$. The set of elements $x \in \mathcal{H}$ expressible as a finite sum of the form $x = \sum_{n=1}^{k} \alpha_n e_n$, with $k \in \mathbb{N}$ and rational (or complex rational) coefficients α_n, is clearly countable, so to show that $\mathcal{H}$ is separable we must show that this set is dense. To do this, choose arbitrary $y \in \mathcal{H}$ and $\epsilon > 0$. Then y can be written in the form $y = \sum_{n=1}^{\cancel{k}\,\infty} \beta_n e_n$, with $\sum_{n=1}^{\infty} |\beta_n|^2 < \infty$, so there exists an integer N such that $\sum_{n=N+1}^{\infty} |\beta_n|^2 < \epsilon^2/2$. Now, for each $n = 1, \ldots, N$, we choose rational coefficients α_n, such that $|\beta_n - \alpha_n|^2 < \epsilon^2/2N$, and let $x = \sum_{n=1}^{k} \alpha_n e_n$. Then

$$\|y - x\|^2 = \sum_{n=1}^{N} |\beta_n - \alpha_n|^2 + \sum_{n=N+1}^{\infty} |\beta_n|^2 < \epsilon^2,$$

which shows that the above set is dense, by part (f) of Theorem 1.25. □

Example 3.53

The Hilbert space ℓ^2 is separable.

It will be shown in Section 3.5 that the space $L^2[a,b]$, a, $b \in \mathbb{R}$, has an orthonormal basis, so is also separable. In addition, by an alternative argument it will be shown that $C[a,b]$ is separable. In fact, most spaces which arise in applications are separable.

EXERCISES

3.21 Use Example 3.46 to find an orthonormal sequence in a Hilbert space $\mathcal{H}$ and a vector $x \in \mathcal{H}$ for which Bessel's inequality holds with strict inequality.

3.22 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in $\mathcal{H}$. Determine whether the following series converge in $\mathcal{H}$:

(a) $\displaystyle\sum_{n=1}^{\infty} n^{-1}e_n$; (b) $\displaystyle\sum_{n=1}^{\infty} n^{-1/2}e_n$.

3.23 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal basis in $\mathcal{H}$. Let $\rho : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$ (so that for all $x \in \mathcal{H}$, $\sum_{n=1}^{\infty} |(x, e_{\rho(n)})|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$). Show that:

(a) $\displaystyle\sum_{n=1}^{\infty}(x, e_{\rho(n)})e_n$ converges for all $x \in \mathcal{H}$;

(b) $\displaystyle\|\sum_{n=1}^{\infty}(x, e_{\rho(n)})e_n\|^2 = \|x\|^2$ for all $x \in \mathcal{H}$.

3.24 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal basis in $\mathcal{H}$. Prove that the *Parseval relation*

$$(x, y) = \sum_{n=1}^{\infty}(x, e_n)(e_n, y)$$

holds for all x, $y \in \mathcal{H}$.

3.25 Show that a metric space M is separable if and only if M has a countable subset A with the property: for every integer $k \geq 1$ and every $x \in X$ there exists $a \in A$ such that $d(x, a) < 1/k$.

Show that any subset N of a separable metric space M is separable. [Note: separability of M ensures that there is a countable dense

Series Fourier

Teo 1.39 (Stone - Weierstrauss). Para $M \subset \mathbb{R}$

$P_{\mathbb{R}}$ (polinomios valores reales) es denso

en $C_{\mathbb{R}}(M)$

Teo 1.61 el conj $C[a,b]$ es denso en

$L^p[a,b]$ $1 \le p < \infty$

subset of M, but none of the elements of this set need belong to N. Thus it is necessary to construct a countable dense subset of N.]

3.26 Suppose that $\mathcal{H}$ is a separable Hilbert space and $Y \subset \mathcal{H}$ is a closed linear subspace. Show that there is an orthonormal basis for $\mathcal{H}$ consisting only of elements of Y and $Y^\perp$.

3.5 Fourier Series

In this section we will prove that the orthonormal sequence in Example 3.39 is a basis for $L^2_{\mathbb{C}}[-\pi,\pi]$, and we will also consider various related bases consisting of sets of trigonometric functions.

Theorem 3.54

The set of functions

$$C = \left\{c_0(x) = (1/\pi)^{1/2},\, c_n(x) = (2/\pi)^{1/2}\cos nx : n \in \mathbb{N}\right\}$$

is an orthonormal basis in $L^2[0,\pi]$.

Proof

We first consider $L^2_{\mathbb{R}}[0,\pi]$. It is easy to check that C is orthonormal. Thus by Theorem 3.47 we must show that $\operatorname{Sp} C$ is dense in $L^2_{\mathbb{R}}[0,\pi]$. We will combine the approximation properties in Theorems 1.39 and 1.61 to do this. $\rightarrow f \in L^2[0,\pi]$

Firstly, by Theorem 1.61 there is a function $g_1 \in C_{\mathbb{R}}[0,\pi]$ with $\|g_1 - f\| < \epsilon/2$ (here, $\|\cdot\|$ denotes the $L^2_{\mathbb{R}}[0,\pi]$ norm). Thus it is sufficient to show that for any function $g_1 \in C_{\mathbb{R}}[0,\pi]$ there is a function $g_2 \in \operatorname{Sp} C$ with $\|g_2 - g_1\| < \epsilon/2$ (it will then follow that there is a function $g_2 \in \operatorname{Sp} C$ such that $\|g_2 - f\| < \epsilon$).

Now suppose that $g_1 \in C_{\mathbb{R}}[0,\pi]$ is arbitrary. We recall that the function $\cos^{-1} : [-1,1] \to [0,\pi]$ is a continuous bijection, so we may define a function $h \in C_{\mathbb{R}}[-1,1]$ by $h(s) = g_1(\cos^{-1} s)$ for $s \in [-1,1]$. It follows from Theorem 1.39 that there is a polynomial p such that $|h(s) - p(s)| \overset{\text{II}}{<} \epsilon/2\sqrt{\pi}$ for all $s \in [-1,1]$, and hence, writing $g_2(x) = p(\cos x)$, we have $|g_2(x) - g_1(x)| < \epsilon/2\sqrt{\pi}$ for all $x \in [0,\pi]$, and so $\|g_2 - g_1\| \overset{\text{I}}{<} \epsilon/2$. But standard trigonometry now shows that any polynomial in $\cos x$ of the form $\sum_{n=0}^m \alpha_n(\cos x)^n$ can be rewritten in the form $\sum_{n=0}^m \beta_n \cos nx$, which shows that $g_2 \in \operatorname{Sp} C$, and so completes the proof in the real case.

II
en $C_{\mathbb{R}}[-1,1]$
norme infinie

In the complex case, for any $f \in L^2_{\mathbb{C}}[0,\pi]$ we let $f_{\mathbb{R}}, f_{\mathbb{C}} \in L^2_{\mathbb{R}}[0,\pi]$ denote the functions obtained by taking the real and imaginary parts of f, and we apply

I $\|g_2 - g_1\| = \left(\int_0^\pi |g_2 - g_1|^2\right)^{\frac{1}{2}} \leq \frac{\epsilon}{2\sqrt{\pi}}$

.) Seen T_n base de pol de chenshev de 1ea clase

$T_0(x)=1$ $\quad T_{n+1}(x)=2xT_n(x)-T_{n-1}(x) \Rightarrow \cos nx = T_n(\cos x)$

$T_1(x)=x$

.) $\cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow \cos 2x = 2\cos^2 x - 1$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$\cos 3x = \cos(2x+x) = \dots = 4\cos(x)^3 - 3\cos(x)$$

the result just proved to these functions to obtain

$$f = f_{\mathbb{R}} + if_{\mathbb{C}} = \sum_{n=0}^{\infty} \alpha_n c_n + i \sum_{n=0}^{\infty} \beta_n c_n = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) c_n,$$

which proves the result in the complex case. □

From Theorems 3.52 and 3.54 we have the following result.

Corollary 3.55

The space $L^2[0, \pi]$ is separable.

Theorem 3.56

The set of functions

$$S = \left\{ s_n(x) = (2/\pi)^{1/2} \sin nx : n \in \mathbb{N} \right\}$$

is an orthonormal basis in $L^2[0, \pi]$.

Proof

The proof is similar to the proof of the previous theorem so we will merely sketch it. This time we first approximate f (in $L^2_{\mathbb{R}}[0, \pi]$) by a function f_δ, with $\delta > 0$, defined by

$$g_1(x) = f_\delta(x) = \begin{cases} 0, & \text{if } x \in [0, \delta], \\ f(x), & \text{if } x \in (\delta, \pi] \end{cases}$$

(clearly, $\|f - f_\delta\|$ can be made arbitrarily small by choosing δ sufficiently small). Then the function $g_1(x)/\sin x$ belongs to $L^2_{\mathbb{R}}[0, \pi]$, so by the previous proof it can be approximated by functions of the form $\sum_{n=0}^{m} \alpha_n \cos nx$, and hence $g_1(x)$ can be approximated by functions of the form

$$\sum_{n=0}^{m} \alpha_n \cos nx \sin x = \frac{1}{2} \sum_{n=0}^{m} \alpha_n (\sin(n+1)x - \sin(n-1)x).$$ (Chevyshev)

The latter function is an element of $\operatorname{Sp} S$, which completes the proof. □

It follows from Theorems 3.54, 3.56, and 3.47, that an arbitrary function $f \in L^2[0, \pi]$ can be represented in either of the forms

$$\overset{\circledast}{f} = \sum_{n=0}^{\infty} (f, c_n) c_n, \quad f = \sum_{n=1}^{\infty} (f, s_n) s_n, \tag{3.6}$$

ojo
⊛ son igualdades en norma $\|f - \sum^{k}\|_2 \xrightarrow{k \to \infty} 0$
pero no necesariamente vale la igualdad puntual

where the convergence is in the $L^2[0,\pi]$ sense. These series are called, respectively, *Fourier cosine* and *sine series* expansions of f. Other forms of Fourier series expansions can be obtained from the following corollary.

Corollary 3.57

The sets of functions

$$E = \{e_n(x) = (2\pi)^{-1/2}e^{inx} : n \in \mathbb{Z}\},$$
$$F = \{2^{-1/2}c_0,\ 2^{-1/2}c_n,\ 2^{-1/2}s_n : n \in \mathbb{N}\},$$

are orthonormal bases in the space $L^2_{\mathbb{C}}[-\pi,\pi]$. The set F is also an orthonormal basis in the space $L^2_{\mathbb{R}}[-\pi,\pi]$ (the set E is clearly not appropriate for the space $L^2_{\mathbb{R}}[-\pi,\pi]$ since the functions in E are complex).

$c_0(x) = (1/\pi)^{1/2}$ $\quad c_n(x) = (2/\pi)^{1/2}\cos nx$ $\quad s_n(x) = (2/\pi)^{1/2}\sin nx$

Proof

Again it is easy to check that the set F is orthonormal in $L^2_{\mathbb{F}}[-\pi,\pi]$. Suppose that F is not a basis for $L^2_{\mathbb{F}}[-\pi,\pi]$. Then, by part (a) of Theorem 3.47, there exists a non-zero function $f \in L^2_{\mathbb{F}}[-\pi,\pi]$ such that $(f,c_0) = 0$, $(f,c_n) = 0$ and $(f,s_n) = 0$, for all $n \in \mathbb{N}$, which can be rewritten as,

$$(f,c_0) \to \quad 0 = \int_{-\pi}^{\pi} f(x)\,dx = \int_0^{\pi} (f(x)+f(-x))\,dx,$$
$$(f,c_n) \to \quad 0 = \int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_0^{\pi} (f(x)+f(-x))\cos nx\,dx,$$
$$(f,s_n) \to \quad 0 = \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_0^{\pi} (f(x)-f(-x))\sin nx\,dx.$$

$\langle f(x)+f(-x), \cos nx\rangle = 0$ $\forall n \in \mathbb{N}$

$\to (f(x)+f(-x)) \in \left\{\sqrt{\tfrac{2}{\pi}}\cos nx : n \in \mathbb{N}\right\}^{\perp}$

$= 0$

tco 3.47

Thus, by part (a) of Theorem 3.47 and Theorems 3.54 and 3.56, it follows that for a.e. $x \in [0,\pi]$,

igualdad en L^2 ppx

ppx

$$f(x)+f(-x) = 0,$$
$$f(x)-f(-x) = 0,$$

$\to$ [illegible]

and hence $f(x) = 0$ for a.e. $x \in [-\pi,\pi]$. But this contradicts the assumption that $f \neq 0$ in $L^2_{\mathbb{F}}[-\pi,\pi]$, so F must be a basis. Next, it was shown in Example 3.39 that the set E is orthonormal in $L^2_{\mathbb{C}}[-\pi,\pi]$, and it follows from the formula $e^{in\theta} = \cos n\theta + i\sin n\theta$ that $\operatorname{Sp} E$ is equal to $\operatorname{Sp} F$, so E is also an orthonormal basis. □

The above results give the basic theory of Fourier series in an L^2 setting. This theory is simple and elegant, but there is much more to the theory of Fourier series than this. For instance, one could consider the convergence of the various series in the pointwise sense (that is, for each x in the interval concerned), or uniformly, for all x in the interval. A result in this area will be obtained in Corollary 7.29, but we will not consider these topics further here.

Finally, we note that there is nothing special about the interval $[0, \pi]$ (and $[-\pi, \pi]$) used above. By the change of variables $x \to \tilde{x} = a + (b-a)x/\pi$ in the above proofs we see that they are valid for a general interval $[a, b]$.

EXERCISES

3.27 Show that for any $b > a$ the set of polynomials with rational (or complex rational) coefficients is dense in the spaces: (a) $C[a, b]$; (b) $L^2[a, b]$.
Deduce that the space $C[a, b]$ is separable.

3.28 (Legendre polynomials) For each integer $n \geq 0$, define the polynomials

$$u_n(x) = (x^2 - 1)^n, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n u_n}{dx^n}$$

(clearly, u_n has order $2n$, while P_n has order n). The polynomials P_n are called *Legendre polynomials.* We consider these polynomials on the interval $[-1, 1]$, and let $\mathcal{H} = L^2[-1, 1]$, with the standard inner product $(\cdot, \cdot)$. Prove the following results.

(a) $d^{2n} u_n / dx^{2n}(x) \equiv (2n)!$.

(b) $(P_m, P_n) = 0$, for $m, n \geq 0$, $m \neq n$.

(c) $\|P_n\|^2 = (2^n n!)^2 \frac{2}{2n+1}$, for $n \geq 0$.

(d) $\left\{ e_n = \sqrt{\frac{2n+1}{2}} P_n : n \geq 0 \right\}$ is an orthonormal basis for $\mathcal{H}$.

[Hint: use integration by parts, noting that u_n, and its derivatives to order $n - 1$, are zero at ± 1.]

4
Linear Operators

4.1 Continuous Linear Transformations

Now that we have studied some of the properties of normed spaces we turn to look at functions which map one normed space into another. The simplest maps between two vector spaces are the ones which respect the linear structure, that is, the linear transformations. We recall the convention introduced in Chapter 1 that if we have two vector spaces X and Y and a linear transformation from X to Y it is taken for granted that X and Y are vector spaces over the same scalar field. Since normed vector spaces have a metric associated with the norm, and continuous functions between metric spaces are in general more important than functions which are not continuous, the important maps between normed vector spaces will be the *continuous linear transformations*.

After giving examples of these, we fix two normed spaces X and Y and look at the space of all continuous linear transformations from X to Y. We show this space is also a normed vector space and then study in more detail the cases when $Y = \mathbb{F}$ and when $Y = X$. In the latter case we will see that it is possible to define the product of continuous linear transformations and therefore, for some continuous linear transformations, the inverse of a continuous linear transformation. The final section of this chapter is devoted to determining when a continuous linear transformation has an inverse.

We start by studying continuous linear transformations. Before we look at examples of continuous linear transformations, it is convenient to give alternative characterizations of continuity for linear transformations. A notational con-

Def Sean X, Y normados (sobre el mismo $\mathbb{F}$)

Decimos que $T: X \to Y$ es Lineal si

$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad \forall x, y \in X$
$\alpha, \beta \in \mathbb{F}$

.) decimos T continua en $x_0 \in X$

si $\forall \varepsilon > 0 \ \exists \delta > 0$ tq $\|x - x_0\| \Rightarrow \|Tx - Tx_0\| < \varepsilon$

.) Si T es cont $\forall x \in X$ decimos T cont

.) T unif cont si $\forall \varepsilon > 0 \ \exists \delta > 0$ tq

$$\|x - y\| < \delta \Rightarrow \|Tx - Ty\| < \varepsilon$$

vention should be clarified here. If X and Y are normed spaces and $T : X \to Y$ is a linear transformation, the norm of an element of X and the norm of an element of Y will frequently occur in the same equation. We should therefore introduce notation which distinguishes between these norms. In practice we just use the symbol $\|\cdot\|$ for the norm on both spaces as it is usually easy to determine which space an element is in and therefore, implicitly, to which norm we are referring. We recall also that if we write down one of the spaces in Examples 2.2, 2.4, 2.5 and 2.6 without explicitly mentioning a norm, it is assumed that the norm on this space is the standard norm.

Lemma 4.1

Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear transformation. The following are equivalent:

(a) T is uniformly continuous;

(b) T is continuous;

(c) T is continuous at 0;

(d) there exists a positive real number k such that $\|T(x)\| \le k$ whenever $x \in X$ and $\|x\| \le 1$;

(e) there exists a positive real number k such that $\|T(x)\| \le k\|x\|$ for all $x \in X$.

Proof

The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) hold in more generality so all that is required to be proved is (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a).

(c) $\Rightarrow$ (d). As T is continuous at 0, taking $\epsilon = 1$ there exists a $\delta > 0$ such that $\|T(x)\| < 1$ when $x \in X$ and $\|x\| < \delta$. Let $w \in X$ with $\|w\| \le 1$. As

$$\|\frac{\delta w}{2}\| = \frac{\delta}{2}\|w\| \le \frac{\delta}{2} < \delta,$$

$\|T\left(\frac{\delta w}{2}\right)\| < 1$ and as T is a linear transformation $T\left(\frac{\delta w}{2}\right) = \frac{\delta}{2}T(w)$. Thus $\frac{\delta}{2}\|T(w)\| < 1$ and so $\|T(w)\| < \frac{2}{\delta}$. Therefore condition (d) holds with $k = \frac{2}{\delta}$.

(d) $\Rightarrow$ (e). Let k be such that $\|T(x)\| \le k$ whenever $x \in X$ and $\|x\| \le 1$. Since $T(0) = 0$ it is clear that $\|T(0)\| \le k\|0\|$. Let $y \in X$ with $y \ne 0$. As $\left\|\frac{y}{\|y\|}\right\| = 1$ we have $\left\|T\left(\frac{y}{\|y\|}\right)\right\| \le k$. Since T is a linear transformation

$$\frac{1}{\|y\|}\|T(y)\| = \left\|\left(\frac{1}{\|y\|}\right)T(y)\right\| = \left\|T\left(\frac{y}{\|y\|}\right)\right\| \le k,$$

and so $\|T(y)\| \leq k\|y\|$. Hence $\|T(x)\| \leq k\|x\|$ for all $x \in X$.

(e) $\Rightarrow$ (a). Since T is a linear transformation,

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq k\|x - y\|$$

for all $x, y \in X$. Let $\epsilon > 0$ and let $\delta = \dfrac{\epsilon}{k}$. Then when $x, y \in X$ and $\|x - y\| < \delta$

$$\|T(x) - T(y)\| \leq k\|x - y\| < k\left(\frac{\epsilon}{k}\right) < \epsilon.$$

Therefore T is uniformly continuous. □

Having obtained these alternative characterizations of continuity of linear transformations, we can now look at some examples. It will normally be clear that the maps we are considering are linear transformations so we shall just concentrate on showing that they are continuous. It is usual to check continuity of linear transformations using either of the equivalent conditions (d) or (e) of Lemma 4.1.

Example 4.2

The linear transformation $T : C_{\mathbb{F}}[0, 1] \to \mathbb{F}$ defined by

$$T(f) = f(0)$$

is continuous.

Solution

Let $f \in C_{\mathbb{C}}[0, 1]$. Then

$$\big\||T(f)|\big\|_{F} = |f(0)| \leq \sup\{|f(x)| : x \in [0, 1]\} = \|f\|_{\infty}$$

Hence T is continuous by condition (e) of Lemma 4.1 with $k = 1$. □

Before starting to check that a linear transformation T is continuous it is sometimes first necessary to check that T is well defined. Lemma 4.3 will be used to check that the following examples of linear transformations are well defined.

Lemma 4.3

If $\{c_n\} \in \ell^\infty$ and $\{x_n\} \in \ell^p$, where $1 \leq p < \infty$, then $\{c_n x_n\} \in \ell^p$ and

$$\sum_{n=1}^{\infty} |c_n x_n|^p \leq \|\{c_n\}\|_\infty^p \sum_{n=1}^{\infty} |x_n|^p. \tag{4.1}$$

Proof

Since $\{c_n\} \in \ell^\infty$ and $\{x_n\} \in \ell^p$ we have $\lambda = \|\{c_n\}\|_\infty = \sup\{|c_n| : n \in \mathbb{N}\} < \infty$ and $\sum_{n=1}^\infty |x_n|^p < \infty$. Since, for all $n \in \mathbb{N}$

$$|c_n x_n|^p \le \lambda^p |x_n|^p,$$

$= |c_n|^p|x_n|^p$

$\sum_{n=1}^\infty |c_n x_n|^p$ converges by the comparison test. Thus $\{c_n x_n\} \in \ell^p$ and the above inequality also verifies (4.1). □

$\lim \sum^N |c_n x_n|^p \le \lim_{N\to\infty} \lambda^p \underbrace{\sum^N |x_n|^p}_{\to 0} = \|\{c_n\}\|_\infty^p \sum |x_n|^p$

$= \sum^\infty |c_n x_n|^p$

Example 4.4

If $\{c_n\} \in \ell^\infty$ then the linear transformation $T : \ell^1 \to \mathbb{F}$ defined by

$$T(\{x_n\}) = \sum_{n=1}^\infty c_n x_n$$

$\sum |c_n x_n| < \infty$

is continuous.

$\Rightarrow \sum c_n x_n < \infty$

Solution

Since $\{c_n x_n\} \in \ell^1$ by Lemma 4.3, it follows that T is well defined. Moreover,

$\nearrow K$

$$|T(\{x_n\})| = |\sum_{n=1}^\infty c_n x_n| \le \sum_{n=1}^\infty |c_n x_n| \le \|\{c_n\}\|_\infty \sum_{n=1}^\infty |x_n| = \|\{c_n\}\|_\infty \|\{x_n\}\|_1.$$

↳ norm on $\mathbb{F}$ ↳ norm on ℓ^1

Hence T is continuous by condition (e) of Lemma 4.1 with $k = \|\{c_n\}\|_\infty$. □

Example 4.5

If $\{c_n\} \in \ell^\infty$ then the linear transformation $T : \ell^2 \to \ell^2$ defined by

$$T(\{x_n\}) = \{c_n x_n\}$$

is continuous.

Solution

Let $\lambda = \|\{c_n\}\|_\infty$. Since $\{c_n x_n\} \in \ell^2$ by Lemma 4.3, it follows that T is well defined. Moreover

$$\|T(\{x_n\})\|_2^2 = \sum_{n=1}^\infty |c_n x_n|^2 \le \lambda^2 \sum_{n=1}^\infty |x_n|^2 = \lambda^2 \|\{x_n\}\|_2^2.$$

Hence T is continuous by condition (e) of Lemma 4.1 with $k = \|\{c_n\}\|_\infty$. □

·) ejemplo operador lineal no continuo

$C_0[0,1]$ P sobre pols. Sea $T: P \to P$ $\quad Tp = p'$.

y uso $x^n \in P$ y se rompe (e)

$$\|Tp\| = \|n x^{n-1}\| = n \qquad \|x^n\| = 1$$

$$n \le K\, 1$$