① b) $((A^TA)^{-1}A^T)^T = A(A^TA)^{-1}$ $\quad m \geq n$

$= A(A^TA)^{-1}(A^TA)^{-1}A^T$ $\quad VV^T = I$

$= A(A^TA)^{-2}A^T$ $\quad V^TV = I$

$= U\Sigma V^T(V\Sigma^T U^T U \Sigma V^T)^{-2} V \Sigma^T U^T$

$= U\Sigma V^T(V\tilde{\Sigma} V^T)^{-2} V\Sigma^T U^T$

I

$= U\Sigma \cancel{V^T V} \tilde{\Sigma}^{-2} \cancel{V^T V} \Sigma^T U^T$ $\quad \tilde{\Sigma} \in \mathbb{R}^{m\times m}$

$= \underbrace{U\Sigma\tilde{\Sigma}^{-2}\Sigma^T}U^T$

$= U\tilde{\tilde{\Sigma}}U^T$ $\quad I$ $\quad \tilde{\tilde{\Sigma}} = \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \frac{1}{\sigma_n^2} & \\ & & \ddots \end{pmatrix}$

obs $\Sigma \in \mathbb{R}^{m\times n}$ ~~pq~~ $A \in \mathbb{R}^{m\times n}$

$\Rightarrow \Sigma^T \in \mathbb{R}^{n\times m}$

$\therefore \tilde{\Sigma} = \Sigma^T\Sigma \in \mathbb{R}^{n\times n}$

y $\tilde{\Sigma} = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} \Rightarrow \tilde{\Sigma}^{-2} = \begin{bmatrix} \frac{1}{\sigma_1^4} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^4} \end{bmatrix}$

$\Rightarrow \tilde{\Sigma}$ es inversible

pq hay n valores singulares

Además $(V\tilde{\Sigma}V^T)^{-2} = \left((V\tilde{\Sigma}V^T)^{-1}\right)^2$

$$= \left(V^{T-1}(V\tilde{\Sigma})^{-1}\right)^2$$

$$= \left(V\tilde{\Sigma}^{-1}V^{-1}\right)^2 = \left(V\tilde{\Sigma}^{-1}V^T\right)^2$$

$$= \left(V\tilde{\Sigma}^{-1}V^T\right)\left(V\tilde{\Sigma}^{-1}V^T\right)$$

$$= V\tilde{\Sigma}^{-2}V^T$$

I Además $\sqrt{\rho(U\Sigma\tilde{\Sigma}^{-1}\Sigma^T U^T)}$

$$= \sqrt{\rho(U^T U\Sigma\tilde{\Sigma}^{-1}\Sigma^T U^T U)}$$

$$= \sqrt{\rho(\Sigma\tilde{\Sigma}^{-1}\Sigma^T)} = \frac{1}{|\sigma_n|} = \sigma_n^{-1}$$

porque cambio de base no cambia avals y $U^T = U^{-1}$

c) $\left(A(A^TA)^{-1}\right)^T = (A^TA)^{-T}A^T$

$= (A^TA)^{-1}A^T$

$(A^TA)^{-1}A^TA(A^TA)^{-1} = (A^TA)^{-1}$

$= (V\Sigma^TU^TU\Sigma V^T)^{-1}$

$(V\Sigma^T\Sigma V^T)^{-1} = (V\breve{\Sigma}V^T)^{-1}$

$= V\breve{\Sigma}^{-1}V^T$

con $\breve{\Sigma} \in \mathbb{R}^{n\times n}$

$$\tilde{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

y $\sqrt{\rho(V\breve{\Sigma}^{-1}U^T)} = \sqrt{\rho(\tilde{\Sigma}^{-1})} = \frac{1}{|\sigma_n|} = \sigma_n^{-1}$

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d) $\left(A(A^TA)^{-1}A^T\right)^T = A\left(A(A^TA)^{-1}\right)^T$

$= A(A^TA)^{-T}A^T$

$= A(A^TA)^{-1}A^T$

$$U\Sigma V^T(\underbrace{V\Sigma^T U^T U\Sigma V^T}_{\tilde{V}\tilde{\Sigma}^{-1}\tilde{V}^T})^{-1} V\Sigma^T U^T$$

$$U\Sigma \cancel{V^T V}\tilde{\Sigma}^{-1}\cancel{V^T V}\Sigma^T U^T$$

$$U\Sigma\tilde{\Sigma}^{-1}\Sigma^T U^T \qquad \Sigma\tilde{\Sigma}^{-1}\Sigma = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 0 \end{bmatrix}$$

$$\sqrt{\rho(U\Sigma\tilde{\Sigma}^{-1}\Sigma^T U^T)} = \sqrt{\rho(\Sigma\tilde{\Sigma}^{-1}\Sigma^T)} = 1$$

2) Se que $A = U\Sigma V^T$

con $\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_j & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{pmatrix}$ caso 1

o $\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_j & & \\ & & & 0 & \\ & & & & \ddots \; 0 \\ & & 0 & & \end{pmatrix}$ caso 2

Sea $A_\varepsilon = U\tilde{\Sigma}V^T$

con $\tilde{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_j & & & 0 \\ & & & \varepsilon & & \\ & & & & \varepsilon & \\ & & & & & \ddots \; \varepsilon \end{bmatrix}$

$$\Rightarrow \| A - A_\varepsilon \|_2 = \| U(\Sigma - \tilde{\Sigma}) V^T \|_2$$

$$= \left\| U \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & 0 \\ & & & \varepsilon & \\ & & & & \ddots \\ & & & & & \varepsilon \end{bmatrix} V^T \right\|_2$$

$$= \| U^T U [\;\cdots\;] V^T \|_2$$

$$= \| [\quad] V^T V \|_2$$

$$\| [\quad] \|_2 = \sqrt{\varepsilon} < \varepsilon$$

(3) $A^T A + \lambda I = (U \Sigma V^T)^T U \Sigma V^T + \lambda I$

$$= V \Sigma^T U^T U \Sigma V^T + \lambda I$$

$$\tilde{\Sigma} = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$= V \tilde{\Sigma} V^T + \lambda \underbrace{V V^T}_{= I \text{ (V ortho)}}$$

$$= V(\tilde{\Sigma} + \lambda I) V^T$$

$$\Rightarrow (A^T A + \lambda I)^{-1} = V \begin{bmatrix} \frac{1}{\sigma_1^2+\lambda} & & \\ & \ddots & \\ & & \frac{1}{\sigma_r^2+\lambda} \end{bmatrix} V^T$$

$$(V^{-1} = V^T)$$

$$(A^T A + \lambda I)^{-1} A^T = \underbrace{V \begin{bmatrix} & \end{bmatrix}}_{\in \mathbb{R}^{n \times n}} \cancel{V^T V} \underbrace{\Sigma^T}_{\in \mathbb{R}^{n \times m}} U^T$$

$$\tilde{\tilde{\Sigma}} = \begin{bmatrix} \frac{\sigma_1}{\sigma_1^2+\lambda} & & & \\ & \ddots \frac{\sigma_r}{(\sigma_r^2+\lambda)} & & \\ & & 0 \ddots & 0 \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix} = V \tilde{\tilde{\Sigma}} U^T \in \mathbb{R}^{n \times m}$$

0

$$\tilde{\tilde{\Sigma}} = \begin{bmatrix} \frac{\sigma_1}{\sigma_1^2+\lambda} & & 0 \\ & \ddots \frac{\sigma_r}{(\sigma_r^2+\lambda)} & \vdots \\ & 0 \ddots & 0 \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$\Rightarrow B(\lambda) - A^+ = V \tilde{\tilde{\Sigma}} U^T - V \begin{bmatrix} \Sigma^{-1} & \\ & 0 \end{bmatrix} U^T$$

$$= V \left(\tilde{\tilde{\Sigma}} - \begin{bmatrix} \Sigma^{-1} & \\ & 0 \end{bmatrix} \right) U^T$$

$$\boxed{\frac{\sigma_1}{\sigma_1^2+\lambda} - \frac{1}{\sigma_1} = \frac{\cancel{\sigma_1^2} - \cancel{\sigma_1^2} + \lambda}{\sigma_1(\sigma_1^2+\lambda)} = \frac{\lambda}{\sigma_1(\sigma_1^2+\lambda)}}$$

$$= V \check{\Sigma} U^T$$

$$\text{con } \check{\Sigma} = \begin{bmatrix} \frac{\lambda}{\sigma_1(\sigma_1^2+\lambda)} & & \\ & \ddots & \\ & & \frac{\lambda}{\sigma_r(\sigma_r^2+\lambda)} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

por ser ortogonal U y V

$$\| V \check{\Sigma} U^T \|_2 = \| \check{\Sigma} \|_2$$

$$= \sqrt{\rho(\check{\Sigma}^T \Sigma)}$$

$$\check{\check{\Sigma}} = \begin{bmatrix} \frac{\lambda}{\sigma_1(\sigma_1^2+\lambda)} & & \\ & \ddots & \\ & & \frac{\lambda}{\sigma_r(\sigma_r^2+\lambda)} \end{bmatrix} \in \mathbb{R}^{n \times n} \qquad = \sqrt{\rho(\check{\check{\Sigma}})}$$

$$= \frac{\lambda}{\sigma_r(\sigma_r^2+\lambda)}$$

11 Por induc

$$\det \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_0 \\ -1 & x & & \\ & \ddots & \ddots & \\ & & & x \\ & & & -1 \end{pmatrix} = a_{n-1} x^{n-1} \cdots + a_0$$

P. induc

$$\det \underbrace{\begin{pmatrix} a_n & \cdots & a_0 \\ -1 & x & \\ & \ddots & \\ & & x \\ & & -1 \end{pmatrix}}^{\in M_{n+1 \times n+1}}$$

$$= a_n \underbrace{\begin{pmatrix} x & & \\ -1 & \ddots & \\ & & x \\ & & -1 \end{pmatrix}}^{\in M_{n\times n}\ = x^n} - (-1)\det \begin{pmatrix} a_{n-1} & \cdots & a_0 \\ -1 & x & \\ & \ddots & \\ & -1 & x \end{pmatrix}$$

$$= a_n x^n + \overbrace{a_{n-1} x^{n-1} \cdots + a_0}^{\text{inducc}}$$

$\Rightarrow \det \begin{bmatrix} x + a_{n-1} & \cdots & & a_0 \\ -1 & x & & \\ & & \ddots & \\ & & -1 & x \end{bmatrix} \in n \times n$

$= (x + a_{n-1}) \det \begin{bmatrix} x & & \\ -1 & \ddots & \\ & -1 & x \end{bmatrix} - (-1) \det \begin{pmatrix} a_{n-2} & \cdots & a_0 \\ -1 & x & \\ & -1 & x \end{pmatrix}$

(n-1)x(n-1)

arriba

$x^n + a_{n-1} x^{n-1} + \overbrace{a_{n-2} x^{n-2} \cdots + a_0}$

$\Rightarrow \chi_A = P(A)$

$\Rightarrow$ raices de χ_A = raices de $P(A)$

12 $H = Q^T A Q \Rightarrow H^T = Q^T A Q$

$A^T = A$

$\Rightarrow$ H es simétrica

$\Rightarrow$ H es tridiagonal