There is another notation for maps which satisfy condition (e) of Lemma 4.1 for some k.

Definition 4.6

Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear transformation. T is said to be *bounded* if there exists a positive real number k such that $\|T(x)\| \leq k\|x\|$ for all $x \in X$.

By Lemma 4.1 we can use the words *continuous* and *bounded* interchangeably for linear transformations. Note, however, that this is a different use of the word bounded from that used for functions from $\mathbb{R}$ to $\mathbb{R}$. For example if $T : \mathbb{R} \to \mathbb{R}$ is the linear transformation defined by $T(x) = x$ then T is bounded in the sense given in Definition 4.6 but, of course, is not bounded in the usual sense of a bounded function. Despite this apparent conflict of usage there is not a serious problem since apart from the zero linear transformation, linear transformations are never bounded in the usual sense of bounded functions so the word may be used in an alternative way. Since the term "bounded" gives a good indication of what has to be shown this compensates for the disadvantage of potential ambiguity. The use of this term also explains the abbreviation used in the following notation for the set of all continuous linear transformations between two normed spaces.

Notation

Let X and Y be normed linear spaces. The set of all continuous linear transformations from X to Y is denoted by $B(X, Y)$. Elements of $B(X, Y)$ are also called *bounded linear operators* or *linear operators* or sometimes just *operators*.

If X and Y are normed linear spaces then $B(X, Y) \subseteq L(X, Y)$.

Example 4.7

Let $a,\ b \in \mathbb{R}$, let $k : [a, b] \times [a, b] \to \mathbb{C}$ be continuous and let

$$M = \sup\{|k(s, t)| : (s, t) \in [a, b] \times [a, b]\}.$$

(a) If $g \in C[a, b]$, then $f : [a, b] \to \mathbb{C}$ defined by

$$f(s) = \int_a^b k(s, t)\, g(t)\, dt$$

is in $C[a, b]$.

$L(X,Y) = \{T : X \to Y ; T$ lineal$\}$

$B(X,Y) = \{T : X \to Y ; T$ " y acotada$\}$

(b) If the linear transformation $K : C[a,b] \to C[a,b]$ is defined by

$$(K(g))(s) = \int_a^b k(s,t)\, g(t)\, dt$$

then $K \in B(C[a,b], C[a,b])$ and (acotadas)

$$\|K(g)\| \leq M(b-a)\|g\|.$$

Solution

① Como t fijo $|t-t'|<\delta$ $\forall \delta > 0 \Rightarrow$ cumple uniforme continua

(a) Suppose that $\epsilon > 0$ and $s \in [a,b]$. We let $k_s \in C[a,b]$ be the function $k_s(t) = k(s,t)$, $t \in [a,b]$. Since the square $[a,b] \times [a,b]$ is a compact subset of $\mathbb{R}^2$, the function k is uniformly continuous and so there exists $\delta > 0$ such that if $|s - s'| < \delta$ then $|k_s(t) - k_{s'}(t)| < \epsilon$ for all $t \in [a,b]$. Hence

$$|f(s) - f(s')| \leq \int_a^b |k(s,t) - k(s',t)||g(t)|\, dt \leq \epsilon(b-a)\|g\|.$$

→ entrar el módulo lo hace menor o igual

Therefore f is continuous.

(b) For all $s \in [a,b]$,

$$|(K(g))(s)| \leq \int_a^b |k(s,t)\, g(t)|\, dt \leq \int_a^b M\|g\|\, dt = M(b-a)\|g\|.$$

Hence $\|K(g)\| \leq M(b-a)\|g\|$ and so $K \in B(C[a,b], C[a,b])$. □

In Example 4.7 there are lots of brackets. To avoid being overwhelmed by these, if $T \in B(X,Y)$ and $x \in X$ it is usual to write Tx rather than $T(x)$.

The examples presented so far may give the impression that all linear transformations are continuous. Unfortunately, this is not the case as the following example shows.

Example 4.8

Let $\mathcal{P}$ be the linear subspace of $C_{\mathbb{C}}[0,1]$ consisting of all polynomial functions. If $T : \mathcal{P} \to \mathcal{P}$ is the linear transformation defined by

$$T(p) = p',$$

where p' is the derivative of p, then T is not continuous.

def X, Y normados $\dim X < \infty$ $T: X \to Y$ cont ⇒ T continua dem definimos $\|x\|_1 = \|x\| + \|Tx\|$ $\|\cdot\|$

es fácil ver que es norma. notar que $\|Tx\| \le \|x\|_1$
Como $\dim X < \infty$ $\|x\|_1 < K\|x\|$ (son equivalentes) $\forall x$
y algún $K > 0$. / Obs $T: X \to Y$ lineal y $\dim Y < \infty$
en general no vale esto $T(p) = p'(0)$
p (poli)

$\Rightarrow \|x\| + \|Tx\| \le K\|x\| \Rightarrow \|Tx\| \le (K-1)\|x\|$

Solution

Let $p_n \in \mathcal{P}$ be defined by $p_n(t) = t^n$. Then

$$\|p_n\| = \sup\{|p_n(t)| : t \in [0,1]\} = 1,$$

while

$$\|T(p_n)\| = \|p_n'\| = \sup\{|p_n'(t)| : t \in [0,1]\} = \sup\{|nt^{n-1}| : t \in [0,1]\} = n.$$

Therefore there does not exist $k \ge 0$ such that $\|T(p)\| \le k\|p\|$ for all $p \in \mathcal{P}$, and so T is not continuous. □

The space $\mathcal{P}$ in Example 4.8 was not finite-dimensional, so it is natural to ask whether all linear transformations between finite-dimensional normed spaces are continuous. The answer is given in Theorem 4.9.

Theorem 4.9

Let X be a finite-dimensional normed space, let Y be any normed linear space and let $T : X \to Y$ be a linear transformation. Then T is continuous.

Proof

To show this we first define a new norm on X. Since this will be different from the original norm, in this case we have to use notation which will distinguish between the two norms. Let $\|\cdot\|_1 : X \to \mathbb{R}$ be defined by $\|x\|_1 = \|x\| + \|T(x)\|$. We will show that $\|\cdot\|_1$ is a norm for X. Let $x, y \in X$ and let $\lambda \in \mathbb{F}$.

(i) $\|x\|_1 = \|x\| + \|T(x)\| \ge 0$.

(ii) If $\|x\|_1 = 0$ then $\|x\| = \|T(x)\| = 0$ and so $x = 0$ while if $x = 0$ then $\|x\| = \|T(x)\| = 0$ and so $\|x\|_1 = 0$.

(iii) $\|\lambda x\|_1 = \|\lambda x\| + \|T(\lambda x)\| = |\lambda|\|x\| + |\lambda|\|T(x)\| = |\lambda|(\|x\| + \|T(x)\|)$.
$= |\lambda|\|x\|_1$

(iv)
$$\begin{aligned}\|x+y\|_1 &= \|x+y\| + \|T(x+y)\| \\ &= \|x+y\| + \|T(x) + T(y)\| \\ &\le \|x\| + \|y\| + \|T(x)\| + \|T(y)\| \\ &= \|x\|_1 + \|y\|_1.\end{aligned}$$

Hence $\|\cdot\|_1$ is a norm on X. Now, as X is finite-dimensional, $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent and so there exists $K > 0$ such that $\|x\|_1 \le K\|x\|$ for all $x \in X$ by Corollary 2.17. Hence $\|T(x)\| \le \|x\|_1 \le K\|x\|$ for all $x \in X$ and so T is bounded. □

$\|x\|_1 = \|x\| + \|Tx\|$

$\Rightarrow \|Tx\| \le \|x\|_1 \Rightarrow \|Tx\| < K\|x\|$

If the domain of a linear transformation is finite-dimensional then the linear transformation is continuous by Theorem 4.9. Unfortunately, if the range is finite-dimensional instead, then the linear transformation need not be continuous as we see in Example 4.10, whose solution is left as an exercise.

Example 4.10

Let $\mathcal{P}$ be the linear subspace of $C_{\mathbb{C}}[0,1]$ consisting of all polynomial functions. If $T : \mathcal{P} \to \mathbb{C}$ is the linear transformation defined by

$$T(p) = p'(1),$$

where p' is the derivative of p, then T is not continuous.

Now that we have seen how to determine whether a given linear transformation is continuous, we give some elementary properties of continuous linear transformations. We should remark here that although the link between matrices and linear transformations between finite-dimensional vector spaces given in Theorem 1.15 can be very useful, any possible extension to linear transformations between infinite-dimensional spaces is not so straightforward since both bases in infinite-dimensional spaces and infinite-sized matrices are much harder to manipulate. We will therefore only use the matrix representation of a linear transformation between finite-dimensional vector spaces.

Lemma 4.11

If X and Y are normed linear spaces and $T : X \to Y$ is a continuous linear transformation then $\operatorname{Ker}(T)$ is closed.

Proof

Since T is continuous, $\operatorname{Ker}(T) = \{x \in X : T(x) = 0\}$ and $\{0\}$ is closed in Y it follows that $\operatorname{Ker}(T)$ is closed, by Theorem 1.28. □

Before our next definition we recall that if X and Y are normed spaces then the Cartesian product $X \times Y$ is a normed space by Example 2.8.

Definition 4.12

If X and Y are normed spaces and $T : X \to Y$ is a linear transformation, the *graph* of T is the linear subspace $\mathcal{G}(T)$ of $X \times Y$ defined by

$$\mathcal{G}(T) = \{(x, Tx) : x \in X\}.$$

Lemma 4.13

If X and Y are normed spaces and $T : X \to Y$ is a continuous linear transformation then $\mathcal{G}(T)$ is closed. on $X \times Y$

Proof

Let $\{(x_n, y_n)\}$ be a sequence in $\mathcal{G}(T)$ which converges to (x, y) in $X \times Y$. Then $\{x_n\}$ converges to x in X and $\{y_n\}$ converges in y in Y by Exercise 2.5. However, $y_n = T(x_n)$ for all $n \in \mathbb{N}$ since $(x_n, y_n) \in \mathcal{G}(T)$. Hence, as T is continuous,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(x_n) = T(x).$$

Therefore $(x, y) = (x, T(x)) \in \mathcal{G}(T)$ and so $\mathcal{G}(T)$ is closed. □

We conclude this section by showing that if X and Y are fixed normed spaces the set $B(X, Y)$ is a vector space. This will be done by showing that $B(X, Y)$ is a linear subspace of $L(X, Y)$, which is a vector space under the algebraic operations given in Definition 1.7.

Lemma 4.14

Let X and Y be normed linear spaces and let $S, T \in B(X, Y)$ with $\|S(x)\| \leq k_1\|x\|$ and $\|T(x)\| \leq k_2\|x\|$ for all $x \in X$. Let $\lambda \in \mathbb{F}$. Then

(a) $\|(S + T)(x)\| \leq (k_1 + k_2)\|x\|$ for all $x \in X$;

(b) $\|(\lambda S)(x)\| \leq |\lambda|k_1\|x\|$ for all $x \in X$;

(c) $B(X, Y)$ is a linear subspace of $L(X, Y)$ and so $B(X, Y)$ is a vector space.

Proof

(a) If $x \in X$ then pq valía por continuidad

$$\|(S + T)(x)\| \leq \|S(x)\| + \|T(x)\| \leq k_1\|x\| + k_2\|x\| = (k_1 + k_2)\|x\|.$$

(b) If $x \in X$ then

$$\|(\lambda S)(x)\| = |\lambda|\|S(x)\| \leq |\lambda|k_1\|x\|.$$

(c) By parts (a) and (b), $S + T$ and λS are in $B(X, Y)$ so $B(X, Y)$ is a linear subspace of $L(X, Y)$. Hence $B(X, Y)$ is a vector space. □

4.2 The Norm of a Bounded Linear Operator

If X and Y are normed linear spaces we showed in Lemma 4.14 that $B(X, Y)$ is a vector space. We now show that $B(X, Y)$ is also a normed space. While doing this we often have as many as three different norms, from three different spaces, in the same equation, and so we should in principle distinguish between these norms. In practice we simply use the symbol $\|\cdot\|$ for the norm on all three spaces as it is still usually easy to determine which space an element is in and therefore, implicitly, to which norm we are referring. To check the axioms for the norm on $B(X,Y)$ that we will define in Lemma 4.15 we need the following consequence of Lemma 4.1

$$\sup\{\|T(x)\| : \|x\| \leq 1\} = \inf\{k : \|T(x)\| \leq k\|x\| \text{ for all } x \in X\}$$

and so in particular $\|T(y)\| \leq \sup\{\|T(x)\| : \|x\| \leq 1\}\|y\|$ for all $y \in X$.

Lemma 4.15

Let X and Y be normed spaces. If $\|\cdot\| : B(X,Y) \to \mathbb{R}$ is defined by α

$$\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\} = \sup\{\|T(x)\| : \|x\| = 1\}$$

then $\|\cdot\|$ is a norm on $B(X,Y)$. β

$\geq$ directo

$\leq$ $\exists x_j$ con $\|x_j\| \leq 1$ tq $\|Tx_j\| \to \beta$

por def sup. Supongo $\alpha < \beta$

ent $\exists x_0 : \|x_0\| \leq 1$ tq $\|Tx_0\| > \alpha$

$\alpha \geq \frac{\|Tx_0\|}{\|x_0\|} \geq \|Tx_0\| > \alpha$ abs!

Proof

Let $S, T \in B(X,Y)$ and let $\lambda \in \mathbb{F}$.

(i) Clearly $\|T\| \geq 0$ for all $T \in B(X,Y)$.

(ii) Recall that the zero linear transformation R satisfies $R(x) = 0$ for all $x \in X$. Hence,

$$\begin{aligned}\|T\| = 0 &\iff \|Tx\| = 0 \text{ for all } x \in X \\ &\iff Tx = 0 \text{ for all } x \in X \\ &\iff T \text{ is the zero linear transformation.}\end{aligned}$$

iii) $\|\lambda T\| = \sup\{\|\lambda T x\| : \|x\| \leq 1\}$
$= |\lambda| \sup\{\|Tx\|\}$
$= |\lambda| \|T\|$

(iv) The final property to check is the triangle inequality.

$$\begin{aligned}\|(S+T)(x)\| &\leq \|S(x)\| + \|T(x)\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\|.\end{aligned}$$

$\rightsquigarrow$ tomando supremo sobre todos $\|x\| \leq 1$

Therefore $\|S+T\| \leq \|S\| + \|T\|$.

$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \|T\| \Rightarrow \|Tx\| \leq \|T\|\|x\|$

Hence $B(X,Y)$ is a normed vector space. □

Definition 4.16

Let X and Y be normed linear spaces and let $T \in B(X,Y)$. The norm of T is defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$.

Using the link between matrices and linear transformations between finite-dimensional spaces we can use the definition of the norm of a bounded linear transformation to give a norm on the vector space of $m \times n$ matrices.

Definition 4.17

Let $\mathbb{F}^p$ have the standard norm and let A be a $m \times n$ matrix with entries in $\mathbb{F}$. If $T : \mathbb{F}^n \to \mathbb{F}^m$ is the bounded linear transformation defined by $T(x) = Ax$ then the norm of the matrix A is defined by $\|A\| = \|T\|$.

Let us now see how to compute the norm of a bounded linear transformation. Since the norm of an operator is the supremum of a set, the norm can sometimes be hard to find. Even if X is a finite-dimensional normed linear space and there is an element y with $\|y\| = 1$ in X such that $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\} = \|T(y)\|$ it might not be easy to find this element y. In the infinite-dimensional case there is also the possibility that the supremum may not be attained. Hence there is no general procedure for finding the norm of a bounded linear transformation. Nevertheless there are some cases when the norm can easily be found. As a first example consider the norm of the linear transformation given in Example 4.2.

Example 4.18

If $T : C_{\mathbb{F}}[0,1] \to \mathbb{F}$ is the bounded linear operator defined by

$$T(f) = f(0)$$

then $\|T\| = 1$.

Solution

It was shown in Example 4.2 that $|T(f)| \leq \|f\|$ for all $f \in C_{\mathbb{F}}[0,1]$. Hence

$$\|T\| = \inf\{k : \|T(x)\| \leq k\|x\| \text{ for all } x \in X\} \leq 1.$$

On the other hand, if $g : [0,1] \to \mathbb{C}$ is defined by $g(x) = 1$ for all $x \in X$ then $g \in C_{\mathbb{C}}[0,1]$ with $\|g\| = \sup\{|g(x)| : x \in [0,1]\} = 1$ and $|T(g)| = |g(0)| = 1$.

Hence

$$1 = |T(g)| \leq \|T\|\|g\| = \|T\|.$$

Therefore $\|T\| = 1$. □

Sometimes it is possible to use the norm of one operator to find the norm of another. We illustrate this in Theorem 4.19.

Theorem 4.19

Let X be a normed linear space and let W be a dense subspace of X. Let Y be a Banach space and let $S \in B(W, Y)$.

(a) If $x \in X$ and $\{x_n\}$ and $\{y_n\}$ are sequences in W such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$ then $\{S(x_n)\}$ and $\{S(y_n)\}$ both converge and $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} S(y_n)$.

(b) There exists $T \in B(X, Y)$ such that $\|T\| = \|S\|$ and $Tx = Sx$ for all $x \in W$.

Proof

(a) Since $\{x_n\}$ converges it is a Cauchy sequence. Therefore, as

$$\|S(x_n) - S(x_m)\| = \|S(x_n - x_m)\| \leq \|S\|\|x_n - x_m\|,$$

$\{S(x_n)\}$ is also a Cauchy sequence and hence, since Y is a Banach space, $\{S(x_n)\}$ converges. (ídem logo $S(y_n)$)

As $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$ we have $\lim_{n\to\infty}(x_n - y_n) = 0$. Since

$$\|S(x_n) - S(y_n)\| = \|S(x_n - y_n)\| \leq \|S\|\|x_n - y_n\|,$$

$\lim_{n\to\infty} S(x_n) - S(y_n) = 0$ and so $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} S(y_n)$. $\Rightarrow$ pq sabemos que existen

(b) We now define $T : X \to Y$ as follows: for any $x \in X$ there exists a sequence $\{x_n\}$ in W such that $\lim_{n\to\infty} x_n = x$ (since W is dense in X) and we define $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} S(x_n)$$

(T is well defined since the value of the limit is independent of the choice of sequence $\{x_n\}$ converging to x by part (a)). In this case it is perhaps not clear that T is a linear transformation so the first step in this part is to show that T is linear.

Let $x, y \in X$ and let $\lambda \in \mathbb{F}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in W such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then $\{x_n\}$ and $\{y_n\}$ are sequences in W such that $\lim_{n\to\infty}(x_n + y_n) = x + y$ and $\lim_{n\to\infty} \lambda x_n = \lambda x$. Hence

$(X_n + y_n) \in W$ pues W subesp análogo λX_n

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} S(x_n + y_n) \\ &= \lim_{n\to\infty} (S(x_n) + S(y_n)) \\ &= \lim_{n\to\infty} S(x_n) + \lim_{n\to\infty} S(y_n) \\ &= T(x) + T(y) \end{aligned}$$

S lineal

and

$$T(\lambda x) = \lim_{n\to\infty} S(\lambda x_n) = \lim_{n\to\infty} \lambda S(x_n) = \lambda \lim_{n\to\infty} S(x_n) = \lambda T(x).$$

Hence T is a linear transformation.

1) Now suppose that $x \in X$ with $\|x\| = 1$ and let $\{x_n\}$ be a sequence in W such that $\lim_{n\to\infty} x_n = x$. Since $\lim_{n\to\infty} \|x_n\| = \|x\| = 1$, if we let $w_n = \dfrac{x_n}{\|x_n\|}$ then $\{w_n\}$ is a sequence in W such that $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \dfrac{x_n}{\|x_n\|} = x$ and $\|w_n\| = \dfrac{\|x_n\|}{\|x_n\|} = 1$ for all $n \in \mathbb{N}$. As

$$\begin{aligned} \|Tx\| &= \lim_{n\to\infty} \|Sw_n\| \\ &\le \sup\{\|Sw_n\| : n \in \mathbb{N}\} \\ &\le \sup\{\|S\|\|w_n\| : n \in \mathbb{N}\} \\ &= \|S\|, \end{aligned}$$

T is bounded and $\|T\| \le \|S\|$. Moreover if $w \in W$ then the constant sequence $\{w\}$ is a sequence in W converging to w and so

$$Tw = \lim_{n\to\infty} Sw = Sw.$$

Thus $\|Sw\| = \|Tw\| \le \|T\|\|w\|$ so $\|S\| \le \|T\|$. Hence $\|S\| = \|T\|$ and we have already shown that if $x \in W$ then $Tx = Sx$. □

The operator T in Theorem 4.19 can be thought of as an extension of the operator S to the larger space X.

We now consider a type of operator whose norm is easy to find.

Definition 4.20

Let X and Y be normed linear spaces and let $T \in L(X, Y)$. If $\|T(x)\| = \|x\|$ for all $x \in X$ then T is called an *isometry*.

notar T iso $\Rightarrow$ T continua e inyectiva (acotada)

-) Si es li ⇒ T isomorfismo isométrico
si dim < ∞ las isometría son li $\dim X = \dim \ker + \dim \mathrm{Im}$
en dim = ∞ no necesariamente (I)

On every normed space there is at least one isometry.

Example 4.21

If X is a normed space and I is the identity linear transformation on X then I is an isometry.

Solution

If $x \in X$ then $I(x) = x$ and so $\|I(x)\| = \|x\|$. Hence I is an isometry. □

As another example of an isometry consider the following linear transformation.

Example 4.22

(a) If $x = (x_1, x_2, x_3, \ldots) \in \ell^2$ then $y = (0, x_1, x_2, x_3, \ldots) \in \ell^2$.

(b) The linear transformation $S : \ell^2 \to \ell^2$ defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots) \tag{4.2}$$

is an isometry. (No sobre) (I)

Solution

(a) Since $x \in \ell^2$,

$$|0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2 + \ldots = |x_1|^2 + |x_2|^2 + |x_3|^2 + \ldots < \infty,$$

and so $y \in \ell^2$.

(b)
$$\begin{aligned}\|S(x)\|^2 &= |0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2 + \ldots \\ &= |x_1|^2 + |x_2|^2 + |x_3|^2 + \ldots \\ &= \|x\|^2,\end{aligned}$$

and hence S is an isometry. □

The operator defined in Example 4.22 will be referred to frequently in the next chapter so it will be useful to have a name for it.

Notation

The isometry $S : \ell^2 \to \ell^2$ defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

is called the *unilateral shift*.

no es bi

It is easy to see that the unilateral shift does not map ℓ^2 onto ℓ^2. This contrasts with the finite-dimensional situation where if X is a normed linear space and T is an isometry of X into X then T maps X onto X, by Lemma 1.12.

We leave as an exercise the proof of Lemma 4.23 which shows that the norm of an isometry is 1.

Lemma 4.23

T lineal $\|T\| = \sup_{\|x\|\le 1} \|Tx\| \overset{\text{iso}}{=} \sup_{\|x\|\le 1} \|x\| = 1$

Let X and Y be normed linear spaces and let $T \in L(X, Y)$. If T is an isometry then T is bounded and $\|T\| = 1$.

The converse of Lemma 4.23 is not true. In Example 4.18, although $\|T\| = 1$ it is not true that $|T(h)| = \|h\|$ for all $h \in C_{\mathbb{F}}[0, 1]$. For example, if $h : [0, 1] \to \mathbb{F}$ is defined by $h(x) = x$ for all $x \in [0, 1]$ then $\|h\| = 1$ while $|T(h)| = 0$. Therefore, saying that a linear transformation is an isometry asserts more than that it has norm 1.

$T: C_{\mathbb{F}}[0,1] \to \mathbb{F}$
$T(f) = f(0)$
$\Rightarrow T(h) = h(0) = 0 \quad \forall h$

Definition 4.24

If X and Y are normed linear spaces and T is an isometry from X onto Y then T is called an *isometric isomorphism* and X and Y are called *isometrically isomorphic.*

If two spaces are isometrically isomorphic it means that as far as the vector space and norm properties are concerned they have essentially the same structure. However, it can happen that one way of looking at a space gives more insight into the space. For instance, $\ell^2_{\mathbb{F}}$ is a simple example of a Hilbert space and we will show in Corollary 4.26 that any infinite-dimensional, separable Hilbert spaces over $\mathbb{F}$ is isomorphic to $\ell^2_{\mathbb{F}}$. We recall that $\{\widetilde{e}_n\}$ is the standard orthonormal basis in $\ell^2_{\mathbb{F}}$.

Theorem 4.25

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space over $\mathbb{F}$ with an orthonormal basis $\{e_n\}$. Then there is an isometry T of $\mathcal{H}$ onto $\ell^2_{\mathbb{F}}$ such that $T(e_n) = \widetilde{e}_n$ for all $n \in \mathbb{N}$.

$\tilde{e}_n = (0, \dots, \underset{n}{1}, 0, \dots)$

(I) Por Bessel la serie converge ⇒ está en ℓ^2

Proof

Let $x \in \mathcal{H}$. Then $x = \sum_{n=1}^{\infty}(x, e_n)e_n$ by Theorem 3.47 as $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. Moreover, if $\alpha_n = (x, e_n)$ then $\{\alpha_n\} \in \ell^2_{\mathbb{F}}$ by Lemma 3.41 (I) (Bessel's inequality) so we can define a linear transformation $T : \mathcal{H} \to \ell^2_{\mathbb{F}}$ by $T(x) = \{\alpha_n\}$. Now

↳ sucesión

parseval (3.47)

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 = \|x\|^2$$

for all $x \in \mathcal{H}$ by Theorem 3.47, so T is an isometry and by definition $T(e_n) = \widetilde{e}_n$ for all $n \in \mathbb{N}$.

↳ ⇒ es inyectiva → $y = \sum^{\infty} \beta_n e_n$

(Sobre) → Finally, if $\{\beta_n\}$ is in $\ell^2_{\mathbb{F}}$ then by Theorem 3.42 the series $\sum_{n=1}^{\infty} \beta_n e_n$ converges to a point $y \in \mathcal{H}$. Since $(y, e_n) = \beta_n$ we have $T(y) = \{\beta_n\}$. Hence T is an isometry of $\mathcal{H}$ onto $\ell^2_{\mathbb{F}}$. □

por 3.47
$y = \sum (y, e_n).e_n$
$\Rightarrow (y, e_n) = \beta_n$

Corollary 4.26

Any infinite-dimensional, separable Hilbert space $\mathcal{H}$ over $\mathbb{F}$ is isometrically isomorphic to $\ell^2_{\mathbb{F}}$. → ⇕

Tiene sucesión ortonormal

Proof

$\mathcal{H}$ has an orthonormal basis $\{e_n\}$ by Theorem 3.52, so $\mathcal{H}$ is isometrically isomorphic to $\ell^2_{\mathbb{F}}$ by Theorem 4.25. □

EXERCISES

4.6 Let $T : C_{\mathbb{R}}[0, 1] \to \mathbb{R}$ be the bounded linear transformation defined by

$$T(f) = \int_0^1 f(x)dx.$$

(a) Show that $\|T\| \leq 1$.

(b) If $g \in C_{\mathbb{R}}[0, 1]$ is defined by $g(x) = 1$ for all $x \in [0, 1]$, find $|T(g)|$ and hence find $\|T\|$.

4.7 Let $h \in L^{\infty}[0, 1]$ and let $T : L^2[0, 1] \to L^2[0, 1]$ be the bounded linear transformation defined by $T(f) = hf$. Show that

$$\|T\| \leq \|h\|_{\infty}.$$

4.8 Let $T : \ell^2 \to \ell^2$ be the bounded linear transformation defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (0, 4x_1, x_2, 4x_3, x_4, \ldots).$$

Find the norm of T.

4.9 Prove Lemma 4.23.

4.10 Let $\mathcal{H}$ be a complex Hilbert space and let $y \in \mathcal{H}$. Find the norm of the bounded linear transformation $f : \mathcal{H} \to \mathbb{C}$ defined by

$$f(x) = (x, y).$$

4.11 Let $\mathcal{H}$ be a Hilbert space and let $y, z \in \mathcal{H}$. If T is the linear transformation defined by $T(x) = (x, y)z$, show that T is bounded and that $\|T\| \leq \|y\|\|z\|$.

4.3 The Space $B(X, Y)$ and Dual Spaces

Now that we have seen some examples of norms of individual operators let us look in more detail at the space $B(X, Y)$ where X and Y are normed linear spaces. Since many of the deeper properties of normed linear spaces are obtained only for Banach spaces it is natural to ask when $B(X, Y)$ is a Banach space. An initial guess might suggest that it would be related to completeness of X and Y. This is only half correct. In fact, it is only the completeness of Y which matters.

Theorem 4.27

If X is a normed linear spaces and Y is a Banach space then the normed space $B(X, Y)$ is a Banach space.

dado $\varepsilon > 0$ $\|T_m\| \leq M$ $\Rightarrow \|T_n\| \leq \|T_n - T_m\| + \|T_m\| \leq \varepsilon + M$ $\forall n \geq n_0$

Proof

We have to show that $B(X, Y)$ is a complete metric space. Let $\{T_n\}$ be a Cauchy sequence in $B(X, Y)$. In any metric space a Cauchy sequence is bounded so there exists $M > 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Let $x \in X$. As (I)

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\|\|x\|$$

and $\{T_n\}$ is Cauchy, it follows that $\{T_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete $\{T_n(x)\}$ converges, so we may define $T : X \to Y$ by

y $\|T_n - T_m\| < K$ $\forall n < n_0$ finitos

y x fijo $\quad T(x) = \lim_{n\to\infty} T_n(x).$

(I) dado $\varepsilon > 0$ $\|T_m\| \leq K_m$ pq $T_m \in B(X,Y)$ $\Rightarrow \|T_n\| \leq \|T_n - T_m\| + \|T_m\| \leq \varepsilon + K_m$ $\forall n, m \geq n_0$ fijo m $\leq \varepsilon + K_m$ $\forall n \geq n_0$

$\|T_n\| < M$ (I) fijo n_0 $< \epsilon$ $B(X,Y)$

usando converg. $\|T_n\| \le \|T_n - T_{n_0}\| + \|T_{n_0}\|$

$\forall n, m \ge n_0$ $\le K$

It is perhaps not clear in this case that T is a linear transformation so the first step to show that T is the required limit is to show that it is linear. As

$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} (T_n x + T_n y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty$$

and

$$T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x) = \lim_{n\to\infty} \alpha T_n x = \alpha \lim_{n\to\infty} T_n x = \alpha T(x),$$

T is a linear transformation.

$\le \sup\{\|T_n(x)\|\}$

Next we show that T is bounded. As $\|T(x)\| = \lim_{n\to\infty} \|T_n(x)\|$,

$$\|T(x)\| \overset{\mathrm{II}}{\le} \sup\{\|T_n(x)\| : n \in \mathbb{N}\} \overset{\mathrm{III}}{\le} \sup\{\|T_n\|\|x\| : n \in \mathbb{N}\} \le M\|x\|.$$

(III) $\|T_n(x)\| \le \|T_n\|\|x\|$

Therefore T is bounded and so $T \in B(X, Y)$. (I) $\|T(x)\| = \lim \|T_n(x)\|$ siempre

Finally, we have to show that $\lim_{n\to\infty} T_n = T$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that when $m, n \ge N$

$$\|T_n - T_m\| < \frac{\epsilon}{2}.$$

Pq T_n es de cauchy

Then for any x with $\|x\| \le 1$ pero

$$\|T_n(x) - T_m(x)\| \le \|T_n - T_m\|\|x\| < \frac{\epsilon}{2},$$

for $m, n \ge N$. As $T(x) = \lim_{n\to\infty} T_n(x)$, there exists $N_1 \in \mathbb{N}$ such that when $m \ge N_1$,

$\to$ en $\|\cdot\|$

$$\|T(x) - T_m(x)\| < \frac{\epsilon}{2}.$$

Then when $n \ge N$ and $m \ge N_1$

$$\|T(x) - T_n(x)\| \le \|T(x) - T_m(x)\| + \|T_n(x) - T_m(x)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}\|x\| \le \epsilon.$$

Thus $\|T - T_n\| \le \epsilon$ when $n \ge N$ and so $\lim_{n\to\infty} T_n = T$. Hence $B(X, Y)$ is a Banach space. □

$\Rightarrow \sup \|(T - T_n)(x)\| \le \epsilon$ $\|x\| \le 1$

One case of the above which occurs sufficiently often to warrant separate notation is when $Y = \mathbb{F}$.

Definition 4.28

Let X be a normed space over $\mathbb{F}$. The space $B(X, \mathbb{F})$ is called the *dual space* of X and is denoted by X'.

The dual space is sometimes also denoted by X^*, but to avoid confusion with notation we will use in Chapter 5, we will not use X^* for the dual space.

Corollary 4.29

If X is a normed vector space then X' is a Banach space.

Proof

The space $\mathbb{F}$ is complete so X' is a Banach space by Theorem 4.27. □

In general it is relatively easy to produce some elements of a dual space but less easy to identify all of them. As an example of what is involved we find all the elements of the dual space of a Hilbert space. We have already identified some of the elements of this dual space in Exercise 4.3.

Example 4.30

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and let $y \in \mathcal{H}$. If $f : \mathcal{H} \to \mathbb{F}$ is defined by $f(x) = (x, y)$ then f is an element of $\mathcal{H}'$.

The harder part of the identification of the dual space of a Hilbert space is to show that all elements of the dual space are of the above form.

Theorem 4.31 (Riesz–Fréchet theorem)

If $\mathcal{H}$ is a Hilbert space and $f \in \mathcal{H}'$ then there is a unique $y \in \mathcal{H}$ such that $f(x) = (x, y)$ for all $x \in \mathcal{H}$. Moreover $\|f\| = \|y\|$.

Proof

(a) (Existence). If $f(x) = 0$ for all $x \in \mathcal{H}$ then $y = 0$ will be a suitable choice. Otherwise, $\operatorname{Ker} f = \{x \in \mathcal{H} : f(x) = 0\}$ is a proper closed subspace of $\mathcal{H}$ so that $(\operatorname{Ker} f)^\perp \neq 0$ by Theorem 3.34. Therefore there exists $z \in (\operatorname{Ker} f)^\perp$ such that $f(z) = 1$. In particular, $z \neq 0$ so we may define $y = \dfrac{z}{\|z\|^2}$. Since f is a linear transformation

$$f(x - f(x)z) = f(x) - f(x)f(z) = 0,$$

and hence $x - f(x)z \in \operatorname{Ker} f$. However, $z \in (\operatorname{Ker} f)^\perp$ so

$$(x - f(x)z, z) = 0.$$

Therefore, $(x,z) - f(x)(z,z) = 0$ and so $(x,z) = f(x)\|z\|^2$. Hence

$$f(x) = (x, \frac{z}{\|z\|^2}) = (x,y).$$

Moreover, if $\|x\| \leq 1$ then by the Cauchy–Schwarz inequality

$$|f(x)| = |(x,y)| \leq \|x\|\|y\| \leq \|y\|.$$

Hence $\|f\| \leq \|y\|$.

On the other hand if $x = \dfrac{y}{\|y\|}$ then $\|x\| = \left\|\dfrac{y}{\|y\|}\right\| = 1$ and so

$$\|f\| \geq |f(x)| = \frac{|f(y)|}{\|y\|} = \frac{(y,y)}{\|y\|} = \|y\|.$$

Therefore $\|f\| \geq \|y\|$.

(b) (Uniqueness). If y and w are such that

$$f(x) = (x,y) = (x,w)$$

for all $x \in \mathcal{H}$, then $(x, y-w) = 0$ for all $x \in \mathcal{H}$. Hence by Exercise 3.1 we have $y - w = 0$ and so $y = w$ as required. □

Another Banach space whose dual is relatively easy to identify is ℓ^1.

Theorem 4.32

(a) If $c = \{c_n\} \in \ell^\infty$ and $\{x_n\} \in \ell^1$ then $\{c_n x_n\} \in \ell^1$ and if the linear transformation $f_c : \ell^1 \to \mathbb{F}$ is defined by $f_c(\{x_n\}) = \sum_{n=1}^\infty c_n x_n$ then $f_c \in (\ell^1)'$ with

$$\|f_c\| \leq \|c\|_\infty.$$

(b) If $f \in (\ell^1)'$ then there exists $c \in \ell^\infty$ such that $f = f_c$ and $\|c\|_\infty \leq \|f\|$.

(c) The space $(\ell^1)'$ is isometrically isomorphic to ℓ^∞.

Proof

(a) This follows from Example 4.4.

(b) Let $\{\widetilde{e}_n\}$ be the sequence in ℓ^1 given in Definition 1.59. Let $c_n = f(\widetilde{e}_n)$ for all $n \in \mathbb{N}$. Then

$$|c_n| = |f(\widetilde{e}_n)| \leq \|f\|\|\widetilde{e}_n\| = \|f\|,$$

for all $n \in \mathbb{N}$, and hence $c = \{c_n\} \in \ell^\infty$ and $\|c\|_\infty \leq \|f\|$.