PR 4

(1) Demostrar que las inclusiones $i : \ell^1 \to \ell^2$, $i : \ell^2 \to \ell^\infty$ e $i : \ell^p \to \ell^q$ para todo $1 \leq p \leq q \leq \infty$ son continuas. ¿Qué ejercicio del Práctico 1 permite asegurarlo?

i) Sea $x \in \ell^p$ $(p \leq q)$ $\quad \|i(x)\|_q = \|x\|_q \leq \|x\|_p$

5) g) PR I

$\Rightarrow$ i continua

obs $\ell^1 \subset \ell^2 \Rightarrow$ i bien def

(2) Sean $\mathcal{N}$ y $\mathcal{M}$ espacios normados. Definimos $\mathcal{B}(\mathcal{N},\mathcal{M}) := \{A : \mathcal{N} \to \mathcal{M} : A$ es lineal y continua$\}$.

(a) Probar que $\mathcal{B}(\mathcal{N},\mathcal{M})$ es un espacio normado con $||A|| = \sup_{x:||x||\leq 1} ||Ax|| = \sup_{x:||x||<1} ||Ax|| = \sup_{x:||x||=1} ||Ax||$.

(b) Probar que $\mathcal{B}(\mathcal{N},\mathcal{M})$ es de Banach si $\mathcal{M}$ lo es.

a) Teórico 5) solo falta $\sup_{||x||<1}$

b) Teórico 5)

(3) Sea X espacio de Banach finitamente dimensional e Y un espacio de Banach. Probar que si $T : X \to Y$ es lineal, entonces T es continua.

(teorico 5)

(4) Sea $X = \ell^1$, e Y un espacio de Banach finitamente dimensional. Probar que no todas las aplicaciones lineales son continuas.

No hace falta que sea $X = \ell^1$

con X dim infinita alcanza

Sea $\{v_i\}$ base algebraica, no numerable pq X dim infinita (ej 10)

↗ siempre existe Zorn

Ahora tomamos un sub conjunto

$$A = \{\tilde{v}_n\}_{n\in\mathbb{N}} \subseteq \{v_i\}_{i\in I}$$

Sea
$$T(v_n) = \begin{cases} \|\tilde{v}_n\|_X \, n & \text{si } v_n \in A \\ 0 & \text{cc} \end{cases}$$

dado K $\exists\, n \in \mathbb{N} > K$ tomo $\tilde{v}_n$

$$\|T(\tilde{v}_n)\| = \big|\, \|\tilde{v}_n\|_X n \,\big|$$

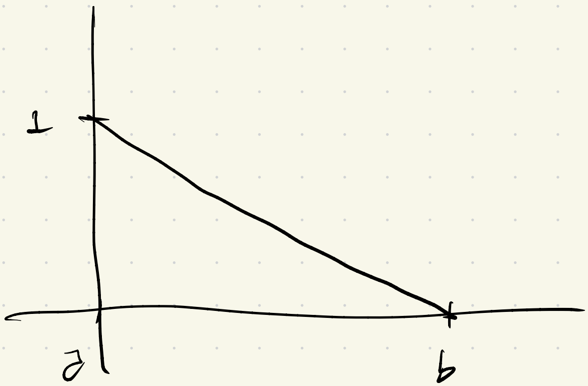
$$= n\, \|\tilde{v}_n\|_X > K \|\tilde{v}_n\|_X$$

(5) (a) Sea $\mathcal{N}_p$ es el espacio $C[a,b]$ con $\|\cdot\|_p$ y definimos el operador $A(f) := f(a)$ para toda f en $C[a,b]$. Probar que $A \in \mathcal{B}(\mathcal{N}_p, \mathbb{K})$ si y sólo si $p = \infty$.

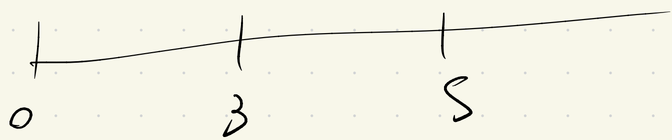
(b) Demostrar que Id: $L^\infty[a,b] \to L^2[a,b] \to L^1[a,b]$ son continuas. Hallar sus normas. Probar que Id: $L^1[a,b] \to L^2[a,b] \to L^\infty[a,b]$ no son continuas.

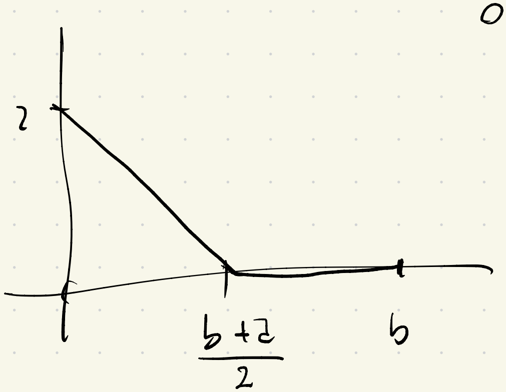
$(\Rightarrow)$ $|A(f)|_{\mathbb{K}} \leq c \, \|f\|_p$ $\qquad A: C[a,b] \to \mathbb{K}$

$|f(a)|_{\mathbb{K}} \leq c \, \|f\|_p \quad \forall f \in C[a,b]$

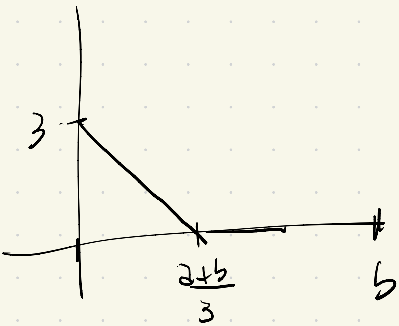


$A = b - a$





$A = b - a$



Algo así

$\Rightarrow |f_n(a)| = n$

supongo $p \neq \infty$ $\|f_n\|_p = A = b-a$

$$\Rightarrow n = |f_n(a)| \leq \|f_n\|_p = b-a$$

$\lim_{n\to\infty} n \leq b-a$ abs!

$\Rightarrow p = \infty$

$(\Leftarrow)$ qvq $|A(f)| = |f(a)|_K \leq C\|f\|_\infty$

$$(A \in B(N_\infty, K))$$

es trivial siempre $|A(a)|_K \leq \|f\|_\infty$
por definición

b) Sea f / $\|f\|_\infty = 1$

$$\Rightarrow \quad \|\mathrm{Id} f\|_2 = \left(\int_a^b |f|^2\right)^{\frac{1}{2}}$$

$$\leq (b-a)^{\frac{1}{2}} = K\|f\|_\infty$$

$\Rightarrow \mathrm{Id} : L^\infty \to L^2$ continua

$$\|\mathrm{Id}\| = \sup\{\|\mathrm{Id} f\|_2 : \|f\|_\infty = 1\}$$

$$\sup\left\{\left(\int |f|^2\right)^2 : \|f\|_\infty = 1\right\} = \sqrt{b-a}$$

$$\|f\|_2 = 1$$

·) $\mathrm{Id} : L^2 \to L^1$, $\|f\|_2 = 1$

$$\|\mathrm{Id} f\|_1 = \underbrace{\int_a^b |f|} = \int_{A_1} |f| + \int_{A_2} |f|$$

$A_1 = \{x \in [a,b] / f(x) < 1\}$ $\qquad \leq \int_{A_1} 1 + \|f\|_2$

$A_2 = \{x \in [a,b] / f(x) \geq 1\}$

$\leq (b-a) + 1$

$= \{(b-a)+1\} \|f\|_2$

$\sup\{ \int |f| : \|f\|_2 = 1 \}$

$= \sup\{ \int |f| : \int_a^b |f|^2 = 1 \}$

$\int_{A_1} |f|^2 + \int_{A_2} |f|^2 = 1$

$\leq \underbrace{\int_{A_1} 1 + \int_{A_2} |f|^2}$

(6) Hacer los ejercicios 4.1, 4.2 y 4.3 de la página 96 del libro Linear Functional Analysis de Rynne y Youngson.

4.1 If $T : C_{\mathbb{R}}[0,1] \to \mathbb{R}$ is the linear transformation defined by

$$T(f) = \int_0^1 f(x)dx$$

show that T is continuous.

$$\left\| \int_0^1 f(x) \right\|_1 \leq \left\| \int_0^1 \sup_{x \in [0,1]} f(x) \right\|_1 = \|f\|_\infty (1-0)$$

$$= \|f\|_\infty \cdot 1$$

$\Rightarrow$ T acotada (continua)

4.2 Let $h \in L^\infty[0,1]$.

(a) If f is in $L^2[0,1]$, show that $fh \in L^2[0,1]$.

(b) Let $T : L^2[0,1] \to L^2[0,1]$ be the linear transformation defined by $T(f) = hf$. Show that T is continuous.

a)

$$\|fh\|_2^2 = \int_0^1 |fh(x)|^2 dx = \int_0^1 |f|^2 |h|^2 dx$$

$$\leq \|h\|_\infty^2 \int_0^1 |f|^2 = \|h\|_\infty^2 \|f\|_2^2$$

$\Rightarrow fh \in L^2[0,1]$

b) T bien def por a)

$$\|T(f)\|_2^2 = \|fh\|_2^2 \le \|h\|_\infty^2 \|f\|_2^2$$
$$= K^2 \|f\|_2^2$$

$$\Rightarrow \|T(f)\|_2 \le K \|f\|_2$$

4.3 Let $\mathcal{H}$ be a complex Hilbert space and let $y \in \mathcal{H}$. Show that the linear transformation $f : \mathcal{H} \to \mathbb{C}$ defined by

$$f(x) = (x, y)$$

is continuous.

$$|f(x)|_{\mathbb{C}}^2 = |(x,y)|_{\mathbb{C}}^2 = \left|\sum^{\infty} x_i \bar{y}_i\right|_{\mathbb{C}}^2 \le \sum |x_i|^2 |\bar{y}_i|^2$$
$$= \sum |x_i|^2 |y_i|^2$$
$$\text{I} \le \|y\|_H^2 \sum |x_i|^2$$
$$\le \|y\|_{\mathbb{C}}^2 \|x\|_H^2$$
$$= \|f\|_H \|x\|_H^2$$
$$= K \|x\|_H^2$$

Αν ο χώρος
[illegible] Hilbert (I)

$$\|\bar{y}\|_H^2 = \sum^{\infty} |(\bar{y}, e_n)|^2 = \sum^{\infty} |\bar{y}_i|^2 \Rightarrow |\bar{y}_i|^2 \leq \|\bar{y}\|_H^2$$

Ανάλογα $\qquad |x|^2 \leq \|x\|_H^2$

(7) Dar un ejemplo de una isometría entre dos espacios de Banach que no sobreyectiva.

$f: \ell^1 \longrightarrow \ell^1$ dada por

$f((a_1, \ldots, a_n, \ldots)) = (0, a_1, \ldots, a_n, \ldots)$

$$\|f(a_1 \ldots, a_n, \ldots)\|_1 = \|(0, a_1 \ldots)\|$$
$$= 0 + \sum_{i=1}^{\infty} |a_i| = \|(a_1, \ldots)\|_1$$

$\Rightarrow f$ isometria

pero $e_1 \in \ell^1$ y no tiene pre imagen

(8) Probar que la composición de isometrías entre espacios normados es isometría.

$$\|T(S(x))\| \underset{T \text{ iso}}{=} \|S(x)\| \underset{S \text{ iso}}{=} \|x\|$$

(9) ¿Cuáles son todas las aplicaciones lineales que son isometrías de $\mathbb{R}^n$ en $\mathbb{R}^n$ con respecto a $\|\cdot\|_2$?

$$\|T(X)\|_2 = \|X\|_2$$

$$\|T\| = \sup\{\|T(X)\|_2 : \|X\| = 1\}$$

$$= \sup\{\|X\|_2 : \|X\| = 1\} = 1$$

$$\left[\sum_{i=1}^{n} (T(X))_i^2\right]^{\frac{1}{2}} = \left(\sum^{n} x_i^2\right)^{\frac{1}{2}} \leq 1$$

$$\sum^{n} x_i^2 \leq 1$$

(10) Hacer los ejercicios 4.6, 4.7, 4.10 y 4.11 de la Sección 4.2 del libro Linear Functional Analysis de Rynne y Youngson.

4.6 Let $T : C_{\mathbb{R}}[0,1] \to \mathbb{R}$ be the bounded linear transformation defined by

$$T(f) = \int_0^1 f(x)dx.$$

(a) Show that $\|T\| \leq 1$.

(b) If $g \in C_{\mathbb{R}}[0,1]$ is defined by $g(x) = 1$ for all $x \in [0,1]$, find $|T(g)|$ and hence find $\|T\|$.

a) $\|T\| = \sup\{ |T(f)| : \|f\|_\infty = 1 \}$

$= \sup\{ |\int_0^1 f(x)| : \|f\|_\infty = 1 \}$

$\leq \sup\{ |\int_0^1 \|f\|_\infty| : \|f\|_\infty = 1 \}$

$= 1$

b) $|T(g)| = |\int_0^1 1| = 1$

$\Rightarrow \|T\| = 1$

4.7 Let $h \in L^\infty[0,1]$ and let $T : L^2[0,1] \to L^2[0,1]$ be the bounded linear transformation defined by $T(f) = hf$. Show that

$$\|T\| \leq \|h\|_\infty.$$

$$\|T\| = \sup\{\|T(f)\|_\infty : \|f\|_\infty = 1\}$$

$$= \sup\{\|h\|_\infty \|f\|_\infty : \|f\|_\infty = 1\}$$

$$\leq \sup\{\|h\|_\infty : \|f\|_\infty = 1\} = \|h\|_\infty$$

4.10 Let $\mathcal{H}$ be a complex Hilbert space and let $y \in \mathcal{H}$. Find the norm of the bounded linear transformation $f : \mathcal{H} \to \mathbb{C}$ defined by

$$f(x) = (x, y).$$

$$\|f\| = \sup\{|f(x)| : \|x\|_H = 1\}$$

$$= \sup\{|(x,y)| : \|x\|_H = 1\}$$

$$(C\text{-}S) \quad \leq \sup\{\|x\|_H \|y\|_H : \|x\|_H = 1\} = \|y\|_H$$

Tomo $x = y \Rightarrow |f(x)|_{\mathbb{C}} = |(y,y)|_{\mathbb{C}} = \|y\|^2_{\mathbb{C}}$

$$\therefore \|f\| = \|y\|_H$$

4.11 Let $\mathcal{H}$ be a Hilbert space and let $y, z \in \mathcal{H}$. If T is the linear transformation defined by $T(x) = (x, y)z$, show that T is bounded and that $\|T\| \leq \|y\|\|z\|$.