1)

3.21 Use Example 3.46 to find an orthonormal sequence in a Hilbert space $\mathcal{H}$ and a vector $x \in \mathcal{H}$ for which Bessel's inequality holds with strict inequality.

Bessel $\quad \sum^{\infty} |(x, e_n)|^2 \leq \|x\|^2 \qquad \mathcal{H} = (\ell^2, \|\cdot\|_2)$

$x = e_1$

$S_n = e_{2n} \qquad \{S_n\}$ ortogonal

$$\Rightarrow \sum_{n=1}^{\infty} |(x, S_n)|^2 = \sum_{n=1}^{\infty} |(x, e_{2n})| = 0$$

$$\|e_1\|_2 = \sum_{i=1}^{\infty} |e_i|^2 = 1$$

$$\Rightarrow 0 = \sum |(e_1, e_n)| < 1 = \|e_1\|_2$$

3.22 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in $\mathcal{H}$. Determine whether the following series converge in $\mathcal{H}$:

(a) $\displaystyle\sum_{n=1}^{\infty} n^{-1} e_n$; (b) $\displaystyle\sum_{n=1}^{\infty} n^{-1/2} e_n$.

Teo 3.42

a) $\sum^{\infty} n^{-1} e_n$ converge $\Leftrightarrow \sum^{\infty} \left|\frac{1}{n}\right|^2$ converge

$\Leftrightarrow \sum^{\infty} \frac{1}{n^2}$ converge y esto sucede ✓

b) $\sum \left|\frac{1}{\sqrt{n}}\right|^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ que no converge

3.23 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal basis in $\mathcal{H}$. Let $\rho : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$ (so that for all $x \in \mathcal{H}$, $\sum_{n=1}^{\infty} |(x, e_{\rho(n)})|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$). Show that:

(a) $\displaystyle\sum_{n=1}^{\infty} (x, e_{\rho(n)}) e_n$ converges for all $x \in \mathcal{H}$;

(b) $\displaystyle\|\sum_{n=1}^{\infty} (x, e_{\rho(n)}) e_n\|^2 = \|x\|^2$ for all $x \in \mathcal{H}$.

a) por bezsel

$$\sum^{\infty} |(x,e_n)|^2 < \infty \quad (\text{pq } \{e_n\} \text{ ortonormal})$$

$$\Rightarrow \sum |(x, e_{p(n)})|^2 < \infty$$

$$\Rightarrow \sum^{\infty} (x, e_{p(n)}) e_n \leq \infty$$

pq $\{e_n\}$ ortonormal
$(a_n = (x, e_{p(n)}))$

b) Más aún $\|\sum^{\infty} (x, e_{p(n)}) e_n\|^2 = \sum |(x, e_{p(n)})|^2$

$$= \sum |(x, e_n)|^2$$

$$= \|x\|^2 \quad \text{teo 3.47 c)}$$

3.24 Let $\mathcal{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal basis in $\mathcal{H}$. Prove that the *Parseval relation*

$$(x, y) = \sum_{n=1}^{\infty} (x, e_n)(e_n, y)$$

holds for all x, $y \in \mathcal{H}$.

$$(x,y) = \left(\sum^{\infty} (x, e_n)\, e_n, \sum^{\infty} (y, e_j)\, e_j\right)$$

$$= \sum_{n=1}^{\infty} (x, e_n) \left(e_n, \sum^{\infty} (y, e_j)\, e_j\right)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (x\, e_n) \overline{(y\, e_j)} (e_n, e_j) = \sum_{n=1}^{\infty} (x, e_n) \overline{(y\, e_j)}$$

$$= \sum_{n=1}^{\infty} (x\, e_n)(e_n\, y)$$

3.26 Suppose that $\mathcal{H}$ is a separable Hilbert space and $Y \subset \mathcal{H}$ is a closed linear subspace. Show that there is an orthonormal basis for $\mathcal{H}$ consisting only of elements of Y and $Y^\perp$.

Como H separable $\Rightarrow$ Y, $Y^\perp$ separables

Si Y no separable $\exists$ U abiertos disjuntos no numerables en $Y \subseteq H$

$\Rightarrow$ H no separable abs!

Además Y cerrado y métrico y sub de un hilbert

$\Rightarrow$ Y hilbert

•) Análogo $Y^\perp$ (es siempre cerrado)

•) Como ambos hilbert separables $\Rightarrow$ tienen BON

•) Uniendo las BON tenemos BON de H

$(Y \cup Y^\perp = H)$

Con Y cerrado dado $x \in H$

$x = y + \tilde{y}$ $\quad y \in Y$, $\tilde{y} \in Y^\perp$

3.27 Show that for any $b > a$ the set of polynomials with rational (or complex rational) coefficients is dense in the spaces: (a) $C[a, b]$; (b) $L^2[a, b]$.
Deduce that the space $C[a, b]$ is separable.

Dado $f \in C[a,b]$ y $\varepsilon > 0$ $\exists p_1$ pol /

$$|f(x) - p_1(x)| \leq \frac{\varepsilon}{2} \quad \forall x \in [a,b] \quad (\text{Stone-Weier})$$

$$\Rightarrow \|f - p_1\|_\infty \leq \frac{\varepsilon}{2}$$

dado $p_1 = \sum_{i=0}^{n} a_i x^i$ $a_i \in \mathbb{R}$ como $\mathbb{Q}$ denso

$$\exists b_i \in \mathbb{Q} / \; d(a_i, b_i) \leq \frac{\varepsilon}{2C}$$

$\tilde{C} = \max\{|a|, |b|\}$

$\tilde{C} \leq K \in \mathbb{N}$

$C = \sum_{i=0}^{n} K^i$

sea $p_2 = \sum^{n} b_i x^i$

$$|p_1(x) - p_2(x)| = \left|\sum^{n} (a_i - b_i) x^i\right|$$

$$= \left|\frac{\varepsilon}{2C} \sum^{n} x^i\right| \leq \left|\frac{\varepsilon}{2C} \sum_{i=0}^{n} K^i\right|$$

$$= \frac{\varepsilon}{2}$$

$$\Rightarrow \|p_1(x) - p_2(x)\|_\infty \leq \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x) - p_2(x)| \leq |f(x) - p_1(x)| + |p_1(x) - p_2(x)|$$

$$\leq \varepsilon/2 \quad + \quad \varepsilon/2 \quad = \varepsilon$$

$$\forall x \in [a,b] \Rightarrow \|f - P_2\|_\infty \leq \varepsilon$$

$\Rightarrow f \in \overline{K[\mathbb{Q}]}$

(dada cualquier bola $B(f,r) \Rightarrow \exists p \in K[\mathbb{Q}]$

tq $\|f-p\| < r \Rightarrow p \in B(f,r)$)

Análogo $B(f,r) \cap K[\mathbb{Q}] \neq \emptyset \quad \forall r > 0$

$\Rightarrow f \in \overline{K[\mathbb{Q}]}$

$\Rightarrow \overline{K[\mathbb{Q}]} = C[0,1]$

$K[\mathbb{Q}]$ es numerable porq es unión de numerables

$S_1 = \{ q \in \mathbb{Q} \}$

$S_2 = \{ (q_1, q_2) \in \mathbb{Q} \times \mathbb{Q} \}$

$S_j = \{ (q_1, \dots q_j) \in \mathbb{Q}^j \}$

$\bigcup_{j \in \mathbb{N}} S_j$ es unión de numerables y se puede dar fácil

biyección entre $K[\mathbb{Q}]$ y $\bigcup_{j \in \mathbb{N}} S_j$

$\Rightarrow (C[0,1], \|\ \|_\infty)$ separable

b) Los pol están en $L^2[a,b]$. $f \in L^2$ $\varepsilon > 0$

por Stone Weier $\exists p_1 \ / \ |f(x) - p_1(x)| \leq \sqrt{\frac{\varepsilon}{2(b-a)}} \quad \forall x \in [a,b]$

$\Rightarrow \|f - p_1\| = \int_a^b (f - p_1)^2 \leq \frac{\varepsilon}{2}$

Análogo $|p_1(x) - p_2(x)| \leq \sqrt{\frac{\varepsilon}{2(b-a)}} \quad \forall x \in [a,b]$

$$\Rightarrow \|p_1 - p_2\| = \int_a^b |p_1(x) - p_2(x)|^2 \leq \frac{\varepsilon}{2}$$

se termina igual que a)

3.28) (Polinomios de Legendre) Para un entero $n \geq 0$, se define los polinomios:

$U_n(x) = (x^2-1)^n$, $P_n(x) = \frac{1}{2^n n!} \frac{d^n u_n}{dx^n}$ ((Claramente U_n tiene orden $2n$, P_n orden n))

Los polinomios P_n se llaman polinomios de Legendre. Consideramos estos polinomios en el intervalo $[-1,1]$, y sea $\mathcal{H} = L^2[-1,1]$ con el P.I estándar.

Probar: a) $d^{2n} u_n / dx^{2n}(x) = (2n)!$ b) $(P_m, P_n) = 0$ para $m, n \geq 0$, $m \neq n$

c) $\|P_n\|^2 = (2^n n!)^2 \frac{2}{2n+1}$, para $n \geq 0$

d) $\{e_n = \sqrt{\frac{2n+1}{2}} P_n : n \geq 0\}$ es b.o de $\mathcal{H}$.

Dem:

a) $\frac{d^{2n} u_n}{dx^{2n}}(x)$, el término de orden $2n$ de u_n es claramente x^{2n}, y la $2n$-ésima derivada es $(2n)!$ //

b) Es suficiente considerar el caso $0 \leq m < n$. Ahora, si $0 < k < n$, entonces integrando k veces por partes:

$$\int_{-1}^{1} x^k \frac{d^n u_n}{dx^n} dx = x^k \frac{d^{n-1} u_n}{dx^{n-1}} \Big|_{-1}^{1} - k \int_{-1}^{1} x^{(k-1)} \frac{d^{n-1} u_n}{dx^{n-1}} dx$$

$$\vdots$$

$$= (-1)^k k! \int_{-1}^{1} \frac{d^{n-k} u_n}{dx^{n-k}} dx$$

$$= (-1)^k k! \left. \frac{d^{n-k-1} u_n}{dx^{n-k-1}} \right|_{-1}^{1} = 0$$

Luego, como P_m tiene orden $m < n$, se sigue que $(P_m, P_n) = 0$ //

3.28 (Legendre polynomials) For each integer $n \geq 0$, define the polynomials

$$u_n(x) = (x^2 - 1)^n, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n u_n}{dx^n}$$

(clearly, u_n has order $2n$, while P_n has order n). The polynomials P_n are called *Legendre polynomials.* We consider these polynomials on the interval $[-1, 1]$, and let $\mathcal{H} = L^2[-1, 1]$, with the standard inner product $(\cdot, \cdot)$. Prove the following results.

(a) $d^{2n}u_n/dx^{2n}(x) \equiv (2n)!$.

(b) $(P_m, P_n) = 0$, for m, $n \geq 0$, $m \neq n$.

(c) $\|P_n\|^2 = (2^n n!)^2 \frac{2}{2n+1}$, for $n \geq 0$.

(d) $\left\{ e_n = \sqrt{\frac{2n+1}{2}} P_n : n \geq 0 \right\}$ is an orthonormal basis for $\mathcal{H}$.

[Hint: use integration by parts, noting that u_n, and its derivatives to order $n - 1$, are zero at ± 1.]

a) $U_n(x)$ es un pol de gr $(2n)$

el término de orden $2n$ es x^{2n}

$\rightarrow \quad \dfrac{d^{2n} u_n}{dx^{2n}} = 2n!$ el resto de los términos desaparecen

(2) Sea $\chi_{[0,1]}$ la función característica del intervalo $[0,1]$. Probar que

$$\{\chi_{[0,1]}(x-n)\, e^{2\pi i m x}\}_{n,m\in\mathbb{Z}}$$

es base ortonormal de $L^2(\mathbb{R})$ (llamada *base de Gabor*). Deducir que $L^2(\mathbb{R})$ es separable.

$$\left(\chi_{[0,1]}(x-n_1)\, e^{2\pi i m_1 x},\ \chi_{[0,1]}(x-n_2)\, e^{2\pi i m_2 x} \right)$$

$$= \int_{\mathbb{R}} \chi_{[0,1]}(x-n_1) e^{2\pi i m_1 x}\ \chi_{[0,1]}(x-n_2)\, e^{-2\pi i m_2 x}$$

$$= \int_{\mathbb{R}} \chi_{[0,1]}(x-n_1)\, \chi_{[0,1]}(x-n_2)\, e^{(m_1-m_2)2\pi i x}$$

$$\chi_{[0,1]}(x-n_1) = \begin{cases} 1 & x\in[n_1, n_1+1] \\ 0 & cc \end{cases}$$

$$\chi_{[0,1]}(x-n_2) = \begin{cases} 1 & x\in[n_2, n_2+1] \\ 0 & cc \end{cases}$$

$C_2 \leq 0$

$\cdot)$ $n_1, n_2 \in \mathbb{Z}$. si $\underline{n_1 \neq n_2}$ $\Rightarrow$ $[n_1, n_1+1] \cap [n_2, n_2+1] = \begin{cases} n_1+1 \\ \emptyset \end{cases}$

$\Rightarrow \int \cdot \_\_\_ = 0$

Caso

·) $n_1 = n_2 = n \Rightarrow \int : \int_{[n,n+1]} e^{(m_1-m_2)2\pi i x} = \dfrac{e^{(m_1-m_2)2\pi i x}}{(m_1-m_2)2\pi i}\Bigg|_n^{n+1}$

$\Rightarrow m_1 \neq m_2$

$$= \frac{e^{(m_1-m_2)(2\pi i)(n+1)} - e^{(m_1-m_2)(2\pi i)(n)}}{(m_1-m_2)2\pi i}$$

$$= \frac{e^{(m_1-m_2)2\pi i n}\left(e^{(m_1-m_2)2\pi i} - 1\right)}{(m_1-m_2)2\pi i}$$

$= 0$

$e^{2k\pi i} = 1$

$\forall k$

$\Rightarrow$ son ortogonales

Si $n_1 = n_2$ y $m_1 = m_2$ sería la norma

$$= \int_{[n,n+1]} e^0 = (n+1) - n = 1$$

$\Rightarrow$ son ortonormales

$$f_n = f\,\chi_{[0,1)}(x-n) = \cancel{f\,\chi_{[n,n+1]}(x)}$$

$\Rightarrow$ Hdz $f \in L^2(\mathbb{R})$

$$f = \sum_{n\in\mathbb{Z}} f_n$$

Usando fourier en $(n, n+1)$ base $\{e^{i2k\pi t} : k \in \mathbb{Z}\}$

$$\xrightarrow{L^2}\qquad\qquad\qquad \to f_n(t) = f(t)\,\chi_{[0,1]}(t-n)$$

$$f_n(t) = \left(\sum_{k\in\mathbb{Z}} a_k e^{i2k\pi t}\right) = \left(\sum_{k\in\mathbb{Z}} a_k e^{i2k\pi t}\right)\chi_{[0,1]}(t-n)$$

con $a_k(t) = \left(f_n(t), e^{-i2k\pi t}\right) = \left(f, e^{-i2k\pi t}\chi_{[0,1]}(t-n)\right)$

$$f(t) = \sum_{n\in\mathbb{Z}} f_n = \sum_{n\in\mathbb{Z}} f_n\,\chi_{[0,1]}(t-n) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_k e^{i2k\pi t}\chi_{[0,1]}(t-n)$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_k e^{i2k\pi t}\chi_{[0,1]}(t-n)$$

$$= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}} \left(f, e^{-i2k\pi t}\chi_{[0,1]}(t-n)\right) e^{i2k\pi t}\chi_{[0,1]}(t-n)$$

)El chiste es que base de fourier sirve de base de las periódicas

Y la base de gabor sirve para cualquier $f \in L^2(\mathbb{R})$

Como? partimos $f = \sum \overbrace{\chi_{[n,n+1]} f}^{f_n}$

y cada una de estas f_n son "periódicas" (se puede pensar que f_n se extiende repitiendo sus valores en $[n, n+1]$)

Como son periódicas las aproximo con base de fourier, ahora sumando todas estas aproximaciones, aproximo f con base de gabor

Base fourier en gral $C[a,b]$

$$\left\{ c^{-\frac{1}{2}} e^{\frac{i2k\pi x}{c}} \right\} \quad c = \frac{1}{b-a}$$

Caso $[a,b) = [0,2\pi]$ $\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\}$

(La que vimos en corolario 3.37)

(3) Sea $f \in C^k(\mathbb{S}^1)$. Probar que

$$\frac{1}{\sqrt{2\pi}}\langle f^{(k)}, e^{inx}\rangle = (in)^k \left(\frac{1}{\sqrt{2\pi}}\langle f, e^{inx}\rangle\right)$$

donde $\langle g, h\rangle := \int_{-\pi}^{\pi} g(x)\overline{h(x)}\, dx$.

$$(f^k, e^{inx}) = \int_{-\pi}^{\pi} f^k e^{-inx} = e^{-inx} f^{k-1}(x)\Big|_{-\pi}^{\pi} + in \int e^{-inx} f^{k-1}$$

$$u = e^{-inx} \quad du = -ine^{-inx}$$

$$v = f^{k-1} \quad dv = f^k$$

$$= \underbrace{e^{-in\pi} f^{k-1}(\pi) - e^{in\pi} f^{k-1}(-\pi)}_{f^{k-1}(\pi) = f^{k-1}(-\pi) \text{ por periodicidad } f \in S^1} + in\,(f^{k-1}, e^{inx})$$

$$= K[\underbrace{e^{-in\pi}}_{=(-1)^n} - \underbrace{e^{in\pi}}_{=(-1)^n}] + in\,(f^{k-1}, e^{inx})$$

$$= in\,(f^{k-1}, e^{inx}) \quad \ldots \quad = in^K (f^{(0)}, e^{inx})$$

with $f^{(0)} = f$.

$\square$

(4) Probar que si $f \in L^1([-\pi,\pi])$ entonces existen sus coeficientes de Fourier

$$a_n(f) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

y se cumple que $(a_n(f))_n \in c_0$.

I) $f \in L^1[-\pi,\pi] \Rightarrow$ medible, e^{-inx} continua $\Rightarrow$ medible

$\Rightarrow f(x)e^{-inx}$ medible $\Rightarrow a_n(f)$ bien def

$$|a_n(f)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f(x)e^{-inx}|\, dx \qquad e^{-inx} \leq 1$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f(x)| < \infty \quad \text{por } f \in L^1$$

$$= \frac{1}{\sqrt{2\pi}} \|f\|_1 \qquad \forall f \in L^1[-\pi,\pi]$$

$$\Rightarrow \|\{a_n(f)\}\|_\infty = \sup\{|a_n(f)| : n \in \mathbb{N}\} \leq \frac{1}{\sqrt{2\pi}} \|f\|_1$$

II) $\{a_n(\alpha f + g)\} = (a_1(\alpha f + g), a_2(\alpha f+g), \dots)$

$= (\alpha a_1(f) + a_1(g), \dots)$

$= (\alpha a_1(f), \dots) + (a_1(g) \dots)$

$= \alpha a(f) + a(g) = \{\alpha a_n(f)\} + \{a_n(g)\}$

$$a_n(\alpha f + g) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\alpha f + g)\, e^{-inx}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \alpha \int f e^{-inx} + \int g\, e^{-inx} \right\} = \alpha a_n f + a_n g$$

III) Sea f continua $\Rightarrow f \in (L^2, \|\cdot\|_2)$

$\mathcal{E} = \{e_n(x) = \sqrt{2\pi}\, e^{inx} : n \in \mathbb{Z}\}$ es bon de $(L^2[-\pi,\pi], \|\cdot\|_2)$

$\Rightarrow \quad f = \sum (f, e_n)\, e_n$ (pq tiene bon)

con $\quad (f, e_n) = \int f \dfrac{e^{-inx}}{\sqrt{2\pi}} = a_n(f)$

por las eq de Bon

$\Rightarrow \infty > \|f\|_2^2 = \sum_{n=1}^{\infty} |(f, e_n)|^2 = \sum^{\infty} |a_n(f)|^2$

$\downarrow$ pq $f \in L^2 \quad \Rightarrow \sum_{n=1}^{\infty} |a_n(f)|^2$ converge

$\Rightarrow \sum_{n=1}^{\infty} |a_n(f)|$ converge

$\Rightarrow |a_n(f)| \xrightarrow{n \to \infty} 0$

$\Rightarrow \{a_n(f)\}_n \in c_0$

a) Ahora sea $f \in L^1[-\pi,\pi]$ como las continuas son densas $\exists f_m$ continuas tq

$f_m \xrightarrow[m\to\infty]{\|\cdot\|_1} f$ además por ser continuas

$\{a_n(f_m)\}_n \in c_0$ probado en III

miramos convergencia $\{a_n(f_m)\}_n$ en $(c_0, \|\cdot\|_\infty)$

$\|\{a_n(f)\}_{n\in\mathbb{N}} - \{a_n(f_m)\}_{n\in\mathbb{N}}\|_\infty \overset{\text{I}}{=} \|\{a_n(f - f_m)\}_{n\in\mathbb{N}}\|$

(I) vale pq f_m son continuas $\Rightarrow$ son L^1 y $f \in L^1$

(II) $\leq \frac{1}{\sqrt{2\pi}} \|f - f_m\|_1 \xrightarrow[m \to \infty]{} 0$

$$\Rightarrow \{a_n(f_m)\}_{n\in\mathbb{N}} \xrightarrow[m\to\infty]{\|\,\|_\infty} \{a_n(f)\}$$

pero $\{a_n(f_m)\}_{n\in\mathbb{N}} \subseteq C_0$

$(C_0, \|\cdot\|_\infty)$ es banach

$\Rightarrow$ es cerrado $\Rightarrow \{a_n(f)\} \subseteq C_0$

(5) (a) Calcular los coeficientes de Fourier de $f(t) = -\chi_{[-\pi,0]}(t) + \chi_{[0,\pi]}(t)$.

(b) Calcular $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

(c) Deducir usando (b) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

a) $\mathcal{E} = \{e_n(x) = 2\pi^{-1/2} e^{inx} : \cancel{k} \in \mathbb{Z}\} = \{e_n(x) = 2\pi^{-1/2} e^{-inx} : n \in \mathbb{N}\}$
$\cup \{\tilde{e}_n(x) = 2\pi^{-1/2} e^{inx} : n \in \mathbb{N}\}$

$$(f, \tilde{e}_n) = \tilde{a}_n(f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f \, e^{-inx} \quad \text{(teo 3.47)}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-\pi}^{0} e^{-inx} + \int_{0}^{\pi} e^{-inx} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-inx}}{in} \Big|_{-\pi}^{0} - \frac{e^{-inx}}{in} \Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{in} - \frac{e^{\pi i n}}{in} - \frac{e^{-\pi i n}}{in} + \frac{1}{in} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2}{in} - \frac{(-1)^n}{in} - \frac{(-1)^n}{in} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2 - 2(-1)^n}{in} \right\} = \begin{cases} 0 & n \text{ par} \\ \frac{1}{\sqrt{2\pi}} \frac{4}{in} & n \text{ impar} \end{cases}$$

$$= \sqrt{\frac{8}{\pi}} \frac{1}{i(2n+1)}$$

$$(f, e_n) = a_n(f) = \frac{1}{\sqrt{2\pi}} \int f \, e^{inx}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-\pi}^{0} e^{inx} + \int_{0}^{\pi} e^{inx} \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[-\frac{e^{inx}}{in}\Big|_{-\pi}^{0} + \frac{e^{inx}}{in}\Big|_{0}^{\pi} \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[-\frac{1}{in} + \frac{e^{-\pi i n}}{in} + \frac{e^{\pi i n}}{in} - \frac{1}{in} \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[-\frac{2}{in} + \frac{2(-1)^n}{in} \right] = \begin{cases} 0 & n \text{ par} \\ -\frac{4}{in} & n \text{ impar} \end{cases}$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{4}{i(2n+1)} \quad n \in \mathbb{N}$$

$$= -\sqrt{\frac{8}{\pi}} \frac{1}{i(2n+1)} \quad n \in \mathbb{N}$$

$\Rightarrow$ por ser coeficientes de $\mathcal{E}$ cmo bon

$\cdot)$ 3.47 e) $\|f\|_2^2 = \sum_{n\in\mathbb{N}} |(f, e_n)|^2 + \sum_{n\in\mathbb{N}} |(f, \tilde{e_n})|^2$ $\to$ coef

$= \sum_{n\in\mathbb{N}} |a_n(f)|^2 + \sum_{n\in\mathbb{N}} |\tilde{a_n}(f)|^2$

$= 2\sum -\frac{8}{\pi}\frac{1}{(2k+1)^2} = -\frac{16}{\pi}\sum (2k+1)^2$

por otro lado

$\|f\|_2^2 = |(f, f)| = |\int f \bar{f}|$

$= \int (-\chi_{[-\pi,0]} + \chi_{[0,\pi]})(-\chi_{[-\pi,0]} + \chi_{[0,\pi]})$

$\int \chi_{[-\pi,0]}^2 - \cancel{2\chi_{[-\pi,0]}\chi_{[0,\pi]}} + \chi_{[0,\pi]}^2$

$= \int_{-\pi}^{0} 1 + \int_0^{\pi} 1 = \int_{-\pi}^{\pi} 1 = 2\pi$

$\Rightarrow 2\pi = -\frac{16}{\pi}\sum \frac{1}{(2n+1)^2}$

$-\frac{\pi^2}{8} = \sum_{n\in\mathbb{N}} \frac{1}{(2n+1)^2}$

c) $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + 1 = \sum_{n=0}^{\infty} \frac{1}{(2n)^2}$

$\sum_{n=1} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$= \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$

c)

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

(6) Sean f y g funciones pertenecientes al $L^2((-\pi,\pi])$. Extenderlas a funciones sobre $\mathbb{R}$ de manera tal que resulten periódicas de período 2π. Mostrar que la convolución

$$(f*g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,dy$$

está en $L^1((-\pi,\pi))$ y se satisface

$$\langle f*g, e^{inx}\rangle = \langle f, e^{inx}\rangle\langle g, e^{inx}\rangle.$$

Fubini

$$\left[\int_{-\pi}^{\pi} |(f*g)(x)|^2\right]^{\frac{1}{2}} = \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(y)|^2 |g(x-y)|^2\,dy\,dx\right]^{\frac{1}{2}} = \left[\int_{-\pi}^{\pi} |f(y)|^2 \left[\int_{-\pi}^{\pi} |g(x-y)|^2 dx\right] dy\right]^{\frac{1}{2}}$$

$u = x-y$

$du = dx$

$$= \left[\int_{-\pi}^{\pi} |f(y)|^2 \left[\int_{-\pi-y}^{\pi-y} |g(u)|^2 du\right] dy\right]^{\frac{1}{2}}$$

$\pi - y - (-\pi - y) = 2\pi$

un período

$$= \left[\int_{-\pi}^{\pi} |f(y)|^2 \int_{-\pi}^{\pi} |g(u)|^2 du\,dy\right]^{\frac{1}{2}}$$

$$= \left[\int_{-\pi}^{\pi} f(y)\, \|g\|_2^2\, dy\right]^{\frac{1}{2}}$$

$$= \|g\|_2 \|f\|_2$$

$\|f*g\|_2 \leq \|f\|_2 \|g\|_2$

$$\Rightarrow \int |f*g(x)|\,dx \leq \int \varepsilon\,dx$$

$$\Rightarrow (f*g)(x) \in L^1$$

$$\left((f*g)(x), e^{inx}\right) = \iint f(x) g(x-y)\, e^{-inx}\,dx\,dy$$

$$= \iint f(x)\, g(x-y)\, e^{-iny} e^{-in(x-y)}\,dx\,dy$$

$$= \int_{-\pi}^{\pi} f(y) e^{-iny}\left[\int g(x-y) e^{in(x-y)}\,dx\right] dy$$

lo mismo que arriba $\rightarrow$ $$= \int_{-\pi}^{\pi} f(y) e^{-iny} \left[\int_{-\pi-y}^{\pi-y} g(u) e^{-inu} du \right] dy$$

$$= \int_{-\pi}^{\pi} f(y) e^{-iny} \left[\int_{-\pi}^{\pi} g(u) e^{-inu} du \right] dy$$

$$= \int_{-\pi}^{\pi} f(y) e^{-iny} \underbrace{\left(g(u), e^{-inu} \right)}_{cte} dy$$

$y \in [-\pi, \pi]$

$u \in [-\pi, \pi)$

$$= \left(f(y), e^{-iny} \right) \left(g(u), e^{-inu} \right)$$

$$= \left(f(x), e^{-inx} \right) \left(g(x), e^{-inx} \right)$$

(7) Sea f una función par en $L^2([-\pi,\pi])$. Probar que $a_n(f) = a_{-n}(f)$ para todo $n \in \mathbb{Z}$.

$$a_n(f) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-inx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{\sqrt{2\pi}} - \int_{\pi}^{-\pi} f(u) e^{inu}\, du$$

$u = -x$

$du = -dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(u) e^{inu}\, du$$

$$= a_{-n}(f)$$

8)

(8) Sea $f \in \mathcal{C}(\mathbb{R})$, periódica de período 2π. Dado $\varepsilon > 0$, probar que existe un polinomio trigonométrico $\varphi(x) = a_0 + \sum_{n=0}^{N} (a_n \cos(nx) + b_n \operatorname{sen}(nx))$, tal que $|f(x) - \varphi(x)| < \varepsilon \ \forall x$.

Sea $K = S^1$ compacto.

Sea $A = \{ \sum_{n=1}^{N} a_n z^n : a_n \in \mathbb{R}, N \in \mathbb{N} \}$ pols complejos

que A $*$-subálgebra $C(S^1)$

tq 1) si $x \neq y \in K \Rightarrow \exists f \in A \ / \ f(x) \neq f(y)$

2) $A \neq 0$

$\Rightarrow A$ es denso en $C(S^1)$

demo 1) Sean $x, y \in S^1$ $x \neq y$

$\Rightarrow x = e^{it}$ $y = e^{i\tilde{t}}$ con $t \neq \tilde{t}$

sea $p(z) = z \Rightarrow p(e^{i\tilde{t}}) = e^{i\tilde{t}} \neq e^{it} = p(e^{it})$

2) $A \neq \phi$

Falta ver que A $*$-subalgebra

$\Rightarrow$ pols complejos son densos en $C(S^1)$

Ahora toda $f \in C(\mathbb{R})$ f

$\Rightarrow \quad f:\mathbb{R}\to\mathbb{R}$ $\quad 2\pi$- periódica

defino $\quad \tilde f : S^1 \to \mathbb{R}$

$$\tilde f(e^{it}) = f(t) \qquad t\in\mathbb{R}$$

está bien def por supongamos

$$f(t) = f(e^{it}) \quad \text{y} \quad f(t) = f(e^{i\tilde t})$$

$$\Rightarrow \quad f(e^{it}) = f(e^{i\tilde t})$$

por tpo

$$\begin{tikzcd} \mathbb{R} \arrow[rr] \arrow[dr] & & \mathbb{R} \\ & S^1 = \mathbb{R}/n\mathbb{Z} \arrow[ur, dashed, "\tilde f"'] & \end{tikzcd}$$

$\Rightarrow \quad \tilde f \in C(S^1) \Rightarrow \exists\, p$ pol complejo

tq $\quad |p(z) - \tilde f(z)| \le \varepsilon \qquad (z = e^{it})$

$\Leftrightarrow \quad |p(e^{it}) - \tilde f(e^{it})| < \varepsilon$

pero notar $\quad p(e^{it}) = \sum a_n e^{int}$

$$= \sum a_n(\cos(nt) + i\,\mathrm{sen}(nt))$$

$$= \sum a_n\cos(nt) + a_n i\,\mathrm{sen}(nt)$$

$(a_0 = 0)$
$$= \sum a_n\cos(nt) + b_n\,\mathrm{sen}(nt)$$

$$= \varphi(t)$$

$y \quad \tilde{f}(e^{it}) = f(t)$

$\Rightarrow \quad |\varphi(t) - f(t)| < \varepsilon$

(9) Sea $\mathcal{P}$ pre-Hilbert con BON, probar que $\mathcal{P}$ separable si y sólo si existe $\{\varphi_i\}$ base numerable.

Tenemos $B = \{e_i\}_{i \in I} \subseteq \mathcal{P}$ cumple $(e_i, e_j) = \delta_{ij}$

y dado $x \in \mathcal{P}$ $x = \sum_{i \in I} \alpha_i e_i$ con $\alpha_i = 0$ salvo numerables $i \in I$

Si $i \neq j$ $(e_i, e_j) = 0$

pre Hilbert

$\Rightarrow \|e_i - e_j\|^2 = (e_i - e_j, e_i - e_j)$

$= \|e_i\|^2 + (e_i, e_j) + (e_j, e_i) + \|e_j\|^2 = 2$

$\Rightarrow \|e_i - e_j\| = \sqrt{2}$ $\forall i, j \in I$ $i \neq j$

Como A denso (y numerable)

$\Rightarrow \forall i \in I$ $\exists\, a_i \in A \,/\, \|e_i - a_i\| \leq \frac{1}{2}$

$\sqrt{2} = \|e_i - e_j\| \leq \|e_i - a_i\| + \|a_i - a_j\| + \|a_j - e_j\|$

$\leq 1 + \|a_i - a_j\|$

$\Rightarrow \|a_i - a_j\| > 0 \Rightarrow a_i \neq a_j$

y como para cada e_i hay un a_j

$\Rightarrow$ tengo numerables e_i

$\Rightarrow e_i$ base numerable

(base de Hilbert

($\Leftarrow$) Recordar $\mathbb{F} = \mathbb{R}$ o $\mathbb{C}$

$$Q = \{ \sum_{i=1}^{n} q_i \varphi_i \mid n \in \mathbb{N} \; \underbrace{q_i \in \mathbb{Q}}_{\mathbb{F}=\mathbb{R}} \text{ o } \underbrace{(\mathbb{Q} + i\mathbb{Q})}_{\mathbb{F}=\mathbb{C}} \}$$

•) es trivialmente numerable

$$Q = \bigcup_{n \in \mathbb{N}} \underbrace{\{ \sum_{i=1}^{n} q_i \varphi_i \mid q_i \in \mathbb{Q} \text{ o } (\mathbb{Q} + i\mathbb{Q}) \}}_{\text{numerable}}$$

Veamos denso. Sea $x \in \mathcal{P}$

como $\{\varphi_i\}$ BON

$$x = \sum_{i=1}^{\infty} c_i \varphi_i$$

por teo 3.42 $c_i \in \ell^2$ $\sum |c_i|^2$ converge

$\Rightarrow$ dado $\varepsilon > 0$ $\exists N_0 \in \mathbb{N} / \sum_{n=N_0}^{\infty} |c_i|^2 < \frac{\varepsilon}{\sqrt{2}}$

y para cada $c_i \in \mathbb{F} = \mathbb{R}$

$$\exists q_i \in \mathbb{Q} / |c_i - q_i|^2 \leq \frac{\varepsilon}{\sqrt{2 N_0}}$$

defino $y = \sum_{i=0}^{N_0} q_i \varphi_i$

$$\|x - y\| = \sum_{i=0}^{\infty} |c_i - q_i|^2 = \sum_{i=0}^{N_0} |c_i - q_i|^2 + \sum_{i=N_0}^{\infty} |c_i|^2$$

$$\leq N_0 \frac{\varepsilon}{2N_0} + \frac{\varepsilon}{2} = \varepsilon$$

(*)

$\phi(5,4)$ por 3.47 c) $\|x-y\|^2 = \sum |(x-y, e_i)|^2$

$$= \sum |(x, e_i) - (y, e_i)|^2$$

por d) $(x, e_i) = c_i \quad (y, e_i) = \eta_i$

(10) Si $\mathcal{H}$ es un espacio de Hilbert de dimensión infinita entonces toda base algebraica es no numerable.

supongo $\{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$ algebraica numerable

G-S $\Rightarrow$ $\{\varphi_n\}_{n\in\mathbb{N}}$ ortonormal base

$$v_N = \sum_{i=1}^{N} \frac{\varphi_n}{2^n} \in H$$

obs v_n es de cauchy en H

$$\underset{(m<n)}{\|v_n - v_m\|} = \left\| \sum_{i=n}^{m} \frac{\varphi_i}{2^i} \right\| \leq \sum_{i=n}^{m} \frac{\|\varphi_i\|}{2^i}$$

$$= \sum_{i=n}^{m} \frac{1}{2^i} < \varepsilon$$

$\Rightarrow \exists \lim v_N := v$ (por ser Hilbert)

$$v = \lim v_n = \lim \sum_{i=1}^{n} \frac{\varphi_i}{2^i}$$

$$\circledast = \sum_{i=1}^{\infty} \frac{\varphi_i}{2^i}$$

Además $v \in H \Rightarrow v = \sum^{\infty} c_n \varphi_n$ con finitos c_n no nulos

$\Rightarrow \quad \frac{1}{2}^{1} = (v, \varphi_k) = c_n \neq 0 \rightsquigarrow$ abs!