2) Sea V - ev y sea $\Sigma \subseteq V$ de subconjuntos de V linealmente indeptes

•) Sea C cadena en Σ

Probemos $T = \bigcup_{S_i \in C} S$ es maximal de C

Supongamos T no es LI

$\Rightarrow \exists\, v_1, \dots, v_n$ no LI

$\exists$ pd $v_i \in S_i$,

(por cadena) $S_i \subseteq S_j \Rightarrow v_i \in S_n$ $i = 1 \dots n$

$\Rightarrow S_j$ no sería LI $\Rightarrow S_j \notin C$ Abs!

$\therefore$ T es LI y es cota superior de C

•) Por Zorn $\exists$ maximal de Σ

•) Afirmo dicho maximal $M(\Sigma)$ es base de V. Supongo que no (es LI)

$\Rightarrow \exists\, v \in V \;/\; \langle v \rangle \notin M(\Sigma)$

$\Rightarrow M(\Sigma) \cup \langle v \rangle$ es $\overline{LI}$

por maximal $M(\Sigma) \cup \langle v \rangle \subseteq \overline{M}(\Sigma)$

$\Rightarrow \langle v \rangle \subseteq M(\Sigma)$ abs!

$\Rightarrow M(\Sigma)$ es base de V

3) $\{X_n\} \subseteq X$ de cauchy $\Rightarrow \|X_n - X_m\| < \frac{1}{2^k}$ $\forall n, m \geq n(k)$

y esto lo hacemos para cada k

y tomamos $n(k)$ crecientes

Sea $y_{n_k} = X_{n(k)}$

<u>notar</u> $\|y_{n_{k+1}} - y_{n_k}\| = \|X_{n(k+1)} - X_{n(k)}\| \leq \frac{1}{2^k}$

como $n(k) \geq n(k-1) \Rightarrow n(k), n(k-1) \geq n(k-1)$ (por cauchy)

$$\Rightarrow \sum^{\infty} \|y_{n_{k+1}} - y_{n_k}\| \leq \sum^{\infty} \frac{1}{2^k} \leq \infty$$

por hipótesis $\sum^{\infty} y_{n_{k+1}} - y_{n_k} < \infty$

$$\Rightarrow \sum_{k=1}^{N} y_{n_{k+1}} - y_{n_k} = y_{n_N} - y_{n_1}$$

$$S = \lim_{N \to \infty} \sum y_{n_{k+1}} - y_{n_k} = \lim_{N \to \infty} y_{n_N} - y_{n_1}$$

$$\Rightarrow S + y_{n_1} = \lim_{N \to \infty} y_{n_N} = \lim_{N \to \infty} X_{n_N}$$

$\Rightarrow X_{n_N}$ sub convergente de X_n

como X_n de cauchy $\Rightarrow X_n$ converge

vale para X normado

$\Rightarrow X$ métrico

4) b) Supongo que $\dim(\ell^2) = n < \infty$

$\Rightarrow$ Ahora tomo $e_1, \dots, e_n, e_{n+1}$

todos pertenecen a ℓ^2 y son li

$\Rightarrow$ tengo $n+1$ elementos li

$\dim(\ell^2) \geq n+1$ abs!

a) Sea X^n de cauchy. Dado $\varepsilon > 0$

$$\exists M \,/\, \left.\begin{array}{l} \| X^n - X^m \| < \varepsilon \\ \displaystyle\sum_{k=1}^{\infty} (X_k^n - X_k^m)^2 \leq \varepsilon^2 \end{array}\right\} \Rightarrow \forall\, n, m \geq M$$

$(k \text{ fijo}) \Rightarrow \quad (X_k^n - X_k^m)^2 \leq \varepsilon^2 \quad \forall\, n, m > M$

$$|X_k^n - X_k^m| \leq \varepsilon \quad \forall\, k \in \mathbb{N}$$

$\Rightarrow (X_k^n)_{n \in \mathbb{N}}$ es de cauchy

$\Rightarrow$ Como es sucesión de reales converge

$/ \; X_k^n \xrightarrow[n \to \infty]{} X_k$

Ahora vemos $\|X^n - X\|^2 = \sum_{k=0}^{\infty} (X_k^n - X_k)^2 < \infty$

$$\sum^{N} (X_k^n - X_k)^2 = \sum_{k=1}^{N} \left(X_k^n - \lim_{m\to\infty} X_k^m\right)^2$$

$$= \lim_{m\to\infty} \sum_{k=1}^{N} (X_k^n - X_k^m)^2$$

$$\leq \lim_{m\to\infty} \sum_{k=1}^{\infty} (X_k^n - X_k^m)^2$$

$$\leq \lim_{m\to\infty} \varepsilon^2 = \varepsilon^2$$

$$\Rightarrow \sum^{N} (X_k^n - X_k)^2 \leq \varepsilon^2$$

$$\lim_{N\to\infty} \sum^{N} (X_k^n - X_k)^2 \leq \varepsilon^2$$

$$\sum_{k=0}^{\infty} (X_k^n - X_k)^2 \leq \varepsilon^2$$

$$\|X^n - X\| = \sqrt{\sum^{\infty} (X_k^n - X_k)^2} \leq \varepsilon$$

$\Rightarrow X^n \to X \Rightarrow \ell^2(\mathbb{N})$ completo

$\therefore \ell^2(\mathbb{N})$ Banach

4) c) $A_1 = \{\{a_n\}_{n\in\mathbb{N}} \ / \ a_n \in \mathbb{Q}, \ a_n = 0 \ \forall n > 1\}$

$\simeq \mathbb{Q}$

$A_j = \{\{a_n\}_{n\in\mathbb{N}} \ / \ a_n \in \mathbb{Q} \quad a_n = 0 \ \forall n > j\}$

$\simeq \mathbb{Q}^j$

estas son numerables

d) $A = \bigcup_{i\in\mathbb{N}} A_i$ es numerable

Veamos denso. Sea $x = \{x_n\} \subseteq \ell^2(\mathbb{N})$

~~q~~ $B(\{x_n\}, \varepsilon) \cap A \neq \emptyset \ \forall \varepsilon > 0$

(notar que $\ell^2(\mathbb{N})$ es banach osea métrico por eso uso ~~bolas~~ bolas como entorno)

) $\sum^{\infty} x_n < \infty \Rightarrow$ Dado $\varepsilon > 0$ $\exists N \in \mathbb{N}$ $\sum_{n=N+1}^{\infty} x_n < \frac{\varepsilon}{\sqrt{2}}$

$$\sum_{n=N+1}^{\infty} x_n^2 \leq \left(\sum_{n=N+1}^{\infty} x_n\right)^2 \leq \frac{\varepsilon^2}{2}$$

$$\Rightarrow \sum_{n=N+1} |x_n|^2 \leq \frac{\varepsilon^2}{2}$$

para cada x_n con $n \leq N$ como $x_n \in \mathbb{R}$ $\xrightarrow{\text{densidad}}$ $\exists q_n \in \mathbb{Q} \ / \ d(q_n, x_n) = |q_n - x_n| \leq \frac{\varepsilon}{\sqrt{2N}}$

definimos $q = q_n \quad \forall n \leq N$

$q = 0 \quad \forall n > N$

$$\|q - x\| = \sqrt{\sum |q_n - x_n|^2}$$

$$= \sqrt{\sum_{n=0}^{N} |q_n - x_n|^2 + \sum_{N+1}^{\infty} |q_n - x_n|^2}$$

$$\leq \sqrt{\sum^{N} \frac{\varepsilon^2}{2N} + \sum_{N+1}^{\infty} |x_n|^2}$$

$$= \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon$$

$\rightarrow \quad q \in B(x, \varepsilon)$

y $q \in A \rightarrow \bar{A} = \ell^2(\mathbb{N})$

d)

$H = \{ \| \| = 1 \}$. Veamos H^c abierto

i) caso $\|b_n\| = k > 1$

$b_n \in H^c$. Afirmo $B(b_n, k-1) \subseteq H^c$

$c_n \in B(b_n, k-1)$ quq $\|c_n\| > 1$

$\|b_n\| = \|b_n + c_n - c_n\| \leq \|b_n - c_n\| + \|c_n\|$

$\Leftrightarrow k - (k-1) < \|b_n\| - \|b_n - c_n\| \leq \|c_n\|$

$\Leftrightarrow 1 < \|c_n\|$

$\Rightarrow c_n \in H^c \Rightarrow B(b_k, k-1) \subseteq H^c$

≥ 0

caso $\|b_n\| = k < 1$ tomo $B(b_k, 1-k)$

$\|c_n\| \leq \|c_n - b_n\| + \|b_n\| < 1 - k + k = 1$

$\Rightarrow c_n \in H^c$

$\Rightarrow B(b_k, 1-k) \subseteq H^c$

e) El conjunto A que di antes está contenido en este y es denso

$\Rightarrow$ este es denso

5) a) $X_k^n = \begin{cases} \frac{1}{\sqrt{k}} & \text{si } k \leq n \\ 0 & cc \end{cases}$ $\quad X^n \in l^2 \ \forall n$

$$\left(\sum^{\infty} (X_k^n)^2\right)^{\frac{1}{2}} = \left(\sum^{n} (X_k^n)^2\right)^{\frac{1}{2}} < \infty$$

$X^n \in l^\infty$

$X^1 = 1, 0 \dots$

$X^2 = 1, \frac{1}{\sqrt{2}}, 0 \dots$

$\vdots$

$X = 1, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{j}}, \dots$

$$\|X_k^n\|_2 = \left(\sum_{k=1}^{n} \left|\frac{1}{n}\right|\right)^{\frac{1}{2}} \longrightarrow \infty \Rightarrow \text{no converg}$$

pero $\|X\|_2 = \left(\sum_{k=1}^{\infty} (X_k)^2\right)^{\frac{1}{2}} = \left(\sum^{\infty} \frac{1}{k}\right)^{\frac{1}{2}} \to \infty$

$\Rightarrow X \notin l^2 \Rightarrow X^n$ no converge en l^2

pero $\|X\|_\infty = 1 \Rightarrow X \in l^\infty$

$\Rightarrow X^n \to X$ en l^∞

(pq sup X^n tiende a 0

i) $X_k^n = \begin{cases} \frac{1}{k} & \text{si } k \leq n \\ 0 & cc \end{cases}$ $\quad \|X^n\|_1$ suma finita

$\Rightarrow X^n \in l^1 \ \forall n \in \mathbb{N}$

Análogo $X^n \in l^2$

$$\|x\|_2 = \left(\sum^{\infty}\left(\frac{1}{k}\right)^2\right)^{\frac{1}{2}} < \infty \Rightarrow x \in \ell^2$$

y $x^n \xrightarrow{\|\|_2} x$ pq la cola $\sum_{n+1}^{\infty}\left(\frac{1}{k}\right)^2 < \infty$

pero $\|x\|_1 = \left(\sum^{\infty} \frac{1}{k}\right)$ diverge

$x \notin \ell^1 \Rightarrow x^n \not\to x$ en ℓ^1

c) $\ell^1(\mathbb{N})$ contiene al conjunto usado en 4)c)

pq son sucesiones con finitos elementos no nulos

$\Rightarrow$ avanza finito

y como era denso en $\ell^2(\mathbb{N}) \Rightarrow \ell^1(\mathbb{N})$ denso en ℓ^2

d) como $\ell^1 \subset \ell^2$ que es un métrico banach

$\Rightarrow \ell^1$ cerrado $\Leftrightarrow$ completo pero si ℓ^1 cerrado $\ell^1 = \underset{\text{dem.}}{\overline{\ell^1}} = \ell^2$

1) vemos $\ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$

Sea $x \,/\, \|x\|_1 = 1 \Rightarrow \sum |x_k| = 1$

$\Rightarrow |x_k| \leq 1$

$$\Rightarrow \|x\|_2 = \sum |x_k|^2 \leq \sum |x_k| = \|x\|_1$$

Sea $x \,/\, \|x\|_1 \neq 1$ tomo $\tilde{x} = \dfrac{x}{\|x\|_1}$

$$\Rightarrow \|\tilde{x}\|_2 \leq \|\tilde{x}\|_1$$

$$\|x\|_2 \leq \|x\|_1 \|\tilde{x}\|_1^{-1}$$

$$\|x\|_2 \leq \|x\|_1 \quad \forall x \in X$$

$$\Rightarrow \ell^1 \subset \ell^2$$

<u>Veamos que es propio</u>

$x_n = (1, \frac{1}{2}, \ldots \quad \frac{1}{n}, \ldots)$, $x_n \in \ell^2$ pq $\sum^{\infty} \left(\frac{1}{n}\right)^2 < \infty$

pero $\sum \left(\frac{1}{n}\right)$ diverge $\Rightarrow x_n \notin \ell^1$

d) $x \in \overline{\ell^1} \Rightarrow \exists \{x^n\}$ sucesiones de ℓ^1

que convergen a $x (= x_n)$ (para $x \in c_0$)

$\|x^n - x\|_\infty < \varepsilon$ Unismo (por convergencia)

$\Rightarrow \quad |X_k^n - X_k| < \varepsilon/2 \quad \forall n \geq n_0 \quad \forall k \in \mathbb{N}$

) pq supremo lo es

X^n es de $\ell^1 \Rightarrow X_k^n \xrightarrow[k\to\infty]{} 0$ (pq $\sum a_n < \infty$ $\Rightarrow a_n \to 0$)

$\Rightarrow |X_k^n| < \frac{\varepsilon}{2} \ \forall k \geq n_1$

$|X_k| \leq |X_k - X_k^n| + |X_k^n|$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq \max\{n_0, n_1\}$

$\Rightarrow |X_k| \to 0 \quad \Rightarrow X \in C_0$

$\Rightarrow \overline{\ell^1} \subset C_0$

.) Sea $|X| \in C_0 \Rightarrow$ tomamos

$X^1 = (X_1, 0, \ldots)$ $\quad X^n \in \ell^1$

$X^2 = (X_1, X_2, 0, \ldots)$

$X^n = (X_1 \ldots X_j, 0, \ldots)$

dado $\varepsilon > 0$ $\exists n_0 / |X_n| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow \|X^n - X\|_\infty = \sup\{|X_j| : j > n\} < \varepsilon \quad \forall n \geq n_0$

$X^n \longrightarrow X \quad \Rightarrow X \in \overline{\ell^1}$

Зел $\{X_n\} \in C_0$

d) Sea $(\underline{X}^n) \in C_0$ de cauchy con ℓ^∞

$$\|X^n - X^m\|_\infty < \varepsilon \quad \forall n,m \geq n_0$$

$$\underline{|X^n_k - X^m_k|} < \varepsilon \quad \forall k \quad \forall n,m \geq n_0$$

$\Rightarrow$ $(X^n_k)_{n\in\mathbb{N}}$ es de cauchy

$$X^n_k \xrightarrow[n\to\infty]{} X_k$$

Veamos $X^n \longrightarrow X$

qvq $\|X^n - X\| < \varepsilon \Rightarrow \sup_{k\in\mathbb{N}} \left(|X^n_k - X_k|\right) < \varepsilon \quad \forall n \geq n_0$

Como $|X^n_k - X^m_k| < \varepsilon \quad \forall k \in \mathbb{N} \ \forall n,m \geq n_0$

$$\Rightarrow \lim_{m\to\infty} |X^n_k - X^m_k| < \varepsilon$$

$$\Rightarrow \quad |X^n_k - \lim_{m\to\infty} X^m_k| < \varepsilon$$

$$|X^n_k - X_k| < \varepsilon \quad \forall k \in \mathbb{N}$$

$\therefore \quad \|X^n - X\|_\infty < \varepsilon$

ii) nota que X está en C_0 pq

$\to X^n$ está en C_0

$\to X^n_k \to 0$

$$|X_k| \leq |X_k - X^n_k| + |X^n_k|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq n_0 \quad \forall k \geq k_0$$

$$\Rightarrow |X_k| < \varepsilon \quad \forall k \geq k_0$$

$$|X_k| \to 0$$

$\to X^n$ converge $\to C_0$ es completo

para la parte del final

Si uno usando que $C_0 \subseteq \ell^\infty$

como ℓ^∞ es completo

C_0 completo $\Leftrightarrow$ C_0 cerrado

Como $C_0 = \overline{\ell^1}$ es cerrado

e) $\ell^1 \subsetneq \ell^2$ c)

·) $\ell^2 \subseteq c_0$ por d)

$\neq$ $\quad x_n = \frac{1}{\sqrt{n}} \Rightarrow \lim |x_n| = 0 \quad \{x_n\} \in c_0$

$$\|x_n\| = \Big(\underbrace{\sum \frac{1}{n}}_{\to \infty}\Big)^{\frac{1}{2}} \Rightarrow \{x_n\} \notin \ell^2$$

·) $c_0 \subseteq c$

·) $c \subseteq \ell^\infty \qquad \{x_n\} \in c \Rightarrow \qquad x_n \to x$

$$\Rightarrow \quad |x_n| < \varepsilon \qquad \forall n \geq n_0$$

$$\Rightarrow \|x_n\|_\infty = \sup \{|x_n| : n \in \mathbb{N}\} = \max\{\varepsilon, \underbrace{\sup\{|x_n| : n < n_0\}}_{\text{son finitos } n}\}$$

$$\Rightarrow \quad x_n \in \ell^\infty$$

$x_n = \begin{cases} 1 & n \text{ par} \\ -1 & n \text{ impar} \end{cases}$ está en ℓ^∞ $\|x\| = 1$

pero $\nexists \lim x_n \Rightarrow x_n \notin c$

f) $\underset{\in C}{X^n} \longrightarrow X$ donde $X^n_k \xrightarrow[k\to\infty]{} L^n$ (por $\{X^n_k\} \in C$) $\forall n \in \mathbb{N}$

qvq $X \in C$ o sea $X_k \to L$

·) qvq L^n converge viendo que es cauchy

$\|X^n - X^m\|_\infty < \varepsilon$ (X^n es de cauchy)

$|X^n_k - X^m_k| < \varepsilon$ $\forall n,m \geq n_0$

$\Rightarrow \left| \lim_{k\to\infty} X^n_k - X^m_k \right| < \varepsilon$

$\Rightarrow |L^n - L^m| < \varepsilon$ $\forall n,m \geq n_0$

$\Rightarrow L^n$ es cauchy en completo

$\Rightarrow L^n$ converge a L

Afirmo $X_k \longrightarrow L$

$|X_k - L| = \left| \lim_{n\to\infty} X^n_k - L^n \right| = \lim_{n\to\infty} |X^n_k - L^n|$

$\leq \lim_{n\to\infty} \varepsilon$ $\forall k \geq n_0$

g) Sea $x / \|x\|_p = 1$

$$\Rightarrow \left(\sum |x_i|^p\right)^{\frac{1}{p}} = 1 \Rightarrow \sum |x_i|^p = 1$$

$$|x_i|^p \leq 1 \Rightarrow |x_i| \leq 1$$

$q > p \quad \Rightarrow |x_i|^q \leq |x_i|^p$

$$\Rightarrow \quad \sum |x_i|^q \leq \sum |x_i|^p$$

$q > p$ $\Rightarrow$ $\left(\sum |x_i|^q\right)^{\frac{1}{q}} \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}}$

$\left(\frac{1}{p} > \frac{1}{q}\right)$

$$\therefore \|x\|_q \leq \|x\|_p$$

Sea $x \in X \Rightarrow \left\| \frac{x}{\|x\|_p} \right\|_p = 1$

$$\Rightarrow \left\| \frac{x}{\|x\|_p} \right\|_q \leq \left\| \frac{x}{\|x\|_p} \right\|_p = 1$$

$$\Rightarrow \|x\|_q \leq \|x\|_p$$

6) a) $\|X\| = 0 \Rightarrow \langle X, X\rangle^{\frac{1}{2}} = 0$

$\langle X, X\rangle = 0 \Rightarrow X = 0$

$\langle X, \lambda X\rangle = \overline{\langle \lambda X, X\rangle}$

$\cdot)\ \|\lambda X\|^2 = \langle \lambda X, \lambda X\rangle = \lambda\bar{\lambda}\langle X, X\rangle$

$= |\lambda|^2 \|X\|$

$\Rightarrow \|\lambda X\| = |\lambda|\|X\|$

$\cdot)\ \|X+y\|^2 = \langle X+y, X+y\rangle$

$= \langle X, X+y\rangle + \langle y, X+y\rangle$

$= \overline{\langle X+y, X\rangle} + \overline{\langle X+y, y\rangle}$

$= \overline{\langle X, X\rangle} + \overline{\langle y, X\rangle} + \overline{\langle X, y\rangle} + \overline{\langle y, y\rangle}$

$= \|X\|^2 + \|y\|^2 + \langle X, y\rangle + \overline{\langle X, y\rangle}$

$= \|X\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle X, y\rangle)$ $\quad |X| = \sqrt{\mathrm{Re}X^2 + \mathrm{Im}(X)^2}$

$\leq \|X\|^2 + \|y\|^2 + 2|\langle X, y\rangle|$ $\quad \geq \sqrt{\mathrm{Re}X^2} = |\mathrm{Re}X|$

$$(C-S) \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

b) $$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x-y \rangle - \langle y, x-y \rangle$$

$$= \overline{\langle x-y, x \rangle} - \overline{\langle x-y, y \rangle}$$

$$= \overline{\langle x, x \rangle} - \overline{\langle y, x \rangle} - \langle \overline{x, y} \rangle + \langle \overline{y, y} \rangle$$

$$= \|x\|^2 - \langle x, y \rangle - \overline{\langle x, y \rangle} + \|y\|^2$$

$$= \|x\|^2 - (\langle x, y \rangle + \overline{\langle x, y \rangle}) + \|y\|^2$$

$$= \|x\|^2 - 2\,\mathrm{Re}\,(\langle x, y \rangle) + \|y\|^2$$

$$\Rightarrow \|x-y\|^2 + \|x+y\|^2$$

$$= 2\|x\|^2 + 2\|y\|^2$$

..) $$\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$$

$$- \quad \|x-y\|^2 = \|x\|^2 - 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$$

$$= 4\,\mathrm{Re}\langle x, y \rangle$$

$$\frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|\right\} = \operatorname{Re}\langle x,y\rangle$$

Si $K = \mathbb{R}$ $\operatorname{Re}\langle x,y\rangle = \langle x,y\rangle$

$\therefore)$ $\|x+iy\|^2 = \langle x+iy, x+iy\rangle$

$$= \langle x, x+iy\rangle + \langle iy, x+iy\rangle$$

$$= \|x\|^2 + \langle x, iy\rangle + \|iy\|^2 + \langle iy, x\rangle$$

$$= \|x\|^2 + \|iy\|^2 + \langle x, iy\rangle + \overline{\langle x, iy\rangle}$$

$$= \|x\|^2 + \|y\|^2 + \bar{i}\langle x,y\rangle + \bar{\bar{i}}\,\overline{\langle x,y\rangle}$$

$$= \|x\|^2 + \|y\|^2 - i\langle x,y\rangle + i\overline{\langle x,y\rangle}$$

$$= \|x\|^2 + \|y\|^2 - i\left[\langle x,y\rangle - \overline{\langle x,y\rangle}\right]$$

$$= \|x\|^2 + \|y\|^2 - i\,2\operatorname{Im}(\langle x,y\rangle)$$

$$\Rightarrow i\|x+iy\|^2 = i\|x\|^2 + i\|y\|^2 + 2\operatorname{Im}\langle x,y\rangle$$

com $-iy$

$$= \|x\|^2 + \langle x, -iy\rangle + \|iy\|^2 - \langle iy, x\rangle$$

$$-\|x\|^2 - \|y\|^2 = \overline{\langle x, iy\rangle} - i\langle x, y\rangle$$

$$= \|x\|^2 + \|y\|^2 - i\overline{\langle x, y\rangle} + i\langle x, y\rangle$$

$2+i \quad -(2-i)$

$$= \|x\|^2 + \|y\|^2 + i\left[\langle x, y\rangle - \overline{\langle x, y\rangle}\right]$$

$$= \|x\|^2 + \|y\|^2 + i\,2\,\mathrm{Im}(\langle x, y\rangle)$$

$$\Rightarrow i\|x - iy\|^2 = i\|x\|^2 + i\|y\|^2 - 2\,\mathrm{Im}\langle x, y\rangle$$

$$\Rightarrow i\|x + iy\|^2 - i\|x - iy\|^2 = 4\,\mathrm{Im}\langle x, y\rangle$$

$$\Rightarrow \|x+y\|^2 - \|x-y\|^2 + i\|x+y\|^2 - i\|x-y\|^2$$

$$= \quad 4\,\mathrm{Re}\langle x, y\rangle + 4\,\mathrm{Im}\langle x, y\rangle$$

dividido $\frac{1}{4}$ $= \mathrm{Re}\langle x, y\rangle + \mathrm{Im}\langle x, y\rangle$

$= \langle x, y \rangle$ si $\mathbb{F} = \mathbb{C}$

c) $2\|(1,0)\|_\infty + 2\|(1,-1)\|_\infty = 4 \neq 3 = \|(0,1)\|_\infty + \|(2,-1)\|_\infty$

$2\|(1,0)\|_1 + 2\|(1,-1)\|_1 = 6 \neq 4 = \|(0,-1)\|_1 + \|(2,-1)\|_1$

no cumplen paralelogramo

d) Veamos: $\|\cdot\|_2$ es norma para ℓ^1

(podría haber usado $\ell^1 \subseteq \ell^2$

$\Rightarrow \|x\|_2$ bien det $\forall \{x\} \in \ell^1$

$\{x\} \subset \ell^1 \qquad \sum_{k=1}^{\infty} |x_k^n| < \infty$

$\Rightarrow \varepsilon = \frac{1}{2} \qquad \sum_{k=M}^{\infty} |x_k^n| < \frac{1}{2}$

$\Rightarrow \quad |x_k^n| < \frac{1}{2} \quad \forall k \geq M$

$\Rightarrow \quad |x_k^n| > |x_k^n|^2$

$\sum_{k=M}^{\infty} |x_k^n|^2 < \sum_{k=M}^{\infty} |x_k^n| < \frac{1}{2}$

$\Rightarrow \left(\sum_{k=1}^{\infty} |x_k^n|^2 \right)^{\frac{1}{2}} < \infty$

$\Rightarrow$ Norma $\|\cdot\|_2$ bien det p/ $\{x\} \subset \ell^1$

$\Rightarrow (\ell^1, \|\cdot\|_2)$ y vimos en 8)b)

que no es completo

Veamos cumple paralelogramo

$$\|X-Y\|_2 + \|X+Y\|_2 = \sum_{k=1}^{\infty} (X_k - y_k)^2 + \sum_{k=1}^{\infty} (X_k + y_k)^2$$

$$= \lim_{N\to\infty} \sum (X_k - y_k)^2 + \lim_{N\to\infty} \sum (X_k + y_k)^2$$

ambos existen $\Rightarrow \lim_{N\to\infty} \sum^{N} (X_k - y_k)^2 + (X_k + y_k)^2$

$$= \lim_{N\to\infty} \sum^{N} X_k^2 - \cancel{2 y_k X_k} + y_k^2 + X_k^2 + \cancel{2 X_k y_k} + y_k^2$$

ambos existen pq son $\|\cdot\|_2$ probamos arriba bien def

$$\lim \sum^{N} 2X_k^2 + 2y_k^2 = 2\lim \sum^{N} X_k^2 + 2 \lim \sum^{N} y_k^2$$

$$= 2\|X\|^2 + 2\|Y\|^2$$

$\Rightarrow$ cumple la regla del paralelogramo

$\Rightarrow$ es pre hilbert

(podemos recuperar el pi $(X,X) = \|X\|^2$)

$\circ$) $(\ell^1, \| \|_\infty)$ está bien def $\| \cdot \|_\infty$

para toda $\{X\} \subset \ell^1$ pq si está

en ℓ^1 $\sum |X_n| < \infty$ $\Rightarrow$ $X_n \to 0$

$\Rightarrow \sum X_n < \infty$

$\Rightarrow \exists M \in \mathbb{R} / |X_n| < M$

Notar si $X = (1, 0, \ldots)$ $y = (0, 1, 0, \ldots)$

$\|X - y\|_\infty^2 + \|X + y\|_\infty^2$

$= \sup\{X_k - y_k : k \in \mathbb{N}\} + \sup\{X_k + y_k : k \in \mathbb{N}\}$

$= 1 + 1 \neq 2 \cdot 1 + 2 \cdot 1$

$= 2 \sup\{X_n\} + 2 \sup\{y_k\}$

No cumple paralelogramo

$\Rightarrow$ No es prehilbert

$\circ$) $(C, \| \|_\infty)$ $\| \cdot \|_\infty$ bien def

$\{X\} \subset C \Rightarrow X_n \to X$

pero el contraejemplo de antes

también cvtz en c $\quad x_k = \frac{1}{k^2}$

$\Rightarrow x_k \to 0 \Rightarrow \{x_k\} \subset C$

Análogo $(c_0, \|\|_\infty)$

7) a) $\langle f, g \rangle = \int f \bar{g}$

$\cdot$) $\langle f, g \rangle = \int f \bar{g} = \int \bar{g} f$

$= \int \bar{g} \bar{\bar{f}}$

$= \int \overline{g \bar{f}} = \overline{\int g \bar{f}}$

$= \overline{\langle g, f \rangle}$

$\cdot$) $\langle af + bg, h \rangle = \int (af + bg) \bar{h}$

$= a \int f \bar{h} + b \int g \bar{h}$

$$= a\langle f,h\rangle + b\langle g,h\rangle$$

i) $\langle f,f\rangle = \int f\bar{f} = \int |f|^2 \geq 0$

($\Leftarrow$)

b) $\|f+g\|^2 = \langle f, f+g\rangle + \langle g, f+g\rangle$

$$= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$

$$= \|f\|^2 + \langle f,g\rangle + \overline{\langle f,g\rangle} + \|g\|^2$$

$$= \|f\|^2 + 2\mathrm{Re}\langle f,g\rangle + \|g\|^2$$

$$= \|f\|^2 + 2\mathrm{Re}\langle f,\alpha f\rangle + \|g\|^2$$

$\alpha \geq 0$

asumo $\alpha \in \mathbb{R}$

$$= \|f\|^2 + 2\alpha\, \mathrm{Re}\langle f,f\rangle + \|g\|^2$$

$$= \|f\|^2 + 2\alpha \|f\|^2 + \|g\|^2$$

$$= \|f\|^2 + 2\alpha \|f\|\|f\| + \|g\|^2$$

$$= \|f\|^2 + 2\|\alpha f\|\|f\| + \|g\|^2$$

$$= \|f\|^2 + 2\|g\|\|f\| + \|g\|^2$$

$$= (\|f\| + \|g\|)^2 \quad \therefore \|f+g\| = \|f\| + \|g\|$$

$(\Leftarrow)$ $\|f+g\|^2 = (f+g, f+g)$

$$= (f,f) + 2\,\mathrm{re}(f,g) + (g,g)$$

$$(\|f\|+\|g\|)^2 = (f,f) + (g,g) + 2\|f\|\|g\|$$

$$\Rightarrow \quad \cancel{2}\,\mathrm{re}(f,g) = \cancel{2}\|f\|\|g\|$$

$$\mathrm{re}(f,g) = \|f\|\|g\|$$

Además por C-S

$$\mathrm{Re}(f,g) \leq |(f,g)| \leq \|f\|\|g\|$$

$$\Rightarrow |(f,g)| = \|f\|\|g\|$$

$\Rightarrow$ son todas igualdades $\Rightarrow$ ($|(f,g)| = \|f\|\|g\|$)

en la desigualdad de C-S la igualdad* vale $\Leftrightarrow$ $\alpha f = g$

c) $\| |f|^p \|_1 \leq \| 1 \|_{q/(q-p)} \| |f|^p \|_{\frac{q}{p}}$

$$\boxed{\frac{p-q}{p} + \frac{q}{p} = 1}$$

$$= \left(\int_X 1^{\frac{p}{q-p}} \right)^{\frac{q-p}{q}} \left(\int_X (|f|^p)^{\frac{q}{p}} \right)^{\frac{p}{q}}$$

$\mu(X) < \infty$ $= \mu(X)^{\frac{q-p}{q}} \left(\int |f|^q \right)^{\frac{p}{q}}$

$$= \mu(X)^{\frac{q-p}{q}} \|f\|_q^p$$

$$\Rightarrow \int |f|^p \leq \mu(X)^{\frac{q-p}{q}} \|f\|_q^p$$

$$\|f\|_p \leq \mu(X)^{\frac{q-p}{pq}} \|f\|_q$$

$$\frac{q-p}{pq} = \frac{1}{p} - \frac{1}{q}$$

d) Sea $f \in L^q \Rightarrow \|f\|_q < \infty$

$\mu(X)^{\frac{1}{p} - \frac{1}{q}}$ es cte

pq $\mu(X) < \infty$

$\Rightarrow \|f\|_p < \infty \Rightarrow f \in L^p$

d) $\|f\|_1 \overset{\text{holder}}{\leq} \|1\|_{p'} \|f\|_p \qquad \frac{1}{p'} + \frac{1}{q} = 1$

$$= \left(\int 1^{p'}\right)^{\frac{1}{p'}} \|f\|_p$$

$$= \mu(X)^{\frac{1}{p'}} \|f\|_p$$

$$\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}} \leq \left(\int |\sup es\, f|^p\right)^{\frac{1}{p}} = |\sup es\, f|\, \mu(X)^{\frac{1}{p}}$$

$$= \|f\|_\infty \mu(X)^{\frac{1}{p}}$$

$$\Rightarrow \quad \mu(X)^{\frac{1}{p'}} \|f\|_p \leq \|f\|_\infty \mu(X)^{\frac{1}{p}+\frac{1}{p'}} = \|f\|_\infty \mu(X)$$

Ademas de $\quad \|f\|_p \leq \|f\|_\infty \mu(X)^{\frac{1}{p}}$

tenemos $\quad \lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty$

de c) $\quad \|f\|_p \leq \mu(X)^{\frac{1}{p}-\frac{1}{p'}} \|f\|_{p'}$

$$\mu(X)^{\frac{1}{p'}} \|f\|_p \leq \mu(X)^{\frac{1}{p}} \|f\|_{p'}$$

$$\mu(X)\|f\|_\infty = \lim_{\substack{p'\to 1 \\ p\to\infty}} \mu(X)^{\frac{1}{p'}} \|f\|_p \leq \lim_{\substack{p'\to 1 \\ p\to\infty}} \mu(X)^{\frac{1}{p}} \|f\|_p = \lim_{p\to\infty} \|f\|_p$$

e) $f \in L^p(X)$ $\qquad f = f\chi_{A_1} + f\chi_{A_2}$

$A_1 = \{x : f(x) \geq 1\} \quad A_2 = \{x : f(x) < 1\}$

$$r \leq p \Rightarrow \int |f\chi_{A_1}|^r < \int |f\chi_{A_1}|^p = \int_{A_1} |f|^p \leq \int_X |f|^p < \infty$$

$$\Rightarrow \left[\int |f\chi_{A_1}|^r\right]^{\frac{1}{r}} < \infty \Rightarrow f\chi_{A_1} \in L^r$$

$$p \leq s \Rightarrow \int |f\chi_{A_2}|^s \leq \int |f\chi_{A_2}|^p \leq \infty$$

$$\Rightarrow f\chi_{A_2} \in L^s$$

$$\Rightarrow f = f\chi_{A_2} + f\chi_{A_1} \in L^s + L^r$$

f) $\Leftarrow)$ $2\|x\|_2^2 + 2\|y\|_2^2 = 2\left(\int |f|^2\right)^{\frac{1}{2}} + \left(\int |g|^2\right)^{\frac{1}{2}}$

$\langle x, x \rangle = \|x\|_2^2 \qquad \langle x, x \rangle = \int |f|^2$

(a) $f(x) = \chi_{[a,b]}$ $\qquad \|2f\|_p = \int_a^b 2 = 2(b-a)^{\frac{1}{p}}$

$$\|f\|_p = \left(\int_a^b 1^p\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}}$$

$$\Rightarrow \left(2\cancel{(b-a)}^{\frac{1}{p}}\right)^2 + \left(2\cancel{(b-a)}^{\frac{1}{p}}\right)^2$$

$$(\text{debería}) = 0 + \|2f\|^2$$

$$= \left(2(b-a)\right)^{\frac{1}{p}}$$

$$= \left(2^{\frac{1}{p}}\cancel{(b-a)}^{\frac{1}{p}}\right)^2$$

$$\to 8 = 2^{\frac{2}{p}}$$

8) a) $f_\alpha(x) = \dfrac{1}{x^\alpha}$ $\quad X = (0,1)$ $\quad \alpha$

$\displaystyle\int_X \frac{1}{|x|^{\alpha p}}$ es impropia

visto en reales $x^{-\alpha+1}$

$$\Rightarrow \int_X \frac{1}{|x|^{\alpha p}} = \lim_{\varepsilon\to 0} \underbrace{\int_\varepsilon^1 \frac{1}{|x|^{\alpha p}}}_{\text{Riemann}} = \lim_{\varepsilon\to 0} \begin{cases} \left.\dfrac{|x|^{1-\alpha p}}{1-\alpha p}\right|_\varepsilon^1 & \alpha p \neq 1 \; \left(\alpha \neq \frac{1}{p}\right) \\ \left.\ln|x|\right|_\varepsilon^1 & \alpha p = 1 \; \left(\alpha = \frac{1}{p}\right) \end{cases}$$

$$= \begin{cases} \displaystyle\lim_{\varepsilon\to 0} \frac{1}{1-\alpha p} - \frac{\varepsilon^{1-\alpha p}}{1-\alpha p} & \alpha p \neq 1 = \begin{cases} -\infty & 1 < \alpha p \\ \dfrac{1}{1-\alpha p} & 1 > \alpha p \end{cases} \\ \displaystyle\lim_{\varepsilon\to 0} -\ln(\varepsilon) & \alpha p = 1 \end{cases}$$

$$= \begin{cases} -\infty & 1 < \alpha p \\ +\infty & 1 = \alpha p \\ \dfrac{1}{1-\alpha p} & 1 > \alpha p \end{cases} \Rightarrow \int \frac{1}{|x|^{\alpha p}} \text{ converge sii } 1 > \alpha p$$

$\therefore f \in L^p(X)$ sii $\alpha < \frac{1}{p}$

b) $f_\alpha(x) = \frac{1}{x^\alpha} \qquad x = [1, \infty)$

$$\int |f|^p = \lim_{\varepsilon \to \infty} \int_1^\varepsilon \left|\frac{1}{x^{\alpha p}}\right| = \lim_{\varepsilon \to \infty} \begin{cases} \left.\frac{|x|^{1-\alpha p}}{1-\alpha p}\right|_1^\varepsilon & \alpha p \neq 1 \quad (\alpha \neq \frac{1}{p}) \\ \left.\ln|x|\right|_1^\varepsilon & \alpha p = 1 \quad (\alpha = \frac{1}{p}) \end{cases}$$

$$\begin{cases} \lim_{\varepsilon \to 0} \frac{|\varepsilon|^{1-\alpha p}}{1-\alpha p} - \frac{1}{1-\alpha p} & \alpha p \neq 1 \\ \lim_{\varepsilon \to 0} \ln|\varepsilon| & \alpha p = 1 \end{cases}$$

$$\begin{cases} \lim_{\varepsilon \to \infty} \frac{|\varepsilon|^{1-\alpha p}}{1-\alpha p} = +\infty & \alpha p < 1 \\ \lim_{\varepsilon \to \infty} \frac{1}{|\varepsilon|^{1-\alpha p}(1-\alpha p)} = 0 & \alpha p > 1 \\ -\infty & \alpha p = 1 \end{cases}$$

$$\Rightarrow \quad \int_X |f|^p < \infty \quad \text{ssi} \quad \alpha > \frac{1}{p}$$

$$f \in L^p(X) \quad \text{ssi} \quad \alpha > \frac{1}{p}$$

a) $f_n(x) = n\chi_{[n,n+1]}(x)$

$f_n(x) \longrightarrow 0$ puntualmente

Sea $\varepsilon > 0$ y $x \in \mathbb{R}$ / $\exists n_0$ / $|f_n(x)| < \varepsilon \ \forall n \geq n_0$

pero $\int_{\mathbb{R}} |f_n| = n \int_{[n,n+1]} 1 = n$

$\Rightarrow \lim_{n\to\infty} \|f_n\|_1 = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n| = +\infty$

10) Por uniforme $|f_n(x) - f(x)| < \varepsilon \ \forall n \geq N \quad \forall x \in [0,1]$

y f_n continua. Sea $\varepsilon > 0$

$$\Rightarrow |f(x) - f(y)|$$

$$= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$\forall n \geq N_0$	$\forall y \in B(x, \delta)$	$\forall n \geq N_1$
$\forall x$	por continuidad	$\forall x$
por conv unif	de f_n y vale	conv unif
	para cualquier n	
	$\Rightarrow$ vale para $n \geq \max\{N_1, N_0\}$	

$$\Rightarrow y \in B(x, \delta) \ (|x - y| < \delta) \Rightarrow |f(x) - f(y)| < \varepsilon$$

f continua

i) dado $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ / $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq n_0$ $\forall x \in [0,1]$

(Uniforme)

$$\|f_n - f\|_1 = \int_0^1 |f_n - f| \leq \int_0^1 \varepsilon = \varepsilon \quad \forall n \geq n_0$$

$$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall n \geq n_0$$

$$\|f_n - f\| \longrightarrow 0$$

$$\therefore f_n \xrightarrow{\|\cdot\|_1} f$$

11) a) todo X e.v. tiene base $\{e_1 \ldots e_n \ldots\}$

Si $x \in X$ $\Rightarrow$ $x = \sum_{i=1}^{N} \lambda_i e_i$ (ej 2)

defino $\|x\| := \|\sum_{i=1}^{N} \lambda_i e_i\| = \max_{1 \leq i \leq N} \{|\lambda_i|\}$

i) $\|\lambda x\| = \|\lambda \sum \lambda_i e_i\| = \|\sum \lambda \lambda_i e_i\|$

$= \|\sum \tilde{\lambda}_i e_i\|$

$= \max\{|\tilde{\lambda}_i|\}$

$= |\tilde{\lambda}_N| = |\lambda \lambda_N|$

$= |\lambda| |\lambda_N|$

$= |\lambda| \|x\|$

$x = \sum \lambda_i e_i$ $y = \sum \beta_i e_i$

ii) $\|x+y\| = \max\{\lambda_i + \beta_i\}$

$\leq \max\{\lambda_i\} + \max\{\beta_i\} = \|x\| + \|y\|$

·) $\|x\| = 0 \Leftrightarrow 0$ trivial

$\Rightarrow$ es norm

b) $$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

$$\Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

$$\|y\| \leq \|y - x\| + \|x\|$$

$$-\|y - x\| \leq \|x\| - \|y\|$$

$$\Rightarrow \quad \big| \|x\| - \|y\| \big| \leq \|x - y\|$$