Teo descomposición cíclica

$T: V \to V$ $\dim V < \infty$ $V = \bigoplus_{i=1}^{k} Z(v_i, T)$

con anuladores $p_1, \dots, p_k$ tales que

$$\left(\overset{\overset{m_T|_{Z(v_i,T)}}{\shortparallel}}{p_i} = m_{v_i,T} = \alpha_1 v_i + \dots + \alpha_{d_1} x^{d_1} \right) \quad d_i = \operatorname{gr}(p_i)$$

$B = \{v_1, T(v_1), \dots T^{d_1-1}(v_1)\} \cup \dots \cup \{v_k, T(v_k), \dots T^{d_k-1}(v_k)\}$

$$[T]_B = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

A_i es la matriz compañera de p_i

($Z(v_i, T)$ son T-inv)

·) A esta matriz se le dice forma racional

Recíprocamente $(T)_B = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$ con A_i matriz compañera de p_i con $p_i | p_{i-1}$

$\Rightarrow v_i, T(v_i), \dots, T^{d_i - 1}(v_i)$ generan $Z(v_i, T)$

Como k y p_i's son únicos es la única forma racional posible. $A \sim B \Leftrightarrow B = PAP^{-1}$

<u>Corolario</u> toda matriz es equivalente a una matriz en forma racional única

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<u>Teorema de Cayley-Hamilton general</u>

$T: V \to V$ T.l. $\dim V < \infty$

(i) $m_T | p_T$ más aún m_T y p_T tienen los mismos factores primos

(ii) Si $m_T = p_1^{v_1} \dots p_k^{v_k}$ (factorización prima)

$d_i = \dim \ker p_i(T)^{r_i} = \dim E_{\lambda_i} := $ autoespacio generalizado

$\Rightarrow p_T = p_1^{d_1} \dots p_k^{d_k}$ donde $d_i = \dfrac{\dim \ker p_i(T)^{r_i}}{gr(p_i)}$

Si estamos en $\mathbb{C}$ $gr(p_i) = 1 \Rightarrow$ es diag

demo (i) Si tomamos $V = \bigoplus_{i=1}^{k} Z(v_i, T)$ con f_i el anulador de v_i $f_i | f_{i-1}$ sabemos que $f_1 = m_T$ y $p_T = f_1 \dots f_k$

$\Rightarrow m_T | p_T$

Si p primo y $p | m_T \Rightarrow p | p_T$

Si $p | p_T = f_1 \dots f_k \Rightarrow \exists i / p | f_i$ y como $f_i | f_1$ tenemos que $p | f_1 = m_T$

∘) Luego p es factor de $m_T \Leftrightarrow$ es factor de p_T

(ii) $m_T = p_1^{r_1} \dots p_k^{r_k} \overset{(i)}{\Rightarrow} p_T = p_1^{d_1} \dots p_k^{d_k}$.

Calculemos d_i con Teo descomp prima

$$V = \bigoplus_{i=1}^{k} V_i \quad \text{con} \quad V_i = \ker p_i(T)^{r_i}$$

Más aún $T_i|_{V_i} : V_i \to V_i$ $m_{T_i} = p_i^{r_i}$

por (i) $\quad p_{T_i} = p_i^{a_i}$ para $a_i \geq r_i$

Como $\quad p_T = p_{T_1} \cdots p_{T_k} = p_1^{a_1} \cdots p_k^{a_k}$

$\Rightarrow a_i = d_i$

$$\underbrace{\dim\left(\operatorname{Ker}\left(p_i(T)^{r_i}\right)\right)}_{\dim'' V_i} = \operatorname{gr}(p_{T_i}) = \operatorname{gr}(p_i^{d_i}) = d_i \operatorname{gr}(p_i)$$

<u>ejemplo</u> Sea $T: V \to V$ diagonalizable

$\rightsquigarrow p_T = (X-\lambda_1)^{d_1} \cdots (X-\lambda_k)^{d_k} \quad d_i = \dim(V_{\lambda_i})$

A menos de reordenar, $d_1 \geq d_2 \geq \cdots \geq d_k$ (los escribo ordenados) $V = \bigoplus_{i=1}^{k} V_\lambda$

$m_T = (X-\lambda_1) \cdots (X-\lambda_k)$. Si queremos hallar la forma racional $\rightsquigarrow f_1 = m_T$ y $f_i | f_1$

$\Rightarrow f_i = \prod_{i \in S_i} (X-\lambda_i) \qquad S_i$ sub de $\{1, \dots, k\}$

Además $\quad S_i \subseteq S_{i-1}$

Ahora $f_1 = (x-\lambda_1) \ldots (x-\lambda_k) = m_T$

$$f_2 = \prod_{i, d_i \geq 2} (x-\lambda_i)$$

$$\vdots$$

$$f_s = \prod_{i, d_i \geq s} (x-\lambda_i)$$

$\Rightarrow \ \chi_T = f_1 \ldots f_k \qquad k = d_1$

Falta hallar $v \in V \,/\, m_{v,T} = f_i \quad \forall i$

$v = v_1 + \ldots + v_k$, $v_i \in V_{\lambda_i}$ de modo único

$$f(T)(v) = f(T)(v_1) + \ldots + f(T)(v_k)$$
$$= f(\lambda_1) v_1 + \ldots + f(\lambda_k) v_k = 0$$

$\Leftrightarrow f(\lambda_i) v_i = 0 \ \forall i$ (v_i son li)

$\Leftrightarrow f(\lambda_i) = 0 \quad \forall i$ tal que $v_i \neq 0$

$\therefore \ m_{v,T} = \prod_{i : v_i \neq 0} (x-\lambda_i) \qquad Z(v,T) = \langle v_1, \ldots, v_k \rangle = \langle v_i \neq 0 \rangle$

$B_{\lambda_1} = \{ v_1^{(1)}, \ldots, v_{d_1}^{(1)} \}$ base de V_{λ_1}

$B_{\lambda_k} = \{ v_1^{(n)}, \ldots, v_{d_n}^{(n)} \}$ base de V_{λ_k}

$$\Rightarrow \quad v_1 = \sum_{i=1}^{k} v_1^{(i)} \Rightarrow m_{v_1,T} = f_1$$

$$v_2 = \sum_{i: d_i \geq 2} v_2^{(i)} \Rightarrow m_{v_2,T} = f_2$$

$$\vdots$$

$$v_k = \sum_{i: d_i \geq k} \Rightarrow m_{v_k,T} = f_k$$

<u>ejemplo</u> $T: \mathbb{R}^3 \to \mathbb{R}^3$ con $p_T = (x-2)^2(x+1)$

$U_2 = \langle (1,0,1)(-1,3,4) \rangle \quad U_1 = \langle (0,1,1) \rangle$

$\Rightarrow T$ diag $\quad m_T = (x-2)(x-1) = f_1$

$1 \neq f_2 \mid f_1$ y $\underbrace{f_1 . f_2}_{gr \geq 3} \mid \underbrace{p_T}_{gr\ 3}$ $\quad f_1 = (x-2)(x+1)$

$f_2 = (x-2)$

$$V = Z(v_1, T) \oplus Z(v_2, T)$$

$B = \left. \begin{array}{l} v_1 = (1,0,1) + (0,1,1) \\ v_2 = (1,3,4) \end{array} \right\}$ usando ejemplo de arriba

$$[T]_B = \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 1 & \\ \hline 0 & & 2 \end{array}\right) \text{ forma racional}$$

ejemplo 2 $T: \mathbb{R}^5 \to \mathbb{R}^5$ $p_T = (x-1)^3(x-4)^2$

$V_1 = \langle e_1, e_2, e_3 \rangle$ $V_4 = \langle e_4, e_5 \rangle$

$f_1 = (x-1)(x-4) = x^2 - 5x + 4$ $v_1 = e_1 + e_4$

$f_2 = (x-1)(x-4)$ $v_2 = e_2 + e_5$

$f_3 = (x-1)$ $v_3 = e_3$

$\underbrace{\qquad\qquad}_{B}$

$$\Rightarrow [T]_B = \begin{pmatrix} \boxed{\begin{matrix} 0 & -4 \\ 1 & 5 \end{matrix}} & & \\ & \boxed{\begin{matrix} 0 & -4 \\ 1 & 5 \end{matrix}} & \\ & & \boxed{1} \end{pmatrix}$$

ejemplo 3 $T: \mathbb{R}^5 \to \mathbb{R}^5$ $p_T = (x-2)^3(x-1)^2$

$\Rightarrow m_T \in \{ (x-1)^a (x-2)^b : a \in \{1,2\} \; b \in \{1,2,3\} \}$

Cayley general

caso I $m_T = (x-1)(x-2)^2 \Rightarrow f_1 = m_T$ $f_1 f_2 \mid p_T$

$\Rightarrow f_2 \mid (x-1)(x-2)$

ejercicio $f_2 = (x-1)(x-2)$

Caso II $\quad m_T = (x-1)^2(x-2) = (x^2-2x+1)(x-2)$

$f_1 = m_T \qquad = x^3 - 2x^2 + x - 2x^2 + 4x - 2$

$\qquad = x^3 - 4x^2 + 5x - 2$

$) \; f_1 \cdot f_2 \mid p_T \Rightarrow f_2 \mid (x-2)^2$

Ademas $f_2 \mid f_1 = m_T \Rightarrow f_2 = (x-2)$

$f_3 \mid f_2 \Rightarrow f_3 = (x-2)$

$(f_1 f_2 . f_3 \mid p_T \Rightarrow f_3 = (x-2)$