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7. Consistencia y estabilidad del esquema a tres niveles temporales y centrado en el espacio.

$$\frac{0'5\,T_j^{n-1} - 2T_j^n + 1'5\,T_j^{n+1}}{\Delta t} - \alpha\left(\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}\right) = 0$$

Reordenando: $T_j^{n+1} = \frac{2}{3}s\left[T_{j+1}^n - 2T_j^n + T_{j-1}^n\right] - \frac{1}{3}T_j^{n-1} + \frac{4}{3}T_j^n$

Los desarrollos en torno a $\Delta t, \Delta x = 0$; nombrando $s = \alpha\Delta t/\Delta x^2$:

$$T_{j\pm1}^n = T(x\pm\Delta x, t) = T_j^n \pm \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n \pm \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n +$$

$$+ \frac{\Delta x^4}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n \pm \mathcal{O}(\Delta x^5)$$

$$T_j^{n\pm1} = T(x, t\pm\Delta t) = T_j^n \pm \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n \pm \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n +$$

$$+ \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_j^n \pm \mathcal{O}(\Delta t^5)$$

Los términos temporales:

$$T_j^{n+1} + \frac{1}{3}T_j^{n-1} - \frac{4}{3}T_j^n = \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n(1-1/3) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n(1+1/3) +$$

$$+ \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n(1-1/3) + \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_j^n(1+1/3) + \mathcal{O}(\Delta t^5) =$$

$$= \frac{2}{3}\Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{2}{3}\Delta t^2\left(\frac{\partial^2 T}{\partial t^2}\right) + \frac{1}{9}\Delta t^3\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \frac{1}{18}\Delta t^4\left(\frac{\partial^4 T}{\partial t^4}\right)_j^n + \mathcal{O}(\Delta t^5)$$

Los espaciales:

$$\frac{2}{3}s\left[T_{j+1}^n - 2T_j^n + T_{j-1}^n\right] = \frac{2}{3}s\Delta x^2\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{1}{18}s\Delta x^4\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n + \mathcal{O}(\Delta x^6)$$

ya que los términos impares se anulan.

La ecuación:

$$\frac{2}{3}\Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{2}{3}\Delta t^2\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \frac{1}{9}\Delta t^3\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \mathcal{O}(\Delta t^4) =$$

$$= \frac{2}{3} s\Delta x^2 \left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{1}{18} s\Delta x^4 \left(\frac{\partial^4 T}{\partial x^4}\right)_j^n + \mathcal{O}(\Delta x^6)$$

$$\Rightarrow \Delta t \left(\frac{\partial T}{\partial t}\right)_j^n - s\Delta x^2 \left(\frac{\partial^2 T}{\partial x^2}\right)_j^n = -\Delta t^2 \left(\frac{\partial^2 T}{\partial t^2}\right)_j^n - \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \frac{s\Delta x^4}{12}\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n + \mathcal{O}(\Delta x^6, \Delta t^4)$$

$$\downarrow$$

$$\alpha = \frac{s\Delta x^2}{\Delta t} \Rightarrow \frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = -\Delta t \left(\frac{\partial^2 T}{\partial t^2}\right) - \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right) + \frac{s\Delta x^4}{\Delta t}\left(\frac{\partial^4 T}{\partial x^4}\right) + \mathcal{O}(\Delta x^6, \Delta t^3)$$

$$E_j^n = -\Delta t \left(\frac{\partial^2 T}{\partial t^2}\right)_j^n - \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \frac{s\Delta x^4}{12\Delta t}\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n + \mathcal{O}(\Delta x^6, \Delta t^3)$$

Además si tenemos en cuenta, permutando parciales:

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(\alpha \frac{\partial^2 T}{\partial x^2}\right) = \alpha \frac{\partial^2}{\partial x^2}\left(\alpha \frac{\partial^2 T}{\partial x^2}\right) = \alpha^2 \frac{\partial^4 T}{\partial x^4}$$

$$\Rightarrow \boxed{\left(-\Delta t \cdot \alpha^2 + \frac{s\Delta x^4}{12\Delta t}\right)\left(\frac{\partial^4 T}{\partial x^4}\right) + \mathcal{O}(\Delta t^2, \Delta x^6) = E}$$

$$\text{Si } -\Delta t \alpha^2 + \frac{s\Delta x^4}{12\Delta t} = -\cancel{\Delta t^2}\frac{s^2 \cancel{\Delta x^4}}{\cancel{\Delta t^2}} + \frac{s\cancel{\Delta x^4}}{12} = 0 \iff \boxed{s = 1/12},$$

el error de consistencia se minimiza hasta $E_j^n = \mathcal{O}(\Delta t^2, \Delta x^6)$

Veamos la estabilidad, $T_j^n \to$ modo arbitrario $\xi_j^n = G^n e^{i\theta j}$

$$G^{n+1} e^{i\theta j} = \frac{2}{3} s \left[G^n e^{i\theta(j+1)} - 2G^n e^{i\theta j} + G^n e^{i\theta(j-1)}\right] - \frac{1}{3} G^{n-1} e^{i\theta j} + \frac{4}{3} G^n e^{i\theta j}$$

$$G = \frac{2}{3} s \left[e^{i\theta} - 2 + e^{-i\theta}\right] - \frac{1}{3} G^{-1} + \frac{4}{3}$$

$$G = \frac{4}{3} s \cos\theta - \frac{4}{3} s - \frac{1}{3} G^{-1} + \frac{4}{3}$$

$$3G = 4s\cos\theta - G^{-1} + 4(1-s)$$

$$3G^2 - 4G\left[s\cos\theta + 1 - s\right] + 1 = 0$$

$$3G^2 - 4G\left[s(\cos\theta - 1) + 1\right] + 1 = 0 \; ; \; \cos\theta - 1 = -2\sin^2\theta/2$$

$$3G^2 - 4G\left[1 - 2s\sin^2\theta/2\right] + 1 = 0$$

$$\Rightarrow G = \frac{4\left[1 - 2s\sin^2\theta/2\right] \pm \sqrt{16(1-2s\sin^2\theta/2)^2 - 12}}{6} =$$

$$= \frac{4 - 8s\sin^2\theta/2 \pm \sqrt{4 + 64s^2\sin^4\theta/2 - 64s\sin^2\theta/2}}{6} =$$

$$= \frac{4 - 8s\sin^2\theta/2 \pm 2\sqrt{1 + 16s\sin^2\theta/2\,(s\sin^2\theta/2 - 1)}}{6}$$

$|G| \leq 1 \quad \forall\theta \Rightarrow$ condición de estabilidad luego:

$$\boxed{-1 \leq \frac{2}{3} - \frac{4}{3}s\sin^2\theta/2 \pm \frac{1}{3}\sqrt{16s\sin^2\theta/2\,(\sin^2\theta/2 - 1) + 1} \leq 1} \quad \forall\theta$$

Si ueamos $s = 1/12$ la cosa no mejora demasiado:

$$-1 \leq \frac{2}{3} - \frac{1}{9}\sin^2\theta/2 \pm \frac{1}{3}\sqrt{\frac{4}{3}\sin^2\theta/2\left(\frac{\sin^2\theta/2}{12} - 1\right) + 1} \leq 1$$