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Upwind advección.

$$\frac{T_i^{n+1} - T_i^{n}}{\Delta t} + u\,\frac{T_i^{n} - T_{i-1}^{n}}{\Delta x} = 0$$

$$T_j^{n+1} = T(x, t+\Delta t) = T_j^{n} + \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \dots$$

$$T_{j-1}^{n} = T(x-\Delta x, t) = T_j^{n} - \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n - \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n$$

Nombrando $C = \frac{u\Delta t}{\Delta x}$

$$T_j^{n+1} = T_j^{n} - C\left(T_j^{n} - T_{i-1}^{n}\right)$$

$$\cancel{T_j^n} + \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n + \dots = T_j^n(\cancel{1} - \cancel{C}) + C\left[\cancel{T_i^n} - \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \right.$$

$$\left. + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n - \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n + \dots\right]$$

$$\Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + C\Delta x\left(\frac{\partial T}{\partial x}\right)_j^n - C\,\frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + C\,\frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n + \mathcal{O}(\Delta x^4, \Delta t^2)$$

$$\left(\frac{\partial T}{\partial t}\right)_j^n + u\left(\frac{\partial T}{\partial x}\right)_j^n - u\,\frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \mathcal{O}(\Delta x^2, \Delta t^2)$$

Teniendo en cuenta $\left(\frac{\partial^2 T}{\partial t^2}\right) = \frac{\partial}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = +u^2\frac{\partial}{\partial x}\frac{\partial T}{\partial x} = +u^2\frac{\partial^2 T}{\partial x^2}$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$$

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$$\Delta t \left(\frac{\partial T}{\partial t}\right)_j^n + C\Delta x \left(\frac{\partial T}{\partial x}\right)_j^n - \frac{C\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + O(\Delta x^3, \Delta t^3) = 0$$

$\downarrow$

$$E_j^n = -\frac{C\Delta x^2}{2\Delta t}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + O(\Delta x^3, \Delta t^2)$$

Además $\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} \Rightarrow \frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = -u\frac{\partial}{\partial x}\left(-u\frac{\partial T}{\partial x}\right) =$

$$= u^2\frac{\partial^2 T}{\partial x^2}$$

$$E_j^n = \underbrace{-\frac{u}{2}\Delta x}\left(\frac{\partial^2 T}{\partial x^2}\right) + \underbrace{\frac{u^2\Delta t}{2}}\left(\frac{\partial^2 T}{\partial x^2}\right) = -\frac{\Delta x^2}{2\Delta t}C\left(\frac{\partial^2 T}{\partial x^2}\right)(1-C) + \ldots$$

$$u\Delta x = \frac{C\Delta x^2}{\Delta t} \qquad u^2\Delta t = \frac{C^2\Delta x^2}{\Delta t}$$

$$= \frac{-\Delta x^2}{2\cancel{\Delta t}}u\frac{\cancel{\Delta t}}{\cancel{\Delta x}}\left(\frac{\partial^2 T}{\partial x^2}\right)(1-C) + \ldots = -u\left(\frac{\Delta x}{2}\right)(1-C)\left(\frac{\partial^2 T}{\partial x^2}\right) + O(\Delta x^2, \Delta t^2)$$

Si $C = 1 \Rightarrow E_j^{n\prime} \ll E_j^n$

Estabilidad

$T_j^n \rightarrow$ modo arbitrario $\xi_j^n = G^n e^{i\theta j}$

$$G^{\cancel{n}+1}\cancel{e^{i\theta j}} = (1-C)\cancel{G^n}\cancel{e^{i\theta j}} + C\cancel{G^n}e^{i\theta(j-1)}$$

$$G = (1-C) + Ce^{-i\theta} = 1 - C + C(\cos\theta - i\sin\theta) =$$

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$G = 1 - C(1-\cos\theta) - iC\sin\theta$

$|G| \leq 1 \iff |G|^2 \leq 1$

$|G|^2 = \left(\sqrt{((1-c)+c\cos\theta)^2 + c^2\sin^2\theta}\right)^2 =$

$= (1-c)^2 + c^2 - 2c(1-c)\cos\theta \; ; \quad c \geq 0$, luego

$|G| \leq 1 \;\forall\theta \iff 0 \leq (1-c)^2 + c^2 \pm 2c(1-c) \leq 1$

- $(1-c)^2 + c^2 + 2c(1-c) \leq 1$

  $\cancel{c^2} + \cancel{c^2} - \cancel{2c^2} + 1 - \cancel{2c} + \cancel{2c} \leq 1 \quad \forall c$

- $(1-c)^2 + c^2 - 2c(1-c) \geq 0$

  $c^2 + 1 - 2c + c^2 - 2c + 2c^2 = 4c^2 - 4c + 1 \geq 0$

  $\cancel{c = \frac{4 \pm \sqrt{16-16}}{8}}$

  que se verifica siempre que $c \leq 1$