Homework 5 Part II

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$$p(\beta \mid \Sigma, X, y) = MVN\left(E[\beta \mid \Sigma, X, y] = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}y, \; Var[\beta \mid \Sigma, X, y] = [X^T\Sigma^{-1}X]^{-1}\right)$$

a) Let $\Sigma = \delta^2 I$:

$$p(\beta \mid \Sigma, X, y) = MVN\left(E[\beta \mid \Sigma, X, y], \; Var[\beta \mid \Sigma, X, y]\right)$$

where $E[\beta \mid \Sigma, X, y] = Var(\beta \mid \Sigma, X, y)\left(X^T\Sigma^{-1}y + \Sigma_\beta^{-1}\beta_0\right)$

so $E[\beta \mid \delta^2 I, X, y] = Var(\beta \mid \delta^2 I, X, y)\left(X^T(\delta^2 I)^{-1}y + \Sigma_\beta^{-1}\beta_0\right)$

$= Var(\beta \mid \delta^2 I, X, y)\left(\delta^2(X^Ty) + \delta_\beta^{-2}\beta_0\right)$

and $Var(\beta \mid \Sigma, X, y) = [X^T\Sigma^{-1}X]^{-1} + \Sigma_\beta^{-1}$

$Var(\beta \mid \delta^2 I, X, y) = [X^T(\delta^{-2})X]^{-1} + \delta_\beta^{-2} I$

$= \delta^2(X^TX)^{-1} + \delta_\beta^{-2} I$

Then $p(\beta \mid \Sigma, X, y) = MVN\left([\delta^2(X^TX)^{-1} + \delta_\beta^{-2} I](\delta^{-2}(X^Ty) + \delta_\beta^{-2}\beta_0), \; [\delta^2(X^TX)^{-1} + \delta_\beta^{-2} I]\right)$

b)

$$E[\beta \mid \Sigma, X, y] = [\delta^2(X^TX)^{-1} + \delta_\beta^{-2} I]\;(\delta^{-2}(X^Ty) + \delta_\beta^{-2}\beta_0)$$

since $E[\beta \mid \Sigma, X, y] = Var(\beta \mid \Sigma, X, y)\left(X^T\Sigma^{-1}y + \Sigma_\beta^{-1}\beta_0\right)$

$= Var(\beta \mid \delta^2 I, X, y)\left(\delta^{-2}X^Ty + \Sigma_\beta^{-1}\beta_0\right)$

$= [\delta^2(X^TX)^{-1} + \Sigma_\beta^{-1}]\;(\delta^{-2}X^Ty + \Sigma_\beta^{-1}\beta_0)$

$= [\delta^2(X^TX)^{-1} + \delta_\beta^{-2} I]^{-1}(\delta^{-2}X^Ty + \delta_\beta^{-2}\beta_0)$

c) WTS: (b) $= (X^TX)^{-1}X^Ty$

Want $Var(\beta \mid \Sigma, X, y) = (X^TX)^{-1}$ and $(X^T\Sigma^{-1}y + \Sigma_\beta^{-1}\beta_0) = X^Ty$

So we need $(X^TX)^{-1} = [\delta^{-2}(X^TX) + \delta_\beta^{-2} I]^{-1}$

$\Rightarrow \delta_\beta^{-2} = 0$

and hence we need $(X^TX)^{-1} = [\delta^{-2}(X^TX)]^{-1}$

$\Rightarrow (X^TX)^{-1} = \delta^2(X^TX)^{-1}$

$\delta^2 = 1$

$\delta = \pm 1$ and $\Sigma_\beta = 0$ so $\delta_\beta^{-2} = 0$

only $\delta = 1$ is legal

But this means $\delta_\beta = \infty$

Now for $(X^T \Sigma^{-1} y + \Sigma_\beta^{-1} \beta_0) = X^T y$

$(\delta^{-2} X^T y + \delta_\beta^{-2} I \beta_0) = X^T y$

since we set $\delta_\beta^{-2} = 0$, β_0 can take any value

and δ is already ± 1

d) Recall: $\hat{y} = X\beta$

Want to set $E[\hat{y} \mid \Sigma, X, y] = X(X^T X)^{-1} X^T y$

$E[\hat{y} \mid \Sigma, X, y] = E[X\beta \mid \Sigma, X, y]$

$= X E[\beta \mid \Sigma, X, y]$ by linearity

$= X \cdot [\delta^2 (X^T X)^{-1} + \delta_\beta^{-2} I] \left(\delta^{-2}(X^T y) + \delta_\beta^{-2} \beta_0 \right)$

so we want this to be $X(X^T X)^{-1} X^T y$

$\Rightarrow [\delta^2 (X^T X)^{-1} + \delta_\beta^{-2} I] = (X^T X)^{-1}$

and $[\delta^{-2}(X^T y) + \delta_\beta^{-2} \beta_0] = X^T y$

Like in part c, we need $\delta^2 = 1$ and $\delta_\beta^{-2} = 0$

e) What is $Var(\beta \mid \Sigma, X, y)$?

$Var[\beta \mid \Sigma, X, y] = [X^T \Sigma^{-1} X]^{-1} + \Sigma_\beta^{-1}$

$= \delta^2 [X^T X]^{-1} + \delta_\beta^{-1} I$