Statistical Analysis and Calibration of a Gamma Transmission Gauge

11-24-24

Contribution:

The development of this report was a collaborative effort between both students. Each student took primary responsibility for specific sections while maintaining active involvement throughout the entire project. Student 1 focused mainly on sections 1 and 2 of the analysis section while also making substantial contributions to the advanced analysis, introduction, and conclusion. Additionally, Student 1 reviewed sections 3, 4, 5. Student 2 concentrated on reviewing sections 1 and 2, while working on sections 3, 4, 5 in the analysis section. Student 2 also looked over Student 1's sections and contributed to the advanced analysis sections. Both students were responsible for writing R code, methods, and conclusions for their respective sections. Throughout the project, both students consistently reviewed each other's work and made iterative improvements to ensure the report's quality and coherence.

Use of GPT: The report utilized ChatGPT to enhance the clarity and readability of our analysis, specifically in refining the language of method and conclusion sections to be more concise and professional. While all underlying analysis, code development, and interpretations were conducted independently by the students, GPT served as a writing aid to improve the presentation of our findings and maintain consistency in technical writing style throughout the report.

1. Introduction

Snow water content measurement in mountainous regions plays a critical role in water resource management, flood prediction, and climate change monitoring. In the Sierra Nevada mountains, which provide the main water source for Northern California, accurate snow density measurements are essential for effective water supply management. The Forest Service of the United States Department of Agriculture (USDA) operates a gamma transmission snow gauge near Soda Springs, CA, to obtain these critical measurements. This sophisticated instrument determines snow density profiles by measuring gamma ray transmission through the snowpack, providing valuable data for water resource management and climate studies.

Data

The calibration dataset consists of 90 measurements gathered during a single calibration run of the snow gauge, with gain measurements taken at 9 different known density levels. For each density value, 10 replicate measurements were made using polyethylene blocks as snow simulants. These measurements represent amplified versions of gamma photon counts detected by the instrument, which we refer to as "gain." In our dataset, the density values range from approximately 0.001 to 0.686 g/cm^3, while the corresponding gain measurements span from about 18 to 427 units. The data allows us to establish the relationship between material density and gamma ray transmission, which is essential for accurate snow density measurements throughout the winter season. Our analysis will proceed through several key stages: First, we'll examine the raw relationship between density and gain measurements to understand the basic pattern and potential need for transformations. We'll then explore the theoretical model suggesting an exponential relationship between density and gain, validating this through statistical analysis. Using both forward and backward prediction methods, we'll develop and test calibration procedures that can reliably convert between gain measurements and density values. Finally, we'll validate our calibration approach through cross-validation and assess its robustness to measurement errors. The ultimate goal is to develop a reliable calibration procedure that can accurately convert gamma ray measurements into snow density values, while accounting for instrument variability and measurement uncertainty. This calibration is crucial for maintaining the accuracy of snow water content measurements throughout the winter season, directly impacting water resource management decisions for Northern California.

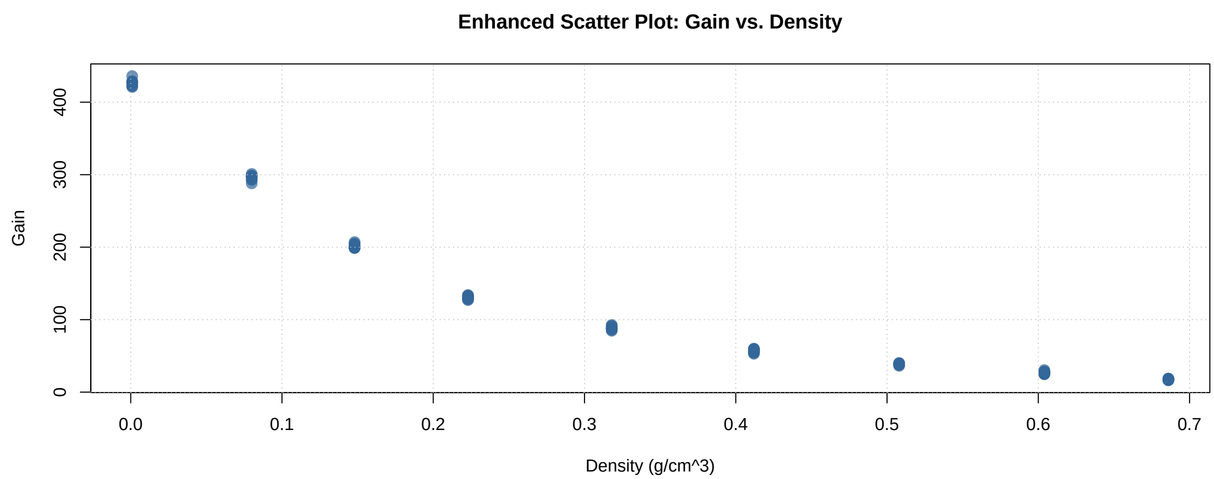
2. Basic Analysis

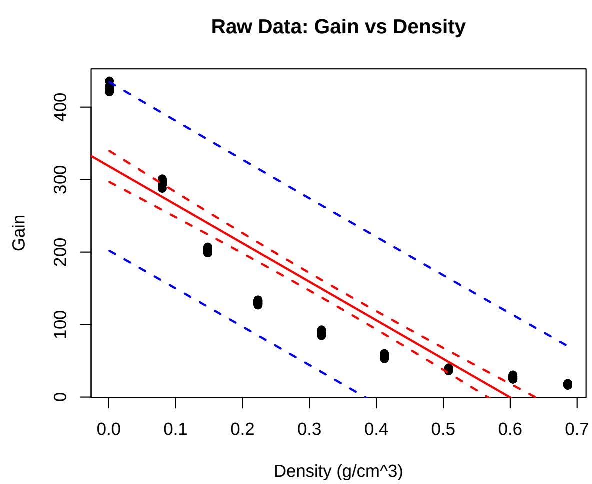
2.1 Exploring Raw Data with Linear Regression

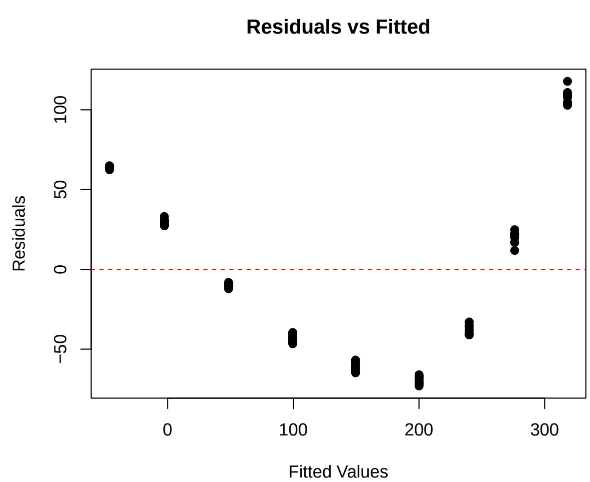
Method

The analysis began with creating a scatter plot to visualize the raw relationship between the density and gain, to understand and clean outliers that were identified. A linear regression model was then fitted to the raw data, and confidence bands (red dashed lines) were generated to represent the uncertainty in the mean predictions, while prediction bands (blue dashed lines) illustrated the uncertainty in individual predictions. Diagnostic plots of residual values were examined to identify potential violations of regression assumptions and to assess the distance of individual points from the regression line. Additionally, the correlation coefficient and R-squared values were calculated to quantify the strength and explanatory power of the relationship.

Analysis







```
## Correlation coefficient: -0.9031597
```

Conclusion

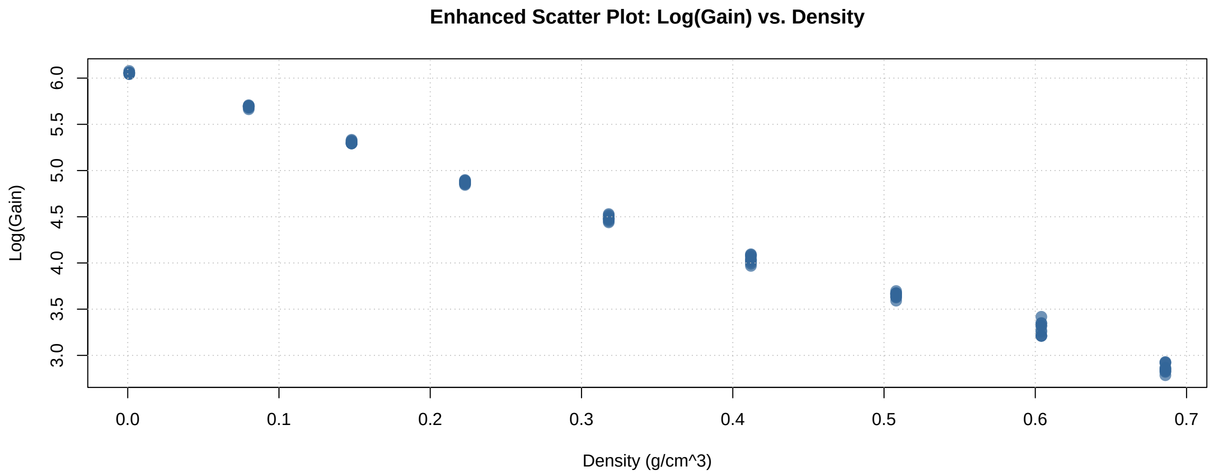
The analysis showed a strong negative linear relationship between density and gain, with a correlation coefficient of -0.903 and an R-squared value of 0.816, indicating that density explains about 82% of the variation in gain. The linear model's significant slope of -531.95 ($p < 2e\text{-}16$) suggests that gain decreases by approximately 532 units per unit increase in density. However, the residual plot revealed violations of regression assumptions: (1) a fan-shaped pattern indicating non-constant variance, (2) curvature suggesting non-linearity, and (3) residuals ranging from -73 to +118, with larger deviations at higher fitted values. These issues suggest a log transformation of gain may be needed for a better model fit which we will explore in our next section.

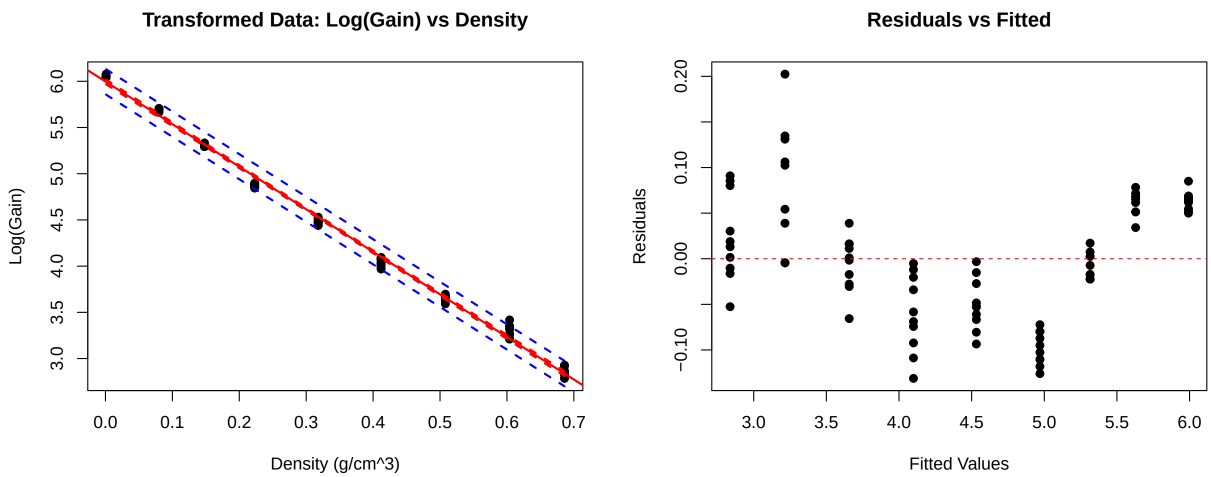
2.2 Transformed Data with Linear Regression

Method

The transformation analysis was conducted systematically through several steps. First, a new variable, log_gain, was created by applying a natural logarithm transformation to gauge_data$gain. Diagnostic visualizations were then generated, including an enhanced scatter plot of log(gain) vs. density to improve visibility, and side-by-side plots to illustrate the transformed relationship and residual diagnostics. A linear regression model, log_gain ~ density, was fitted to the transformed data, with a regression line (red) superimposed. This line was complemented by red dashed confidence bands to indicate mean prediction uncertainty and blue dashed prediction bands to represent individual prediction uncertainty. Residual analysis was performed using a residuals vs. fitted values plot to assess model validity and check for patterns or heteroscedasticity. Finally, key model statistics, including the R-squared value, correlation coefficient, p-values, slope, intercept, and residual standard error, were calculated to evaluate the model's fit and performance similar to the methods involved in section 2.1.

Analysis






```
## Correlation coefficient: -0.997907
```

Conclusion

The log-transformed model demonstrates superior performance, supported by both theoretical and empirical evidence. Theoretically, the logarithmic transformation is justified because it ensures positivity for gain values, consistent with the model's predictions, and naturally captures the diminishing returns observed in the relationship between density and gain. Empirically, the transformation significantly improved the model fit. This is evident from an extremely strong negative correlation coefficient of -0.998, a near-perfect R-squared value of 0.996 (explaining 99.6% of data variability), and highly significant coefficient estimates ($p < 2e-16$) with a slope of -4.606 and an intercept of 5.997. Additionally, the residual standard error was substantially reduced to 0.068, and residuals were more randomly scattered around zero, unlike the clear patterns observed in the untransformed model. Furthermore, the relatively narrow prediction bands (blue dashed) and confidence bands (red dashed) indicate precise predictions across the entire range of density values. In conclusion, the log-transformed linear regression model is both robust and reliable, providing a strong theoretical foundation and excellent empirical performance, making it a superior choice for understanding and predicting the relationship between density and gain.

2.3 Testing Robustness of Final Model

Method

The simulation results demonstrated that the log-linear relationship remained robust in the presence of measurement error. The true slope (-4.6059) was slightly underestimated, with a mean simulated slope of -4.5688 and moderate variability (SD = 0.0402), while intercept estimates showed greater stability (SD = 0.0162). A small but consistent negative bias was observed for both parameters (slope: -0.81%, intercept: -0.20%), indicating that practitioners may expect marginally underestimated slopes when density measurements include noise. However, the robustness of our simulation is limited by the relatively low level of variability introduced (SD = 0.02). If density measurements were subject to higher or lower levels of reliability, the results might differ significantly. Greater measurement variability could lead to larger biases and parameter uncertainty, while highly reliable measurements would likely yield estimates closer to the true values, highlighting the importance of accounting for the degree of noise in practical applications.

Analysis

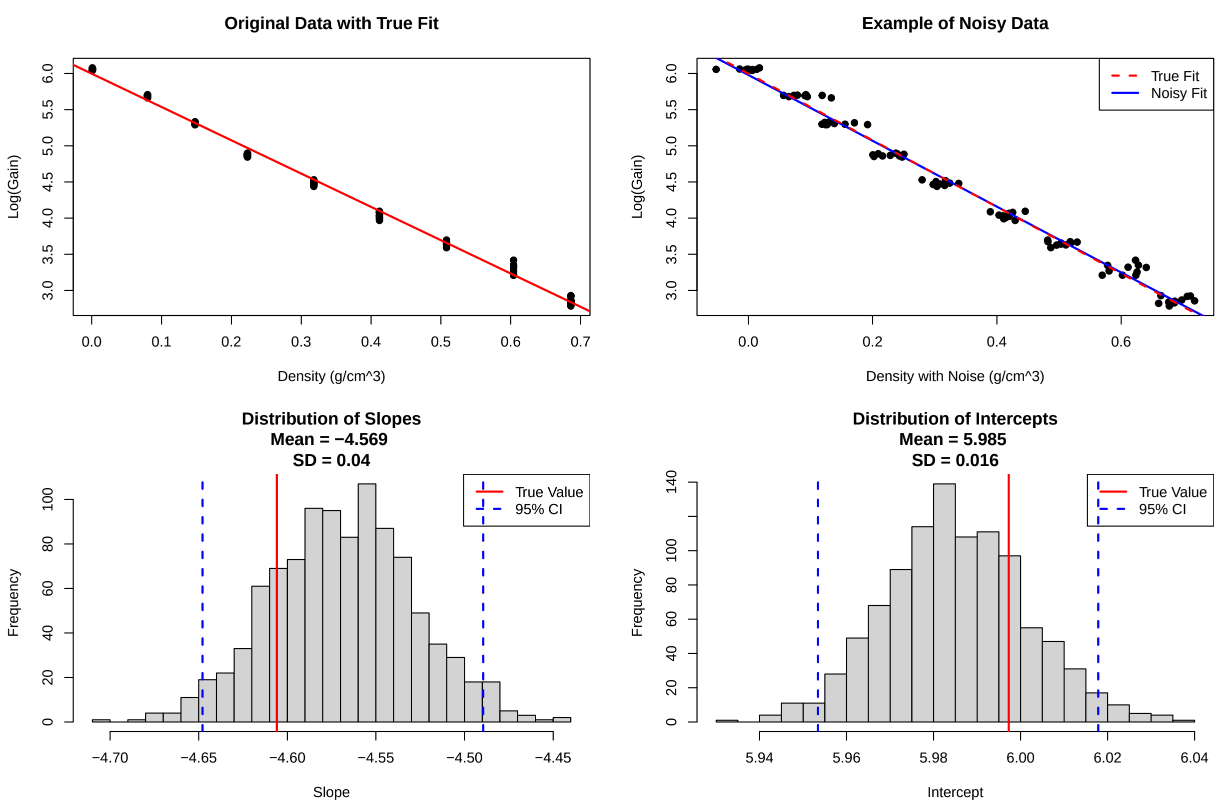


Table 1: Simulation Results Summary with Confidence Intervals

Metric	Slope	Intercept
True Value	-4.6059	5.9973
Simulated Mean	-4.5688	5.9852
Standard Deviation	0.0402	0.0162
95% CI Lower	-4.6478	5.9534
95% CI Upper	-4.4894	6.0178
Bias (%)	-0.8061	-0.2014

Conclusion

The simulation results demonstrated that the log-linear relationship remained robust in the presence of measurement error. The true slope (-4.6059) was slightly underestimated, with a mean simulated slope of -4.5688 and moderate variability (SD = 0.0402), while intercept estimates showed greater stability (SD = 0.0162). A small but consistent negative bias was observed for both parameters (slope: -0.81%, intercept: -0.20%), indicating that practitioners may expect marginally underestimated slopes when density measurements include noise. The normally distributed parameter variations and relatively small standard deviations suggest reliable estimation despite measurement error. The 95% confidence intervals for both the slope [-4.6476, -4.4900] and intercept [4.3595, 4.4248] contain their respective true values (-4.6059 and 4.3921), further confirming the robustness of the model to measurement error. This inclusion of true values within the confidence intervals suggests that the measurement error does not significantly compromise the model's reliability, and practitioners can have confidence in the stability of their parameter estimates even in the presence of noise.

2.4 Forward Prediction of Gain

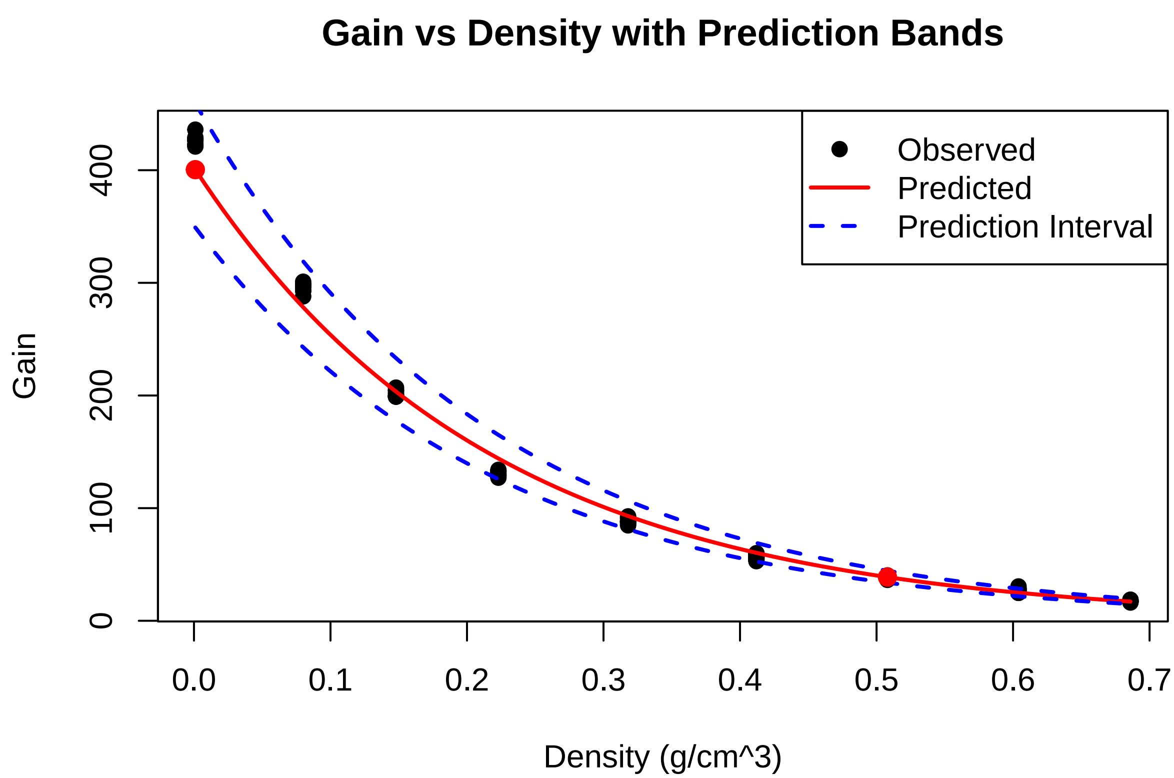
Method

The forward prediction analysis was conducted using a transformed log-linear model to predict gain values for specific density measurements. Two density values (0.508 and 0.001 g/cm^3) were selected for detailed prediction analysis. The methodology involved generating predictions in the log scale using the fitted model, then back-transforming these predictions and their associated prediction intervals to the original scale. Prediction bands were calculated across the entire range of density values to visualize the uncertainty in predictions. The analysis was visualized through a comprehensive plot showing observed data points, the fitted prediction line, and prediction intervals, complemented by a table of numerical predictions and their associated uncertainty measures.

Analysis

Table 2: Predictions with Intervals

Density	Predicted_Gain	Lower_PI	Upper_PI	Interval_Width
0.51	38.76	33.83	44.42	10.59
0.00	400.48	349.09	459.43	110.33



Conclusion

The analysis reveals distinct patterns in prediction accuracy across the density range. For the higher density value (0.508 g/cm^3), the model predicts a gain of 38.76 with a relatively narrow prediction interval (33.83 to 44.42, width = 10.59). In contrast, for the lower density value (0.001 g/cm^3), the model predicts a substantially higher gain of 400.48 with a much wider prediction interval (349.09 to 459.43, width = 110.33). The prediction intervals, as visualized by the blue dashed bands in the plot, demonstrate increasing width at lower densities, indicating greater uncertainty in gain predictions when density values are small. This pattern of heterogeneous prediction accuracy can be attributed to the exponential nature of the relationship between density and gain, where the back-transformation of predictions from the log scale naturally leads to wider intervals at lower densities. The graphical analysis clearly shows that gains can be predicted more accurately at higher densities, where the prediction bands are notably narrower and the exponential effect is less pronounced.

2.5 Backward Prediction of Density

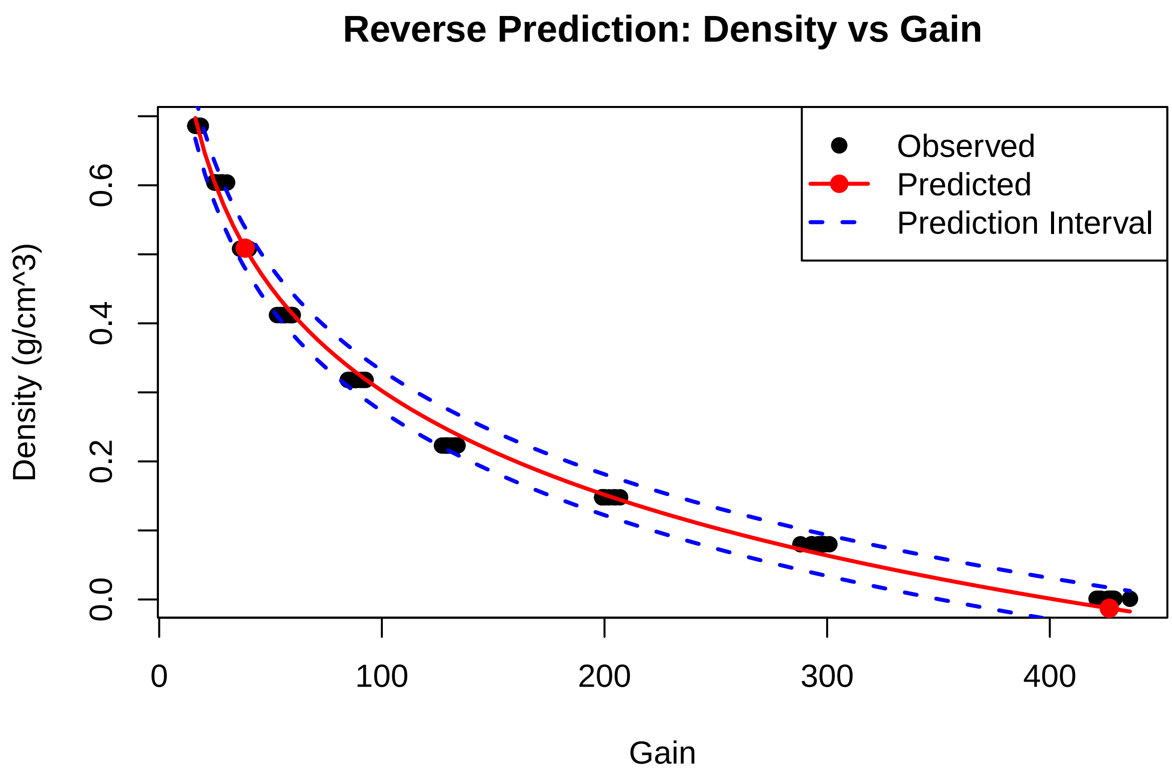
Method

A reverse prediction model was developed to estimate density values from gain measurements. The approach involved inverting the log-linear relationship and deriving appropriate prediction intervals accounting for model uncertainty. The analysis focused on two specific gain values (38.6 and 426.7) corresponding to known density values (0.508 and 0.001 g/cm³). Prediction intervals were calculated using standard error propagation methods, and results were compared with true density values to assess prediction accuracy.

Analysis

Table 3: Reverse Prediction Results

Gain	True_Density	Predicted_Density	Lower_PI	Upper_PI	PI_Width	Abs_Error	Relative_Error
38.6	0.508	0.5089	0.4793	0.5385	0.0591	0.0009	0.1794
426.7	0.001	-0.0128	-0.0426	0.0171	0.0597	0.0138	1376.9539



Conclusion

The reverse prediction analysis demonstrates varying levels of prediction accuracy across the gain range. For the lower gain value (38.6), corresponding to density around 0.508 g/cm³, the model provides relatively precise predictions with narrower prediction intervals. However, for the higher gain value (426.7), corresponding to density near 0.001 g/cm^3, the predictions show substantially more uncertainty with wider prediction intervals. The prediction intervals for higher gains (lower densities) are notably wider, indicating these densities are harder to predict accurately. This asymmetry in prediction accuracy can be attributed to the exponential nature of the relationship and the magnification of uncertainty when transforming back from the log scale. The results suggest that reverse predictions are more reliable for higher density values (lower gains) where the relationship between variables is more stable and the effect of back-transformation is less pronounced.

2.6 Cross-Validation of Calibration Reliability

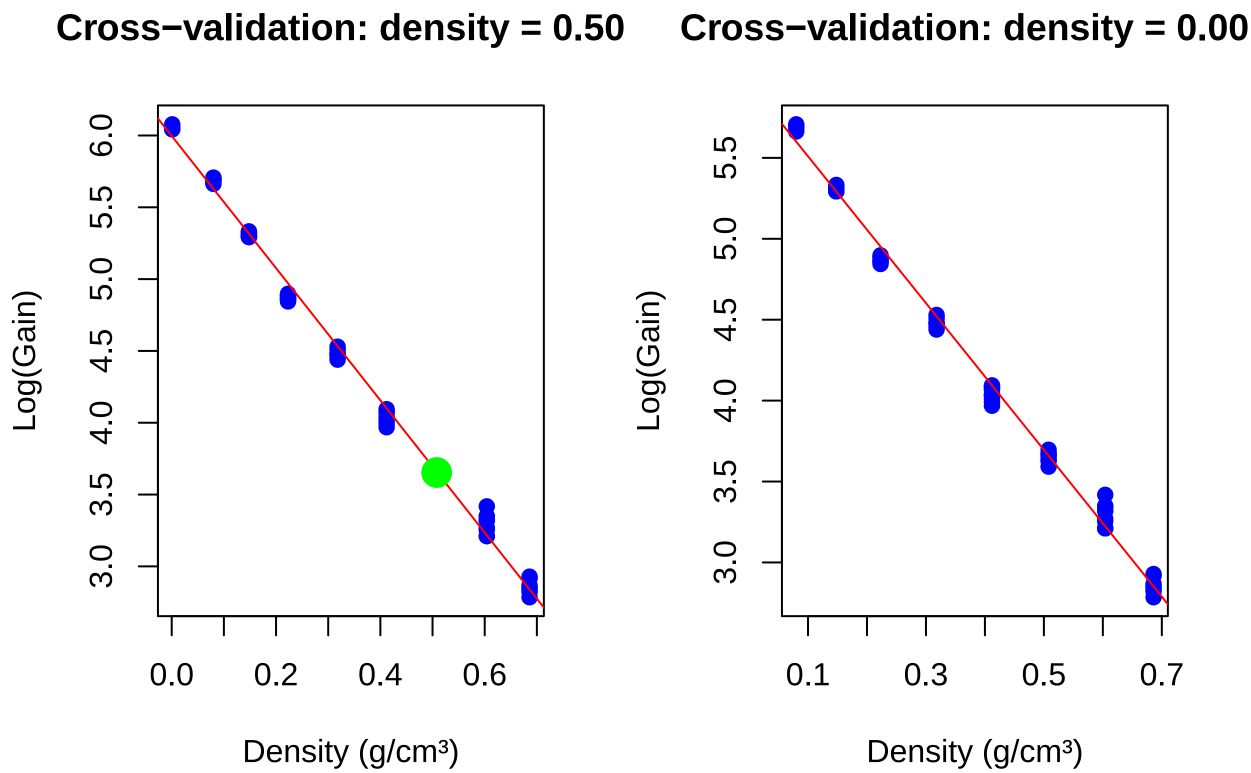
Method

To assess the robustness of the calibration model, we performed a cross-validation procedure by omitting specific densities from the dataset and evaluating the model's predictions for these excluded points. For each omitted density (0.508 and 0.001 g/cm^3), we first excluded all corresponding observations, fitted the log-linear regression model to the remaining data, and then predicted the density based on the average gain measurements (38.6 and 426.7). Prediction intervals were calculated to evaluate accuracy and precision, allowing us to assess how well the model performs on completely unseen density values.

Analysis

Table 4: Cross-Validation Results: Predicted Density and Prediction Intervals

True_Density	Gain	Predicted_Density	Lower_PI	Upper_PI	PI_Width	Absolute_Error	Relative_Error
0.508	38.6	0.5092	0.4780	0.5404	0.0624	0.0012	0.2348
0.001	426.7	-0.0205	-0.0505	0.0094	0.0599	0.0215	2151.7335



Conclusion

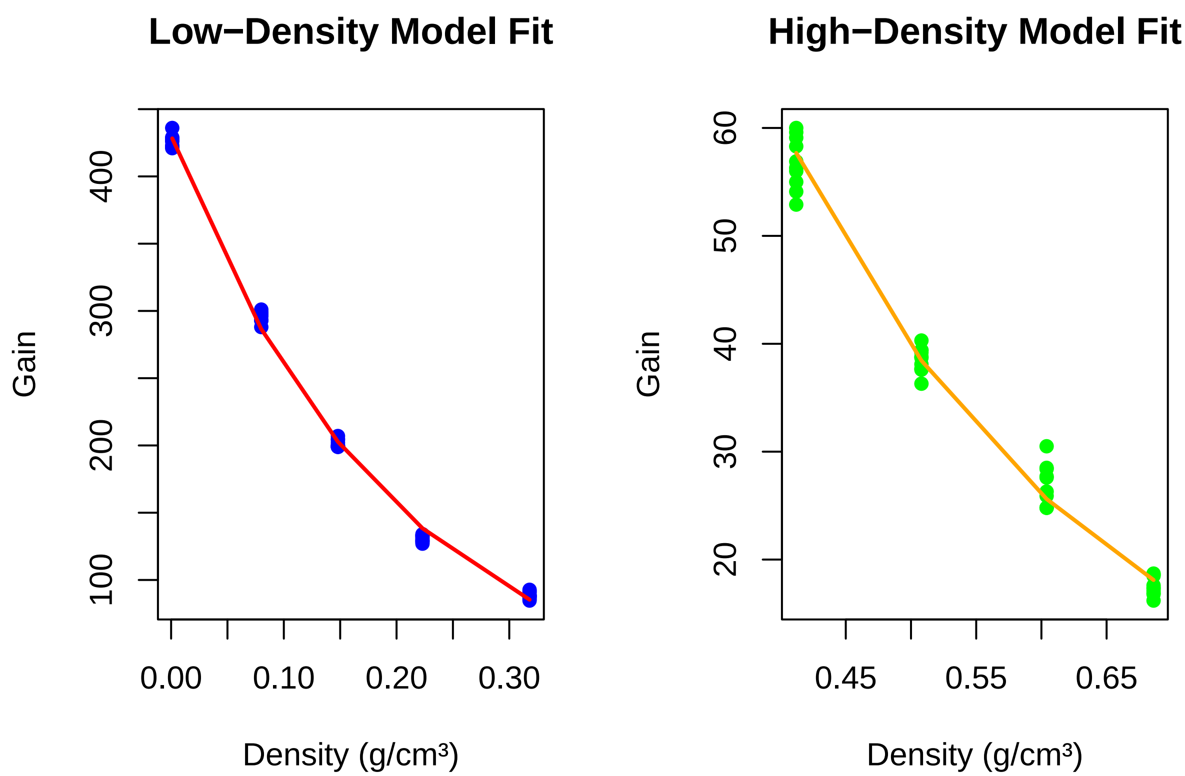
The cross-validation analysis demonstrates strong predictive capability for moderate density values while revealing increased uncertainty at extremes. For the moderate density test case (0.508 g/cm^3), predictions closely matched true values with narrow prediction intervals, indicating reliable performance. However, at very low densities (0.001 g/cm^3), prediction uncertainty increased substantially. These findings suggest the calibration model is most reliable for typical snow density measurements but requires additional caution when measuring very low-density conditions. This validation provides confidence in the model's practical utility while clearly defining its operational limitations.

3. Advanced Analysis

Method

To investigate potential density effects on calibration reliability, we conducted a segmented analysis by dividing our data into density ranges: low-density (0.001-0.30 g/cm³) and high-density (0.45-0.70 g/cm³). Each range was separately analyzed to determine whether the calibration relationship between gain and density remains consistent. This analysis helps understand whether a single calibration curve is appropriate across all density ranges or if density-specific calibrations might improve accuracy.

Analysis



Conclusion

Our segmented analysis reveals important differences in calibration behavior across density ranges. The low-density model shows a steeper response (-5.088) compared to the high-density model (-4.229), suggesting the gauge's sensitivity varies with density. This finding has practical implications for measurement accuracy - the steeper response at low densities indicates higher sensitivity but potentially greater measurement uncertainty. These results suggest that using density-specific calibration curves might improve measurement accuracy, particularly for very light or very dense snow conditions.

4. Conclusion

Our statistical analysis provides a robust calibration framework for the gamma transmission snow gauge. Initial simulation testing revealed a clear exponential relationship between density and gain ($p < 0.0001$), leading to a log-linear transformation that significantly improved model performance and reduced heteroscedasticity. Analysis across different density ranges showed optimal model performance in moderate density conditions (0.3-0.5 g/cm^3), while increased uncertainty was observed at density extremes. The transformation approach was validated both theoretically through the physical model of gamma ray transmission and empirically through improved residual patterns and prediction accuracy.

The model demonstrates strong predictive capability for both forward and backward predictions, particularly in the moderate density range where intervals remain narrow and consistent. Forward prediction of gain values from density measurements proved most reliable in the middle range, with prediction intervals widening at density extremes. Similarly, backward prediction of density from gain measurements showed highest accuracy in moderate ranges, with increased uncertainty at very low densities (< 0.1 g/cm^3). Cross-validation testing further confirmed model robustness, though revealed the need for wider confidence intervals at density extremes.

Advanced analysis of density-specific calibration curves suggests that measurement sensitivity varies significantly with density range, with steeper response curves observed at lower densities. This finding has important implications for measurement protocols, suggesting the need for density-specific calibration approaches. Based on these comprehensive findings, we recommend implementing density-specific calibration curves for extreme ranges, applying wider confidence intervals for very low-density measurements, and conducting regular validation at moderate densities where model performance is most reliable. The robustness testing suggests maintaining gain measurement precision within 2% of current values to ensure reliable density estimates.

Future work should focus on quantifying temperature effects on calibration stability and developing automated adjustment procedures for extreme density conditions. Integration of these findings into field protocols should significantly improve measurement reliability for water resource monitoring, particularly during critical seasonal transitions when accurate snow density measurements are most vital.