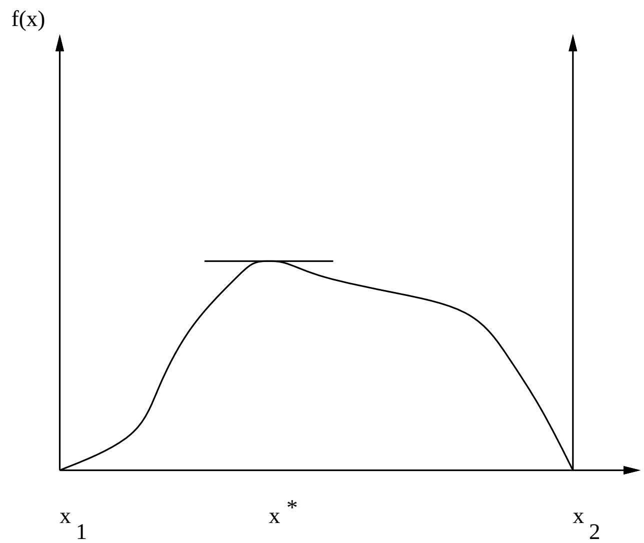
Mathematics 25: Numerical Analysis

Numerical analysis of most differential algorithms is founded on *three* major mathematical results:
(1) Rolle's Theorem
(2) Taylor's Theorem
(3) Interpolation Formulae
Rolle's Theorem is the fundamental, and the other two follow fairly directly.

(1) Rolle's Theorem
If a continuously differentiable function, $f(x)$ say, vanishes at two points, x_1 and x_2, then $f^{(1)}(x^*) = 0$, for some $x^* \in (x_1, x_2)$.



$f(x)$ starts out either up or down. Eventually this initial direction must reverse, and this point marks $f^{(1)}(x^*) = 0$.

(2) Taylor's Theorem

$$f(x+h) = \sum_{m=0}^{n} \frac{h^m}{m!} f^{(m)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x+\theta h)$$

for some $\theta \in (0,1)$. To prove this result we set:

$$g(y) = f(x+y) - \sum_{m=0}^{n} \frac{y^m}{m!} f^{(m)}(x) - Ay^{n+1}$$

and *choose* A so that $g(h) = 0$. Note that:

$$g(0) = g^{(1)}(0) = ... = g^{(n)}(0) = 0$$

We now employ Rolle's Theorem n times. Now $g(0) = 0 = g(h)$ and so there is a point where $g^{(1)}(\theta_1 h) = 0$, for some $\theta_1 \in (0,1)$. Now $g^{(1)}(0) = 0 = g^{(1)}(\theta_1 h)$ and so there is a point where $g^{(2)}(\theta_2 \theta_1 h) = 0$ for some $\theta_2 \in (0,1)$. This argument proceeds one at a time until, finally, we find a point where, $g^{(n+1)}(\theta_n\theta_{n-1}....\theta_2\theta_1 h) = 0$ for some $\theta_n \in (0,1)$. We set $\theta = \theta_n\theta_{n-1}....\theta_2\theta_1$ and then we immediately see that:

$$g^{(n+1)}(\theta h) = 0 = f^{(n+1)}(x+\theta h) - (n+1)!A$$

and Taylor's Theorem is proven.

(3) Interpolation Formulae

The basic idea here is to use *difference formulae* to construct the interpolating polynomial. A function, $f(x)$, is central and then subsequent differences and their inverses are defined by:

$$[x_0x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} \qquad f(x_0) = f(x_1) + (x_0 - x_1)[x_0x_1]$$

$$[x_0x_1x_2] = \frac{[x_0x_1] - [x_1x_2]}{x_0 - x_2} \qquad [x_0x_1] = [x_1x_2] + (x_0 - x_2)[x_0x_1x_2]$$

$$[x_0x_1x_2x_3] = \frac{[x_0x_1x_2] - [x_1x_2x_3]}{x_0 - x_3} \qquad [x_0x_1x_2] = [x_1x_2x_3] + (x_0 - x_3)[x_0x_1x_2x_3]$$

and so on. If we include a special, *variable* point, x into the description, then:

$$f(x) = f(x_0) + (x - x_0)([x_0x_1] + (x - x_1)([x_0x_1x_2] + (......(x - x_{n-1})[x_0x_1...x_n]))...))$$

$$+(x - x_0)(x - x_1)....(x - x_n)[xx_0x_1....x_n] \equiv P(x) + R(x)$$

where $P(x)$ is the initial polynamial, depending only on the values of $f(x)$ evaluated at the points x_m, and $R(x)$ is the final term:

$$R(x) = (x - x_0)(x - x_1)....(x - x_n)[xx_0x_1....x_n]$$

Each difference reduces a polynomial by one degree, so if the original function were a polynomial of degree n, the corresponding $R(x)$ would vanish. This shows that $P(x)$ is the *unique* interpolating polynomial which agrees with the function $f(x)$ at the special points x_m.

Unless the function is 'pathological', then $R(x_m) = 0$ for each x_m. This enables us to apply Rolle's Theorem. There are initially $n+1$ zeroes, and so between each neighbouring pair there is a point for which $R^{(1)}(x^*)$ vanishes. There are now n zeroes in $R^{(1)}(x)$, and so between each neighbouring pair there is a point for which $R^{(2)}(x^*) = 0$. We can sequentially proceed with this argument, until eventually we find a single point for which, $R^{(n)}(x^*) = 0$. A direct differentiation of the explicit representation then provides us with:

$$R^{(n)}(x^*) = 0 = f^{(n)}(x^*) - n![x_0x_1x_2...x_n]$$

We finally employ a rather subtle idea, and note that $R(x)$ itself is composed of an $n+1$'th difference, and so if we include the point x itself in amongst the x_m and repeat the argument we find:

$$[xx_0x_1x_2.....x_n] = \frac{f^{(n+1)}(x^*)}{(n+1)!}$$

for some x^* contained in the region covered by the x_m *and* x! The final interpolation formula therefore becomes:

$$f(x) = f(x_0) + (x - x_0)([x_0 x_1] + (x - x_1)([x_0 x_1 x_2] + (......(x - x_{n-1})[x_0 x_1 ... x_n]))...))$$

$$+(x - x_0)(x - x_1)....(x - x_n)\frac{f^{(n+1)}(x^*)}{(n+1)!}$$

For equal spaced points at $x_n = nh$, we find:

$$[x_0 x_1 ... x_r] = \frac{[x_1 x_2 ... x_r] - [x_0 x_1 ... x_{r-1}]}{rh} \equiv \frac{(\Delta - 1)[x_0 x_1 ... x_{r-1}]}{rh} = \frac{(\Delta - 1)^r}{r! h^r} f_0$$

in terms of the operator Δ, which raises all the indicies by one. In these terms:

$$f(x) = f_0 + h\frac{x}{h}(\Delta - 1) f_0 + \frac{h^2}{2!}\frac{x}{h}\left(\frac{x}{h} - 1\right)(\Delta - 1)^2 f_0 + \frac{h^3}{3!}\frac{x}{h}\left(\frac{x}{h} - 1\right)\left(\frac{x}{h} - 2\right)(\Delta - 1)^3 f_0 + ...$$

in terms of:

$$(\Delta - 1)^n f_0 = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(-1)^{n-m} f_m$$

Since all errors from a truncation to this sequence at order n, are order h^{n+1}, we can form approximations to derivatives of arbitrary accuracy by using as many more terms than derivatives as is required to achieve our desired accuracy.