Mathematics 21: Dynamical Systems and Lagrangians

In this section we will review *Lagrangian Mechanics*, recalling the main ideas. The main thrust is towards an investigation of a *non-integrable* dynamical system.

The equations of motion for simple dynamical systems are best found using the *Lagrangian*, $\hat{L}$:

$$\hat{L} = \hat{T} - \hat{V}$$

in terms of $\hat{T}$, the Kinetic energy, and $\hat{V}$, the potential energy. These quantities are functions of the variables, q_i and their time derivatives, $\dot{q}_i$:

$$\hat{L}[\dot{q}_i, q_i]$$

The equations of motion are simply:

$$\frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \dot{q}_i}\right] = \frac{\partial \hat{L}}{\partial q_i}$$

one for each variable, q_i. For the particular case that $\hat{L}$ does not depend explicitly on time, t, then there is a conserved quantity, the 'energy' $\hat{H}$:

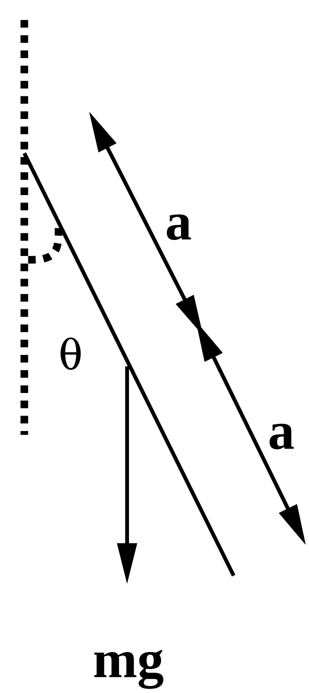
$$\hat{H} = \sum_i \dot{q}_i \frac{\partial \hat{L}}{\partial \dot{q}_i} - \hat{L}$$

which is immediately deduced via:

$$\frac{d}{dt}\hat{H} = \sum_i \left(\ddot{q}_i \frac{\partial \hat{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \dot{q}_i}\right]\right) - \frac{d}{dt}\hat{L} = \sum_i \left(\ddot{q}_i \frac{\partial \hat{L}}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial \hat{L}}{\partial q_i}\right) - \frac{d}{dt}\hat{L} = 0$$

where we have used the equations of motion.

Armed with these ideas, let us look at a rod forming a simple pendulum as an example:



There is a single variable, θ. There is also an immediate problem in that the Kinetic energy of our system is angular and we therefore need to be able to deal with rotations. The first issue is that of separating linear motion from rotational motion, and this is best done in the *centre of mass* frame. The Kinetic energy of a body is the sum of the linear motion of the centre of mass to the rotational energy about the centre of mass:

$$\begin{aligned} M &= \sum_i \delta m_i \\ M\mathbf{R} &= \sum_i \delta m_i \mathbf{r}_i \\ \mathbf{r}_i &= \mathbf{R} + \mathbf{\Delta r}_i \end{aligned}$$

formalises the total mass, M, the centre of mass $\mathbf{R}$ and using the centre of mass as the origin, all in terms of a label i which runs over some decomposition of the 'body'. The Kinetic energy is then:

$$\hat{T} = \frac{1}{2}\sum_i \delta m_i \left[\dot{\mathbf{R}} + \mathbf{\Delta}\dot{\mathbf{r}}_i\right] . \left[\dot{\mathbf{R}} + \mathbf{\Delta}\dot{\mathbf{r}}_i\right] = \frac{1}{2}M\dot{\mathbf{R}}.\dot{\mathbf{R}} + \frac{1}{2}\sum_i \delta m_i \mathbf{\Delta}\dot{\mathbf{r}}_i.\mathbf{\Delta}\dot{\mathbf{r}}_i$$

which is clearly a sum of the motion of the centre of mass to the motion about the centre of mass. We must also deal with the motion of a rigid body, and for this we need to understand *moments of inertia.* The 'inertia' opposing the rotation of a body may be derived by noting that motion of a rigid body may only be rotation about some particular axis at any particular time. The local velocities at that time are simply:

$$\mathbf{\Delta}\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{\Delta r}_i$$

in terms of this local angular momentum, $\boldsymbol{\omega}$. Substitution reveals:

$$\frac{1}{2}\sum_i \delta m_i \mathbf{\Delta}\dot{\mathbf{r}}_i.\mathbf{\Delta}\dot{\mathbf{r}}_i = \frac{1}{2}\sum_i \delta m_i \left[\boldsymbol{\omega} \times \mathbf{\Delta r}_i\right].\left[\boldsymbol{\omega} \times \mathbf{\Delta r}_i\right] = \frac{1}{2}\delta m_i \left[\omega^2 \mid \mathbf{\Delta r}_i \mid^2 - (\boldsymbol{\omega}.\mathbf{\Delta r}_i)^2\right]$$

$$= \frac{1}{2}\sum_{\alpha\beta} \omega_\alpha I_{\alpha\beta}\omega_\beta$$

leading to the *moment of inertia* tensor:

$$I_{\alpha\beta} = \sum_i \delta m_i \left[\delta_{\alpha\beta}\mathbf{\Delta r}_i.\mathbf{\Delta r}_i - \Delta r_{i\alpha}\Delta r_{i\beta}\right]$$

For the current example, the angular momentum is perpendicular to the object, so we are left to find:

$$I = \sum_i \delta m_i \mathbf{\Delta r}_i.\mathbf{\Delta r}_i$$

For a simple rod of length $2a$ we have:

$$I = \int_{-a}^{a} dm x^2 = \rho \int_{-a}^{a} dx x^2 = \frac{m}{2a}\frac{2a^3}{3} = \frac{ma^2}{3}$$

and so the Lagrangian for our system is:

$$\begin{aligned}\hat{T} &= \frac{1}{2}m\left(a\dot{\theta}\right)^2 + \frac{1}{6}ma^2\dot{\theta}^2 \\ \hat{V} &= mg(1-\cos\theta)a \\ \hat{L} &= \frac{2}{3}ma^2\dot{\theta}^2 - mga(1-\cos\theta)\end{aligned}$$

At this point it is valuable to reparameterise the problem via:

$$\begin{aligned}A &= \frac{4}{3}ma^2 \\ C &= mga\end{aligned}$$

in terms of which:

$$\hat{L} = \frac{1}{2}A\dot{\theta}^2 - C(1-\cos\theta)$$

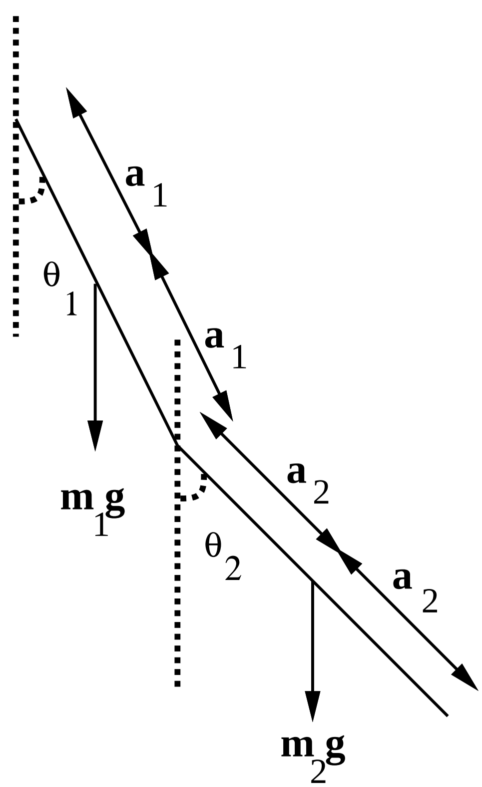
leading to the equation of motion:

$$A\ddot{\theta} + C\sin\theta = 0$$

together with a conserved energy:

$$E = \frac{1}{2}A\dot{\theta}^2 + C(1-\cos\theta)$$

The problem of most interest to us is that of two coupled rods, a genuinely *non-integrable* system:



which depends on two variables, θ_1 and θ_2. Once again it is best to find the Langrangian, and further, to split the Kinetic energy up into centre of mass and rotation about the centre of mass pieces:

$$
\begin{aligned}
\hat{T}_1 &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_1\dot{\theta}_1^2 = \frac{1}{2}m_1 a_1^2\dot{\theta}_1^2 + \frac{1}{6}m_1 a_1^2\dot{\theta}_1^2 = \frac{2}{3}m_1 a_1^2\dot{\theta}_1^2 \\
\hat{T}_2 &= \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\left[(2a_1\sin\theta_1\dot{\theta}_1 + a_2\sin\theta_2\dot{\theta}_2)^2 + (2a_1\cos\theta_1\dot{\theta}_1 + a_2\cos\theta_2\dot{\theta}_2)^2\right] + \frac{1}{6}m_2 a_2^2\dot{\theta}_2^2 \\
&= 2m_2 a_1^2\dot{\theta}_1^2 + \frac{2}{3}m_2 a_2^2\dot{\theta}_2^2 + 2m_2 a_1 a_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)
\end{aligned}
$$

and a simple potential energy:

$$
\hat{V} = m_1 g a_1(1 - \cos\theta_1) + m_2 g(2a_1 + a_2 - 2a_1\cos\theta_1 - a_2\cos\theta_2)
$$

where we are measuring potential with respect to static equilibrium. Once again it is useful to reparameterise using:

$$
\begin{aligned}
A_1 &= \frac{4}{3}m_1 a_1^2 + 4m_2 a_1^2 \\
A_2 &= \frac{4}{3}m_2 a_2^2 \\
B &= 2m_2 a_1 a_2 \\
C_1 &= g(m_1 + 2m_2)a_1 \\
C_2 &= g m_2 a_2
\end{aligned}
$$

in terms of which:

$$
\hat{L} = \frac{1}{2}\left[A_1\dot{\theta}_1^2 + A_2\dot{\theta}_2^2 + 2B\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] - C_1(1 - \cos\theta_1) - C_2(1 - \cos\theta_2)
$$

The explicit equations of motion are:

$$
\begin{aligned}
A_1\ddot{\theta}_1 + B\ddot{\theta}_2\cos(\theta_1 - \theta_2) + B\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + C_1\sin\theta_1 &= 0 \\
A_2\ddot{\theta}_2 + B\ddot{\theta}_1\cos(\theta_1 - \theta_2) - B\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + C_2\sin\theta_2 &= 0
\end{aligned}
$$

combined with an energy:

$$
E = \frac{1}{2}\left[A_1\dot{\theta}_1^2 + A_2\dot{\theta}_2^2 + 2B\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + C_1(1 - \cos\theta_1) + C_2(1 - \cos\theta_2)
$$

A major component of the investigation involves *Poincare sections*: A way of picturing these complicated dynamics. If the system has a sort of 'periodic' behaviour, involving an 'angle', then we can consider a fixed value of this 'angle' and freeze the system every time the chosen angle takes its chosen value. In more technical jargon, we may consider the dynamics of the system to be a *trajectory* in $\{q_i, \dot{q}_i\}$ space, or *phase space* as it is usually known. A Poincare section is then a lower dimensional surface embedded in phase space, punctured every time the trajectory crosses it. One of the main uses for Poincare sections is the existence of conservation laws. Each conservation law is a *constraint* on the allowed phase space that the trajectory may explore. Let us look at an example:

The simple rod-pendulum has a two-dimensional phase-space: $\{\theta, \dot{\theta}\}$. There is an angle in the problem, and it is natural to consider $\theta = 0$ for the section. The remaining space is then one-dimensional, the value of $\dot{\theta}$ when $\theta = 0$. Due to the conservation-law, however, there are only two possible values of $\dot{\theta}$, since we know the potential energy and hence can deduce the kinetic energy from total-energy conservation.

The coupled pair of pendula have a four-dimensional phase space: $\{\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\}$. The natural section to consider is $\theta_2 = 0$, and then we would expect to have a three-dimensional subspace remaining. However, we know that the energy is conserved, and so we can use the energy to find θ_1 (modulo 2π), leaving us with a two-dimensional space remaining: $\{\dot{\theta}_1, \dot{\theta}_2\}$. Even then, the conservation of energy leads to restrictions on the permitted two-dimensional phase-space. It is easy to verify that the permitted area of phase space lies between the two ellipses:

$$
\begin{aligned}
E =& \frac{1}{2}\left[A_1\dot{\theta}_1^2 + A_2\dot{\theta}_2^2 + 2B\dot{\theta}_1\dot{\theta}_2\right] \\
E =& \frac{1}{2}\left[A_1\dot{\theta}_1^2 + A_2\dot{\theta}_2^2 - 2B\dot{\theta}_1\dot{\theta}_2\right] + 2C_1
\end{aligned}
$$