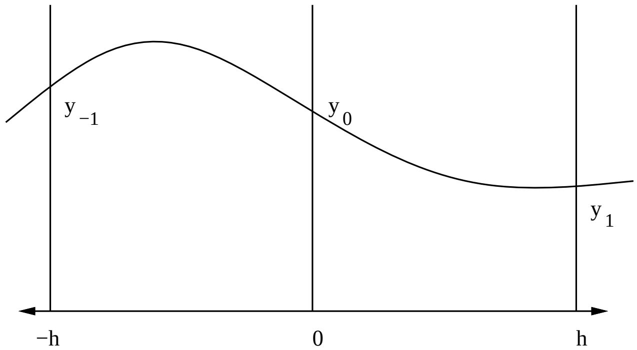
Mathematics 1: Simpson's Rule

This is a simple method for performing numerical integration. Given a set of equally spaced points and the function to be integrated evaluated at these points, a quadratic is fitted through neighbouring triples and then integrated exactly. Using the notation in the diagram:



The interpolation is:

$$y(x) = y_0 + \alpha \left[\frac{x}{h}\right] + \beta \left[\frac{x}{h}\right]^2$$

from which:

$$y_{-1} + y_1 = 2y_0 + 2\beta$$

The integration is elementary:

$$\int_{-h}^{h} dx y(x) = 2hy_0 + \frac{2h}{3}\beta = \frac{h}{3}\left[y_{-1} + y_1 + 4y_0\right]$$

The application of this rule is best done by *bisection.* The interval to be integrated is bisected making two intervals and Simpson's rule is applied for a first estimate. The two new intervals are then bisected again, leading to four intervals and a second estimate based on two applications of Simpson. This procedure is repeated until the result converges. Note that all the previous calculations can be used for the next estimate, so all the work is used in the final estimate. This idea is expressed mathematically by:

$$r_1 = \frac{h}{3}\left[y(a) + y(a + 2Nh)\right] + \frac{2h}{3}\sum_{n=1}^{N-1} y(a + 2nh)$$

$$r_2 = \frac{4h}{3}\sum_{n=1}^{N} y(a - h + 2nh)$$

where N is the number of double intervals, $b = a + 2Nh$ and the estimate from Simpson's rule is obtained from $r = r_1 + r_2$. The next estimate is provided by:

$$h \mapsto \frac{h}{2}$$

$$r_1 \mapsto \frac{r_1}{2} + \frac{r_2}{4}$$

and then the new r_2 is calculated from the values of $y(x)$ at the new bisecting points.