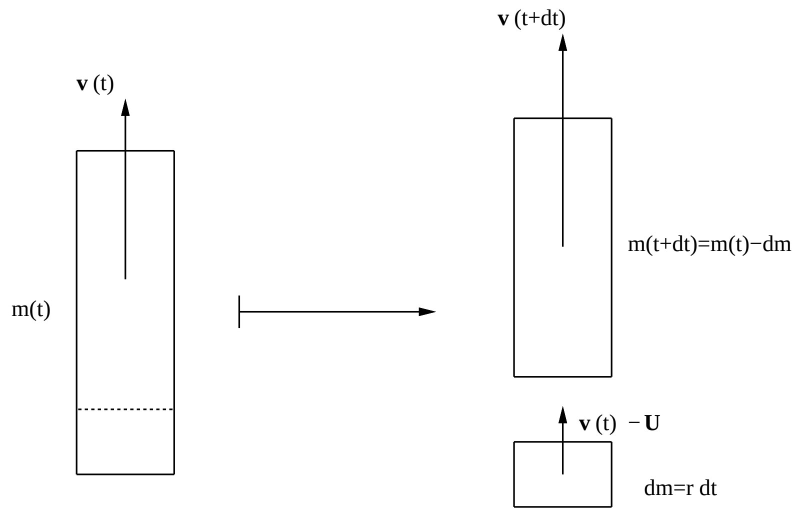
Mathematics 28: Payloads

In this section we will investigate a rudimentary model for sending a satellite into space. The basic idea behind a 'rocket propulsion system' is that of a uniform stream of the fuel, usually composing the major initial component of the mass of the rocket, being expelled at a constant relative velocity. Pictorially:



where $\mathbf{U}$ is the relative velocity of the expelled fuel and r is the rate at which the fuel is expelled. The equations of motion for such a rocket come from Newton's Laws:

$$m(t+\delta t)\mathbf{v}(t+\delta t) - m(t)\mathbf{v}(t) + \delta m\left[\mathbf{v}(t) - \mathbf{U}\right] = \delta t\mathbf{F}(t)$$

in terms of the force, $\mathbf{F}(t)$, acting on the rocket, yielding:

$$m\frac{d\mathbf{v}}{dt} = r\mathbf{U} + \mathbf{F} \qquad \frac{dm}{dt} = -r$$

The uniform flow of mass yields:

$$m(t) = m_0 - rt$$

a mass which reduces as the fuel is spent. The gravitational potential of the Earth may be extracted leading to:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{|\,\mathbf{r}\,|^3} + \frac{1}{m_0 - rt}\left[r\mathbf{U} + \mathbf{f}\right]$$

where $\mathbf{f}$ is the residual force from air-resistance and any other effects. If we ignore the force, $\mathbf{f}$, on the grounds that the Earth's atmosphere is rather thin on the length scales of interest, then we are left with:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{|\,\mathbf{r}\,|^3} + \frac{r\mathbf{U}}{m_0 - rt}$$

for the motion of the rocket. It is this problem that you will investigate.

In order that the rocket starts off upwards, we require that:

$$\hat{\mathbf{r}}_0.\mathbf{U}\frac{r}{m_0} \geq \frac{GM}{\mid \mathbf{r}_0 \mid^2} \equiv g$$

and since $\mid \mathbf{U} \mid$ is 'fixed' by the style of fuel and the planet's gravitational field is fixed (at the surface), the only real variable in the equation is:

$$t_b = \frac{m_0}{r} \leq \frac{\mid \mathbf{U} \mid}{g}$$

where t_b is the duration of a *complete* burn of the rocket and g is the acceleration due to gravity field at the surface of the planet.

We need to rescale our equation, and to do this we need to think about boundary conditions: There is an implicit time before which the burning must stop, t_b, the time at which the rocket would have disappeared! This is the natural time scale, and we can solve for all possible *final* masses of the rocket by solving for all times and treating each possible time as the time at which the burning stops. The rocket starts off at the surface of the planet, and so it makes sense to use this as the natural length-scale. Under these assumptions we rescale with:

$$t \mapsto \frac{m_0}{r}\tau = t_b\tau \qquad \mathbf{r} \mapsto \mid \mathbf{r}_0 \mid \mathbf{s} = r_0\mathbf{s}$$

in terms of which:

$$\frac{d^2\mathbf{s}}{d\tau^2} = \frac{t_b\mathbf{U}}{(1-\tau)r_0} - \frac{g}{r_0}\frac{t_b^2\mathbf{s}}{\mid \mathbf{s} \mid^3} \equiv \frac{t_b}{t_g^2}\left[\frac{t_c\hat{\mathbf{U}}}{1-\tau} - \frac{t_b\mathbf{s}}{\mid \mathbf{s} \mid^3}\right]$$

where:

$$t_c = \frac{\mid \mathbf{U} \mid}{g}$$

is the critical time which must be greater than the burn time, t_b, in order to force take-off,

$$t_g = \left[\frac{r_0}{g}\right]^{1/2}$$

is the natural gravitational time-scale, and corresponds to the period that a mass would have if it oscillated through the Earth under the action of the Earth's gravitational field. The actual values that these quantities take can be assessed from the following data:

Gravitational Constant =	$0.667 \times 10^{-10} m^3 kg^{-1} s^{-2}$
Mass of the Earth =	$0.5976 \times 10^{25} kg$
Radius of the Earth's Orbit =	$0.1495 \times 10^{12} m$
Radius of the Earth =	$0.6368 \times 10^{7} m$
Earth's Surface Gravitational accelleration =	$0.9829 \times 10^{1} ms^{-2}$
The gravitational time-scale, t_g=	$0.8049 \times 10^{3} s$

The basic problem is therefore, given a type of fuel, corresponding to an expulsion rate and hence t_c, one can investigate the role of the burn time, t_b, and implicitely the *payload*:

$$\frac{m_\infty}{m_0} = 1 - \tau_{final}$$

as parameters used to maximise the payload and the cost of the procedure. Clearly, there is a natural limit to the expulsion rate, since one is not likely to obtain relativistic expulsion of fuel and hence, $t_c \ll 10^7 s$, with a more reasonable estimate of $t_c \sim 10^4 s$. This sets a strong limit on the burn-time, t_b, to be hours or less and not days, and explains to some extent why rockets are usually at least two-stage processes.

Once the burn has finishes, the equations of motion revert to those of a simple trajectory under the action of the Earth:

$$\frac{d^2\mathbf{s}}{d\tau^2} = -\frac{t_b^2}{t_g^2}\frac{\mathbf{s}}{|\mathbf{s}|^3}$$

in our new units. This problem is solved in 'Maths27', and escape from orbit is only achieved if the energy:

$$E = \frac{1}{2}\left|\frac{d\mathbf{s}}{d\tau}\right|^2 - \frac{t_b^2}{t_g^2}\frac{1}{|\mathbf{s}|}$$

which is conserved in the new motion, is *positive*. We can calculate this quantity during the burn, where it is *not* conserved, and can investigate when it becomes positive.

If the burn were so short that the gravitational field of the Earth had not had time to change significantly, then we can remodel the burn with:

$$\frac{d^2 s}{d\tau^2} = \frac{t_b^2}{t_g^2}\left[\frac{t_c}{t_b}\frac{\hat{\mathbf{U}}}{1-\tau} - 1\right]$$

which, for linear motion, can be directly integrated to provide:

$$\frac{ds}{dt} = -\frac{t_b t_c}{t_g^2}\log(1-\tau) - \frac{t_b^2}{t_g^2}\tau$$

under the assumption that the rocket starts at rest and hence:

$$s = \frac{t_b t_c}{t_g^2}\left[(1-\tau)\log(1-\tau) + \tau\right] - \frac{t_b^2}{t_g^2}\frac{\tau^2}{2} + 1$$

This problem may be used to assess a solution to the more general problem.