How can we do better than Euler's method? From our experience with derivatives, we know that the central difference formula is much better than the forward difference formula, with the same number of function evaluations. If we directly applied it to t_i, we'd have

$$\left.\frac{\mathrm{dy}}{\mathrm{dt}}\right|_{ti} \simeq \frac{y(t_i + h/2) - y(t_i - h/2)}{h} + \mathcal{O}(h^2)$$

but we want to evaluate $y(t_i + h)$ so let us shift all values by $h/2$:

$$\left.\frac{\mathrm{dy}}{\mathrm{dt}}\right|_{ti+h/2} \simeq \frac{y(t_i + h) - y(t_i)}{h} + \mathcal{O}(h^2) = f(y_{i+1/2}, t_i + h/2)$$

rearranging:

$$y_{i+1} = y_i + hf(y_{i+1/2}, t_{i+1/2})$$

The problem is that we now need to know what $y_{i+1/2}$ is. For that we will use Euler:

$$y_{i+1/2} = y_i + \frac{1}{2}hf(y_i, t_i) + \mathcal{O}(h^2) \equiv y_i + \frac{1}{2}k_1$$

Substituting this in the final rule is

$$y_{i+1} \simeq y_i + k_2 \qquad \text{with} \qquad k_1 = hf(y_i, t_i) \qquad \text{and} \qquad k_2 = hf(y_i + k_1/2, t_i + h/2)$$

This is called the **second order Runge-Kutta Method**.

For a more general derivation now see "maths 20".