Physics 4: Flux lines and Bessel's Functions

In this section we briefly investigate the simplest model of superconductivity: The London model. The basic physical idea is that a certain fraction of the electrons, with density n_s, do not move subject to electrical resistance and are free to move without dissipation. Although they will move so as to build up charge and expel any electric field, the magnetic field problem is more subtle and it is not obvious how they will respond to a magnetic field. Since this is an electromagnetic problem, we first need to recall the laws of electromagnetism: Maxwell's equations:

$$\nabla.\mathbf{B} = 0 \tag{$M1$}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{$M2$}$$

$$\nabla.\mathbf{D} = \rho \Rightarrow \nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{$M3$}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \Rightarrow \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} \tag{$M4$}$$

in free space. Together with this we need to know how the sources: the charge density, ρ, and the current density, $\mathbf{j}$, react to the fields. Usually we have Ohm's Law:

$$\mathbf{j} = \sigma \mathbf{E}$$

but in superconductors $\sigma \mapsto \infty$ and so $\mathbf{j}$ flows so as to cancel $\mathbf{E}$. Without any $\mathbf{E}$, in principle we could still have $\mathbf{B}$ and then a current, $\mathbf{j} = \mu_0^{-1} \nabla \times \mathbf{B}$, would steadily flow. In practice, however, superconductors *expel* field: The Meissner effect. To motivate London theory, it is usual to consider a small applied electric field and then deduce and investigate the response of the superconducting electrons. If these electrons move with velocity $\mathbf{v}_s$, then:

$$\mathbf{j}_s = -e n_s \mathbf{v}_s$$

$$m \frac{\partial \mathbf{v}_s}{\partial t} = -e\mathbf{E} \Rightarrow \frac{\partial \mathbf{j}_s}{\partial t} = \frac{e^2 n_s}{m} \mathbf{E}$$

under the action of the field. Considering the induction from this field we find from (M2):

$$\frac{\partial}{\partial t} \left[\mathbf{B} + \frac{m}{e^2 n_s} \nabla \times \mathbf{j}_s \right] = \mathbf{0}$$

and so we would expect:

$$\mathbf{B} + \frac{m}{e^2 n_s} \nabla \times \mathbf{j}_s = \mathbf{B}^*$$

with $\mathbf{B}^*$ independent of time. In fact, this relation is the fundamental result of the analysis, and corresponds to the response of the superconducting electrons to applied fields. One should consider this result as analogous to '$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$': The field in free-space, $\mathbf{H}$, being analogous to $\mathbf{B}^*$ (up to a constant applied field) in this derivation, the induced magnetisation, $\mathbf{M}$, being analogous to the term involving the superconducting currents, and the measured field, $\mathbf{B}$, being analogous to $\mathbf{B}$. The 'content' of

London theory is that in a superconductor $\mathbf{B}^* = \mathbf{0}$. Eliminating $\mathbf{j}_s$ from (M4) yields an equation for the field induced by $\mathbf{B}^*$, considered as a source:

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} + \mu_0 \nabla \times \mathbf{j}_s \Rightarrow$$

$$-\nabla^2 \mathbf{B} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{\mu_0 e^2 n_s}{m} \mathbf{B} = \frac{\mu_0 e^2 n_s}{m} \mathbf{B}^*$$

where we used both (M1) and (M2) in the derivation. Setting:

$$\lambda^2 = \frac{m}{\mu_0 e^2 n_s} = \frac{m c^2 \epsilon_0}{e^2 n_s}$$

where λ is the so-called penetration depth, we find that the *static* fields satisfy:

$$-\nabla^2 \mathbf{B} + \frac{\mathbf{B}}{\lambda^2} = \frac{\mathbf{B}^*}{\lambda^2}$$

The physical interpretation and use of this equation is that $\mathbf{B}^* = \mathbf{0}$ in the superconductor, but not necessarily outside, nor in the 'normal' conductor. The equation therefore describes the *decay* of the field from *outside* the superconductor into the interior.

In the case of a type II superconductor, the external field can, if strong enough, enter the superconductor. The superconductor *requires* $\mathbf{B}^* = \mathbf{0}$, and so the field that enters the superconductor drives small regions of the system *normal.* In most cases these regions are very small and may be modelled by δ-functions. In this case we obtain:

$$-\nabla^2 B + \frac{B}{\lambda^2} = \sum_{\mathbf{R}} \delta[\mathbf{r} - \mathbf{R}]$$

for the component of the field parallel to the applied field, where $\mathbf{R}$ denote the positions of the small normal regions. Note that the equation has been *rescaled.*

We now move on to a brief mathematical analysis of how we deal with a single isolated flux-line. The equation we need to solve is:

$$-\nabla^2 B(\mathbf{r}) + \frac{B(\mathbf{r})}{\lambda^2} = \delta(\mathbf{r})$$

which is a source for the field, the δ-function, combined with some 'smearing'.

The first thing to do is to use spherical polar coordinates:

$$-\frac{\partial^2 B}{\partial r^2} - \frac{1}{r} \frac{\partial B}{\partial r} + \frac{B}{\lambda^2} = 0$$

yields a cylindrically symmetric solution for $r \geq 0$. Our first task is to establish the boundary condition at $r = 0$ which ensures the source. This is done using the two-dimensional analogue of the divergence theorem:

$$1 = \int_A d^2\mathbf{r} \delta(\mathbf{r}) = -\int_A d^2\mathbf{r} \nabla^2 B(\mathbf{r}) = -\int_{\delta A} \mathbf{dS}.\nabla B(\mathbf{r})$$

where A is the area of integration δA is the 'edge' of this region, and $\mathbf{dS}$ is a vector element of this 'edge' *perpendicular* to the 'edge'.For the current problem, we may choose to use a *tiny* circle surrounding the δ-function with radius ϵ:

$$1 = \int_0^{2\pi} d\theta \epsilon \hat{\mathbf{r}}.\nabla B(\mid \mathbf{r} \mid) \mid_{|\mathbf{r}|=\epsilon} = -2\pi \left[r \frac{\partial B}{\partial r} \right]_{r=\epsilon}$$

and so:

$$B(r) \mapsto \frac{1}{2\pi} \log \frac{\lambda}{r}$$

in the limit that $r \mapsto 0$.

The solution to this problem is:

$$B^*(r) = \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{r}{\lambda}\cosh u\right]$$

since:

$$-\frac{\partial^2 B^*}{\partial r^2} - \frac{1}{r}\frac{\partial B^*}{\partial r} + \frac{B^*}{\lambda^2} = \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{r}{\lambda}\cosh u\right] \left[\frac{1}{\lambda^2} - \frac{\cosh^2 u}{\lambda^2} + \frac{\cosh u}{\lambda r}\right]$$

and then integrating by parts (and using $\cosh^2 u = 1 + \sinh^2 u$):

$$-\frac{\partial^2 B^*}{\partial r^2} - \frac{1}{r}\frac{\partial B^*}{\partial r} + \frac{B^*}{\lambda^2} = \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{r}{\lambda}\cosh u\right] \left[\frac{\cosh u}{\lambda r} - \frac{\cosh u}{\lambda r}\right]$$

$$+ \left[\exp\left[-\frac{r}{\lambda}\cosh u\right] \frac{\sinh u}{\lambda r}\right]_0^\infty = 0$$

which vanishes. All we need do, is therefore to verify that the behaviour is correct near $r = 0$.

Using the substitution $v = \cosh u$, followed by $vr/\lambda = s$, we are led to:

$$B^*(r) = \int_1^\infty \frac{dv}{2\pi} \frac{\exp\left[-\frac{r}{\lambda}v\right]}{\sqrt{(v^2-1)}} = \int_{r/\lambda}^\infty \frac{ds}{2\pi} \frac{\exp(-s)}{\sqrt{\left[s^2 - \frac{r^2}{\lambda^2}\right]}}$$

which can be expanded via:

$$B^*(r) = \int_{r/\lambda}^\infty \frac{ds}{2\pi} \exp(-s) \left[\frac{1}{\sqrt{\left[s^2 - \frac{r^2}{\lambda^2}\right]}} - \frac{1}{s}\right] + \int_1^\infty \frac{ds}{2\pi} \frac{\exp(-s)}{s}$$

$$+ \int_{r/\lambda}^1 \frac{ds}{2\pi} \frac{1}{s} \left[\exp(-s) - 1\right] + \int_{r/\lambda}^1 \frac{ds}{2\pi} \frac{1}{s}$$

where all the terms bar the last one have been chosen to be regular as $r \mapsto 0$, and the last one is:

$$B^*(r) \mapsto \frac{1}{2\pi} \log\left[\frac{\lambda}{r}\right]$$

as required.

It is also useful to understand the asymptotic behaviour of the field as $|\,\mathbf{r}\,| \mapsto \infty$. This is totally controlled by the region of the u integration around the origin:

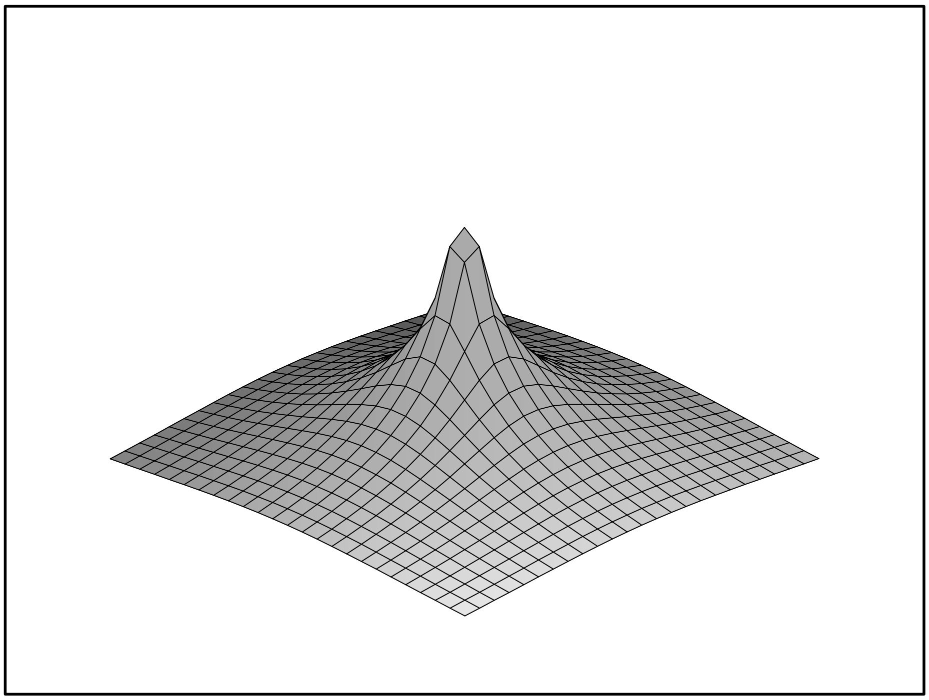
$$B^*(r) = \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{r}{\lambda}\cosh u\right] \mapsto \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{r}{\lambda}\left(1+\frac{u^2}{2}\right)\right]$$

which is exactly integrable, leading to:

$$B^*(r) \mapsto \left[\frac{\lambda}{8\pi r}\right]^{1/2} \exp\left[-\frac{r}{\lambda}\right]$$

so the field *exponentially* decays on the penetration depth length-scale.

The field distribution around a single flux-line is:



Since the flux-lattice is simply a sum over δ-function sources:

$$-\nabla^2 B(\mathbf{r}) + \frac{1}{\lambda^2} B(\mathbf{r}) = \sum_{\mathbf{R}} \delta\left[\mathbf{r} - \mathbf{R}\right]$$

the field distribution is likewise such a sum, and may be calculated via:

$$B(r) = \sum_{\mathbf{R}} B^*(\mathbf{r}-\mathbf{R}) = \sum_{\mathbf{R}} \int_0^\infty \frac{du}{2\pi} \exp\left[-\frac{|\,\mathbf{r}-\mathbf{R}\,|}{\lambda}\cosh u\right]$$

Only the most studious student should proceed beyond this point! We will now briefly review these most simple of Bessel's functions, deriving them and analysing the regular solution as well.

First of all we will solve the differential equation by 'Complex Fourier transform': The idea is to try to represent the answer as:

$$B(r) = \int_C dz \exp(rz) b(z)$$

where C is a contour in the complex plane which must be determined. Substituting this Ansatz into the equation yields:

$$-\frac{\partial^2 B}{\partial r^2} - \frac{1}{r}\frac{\partial B}{\partial r} + \frac{B}{\lambda^2} = \int_C dz \exp(rz) b(z) \left[\frac{1}{\lambda^2} - \frac{z}{r} - z^2\right] = 0$$

integration by parts then yields:

$$\left[\exp(rz)\frac{b(z)}{r}\left(\frac{1}{\lambda^2} - z^2\right)\right]_{\partial C} - \int_C dz \exp(rz)\frac{1}{r}\left[zb(z) + \frac{d}{dz}\left[\left(\frac{1}{\lambda^2} - z^2\right)b(z)\right]\right] = 0$$

leading to two requirements: Firstly that the contour should either be closed or start and finish at places where the surface term vanishes, and secondly that the integrand should vanish:

$$\frac{d}{dz}\left[\left(\frac{1}{\lambda^2} - z^2\right)b(z)\right] = -\frac{z}{\left(\frac{1}{\lambda^2} - z^2\right)}\left(\frac{1}{\lambda^2} - z^2\right)b(z)$$

and so from direct integration:

$$b(z) = \frac{const}{\left(\frac{1}{\lambda^2} - z^2\right)^{1/2}}$$

the ends of the contour must therefore be at two from $z \in \{\frac{1}{\lambda}, -\frac{1}{\lambda}, -\infty\}$. The solution to the isolated flux-line is simply:

$$B^*(r) = \int_{-\infty}^{-1/\lambda} \frac{dz}{2\pi} \exp(zr) \frac{1}{\left(\frac{1}{\lambda^2} - z^2\right)^{1/2}}$$

but there is a second 'regular' solution:

$$J(r) = \int_{-1/\lambda}^{1/\lambda} \frac{dz}{\pi} \exp(zr) \frac{1}{\left(\frac{1}{\lambda^2} - z^2\right)^{1/2}}$$

which transforms under $\lambda z = \cos\theta$ to:

$$J(z) = \int_0^\pi \frac{d\theta}{\pi} \exp\left[\frac{r}{\lambda}\cos\theta\right]$$

As well as the asymptotics:

$$J(r) = \int_0^\pi \frac{d\theta}{\pi} \exp\left[\frac{r}{\lambda}\cos\theta\right] \mapsto \int_0^\infty \frac{d\theta}{\pi} \exp\left[\frac{r}{\lambda}\left(1 - \frac{\theta^2}{2}\right)\right] \mapsto \left[\frac{\lambda}{2\pi r}\right]^{1/2} \exp\left[\frac{r}{\lambda}\right]$$

there is a 'Taylor's theorem' expansion:

$$J(r) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{r}{\lambda}\right)^{2n} \int_0^\pi \frac{d\theta}{\pi} \cos^{2n}\theta = \sum_{n=0}^{\infty} \frac{1}{n!n!}\left(\frac{r}{2\lambda}\right)^{2n}$$