Physics 5: The Evolution of an Accretion Disc around a Black-Hole

The full theory concerning the basic derivation of the governing equations will only be outlined here: For a fuller discussion see the book 'Accretion Power in Astrophysics' by Frank, King and Raine.

The basic idea involves a massive star or black-hole, with mass M, and a relatively small mass of gas, with mass $m \ll M$, in orbit around the star in the form of a very thin disk, in a plane perpendicular to the orbital angular momentum. If we describe the problem in terms of cylindrical polar coordinates (R, ϕ, z), then we will presume that no appreciable spreading in z occurs ($z = 0$), and that the gas starts out and remains cylindrically symmetric (and hence does not depend on the angle ϕ). We are left to find the evolution of this mass of gas, a problem depending only on R and t.

There are *three* main quantities central to the problem; The surface density $\Sigma(R,t)$, which is the mass per unit surface area of the disk; $\Omega(R,t)$, the orbital angular velocity of the material (about the black-hole) and; $v_R(R,t)$, the *radial* drift velocity of the disk, caused by the 'drag' from the viscosity in the self-interactions of the gas, between particles moving at different speeds in neighbouring radial orbits.

Due to the assumption that the star is very massive, we will force Keplerian motion on the gas and work 'only' with the slower motion induced by the drag. Initially we have conservation of mass and conservation of angular momentum to contend with:

For both of these laws, we may consider a fixed, thin, narrow, annulus of space and demand that the decay of the conserved quantity is controlled by the amount leaving. For the case of mass conservation:

$$2\pi R \Delta R \left[\Sigma(R, t + \Delta t) - \Sigma(R,t)\right] =$$

$$\Delta t \left[2\pi R v_R(R) \Sigma(R,t) - 2\pi (R + \Delta R) v_R(R + \Delta R) \Sigma(R + \Delta R, t)\right]$$

the gain in mass is equal to the amount drifting in at the inner radius minus the amount drifting out at the lower radius. In the limit we find:

$$\frac{\partial}{\partial t}\left[2\pi R \Sigma\right] + \frac{\partial}{\partial R}\left[2\pi R v_R \Sigma\right] = 0 \tag{1}$$

For the case of angular momentum conservation:

$$\Delta R 2\pi R \left[\Sigma(R, t+\Delta t) R^2 \Omega(R, t + \Delta t) - \Sigma(R,t) R^2 \Omega(R,t)\right] =$$

$$\Delta t \left[2\pi R v_R(R) \Sigma(R,t) R^2 \Omega(R,t) - 2\pi R v_R(R+\Delta R) \Sigma(R + \Delta R, t)(R+\Delta R)^2 \Omega(R + \Delta R, t)\right]$$
$$+ \Delta t \left[H(R + \Delta R, t) - H(R,t)\right]$$

where we must consider the change in *torque*, $H(R,t)$, acting on the gas as well as the drift of angular momentum. This leads to:

$$\frac{\partial}{\partial t}\left[2\pi R \Sigma R^2 \Omega\right] + \frac{\partial}{\partial R}\left[2\pi R v_R \Sigma R^2 \Omega\right] = \frac{\partial H}{\partial R} \tag{2}$$

The torque satisfies:

$$H(R,t) = 2\pi R\nu\Sigma(R,t)R^2\frac{\partial\Omega}{\partial R}(R,t) \tag{3}$$

in terms of the viscosity, ν. To understand where this expression for the torque comes from, consider two neighbouring regions of width λ (where λ is the mean particle interaction distance) on either side of a surface of radius R, $(R \gg \lambda)$. Because of thermal or turbulent motion, gas will constantly cross this radius, with say an average speed $\bar{v}$ and interact with particles on the other side of this radius. The important point is that while there will be no net transfer of matter across this radius there will be a net transfer of angular momenta.

Define the coefficient of kinematic viscosity as $\nu = \lambda\bar{v}$. Then consider an "average" parcel of gas originating from a radius $R - \lambda/2$ which conserves its ϕ velocity, but moves a distance λ out to a radius $R + \lambda/2$. Its angular momentum at this radius will then be:

$$(R + \lambda/2)\,(R - \lambda/2)\,\Omega\,(R - \lambda/2)$$

Then consider a corresponding parcel of gas originating at a radius $R + \lambda/2$ which conserves its ϕ velocity, but moves into a radius $R - \lambda/2$. Its angular momentum at this radius will then be

$$(R - \lambda/2)\,(R + \lambda/2)\,\Omega\,(R + \lambda/2)$$

For an interchange of these two "average" parcels of gas the net transfer of angular momentum from the outer ring to the inner ring will then be

$$(R - \lambda/2)\,(R + \lambda/2)\,[\Omega\,(R + \lambda/2) - \Omega\,(R - \lambda/2)]\,.$$

To first order in λ this simply reduces to $R^2\lambda\partial\Omega/\partial R$, where

$$\frac{\partial\Omega}{\partial R} = \frac{\Omega\,(R + \lambda/2) - \Omega\,(R - \lambda/2)}{\lambda}$$

assuming that $\Omega(R)$ varies only slowly over the length-scale λ.

This is an expression for the angular momentum transfer per unit mass per unit arc length. The net transfer of angular momentum (viscous torque) will then be $R^2\lambda\partial\Omega/\partial R$, multiplied by the rate that mass crosses the boundary for the entire circular ring under consideration ($= 2\pi R\Sigma\bar{v}$). Then the torque exerted by the outer ring on the inner ring is then:

$$H(R,t) = 2\pi R\Sigma\bar{v}R^2\lambda\frac{\partial\Omega}{\partial R} = 2\pi R\Sigma\nu R^2\frac{\partial\Omega}{\partial R}$$

Both of these results, (2) and (3), are fairly general. Our assumption of a *very* massive star and hence Keplerian motion for the gas cloud provides us with:

$$\Omega(R) = \left[\frac{GM}{R^3}\right]^{1/2} \tag{4}$$

and then we can use (4) to eliminate $\Omega(R)$ and use (1) and (2) to eliminate $v_R(R,t)$. This is best done by rewriting (2) using (1) as:

$$2\pi R v_R \Sigma \frac{\partial}{\partial R}(R^2\Omega) = \frac{\partial H}{\partial R} \tag{2'}$$

and then substituting (2') and (3) into (1), finally provides:

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial}{\partial R}(R^2\Omega)\right)^{-1}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right)\right]$$

and the final elimination of Ω generates:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\nu\Sigma)\right]$$

If we further assume that ν is a constant, then this reduces to:

$$\frac{\partial \Sigma}{\partial t} = 3\nu\left[\frac{\partial^2\Sigma}{\partial R^2} + \frac{3}{2}\frac{1}{R}\frac{\partial\Sigma}{\partial R}\right]$$

and we are free to rescale our variables as we see fit. In fact:

$$\Sigma = \frac{m_0 z}{\pi R_0^2} \qquad R = R_0 x \qquad t = \frac{R_0^2}{3\nu}\tau$$

gives:

$$\frac{\partial \Sigma}{\partial \tau} = \frac{\partial^2\Sigma}{\partial x^2} + \frac{3}{2}\frac{1}{x}\frac{\partial\Sigma}{\partial x}$$

and a judicious choice of m_0 and R_0 allows:

$$\int_0^\infty dx x^p z(x,0) = 1$$

for a judicious choice of p, a unit particle number and a special length to be unity; either the black-hole radius, if present, or the average initial radius of the dust cloud.

There are *two* useful representations for this partial differential equation:

$$\frac{\partial z}{\partial t} = x^{-1}\frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial x} + \frac{1}{2}z\right] \tag{E1}$$

$$\frac{\partial z}{\partial t} = x^{-3/2}\frac{\partial}{\partial x}\left[x^{3/2}\frac{\partial z}{\partial x}\right] \tag{E2}$$

Each of these representations has an underlying conservation law:

The conservation of mass is represented by (E1). Using the 'normal' measure in two dimensions:

$$M_2(f) = 2\pi\int_{x_1}^{x_2} dx x f(x)$$

we can immediately find a particle number:

$$N(x_1, x_2; t) = 2\pi \int_{x_1}^{x_2} dx x z(x, t)$$

and an associated particle flux:

$$\phi(x, t) = 2\pi \left[x \frac{\partial z}{\partial x}(x, t) + \frac{1}{2} z(x, t) \right]$$

The change in number in any interval is then directly related to the incoming and outgoing particle fluxes via:

$$\frac{\partial N}{\partial t}(x_1, x_2; t) = 2\pi \int_{x_1}^{x_2} dx x \frac{\partial z}{\partial t} = \phi(x_2, t) - \phi(x_1, t)$$

For a system with a vanishing number of particles at ∞, and a boundary condition at x^*, the *total* particle number is simply:

$$N(x^*, \infty; t) = N(x^*, \infty; 0) - \int_0^t dt' \phi(x^*, t')$$

The natural 'mass' boundary conditions are:

$$\phi(x^*, t) = \alpha z(x^*, t)$$

with $\alpha = 0$ corresponding to *reflecting* boundary conditions and conservation of mass. These boundary conditions may be rescaled to:

$$z(x^*, t) = \beta \frac{\partial z}{\partial x}(x^*, t)$$

with $\beta = 2\pi x^*/(\alpha - \pi)$.

The conservation of angular momentum is represented by (E2). Using a 'scaled' measure:

$$M_{5/2}(f) = S(5/2) \int_{x_1}^{x_2} dx x^{3/2} f(x)$$

we can find an angular momentum variable:

$$A(x_1, x_2; t) = S(5/2) \int_{x_1}^{x_2} dx x^{3/2} z(x, t)$$

and an associated flux:

$$\psi(x, t) = S(5/2) x^{3/2} \left[\frac{\partial z}{\partial x}(x, t) \right]$$

The change in angular momentum in any interval is then directly related to the outgoing and incoming fluxes by:

$$\frac{\partial A}{\partial t}(x_1, x_2; t) = S(5/2) \int_{x_1}^{x_2} dx x^{3/2} \frac{\partial z}{\partial t} = \psi(x_2, t) - \psi(x_1, t)$$

For a system with a vanishing density of particles at ∞, and a boundary condition at x^*, the *total* angular momentum is simply:

$$A(x^*, \infty; t) = A(x^*, \infty; 0) - \int_0^t dt' \psi(x^*, t')$$

The natural 'angular momentum' boundary conditions are:

$$\psi(x^*, t) = \gamma z(x^*, t)$$

with $\gamma = 0$ corresponding to *reflecting* boundary conditions and conservation of angular momentum. These boundary conditions may be rescaled to:

$$z(x^*, t) = \beta \frac{\partial z}{\partial x}(x^*, t)$$

with $\beta = S(5/2)x^{*3/2}/\gamma$. Note that:

$$\gamma = \frac{S(5/2)x^{*1/2}}{2\pi}(\alpha - \pi)$$

and that we *cannot* achieve $\alpha = 0 = \gamma$ unless x^* is pathalogical.