Mathematics 27: The Two (and Three) Body Problem(s)

In this section we will develop the simple classical gravitational problem, solving the two-body problem and reformulating the three body problem into it's natural limits. We use Lagrange's formulation to establish the equations of motion; $\hat{\mathbf{L}} = \hat{\mathbf{T}} - \hat{\mathbf{V}}$:

$$\hat{\mathbf{T}} = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i.\dot{\mathbf{r}}_i$$

$$\hat{\mathbf{V}} = -\frac{1}{2}\sum_{ij} \frac{G m_i m_j}{\mid \mathbf{r}_i - \mathbf{r}_j \mid}$$

and then the equations of mation are:

$$\frac{d}{dt}\left[\frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i}\right] = \frac{\partial \hat{\mathbf{L}}}{\partial \mathbf{r}_i} \qquad \Rightarrow \qquad m_i \ddot{\mathbf{r}}_i = -\sum_j \frac{G m_i m_j (\mathbf{r}_i - \mathbf{r}_j)}{\mid \mathbf{r}_i - \mathbf{r}_j \mid^3} \tag{1}$$

There are 'two' obvious conservation laws: Conservation of *total* angular momentum:

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$$

with:

$$\frac{d\mathbf{L}}{dt} = \sum_i m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_{ij} \frac{G m_i m_j (\mathbf{r}_i \times \mathbf{r_j})}{\mid \mathbf{r}_i - \mathbf{r}_j \mid^3} = \mathbf{0}$$

and conservation of energy:

$$E = \sum_i \dot{\mathbf{r}}_i . \frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i} - \hat{\mathbf{L}} = \hat{\mathbf{T}} + \hat{\mathbf{V}}$$

with:

$$\frac{dE}{dt} = \frac{d}{dt}\left[\sum_i \dot{\mathbf{r}}_i . \frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i} - \hat{\mathbf{L}}\right] = \sum_i \ddot{\mathbf{r}}_i . \frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i} + \sum_i \dot{\mathbf{r}}_i . \frac{d}{dt}\left[\frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i}\right] - \frac{d\hat{\mathbf{L}}}{dt}$$

$$= \sum_i \ddot{\mathbf{r}}_i . \frac{\partial \hat{\mathbf{L}}}{\partial \dot{\mathbf{r}}_i} + \sum_i \dot{\mathbf{r}}_i . \frac{\partial \hat{\mathbf{L}}}{\partial \mathbf{r}_i} - \frac{d\hat{\mathbf{L}}}{dt} = 0$$

The Two-Body Problem

The total momentum is best separated via:

$$M\mathbf{R}_1 = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \qquad \mathbf{R}_2 = \mathbf{r}_2 - \mathbf{r}_1$$

in terms of the total mass, $M = m_1 + m_2$, and then:

$$M\ddot{\mathbf{R}}_1 = \mathbf{0}$$

for the centre of mass motion, combined with:

$$\ddot{\mathbf{R}}_2 = -\frac{G(m_1+m_2)\mathbf{R}_2}{\mid \mathbf{R}_2 \mid^3} \tag{2}$$

for the relative motion. The generic problem to solve is therefore:

$$\ddot{\mathbf{R}} = -\lambda \frac{\mathbf{R}}{\mid \mathbf{R} \mid^3}$$

Firstly;

$$\frac{d}{dt}(\mathbf{R} \times \dot{\mathbf{R}}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{R} \times \dot{\mathbf{R}} = \mathbf{h}$$

is conservation of angular momentum with constant $\mathbf{h}$. Consequently,

$$\mathbf{h}.\ddot{\mathbf{R}} = 0 \quad \Rightarrow \quad \mathbf{h}.\mathbf{R}(t) = \mathbf{h}.\mathbf{R}(0) + \mathbf{h}.\dot{\mathbf{R}}(0)t \equiv \mathbf{0}$$

and $\mathbf{R}(t)$ is restricted to a plane perpendicular to $\mathbf{h}$ and passing through $\mathbf{R} = \mathbf{0}$. This problem is readily solved in polar coordinates confined to that plane:

$$\mathbf{R} = r\hat{\mathbf{e}}_r \qquad \dot{\mathbf{R}} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta \qquad \ddot{\mathbf{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta$$

and so:

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0 \quad \Rightarrow \quad r^2\dot{\theta} = h$$

combined with:

$$\ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{h^2}{r^3} = -\frac{\lambda}{r^2}$$

Conservation of energy is the first integral of this, providing us with:

$$\dot{r}^2 = \frac{2\lambda}{r} - \frac{h^2}{r^2} + E$$

where E is a constant. The 'best' way to represent this equation is in terms of *two* parameters, r_0 and r_1, via:

$$\left[\frac{dr}{dt}\right]^2 = \dot{r}^2 = h^2\frac{(r_1 - r)(r - r_0)}{r^2 r_0 r_1}$$

and $r_1 > 0$, $r_0 > 0$ (bound) $r_0 < 0$ (unbound), and subject to:

$$h^2\left[\frac{1}{r_0} + \frac{1}{r_1}\right] = 2\lambda$$

For a bound orbit, $r_0 \le r \le r_1$, and hence the parameters r_0 and r_1 denote the *limits* of the orbit.

We are left to solve:

$$\frac{dr}{dt} = \pm\frac{h}{r}\left(\left[\frac{r}{r_0} - 1\right]\left[1 - \frac{r}{r_1}\right]\right)^{1/2}$$

for the radius in terms of time, and:

$$\frac{dr}{d\theta} = \pm r \left(\left[\frac{r}{r_0} - 1 \right] \left[1 - \frac{r}{r_1} \right] \right)^{1/2}$$

for the radius in terms of the angle. The first is solvable by substitution:

$$r = \frac{1}{2}(r_0 + r_1) - \frac{1}{2}(r_1 - r_0)\cos a$$

in terms of which:

$$\pm h \frac{dt}{da} = \frac{1}{2}\sqrt{r_0}\sqrt{r_1}\left[(r_0 + r_1) - (r_1 - r_0)\cos a\right]$$

and so:

$$\pm 2ht = \sqrt{r_0}\sqrt{r_1}\left[(r_0 + r_1)a - (r_1 - r_0)\sin a\right]$$

The second is solvable via:

$$\frac{d[1/r]}{d\theta} = \pm \left(\left[\frac{1}{r_0} - \frac{1}{r} \right] \left[\frac{1}{r} - \frac{1}{r_1} \right] \right)^{1/2}$$

and so:

$$\frac{1}{r} = \frac{1}{2}\left(\frac{1}{r_0} + \frac{1}{r_1} - \left[\frac{1}{r_1} - \frac{1}{r_0} \right] \cos(\theta - \theta_0) \right)$$

which solves the angular dependence completely. A half-period occurs when a ranges from 0 to π and then:

$$2hT = (r_0 + r_1)\pi\sqrt{r_0}\sqrt{r_1} \quad \Rightarrow \quad T = \frac{(r_0 + r_1)\pi\sqrt{r_0}\sqrt{r_1}}{2h} = \left(\frac{1}{2}(r_0 + r_1) \right)^{3/2} \frac{\pi}{\sqrt{\lambda}}$$

Note that we can transform between representations, ie boundary conditions, via:

$$v_0 = r_0\dot{\theta}_0 = \frac{h}{r_0} = \left(\frac{2\lambda r_1}{r_0(r_0 + r_1)} \right)^{1/2} \quad \Rightarrow \quad \frac{1}{r_1} = \frac{2\lambda}{r_0^2 v_0^2} - \frac{1}{r_0}$$

and hence escape velocity is $v_e^2 = \frac{2\lambda}{r_0}$, which is *twice* the square velocity for *circular* motion, $v_c^2 = \frac{\lambda}{r_0}$.

Two-Body Conservation Laws

The trajectory involved in two-body motion can be well described using solely the two conservation laws of: Angular momentum and Energy:

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \tag{A}$$

$$E = \frac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{G(M + m)}{\mid \mathbf{r} \mid} \tag{E}$$

At the exremes of the radial motion, the radial velocity vanishes and $(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{h})$ are all mutually orthogonal. In terms of the minimal radius, subscript 0, and maximal radius, subscript 1, the conservation laws become:

$$h = r_0 \dot{r}_0 = r_1 \dot{r}_1$$

$$E = \frac{1}{2}\dot{r}_0^2 - \frac{G(M+m)}{r_0} = \frac{1}{2}\dot{r}_1^2 - \frac{G(M+m)}{r_1}$$

which can be solved to provide:

$$\dot{r}_0^2 = \frac{2r_1 G(M+m)}{r_0(r_0+r_1)} \qquad \dot{r}_1^2 = \frac{2r_0 G(M+m)}{r_1(r_0+r_1)}$$

and hence:

$$E = -\frac{G(M+m)}{(r_0+r_1)} \qquad h^2 = \frac{2r_0 r_1 G(M+m)}{(r_0+r_1)}$$

in terms of the extremal radii. To escape from a trajectory, sufficient energy must be given to make the total energy *positive.*

To escape from the Earth, the satellite is *not* in a trajectory. First one need establish that the velocity of the Earth's surface is negligible in comparison to the velocity required to *orbit* at the Earth's surface. Next one can consider the satelite to have a 'standing start' and so:

$$E_{Earth-Escape} \sim -\frac{G(M_{Earth} + m_{Satellite})}{r_{Earth-Surface}} \sim -0.626 \times 10^8 m^2 s^{-2}$$

Last, the details of air-resistance need also be considered, but this is a separate issue. To escape from the new orbit just 'outside' the pull of the Earth, an additional energy of:

$$E_{Sun-Escape} \sim -\frac{G(M_{Sun} + m_{Satellite})}{2r_{Earth-Orbit}} \sim -0.4425 \times 10^9 m^2 s^{-2}$$

need be provided. If we seek to gain some of this energy from an interaction with Jupiter, then we need only provide the difference between $E_{Sun-Escape}$ and:

$$E_{Jupiter} \sim -\frac{G(M_{Sun} + m_{Satellite})}{(r_{Earth-Orbit} + r_{Jupiter-Orbit})} \sim -0.1427 \times 10^9 m^2 s^{-2}$$

a clear saving of a sizeable fraction of the total. This final energy, $E_{Jupiter}$, must be provided by the interaction with Jupiter.

Data

$G =$	$0.667 \times 10^{-10} m^3 kg^{-1} s^{-2}$
$M_{Earth} =$	$0.5976 \times 10^{25} kg$
$M_{Jupiter} =$	$0.1903 \times 10^{28} kg$
$M_{Sun} =$	$0.1984 \times 10^{31} kg$
$r_{Earth-Surface} =$	$0.6368 \times 10^7 m$
$r_{Jupiter-Surface} =$	$0.6985 \times 10^8 m$

$r_{Earth-Orbit} =$ $0.1495 \times 10^{12} m$
$r_{Jupiter-Orbit} =$ $0.7778 \times 10^{12} m$
$GM_{Earth} =$ $0.3986 \times 10^{15} m^3 s^{-2}$
$GM_{Sun} =$ $0.1323 \times 10^{21} m^3 s^{-2}$

The Three-Body Problem

There are three equations of motion:

$$m_1 \ddot{\mathbf{r}}_1 = -\frac{Gm_1 m_2}{\mid \mathbf{r}_1 - \mathbf{r}_2 \mid^3}(\mathbf{r}_1 - \mathbf{r}_2) - \frac{Gm_1 m_3}{\mid \mathbf{r}_1 - \mathbf{r}_3 \mid^3}(\mathbf{r}_1 - \mathbf{r}_3)$$

$$m_2 \ddot{\mathbf{r}}_2 = -\frac{Gm_2 m_3}{\mid \mathbf{r}_2 - \mathbf{r}_3 \mid^3}(\mathbf{r}_2 - \mathbf{r}_3) - \frac{Gm_2 m_1}{\mid \mathbf{r}_2 - \mathbf{r}_1 \mid^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

$$m_3 \ddot{\mathbf{r}}_3 = -\frac{Gm_3 m_1}{\mid \mathbf{r}_3 - \mathbf{r}_1 \mid^3}(\mathbf{r}_3 - \mathbf{r}_1) - \frac{Gm_3 m_2}{\mid \mathbf{r}_3 - \mathbf{r}_2 \mid^3}(\mathbf{r}_3 - \mathbf{r}_2)$$

from which the centre-of-mass mation may be extracted:

$$M\mathbf{R}_1 = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 \qquad \mathbf{R}_2 = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{R}_3 = \mathbf{r}_3 - \mathbf{r}_1$$

and inverting:

$$\mathbf{r}_1 = \mathbf{R}_1 - \frac{m_2}{M}\mathbf{R}_2 - \frac{m_3}{M}\mathbf{R}_3 \quad \mathbf{r}_2 = \mathbf{R}_1 + \frac{m_1 + m_3}{M}\mathbf{R}_2 - \frac{m_3}{M}\mathbf{R}_3 \quad \mathbf{r}_3 = \mathbf{R}_1 - \frac{m_2}{M}\mathbf{R}_2 + \frac{m_1 + m_2}{M}\mathbf{R}_3$$

in terms of the total-mass; $M = m_1 + m_2 + m_3$, we find:

$$M\ddot{\mathbf{R}}_1 = \mathbf{0}$$

$$\ddot{\mathbf{R}}_2 = -\frac{G(m_1 + m_2)}{\mid \mathbf{R}_2 \mid^3}\mathbf{R}_2 - \frac{Gm_3}{\mid \mathbf{R}_3 \mid^3}\mathbf{R}_3 - \frac{Gm_3}{\mid \mathbf{R}_2 - \mathbf{R}_3 \mid^3}(\mathbf{R}_2 - \mathbf{R}_3) \tag{3}$$

$$\ddot{\mathbf{R}}_3 = -\frac{G(m_1 + m_3)}{\mid \mathbf{R}_3 \mid^3}\mathbf{R}_3 - \frac{Gm_2}{\mid \mathbf{R}_2 \mid^2}\mathbf{R}_2 - \frac{Gm_2}{\mid \mathbf{R}_3 - \mathbf{R}_2 \mid^3}(\mathbf{R}_3 - \mathbf{R}_2) \qquad (3)$$

This provides us with *six* second-order differential equations and even after extracting the total-angular momentum and total-energy, we are still left with *eight* degrees of freedom to solve: A sizeable problem.

Although this problem is numerically tractable, there are a wide range of limits of physical interest to astrophysical problems.

The first simplification is to two-dimensional planar motion, in which case polar-coordinates are useful:

$$\frac{d}{dt}\left(R_2^2\dot{\theta}_2\right) = Gm_3 \sin(\theta_3 - \theta_2) R_2 R_3 \left[\frac{1}{X^3} - \frac{1}{R_3^3}\right]$$

$$\frac{d}{dt}\left(R_3^2\dot{\theta}_3\right) = Gm_3 \sin(\theta_2 - \theta_3) R_2 R_3 \left[\frac{1}{X^3} - \frac{1}{R_2^3}\right]$$

$$\ddot{R}_2 - R_2\dot{\theta}_2^2 = -G\left[\frac{m_3}{X^3} + \frac{m_1+m_2}{R_2^3}\right]R_2 + Gm_3\cos(\theta_3-\theta_2)R_3\left[\frac{1}{X^3} - \frac{1}{R_3^3}\right]$$
$$\ddot{R}_3 - R_3\dot{\theta}_3^2 = -G\left[\frac{m_2}{X^3} + \frac{m_1+m_3}{R_3^3}\right]R_2 + Gm_2\cos(\theta_2-\theta_3)R_2\left[\frac{1}{X^3} - \frac{1}{R_2^3}\right]$$
$$X^2 = R_2^2 + R_3^2 - 2R_2R_3\cos(\theta_2-\theta_3)$$

For the case $m_3 \mapsto 0$, viz a man-made satellite in a planetary system, the motion of the planets is unaffected by the satellite to leading order, yielding a two-body problem:

$$\ddot{\mathbf{R}}_2 = -G(m_1+m_2)\frac{\mathbf{R}_2}{\mid \mathbf{R}_2 \mid^3}$$

solved previously, and:

$$\ddot{\mathbf{R}}_3 = -Gm_1\frac{\mathbf{R}_3}{\mid \mathbf{R}_3 \mid^3} - Gm_2\left[\frac{\mathbf{R}_2}{\mid \mathbf{R}_2 \mid^3} + \frac{\mathbf{R}_3 - \mathbf{R}_2}{\mid \mathbf{R}_3 - \mathbf{R}_2 \mid^3}\right]$$

for the motion of the satellite.

The corresponding conservation laws are:

$$\mathbf{h} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 + m_3\mathbf{r}_3 \times \dot{\mathbf{r}}_3$$
$$= m_2\left[\frac{m_1+m_3}{M}\right]\mathbf{R}_2 \times \dot{\mathbf{R}}_2 + m_3\left[\frac{m_1+m_2}{M}\right]\mathbf{R}_3 \times \dot{\mathbf{R}}_3 - \left[\frac{m_2m_3}{M}\right]\left(\mathbf{R}_3 \times \dot{\mathbf{R}}_2 + \mathbf{R}_2 \times \dot{\mathbf{R}}_3\right)$$
$$\mapsto \frac{m_1m_2}{m_1+m_2}\left[\mathbf{R}_2 \times \dot{\mathbf{R}}_2 + \boldsymbol{\delta}\mathbf{R}_2 \times \dot{\mathbf{R}}_2 + \mathbf{R}_2 \times \boldsymbol{\delta}\dot{\mathbf{R}}_2\right]$$
$$+m_3\left[\mathbf{R}_3 - \frac{m_2}{m_1+m_2}\mathbf{R}_2\right] \times \left[\dot{\mathbf{R}}_3 - \frac{m_2}{m_1+m_2}\dot{\mathbf{R}}_2\right]$$

in the limit that $m_3 \mapsto 0$, where $\boldsymbol{\delta}\mathbf{R}_2$ is the leading order correction to the motion of the planet caused by the satellite, and:

$$E = \frac{1}{2}m_1\dot{\mathbf{r}}_1.\dot{\mathbf{r}}_1 + \frac{1}{2}m_2\dot{\mathbf{r}}_2.\dot{\mathbf{r}}_2 + \frac{1}{2}m_3\dot{\mathbf{r}}_3.\dot{\mathbf{r}}_3 - \frac{Gm_1m_2}{\mid \mathbf{r}_1 - \mathbf{r}_2 \mid} - \frac{Gm_1m_3}{\mid \mathbf{r}_1 - \mathbf{r}_3 \mid} - \frac{Gm_2m_3}{\mid \mathbf{r}_2 - \mathbf{r}_3 \mid}$$
$$= \frac{1}{2}\left[m_2\frac{m_1+m_3}{M}\dot{\mathbf{R}}_2.\dot{\mathbf{R}}_2 + m_3\frac{m_1+m_2}{M}\dot{\mathbf{R}}_3.\dot{\mathbf{R}}_3 - 2\frac{m_2m_3}{M}\dot{\mathbf{R}}_2.\dot{\mathbf{R}}_3\right]$$
$$-\frac{Gm_1m_2}{\mid \mathbf{R}_2 \mid} - \frac{Gm_1m_3}{\mid \mathbf{R}_3 \mid} - \frac{Gm_2m_3}{\mid \mathbf{R}_2 - \mathbf{R}_3 \mid}$$
$$\mapsto \frac{m_1m_2}{m_1+m_2}\left[\frac{1}{2}\dot{\mathbf{R}}_2.\dot{\mathbf{R}}_2 + \boldsymbol{\delta}\dot{\mathbf{R}}_2.\dot{\mathbf{R}}_2\right] - \frac{Gm_1m_2}{\mid \mathbf{R}_2 \mid} - Gm_1m_2\boldsymbol{\delta}\mathbf{R}_2.\nabla\frac{1}{\mid \mathbf{R}_2 \mid}$$
$$+\frac{1}{2}m_3 \mid \dot{\mathbf{R}}_3 - \frac{m_2}{m_1+m_2}\dot{\mathbf{R}}_2 \mid^2 - \frac{Gm_1m_3}{\mid \mathbf{R}_3 \mid} - \frac{Gm_3m_2}{\mid \mathbf{R}_3 - \mathbf{R}_2 \mid}$$

in the limit $m_3 \mapsto 0$.

The role of the correction to the planetary motion is *relevant* to these conservation laws:

$$\boldsymbol{\delta}\ddot{\mathbf{R}}_2 + G(m_1+m_2)\boldsymbol{\delta}\mathbf{R}_2.\nabla\frac{\mathbf{R}_2}{\mid \mathbf{R}_2 \mid^3} = -Gm_3\left[\frac{\mathbf{R}_3}{\mid \mathbf{R}_3 \mid^3} + \frac{\mathbf{R}_2 - \mathbf{R}_3}{\mid \mathbf{R}_2 - \mathbf{R}_3 \mid^3}\right]$$

which is another orbital problem to solve, involving the previously solved satellite motion.