Mathematics 7: Laplace's Equation

We solve Laplace's equation on a square with a function provided on the boundary: The soap bubble problem. Mathematically:

$$\nabla^2 \phi(x,y) = 0$$

with $\phi(0,y)$, $\phi(1,y)$, $\phi(x,0)$, $\phi(x,1)$ provided. Since the equation is *linear*, we can solve using the superposition principle:

$$\phi(0,y) = \phi(1,y) = \phi(x,0) = 0 \qquad \phi(x,1) = f(x)$$

These boundary conditions are solved by:

$$\phi(x,y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}$$

where:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

being a Fourier series, with inverse:

$$a_n = 2\int_0^1 dx f(x) \sin n\pi x$$

We consider two cases:

(i) $f(x) = 1$

$$a_n = 2\int_0^1 dx \sin n\pi x = \frac{2}{\pi n}\Big[(-1)\cos n\pi x\Big]_0^1 = \frac{2[1-(-1)^n]}{n\pi}$$

and so:

$$\phi(x,y) = \sum_{m=0}^{\infty} \frac{4}{\pi[1+2m]} \sin[1+2m]\pi x \frac{\sinh[1+2m]\pi y}{\sinh[1+2m]\pi}$$

(ii) $f(x) = 2x\theta\left[\frac{1}{2} - x\right] + 2(1-x)\theta\left[x - \frac{1}{2}\right]$

$$a_n = 2\int_0^{1/2} dx 2x \sin n\pi x + 2\int_{1/2}^1 dx 2(1-x)\sin n\pi x = 2(1-(-1)^n)\int_0^{1/2} dx 2x \sin n\pi x$$

$$= \frac{4(1-(-1)^n)}{n\pi}\Big[(-1)x\cos n\pi x\Big]_0^{1/2} + \frac{4(1-(-1)^n)}{n\pi}\int_0^{1/2} dx \cos n\pi x$$

$$= \frac{4(1-(-1)^n)}{n^2\pi^2}\Big[\sin n\pi x\Big]_0^{1/2} = \frac{4(1-(-1)^n)}{n^2\pi^2} \sin\frac{n\pi}{2}$$

and so:

$$\phi(x,y) = \sum_{m=0}^{\infty} \frac{8(-1)^m}{\pi^2[1+2m]^2} \sin[1+2m]\pi x \frac{\sinh[1+2m]\pi y}{\sinh[1+2m]\pi}$$