Mathematics 33: Transfer Matrices

Statistical mechanics can be formulated in terms of the partition function:

$$Z(\beta) = \sum_n \exp\left(-\beta \epsilon_n\right)$$

where $\beta = 1/k_B T$ and ϵ_n are the energies of the system. Measurements are described by statistical averages:

$$< \hat{O} >= \frac{1}{Z} \sum_n < n \mid \hat{O} \mid n > \exp\left(-\beta \epsilon_n\right)$$

where $\hat{O}$ is the operator describing the quantity to be measured.

Perhaps the simplest model of magnetism is the Ising model:

$$H = \sum_{ij} J_{i-j} \sigma_i \sigma_j$$

which describes spin-half atoms which have two possible states denoted by $\sigma = \pm 1$ and which only interact with each-other in one spin direction. The matrix elements J_n denote the strengths of these interactions. The partition function becomes:

$$Z(\beta) = \sum_{\{\boldsymbol{\sigma}\}} \exp\left[-\beta \sum_{in} J_n \sigma_i \sigma_{i+n}\right]$$

and in one-dimension with a *finite-range* for the interactions this quantity and the spin-spin correlation function:

$$< \sigma_j \sigma_{j+m} >= \frac{1}{Z} \sum_{\{\boldsymbol{\sigma}\}} \sigma_j \sigma_{j+m} \exp\left[-\beta \sum_{in} J_n \sigma_i \sigma_{i+n}\right]$$

can be evaluated with the help of *transfer matrices.*

The basic idea is to consider the quantity:

$$A_m\left[\sigma_m, \sigma_{m+1}, ..., \sigma_{m+N-1}\right] = \sum_{\{\sigma_1, ..., \sigma_{m-1}\}} \exp\left[-\beta \sum_{i=1}^{m-1} \sum_{n=1}^{N} J_n \sigma_i \sigma_{i+n}\right]$$

which includes all contributions involving $\{\sigma_1, ..., \sigma_{m-1}\}$. The next contribution can be included with:

$$A_{m+1}\left[\sigma_{m+1}, \sigma_{m+2}, ..., \sigma_{m+N}\right] = \sum_{\sigma_m = \pm} \exp\left[-\beta \sigma_m \sum_{m=1}^{N} J_n \sigma_{m+n}\right] A_m\left[\sigma_m, \sigma_{m+1}, ..., \sigma_{m+N-1}\right]$$

which, in terms of the basis of 2^N possible states, can be represented by a matrix equation:

$$A_{m+1} = T A_m = T^{m+1} A_0$$

where T does *not* depend explicitely on m. In the diagonal basis:

$$T = \sum_\alpha R_\alpha t_\alpha L_\alpha^\dagger$$

in terms of the eigenvectors:

$$L_\alpha^\dagger T = L_\alpha^\dagger t_\alpha \qquad T R_\alpha = t_\alpha R_\alpha \qquad L_\alpha^\dagger R_\beta = \delta_{\alpha\beta}$$

and then when $m \mapsto \infty$:

$$A_m \mapsto \tilde{R}\tilde{t}^m \tilde{L}^\dagger A_0$$

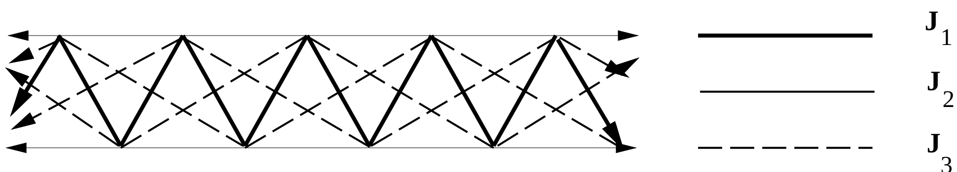
where $\tilde{t}$ is the *maximum* eigenvalue.

Provided that the boundary effects are negligible then:

$$< \sigma_j \sigma_{j+m} > \mapsto \frac{A_\infty^\dagger \tilde{R}\tilde{t}^j \tilde{L}^\dagger \sigma_j T^m \sigma_{j+m} \tilde{R}\tilde{t}^{R-j-m}\tilde{L}^\dagger A_0}{A_\infty^\dagger \tilde{R}\tilde{t}^R \tilde{L}^\dagger A_0}$$

$$\mapsto \tilde{L}^\dagger \sigma_j \left[\frac{T}{t}\right]^m \sigma_{j+m}\tilde{R} \mapsto \sum_\alpha \tilde{L}^\dagger \sigma_j R_\alpha \left[\frac{t_\alpha}{\tilde{t}}\right]^m L_\alpha^\dagger \sigma_{j+m}\tilde{R}$$

Example: One-dimensional chain



$$z_1 = \exp(\beta J_1) \qquad z_2 = \exp(\beta J_2) \qquad z_3 = \exp(\beta J_3)$$

(i) $J_2 = 0 = J_3$

$$\begin{bmatrix} + \\ - \end{bmatrix} \quad T = \begin{bmatrix} \frac{1}{z_1} & z_1 \\ z_1 & \frac{1}{z_1} \end{bmatrix} \quad \sigma \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \tilde{L}^\dagger = \frac{1}{2}[1 \quad 1] \quad \tilde{R} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and in the diagonal basis:

$$T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\frac{1}{z_1} + z_1}{2} [1 \quad 1] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{\frac{1}{z_1} - z_1}{2} [1 \quad -1]$$

The correlation functions are:

$$< \sigma_j \sigma_{m+j} > = \left[\frac{\frac{1}{z_1} - z_1}{\frac{1}{z_1} + z_1}\right]^m$$

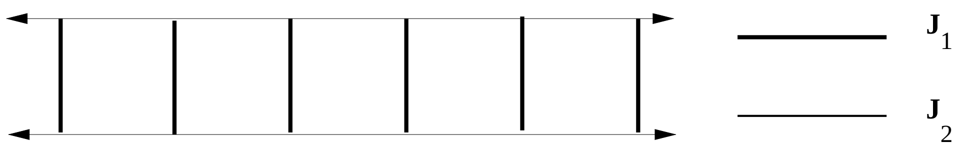
(ii) $J_3 = 0$

$$\begin{bmatrix} ++ \\ +- \\ -+ \\ -- \end{bmatrix} \qquad T = \begin{bmatrix} \frac{1}{z_1 z_2} & 0 & z_1 z_2 & 0 \\ \frac{z_2}{z_1} & 0 & \frac{z_1}{z_2} & 0 \\ 0 & \frac{z_1}{z_2} & 0 & \frac{z_2}{z_1} \\ 0 & z_1 z_2 & 0 & \frac{1}{z_1 z_2} \end{bmatrix}$$

(iii)

$$\begin{bmatrix} +++ \\ ++- \\ +-+ \\ +-- \\ -++ \\ -+- \\ --+ \\ --- \end{bmatrix} \qquad T = \begin{bmatrix} \frac{1}{z_1 z_2 z_3} & 0 & 0 & 0 & z_1 z_2 & 0 & 0 & 0 \\ \frac{z_3}{z_1 z_2} & 0 & 0 & 0 & \frac{z_1 z_2}{z_3} & 0 & 0 & 0 \\ 0 & \frac{z_2}{z_1 z_3} & 0 & 0 & 0 & \frac{z_1 z_3}{z_2} & 0 & 0 \\ 0 & \frac{z_2 z_3}{z_1} & 0 & 0 & 0 & \frac{z_1}{z_2 z_3} & 0 & 0 \\ 0 & 0 & \frac{z_1}{z_2 z_3} & 0 & 0 & 0 & \frac{z_2 z_3}{z_1} & 0 \\ 0 & 0 & \frac{z_1 z_3}{z_2} & 0 & 0 & 0 & \frac{z_2}{z_1 z_3} & 0 \\ 0 & 0 & 0 & \frac{z_1 z_2}{z_3} & 0 & 0 & 0 & \frac{z_3}{z_1 z_2} \\ 0 & 0 & 0 & z_1 z_2 & 0 & 0 & 0 & \frac{1}{z_1 z_2 z_3} \end{bmatrix}$$

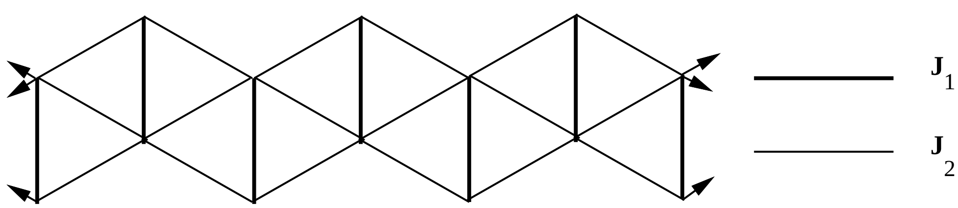
Example: Ladder geometry:



$$z_1 = \exp(\beta J_1) \qquad z_2 = \exp(\beta J_2)$$

$$\begin{bmatrix} ++ \\ +- \\ -+ \\ -- \end{bmatrix} \qquad T = \begin{bmatrix} \frac{1}{z_1 z_2^2} & z_1 & z_1 & \frac{z_2^2}{z_1} \\ \frac{1}{z_1} & \frac{z_1}{z_2^2} & z_1 z_2^2 & \frac{1}{z_1} \\ \frac{1}{z_1} & z_1 z_2^2 & \frac{z_1}{z_2^2} & \frac{1}{z_1} \\ \frac{z_2^2}{z_1} & z_1 & z_1 & \frac{1}{z_1 z_2^2} \end{bmatrix}$$

Example: Tricky Geometry:



$$z_1 = \exp(\beta J_1) \qquad z_2 = \exp(\beta J_2)$$

$$\begin{bmatrix} ++ \\ +- \\ -+ \\ -- \end{bmatrix} \qquad T = \begin{bmatrix} \frac{1}{z_1^2 z_2^2} & z_1^2 & 1 & z_2^2 \\ 1 & z_2^2 & \frac{z_1^2}{z_2^2} & \frac{1}{z_1^2} \\ \frac{1}{z_1^2} & \frac{z_1^2}{z_2^2} & z_2^2 & 1 \\ z_2^2 & 1 & z_1^2 & \frac{1}{z_1^2 z_2^2} \end{bmatrix}$$