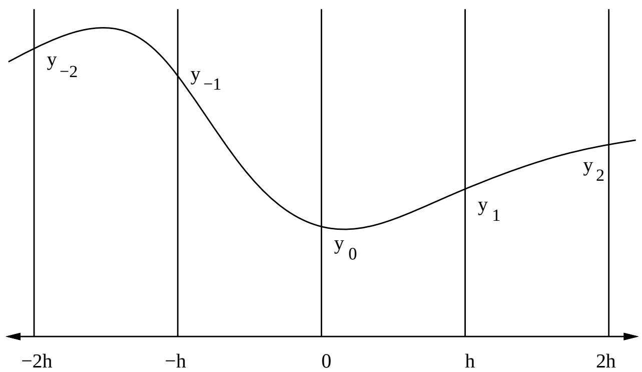
Mathematics 2: Long's Rule

This is another simple method for performing numerical integration. It is a direct generalisation of Simpson's rule, using four intervals and a quartic fit. Using the notation in the diagram:



The interpolation is:

$$y(x) = y_0 + \alpha\left[\frac{x}{h}\right] + \beta\left[\frac{x}{h}\right]^2 + \gamma\left[\frac{x}{h}\right]^3 + \delta\left[\frac{x}{h}\right]^4$$

from which:

$$\begin{aligned} y_{-1} + y_1 &= 2y_0 + 2\beta + 2\delta \\ y_{-2} + y_2 &= 2y_0 + 8\beta + 32\delta \end{aligned}$$

which may be solved to provide:

$$\begin{aligned} 24\delta &= y_{-2} + y_2 - 4(y_{-1} + y_1) + 6y_0 \\ 24\beta &= 16(y_{-1} + y_1) - y_{-2} - y_2 - 30y_0 \end{aligned}$$

The integration is elementary:

$$\int_{-2h}^{2h} dx y(x) = 4hy_0 + \frac{16h}{3}\beta + \frac{64h}{5}\delta = \frac{h}{45}\left[14(y_{-2} + y_2) + 64(y_{-1} + y_1) + 24y_0\right]$$

The application of this rule is also best done by *bisection*. The idea follows the same path as before, but is slightly more complicated. Note that all the previous calculations can be used for the next estimate, so all the work is used in the final estimate. This idea is expressed mathematically by:

$$\begin{aligned} r_1 &= \frac{14h}{45}\left[y(a) + y(a + 4Nh)\right] + \frac{28h}{45}\sum_{n=1}^{N-1} y(a + 4nh) \\ r_2 &= \frac{24h}{45}\sum_{n=1}^{N} y(a - 2h + 4nh) \\ r_3 &= \frac{64h}{45}\sum_{n=1}^{2N} y(a - h + 2nh) \end{aligned}$$

where $b = a + 4Nh$ and the estimate from this approximation is obtained from $r = r_1 + r_2 + r_3$. The next estimate is provided by:

$$
\begin{aligned}
h &\mapsto \frac{h}{2} \\
r_1 &\mapsto \frac{r_1}{2} + \frac{7r_2}{12} \\
r_2 &\mapsto \frac{3r_3}{16}
\end{aligned}
$$

and then the new r_3 is calculated from the values of $y(x)$ at the new bisecting points.