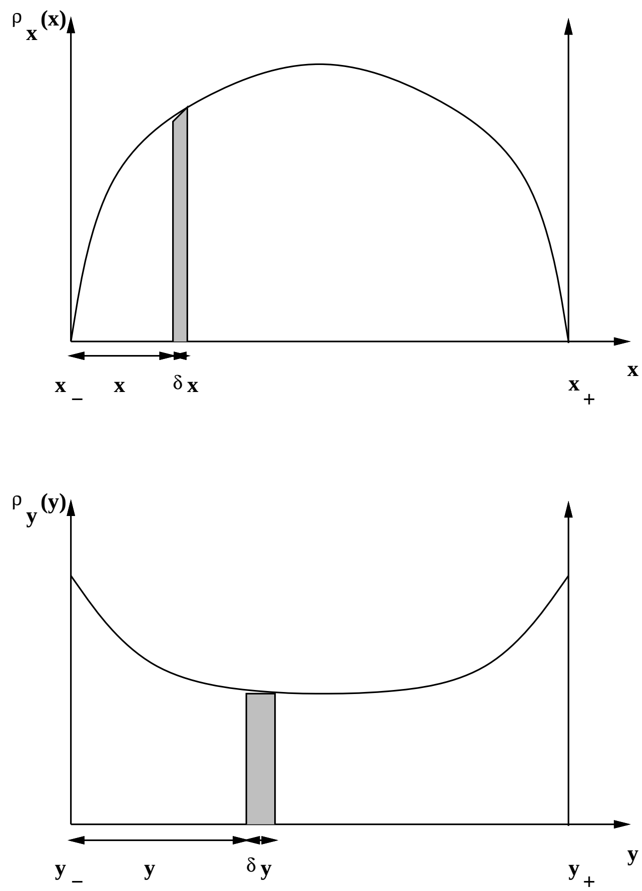
Mathematics 22: Probability Distributions

A probability distribution is a positive function, $\rho_x(x)$, which controls the likelihood of finding the continuous variable, x. The probability of the variable x being in the interval $x \in (x, x+dx)$ is $\rho_x(x)dx$, in the limit that $dx \mapsto 0$. The probability distribution is usually *normalised* to satisfy:

$$\int_{x_-}^{x_+} dx \rho_x(x) = 1$$

in order that the variable x must take some value, where $x \in (x_-, x_+)$ may have a restricted domain.

In this section we want to use sampling from one distribution to yield a second distribution, where both distributions are known. This problem amounts to finding a function $y(x)$ which maps one variable onto the other in such a way that the areas are maintained constant:



Given δx, we need to choose δy so that the two areas depicted are equal and hence that the two probabilities match. This is true if:

$$\delta x \rho_x(x) = \delta y \rho_y(y)$$

which in the limit becomes the differential equation:

$$\frac{dy}{dx} = \frac{\rho_x(x)}{\rho_y(y)}$$

In fact, the form of this problem immediately leads to a solution:

$$\int_{y_-}^{y} dy' \rho_y(y') = \int_{x_-}^{x} dx' \rho_x(x')$$

an implicit relation for y in terms of x.

The particular case of interest is that of forming a new probability distribution from a uniform distribution on $(0,1)$. The uniform distribution is:

$$\rho_y(y) = \theta(y)\theta(1-y)$$

viz unity on the relevant interval. The mapping is therefore:

$$y(x) = \int_{x_-}^{x} dx' \rho_x(x')$$

Given a sequence of y's; $y_1, y_2, y_3 \ldots$ which are uniformly distributed, then the sequence of x's; $x_1, x_2, x_3 \ldots$ which satisfy:

$$y_i = \int_{x_-}^{x_i} dx \rho_x(x)$$

are distributed according to $\rho_x(x)$. The solution is unique, because $\rho_x(x)$ is positive definite.

To employ Newton-Raphson on this problem (see 'Maths 16'), involves using:

$$f(x) = y - \int_{x_-}^{x} dx' \rho_x(x')$$

from which we find:

$$x \mapsto x - \frac{f(x)}{\frac{df}{dx}(x)} = x + \frac{1}{\rho_x(x)} \left[y - \int_{x_-}^{x} dx' \rho_x(x') \right]$$