1 Numerical Integration

Answers to $\star$ question must be submitted to WebCT, as an extensively commented program and any additional text by Wedesday 19/10/11 5pm (usual late penalties will apply).

In solving physics problems, it is frequently the case that once an integration is involved it is no longer possible to carry on solving analytically and we need to turn to numerical integration. There are many functions available in C++ libraries such as GSL or from Numerical Recipes to do this - but before relying on a "black box" integration we will use some simple methods which can be readily programmed by you!

We will start with using the Trapezium rule which approximates the area under a curve between two given points by a trapezium -to remind yourself of this see, for example:
http://mathworld.wolfram.com/TrapezoidalRule.html
http://numericalmethods.eng.usf.edu/topics/trapezoidal_rule.html

1. When testing a computational method it is a good idea to start with a problem for which the answer is known from another source.

   Consider the integral

$$\int_0^1 \frac{1}{(1+x)^2}\mathrm{d}x$$

   (a) Work out the answer analytically.

   (b) Write a program to evaluate this integral which uses the Trapezium rule. It should work over a range of n where n is the number of intervals you have used to divide your integral up.

      i. Print to the screen the width (h) of the interval and the result of the integration, and the difference from the exact result.

      ii. Determine how the error (approximately) varies with the value of h.

2. $\star$

   (a) Evaluate analytically the integral

   $$\int_0^2 \mathrm{e}^{-x}\sin(x)\,\mathrm{dx}$$

   (*the working should be submitted to WebCT as an electronic document in pdf format. For this, and later work, you may find it useful to use Latex, see links under Latex on WebCT, in particular the report template LatexExample.zip.*)

   (b) Generate a set of data for use with xmgrace which will enable you to plot the function $\mathrm{e}^{-x}\sin(x)$ in the range $[0, \pi]$. You should include a labeled figure with your submission to WebCT.

   (c) Write a program that uses the Trapezium rule to evaluate the integral, and determine how the accuracy depends on the number of intervals that you break the integral in to.

   (d) Now using the resource *math1* on WebCT or otherwise, write a program that uses Simpson's rule to evaluate the integral. The program should be accurate to 3,4,5 decimal spaces - where this is a screen input parameter. You must make it clear how you have checked the accuracy!

3. ♠ (♠ fourth years must submit this question in addition). This is a physically motivated problem which involves a change of variables to evaluate the integral numerically . Consider a simple pendulum of length l. It is oscillating with a maximum angle of from the vertical of θ_m. You know that if θ_m is small the pendulum undergoes simple harmonic motion with period $T_0 = 2\pi\sqrt{l/g}$, where g is the acceleration due to gravity. Here we investigate how the period changes when the amplitude is no longer small.

   (a) From conservation of the energy, E, where

   $$E = \frac{1}{2}m(l\dot{\theta})^2 + mgl(1 - \cos\theta),$$

   show that the period of oscillation $T(\theta_m)$ can be expressed as

   $$\frac{T(\theta_m)}{T_0} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_m}}$$

   Your working should be coherent in your WebCT submission.

   (b) Why is this form not a good one to choose for the computational integration? (You may find it helpful to plot the integrand.)

(c) Show in your notebook that it is possible to re-express the integral as

$$\frac{T(\theta_m)}{T_0} = \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\psi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2(\psi)}},$$

Why is this form preferable?

(d) This last integral is actually a function known in the literature as an elliptical integral. However, we will not use this information here. Expand the integrand in powers of $\sin^2(\theta_m/2)$ and integrate it term by term, to obtain the coefficients a and b in the expansion:

$$T(\theta_m)/T_0 = 1 + a\sin^2(\theta_m/2) + b\sin^4(\theta_m/2) + \cdots$$

(e) Evaluate $T(\theta_m)/T_0$ numerically using Simpson's Rule for $\theta_m = 0.1; 0.2; \pi/4; \pi/2$ and $3\pi/4$. Compare your answers in *tabulated form* with the results of the series expansion in the last section. Which works better?