Mathematics 17: The Logistic Equation

The logistic equation:

$$x_{n+1} = f(x_n) \equiv r x_n (1 - x_n)$$

is a very simple mapping which is easy to analyse and has behaviour which is believed to be representative of a large class of more sophisticated mappings. The issue of interest is the behaviour of the mapping in the limit that $n \mapsto \infty$. The parameter r is then analysed with a view to classifying the styles of limit permissible.

The *bifurcation tree* is the initial investigation: There are various possible styles of solution in the limit $n \mapsto \infty$:
(i) The mapping converges to a single limit, x^* say, for which:

$$f(x^*) = x^*$$

(ii) The mapping converges to a 'limit-cycle', $x_1^*, x_2^*, \ldots, x_N^*$ say, for which:

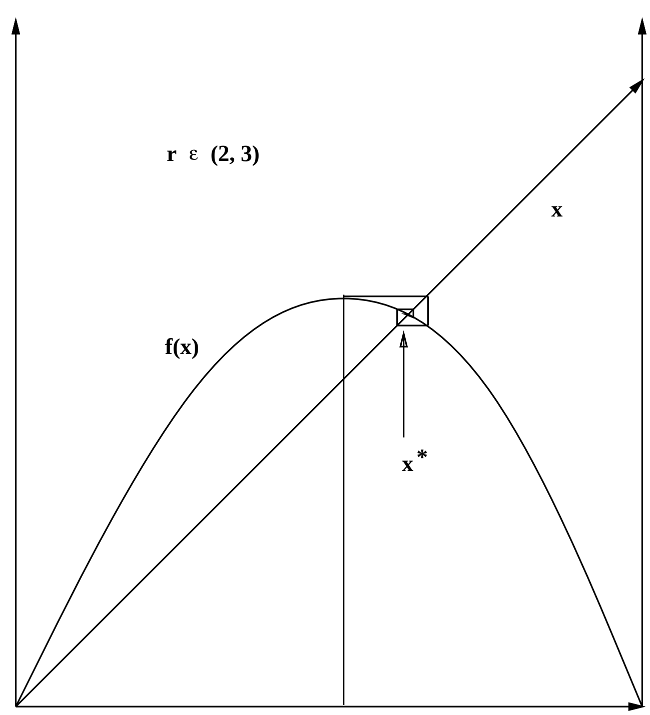
$$x_{n+1}^* = f(x_n^*)$$

and $x_{N+1}^* = x_1^*$ and the cycle closes and repeats after N *different* intermediate steps.
(iii) The mapping *never* repeats and is *chaotic*.

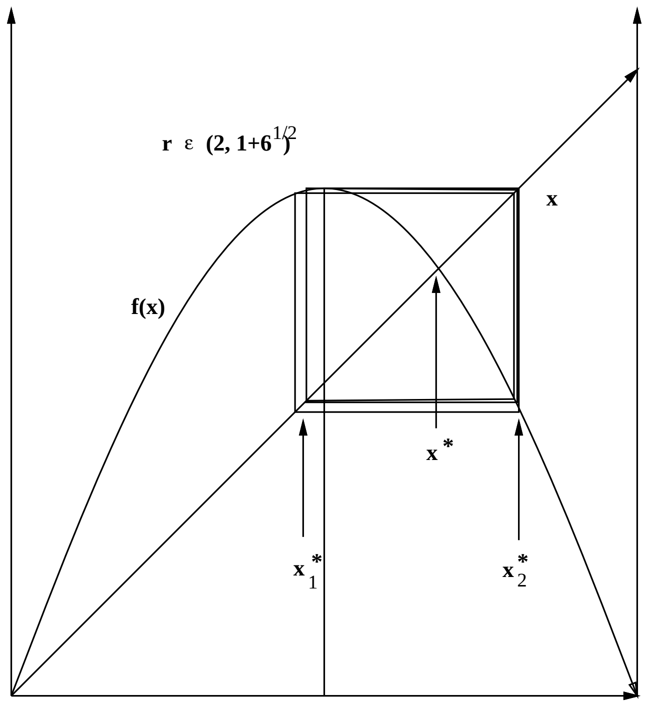
The values of x^* may be found by solving the equations: The possible unique solutions are:

$$x^* = r x^* (1 - x^*) \quad \Longrightarrow \quad x^* = 0, 1 - \frac{1}{r}$$

but these solutions become *unstable*. These instabilities may be understood by plotting $y = x$ and $y = f(x)$ and analysing the mapping, which is:



when stable and:



when initially unstable. The instability is towards a *bifurcation* and a pair of roots.

The instability is controlled by the slope of $f(x)$ at x^*. If:

$$
\begin{aligned}
1 > f'(x^*) > -1 &\Longrightarrow \mathit{Stable} \\
f'(x^*) = 0 &\Longrightarrow \mathit{Supercycle} \\
f'(x^*) = -1 &\Longrightarrow \mathit{Bifurcation} \\
f'(x^*) < -1 &\Longrightarrow \mathit{Unstable}
\end{aligned}
$$

Once the first bifurcation has occurred, then we need to consider:

$$x_{n+1} = f^{(2)}(x_n) = f(f(x_n))$$

which remains stable. Clearly, the next instability occurs by analogy when:

$$f^{(2)}(x^*) = x^*$$

$$\frac{\partial}{\partial x} f^{(2)}(x^*) = -1$$

but note that simultaneously:

$$f^{(4)}(x^*) = x^*$$

$$\frac{\partial}{\partial x} f^{(4)}(x^*) = +1$$

since there are two roots to the four-fold cycle at this point waiting to become the non-degenerate roots. We can generalise to a *sequence* of bifurcations, which occur when:

$$f^{(2^N)}(x^*) = x^*$$

$$\frac{\partial}{\partial x} f^{(2^N)}(x^*) = -1$$

which are equations to be solved for x^* and r. Supercycles occur when:

$$f^{(2^N)}(x^*) = x^*$$

$$\frac{\partial}{\partial x} f^{(2^N)}(x^*) = 0$$

and, as we shall see, are much easier to solve for. Between any two bifurcations is a supercycle and vice-versa.

To make use of these ideas we need to be able to differentiate $f^{(n)}(x)$ with respect to both x and r:

$$\frac{\partial}{\partial x} f^{(n)}(x) = \frac{\partial}{\partial x} f\left[f^{(n-1)}(x)\right] = r(1 - 2f^{(n-1)}(x))\frac{\partial}{\partial x} f^{(n-1)}(x)$$

and so:

$$\frac{\partial}{\partial x} f^{(n)}(x) = \prod_{m=0}^{n-1} r\left[1 - 2f^{(m)}(x)\right]$$

where $f^{(0)}(x) = x$. This result immediately explains why supercycles are easy to find, since:

$$\frac{\partial}{\partial x} f^{(n)}(x) = 0$$

may only be true if one of the values in sequence is equal to a half. Therefore to find a supercycle we need only find the value of r at which:

$$f^{(2^N)}\left(\frac{1}{2}\right) = \frac{1}{2}$$

The additional derivatives may be found quite easily:

$$\frac{\partial}{\partial r} f^{(n)}(x) = \frac{\partial}{\partial r} f\left[f^{(n-1)}(x)\right] = \frac{f^{(n)}(x)}{r} + r(1 - 2f^{(n-1)}(x))\frac{\partial}{\partial r} f^{(n-1)}(x)$$

which may be found iteratively using $\frac{\partial}{\partial r} f^{(0)}(x) = 0$ to start.

$$\ln\left[\frac{\partial}{\partial x} f^{(n)}(x)\right] = \sum_{m=0}^{n-1} \ln\left[1 - 2f^{(m)}(x)\right] + n\ln r$$

and so the final derivatives are:

$$\frac{\partial^2}{\partial x^2} f^{(n)}(x) = \frac{\partial}{\partial x} f^{(n)}(x)\left[\sum_{m=0}^{n-1} \frac{(-2)\frac{\partial}{\partial x} f^{(m)}(x)}{1 - 2f^{(m)}(x)}\right]$$

$$\frac{\partial^2}{\partial r \partial x} f^{(n)}(x) = \frac{\partial}{\partial x} f^{(n)}(x)\left[\frac{n}{r} + \sum_{m=0}^{n-1} \frac{(-2)\frac{\partial}{\partial r} f^{(m)}(x)}{1 - 2f^{(m)}(x)}\right]$$

both of which may be found iteratively using $\frac{\partial}{\partial x} f^{(0)}(x) = 1$ and $\frac{\partial}{\partial r} f^{(0)}(x) = 0$ to start.