Mathematics 8: Relaxing Laplace

Numerical treatments of problems based on the Laplacian operator, ∇^2, are usually tackled by relaxation. Once one has chosen a 'grid' to work with, an approximate value for ∇^2 may be obtained from the values of the function at the 'cage' of nearest-neighbours:

$$y(\mathbf{a}+\mathbf{x}_j) = y(\mathbf{a}) + \mathbf{x}_j.\nabla y(\mathbf{a}) + \frac{1}{2}\left[\mathbf{x}_j.\nabla\right]^2 y(\mathbf{a}) + \frac{1}{6}\left[\mathbf{x}_j.\nabla\right]^3 y(\mathbf{a}) + O\left(\mid \mathbf{x}_j \mid^4\right)$$

where $\mathbf{x}_j$ is a vector joining nearest-neighbours, which is small if the grid is fine. When we average over nearest-neighbours, since both $\mathbf{x}_j$ and $-\mathbf{x}_j$ are present in the sum, the odd terms cancel, leaving:

$$\frac{1}{Z}\sum_j y(\mathbf{a}+\mathbf{x}_j) = y(\mathbf{a}) + \frac{1}{2Z}\sum_j \left[\mathbf{x}_j.\nabla\right]^2 y(\mathbf{a}) + O\left(\mid \mathbf{x} \mid^4\right)$$

where Z is the number of nearest-neighbours, the so-called *coordination number*.

For most choices of *symmetric* grid, the second order term is proportional to the Laplacian.

Square or Triangular:

$$\frac{1}{2Z}\sum_j \left[\mathbf{x}_j.\nabla\right]^2 y(\mathbf{a}) = \frac{\mid \mathbf{x} \mid^2}{4}\nabla^2 y(\mathbf{a})$$

so for either case:

$$\mid \mathbf{x} \mid^2 \nabla^2 y(\mathbf{a}) = 4\left[\frac{1}{Z}\sum_j y(\mathbf{a}+\mathbf{x}_j) - y(\mathbf{a})\right] + O\left(\mid \mathbf{x} \mid^4\right)$$

Relaxation involves updating the central site to be the average of its neighbours according to this quadratic approximation for $\nabla^2 y(\mathbf{a})$. So if we want to solve $\nabla^2 y(\mathbf{a}) = 0$ and we have a grid set up denoted by a label i, then we set:

$$y_i \mapsto \frac{1}{Z}\sum_{<ij>} y_j$$

where $< ij >$ denotes all nearest-neighbours. It is often useful to 'overshoot':

$$y_i \mapsto (1+\alpha)\frac{1}{Z}\sum_{<ij>} y_j - \alpha y_i$$

to improve convergence.