Questions

1. Finding Solution of Given Linear Equation

1.1. Finding the Solution of System of Linear Equations

$$3x_1 - 2x_2 - x_3 = 6$$
$$x_1 + 10x_2 - x_3 = 2$$
$$3x_1 - 2x_2 + x_3 = 0$$

$$\begin{bmatrix} 3 & -2 & 1 & : & 6 \\ 1 & 10 & -1 & : & 2 \\ 3 & -2 & 1 & : & 0 \end{bmatrix}$$

- $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 3 & -2 & -1 & : & 6 \\ 3 & -2 & 1 & : & 0 \end{bmatrix}$$

- $R_2 = R_2 - 3R_1$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 0 & -32 & 2 & : & 0 \\ 3 & -2 & 1 & : & 0 \end{bmatrix}$$

- $R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 0 & -32 & 2 & : & 0 \\ 0 & -32 & 4 & : & -6 \end{bmatrix}$$

- $R_2 = \frac{R_2}{-32}$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 0 & 1 & \frac{-1}{16} & : & 0 \\ 0 & -32 & 4 & : & -6 \end{bmatrix}$$

- $R_3 = R_3 + 32R_2$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 0 & 1 & \frac{-1}{16} & : & 0 \\ 0 & 0 & 2 & : & -6 \end{bmatrix}$$

- $R_3 \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 10 & -1 & : & 2 \\ 0 & 1 & \frac{-1}{16} & : & 0 \\ 0 & 0 & 1 & : & -3 \end{bmatrix}$$

- $R_1 = R_1 - 10R_2$

$$\begin{bmatrix} 1 & 0 & \frac{-3}{8} & : & 2 \\ 0 & 1 & \frac{-1}{16} & : & 0 \\ 0 & 0 & 1 & : & -3 \end{bmatrix}$$

- $R_1 = R_1 + \frac{3}{8}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & \frac{7}{8} \\ 0 & 1 & \frac{-1}{16} & : & 0 \\ 0 & 0 & 1 & : & -3 \end{bmatrix}$$

- $R_2 = R_2 + \frac{1}{16}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & \frac{7}{8} \\ 0 & 1 & 0 & : & -\frac{3}{16} \\ 0 & 0 & 1 & : & -3 \end{bmatrix}$$

Now from Matrix we have followings:

$$\boxed{\begin{aligned} x_1 &= \frac{7}{8} \\ x_2 &= -\frac{3}{16} \\ x_3 &= -3 \end{aligned}}$$

Also we have the Following:

$$\boxed{\begin{aligned} \text{Rank (A)} &= 3 \\ \text{Rank (Ab)} &= 3 \\ \text{Number of Unknowns} &= 3 \end{aligned}}$$

So as $\text{Rank (A)} = \text{Rank (Ab)} = \text{Number of Unknowns i.e.} 3 = 3 = 3$

So Solution is Unique!

Also,

Putt x_1, x_2 & x_3 in First Equation

$3(\frac{7}{8}) - 2(-\frac{3}{16}) - (-3) = 6$

$\frac{21}{8} + \frac{6}{16} + 3 = 6$

$\frac{42+6+48}{16} = 6$

$\frac{96}{16} = 6$

$6 = 6$

So it is verified, that the Solution is correct!

1.2. Finding the Solution of System of Linear Equation

$$\begin{aligned} x + y + z &= 2 \\ 2x + 2y + 2z &= 4 \\ 3x + 3y + 3z &= 6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 2 & 2 & 2 & : & 4 \\ 3 & 3 & 3 & : & 6 \end{bmatrix}$$

- $R_2 = R_2 - 2R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 0 & 0 & 0 & : & 0 \\ 3 & 3 & 3 & : & 6 \end{bmatrix}$$

- $R_3 = R_3 - 3R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Now from Matrix we have followings:

$$\boxed{x = 2}$$

Also we have the Following:

$$\boxed{\begin{aligned} \text{Rank (A)} &= 1 \\ \text{Rank (Ab)} &= 1 \\ \text{Number of Unknowns} &= 3 \end{aligned}}$$

So as $\text{Rank (A)} = \text{Rank (Ab)} < \text{Number of Unknowns i.e. } 1 = 1 < 3$

So Solution is Infinite!

Now,

From first row:

$$x + y + z = 2$$

Let:

- $\boxed{y = s}$
- $\boxed{z = t}$

Then:

$$\boxed{x = 2 - s - t}$$

Putt $x = 2, z = t$ in Equation (1)

$$2 + y + t = 2 \implies 2 + y = 2 - t \implies y = 2 - t - 2 \implies y = -t$$

For Particular Solution

Putt $t = 1$

$$\boxed{\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 1 \end{aligned}}$$

Putt $in(1)$

$2 - 1 + 1 = 2 \implies 2 = 2$

So it is verified, that the Solution is correct!

1.3. Finding the Solution of System of Linear Equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 2 \\ x + y + z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 2 & 2 & : & 2 \\ 1 & 1 & 1 & : & 3 \end{bmatrix}$$

- $R_2 = R_2 - 2R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 0 & 0 & : & 0 \\ 1 & 1 & 1 & : & 3 \end{bmatrix}$$

- $R_3 = R_3 - R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 2 & 2 & : & 2 \\ 0 & 0 & 0 & : & 2 \end{bmatrix}$$

Now from Matrix we have followings:

$$\boxed{\begin{aligned} \text{Rank (A)} &= 2 \\ \text{Rank (Ab)} &= 3 \end{aligned}}$$

So as $\text{Rank (A)} < \text{Rank (Ab)}\ 2 < 3$

So No Solution Exists!

It means, System is **inconsistent** — **No solution exists**.

1.4. Finding the Solution of System of Linear Equations

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 4 \\ 3x_1 - 2x_2 + 2x_3 &= 2 \\ 5x_1 - 8x_2 + 4x_3 &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 5 & : & 4 \\ 3 & -2 & 2 & : & 2 \\ 5 & -8 & 4 & : & 1 \end{bmatrix}$$

- $R_2 = R_2 - 3R_1$:

$$\begin{bmatrix} 1 & 1 & 5 & : & 4 \\ 0 & -5 & -13 & : & -10 \\ 5 & -8 & 4 & : & 1 \end{bmatrix}$$

- $R_3 = R_3 - 5R_1$:

$$\begin{bmatrix} 1 & 1 & 5 & : & 4 \\ 0 & -5 & -13 & : & -10 \\ 0 & -13 & -21 & : & -19 \end{bmatrix}$$

- $R_2 = \frac{R_2}{-5}$:

$$\begin{bmatrix} 1 & 1 & 5 & : & 4 \\ 0 & 1 & \frac{13}{5} & : & 2 \\ 0 & -13 & -21 & : & -19 \end{bmatrix}$$

- $R_3 = R_3 + 13R_2$:

$$\begin{bmatrix} 1 & 1 & 5 & : & 4 \\ 0 & 1 & \frac{13}{5} & : & 2 \\ 0 & 0 & \frac{64}{5} & : & 7 \end{bmatrix}$$

From Matrix we have:

$$\frac{64}{5}x_3 = 7 \implies 64x_3 = 35 \implies \boxed{x_3 = \frac{35}{64}}$$

Substitute in second row:

$$x_2 + \frac{13}{5}(\frac{35}{64}) = 2 \implies x_2 + \frac{455}{320} = 2 \implies x_2 = 2 - \frac{455}{320} \implies \boxed{x_2 = \frac{37}{64}}$$

Substitute x_2 and x_3 in first row:

$$x_1 + \frac{37}{64} + 5(\frac{35}{64}) = 4 \implies x_1 + \frac{37}{64} + \frac{175}{64} = 4 \implies 4 \implies x_1 + \frac{37+175}{64} = 4$$
$$x_1 + \frac{212}{64} = 4 \implies x_1 + \frac{53}{16} = 4 \implies x_1 = 4 - \frac{53}{16} \implies \boxed{x_1 = \frac{11}{16}}$$

Now, we have followings:

$$\boxed{\begin{aligned} x_1 &= \frac{11}{16} \\ x_2 &= \frac{37}{64} \\ x_3 &= \frac{35}{64} \end{aligned}}$$

Also we have the Following:

$$\boxed{\begin{aligned} \text{Rank (A)} &= 3 \\ \text{Rank (Ab)} &= 3 \\ \text{Number of Unknowns} &= 3 \end{aligned}}$$

So as $\text{Rank (A)} = \text{Rank (Ab)} = \text{Number of Unknowns i.e. } 3 = 3 = 3$

So Solution is Unique!

Also,

Putt x_1, x_2 & x_3 in First Equation

$\frac{11}{16} + \frac{37}{64} + 5(\frac{35}{64}) = 4$

$\frac{11}{16} + \frac{37}{64} + \frac{175}{64} = 4$

$\frac{44+37+175}{64} = 6$

$\frac{256}{64} = 4$

$4 = 4$

So it is verified, that the Solution is correct!

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2. Finding if a vector is independent or dependent (Linearly)

2.1. Show that the vector $v_1 = (1,1), v_2(1,2) \in \mathbb{R}^2$ are linearly dependent or independent vectors

let

$$\alpha_1, \alpha_2 \in F$$

So Equation Become as:

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

So by putting values of v_1 and v_2:

$$\alpha_1(1,1) + \alpha_2(1,2) = (0,0)$$

$$(\alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2) = (0,0)$$

Expand:

$$(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) = (0,0)$$

Now set up the system of equations:

$$\alpha_1 + \alpha_2 = 0 \tag{1}$$

$$\alpha_1 + 2\alpha_2 = 0 \tag{2}$$

Subtract (1) from (2)

Equation (2) − Equation (1):

$\alpha_1 + 2\alpha_2 = 0$
$-(\alpha_1 + \alpha_2) = 0$

So,

$\alpha_1 + 2\alpha_2 = 0$
$-\alpha_1 - \alpha_2 = 0$

Simplify:

$$(0) + (\alpha_2) = 0$$

$$\alpha_2 = 0$$

Substitute $\alpha_2 = 0$ into (1)

From (1):

$$\alpha_1 + 0 = 0$$

Thus:

$$\alpha_1 = 0$$

So, $\alpha_1 = 0$, $\alpha_2 = 0$.
Thus, the vectors are **linearly independent**.

2.2. Show that the Vectors $v_1 = (3, 2, 1)\ ,\ v_2 = (1, 2, 0)\ and\ v_3 = (-1, 2, -1) \in \mathbb{R}^3 \text{are linearly dependent or independent.}$

let

$$\alpha_1, \alpha_2, \alpha_3 \in F$$

So Equation Become as:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

So by putting values of v_1, v_2 and v_3:

$$\alpha_1(3, 2, 1) + \alpha_2(1, 2, 0) + \alpha_3(-1, 2, -1) = (0, 0, 0)$$

$$(3\alpha_1, 2\alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 0) + (-\alpha_3, 2\alpha_3, -\alpha_3) = (0, 0, 0)$$

Expand:

$$(3\alpha_1 + \alpha_2 - \alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 0 - \alpha_3) = (0, 0, 0)$$

Now set up the system of equations:

$$\begin{gathered} 3\alpha_1 + \alpha_2 - \alpha_3 = 0(3) \\ 2(\alpha_1 + \alpha_2 + \alpha_3) = 0 \implies \alpha_1 + \alpha_2 + \alpha_3 = 0(4) \\ \alpha_1 - \alpha_3 = 0(5) \end{gathered}$$

Subtract (3) from (4)

Equation (3) − Equation (4):

$3\alpha_1 + \alpha_2 - \alpha_3 = 0$
$-(\alpha_1 + \alpha_2 + \alpha_3) = 0$

So,

$3\alpha_1 + \alpha_2 - \alpha_3 = 0$
$-\alpha_1 - \alpha_2 - \alpha_3 = 0$

So,

$2\alpha_1 - 2\alpha_3 = 0$
$2(\alpha_1 - \alpha_3) = 0 \implies \alpha_1 - \alpha_3 = 0$

Now putt $\alpha_3 = t$

So,

$\alpha_1 - t = 0 \implies \alpha_1 = t$

Step 2: Substitute $\alpha_1 = t$ & $\alpha_3 = t$ into (4)

From (1):

$$t + \alpha_2 + t = 0 \implies \alpha_2 + 2t = 0 \implies \alpha_2 = -2t \tag{6}$$

Thus:

$$\boxed{\begin{aligned} \alpha_1 &= t \\ \alpha_2 &= -2t \\ \alpha_3 &= t \end{aligned}}$$

Putt $t = 1$

$$\boxed{\begin{array}{l}\alpha_1 = 1 \\ \alpha_2 = -2 \\ \alpha_3 = 1\end{array}}$$

Thus, the vectors are **linearly dependent**.

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3. Finding Linear Span

3.1. Prove that $\mathbb{R}^2$ is spaned by the following vectors $v_1(1,2), v_2(-1,1)$

Let $\mathbb{R}^2 = \alpha_1 v_1 + \alpha_2 v_2$

$(x,y) = \alpha_1(1,2) + \alpha_2(-1,1)$
$(x,y) = (\alpha_1, 2\alpha_1) + (-\alpha_2, \alpha_2)$

$$(x,y) = \alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2 \tag{7}$$

$$\alpha_1 - \alpha_2 = x \tag{8}$$

$$2\alpha_1 + \alpha_2 = y \tag{9}$$

$Adding\ (2) \&\ (3)$

$\alpha_1 - \alpha_2 = x$
$+2\alpha_1 + \alpha_2 = y$

$3\alpha_1 = x + y$

$\alpha_1 = \frac{x+y}{3}$

Putt $\alpha_1\ in\ Equation(2)$

$\frac{x+y}{3} - \alpha_2 = x$

$\frac{x+y}{3} - x = \alpha_2$

$\frac{x+y-3x}{3} = \alpha_2$

$\alpha_2 = \frac{-2x+y}{3}$

Now,

$\alpha_1 = \frac{x+y}{3}$ and $\alpha_2 = \frac{-2x+y}{3}$

$\text{Putt the back the value in the equation from where you have started finding its values:}$

Putt $\alpha_1\ \&\ \alpha_2\ in\ Equation(1)$

$(x,y) = (\frac{x+y}{3} - \frac{-2x+y}{3}, 2(\frac{x+y}{3}) + \frac{-2x+y}{3})$

$(x,y) = (\frac{x+y+2x-y}{3}, \frac{2x+2y-2x+y}{3})$

$(x,y) = (\frac{3x}{3}, \frac{3y}{3})$

$(x,y) = (x,y)$

If we got this $(x,y) = (x,y)$ then they are spanned and if not then they are not spanned

$So \mathbb{R}^2 \text{ is spaned by vectors } v_1 \& v_2$

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4. Finding if a space is a subspace of a Vector

4.1. If $R^3 = \{(x, y, z)|x, y, z \in R\}$

then show that $W = \{(x, y, z)|x + y + z = 0\}$ is a subspace of R^3

Solution:

Let $\alpha_1, \alpha_2 \in R$

So,

$w_1 = \{(x_1, y_1, z_1)|x_1 + y_1 + z_1 = 0\}$

$w_2 = \{(x_2, y_2, z_2)|x_2 + y_2 + z_2 = 0\}$

$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2)$

= $(\alpha_1 x_1, \alpha_1 y_1, \alpha_1 z_1) + (\alpha_2 x_2, \alpha_2 y_2, \alpha_2 z_2)$

= $(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$

Now $\alpha_1 w_1 + \alpha_2 w_2 \in W$ if
$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_1 z_1 + \alpha_2 z_2 = 0$

Taking L.H.S = $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_1 z_1 + \alpha_2 z_2$

= $\alpha_1(x_1 + y_1 + z_1) + \alpha_2(x_2 + y_2 + z_2)(2)$

So, according to the condition given i.e. $x + y + z = 0$

Putting the values in equation (2)

= $\alpha_1(0) + \alpha_2(0)$
= $0 + 0$
= 0

So given condition is met and we have $L.H.S = R.H.S$

So the R^2 is a subspace of a vector V

4.2. If $R^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

then show that $W = \{(x, y, z) \mid y^2 = x + z\}$ is a subspace of R^3

Solution:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

So,

$$w_1 = \{(x_1, y_1, z_1) \mid y_1^2 = x_1 + z_1\}$$
$$w_2 = \{(x_2, y_2, z_2) \mid y_2^2 = x_2 + z_2\}$$

Now,

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2)$$
$$= (\alpha_1 x_1, \alpha_1 y_1, \alpha_1 z_1) + (\alpha_2 x_2, \alpha_2 y_2, \alpha_2 z_2)$$
$$= (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$$

Now, to check whether $\alpha_1 w_1 + \alpha_2 w_2 \in W$, we need:

$$(y)^2 = (x) + (z)$$

i.e.,

$$(\alpha_1 y_1 + \alpha_2 y_2)^2 = (\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 z_1 + \alpha_2 z_2)$$

Taking L.H.S:

$$(\alpha_1 y_1 + \alpha_2 y_2)^2 = \alpha_1^2 y_1^2 + 2\alpha_1\alpha_2 y_1 y_2 + \alpha_2^2 y_2^2$$

But R.H.S is:

$$(\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1(x_1 + z_1) + \alpha_2(x_2 + z_2)$$

Given:

$$y_1^2 = x_1 + z_1, \quad y_2^2 = x_2 + z_2$$

Substituting:

$$= \alpha_1(y_1^2) + \alpha_2(y_2^2)$$

Thus R.H.S becomes:

$$= \alpha_1 y_1^2 + \alpha_2 y_2^2$$

Clearly,
L.H.S = $\alpha_1^2 y_1^2 + 2\alpha_1\alpha_2 y_1 y_2 + \alpha_2^2 y_2^2$
R.H.S = $\alpha_1 y_1^2 + \alpha_2 y_2^2$

They are **not equal** because of the extra term $2\alpha_1\alpha_2 y_1 y_2$ in the L.H.S.

Thus, the condition is not satisfied.

Therefore, W **is NOT a subspace of the vector space** R^3.

4.3. If $R^2 = \{(u, v) \mid u, v \in \mathbb{R}\}$

then show that $W = \{(u, v) \mid u = v\}$ is a subspace of R^2

Solution:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

So,

$$w_1 = \{(u_1, v_1) \mid u_1 = v_1\}$$

$$w_2 = \{(u_2, v_2) \mid u_2 = v_2\}$$

Now,

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1(u_1, v_1) + \alpha_2(u_2, v_2)$$

$$= (\alpha_1 u_1, \alpha_1 v_1) + (\alpha_2 u_2, \alpha_2 v_2)$$

$$= (\alpha_1 u_1 + \alpha_2 u_2, \alpha_1 v_1 + \alpha_2 v_2)$$

Now, to check whether $\alpha_1 w_1 + \alpha_2 w_2 \in W$, we need:

$$u = v$$

i.e.,

$$\alpha_1 u_1 + \alpha_2 u_2 = \alpha_1 v_1 + \alpha_2 v_2$$

Given:

$$u_1 = v_1, \quad u_2 = v_2$$

Substituting:

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 v_1 + \alpha_2 v_2$$

Which is **obviously true** because we are just adding and scaling the same numbers on both sides..

Therefore, the condition is satisfied and
W **is a subspace** of R^2.

4.4. If $R^2 = \{(u, v) \mid u, v \in \mathbb{R}\}$

then show that $W = \{(u, v) \mid u + v = 0\}$ is a subspace of R^2

Solution:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

So:

$$w_1 = \{(u_1, v_1) \mid u_1 + v_1 = 0\}$$

$$w_2 = \{(u_2, v_2) \mid u_2 + v_2 = 0\}$$

Now:

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1 (u_1, v_1) + \alpha_2 (u_2, v_2)$$

Expanding:

$$= (\alpha_1 u_1, \alpha_1 v_1) + (\alpha_2 u_2, \alpha_2 v_2)$$

$$= (\alpha_1 u_1 + \alpha_2 u_2, \alpha_1 v_1 + \alpha_2 v_2)$$

Now, for this new vector to be in W, it must satisfy:

$$(u) + (v) = 0$$

i.e.,

$$(\alpha_1 u_1 + \alpha_2 u_2) + (\alpha_1 v_1 + \alpha_2 v_2) = 0$$

Simplify:

$$= \alpha_1 (u_1 + v_1) + \alpha_2 (u_2 + v_2)$$

Given:

$$u_1 + v_1 = 0 \quad \text{and} \quad u_2 + v_2 = 0$$

because both w_1 and w_2 are elements of W.

Thus:

$$= \alpha_1 (0) + \alpha_2 (0)$$

$$= 0 + 0$$

$$= 0$$

So the condition is satisfied.

Therefore, W **is a subspace** of R^2.

4.5. If $R^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

then show that $W = \{(x, y, z) \mid x = z\}$ is a subspace of R^3

Solution:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

So,

$$w_1 = \{(x_1, y_1, z_1) \mid x_1 = z_1\}$$

$$w_2 = \{(x_2, y_2, z_2) \mid x_2 = z_2\}$$

Now,

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2)$$
$$= (\alpha_1 x_1, \alpha_1 y_1, \alpha_1 z_1) + (\alpha_2 x_2, \alpha_2 y_2, \alpha_2 z_2)$$
$$= (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$$

Now, to check whether $\alpha_1 w_1 + \alpha_2 w_2 \in W$, we need:

$$x = z$$

i.e.,

$$\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 z_1 + \alpha_2 z_2$$

Given:

$$x_1 = z_1, \quad x_2 = z_2$$

Substituting:

$$\alpha_1 z_1 + \alpha_2 z_2 = \alpha_1 z_1 + \alpha_2 z_2$$

$\text{Hence, L.H.S} = \text{R.H.s}$

Therefore, the condition is satisfied and
W **is a subspace** of R^2.

4.6. If $R^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

then show that $W = \{(x, y, z) \mid y = x + z\}$ is a subspace of R^3

Solution:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$

So,

$$w_1 = \{(x_1, y_1, z_1) \mid y_1 = x_1 + z_1\}$$
$$w_2 = \{(x_2, y_2, z_2) \mid y_2 = x_2 + z_2\}$$

Now,

$$\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2)$$
$$= (\alpha_1 x_1, \alpha_1 y_1, \alpha_1 z_1) + (\alpha_2 x_2, \alpha_2 y_2, \alpha_2 z_2)$$
$$= (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$$

Now, to check whether $\alpha_1 w_1 + \alpha_2 w_2 \in W$, we need:

$$(y) = (x) + (z)$$

i.e.,

$$(\alpha_1 y_1 + \alpha_2 y_2) = (\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 z_1 + \alpha_2 z_2)$$

Taking R.H.S:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 z_1 + \alpha_2 z_2 = \alpha_1 x_1 + \alpha_1 z_1 + \alpha_2 x_2 + \alpha_2 z_2 = \alpha_1(x_1 + z_1) + \alpha_2(x_2 + z_2)$$

Given:

$$y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2$$

Substituting:

$$= \alpha_1(y_1) + \alpha_2(y_2)$$

Thus R.H.S becomes:

$$= \alpha_1 y_1 + \alpha_2 y_2$$

Clearly,

$\text{L.H.S} = \text{R.H.S}$

Thus, the condition is satisfied.

Therefore, W **is a subspace of the vector space** R^3.

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5. Finding Norm of a Vector

5.1. Find the Norm of a Vector $V = (1, 3)$

$$\|v\| = \sqrt{1^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$$

5.2. Find the Norm of a Vector $V = (2, 4, 5)$

$$\|v\| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$$

5.3. Find the Norm of a Vector $V = (1, 2, 3, 4)$

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1+4+9+16} = \sqrt{30}$$

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6. Finding if one vector is orthagonal of another vector or not.

6.1. $u = (1, -1, 2), v = (-1, 1, 1)$

$< u, v >= (1)(-1) + (-1)(1) + (2)(1) = -1 - 1 + 2 = -2 + 2 = 0$

$So\, u \perp v$

u orthagonal to v

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7. Finding Orthogonal Matrix

7.1. Prove that A is Orthogonal Matrix, If

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $A^t A = A$, So we can say that A is an **Orthogonal Matrix**.

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8. Least Square Method

$$\boxed{x = (A^T A)^{-1} A^T b}$$

8.1. Find x if:

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ 11 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

Now,

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 2 + 0 \cdot 6 + 1 \cdot 11 \\ 0 \cdot 2 + 2 \cdot 6 + 1 \cdot 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 11 \\ 12 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 23 \end{bmatrix}$$

Now finding $(A^T A)^{-1}$:
As

$$(A^T A)^{-1} = \frac{\text{adj}(A^T A)}{\det(A^T A)}$$

So determinant of $A^T A$:

$$\det = (17)(5) - (1)(1) = 85 - 1 = 84$$

And for Adjoint:

$$\text{adj}\,(A^T \cdot A) = \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

The inverse:

$$(A^T A)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

Now finding x:

$$x = (A^T A)^{-1} A^T b$$

$$x = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 23 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} (5 \cdot 19 - 1 \cdot 23) \\ -(1 \cdot 19 + 17 \cdot 23) \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 95 - 23 \\ -19 + 391 \end{bmatrix}$$
$$= \frac{1}{84} \begin{bmatrix} 72 \\ 372 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{72}{84} \\ \frac{372}{84} \end{bmatrix}$$
$$x = \frac{1}{84} \begin{bmatrix} 72 \\ 372 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{6}{7} \\ \frac{31}{7} \end{bmatrix}$$
