Optics Problem Set 1

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Tuesday, February 12, 2019

Chapter 2

2.40) Given the traveling wave $\Psi(x,t) = 5.0\exp(-ax^2 - bt^2 - 2\sqrt{ab}xt)$, determine its direction of propagation. Calculate a few values of Ψ and make a sketch of the wave at $t = 0$, taking $a = 25m^{-2}$ and b = $b = 9.0s^{-2}$. What is the speed of the wave?

After factoring notice that $\Psi = 5.0\exp([-(x + \sqrt{\frac{b}{a}}t^2)^2])$. Thus, the direction of propagation is in the **negative x direction**.

The following is Ψ as x varies.

x	Ψ
0.6	0.0006
0.4	0.09
0.2	1.8
0.0	5.0
-0.2	1.8
-0.4	0.09
-0.6	0.0006

Let $t = 0$, $a = 25m^{-2}$, and $b = 3s^2$. Then $\Psi = -5.0\exp 25x^2$.

Below is a graph of Ψ at time $t = 0$.

Ψ is a function of $(x + vt)$, where $x + vt = (x + \sqrt{\frac{a}{b}}t)$. Thus, the speed of the wave is $v = \sqrt{\frac{a}{b}} = \sqrt{\frac{9m^2}{25s^2}} = \frac{3m}{5s} = 0.6ms^{-1}$.

2.44) Working with exponentials directly, show that the magnitude of $\Psi = Ae^{i\omega t}$ is A. Then, rederive the same result using Euler's formula. Prove that $e^{i\alpha}e^{i\beta} = e^{(\alpha+\beta)}$

Let $\tilde{z} = A^{i\omega t}$ and its congugate $\tilde{z}^* = Ae^{-i\omega t}$. Let the modulus, or magnitude, of $\tilde{z}$ be $\|\tilde{z}\| = (\tilde{z}\tilde{z}^*)^{\frac{1}{2}}$.
Thus, the magnitude of Ψ is $\|\tilde{z}\| = (\tilde{z}\tilde{z}^*)^{\frac{1}{2}} = [(Ae^{i\omega t})(Ae^{-i\omega t})]^{\frac{1}{2}} = [A^2e^{i\omega t - i\omega t}]^{\frac{1}{2}} = \sqrt{A^2} = A$.

Euler's formula states that $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Notice that $\omega = 2\pi\nu = \frac{2\pi}{t}$. Plugging this in yeilds $Ae^{i\frac{2\pi}{\tau}t} = A[\cos\frac{2\pi}{\tau}t + i\sin\frac{2\pi}{\tau}t]$.
If $t = \tau$, then $Ae^{i\frac{2\pi}{t}t} = A[\cos\frac{2\pi}{t}t + i\sin\frac{2\pi}{t}t] = Ae^{2\pi} = A[\cos 2\pi + i\sin 2\pi] = A[1+0] = A$.

By law of exponents, $e^x + e^y = e^{x+y}$.
Thus, $e^{i\alpha}e^{i\beta} = e^{i\alpha + i\beta} = e^{i(\alpha+\beta)}$.

2.49) Show that $\Psi(x,y,z,t) = f(\alpha x + \beta y + \gamma z - vt)$ and $\Psi(x,y,z,t) = g(\alpha x + \beta y + \gamma z - vt)$ which are plane waves of arbitrary form satisfy the three-dimensional wave equation.

Since f and g are functions which are twice differentiable, compute the partial derivatives of Ψ

$\frac{\partial^2\Psi}{\partial x^2} = -\alpha^2 k^2 \Psi$ $\frac{\partial^2\Psi}{\partial y^2} = -\beta^2 k^2 \Psi$ $\frac{\partial^2\Psi}{\partial z^2} = -\gamma^2 k^2 \Psi$ $\frac{\partial^2\Psi}{\partial t^2} = -k^2 v^2 \Psi$

Add the three spatial derivatives $\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = -k^2(\alpha+\beta+\gamma)\Psi$ and recall that $\alpha+\beta+\gamma = 1$. Thus $-k^2(\alpha+\beta+\gamma)\Psi = -k^2\Psi$.

Plugging the second partial of Ψ with respect to t into the 3-d wave function yeilds $\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = -k^2\Psi$.

2.46) Take the complex quantities $\tilde{z}_1 = (x_1 + iy_1)$ $\tilde{z}_2 = (x_2 + iy_2)$ and show that

$$\mathbf{Re}(\tilde{z}_1 + \tilde{z}_2) = \mathbf{Re}(\tilde{z}_1) + \mathbf{Re}(\tilde{z}_2)$$

If **Re** means to extract the real components of a complex number, then since the argument $\tilde{z}_1 + \tilde{z}_2$ resolves to $x_1 + iy_1 + x_2 + iy_2$, another complex number, simply extract the real component. Thus, the real component from $x_1 + x_2 + (y_1 + y_2)i$ is $x_1 + x_2$. This result is the same as $\mathbf{Re}(\tilde{z}_1) + \mathbf{Re}(\tilde{z}_2) = \mathbf{Re}(x_1 + iy_1) + \mathbf{Re}(x_2 + iy_2)$.

2.47) Take the complex quantities $\vec{z}_1 = (x_1 + iy_1)$ and $\tilde{z}_2 = (x_2 + iy_2)$ and show that $\mathbf{Re}(\tilde{z}_1) \times \mathbf{Re}(\tilde{z}_2) \neq \mathbf{Re}(\tilde{z}_1) \times \mathbf{Re}(\tilde{z}_2)$.

Start with the right hand side and notice that
$\mathbf{Re}(\tilde{\mathbf{z}}_\mathbf{1}) \times \mathbf{Re}(\tilde{\mathbf{z}}_\mathbf{2})$
$= \mathbf{Re}(\mathbf{x_1} + \mathbf{iy_1}) \times \mathbf{Re}(\mathbf{x_2} + \mathbf{iy_2})$
$= x_1 \times x_2 = x_1x_2$.

Then the lefthand side resolves to
$\mathbf{Re}(\tilde{z}_1 \times \tilde{z}_2)$
$= \mathbf{Re}((x_1 + iy_1) \times (x_2 + iy_2))$
$= \mathbf{Re}(x_1x_2 + x_1iy_2 + x_2iy_1 + i^2y_1y_2)$
$= \mathbf{Re}(x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1))$
$= x_1x_2 - y_1y_2 \neq x_1x_2$.

2.56) Show explicitly that the function $\Psi(\vec{r}, t) = A\exp[i(\vec{k}\cdot\vec{r} + \omega t + \epsilon)]$ describes a wave provided that $v = \frac{\omega}{k}$.

In order for Ψ to describe a wave, it needs to hold true that $v = \frac{\omega}{k}$. Let's assume that Ψ is a wave function. Utilize the 3-D wave equation.

$\nabla^2\Psi - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t} = 0$

$\nabla^2[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]] - \frac{1}{v^2}\frac{\partial^2}{\partial t}[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]] = 0$

$\frac{\partial}{\partial r}[\frac{\partial}{\partial r}[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]] - \frac{1}{v^2}[\frac{\partial}{\partial t}[\frac{\partial}{\partial t}[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]]] = 0$

$\frac{\partial}{\partial r}[ik[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]] - \frac{1}{v^2}[\frac{\partial}{\partial t}[i\omega[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]]] = 0$

$(ik)^2[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]] - \frac{1}{v^2}[(i\omega)^2[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]] = 0$

$-k^2[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]] + \frac{1}{v^2}[\omega^2[A\text{exp}[i[\vec{k}\cdot\vec{r}+\omega t+\epsilon]]]] = 0$

$-k^2 = -\frac{1}{v^2}(\omega^2)$

$k^2 = \frac{\omega^2}{v^2}$

$v = \frac{\omega}{k}$

Chapter 3

3.19) A 1.0 mW laser produces a nearly parallel beam $1.0cm^2$ in cross-sectional area at a wavelength of 650nm. Determine the amplitude of the electric field in the beam, assuming the wavefronts are homogeneous and the light travels in a vacuum.

$P = 1.0\times10^{-3}W$

$A = 1.0\times10^{-4}m^2$

$\lambda = 650\times10^{-9}m$

$P = \frac{E_{max}^2}{2\mu_0 c} \rightarrow E = \sqrt{\frac{2\mu_0 cP}{A}} = 2.30\times10^{-1}Vm^{-1}$

3.21) The following is the expression for the $\vec{E}$-feild of an electromagnetic wave traveling in a homogeneous dielectric: $\vec{E} = (-100Vm^{-1})\hat{i}e^{i(kz-\omega t)}$. Here $\omega = 1.80\times10^{15}\frac{rad}{s}$ and $k = 1.20\times10^{7}\frac{rad}{m}$.
(a)Determine the associated $\vec{B}$-field.
$B = \frac{Em}{c} = \frac{-100Vm^{-1}}{3\times10^8 ms^1} = -3.33\times10^{-7}T$

(b)Find the index of refraction
$v = f\lambda = 2\pi\frac{f\lambda}{2\pi} = \frac{\omega}{k} = \frac{1.8\times10^{15}s^{-1}rad}{1.2\times10^7 m^{-1}} = 1.5\times10^8 ms^{-1}$

$n = \frac{c}{v} = \frac{3\times10^8 ms^{-1}}{1.5\times10^8 ms^{-1}} = 2$

(c)Compute the permiativity
$\mu = \sqrt{(\frac{\mu_0\epsilon_0}{\mu_0})^{-1}} \rightarrow \epsilon^{\frac{1}{2}} = 2\sqrt{8.85\times10^{-12}} \rightarrow k = 3.5\times4\times10^{-12}$

(d)Find the irradiance
$I = \frac{C\epsilon_0 E_0^2}{2} == \frac{3\times10^8\times8.85\times10^{-12}\times(100)^2}{2} = 13.275Wm^{-2}$

3.23) Consider a linearly polarized plane electromagnetic wave traveling in the positive x direction in free space having as its plane of vibration the xy-plane. Given that its frequency is 10MHz and its amplitude is $E_0 = 0.08Vm^{-1}$,
(a) find the period of the wave
(b) write an expression for $E(t)$ and $B(t)$
(c) find the flux density, $\langle S \rangle$, of the wave.

(a) The period of an electromagnetic wave can be calculated as $\tau = \frac{1}{\nu} = \frac{1}{10^7 s^{-1}} = 10^{-7}s$.

$(b)\lambda = \frac{v}{\nu} = \frac{c}{\nu} = \frac{3\times 10^8 ms^{-1}}{10^{-7}s^{-1}} = 30m.$

Since $E(x,t) = \hat{y}E_0 \cos(k_x x - \omega t) = \hat{y}E_0 \cos 2\pi(\frac{x}{\lambda} - \nu t) = \hat{y}(0.08Vm^{-1})\cos(2\pi(\frac{x}{30} - 10^7 t))$.

Recall that $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E} = \frac{1}{c}\hat{x} \times \hat{y}\|\vec{E}\|$. Since the wave is propagating in the x direction, $\hat{k} = \hat{x}$.

Thus $\vec{B} = \hat{z}\frac{0.08Vm^{-1}}{2.998\times 10^8 ms^{-1}}\cos[2\pi(\frac{x}{30} - 10^7 t)]$

$= \hat{z}(2.67 \times 10^{-10}Vsm^{-2})\cos[2\pi(\frac{x}{30} - 10^7 t)]$

$= \hat{z}(2.67 \times 10^{-10}T)\cos[2\pi(\frac{x}{30} - 10^7 t)]$.

(c) The flux density is given by $u = \epsilon_0 E_0^2 = (8.825 \times 10^{-12}\frac{C^2}{Nm^2})(0.08Vm^{-1})^2 = 5.648 \times 10^{-14}Jm^{-3}$.

The average magnitude of the Poynting vector is

$\langle S \rangle = \frac{c\epsilon_0}{2}E_0^2 = \frac{(3\times 10^8 ms^1)(5.648\times 10^{-14}Jm^{-3})}{2} = 8.472 \times 10^{-6}Wm^{-2}$.