

Toward an Understanding of Skewed Top Corridors

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Introduction

Lattice paths have been studied extensively over the course of several centuries. For the purpose of this study, a lattice consists of points \mathbb{Z}^2 with certain restrictions and with only two allowable moves, up-right and down-right. Movements in various directions on the lattice are called paths.

The lattice path enumeration model that we propose consists of a starting point, an upper and lower bound, and all possible paths from said starting point to some end point. The area in which the paths are propagated is referred to as a corridor. The number of paths within the corridor depend on the initial values of the starting point, the nature of the upper and lower bounds, and the value placed at the starting point. In our model, the lower bound is a line with zero slope and the upper bound is a line with a variable slope. These conditions seem to present a problem when attempting to systematically generate the values contained in the corridors. Due to the nature of the upper bound, which is a line with slope not parallel to that of the lower bound, the paths bounce off of the upper diagonal line, rippling into and distorting the data below it. One would think that calculating the error caused by each interruption would be somewhat intuitive, but the impacts of each diagonal-boundary disturbance grow larger as time in the corridor progresses. Although the data changes because of the upper bound's slope, intriguing patterns and characteristics have been observed in the configurations of this environment. We call the model which has been briefly introduced a skewed top corridor.

The corridor exists on a two dimensional plane which involves the elapse of time on the horizontal axis and the amount of paths on the vertical axis. A gap extends vertically from the starting point and contains the initial value which is placed at the starting point. The gap contains all zeros except for the initial value which resides at the bottom of the gap.. A diagonal line, which the paths can touch but not surpass, begins just above the gap and extends to a finite point on the grid. Paths can move in an upward or downward direction in the corridor.

Symbols and Definitions

Let \mathbb{N} represent the set natural numbers not including zero and \mathbb{N}_0 represent the set natural numbers including zero.

Path counts will be in \mathbb{N}_0 .

The upper boundary restricting the corridor from above has a constant slope with a rise of 1 over some variable in the set of natural numbers not including zero, $r \in \mathbb{N}$. Let m represent slope, $m = \frac{1}{r}$ which, for now, is a fixed number, $m = \frac{1}{2}$.

A gap of allotted space is initialized upon creation of the corridor area. This is a column with a height of some value in the set of natural numbers. This gap should be greater than or equal to one. Let $g \in \mathbb{N}$. This represents a gap above the initial point and contains g elements.

Let $a \in \mathbb{N}$ be the location on the horizontal axis where the paths begin. The starting point will always be a position a on the horizontal and at position 0 on the vertical. This value needs to reside within the gap, so $a \leq g$.

The paths that are propagated from the start point, $(x, y) \in \mathbb{N}_0^2$ with $(x, 0) = 0$ to some other point begin

with some initial number of paths taken. Let $v \in \mathbb{N}$ dictate how many paths exist at the start point.

The model considers the elapse of time at each point in the lattice corridor. For some moment in time, a specific number of paths exists at some point in the grid space. Let $n \in \mathbb{N}_0$ model the time period in which the paths can be modeled where the corridor only extends for n inclusive columns.

The corridor containing all possible path combinations is a subset on an infinite plane of integers. Consider a function, $V: \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ called a corridor state function. Then $V(x, n)$ represents the number of paths at any point (x, n) in the corridor, where $x \in \mathbb{N}_0$. While the V 's range consists of only natural numbers including zero, the two dimensional grid space contains all \mathbb{Z} values. The infinite plane of integers, of which the corridor is a subset, is the span of unit vectors $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In other words, it is the \mathbb{R}^2 plane with \hat{i} and \hat{j} basis vectors.

Skewed top corridors may be referenced by their g value. Let C_g be the corridor with gap = g .

The diagonals, a concepts discussed later, may also be referenced in the corridor. Let $C_g d_j$ where $j \in \mathbb{N}_0$. Then $C_g d_j$ is the diagonal j in corridor g .

Propagating Values

When initializing the skewed top corridor and its values, it is important to follow a specific algorithm. The following is an ordered description of setting up the corridor boundaries, placing initial values, and generating paths in the corridor structure.

Initialize all numbers on the plane to zero. Choose some value for a and some value for v . For initial analysis, we choose to set static values for v and a . Assume, for now, $v = 1$ and $a = 1$. Insert the initial value v into position $(0, a)$, that is, place an initial value of 1 into position $(0, 1)$ on the plane.

Choose some value for g , which represents the size of a gap in the first corridor column. More specifically, g is an indication of where the upper boundary of the corridor begins. If the corridor were placed in an array, the gap column would consist of g entries: the initial value, and $g - 1$ zeros above v . On the plane, the gap begins at inclusive point $(0, 2)$ and extends upward to non-inclusive coordinate $(0, g + 1)$. A sentinel marker is placed at $(0, g + 1)$ indicating the first establishment of the corridor's upper boundary. From now on, the upper boundary of the corridor may be referred to as the sentinel line.

Assign a value to $m = \frac{1}{r}$. For now, let $r = 2$, so that $m = \frac{1}{2}$. From the initial sentinel point at $(0, g + 1)$, continue establishing such points according to the following procedure. Continually move one unit rightward, keeping track of the current column number, or, horizontal coordinate. Record the current position on the vertical axis as well. If the current column is divisible by r , place a sentinel marker at the $(horizontal, vertical)$ position on the grid, move up one unit on the horizontal axis, and move rightward one unit on the vertical axis. If the current column is not divisible by r , do not place a sentinel marker, move up zero units on the horizontal axis, and move rightward one unit on the vertical axis. Do this until the current vertical position is n .

Generally, the procedure for initializing the sentinel line is as follows:

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place first sentinel point at  $(0, g + 1)$ 

while current vertical is  $< n$ ,

    increase horizontal position by one

    note current position  $(horizontal, vertical)$ 

    if current vertical position  $mod\ r = 0$ 

        place sentinel marker at current position

        increase vertical position by one

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Another way to express this is $(x, y) \in \mathbb{N}_\neq^2$ with $0 < y < mx + g_1$. Since $m = \frac{1}{2}$, move up one row in the plane when an even column is encountered. This is continued until one reaches the n^{th} column. A sentinel

line is established with a slope of $\frac{1}{2}$. For now, assume that every value on or above the sentinel line is equal to zero and assume that every value below the lower boundary is equal to zero. The following recursive formula can be used to calculate each value in the array, where $(0, k) = \text{initial state}$:

$$V(x, y) = V(x - 1, y - 1) + V(x - 1, y + 1), x \in \mathbb{N}, 0 < y < mx + g + 1, V(0, y) = \begin{cases} v, & \text{if } y = a \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Notice that this forms an area of zero and nonzero values which reside on and above row = 1 on the vertical axis and below the sentinel line. The area in which numbers are propagated, having chosen some value for n , consists of n columns and $g + \frac{n+1}{2}$ rows. The values that exist in this area are all natural numbers including zero. After completing this configuration,

- with $a = 1$ and $v = 1$,
- having chosen some n ,
- having chosen some g ,
- and with $m = \frac{1}{2}$,

various observations can be made.

Observations

The corridor will contain values beginning at $(0, 1)$, with rows starting from 1 extending to $g + \frac{n+1}{2}$ inclusive and columns starting from 0 extending to n inclusive.

The sentinel points implicitly form a line with a slope of m , which we have set to $\frac{1}{2}$. The following statements build upon the characteristics of this line, the sentinel line.

From point $(0, 1)$, a line of 1s with slope one is generated until the sentinel line is encountered at $(2g - 1, 2g)$. Building on the idea of where the line of ones will encounter the sentinel line, it can be proven that the line of ones will always contain $2g$ paths and will always intersect the sentinel line at $(2g - 1, 2g + 1)$. The line of ones will exist due to the nature of how the numbers in the corridor are generated, see (1). The value $V(k, x)$ accepts an upper contribution of 0 and a lower contribution from the initial value 1. Because of this initial state, a line of 1s will be generated because numbers outside of the corridor are 0 and contributions from the previous 1 will calculate more 1s. An equation for the sentinel line is $y = \frac{1}{2}x + g + 1$ and an equation for the line of ones is $y = x + 1$. To see where these two lines intersect, simply set the two equations equal to each other and solve for the x and y coordinates of the intersection.

$$\frac{1}{2}x + g + 1 = x + 1$$

$$x = 2g$$

$$y = 2g + 1$$

$$(2g, 2g + 1)$$

This also addresses the fact that if $n > 2g$, the one diagonal will never hit the sentinel line before row n is reached.

Beyond this point, it seems that diagonals containing four of the same elements repeat until the end of the corridor is reached. Upon reaching the sentinel line, a new diagonal begins just below the diagonal line of ones. The point at which the first diagonal of four elements begins is $(2g + 1, 2g)$. The first repeating diagonals will contain 4 elements and if g is odd, these values will be $2g + 1$. If g is even, numbers in the

first four-diagonal are $2g - 1$.

There are diagonals consisting of only zeros between each diagonal of non-zeros, where a diagonal (for these purposes) begins at some point, $(b \in V, c \in V)$, consists of points $(b + 1, c + 1) \dots$, and extends to some distant point below or on the sentinel line. From any point on the grid, moving in a horizontal line to the n^{th} column, the values on the grid will alternate between zeros and non-zeros.

The diagonal beginning at point $(2, 0)$ and ending at $(2g, 2g - 1)$ consists of counting numbers from 1 to $2g$. This diagonal gets disturbed by the sentinel line and gets added up with the final 1 from the first diagonal line of ones. The resulting value is the first value of the 4 element repeating diagonals.

It can be proven that the second diagonal, the diagonal beginning at $(2, 1)$ inclusive and extending to $(2g, 2g - 1)$ inclusive, is indeed the counting numbers. The following is an inductive proof based on the fact that the line of 1s exists and formula (1).

Prove $V(k + 1, k) = k$, where $2 \leq k \leq 2g$.

Base Case: $k = 2$

$$V(k + 1, k) = V(k, k - 1) + V(k, k + 1)$$

$$V(3, 2) = V(2, 1) + V(2, 3)$$

$$2 = 2$$

Assume $V(k + 1, k) = k$ for all $2 \leq k \leq 2g$. Use $k + 1$.

Goal: $V(k + 2, k + 1) = k + 1$

$$V(k + 2, k + 1)$$

$$V(k + 1, k) + V(k + 1, k + 2)$$

$$k + 1^*$$

$$* \text{while } k \leq 2g \quad (2)$$

The diagonal beginning at point $(0, 4)$ and ending at $(2g - 1, 2g - 5)$ contains numbers that are on the outer edge of a triangular arrangement of \mathbb{Z} .

In a horizontal traversal with slope 0 from point $(1, 0)$ to point $(1, 2g - 1)$, one can observe the Catalan numbers 0 to g , however, this pattern quickly gets destroyed at row $2g - 1$ when the first diagonal of ones hits the sentinel line. This causes what should be $g + 1^{th}$ Catalan number to be off by negative one or positive one, thus this pattern disappears.

The various values that exist in the 4 element repeating diagonals do not seem to have any known relationship. No results were found by performing web searches or by entering the numbers into the OEIS. Although there does not seem to be an apparent formula for traversing among diagonals within one corridor, an analysis of diagonals among corridors as g increases has shown that there does seem to be a relationship among the numbers. Specifically, the diagonals among corridors as g increases are degree j .

Conjecture

It appears that d_j^{th} diagonals among corridors make arithmetic sequences to the j^{th} degree. The procedure for assembling the list of diagonals among corridors is the following:

for corridors 1 to g

for diagonals 0 to j

add $C_g d_j$ to list

Generating corridors starting at C_1 to some finite number, note the relationships between the $C_g d_j^{th}$ diagonals across configurations.

The formula for the d_0 diagonal with $x_0 = 1$ is the following:

$$f(x) = 1$$

The formula for the d_1 diagonal with $x_0 = 1$ is the following:

$$f(x) = 2x - 1$$

The formula for the d_2 diagonal with $x_0 = 1$ is the following:

$$f(x) = 2x^2 + 9x + 3$$

The formula for the d_3 diagonal with $x_0 = 1$ is the following:

$$f(x) = \frac{4}{3}x^3 + 16x^2 + \frac{149}{3}x + 13$$

The formula for the d_4 diagonal with $x_0 = 1$ is the following:

$$f(x) = \frac{2}{3}x^4 + \frac{46}{3}x^3 + \frac{707}{6}x^2 + \frac{1867}{6}x + 68$$

These were calculated from a common difference that appeared among the diagonal numbers in corridors as g increases. As the values for g increment for a new configuration of C_g , it seems that the arithmetic sequences continue to follow the pattern of being sequences of the j^{th} degree. These values are considered by first indexing "default" corridors C_1 to C_g , where C_g is the corridor with g gap. This presents the question of how to prove that this really occurs, rather, how to navigate among diagonals within a single corridor. Although this pattern of arithmetic sequences to the j^{th} degree has occurred among diagonals of the same index among subsequent corridors, how can one be certain that this most definitely occurs as g increases? We begin answering this question by observing the relationship among diagonals in one corridor; moreover, how can one traverse diagonals within one corridor? What is the pattern to calculate the next diagonal value in the same corridor and how can we be certain that this pattern holds? These are the questions that will lead to proving that the corridor diagonals are certainly arithmetic sequences of the j^{th} degree.

Bibliography

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