An Exploration of Skewed Top Corridors

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Lattice paths have been studied extensively over the course of several centuries. For the purpose of this study, a lattice consists of points $\mathbb{Z}^2$ with certain restrictions and with only two allowable moves, up-right and down right. Movements in various directions on the lattice are called paths.

We are studying a lattice path model which consists of a point at which all paths begin, upper and lower bounds which mark areas beyond which non-zero paths cease to exist, and an ending point which marks the end of non-zero path values. The area of the lattice contained by the upper and lower bound is referred to as a corridor. The number of paths within the corridor depend on the initial value which is placed at the starting position of the corridor, the nature of the upper and lower bounds, and the position of the starting value in the corridor. The upper and lower boundary lines mark the areas where non-zero path values cease to exist. Paths can exist on the two boundary lines, but they cannot exist beyond the lines. The lower boundary is essentially a horizontal line with zero slope. The upper boundary line is a line with variable slope. These conditions seem to present a problem when attempting to systematically generate values contained in the corridors. Due to the nature of the upper bound, which is a line with slope not parallel to that of the lower bound, the paths bounce off of the upper diagonal line, rippling into and distorting the data below it. One might propose that calculating the error caused by each interruption would be somewhat intuitive, but the impacts of each diagonal-boundary disturbance grow larger as time in the corridor progresses. Although the data changes because of the upper bound's slope, intriguing patterns and characteristics have been observed in the configurations of this environment. The corridor model which has varying boundary lines is referred to as a skewed top corridor.

We interpret the corridor, for now, on the two dimensional Cartesian coordinate plane which is the span of unit vectors $\hat{\imath} = \vec{(1,0)}$ and $\hat{\jmath} = \vec{(0,1)}$. We refer to the vertical axis as the y-axis and the horizontal axis as the x-axis. The elapse of time is represented along the x-axis. The amount of paths is represented along the y-axis which is perpendicular to the x-axis. At a glance, the structure of the corridor consists of an initial value placed at a starting position in the corridor. A gap of predetermined size exists in the same column as the starting position. The size of the gap is equal to the number of lattice points extending vertically from the starting position and includes the starting path value. The gap, for now, extends vertically in the positive direction of the y-axis, for which our convention is upward. Values in this gap are zeros except for the starting value which is also contained in the gap. Our exploration consists of manipulating the initial conditions of the corridors, observing and quantifying the results, and assembling various formulas to express relationships that we notice. We want to learn as much as we can about these corridor structures and what the relationships within them might encode.

Symbols and Definitions

We start our journey of exploring skewed top corridors by establishing our definitions. We also need a symbolic method for referring to specific aspects of the corridors. We will use these specific symbols and definitions not only to discuss corridors but to also make mathematical conclusions about the structures.

First off, we will be using the set of numbers which are integers larger than one.

Definition 1. Let the set of natural numbers be represented by $\mathbb{N}$.

We will also discuss integers which are positive and greater than zero.

Definition 2. Let $\mathbb{N}_0$ represent the set of natural numbers including zero.

At times, we will need to utilize the set of integer values which are positive, negative, and include zero.

Definition 3. Let $\mathbb{Z}$ represent the set of integers.

We can now begin defining some of the parameters for establishing the skewed-top corridor. Let us begin by defining the corridor gap. This will be an important feature later in our journey.

Definition 4. Let $g \in \mathbb{N}$ equal the gap size in the corridor.

This value g is the number of points in the column of the initial starting path value in the corridor. It simply counts the number of points in the starting column including the starting value. The gap is one of the parameters which is required to establish a skewed top corridor. We will allow the gap to vary over time and will later discuss this in more detail.

Another important parameter for the skewed top corridor is the value that is placed at the beginning of the structure. This value exists at a position in the gap. We will define this position momentarily.

Definition 5. Let $v \in \mathbb{N}$ represent the initial number of paths at the beginning of the corridor.

The position at which the initial value v is placed is an important factor.

Definition 6. Let $a \in \mathbb{N}$ be the location on the horizontal axis where the paths begin. The starting point will always be $(a, 0)$.

The corridor containing all possible path combinations is a subset on an infinite plane of integers. Consider a function whose input is two natural numbers, including zero, and whose output is a natural number or zero.

Definition 7. Let $V: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ and let us refer to this as a corridor state function. The output of V is $\in \mathbb{N}_0$ and denotes the number of paths at some point in the corridor.

As we vary these parameters, different corridors are produced. We've developed the following notation for characterizing corridors.

Definition 8. Let C_g be the corridor with gap $= g$.

The diagonal values in said corridors are one of the features of interest. Diagonals are characterized by their start and end points. We will learn more about the characteristics of corridor diagonals and will develop more definitions later in the paper.

Definition 9. Let D_y, $y \in \mathbb{Z}$, be the diagonal beginning at position $(0, y)$ and ending at an inclusive point on corridor's upper boundary.

Propagating Values

When initializing the skewed top corridor and its values, it is important to follow a specific algorithm. The following is an ordered description of setting up the corridor boundaries, placing initial values, and generating paths in the corridor structure.

Initialize all numbers on the plane to zero. Choose some value for a and some value for v. For initial analysis, we choose to set static values for v and a. Assume, for now, $v = 1$ and $a = 1$. Insert the initial value v into position $(0, a)$, that is, place an initial value of 1 into position $(0, 1)$ on the plane.

Choose some value for g, which represents the size of a gap in the first corridor column. More specifically, g is an indication of where the upper boundary of the corridor begins. If the corridor were placed in an array, the gap column would

consist of g entries: the initial value, and $g-1$ zeros above v. On the plane, the gap begins at inclusive point $(0, 2)$ and extends upward to non-inclusive coordinate $(0, g+1)$. A sentinel marker is placed at $(0, g+1)$ indicating the first establishment of the corridor's upper boundary. From now on, the upper boundary of the corridor may be referred to as the sentinel line.

Assign a value to $m = \frac{1}{r}$. For now, let $r = 2$, so that $m = \frac{1}{2}$. From the initial sentinel point at $(0, g+1)$, continue establishing such points according to the following procedure. Continually move one unit rightward, keeping track of the current column number, or, horizontal coordinate. Record the current position on the vertical axis as well. If the current column is divisible by r, place a sentinel marker at the $(horizontal, vertical)$ position on the grid, move up one unit on the horizontal axis, and move rightward one unit on the vertical axis. If the current column is not divisible by r, do not place a sentinel marker, move up zero units on the horizontal axis, and move rightward one unit on the vertical axis. Do this until the current vertical position is n.

Generally, the procedure for initializing the sentinel line is as follows:

place first sentinel point at $(0, g+1)$

while current vertical is $< n$,

increase horizontal position by one

note current position $(horizontal, vertical)$

if current vertical position $mod\ r = 0$

place sentinel marker at current position

increase vertical position by one

Another way to express this is $(x, y) \in \mathbb{N}_0^2$ with $0 < y < mx + g_1$. Since $m = \frac{1}{2}$, move up one row in the plane when an even column is encountered. This is continued until one reaches the nth column. A sentinel line is established with a slope of $\frac{1}{2}$. For now, assume that every value on or above the sentinel line is equal to zero and assume that every value below the lower boundary is equal to zero. The following recursive formula can be used to calculate each value in the array, where $(0, k) = $ initial state:

$$V(x,y) = V(x-1, y-1) + V(x-1, y+1) \ , \ x \in \mathbb{N} \ , \ 0 < y < mx + g + 1 \ , \ V(0, y) = \begin{cases} v, \text{ if } y = a \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

Observations

The corridor will contain values beginning at $(0, 1)$, with rows starting from 1 extending to $g + \frac{n+1}{2}$ inclusive and columns starting from 0 extending to n inclusive.

The sentinel points implicitly form a line with a slope of m, which we have set to $\frac{1}{2}$. The following statements build upon the characteristics of this line, the sentinel line.

From point $(0, 1)$, a line of 1s with slope one is generated until the sentinel line is encountered at $(2g - 1, 2g)$. Building on the idea of where the line of ones will encounter the sentinel line, it can be proven that the line of ones will always contain $2g$ paths and will always intersect the sentinel line at $(2g - 1, 2g + 1)$.

The line of ones will exist due to the nature of how the numbers in the corridor are generated, see (1). The value $V(k, x)$ accepts an upper contribution of 0 and a lower contribution from the initial value 1. Because of this initial state, a line of 1s will be generated because numbers outside of the corridor are 0 and contributions from the previous 1 will calculate more 1s. An equation for the sentinel line is $y = \frac{1}{2}x + g + 1$ and an equation for the line of ones is $y = x + 1$. To see where these two

lines intersect, simply set the two equations equal to each other and solve for the x and y coordinates of the intersection.

$$\frac{1}{2}x + g + 1 = x + 1$$

$$x = 2g$$

$$y = 2g + 1$$

$$(2g, 2g + 1) \tag{2}$$

The D_{-1} diagonal is the counting numbers. By setup and (1), D_1 is a sequence consisting of $2g$ elements. The following is an inductive proof based on the fact that the line of 1s exists and formula (1).

$$\text{Prove } V(k+1, k) = k \text{ , where } 2 \leq k \leq 2g.$$
$$\text{Base Case: } k = 2$$
$$V(k+1, k) = V(k, k-1) + V(k, k+1)$$
$$V(3, 2) = V(2, 1) + V(2, 3)$$
$$2 = 2$$
$$\text{Assume } V(k+1, k) = k \text{ for all } \leq k \leq 2g. \text{ Use } k+1.$$
$$V(k+2, k+1)$$
$$V(k+1, k) + V(k+1, k+2)$$

$$k + 1^* \qquad\qquad ^*\text{while } k \leq 2g \tag{3}$$

Conjecture

Four-element arithmetic sequences exist in the corridors along with additional sequences of greater length. If D_{-3} is an arithmetic sequence with degree N, where $N \in \mathbb{N}_0$, then D_{-1} is a sequence with degree $N+1$ and D_{-1} is a sequence with degree $N - 1$. Because the values within the corridors are defined by formula (1), D_1 is a degree 0 sequence by setup, and D_{-1} is a degree 1 sequence by (2), sequences contribute to each other linearly via addition. This results in sequences and subsequences in the corridor that are of increasing degree as one approaches $-\infty$ on the y axis.

$2g$-element sequences begin at coordinate $(0, 2y - 1)$, end at coordinate $(2g - 1, 2g - |y|)$, and have degree $N = -\frac{1}{2}(y-1)$. For $(2y-1) < 1$, the first instance four-element subsequences begins at coordinate $(2g, 2g - y)$, ends at coordinate $(2g + 3, 2g - y + 3)$, and is of degree $N = -\frac{1}{2}(y + 1)$.

Subsequence $\sigma_{\hat{y}}^{\hat{x}}$, where $\hat{y} \in \mathbb{Z}$ and $\hat{x} \in \mathbb{N}_0$, denotes the $\hat{x}^{th}$ subsequence of diagonal $D_{\hat{y}}$. Four-element subsequences of $D_{\hat{y}}$ are classified as all $\sigma_{\hat{y}}^{\hat{x}}$ where $\hat{x} > 0$. For all 4-element subsequences after $\hat{x} = 1$, the beginning coordinate is $(2g+3\hat{x}, 2g+3\hat{x})$ and the ending coordinate is $(2g + 3\hat{x} + 3, 2g + 3\hat{x} + 3)$. $\sigma_{\hat{y}}^{\hat{x}}$ with $\hat{x} > 0$ have a degree of $N = -\frac{1}{2}(\hat{y} - 1) - \hat{x}$.

Conclusions