Toward an Understanding of Skewed Top Corridors

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Introduction

Lattice paths have been studied extensively over the course of several centuries. For the purpose of this study, a lattice consists of points $\mathbb{Z}^2$ with certain restrictions and with only two allowable moves, up-right and down-right. Movements in various directions on the lattice are called paths.

A corridor is a rectangular region within a lattice The lattice path enumeration model that we propose consists of a starting point, an upper and lower bound, and all possible paths from said starting point to some end point. The area in which the paths are propagated is referred to as a corridor. The number of paths within the corridor depend on the initial values of the starting point, the nature of the upper and lower bounds, and the value placed at the starting point. In our model, the lower bound is a line with zero slope and the upper bound is a line with a rise one, variable run, slope. These conditions seem to present a problem when attempting to systematically generate the values contained in the corridors. Due to the nature of the upper bound, which is a line with slope non-parallel to that of the lower bound, the paths bounce off of the upper diagonal line, rippling into and distorting the data below it. One would think that calculating the error caused by each interruption would be somewhat intuituve, but the impacts of each diagonal-boundary disturbance grow larger and more disturbing as time in the corridor progresses. Although the data changes because of the upper bound's slope, intruiguing patterns and characteristics have been observed in the configurations of this environment. We call the model which has been breifly introduced a skewed top corridor.

The corridor exists on a two demensional plane which involves the elapse of time on the horizonal axis and the amount of paths on the vertical axis. A gap extends vertically from the starting point and contains the initial value which is placed at the starting point. The gap contains elements, other than the value initialized at the start point, of zero value. A diagonal line, which the paths can touch but not surpass, begins just above the gap and extends to a finite point on the grid. Paths can move in an upward or downward direction in the corridor.

Symbols and Definitions

The paths in this model exist in a space composed of natural numbers including zero. The natural numbers not including zero may also be referenced. Let $\mathbb{N}$ represent the set natural numbers not including zero and $\mathbb{N}_0$ represent the set natural numbers including zero.

Let $\mathbb{Z}$ represent the set of integers which may also be used in describing the system of paths.

The upper boundary restricting the corridor from above has a slope. The slope has a rise of 1 over some variable in the set of natural numbers not including zero, $r \in \mathbb{N}$. Let m represent slope, $m = \frac{1}{r}$ which, for now, is a fixed number, $m = \frac{1}{2}$.

A gap of alloted space is initialized upon creation of the corridor area. This is a column with a height of some value in the set of natural numbers. This gap should be greater than or equal to one. Let $g \in \mathbb{N}$ where $g \geq 1$. This represents a gap above the initial point and contains g elements.

Let $a \in \mathbb{N}$ be the location on the horizontal axis where the paths begin. The starting point will always be a position a on the horizontal and at position 0 on the vertical. This value needs to reside within the gap, so

$a \geq g$.

The paths that are propigated from the start point to some other point begin with some initial number of paths taken. Let $v \in \mathbb{N}$ dictate how many paths exist at the start point.

The model considers the elapse of time at each point in the lattice corridor. For some moment in time, a specific number of paths exits at some point in the grid space. Let $n \in \mathbb{N}_0$ model the time period in which the paths can be modeled where the corridor only extends for n inclusive columns.

The corridor containing all possible path combinations is a subset on an infinite plane of integers. Consider a function, $V\colon \mathbb{N}_0 \times \mathbb{N}_0$. Then $V(x, n)$ represents the number of paths at any point (x, n) in the corridor, where $x \in \mathbb{N}_0$. While the corridor consists of only natural numbers including zero, the two demensional grid space contains all $\mathbb{Z}$ values.

The skewed top corridors may be referenced in relation to their g value. Let C_i, where $i \in \mathbb{N}_0$, be the corridor with $g = i$, $m = \frac{1}{2}$, $a = 1$, and $v = 1$.

The diagonals, a concepts discussed later, may also be referenced in the corridor. Let $C_i d_j$ where $j \in \mathbb{N}_0$. Then $C_i d_j$ is the diagonal j in corridor i.

Propagating Values

Choose some value for a and some value for v. For now, $v = 1$ and $a = 1$. Insert v into position $(0, a)$ in the array. Note that the starting point of all paths is $(0, a)$ and the value at said point is v. The position on the vertical axis of the grid in this case is a.

Choose some value for g, which represents the size of a gap in column 1 of the array. This gap consists of g elements and begins at point $(0, 1)$ in the array. The gap extends upward to $(g + 1, 0)$ and a sentinel boundary is placed at $(g + 1, 0)$. Continue establishing sentinel areas from $(g + 1, 0)$, progressing through each column. Since $m = \frac{1}{2}$, move up one row in the array when an even column is encountered. This is conintued until one reaches the n^{th} column. A sentinel line is established with a slope of $\frac{1}{2}$. For now, assume that every value in the array on or above the sentinel line is equal to zero and assume that every value below row one in the array is equal to zero. The following recursive formula can be used to calulate each value in the array:

$$V(x, k) = V(x - 1, k - 1) + V(x + 1, k - 1)$$

where , $x \in \mathbb{N}$ and $k \in \mathbb{N}$.

Notice that this forms an area of zero and nonzero values which reside above row one and below the sentinel line. The area in which numbers are propigated, having chosen some value for n, consists of n columns and $g + \frac{n+1}{2}$ rows. The values that exist in this area are all natural numbers including zero. With this configuration complete,

- with $a = 1$ and $v = 1$,
- having chosen some n,
- having chosen some g,
- and with $m = \frac{1}{2}$,

various observations can be made.

Observations

The corridor will contain values beginning at $(0, a = 1)$, with rows starting from 1 extending to $g + \frac{n+1}{2}$ and columns starting from 0 extending to n.

From point $(0, a)$, a line of 1s with slope one is generated until the sentenel line is hit at $(2g, 2g + 1)$. Beyond this point, diagonals containing four of the same elements repeat until the end of the corridor is reached. Upon reaching the sentinel line, a new diagonal begins just below the diagonal line of ones. The point at which the first diagonal of four elements begins is $(2g+1, 2g)$. The first repeating diagonals will contain 4 elements and if g is odd, these values will be $2g+1$. If g is even, numbers in the first four-diagonal are $2g-1$.

If $n > 2g$, the one diagonal will never hit the sentinel line before row n is reached.

There are diagonals consisting of only zeros between each diagonal of nonzeros, where a diagonal (for these purposes) begins at some point, $(b \in V,\ c \in V)$, consists of points $(b+1, c+1)$ ... , and extends to some distant point below or on the sentinel line. From any point on the grid, moving in a horizontal line to the n^{th} column, the values on the grid will alternate between zeros and nonzeros.

The diagonal beginning at point $(2,0)$ and ending at $(2g, 2g-1)$ consists of counting numbers from 1 to $2g$. This diagonal gets disturbed by the senintel line and gets added up with the final 1 from the first diagonal line of ones. The resulting value is the first value of the 4 element repeating diagonals.

The diagonal beginning at point $(0,4)$ and ending at $(2g-1, 2g-5)$ contains numbers that are on the outer edge of a triangluar arrangement of $\mathbb{Z}$.

In a horizontal traversal with slope 0 from point $(1,0)$ to point $(1, 2g-1)$, one can observe the Catalan numbers C_0 to C_g, however, this pattern quickly gets destroyed at row $2g-1$ when the first diagonal of ones hits the sentinel line. This causes what should be $g+1^{th}$ Catalan number to be off by negetive one or positive one, thus this pattern disappears.

The various values that exist in the 4 element repeating diagonals do not seem to have any known relationship.

Conjecture

Generating corridors starting at C_1 to some finite number, note the relationships between the $C_i d_j^{th}$ diagonals across configurations. It appears that d_j^{th} diagonals among corridors make arithmetic sequences to the j^{th} degree.

The formula for the d_0 diagonal with $x_0 = 1$ is the following:

$$f(x) = 1$$

The formula for the d_1 diagonal with $x_0 = 1$ is the following:

$$f(x) = 2(x-1) + 1$$

The formula for the d_2 diagonal with $x_0 = 1$ is the following:

$$f(x) = 2x^2 + 9x + 3$$

The formula for the d_3 diagonal with $x_0 = 1$ is the following:

$$f(x) = \frac{4}{3}x^3 + 16x^2 + \frac{149}{3}x + 13$$

The formula for the d_4 diagonal with $x_0 = 1$ is the following:

$$f(x) = \frac{2}{3}x^4 + \frac{46}{3}x^3 + \frac{707}{6}x^2 + \frac{1867}{6}x + 68$$

As the values for g increment for a new configuration of C_i, it seems that the arithmetic sequences continue to follow the pattern of being sequences of the j^{th} degree.

Bibliography

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