Homework 2

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Prove the following statements. Show and explain **ALL** your work. Unless otherwise specified, give direct proofs.

Problem 1. The sum of two even integers is always even.

Proof. If a and b are even integers, then $a + b$ is an even integer.

Assume a and b are even integers.

Then by definition,
$a = 2m$ where $m \in \mathbb{Z}$,
$b = 2n$ where $n \in \mathbb{Z}$.

Then,

$$\begin{aligned} a + b &= (2m) + (2n) \\ &= 2(m + n), \end{aligned}$$

and $2(m + n)$ is an even integer by definition.

Therefore, $a + b$ is even by definition. □

Problem 2. State the contrapositive of (for all integers n, if n^2 is odd, then n is odd) and prove it.

$$\begin{aligned} \text{Given}\colon & \forall n \in \mathbb{Z}\,(n^2 \text{ odd} \rightarrow n \text{ odd}) \\ \text{Contrapositive}\colon & \forall n \in \mathbb{Z}\,(n \text{ even} \rightarrow n^2 \text{ even}) \end{aligned}$$

Proof. $\forall n \in \mathbb{Z}\,(n \text{ even} \rightarrow n^2 \text{ even})$

Let n be an integer.

Assume n is even.

Then by definition, $n = 2k$ where $k \in \mathbb{Z}$.

So,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= (4)(k \cdot k) \\ &= 2k \cdot 2k. \end{aligned}$$

And two even integers multiplied are always even.

Therefore, n^2 is even. □

Problem 3. Prove that if the sum of the digits of a 3-digit number n is divisible by 9, then n is divisible by 9.

Proof. If the sum of the digits of a 3-digit number n is divisible by 9, then n is divisible by 9.

Let n be an integer.

Assume that n is a 3-digit number such that $n \geq 100 \land n \leq 999$.

n can be represented by its digits as,

$$n = n_3(100) + n_2(10) + n_1(1).$$

Then, by rearranging the previous expression,

$$\begin{aligned} n &= n_3(1+99) + n_2(1+9) + n_1 && 100 = 1 + 99 \\ &= n_3 + n_3(99) + n_2 + n_2(9) + n_1 && \text{Distributive property of multiplication} \\ &= n_3(99) + n_2(9) + n_3 + n_2 + n_1 && \text{Associative property of addition.} \end{aligned}$$

Since divisibility is defined as $a/b = c$ where $a, b, c \in \mathbb{Z}$, and

$$\frac{n_3(99)}{9} = n_3(11) \text{ and } n_3(11) \in \mathbb{Z},$$
$$\frac{n_2(9)}{9} = n_2(1) \text{ and } n_2(1) \in \mathbb{Z},$$

so the terms $n_3(99)$ and $n_2(9)$ must be divisible by 9.

Then, n is divisible by 9 if the remaining terms (n_3, n_2, n_1) sum such that,

$$\frac{n_3 + n_2 + n_1}{9} = k \text{ (where } k \in \mathbb{Z})$$

Then, if all terms are divisible by 9, the sum of those terms must be divisible by 9.

Therefore, if $(n_3 + n_2 + n_1)/9 = k$ (where $k \in \mathbb{Z}$), then n is divisible by 9. $\square$

Problem 4. Prove by contradiction that the product of two odd numbers is odd.

Proof. $(\forall m \in \mathbb{Z})(\forall n \in \mathbb{Z})\,((m \text{ odd} \wedge n \text{ odd}) \rightarrow ((m \cdot n) \text{ odd}))$

Let m and n be integers.

Assume m and n are odd.

To prove by contradiction, assume $m \cdot n$ is even.

Then by definition, $mn = 2k$ for some integer k.

So,

$$\begin{aligned} 2k + 1 &= mn \\ 2k &= mn - 1 \\ k &= \frac{mn - 1}{2} \\ k &= mn - \frac{1}{2} \end{aligned}$$

Then, k is a rational number and not an integer $(k \in \mathbb{Q} \wedge k \notin \mathbb{Z})$,
since mn is an integer, given that the product of two integers is an integer,
and subtracting $1/2$ from an integer will result in a rational number.

This is a contradiction since k was assumed to be an integer.

Therefore, $m \cdot n$ cannot be even. $\square$

Problem 5. Prove that the product of two rational numbers is rational.

Proof. $(\forall p \in \mathbb{Q})(\forall q \in \mathbb{Q})\,((p \cdot q) \in \mathbb{Q})$

Let p and q be rational numbers.

Then by definition,

$$\begin{aligned} p &= \frac{a}{b} \\ q &= \frac{c}{d}, \end{aligned}$$

where $\{a, b, c, d\} \in \mathbb{Z}$.

Then,

$$\begin{aligned} p \cdot q &= \frac{a}{b} \cdot \frac{c}{d} \\ &= \frac{ab}{cd}. \end{aligned}$$

Then $\{ab, cd\} \in \mathbb{Z}$ since the product of two integers is an integer.

So then, $\frac{ab}{cd}$ is a rational number.

Therefore, the product of two rational numbers is rational. $\square$