# CS330 Activities for Lecture 15 *

## 6.1 Linear Recurrence Relations

In a divide and conquer recurrence relation $T(n) = a\,T(n/b) + f(n)$

"$a$" represents ____________ "$n/b$" represents ____________ "$f(n)$" represents ____________

### Linear Homogeneous Recurrence Relation

- We'll study how to solve one class of recurrence relations, the linear homogeneous ones. In this kind of relation, each number in a sequence is a linear combination of numbers that occur earlier in the sequence.
- Specifically, a **linear homogeneous recurrence relation** of degree $k$ has the following form: $a_n = c_1\,a_{n-1} + c_2\,a_{n-2} + \cdots + c_k\,a_{n-k}$, where $c_1, c_2, \ldots$ are constants that do not depend on $n$ and $c_k \neq 0$.
  - The "linear" part of the definition refers to only using constants as coefficients.
  - The "homogeneous" part of the definition refers to only using previous terms in the equation.
- Some examples:
  - $a_n = 1.02\,a_{n-1}$ Linear (constant coefficients), homogenous, degree 1
  - $a_n = 1.02\,a_{n-1} + 2^{n-1}$ Linear, non-homogenous (due to term $2^{n-1}$), degree 1
  - $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$ Linear, non-homogenous (due to term $2^{n-3}$), degree 3
  - $a_n = n\,a_{n-1} + n\,a_{n-2} + a_{n-1}\,a_{n-2}$ Nonlinear (due to non-constant coefficient $a_{n-1}$), homogenous, degree 2

- To summarize, the special form we will be solving has constant coefficients, is homogeneous and has some degree $k$.
  - $a_n = c_1\,a_{n-1} + c_2\,a_{n-2} + \ldots + c_k\,a_{n-k}$
- The solution uses exponentials: We assume $a_n = b^n$ for some $b$. Treating $b$ as a variable, we get a polynomial and solve it for $b$. We find the constants $c_1, c_2, \ldots$ with the help of the initial conditions.

### Solution Process

1. Put all $a_i$'s on the left side of the equation, everything else on the right. If right side is not "= 0", the recurrence is non-homogeneous: stop.
   $$a_n - c_1\,a_{n-1} - c_2\,a_{n-2} - \ldots - c_k\,a_{n-k} = 0$$
2. Assume a solution of the form $a_n = b^n$ and substitute the solution into the equation.
   $$b^n - c_1\,b^{n-1} - c_2\,b^{n-2} - \ldots - c_k\,b^{n-k} = 0$$
3. Factor out the lowest power of $b$ and eliminate it to obtain the *characteristic polynomial*.
   $$b^{n-k}\,(b^k - c_1\,b^{k-1} - c_2\,b^{k-2} - \ldots - c_k) = 0$$
   $$b^k - c_1\,b^{k-1} - c_2\,b^{k-2} - \ldots - c_k = 0$$
4. The characteristic polynomial has degree $k$. Find its $k$ roots, $r_1, r_2, \ldots, r_k$ (we assume they are all reals).
5. If the roots are distinct, the general solution is $a_n = c_1\,r_1^n + c_2\,r_2^n + \ldots + c_k\,r_k^n$
6. The coefficients $c_1, c_2, \ldots, c_k$ are found by enforcing the initial conditions.

**Example 1**: $a_{n+2} = 3\,a_{n+1}$, $a_0 = 4$

- Bring subscripted variables to one side; we get $a_{n+2} - 3\,a_{n+1} = 0$
- Substitute $a_n = b^n$ to get $b^{n+2} - 3\,b^{n+1} = 0$ and factor out $b^{n+1}$ to get $b^2 - 3 = 0$
- Find the root of the characteristic polynomial; we get $r_1 = 3$
- Compute the general solution: $a_n = c\,3^n$

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- Find the constant based on the initial conditions: $a_0 = 4$ and $a_0\ c\ 3^0$, so $c = 4$
- Produce the specific solution: $a_n = 4(3^n)$

***Multiple roots***

- We've seen that distinct roots produce different terms, as in $c_1\ r_1{}^n + c_2\ r_2{}^n$
- If a root $r$ has multiplicity $m$, then we still get $m$ terms but they become linear combinations of $r^n$, $n\ r^n$, $n^2\ r^n$, …, $n^{m-1}\ r^n$.
- E.g. for the characteristic polynomial $(x - 5)^3$, we get $a_n = c_1\ 5^n + c_2\ n\ 5^n + c_3\ n^2\ 5^n$.

**Example 2**: $a_n = 6a_{n-1} - 9a_{n-2}$ , $a_0 = a_1 = 1$

- Recurrence system: $a_n - 6\ a_{n-1} + 9a_{n-2} = 0$
- Find roots of characteristic polynomial $b^2 - 6b + 9 = 0$: $(b-3)^2 = 0$
- Roots are equal: $b_1 = b_2 = 3$
- General solution is $a_n = c_1\ 3^n + c_2\ n\ 3^n$
- Solve for coefficients:
  - $a_0 = 1 = c_1 + 0$ $c_1 = -1$
  - $a_1 = 1 = 1(3^1) + c_2\ (1)(3^1)$ $c_2 = -2/3$
- Solution: $a_n = (-1)3^n + (-2/3)\ n\ 3^n$

1. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$?

2. What is the solution of the recurrence relation $a_n = 6\ a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

3. What is the solution of the recurrence relation $a_n = 6\,a_{n\text{-}1} - 11a_{n\text{-}2} + 6a_{n\text{-}3}$ with $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$?