

Homework 2

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CS 330 - Discrete Structures

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Prove the following statements. Show and explain **ALL** your work. Unless otherwise specified, give direct proofs.

Problem 1. The sum of two even integers is always even.

Proof. If a and b are even integers, then $a + b$ is an even integer.

Assume a and b are even integers.

Then by definition,

$$a = 2m \text{ where } m \in \mathbb{Z},$$

$$b = 2n \text{ where } n \in \mathbb{Z}.$$

Then,

$$\begin{aligned} a + b &= (2m) + (2n) \\ &= 2(m + n), \end{aligned}$$

and $2(m + n)$ is an even integer by definition.

Therefore, $a + b$ is even by definition. □

Problem 2. State the contrapositive of (for all integers n , if n^2 is odd, then n is odd) and prove it.

$$\text{Given : } \forall n \in \mathbb{Z} (n^2 \text{ odd} \rightarrow n \text{ odd})$$

$$\text{Contrapositive : } \forall n \in \mathbb{Z} (n \text{ even} \rightarrow n^2 \text{ even})$$

$$\text{Proof. } \forall n \in \mathbb{Z} (n \text{ even} \rightarrow n^2 \text{ even})$$

Let n be an integer.

Assume n is even.

Then by definition, $n = 2k$ where $k \in \mathbb{Z}$.

So,

$$n^2 = (2k)^2 \qquad \qquad \qquad = (4)(k \cdot k) = 2k \cdot 2k.$$

And two even integers multiplied are always even.

Therefore, n^2 is even. □

Problem 3. Prove that if the sum of the digits of a 3-digit number n is divisible by 9, then n is divisible by 9.

Proof. If the sum of the digits of a 3-digit number n is divisible by 9, then n is divisible by 9.

Let n be an integer.

Assume that n is a 3-digit number such that $n \geq 100 \wedge n \leq 999$.

n can be represented by its digits as,

$$n = n_3(100) + n_2(10) + n_1(1).$$

Then, by rearranging the previous expression,

$$\begin{aligned} n &= n_3(1 + 99) + n_2(1 + 9) + n_1 & 100 &= 1 + 99 \\ &= n_3 + n_3(99) + n_2 + n_2(9) + n_1 & &\text{Distributive property of multiplication} \\ &= n_3(99) + n_2(9) + n_3 + n_2 + n_1 & &\text{Associative property of addition.} \end{aligned}$$

Since divisibility is defined as $a/b = c$ where $a, b, c \in \mathbb{Z}$, and

$$\begin{aligned} \frac{n_3(99)}{9} &= n_3(11) \text{ and } n_3(11) \in \mathbb{Z}, \\ \frac{n_2(9)}{9} &= n_2(1) \text{ and } n_2(1) \in \mathbb{Z}, \end{aligned}$$

so the terms $n_3(99)$ and $n_2(9)$ must be divisible by 9.

Then, n is divisible by 9 if the remaining terms (n_3, n_2, n_1) sum such that,

$$\frac{n_3 + n_2 + n_1}{9} = k \text{ (where } k \in \mathbb{Z}\text{)}$$

Then, if all terms are divisible by 9, the sum of those terms must be divisible by 9.

Therefore, if $(n_3 + n_2 + n_1)/9 = k$ (where $k \in \mathbb{Z}$), then n is divisible by 9. □

Problem 4. Prove by contradiction that the product of two odd numbers is odd.

Proof. $(\forall m \in \mathbb{Z})(\forall n \in \mathbb{Z}) ((m \text{ odd} \wedge n \text{ odd}) \rightarrow ((m \cdot n) \text{ odd}))$

Let m and n be integers.

Assume m and n are odd.

To prove by contradiction, assume $m \cdot n$ is even.

Then by definition, $mn = 2k$ for some integer k .

So,

$$2k + 1 = mn$$

$$2k = mn - 1$$

$$k = \frac{mn - 1}{2}$$

$$k = mn - \frac{1}{2}$$

Then, k is a rational number and not an integer ($k \in \mathbb{Q} \wedge k \notin \mathbb{Z}$), since mn is an integer, given that the product of two integers is an integer, and subtracting $1/2$ from an integer will result in a rational number.

This is a contradiction since k was assumed to be an integer.

Therefore, $m \cdot n$ cannot be even. □

Problem 5. Prove that the product of two rational numbers is rational.

Proof. $(\forall p \in \mathbb{Q})(\forall q \in \mathbb{Q}) ((p \cdot q) \in \mathbb{Q})$

Let p and q be rational numbers.

Then by definition,

$$p = \frac{a}{b}$$

$$q = \frac{c}{d},$$

where $\{a, b, c, d\} \in \mathbb{Z}$.

Then,

$$\begin{aligned} p \cdot q &= \frac{a}{b} \cdot \frac{c}{d} \\ &= \frac{ac}{bd}. \end{aligned}$$

Then $\{ac, bd\} \in \mathbb{Z}$ since the product of two integers is an integer.

So then, $\frac{ac}{bd}$ is a rational number.

Therefore, the product of two rational numbers is rational. □