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Professor Sasaki
CS 330 – Discrete Structures
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Homework 9

1) *Points = 4. What kind of graph can be used to model a highway system between major cities where*
    a) *There is an edge between the vertices representing cities if there is an interstate highway between them?*

    undirected simple graph

    undirected – since the edge is assumed to represent a two-way road ("connection") between cities
    simple – since there is only ever at most one edge between cities, there cannot be any parallel edges or self-loops

    b) *There is an edge between the vertices representing cities for each interstate highway between them?*

    undirected graph

    undirected – since the edge is assumed to represent a two-way road ("connection") between cities
    not simple – since there could be more than one interstate highway been a given city, hence parallel edges which cannot be part of a simple graph

2) *Points = 2.*
    a) *Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?*
    To model email messages in a network, a directed graph could be used to model the sender (*initial vertex*) and the receiver (*terminal vertex*) where each vertex is a machine able to send and receive emails while being attached to the network.
    Likewise, multiple edges would be allowed since a given machine (*vertex*) could repeatedly send emails to the same other machine (*vertex)*, also hence *parallel edges*.
    Loops would be allowed since the same message could be sent to the same machine, for example, if the sender and the receiver's mailbox were stored on the same machine.

    b) *Describe a graph that models the electronic mail sent in a network in a particular week.*
    The same graph can be used as outlined in 2a with the exception that there would only ever be an edge connected to other machines (*vertices*) only if there was ever an email send or received to that machine (*vertices*).

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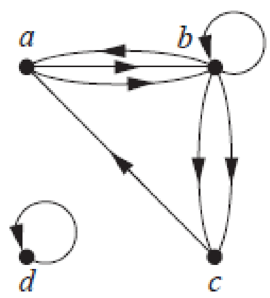
3) Points = 2.

   a) List the in-degrees and out-degrees of each vertex for the graph below. Give the sums of the in-degrees and the out-degrees (they should be equal and also equal to the number of edges in the graph).



Vertex	In-degree	Out-degree
a	2	2
b	3	4
c	2	1
d	1	1
Sum	**8**	**8**

4) Points = 2. In a Hollywood graph, actors are vertices; an edge between actors indicates they've been in one or more movies together. What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?

   i) *degree of a vertex* – represents the number of other actors that the given actor (*vertex*) has been in movies together with
   ii) *neighborhood of a vertex* – represents the other actors that the given actor (*vertex*) has been in movies together with
   iii) *isolated vertex* – represents a given actor (*vertex*) who has never been together in any movies with any other actors
   iv) *pendant vertex* – represents a given actor (*vertex*) who has only been together in any movies with only one other actor

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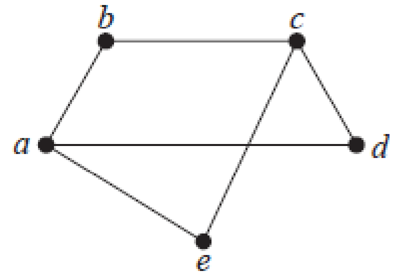
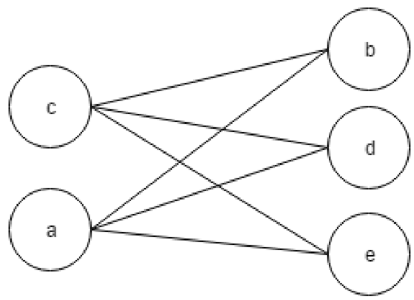
5) Points = 2. Is this graph bipartite? If it is, give a partition.





Yes, the graph is bipartite (as shown above).
A valid partition is of vertices $U = \{c, a\}$ and vertices $V = \{b, d, e\}$

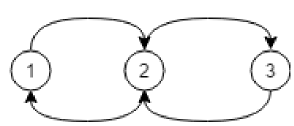
6) Points = 2. Is 6, 5, 4, 3, 2, 1 a possible degree sequence for a simple graph? If it is, draw the graph.

No, the total degree of a graph must be even.
$6 + 5 + 4 + 3 + 2 + 1 = 21$, and 21 is not even.

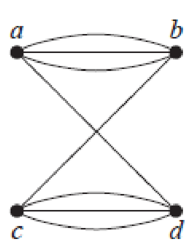
7) Points = 8.

a. Draw a directed graph for this adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



b. Give an adjacency matrix representation for this multigraph:



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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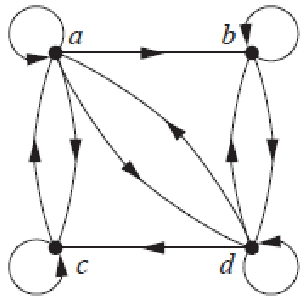
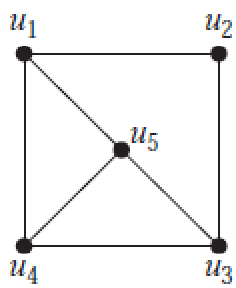
c. *Give an adjacency matrix for this graph:*

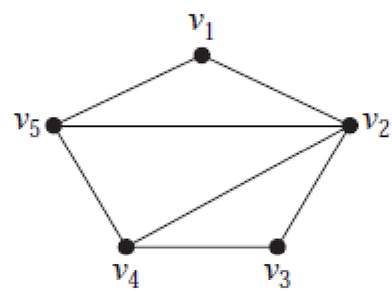


$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

8) *Points = 3. What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?*
   a) For undirected graph – number of vertices connected to the given vertex
   b) For directed graph – the number of terminal vertices connected at the given vertex

9) *Points = 3. Are these graphs isomorphic? If yes, give a correspondence of the vertices. If not, give a rigorous argument why not.*





No, the graphs are not isomorphic.

Rigorously, in a proof by contradiction, given that degree is preserved under isomorphism, then the degree sequence of each graph must be the same for them to be isomorphic. Let the graph on the left be graph A and the graph on the right be graph B. Assume the degree sequence of graph A and graph B are the same.

The degree sequence for graph A is $3, 3, 3, 3, 2,$
but the degree sequence of graph B is $4, 3, 3, 2, 2$.

Since the degree sequences differ, this is a contradiction since we assumed the degree sequence for graph A and graph B were the same.

Therefore, graph A and graph B are not isomorphic.

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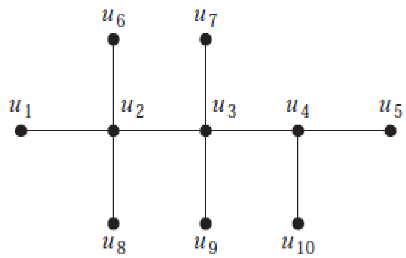
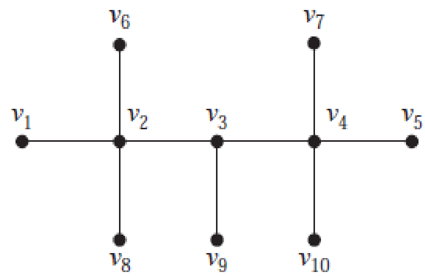
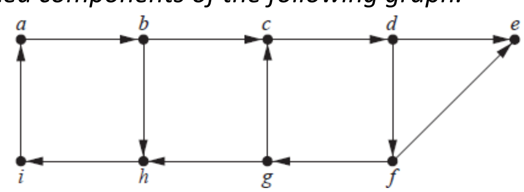
10) *Points = 3. Repeat, for these two graphs.*





Proof by contradiction. Given that degree is preserved under isomorphic, assume that each vertex u_i has the same degree as the corresponding vertex v_i. Vertex u_3 is of degree 4 while vertex u_3 is of degree 3. This is a contradiction since we state each vertex u_i must have the same degree as the corresponding vertex v_i. Therefore, the two graphs are not isomorphic.

11) *Points = 3. In a directed graph, a* strongly-connected component *is a subgraph where for every pair of vertices v and w, there is a directed path from v to w and another from w to v. The subgraph should be* maximal *(you can't add another vertex without losing strong connectedness. Find the strongly-connected components of the following graph.*



The strongly-connected components of the graph are:

- Sub-graph $G_1 = (V_1, E_1)$
  where $V_1 = \{a, b, c, d, f, g, h, i\}$
  $E_1 = \{(a, b), (b, c), (c, d), (d, f), (f, g), (g, h), (h, i), (i, a)\}$

- Sub-graph $G_2 = (V_2, E_2)$
  where $V_2 = \{e\}$
  $E_2 = \emptyset$

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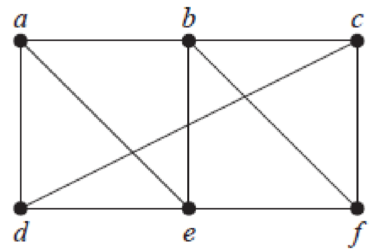
12) Points = 6. Consider this graph.



a. Give its adjacency matrix representation, call it *A*.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b. *Calculate and give A^2. What is the value of $A^2[c, d]$; what does it value represent?*

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 3 & 2 & 2 \\ 2 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 & 4 & 1 \\ 2 & 2 & 1 & 2 & 1 & 3 \end{bmatrix}$$

$A^2[c, d] = 0$, where 0 represents the number of walks of length 2 from c to *d.*

c. *Repeat for A^3 and $A^3[c , d]$.*

$$A^3 = A * A * A = \begin{bmatrix} 4 & 9 & 4 & 7 & 7 & 5 \\ 9 & 6 & 9 & 4 & 10 & 7 \\ 4 & 9 & 2 & 8 & 4 & 7 \\ 7 & 4 & 8 & 2 & 9 & 4 \\ 7 & 10 & 4 & 9 & 6 & 9 \\ 5 & 7 & 7 & 4 & 9 & 4 \end{bmatrix}$$

$A^3[c , d] = 8$, where 8 represents the number of walks of length 3 from c to *d.*