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New mathematical models of the generalized vehicle routing problem and extensions

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ABSTRACT

The generalized vehicle routing problem (GVRP) is an extension of the vehicle routing problem (VRP) and was introduced by Ghiani and Improta [1]. The GVRP is the problem of designing optimal delivery or collection routes from a given depot to a number of predefined, mutually exclusive and exhaustive node-sets (clusters) which includes exactly one node from each cluster, subject to capacity restrictions. The aim of this paper is to provide two new models of the GVRP based on integer programming. The first model, called the node formulation is similar to the Kara–Bektaş formulation [2], but produces a stronger lower bound. The second one, called the flow formulation, is completely new. We show as well that under specific circumstances the proposed models of the GVRP reduces to the well known routing problems. Finally, the GVRP is extended for the case in which the vertices of any cluster of each tour are contiguous. This case is defined as the clustered generalized vehicle routing problem and both of the proposed formulations of GVRP are adapted to clustered case.

1. Introduction

Problems associated with determining optimal routes for vehicles from one or several depots to a set of locations/customers, subject to various constraints, such as vehicle capacity, route length, time windows, etc., are known as vehicle routing problems (VRPs). These problems have a significant economic importance due to the many practical applications in the field of distribution, collection, logistics, etc. A wide body of literature exists on the VRP problem (for an extensive bibliography, see Laporte and Osman [3], Laporte [4] and the book edited by Ball et al. [5]).

The generalized vehicle routing problem (GVRP) is a generalization of the VRP, in which the nodes of a graph are partitioned into a given number of nodes sets, called clusters, and we are interested in finding the optimal routes from the given depot to the number of predefined clusters which include exactly one node from each cluster.

The GVRP belongs to the class of generalized combinatorial optimization problems. Classical combinatorial optimization problems can be generalized in a natural way by considering a related problem relative to a given partition of the nodes of the graph into node sets. In the literature one finds generalized problems such as the generalized minimum spanning tree problem (GMSTP), the generalized traveling salesman problem (GTSP), the generalized vehicle routing problem (GVRP), the generalized Steiner tree problem, the generalized (subset) assignment problem, etc. These generalized problems typically belong to the class of NP-complete problems, are harder than the classical ones and nowadays are intensively studied due to the interesting properties and applications in the real world.

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The GVRP can be seen as well as a variant of the generalized traveling salesman problem (GTSP), which is an extension of the well known traveling salesman problem. An extensive research exists on the GTSP (see for example [6–9], etc). Integer linear programming formulations for GTSP are presented by Laporte and Nobert [6] and Fischetti et al. [8]. In these formulations, the number of the constraints grows exponentially with the number of the nodes of the graph. Recently, Pop [10] proposed six new integer programming formulations four of them are polynomial size formulations for GTSP. We could not observe any formulation for the multiple traveler case of the GTSP, namely the generalized multiple traveling salesman problem (GmTSP). In fact, the GVRP can be considered as an extension of the GmTSP where travelers are turn to be vehicles having limited capacities and clusters have a demand to be satisfied.

The GVRP has been introduced by Ghiani and Improta [1] in 2000. They proposed as well a solution procedure by transforming the GVRP into a capacitated arc routing problem for which an exact algorithm and several approximate procedures are reported in literature. In 2003, Kara and Bektaş [2] proposed an integer programming formulation for GVRP with a polynomially increasing number of binary variables and constraints. Pop et al. proposed an ant colony based algorithm [11] and recently a genetic algorithm [12] for solving the GVRP.

In this paper we present two integer linear programming formulations for GVRP with $O(n^2)$ binary variables and $O(n^2)$ constraints. The first model, named as node based formulation, is structurally similar to the Kara–Bektaş [2] formulation, but we show that it produces stronger lower bounds than Kara–Bektaş formulation. The second one, named as flow based formulation is completely new. Under specific circumstances, we show that, the proposed models reduce to the GmTSP and GTSP. Finally, the GVRP is extended for the case where a vehicle follows a Hamiltonian path within each cluster. This case is defined as the clustered generalized vehicle routing problem and the proposed formulations for the GVRP are adapted to this case.

The remainder of this paper is organized as follows. In the second section we define the generalized vehicle routing problem (GVRP). Section 3 contains the new integer linear programming formulations of the GVRP. We show in Section 4 that special cases of the proposed formulations reduce to the integer programming models of the GmTSP and GTSP. Finally, the GVRP is extended for the case in which the vertices of any cluster of each tour are contiguous. This case is defined as the clustered generalized vehicle routing problem and both of the proposed formulations of GVRP are extended to clustered case in Section 5. In Section 6, we considered the numerical example described by Ghiani and Improta [1] for which we provided the optimal solution using our novel integer programming based formulations of the GVRP. The paper concludes with some remarks and further suggestions given in Section 7.

2. Definition of the GVRP

Let $G = (V,A)$ be a directed graph with $V = \{0,1,2,\dots,n\}$ as the set of vertices and the set of arcs $A = \{(i,j)|i, j \in V, i \neq j\}$. A nonnegative cost c_{ij} associated with each arc$(i,j) \in A$. The set of vertices (nodes) is partitioned into $k+1$ mutually exclusive nonempty subsets, called clusters, $V_0, V_1, \dots, V_k$ (i.e. $V = V_0 \cup V_1 \cup \cdots \cup V_k$ and $V_l \cap V_p = \emptyset$ for all $l, p \in \{0,1,\dots,k\}$ and $l \neq p$). The cluster V_0 has only one vertex 0, which represents the depot, and remaining n nodes belonging to the remaining k clusters

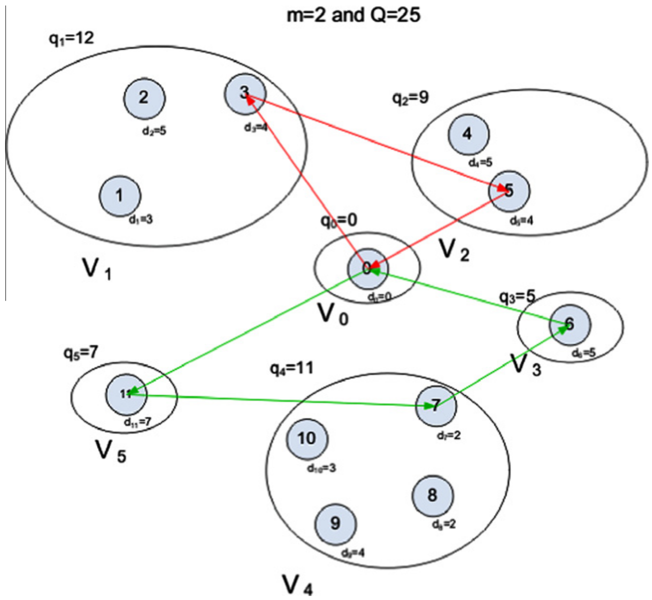

Fig. 1. An example of a feasible solution of the GVRP.

represent geographically dispersed customers. Each customer has a certain amount of demand and the total demand of each cluster can be satisfied via any of its nodes. There exist m identical vehicles, each with a capacity Q.

The generalized vehicle routing problem (GVRP) consists in finding the minimum total cost tours of starting and ending at the depot, such that each cluster should be visited exactly once, the entering and leaving nodes of each cluster is the same and the sum of all the demands of any tour (route) does not exceed the capacity of the vehicle Q. An illustrative scheme of the GVRP and a feasible tour is shown in Fig. 1.

The GVRP reduces to the classical vehicle routing problem (VRP) when all the clusters are singletons and to the generalized traveling salesman problem (GTSP) when $m = 1$ and $Q = \infty$.

The GVRP is *NP*-hard because it includes the generalized traveling salesman problem as a special case when $m = 1$ and $Q = \infty$.

Several real-world situations can be modeled as a GVRP. The post-box collection problem described in Laporte et al. [13] becomes an asymmetric GVRP if more than one vehicle is required. Furthermore, the GVRP is able to model the distribution of goods by sea to a number of customers situated in an archipelago as in Philippines, New Zealand, Indonesia, Italy, Greece and Croatia. In this application, a number of potential harbors is selected for every island and a fleet of ships is required to visit exactly one harbor for every island.

Several applications of the GTSP may be extended naturally to GVRP. In addition, several other situations can be modeled as a GVRP, these include:

- the traveling salesman problem (TSP) with profits [14];
- a number of vehicle routing problem (VRP) extensions: the VRP with selective backhauls, the covering VRP, the periodic VRP, the capacitated general windy routing problem, etc.;
- the design of tandem configurations for automated guided vehicles [15].

3. Integer linear programming formulations

In this section we first propose a general semi closed formulation for GVRP, then we specify the explicit forms of this model and finally we present two new formulations of the problem.

3.1. A general formulation

In order to model the GVRP as an integer programming problem, we define the related sets, decision variables and parameters as follows:

Sets:

$V = \{0,1,2,\ldots,n\}$ the set of nodes corresponding to customers, where 0 represents the origin (depot). Let V be partitioned into mutually exclusive and exhaustive non-empty sub-sets $V_0, V_1, \ldots, V_k$, each of which represents a cluster of customers, where $V_0 = \{0\}$ is the origin (home city, depot).
$K = \{0,1,2,\ldots,k\}$ the set of clusters.
$A = \{(i,j) | i, j \in V, i \neq j\}$ the set of arcs.

Decision variables:
We define the following binary variables:

$$x_{ij} = \begin{cases} 1 & \text{if arc}(i,j) \text{ is included in the tour of a vehicle}, \quad i \in V_p,\ j \in V_r,\ p, r \in K, \\ 0 & \text{otherwise}, \end{cases}$$

$$w_{pr} = \begin{cases} 1 & \text{if there is a path from cluster } V_p \text{ to cluster } V_r, p, r \in K, \\ 0 & \text{otherwise}. \end{cases}$$

Parameters:
Let,

c_{ij} be the cost of traveling from node i to node j, $i \neq j$, $i \in V_p$, $j \in V_r$, $p \neq r$, p, $r \in K$;
d_i be the demand of customer i, $i = 1,2,\ldots,n$;
q_r be the demand of the cluster V_r, $q_r = \sum_{i \in V_r} d_i$, $r \in K$;
$M = \{1,\ldots,m\}$ be the set of the uniform vehicles (tours);
Q be the capacity of each vehicle.

Through out this paper, we assume that $k \geqslant m$.
Constraints of the problem, under the heading to which they correspond to, are given below:

Cluster degree constraints

For each cluster excluding V_0, there can be only a single outgoing arc to any other node belonging to other clusters. This condition is implied by the following constraints:

$$\sum_{i \in V_p} \sum_{j \in V \setminus V_p} x_{ij} = 1, \quad p \neq 0, \ p \in K. \tag{1}$$

There can be only a single incoming (entering) arc to a cluster from any other node belonging to other clusters, excluding V_0. This condition is implied by the following constraints:

$$\sum_{i \in V \setminus V_p} \sum_{j \in V_p} x_{ij} = 1, \quad p \neq 0, \ p \in K. \tag{2}$$

There should be at most m leaving arcs from and at most m entering arcs to the home city (origin), conditions which are implied by the following constraints:

$$\sum_{i=1}^{n} x_{0i} \leqslant m, \tag{3}$$

$$\sum_{i=1}^{n} x_{i0} \leqslant m. \tag{4}$$

Cluster connectivity constraints

The entering and leaving nodes should be the same for each cluster, which is satisfied by:

$$\sum_{i \in V \setminus V_p} x_{ij} = \sum_{i \in V \setminus V_p} x_{ji}, \quad j \in V_p, \ p \in K. \tag{5}$$

Flows from cluster V_p to the cluster V_r are defined by w_{pr}. Thus, w_{pr} should be equal to the sum of x_{ij}'s from V_p to V_r. Hence, we write:

$$w_{pr} = \sum_{i \in V_p} \sum_{j \in V_r} x_{ij}. \tag{6}$$

With the above definitions and constraints, a general integer programming formulation for the GVRP may be given as:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}, \\
\text{subject to} \quad & (1)-(6), \\
& \text{Capacity bounding constraints}, && (7) \\
& \text{Subtour elimination constraints}, && (8) \\
& x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A.
\end{aligned}$$

Note that w_{pr} will automatically be 0 or 1 by the degree constraints given by (1)–(4). Therefore we do not need to impose integrality constraints for w_{pr}.

The constraints given in (1)–(5) are similar to those given by Noon and Bean [7] for the generalized traveling salesman problem. Constraints given in (6) have been introduced by Kara and Bektaş [2].

Model formulations of the GVRP may be differ from each other with respect to the "capacity bounding" and/or "subtour elimination" constraints given in (7) and (8) above.

In the case of an integer programming formulation, if the number of the constraints of the formulation grows exponentially with respect to the number of the nodes of the graph, it is called as *exponential size formulation*; and if the number of the constraints grows polynomially, it is called *polynomial size formulation*. In addition, if a formulation has additional auxiliary decision variables, it may be classified with respect to the new decision variables as, *node based formulation* if the additional variables are relative to the nodes of the graph, and *flow based formulation* if the new variables are relative to the arcs of the graph.

By defining new auxiliary variables, we propose integer programming formulations for the GVRP in the following subsections.

3.2. A node based formulation

In addition to the decision variables defined before, let us define the following auxiliary variables:

u_p: be the load of a vehicle just after leaving the cluster V_p (collection case) or delivered amount of the goods from a vehicle just after leaving the cluster V_p (delivery case), $p \neq 0$, $p \in K$.

3.2.1. Capacity bounding constraints

The following proposition gives capacity bounding constraints for a node based formulation.

Proposition 1. *The following inequalities are valid capacity bounding constraints for the GVRP:*

$$u_p - \sum_{r\in K, r\neq p} q_r w_{rp} \geqslant q_p, \quad p \neq 0,\ p \in K, \tag{9}$$

$$u_p + (Q - q_p)w_{0p} \leqslant Q, \quad p \neq 0,\ p \in K, \tag{10}$$

$$u_p + \sum_{r\in K, r\neq p} q_r w_{pr} \leqslant Q, \quad p \neq 0,\ p \in K, \tag{11}$$

where $q_0 = 0$.

Proof. Any cluster V_p may be on the first, or last or intermediate position of a vehicle tour, i.e., either $w_{0p} = 1$ and/or $w_{p0} = 1$ or there exist some cluster V_s and V_t such that $w_{sp} = w_{pt} = 1$.

If $w_{0p} = 1$ (or $w_{0p} = w_{p0} = 1$), the inequalities (10) and (11) imply that $u_p = q_p$, and when $w_{p0} = 1$, the bounding constraints imply $q_p \leqslant u_p \leqslant Q$. If a vehicle comes from the cluster V_s to the cluster V_p and then goes from V_p to the cluster V_t, which means that $w_{rp} = 0$ and $w_{pr} = 0$ for all $r \neq s$ and $r \neq t$, from the bounding constraint we get $q_p + q_s \leqslant u_p \leqslant Q - q_t$. □

3.2.2. Subtour elimination constraints

Formation of any subtour between clusters excluding V_0 will not allowed by the following constraint:

$$u_p - u_r + Qw_{pr} + (Q - q_p - q_r)w_{rp} \leqslant Q - q_r, \quad p \neq r \neq 0,\ p, r \in K, \tag{12}$$

where $w_{pr} = 0$, whenever $q_p + q_r > Q$ with $p \neq r$ and $p, r \in K$.

These constraints are the adaptation of the subtour elimination constrains of the capacitated VRP (see [16,17]).

Note that, the constraints given in (11) guarantee that $u_p \geqslant 0$ for all $p \in K$, therefore we do not need nonnegativity constraints.

The first integer linear programming formulation of the GVRP is given by:

$$\begin{aligned} (NB) \quad \text{minimize} \quad & \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}, \\ \text{subject to} \quad & (1)-(6) \text{ and } (9)-(12), \\ & x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A, \end{aligned}$$

where, $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$ and $w_{pr} = 0$ whenever $q_p + q_r > Q$.

In the above formulation, the total number of binary variables is $n(n+1)$, i.e. $O(n^2)$. The variables w_{pr} take binary values automatically. The number of constraints implied by (1)–(6), (10)–(12) and (13) are k, k, 1, 1, $(n+1)$, $k(k+1)$, k, k, k and $k(k-1)$, respectively, where n is the number of customers, k is the number of clusters and $k \leqslant n$. Hence the number of constraints is $2k^2 + n + 5k + 3$, i.e. $O(k^2) \leqslant O(n^2)$.

The node based formulation (NB) of the GVRP is structurally similar to Kara–Bektaş [2] formulation, but it contains completely different bounding constraints. We show below that node based formulation produces stronger lower bound than the formulation described by Kara and Bektaş [2].

We denote the feasible set of the linear programming relaxation of the node based formulation of the GVRP by P_{NB}, where we replace the integrality constraints $x_{ij} \in \{0,1\}$ by $0 \leqslant x_{ij} \leqslant 1$. If we denote the feasible set of the linear programming relaxation of the Kara–Bektaş formulation by P_{K-B} then the following result holds:

Proposition 2. $P_{NB} \subseteq P_{K-B}$.

Proof. The degree constraints, subtour elimination constraints and route continuity constraints of the two formulations are the same, they differ from each other with respect to their bounding constraints. As we discussed in the proof of previous proposition, any cluster V_p may be on the first, or last or intermediate position of a vehicle tour, i.e., either $w_{0p} = 1$ or $w_{p0} = 1$ or there exist some clusters V_s and V_t such that $w_{sp} = w_{pt} = 1$.

If $w_{0p} = 1$ or $w_{p0} = 1$ both formulations produce the same u_p values. If a vehicle comes from the cluster V_s to V_p and then goes from V_p to cluster V_t, which means that $w_{rp} = 0$ and $w_{pr} = 0$ for all $r \neq s$ and $r \neq t$, from the bounding constraints given in (10) and (12) we get:

$$q_p + q_s \leqslant u_p \leqslant Q - q_t.$$

For the same case, the formulation described by Kara and Bektaş [2] implies:

$$q_p + \bar{q}_p \leqslant u_p \leqslant Q - \bar{q}_p,$$

where $\bar{q}_p = \min\{q_r | r \in K, r \neq p\}$.

Since $q_s \geqslant \bar{q}_p$ and $q_t \geqslant \bar{q}_p$ our formulation produces narrower bounds for auxiliary variables, so every feasible solution of the proposed node based model is also a feasible solution to Kara–Bektaş model. □

3.3. A flow based formulation

In addition to the defined decision variables, let us define the following auxiliary variables:

y_{pr}: be the amount of goods picked up (or delivered in the case of delivery) on the route of a vehicle just after leaving the cluster V_p, if the vehicle goes from the cluster V_p to the cluster V_r, and zero otherwise.

Proposition 3. *The following inequalities are valid capacity bounding and subtour elimination constraints for the GVRP:*

$$y_{rp} \leqslant (Q - q_p) w_{rp}, \quad r \neq p,\ r, p \in K, \tag{13}$$

$$y_{rp} \geqslant q_r w_{rp}, \quad r \neq p,\ r, p \in K, \tag{14}$$

$$\sum_{p=1}^{k} y_{p0} = \sum_{p=1}^{k} q_p, \tag{15}$$

$$\sum_{p=1}^{k} y_{rp} - \sum_{p=1}^{k} y_{pr} = q_r, \quad r \neq 0,\ r, p \in K, \tag{16}$$

where $y_{0p} = 0$ for all $p \in K$ and $q_0 = 0$.

Proof. When any arc(r,s) is on the tour of a vehicle, i.e., $w_{rs} = 1$, the constraints given in (14) and (15) imply that

$$q_r \leqslant y_{rs} \leqslant Q - q_s.$$

If the arc(r,s) is the final arc of a vehicle tour, i.e., $w_{r0} = 1$, then $q_r \leqslant y_{rs} \leqslant Q$. These constraints also guarantee that if there is no flow from cluster V_r to the cluster V_p then $y_{rp} = 0$. Therefore, the constraints (14) and (15) produce upper and lower bounds of the flow variables.

The constraints given in (16) guarantee that the sum of the in flow to the origin is equal to the total demand.

Finally, the constraints given in (17) are the classical conservation of flow equation which also eliminates illegal tours, i.e., they are the subtour elimination constraints of the flow based formulation. □

The second integer linear programming formulation of the GVRP is given by:

$$\begin{aligned} (FF) \quad \text{minimize} \quad & \sum_{i=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}, \\ \text{subject to} \quad & (1){-}(6) \text{ and } (13){-}(16), \\ & x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A, \end{aligned}$$

where, $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$ and $w_{pr} = 0$ whenever $q_p + q_r > Q$. In addition $y_{0p} = 0$ for all $p \in K$ and $q_0 = 0$. Note that, the constraints given in (15) guarantee that $y_{pr} \geqslant 0$, for all $p \neq r$, p, $r \in K$.

In this formulation, as we observed in the previous formulation, the total number of binary variables is $n(n + 1)$, i.e. $O(n^2)$. The variables w_{pr} take binary values automatically. The number of constraints implied by (1)–(6), (14)–(16) and (17) are k, k, 1, 1, $(n + 1)$, $k(k + 1)$, $k(k + 1)$, $k(k + 1)$, 1 and k, respectively, where n is the number of customers, k is the number of clusters and $k \leqslant n$. Hence the number of constraints is $3k^2 + n + 6k + 4$, i.e., $O(k^2) \leqslant O(n^2)$.

4. Special cases of the proposed formulations

In this section, we show that one can obtain integer linear programming formulations for some routing problems as special cases of the proposed formulations for GVRP.

4.1. The generalized multiple traveling salesman problem

In the GVRP, let the total demand of each cluster to be equal to 1 and there is no capacity restriction for the vehicles. We will call this version of the GVRP as the generalized multiple traveling salesman problem (GmTSP).

For GmTSP, the meaning of auxiliary variables u_i's and y_{rp}'s and parameters of the model will be as follows:

u_p: the rank order of cluster V_p on the tour of a vehicle (visit number), $p \in K$;
y_{pr}: is the total number of the arcs on the route of a vehicle traveled just after leaving the cluster V_p if a vehicle goes from cluster V_p to the cluster V_r and zero otherwise;

$q_p = 1$, for all $p \in K$, $q_0 = 0$ and $Q = k - m + 1$ is the maximum number of clusters that a vehicle can visit.

Substituting these conditions into the constraints (10), (11) and (13) of the node based formulation (NB), we obtain the following polynomial size node based formulation for the generalized multiple traveling salesman problem (GmTSP):

$$\text{minimize} \quad \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij},$$

$$\text{subject to} \quad (1)-(6),$$

$$u_p - \sum_{r=1,r\neq p}^{k} w_{rp} \geqslant 1, \quad p \neq 0,\ p \in K, \tag{17}$$

$$u_p + (k-m)w_{0p} \leqslant k - m + 1, \quad p \neq 0,\ p \in K, \tag{18}$$

$$u_p - u_r + (k-m+1)w_{pr} + (k-m-1)w_{rp} \leqslant k - m,$$
$$p \neq r \neq 0, \quad p,r \in K, \tag{19}$$

$$x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A,$$

where $k - m - 1 \geqslant 0$ and $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$.

Substituting the above conditions in the constraints (14)–(17) of the flow based formulation (FF), we obtain a polynomial size flow based formulation for the generalized multiple traveling salesman problem (GmTSP), as follows:

$$\text{minimize} \quad \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij},$$

$$\text{subject to} \quad (1)-(6),$$

$$y_{rp} \leqslant (k-m)w_{rp}, \quad r \neq p,\ r,p \in K, \tag{20}$$

$$y_{rp} \geqslant w_{rp}, \quad r \neq p,\ r,p \in K, \tag{21}$$

$$\sum_{p=1}^{k} y_{p0} = k, \tag{22}$$

$$\sum_{p=1}^{k} y_{rp} - \sum_{p=1}^{k} y_{pr} = 1, \quad \forall r \in K, \tag{23}$$

$$x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A,$$

where $k \geqslant m$ and $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$. In addition, $y_{0p} = 0$ for all $p \in K$ and $q_0 = 0$.

4.2. The generalized traveling salesman problem

The GmTSP reduces to the generalized traveling salesman problem (GTSP) when $m = 1$, i.e., when there is a single traveler.

If we substitute $m = 1$ in the previous two integer programming formulations of the (GmTSP), we obtain the following polynomial size integer linear programming models for the generalized traveling salesman problem:

$$\text{minimize} \quad \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to} \quad (1),\ (2),\ (5),\ (6) \text{ and } (17),$$

$$\sum_{i=1}^{n} x_{i0} = 1, \tag{24}$$

$$\sum_{i=1}^{n} x_{0i} = 1, \tag{25}$$

$$u_p + (k-1)w_{0p} \leqslant k, \quad p \neq 0,\ p \in K, \tag{26}$$

$$u_p - u_r + kw_{pr} + (k-2)w_{rp} \leqslant k - 1, \quad p \neq r \neq 0,\ p,r \in K, \tag{27}$$

$$x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A,$$

where $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$ and $k \geqslant 2$.

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}, \\ \text{subject to} \quad & (1),\ (2),\ (5),\ (6),\ (17) \text{ and } (21)-(25), \\ & y_{rp} \leqslant (k-1) w_{rp}, \quad p \neq r,\ p, r \in K, \\ & x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A, \end{aligned} \tag{28}$$

where $x_{ij} = 0$ whenever $i, j \in V_r$, $r \in K$.

The GTSP has been introduced without a fixed depot in most of the previous studies. The above formulation can easily be adapted to the non-fixed depot case as follows: select any cluster as the starting and ending cluster (depot) and denote it as V_0 ($|V_0| > 1$) and rewrite the constraints (1) and (2) including as well the cluster V_0 and omit the constraints (25) and (26).

5. The clustered generalized vehicle routing problem

Consider the GTSP with the restriction that the traveler must visit all the nodes of each cluster consecutively. According to Laporte and Palekar [18], this variant of the GTSP is defined as the cumulative traveling salesman problem and we will denote it by CTSP. Several applications of CTSP are outlined by Laporte and Palekar [18]. Recently, Ding et al. [19] proposed a two level genetic algorithm for CTSP. As far as we are aware, there is not such an extension for GVRP.

We define in what it follows a similar extension of the GVRP as clustered generalized vehicle routing problem, denoted by CGVRP, where all the nodes of each cluster must be on a route of a vehicle consecutively. Thus, for the case of CGVRP, we are interested in finding a collection of m tours of minimum cost, starting and ending at the depot such that, each node of the entire graph is visited exactly once by performing a Hamiltonian path within each cluster, and each cluster should be visited by exactly one vehicle at any of its nodes, and the load of each vehicle does not exceed its capacity Q. A clustered GVRP and a feasible solution of the problem are presented in the Fig. 2.

The proposed formulations of the GVRP can be easily adapted to the CGVRP by adding additional constraints to each formulation, as we will shown below. There is no need to define new parameters and new decision variables for CGVRP. It is enough do redefine the cost parameters c_{ij} for all $i \neq j$, $i, j \in V$.

5.1. Node degree constraints for CGVRP

In the case of the CGVRP each node should be on the tour of a vehicle. Therefore, in addition to the cluster degree constraints given in (1) and (2), we have to write the degree constraints for all nodes. Therefore, we omit the connectivity constraints given in (5) and we add the following node degree constraints to both formulations:

$$\sum_{i=0}^{n} x_{ij} = 1, \quad j = 1, \dots, n, \tag{29}$$

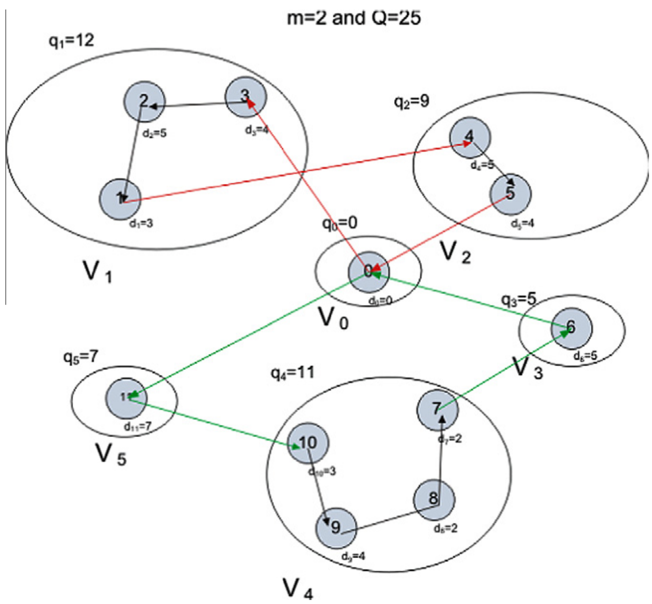

Fig. 2. An example of a feasible solution of the CGVRP.

$$\sum_{j=0}^{n} x_{ij} = 1, \quad i = 1, \ldots, n. \tag{30}$$

In what it follows we present the node based and flow based formulations in the case of the clustered generalized vehicle routing problem.

5.2. Node based formulation for CGVRP

We define a new variable as follows: v_i is the rank order of node i on the tour of a vehicle, $i \in \{1, \ldots, n\}$, then the CGVRP can be modeled as an integer program in the following way:

$$\begin{aligned}
&\text{minimize} \quad \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij},\\
&\text{subject to} \quad (1)–(4),\ (6),\ (9)–(12),\ (29) \text{ and } (30),\\
&\qquad v_i - v_j + |V_p|x_{ij} + (|V_p - 2)x_{ji} \leqslant |V_p| - 1, \quad p \in K,\ i,j \in V_p, && (31)\\
&\qquad v_i \geqslant 1, \quad i \in V_p,\ p \in K, && (32)\\
&\qquad x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A,
\end{aligned}$$

where as in the other node based formulations, $x_{ij} = 0$, whenever $q_i + q_j > Q$ for all $i \neq j$, and $c_{ii} = 0$ for all $i \in V$.

The constraints presented in (31) were introduced by Desrochers and Laporte [16] in the case of the traveling salesman problem in order to eliminate the subtours. These constraints avoid the formation of any subtours in each of the clusters. The constraints (32) initialize the variables v_i's for each cluster.

5.3. Flow based formulation for CGVRP

We introduce a new flow variable as follows: t_{ij} is the number of the arcs visited within a given cluster on the route of a vehicle just after leaving the ith node if the vehicle goes from node i to node j and zero otherwise. Then, the flow based formulation for CGVRP may be given as:

$$\begin{aligned}
&\text{minimize} \quad \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij},\\
&\text{subject to} \quad (1)–(4),\ (6),\ (13)–(16),\ (29) \text{ and } (30),\\
&\qquad \sum_{j=0}^{n} t_{ij} - \sum_{j=0}^{n} t_{ji}1, \quad \forall i \in V_p,\ p \in K, && (33)\\
&\qquad t_{ij} \leqslant (|V_k| - 1)x_{ij}, \quad i,j \in V_k,\ k \in K, && (34)\\
&\qquad x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A,
\end{aligned}$$

where $k \geqslant m$.

In the above formulation, the constraints presented in (33) are the standard conservation of flow equations, they increase the t_{ij}'s of the consecutive arc of any path by one after each node, so then they are the subtour elimination constraints of the formulation. Constraints given in (34) are the bounding constraints of the flow variables. They guarantee that if the arc(i,j) is not on any route, i.e. $x_{ij} = 0$, then corresponding flow variables $t_{ij} = 0$.

6. A numerical example

In this section we solve optimally the numerical example described by Ghiani and Improta [1] using our novel integer programming formulations of the GVRP. The example was derived from an VRP instance, namely test problem 7, introduced by Araque et al. [20] and has 50 vertices, 25 clusters and 4 vehicles. This instance is a randomly generated problem and was generated by placing the customers randomly on a square area. The vertex coordinates of the depot are (50, 50) and of the customers are:

1.	(10,42)	2.	(23,6)	3.	(8,46)	4.	(51,29)	5.	(64,24)
6.	(28,6)	7.	(30,69)	8.	(87,3)	9.	(13,56)	10.	(80,87)
11.	(95,41)	12.	(92,21)	13.	(86,10)	14.	(39,45)	15.	(76,44)
16.	(30,34)	17.	(47,3)	18.	(84,70)	19.	(65,5)	20.	(98,18)
21.	(3,13)	22.	(52,98)	23.	(73,17)	24.	(48,82)	25.	(28,32)
26.	(16,86)	27.	(50,56)	28.	(53,72)	29.	(75,89)	30.	(41,38)
31.	(11,28)	32.	(76,57)	33.	(86,18)	34.	(34,19)	35.	(70,25)
36.	(79,50)	37.	(25,13)	38.	(55,94)	39.	(4,33)	40.	(8,66)
41.	(51,70)	42.	(3,22)	43.	(93,17)	44.	(96,45)	45.	(71,85)
46.	(39,32)	47.	(37,71)	48.	(79,45)	49.	(96,66)	50.	(69,92)

The distances between the customers are the euclidean distances and have been rounded to obtain integer values. The set of vertices is partitioned into 25 clusters as follows:

$V_0 = \{0\}$ $V_1 = \{22, 38\}$ $V_2 = \{26\}$ $V_3 = \{24\}$
$V_4 = \{10, 29, 45, 50\}$ $V_5 = \{40\}$ $V_6 = \{7, 47\}$ $V_7 = \{28, 41\}$
$V_8 = \{18, 49\}$ $V_9 = \{1, 3, 9\}$ $V_{10} = \{14\}$ $V_{11} = \{32\}$
$V_{12} = \{21, 42, 39, 31\}$ $V_{13} = \{16, 25\}$ $V_{14} = \{30, 46\}$ $V_{15} = \{4\}$
$V_{16} = \{15, 36, 48\}$ $V_{17} = \{11, 44\}$ $V_{18} = \{2, 6, 37\}$ $V_{19} = \{34\}$
$V_{20} = \{5, 23, 35\}$ $V_{21} = \{12, 20, 33, 43\}$ $V_{22} = \{17\}$ $V_{23} = \{8, 13, 19\}$
$V_{24} = \{27\}$

In the Fig. 3 we represent the vertices (customers) and their partitioned into clusters.

Each customer has a unit demand and the demand of a cluster is given by the cardinality of that cluster. The capacity of each vehicle is equal to 15.

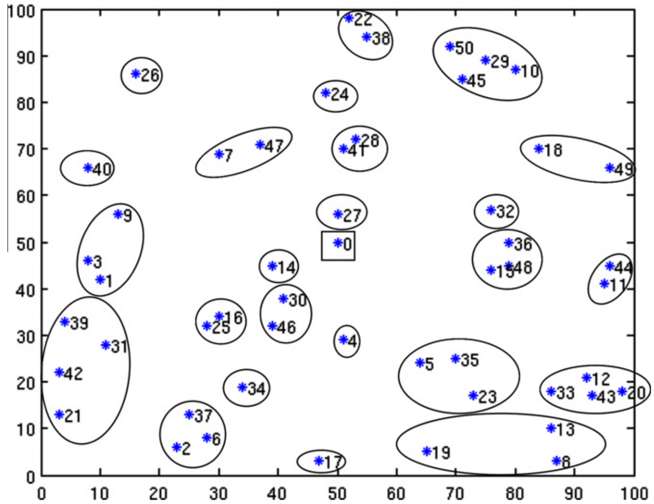

Fig. 3. Representation of the 51 vertices and their partitioned into 25 clusters.

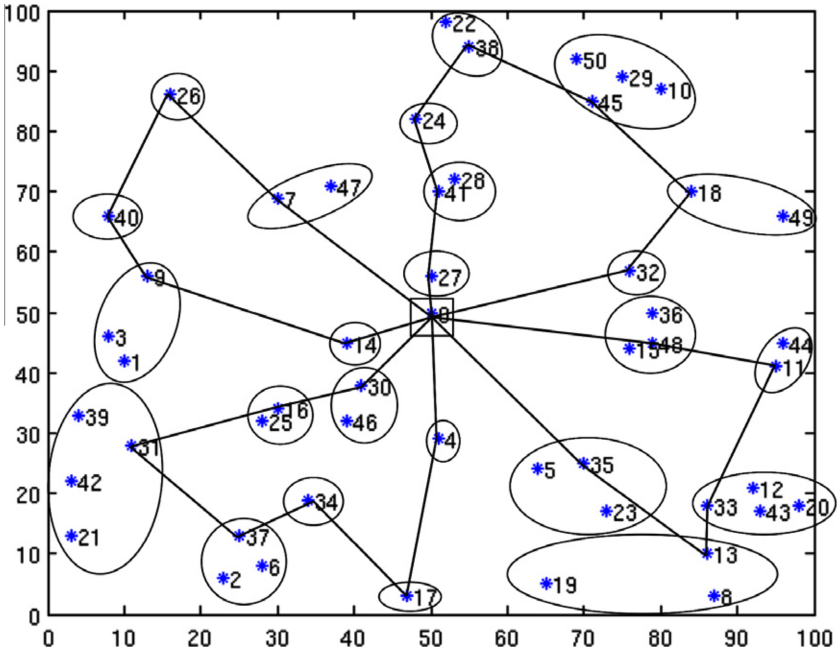

Fig. 4. The optimal solution of the GVRP instance described by Ghiani and Improta.

The solution reported by Ghiani and Improta [1], was obtained by transforming the GVRP into a Capacitated Arc Routing Problem (CARP), which was then solved using the heuristic proposed by Hertz et al. [21], to yield the objective value 532.73. The same instance was solved to optimality by Kara and Bektaş [2] using their proposed integer programming formulation by CPLEX 6.0 on a Pentium 1100 MHz PC with 1 GB RAM in 17600.85 CPU seconds.

Using our proposed formulations for the GVRP, we solved the same instance to optimality using CPLEX 12.2, getting the same value as reported by Kara and Bektaş [2]. The required computational times were 1210.01 CPU seconds in the case of the node based formulation, respectively 1002.87 CPU seconds in the case of the flow based formulation. The computations were performed on a Intel Core 2 Duo 2.00 GHz with 2 GB RAM.

In the Fig. 4 we point out the optimal solution obtained in the case of the instance described by Ghiani and Improta [1].

These results prove that we are able to cope with the considered instance and as well the superiority of our novel compact integer programming based models in comparison to the existing ones.

7. Conclusions

In this paper we present two integer programming formulations of the generalized vehicle routing problem (GVRP). The formulations that we introduced are compact in the sense that the number of variables and constraints is a polynomial function of the number of nodes of the problem. The computational results obtained in the case of the instance provided by Ghiani and Improta [1] prove the efficiency of our novel models of the GVRP in comparison with the existing ones.

We define the generalized multiple traveling salesman problem (GmTSP) and, as a special case, it is shown that both of the proposed formulation of the GVRP reduce to the formulation of the GmTSP. We define as well the clustered generalized vehicle routing problem (CGVRP) where all the nodes of each cluster must be on a route of a vehicle consecutively. The formulations proposed for the GVRP are extended and presented also in the case of the CGVRP.

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