Node-based and flow-based formulations for the Generalized Vehicle Routing Problem

06-815 Course Project

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1 Introduction

The Generalized Vehicle Routing Problem (GVRP) is an extension of the Vehicle Routing Problem (VRP) defined on a graph in which the customers are grouped into a given number of mutually exclusive and exhaustive clusters. In this study, two MILP formulations of the GVRP described by Pop et al. (2012) are implemented. The computational performance of the models solved using CPLEX on test problems is presented. New test problems are generated in addition to the ones available in literature by clustering test instances for the Capacitated VRP.

1.1 Definition of the GVRP

Let $G = (V, A)$ be a directed graph with $V = \{0, 1, 2, \ldots, n\}$ as the set of vertices and $A = \{(i, j) | i, j \in V, i \neq j\}$ as the set of arcs. A non-negative cost c_{ij} is associated with each arc $(i, j) \in A$.

The set of vertices V is partitioned into $k+1$ mutually exclusive subsets (clusters) $V_0, V_1, \ldots, V_k$. The cluster V_0 has only one vertex 0 which represents the depot and the remaining n nodes belong to the remaining k clusters. These can represent geographically distributed customers or islands. Each customer has a demand and the demand of each cluster (denoted by q) is the total demand of customers in that cluster. The demand of each cluster can be satisfied via any of its vertices. There exist m vehicles, each with capacity Q.

The generalized vehicle routing problem (GVRP) consists of finding the minimum total cost tours starting and ending at the depot, such that each cluster is visited exactly once, the entering and leaving node of each cluster is the same, and the sum of all the demands of any tour (route) does not exceed the capacity of the vehicle Q. An example of a GVRP with a feasible tour is shown in Fig. 1.

2 MILP formulations

The first mixed integer linear formulation of the GVRP was presented by Kara and Bektaş (2003), with a polynomially increasing number of binary variables and constraints. Pop

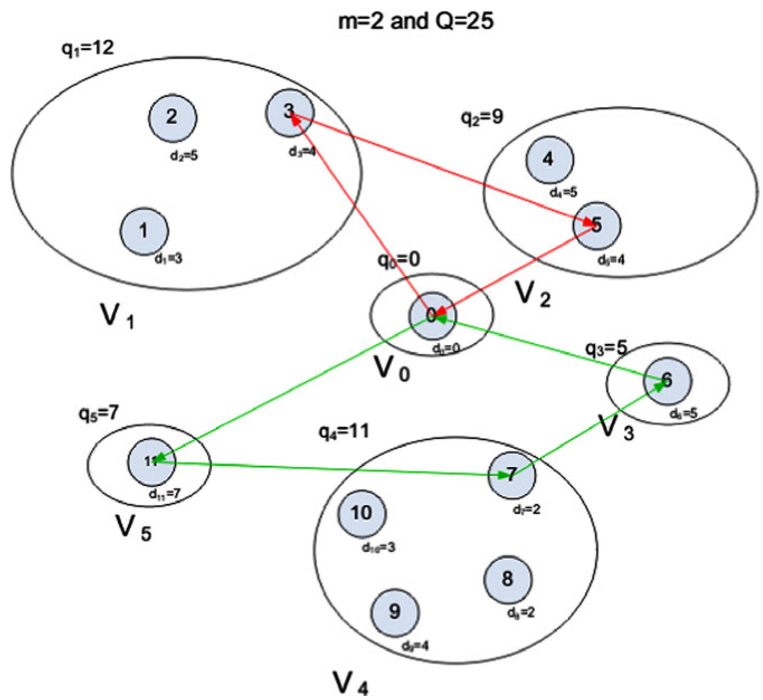


Figure 1: An example of a GVRP and a feasible solution (Pop et al., 2012)

et al. (2012) presented two different formulations based on the additional auxiliary decision variables. These are called *node-based* and *flow-based* formulations respectively. The node-based formulation is structurally similar to the formulation by Kara and Bektaş (2003), but with stronger lower bounds. The flow-based formulation is completely new. In this section, a general semi-closed formulation for the GVRP is first presented and followed by the two formulations proposed by Pop et al. (2012).

2.1 General formulation

The related sets, decision variables and parameters for the GVRP are defined as follows:

Sets:

$V = \{0, 1, 2, \ldots, n\}$ is the set of nodes corresponding to the customers, where 0 represents the depot. V is partitioned into mutually exclusive and exhaustive non-empty sub-sets V_0, V_1, $\ldots V_k$, each of which represents a cluster of customers, where $V_0 = \{0\}$ is the depot.

$K = \{0, 1, 2, \ldots, k\}$ is the set of clusters.

$A = \{(i, j)|\ i, j \in V, i \neq j\}$ is the set of arcs.

Decision variables:

We define the following binary variables:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is included in the tour of a vehicle, } i \in V_p, j \in V_r, p, r \in K, \\ 0, & \text{otherwise} \end{cases}$$

$$W_{pr} = \begin{cases} 1 & \text{if there is a path from cluster } V_p \text{ to cluster } V_r,\ p, r \in K, \\ 0, & \text{otherwise} \end{cases}$$

Parameters:

c_{ij} is the cost of traveling from node i to node j, $i \neq j$, $i \in V_p$, $j \in V_r$, $p \neq r$, $p, r \in K$;

d_i is the demand of customer i, $i = 1, 2, \ldots, n$;

q_r is the demand of the cluster V_r, $q_r = \sum_{i \in V_r} d_i$, $r \in K$;

$M = \{1, \ldots, m\}$ is the set of identical vehicles;

Q is the capacity of each vehicle.

We assume that $k \geq m$ ie. the number of vehicles is no greater than the number of clusters.

2.1.1 Cluster degree constraints

For each cluster except V_0, only a single outgoing arc to any other node belonging to other clusters can exist. This condition is expressed by:

$$\sum_{i \in V_p} \sum_{j \in V \setminus V_p} x_{ij} = 1, p \neq 0, p \in K \tag{1}$$

There can be exactly one incoming arc to a cluster from any other node belonging to other clusters, excluding V_0. This condition is implied by:

$$\sum_{i \in V \setminus V_p} \sum_{j \in V_p} x_{ij} = 1, p \neq 0, p \in K \tag{2}$$

There should be at most m leaving arcs from and at most m entering arcs to the depot:

$$\sum_{i=1}^{n} x_{0i} \leq m \tag{3}$$

$$\sum_{i=1}^{n} x_{i0} \leq m \tag{4}$$

For the purposes of this study, the constraints (3) & (4) have been taken to be equality constraints.

2.1.2 Cluster connectivity constraints

The entering and leaving nodes should be the same for each cluster, which is satisfied by:

$$\sum_{i \in V \setminus V_p} x_{ij} = \sum_{i \in V \setminus V_p} x_{ji}, j \in V_p, p \in K \tag{5}$$

Flows from cluster V_p to the cluster V_r are defined by w_{pr}. Thus, w_{pr} should be equal to the sum of x_{ij}'s from V_p to V_r:

$$w_{pr} = \sum_{i \in V_p} \sum_{j \in V_r} x_{ij} \tag{6}$$

With the above definitions and constraints, a general integer programming formulation for the GVRP is given as:

$$\begin{aligned} \min \ & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t. } & (1) \text{ - } (6) \\ & \text{Capacity bounding constraints} && (7) \\ & \text{Subtour elimination constraints} && (8) \\ & x_{i,j} \in \{0,1\}, \forall (i,j) \in A \end{aligned}$$

Integrality constraints do not need to be imposed on w_{pr} because Eqs. (1) - (4) and (6) automatically imply that w_{pr} is either 0 or 1.

Model formulations of the GVRP are thus constrained by (1) - (6) and "capacity bounding" and "subtour elimination" constraints given in (7) and (8) above.

2.2 Node-based formulation

In this case, the additional auxiliary variables are defined with respect to the vertices of the graph:

u_p: the load of a vehicle just after leaving the cluster V_p, or delivered amount of goods just after leaving the cluster V_p, $p \neq 0$, $p \in K$.

2.2.1 Capacity bounding constraints

The capacity bounding constraints for the node-based formulation are given by:

$$u_p - \sum_{r \in K, r \neq p} q_r w_{rp} \geq q_p,\ p \neq 0, p \in K \tag{9}$$

$$u_p + (Q - q_p) w_{0p} \leq Q,\ p \neq 0, p \in K \tag{10}$$

$$u_p + \sum_{r \in K, r \neq p} q_r w_{pr} \leq Q,\ p \neq 0, p \in K \tag{11}$$

2.2.2 Subtour elimination constraints

The formation of any subtour between clusters excluding V_0 will be prevented by the following constraint:

$$u_p - u_r + Q w_{pr} + (Q - q_p - q_r) w_{rp} \leq Q - q_r,\ p \neq r \neq 0; p, r \in K \tag{12}$$

where $w_{pr} = 0$, whenever $q_p + q_r > Q$ with $p \neq r$ and $p, r \in K$.

Hence the complete node based formulation is given by replacing Eqs. (7) and (8) in the generalized formulation by Eqs. (9) - (11) and Eq. (12) respectively.

2.3 Flow-based formulation

For the flow-based formulation, the additional auxiliary variables are defined with respect to the arcs of the graph:

y_{pr}: amount of goods picked up (or delivered in the case of delivery) on the route of a vehicle just after leaving the cluster V_p if the vehicle goes from the cluster V_p to the cluster V_r.

2.3.1 Capacity bounding and subtour elimination constraints

Capacity bounding and subtour elimination constraints for the flow-based formulation are given by:

$$y_{rp} \leq (Q - q_p) w_{rp},\ r \neq p; r, p \in K \tag{13}$$

$$y_{rp} \geq q_r w_{rp},\ r \neq p; r, p \in K \tag{14}$$

$$\sum_{p=1}^{k} y_{p0} = \sum_{p=1}^{k} q_p \tag{15}$$

$$\sum_{p=0}^{k} y_{rp} - \sum_{p=0}^{k} y_{pr} = q_r, \; r \neq 0; r, p \in K \tag{16}$$

where $y_{0p} = 0$ for all $p \in K$ and $q_0 = 0$. Hence the complete flow based formulation is given by replacing Eqs. (7) and (8) in the generalized formulation by Eqs. (13) - (16).

3 Test examples

Table 1: Details of GVRPs under consideration

Instance	Customers	Clusters	Vehicles	Capacity	Total demand
GVRP1	50	24	4	15	50
GVRP2	60	16	6	15	60
P-n50-k10	49	30	5	200	951
A-n69-k9	68	31	6	150	845
P-n76-k5	75	38	5	280	1364

Unfortunately, there are only two available test problems for the GVRP. The first, given by Ghiani and Improta (2000) (denoted in this study by `GVRP1`), is originally a VRP taken from Araque et al. (1994) with 50 customers and 4 vehicles. Each customer has a unit demand and the demand of a cluster is equal to the cardinality of that cluster.

Kara and Bektaş (2003) generated another test problem by clustering Problem 13 from Araque et al. (1994). However, in the formulation under consideration in their work, an additional parameter K is involved which is the minimum load of each vehicle when it returns to the depot (collection case) or minimum starting load of a vehicle just before starting its trip (delivery case). In this study, that parameter has not been considered while solving the instance. This particular problem is denoted by `GVRP2`.

3.1 Generation of additional test examples

Three additional test examples were generated using capactitated vehicle routing problem (CVRP) data obtained from CVRPLIB (`http://vrp.atd-lab.inf.puc-rio.br`). Instances `P-n50-k10`, `A-n69-k9` and `P-n76-k5` were chosen. The clusters were obtained by implementation of a simple clustering algorithm (K-means), with the number of clusters chosen such that the number of customers in any cluster is no more than 5. The demand of each cluster was calculated by summing the demands of each customer in that cluster. Details of each of the problems under consideration is given in Table 1.

4 Computational study

Both node-based and flow-based formulations described above were implemented in CPLEX and solved using a Windows PC with an Intel i7 processor (2.90 GHz).

4.1 Tightness of root node relaxation

Table 2: Deviation between root node relaxation and optimal values

Instance	Best known objective	Formulation	Root node relaxation	% Deviation
GVRP1	527.813	flow	498.594	5.54
		node	449.358	14.86
GVRP2	557.564	flow	533.9	4.24
		node	545.365	2.19
P-n50-k10	417.742	flow	384.597	7.93
		node	338.098	19.07
A-n69-k9	756.131	flow	692.212	8.45
		node	560.44	25.88
P-n76-k5	500.955	flow	450.746	11.27
		node	399.788	21.30

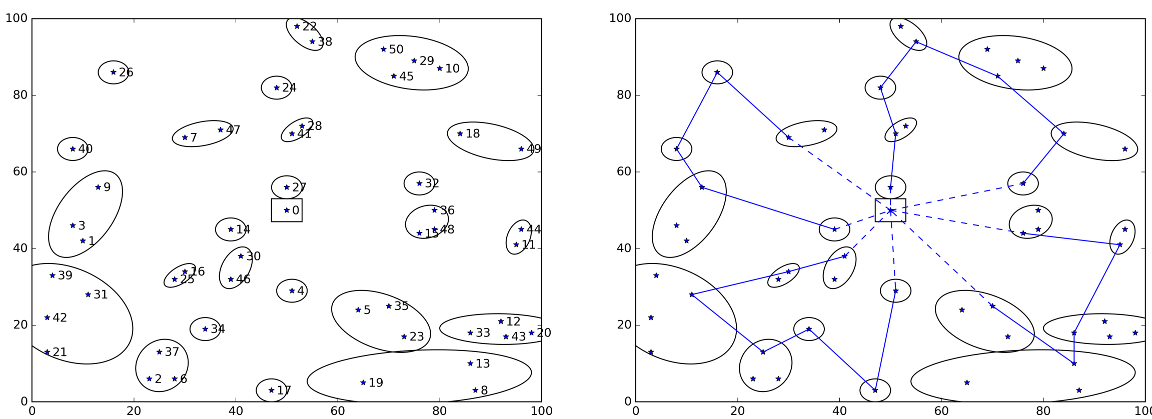
Table 2 shows a comparision between the root node relaxations for the flow- and node-based formulations. In all but one of the instances under consideration, the flow-based formulation shows a tighter LP lower bound. Thus we would expect the flow-based formulation to be more efficient than the node-based formulation.

This is indeed the case as seen in Table 3: the flow-based formulation finds a better objective value or closes the relative gap more than the node-based formulation. A one hour time limit for each problem was set for each of these calculations. Both formulations found the proven optimum within the time limit only for `GVRP2`. For `GVRP1`, only the flow-based formulation reached zero gap in 85 seconds. In all the other instances, the realtive gap at time limit was significantly smaller for the flow-based formulation.

The obtained objective value for `GVRP1` (Fig. 2) is the same as the optimum reported by Pop et al. (2012). Since different constraints were used for `GVRP2`, the obtained solution does not match that reported by Kara and Bektaş (2003).

Table 3: Model comparision for GVRP instances (time limit = 1 hour)

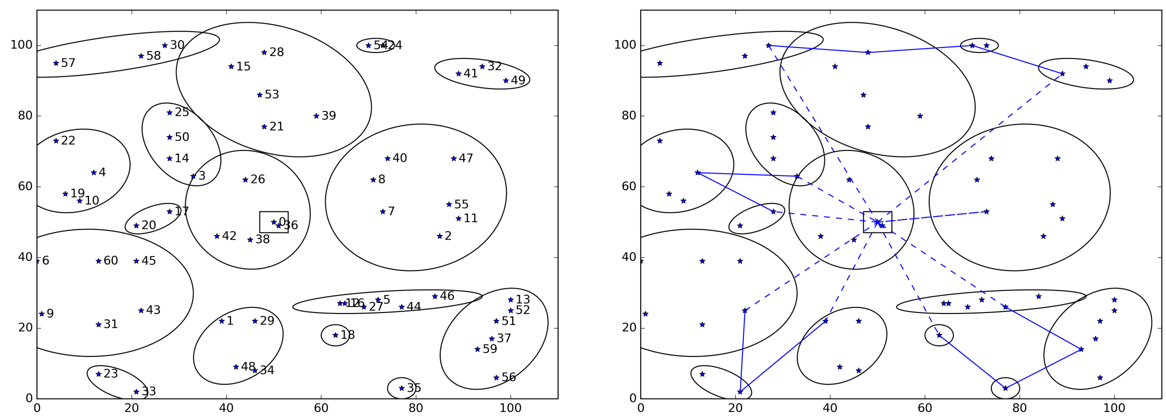
Instance	**Formulation**	**Status**	**Objective value**	**Relative gap (%)**	**Number of constraints**
`GVRP1`	flow	Optimal	527.813	0.01	2108
	node	Feasible	527.813	5.42	1482
`GVRP2`	flow	Optimal	557.564	0.00	1242
	node	Optimal	557.564	0.00	952
`P-n50-k10`	flow	Feasible	417.742	0.75	3094
	node	Feasible	417.742	14.69	2132
`A-n69-k9`	flow	Feasible	756.131	1.25	3385
	node	Feasible	763.045	18.59	2360
`P-n76-k5`	flow	Feasible	508.194	6.04	4896
	node	Feasible	595.149	31.20	3374



(a) Customer clusters

(b) Optimum solution (objective = 527.813)

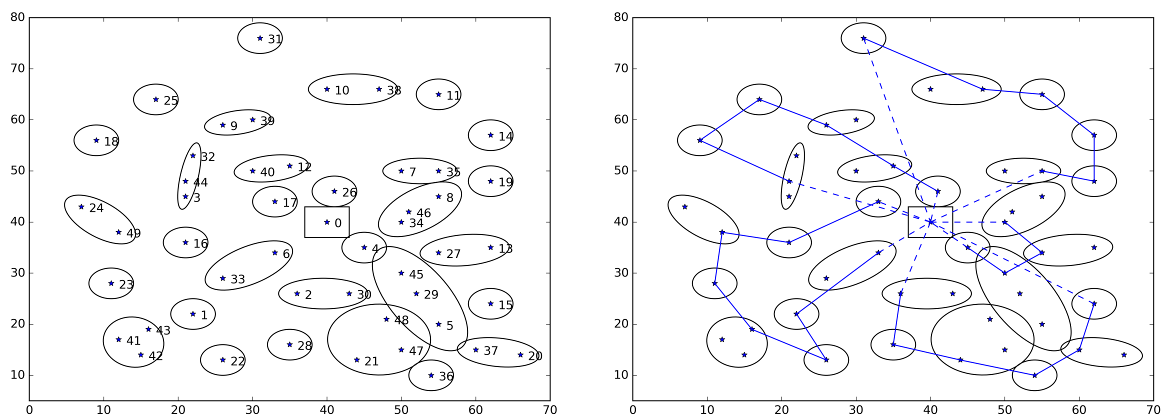
Figure 2: `GVRP1`



(a) Customer clusters (b) Optimal solution (objective = 557.564)

Figure 3: `GVRP2`

Instances `P-n50-k10` and `A-n69-k9`, were solved to optimality with the flow-based formulation in 4100 s and 9120 s respectively. The instances and obtained solutions are shown in Fig. 4 and 5 respectively. Instance `P-n76-k5` was solved to 2.55% relative gap with an 8 hour time limit. The solution is shown in Fig. 6.

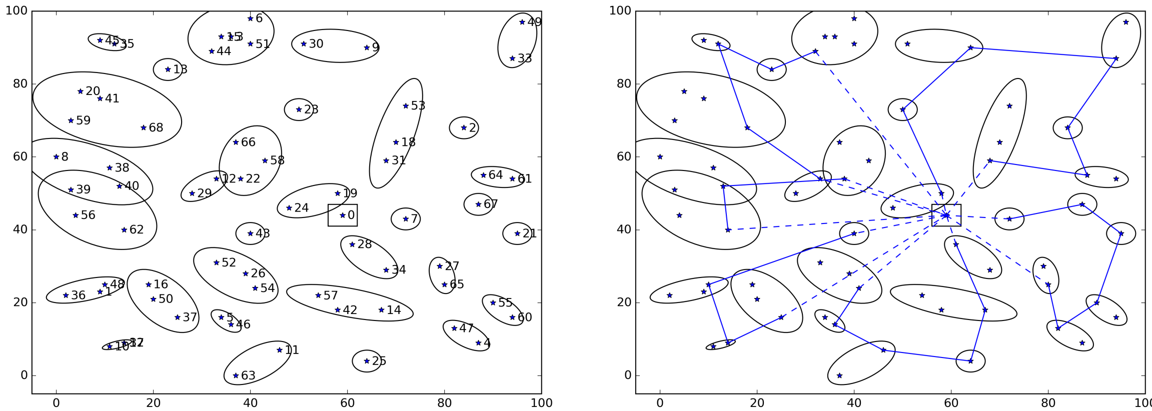


(a) Customer clusters (b) Optimal solution (objective = 417.742)

Figure 4: `P-n50-k10` (clustered)

5 Conclusions

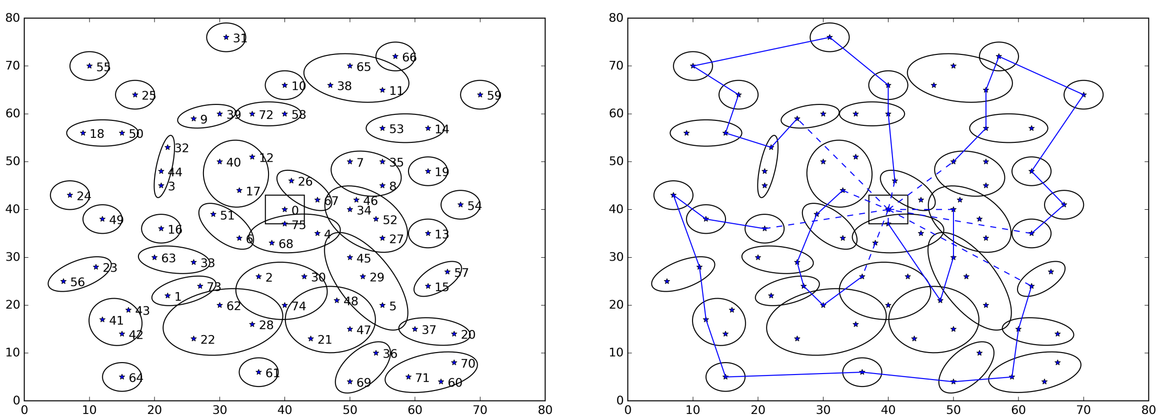
Two MILP formulations of the generalized vehicle routing problem (GVRP) were studied. The number of variables and constraints is a polynomial function of the number of customers for both the formulations. Two formulations were defined: the first had auxiliary variables defined based on the customers (node-based) where as the second had auxiliary variables defined based on arcs between customers (flow-based).



(a) Customer clusters (b) Optimal solution (objective = 756.131)

Figure 5: `A-n69-k9` (clustered)

Three new instances were generated by clustering benchmark instances for the capacitated vehicle routing problem (CVRP). Computational studies were conducted to compare the node-based and flow-based formulations. The flow-based formulation was proved to have better computational properties as compared to the node-based formulation for all instances under consideration. Two of the new instances were solved to optimality using the superior flow-based formulation.



(a) Customer clusters (b) Best known solution (objective = 500.955)

Figure 6: `P-n76-k5` (clustered)

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