CHROMATIC HOMOTOPY THEORY AND THE MODULI STACK OF FORMAL GROUPS

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Contents

1. Introduction

1.1. **Outline and motivation**

This document, which was written to fulfill the second year Writing Milestone requirement for math Ph.D. students at the University of Washington, is an attempt for me to track how objects and theorems from classical chromatic homotopy theory arise from and can be described in terms of the moduli stack of formal groups, $\mathcal{M}_{\mathrm{fg}}$. The main thesis and motivation of this document is to illustrate how this stacky picture offers some very rich and elegant perspectives on classical chromatic phenomena. After a discussion of $\mathcal{M}_{\mathrm{fg}}$ itself and its height stratification (Section 3), we will discuss how the Miller-Ravenel change of rings theorem

(Section 4), the Landweber exact functor theorem (Section 5), and the smashing localization and chromatic convergence theorems (Section 6) can all be described and proven in this language.

Since starting to write, though, plenty of other motivations have arisen.

One is of course to understand classical chromatic homotopy theory itself at a richer level. This has manifested in some longer discussions about the classical chromatic spectral sequence (Section 4.1), applications of the Landweber exact functor theorem (Section 5.2), and the various chromatic localizations (Section 6.1).

Another motivation has been to learn about algebraic geometry in a modern, functorial language. This has resulted in the rather long Section A which records some of this language. To a lesser extent, I have attempted to indicate how some of this can be understood ∞-categorically. For example, our proof of the Miller-Ravenel change of rings theorem uses the theory of homotopy groups of ∞-stacks. This document should largely be read in 1-categorical language, however.

One more goal is understanding how stable homotopy theory itself—not just the chromatic picture—can be understood in terms of stacks, descent, and derived algebraic geometry. In Section 2, we begin with a vignette showing how almost all homotopical questions can be phrased in this language.

A result of this mix of motivations is the particular tone this document takes. While we will discuss many topics, we are not trying to be encyclopedic. This is more like an attempt to compile the firehose of intertwined stories in homotopy theory and algebraic geometry which have caught my interest in the last several months into a single place where I can understand them. In this vein, another one of my main goals has been to write down (as much as possible) the remarks and perspectives which I have encountered lately and found particularly insightful or which have been especially key to my understanding. I have also tried to express how interrelated the topics here are, particularly in the translation process back and forth between homotopy theory and algebraic geometry. The result is many extended discussions and remarks.

We assume the reader is generally familiar with the basic players in chromatic homotopy theory (e.g. BP and the Morava K-theories, the Adams-Novikov spectral sequence, Quillen's theorem, chromatic localizations; generally anything in [Rav86]) and will refer to facts about them freely. On the other hand, for some topics we develop the theory from the beginning. In particular, our treatments of the height filtration of $\mathcal{M}_{\mathrm{fg}}$ (Section 3.2), the Morava stabilizer group (Section 3.4), and of formal groups themselves (Section B.1) are relatively complete.

My interest in this topic especially grew out of reading [Pst21]. This, along with [Rav86], [Goe08], and [Pet19] comprise the main sources I rely on in this document.

1.2. **Acknowledgments**

I would like to thank John Palmieri who has guided me not only through the process of writing this document, but in learning homotopy theory from the beginning. I would also like to thank my friends Daniel Rostamloo, Jackson Morris, and Mal Dolorfino for many helpful conversations and insights while learning this material. Thank you to Danny Shi for reviewing the final draft of this project. Also, thanks to my cat, Scully, for her support.

2. Prologue: derived descent and the Adams spectral sequence

Let us first discuss an extended example from the classical theory of faithfully-flat descent which motivates the meeting of spectra and stacks in the first place. This will help us develop the language to pass from the classical homotopy theory of comodules over Hopf algebroids to an equivalent algebro-geometric formulation in terms of quasi-coherent sheaves over stacks. This section will primarily follow [Pet19, Seciton 3.1].

The fundamental question in descent theory is whether given a ring map $f : R \to S$ and an S-module N we find an R module M such that $M \otimes_R S \cong N$. What data would we need to know about N to do this? And if we can make such a construction, can we do it uniquely up to isomorphism? Up to unique isomorphism?

One answer to this question due to Grothendieck is as follows.

Definition 2.0.1. Define the *descent object* of the ring map $f : R \to S$ to be the simplicial scheme

$$\mathcal{D}_f = \left\{ \mathrm{Spec}(S) \leftleftarrows\!\!\!\!\rightarrow \mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \cdots \right\}.$$

Equivalently, $\mathcal{D}_f$ is obtained by applying Spec to the Čech nerve of f in the category of rings. For our purposes, it will be most convenient to consider the scheme-theoretic picture.

A simplicial quasi-coherent sheaf $\mathcal{F}_*$ over $\mathcal{D}_f$ is a collection of sheaves

$$\mathcal{F}_{[n]} \in \mathsf{QCoh}(\mathrm{Spec}(S)^{\times[n]})$$

which are naturally compatible with the simplicial structure on $\mathcal{D}_f$. Note in particular that from any R-module M, the corresponding quasi-coherent sheaf $\tilde{M}$ over $\mathrm{Spec}(R)$ yields a simplicial quasi-coherent sheaf over $\mathcal{D}_f$ defined level-wise by pulling back along the maps $\mathrm{Spec}(S)^{\times[n]} \to \mathrm{Spec}(R)$.

A result of Grothendieck is that if f is faithfully-flat, this functorial construction

$$\mathsf{QCoh}(\mathrm{Spec}(R)) \to \mathsf{QCoh}(\mathcal{D}_f) \tag{2.0.2}$$

is an equivalence of categories with inverse given by taking colimits. In particular, when f is faithfully-flat, the natural map $\operatorname{colim} \mathcal{D}_f \to \mathrm{Spec}(R)$ is an isomorphism (see, e.g. [Pet19, Theorem 3.1.4]). This theorem commonly goes by the name *faithfully-flat descent*.

To motivate our discussion of a derived generalization of faithfully-flat descent, let us look and an example of how descent can fail along a map which is not flat.

Example 2.0.3. Consider the quotient map $f : \mathbb{Z} \to \mathbb{F}_p$ which is not flat. In this case we have

$$\mathcal{D}_f = \left\{ \mathrm{Spec}(\mathbb{F}_p) \leftleftarrows\!\!\!\!\rightarrow \mathrm{Spec}(\mathbb{F}_p) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} \cdots \right\}.$$

which degenerates to

$$\mathcal{D}_f = \left\{ \mathrm{Spec}(\mathbb{F}_p) = \mathrm{Spec}(\mathbb{F}_p) = \mathrm{Spec}(\mathbb{F}_p) \ldots \right\}$$

since $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p$. Consider the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{F}_p$ and their corresponding quasi-coherent sheaves $\tilde{\mathbb{Z}}$ and $\tilde{\mathbb{F}}_p$ over $\mathrm{Spec}(\mathbb{Z})$. Note that both pull back to the *same* simplicial sheaf over $\mathcal{D}_f$, and so after taking colimits, we are left with two identical copies of $\tilde{\mathbb{F}}_p$ over $\mathrm{Spec}(\mathbb{Z})$. This all comes from the fact that

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p\,.$$

This is the sense in which descent fails along the map $\mathbb{Z} \to \mathbb{F}_p$: base-changing $\mathbb{Z}$-modules to $\mathbb{F}_p$ does not have an inverse operation.

There is a way that base changing along $\mathbb{Z} \to \mathbb{F}_p$ remembers some distinction between $\mathbb{Z}$ and $\mathbb{F}_p$ though, but it only appears at the derived level. Remaining in the world of $\mathbb{Z}$-modules for a moment, consider the derived base change

$$- \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p : D(\mathbb{Z}) \to D(\mathbb{F}_p).$$

Here, $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p$ and $\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p$ are different as elements of $D(\mathbb{F}_p)$ since they have different homology in degree 1:

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_p) = 0$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\,.$$

The point is that we can hope somehow obtain a richer notion of descent by passing to the derived setting. To make the notion of "derived descent" precise, we leverage the following theorem, which will take us into the world of homotopy theory.

Theorem 2.0.4 (Stable Dold-Kan Correspondence [Pet19, Lemma 3.1.5])**.** *There is an equivalence of symmetric monoidal triangulated categories*

$$D(\mathrm{Spec}(R)) \simeq \mathsf{Mod}(HR),$$

where $\mathsf{Mod}(HR)$ *is the category of* HR*-module spectra.*

Accordingly, to a ring map $f : R \to S$, we now define the *derived descent object* to be the cosimplicial spectrum

$$\mathcal{D}_{Hf} := \left\{ HS \rightrightarrows HS \otimes_{HR} HS \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} HS \otimes_{HR} HS \otimes_{HR} HS \;\; \cdots \right\}. \tag{2.0.5}$$

And following the story above, to an R-module M, we associate a cosimplicial left $\mathcal{D}_{Hf}$-module $\mathcal{D}_{Hf} \otimes_{HR} HM$. By construction, $\lim(\mathcal{D}_{Hf} \otimes_{HR} HM)$ is the *nilpotent completion* of HM with respect to HS, denoted $HM^\wedge_{HS}$ (as in [Rav84, Definition 1.12]), and there is a natural comparison map

$$HM \to HM^\wedge_{HS}.$$

As a corollary of ordinary faithfully-flat descent, this comparison map is an equivalence whenever the map $R \to S$ is faithfully-flat. But as hoped, we do get a little more in this derived setting. Indeed, $H\mathbb{Z}^\wedge_{H\mathbb{F}_p} = H\mathbb{Z}_p$, while $(H\mathbb{F}_p)^\wedge_{H\mathbb{F}_p} = H\mathbb{F}_p$ [Rav84, Example 1.16]. That is, derived descent along $\mathbb{Z} \to \mathbb{F}_p$ "distinguishes the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{F}_p$," while ordinary faithfully-flat descent does not.

The example of derived descent which is most interesting to homotopy theorists comes from considering a very special class of examples that do not arise in classical commutative algebra: descent along the unit map $\mathbb{S} \to E$ of an $\mathbb{E}_1$-ring spectrum E. Before going further, let us state some assumptions on E:

Assumptions 2.0.6.

(i) E has an $\mathbb{E}_1$-structure.
(ii) E is even*, that is, E_* is concentrated in even degrees.*
(iii) The ring $\pi_*(E^{\otimes n})$ is commutative for all $n \geq 1$.
(iv) E_*E is flat as a left E_* module (and hence as a right E_*-module as well).

Remark 2.0.7. The assumption that our spectra E are even (or *even-graded*) may seem restrictive at first. Indeed, $\mathbb{S}$ itself is not even-graded. All of the other important spectra in chromatic homotopy theory are though, for example MU, BP, $E(n)$, $K(n)$, $H\mathbb{F}_p$, KU, etc.

The motivation for restricting our attention to even-graded ring spectra E is that under this assumption, E_* is "honestly" commutative, as opposed to graded-commutative. This allows us to speak of the schemes $\mathrm{Spec}(E_*)$ and $\mathrm{Spec}(E_*E)$ in a natural way, and remember the grading information via by equipping our schemes with $\mathbb{G}_m$-actions (see Lemma A.4.17) and working in a $\mathbb{G}_m$-equivariant setting.

Let E be a ring spectrum of satisfying Assumptions 2.0.6. As with the descent object of a ring map $R \to S$ above, we can use the unit and multiplication maps of E to define the *descent object* of E [Pet19, Definition 3.1.7]:

$$\mathcal{D}_E := \left\{ \begin{tikzcd} E \arrow[r, shift left=2] \arrow[r, shift right=2] & E \otimes E \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & E \otimes E \otimes E \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd} \right\}.$$

where all smash products are taken over $\mathbb{S}$. After applying π_*, we obtain

$$\pi_* \mathcal{D}_E = \left\{ \begin{tikzcd} \pi_* E \arrow[r, shift left=2] \arrow[r, shift right=2] & \pi_*(E \otimes E) \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \pi_*(E \otimes E \otimes E) \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd} \right\}$$

$$= \left\{ \begin{tikzcd} E_* \arrow[r, shift left=2] \arrow[r, shift right=2] & E_*E \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & E_*E \otimes_{E_*} E_*E \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd} \right\}$$

where the latter equality makes use of the Segal isomorphism

$$\pi_* E \otimes E \otimes E \cong E_*E \otimes_{E_*} E_*E,$$

which relies on the fact that E_*E is flat as a left E_* module [Rav86, Lemma 2.2.7]. Since E is even-graded, following Remark 2.0.7, we can apply Spec to get a ($\mathbb{G}_m$-equivariant) simplicial scheme

$$\mathrm{Spec}(\pi_* \mathcal{D}_E) = \left\{ \begin{tikzcd} \mathrm{Spec}(E_*) \arrow[r] & \mathrm{Spec}(E_*E) \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Spec}(E_*E) \times_{\mathrm{Spec}(E_*)} \mathrm{Spec}(E_*E) \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots \end{tikzcd} \right\}.$$

Moreover, the Hopf algebroid structure of (E_*, E_*E) ensures that for any ring R, the simplicial set $\mathrm{Spec}(\pi_* \mathcal{D}_E)(R)$ is the nerve of a groupoid. It follows that the colimit of this simplicial scheme has the structure of a stack (see Definition A.3.4), allowing us to make the following definition.

Definition 2.0.8. The *associated stack* of an ring spectrum E satisfying Assumptions 2.0.6 is

$$\mathcal{M}_E = \varinjlim \mathrm{Spec}(\pi_* \mathcal{D}_E).$$

Remark 2.0.9. Following Remark 2.0.7, $\mathcal{M}_E$ is naturally equipped with a $\mathbb{G}_m$ action, coming from the even grading on $\pi_* \mathcal{D}_E$.

In the setting of classical faithfully-flat descent of 2.0.2, given a faithfully-flat ring map $f : R \to S$, we are able to recover an R-module M after tensoring with the descent object $\mathcal{D}_f$. We make a similar construction here.

Definition 2.0.10 ([Pet19, Definition 3.1.7])**.** The *descent object* for X along $\mathbb{S} \to E$ is the cosimplicial spectrum obtained by smashing X with $\mathcal{D}_E$:

$$\mathcal{D}_E(X) := \left\{ E \otimes X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; E \otimes E \otimes X \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; E \otimes E \otimes E \otimes X \;\; \cdots \right\}.$$

We define the *E-nilpotent completion of X* to be the totalization

$$X_E^\wedge = \varprojlim \mathcal{D}_E(X).$$

Remark 2.0.11. Note that if E has an $\mathbb{E}_1$ structure, then this construction does in fact yield a cosimplicial object $\mathcal{D}_E(X)$ in the ∞-category Sp [Lur17, Theorem 4.4.2.8]. We will not concern ourselves with this so much except to know that it is true; our goal is to apply $\pi_*(-)$ and pass back to the world of ordinary rings.

After applying π_* to $\mathcal{D}_E(X)$ we can define differentials to obtain a chain complex called the *cobar resolution* [Rav86, A1.2.11], and the skeletal filtration on X yields a spectral sequence converging to

$$\mathrm{tot}(\pi_* \mathcal{D}_E) = \pi_* X_E^\wedge :$$

this is the E-based *Adams spectral sequence.*

This construction of the Adams spectral sequence is "the same" as the usual one involving Adams resolutions (discussed, for example, in [Rav86, Chapter 2.2]) repackaged to emphasize the connection to classical faithfully flat-descent. For details on the similarities, see the proof of [Pet19, Lemma 3.1.15].

Remark 2.0.12. Our algebro-geometric setup affords us an additional way to phrase this spectral sequence. Indeed, for each n, the $\pi_* E^{\otimes n}$-modules $\pi_*(E^{\otimes n} \otimes X)$ assemble to make $\mathcal{D}_E(X)$ into a cosimplical $\mathcal{D}_E$-module, and so we can alternatively consider the associated simplicial quasi-coherent sheaf over $\mathrm{Spec}(\pi_* \mathcal{D}_E)$. After taking colimits, this construction yields a quasi-coherent sheaf over $\mathcal{M}_E$, which we will denote $\mathcal{M}_E(X)$ (see Definition A.2.13 and [Pet19, p.105]). Note that if the E-homology of X is finitely presented over E_*, then $\mathcal{M}_E(X)$ is in fact a coherent sheaf. With this language, we can make the following characterization of the E_2-page of the E-Adams spectral sequence.

Theorem 2.0.13 (E-Adams Spectral Sequence [Pet19, 3.1.15])**.** *For E a ring spectrum satisfying Assumptions 2.0.6, the E_2-page of the E-based Adams spectral sequence converging to the homotopy groups of the spectrum $X_E^\wedge$ is given by*

$$E_2^{s,2t} = \mathrm{Ext}^s_{E_*E}(E_*, E_*(\Sigma^{-2t}X)) \cong H^s(\mathcal{M}_E; \mathcal{M}_E(X) \otimes \mathcal{M}_E(\mathbb{S}^{-2t})) \implies \pi_{s+2t} X_E^\wedge,$$

with $E_2^{s,2t+1} = 0$ for all s and t since E is even.

3. Moduli of formal groups

3.1. Complex cobordism

In this section, we will define the moduli of formal groups, $\mathcal{M}_{\mathrm{fg}}$, and study its intimate relationship with the spectrum representing complex cobordism, MU. Our discussion will closely follow [Pst21, Section 12]. (See Section A.3 and Section B.1 for the relevant background on stacks and formal groups, respectively.)

Remark 3.1.1. There are many reasons to study the MU-based Adams spectral sequence, commonly called the *Adams-Novikov spectral sequence*:

(i) MU is a well-understood ring spectrum satisfying Assumptions 2.0.6. Indeed, after Eilenberg-MacLane spectra, Thom spectra are some of the most "accessible" to study—this is what originally led Novikov to consider the MU-based Adams spectral sequence.
(ii) Any spectrum X is MU-complete, i.e., the natural map $X \to X_{MU}^\wedge$ is an equivalence [Rav84, Example 1.16]. From a computational perspective, this is a convenient (but hardly necessary) property.
(iii) The E_2-page admits a filtration into pieces which are torsion with respect to certain interesting elements of BP_*. This filtration yields a spectral sequence which reveals rich qualitative structure in the Adams-Novikov spectral sequence itself.

(iv) Finally, we have a very concrete model for the stack $\mathcal{M}_{MU}$ associated to MU: it is the moduli stack of formal groups. This fact allows us to study stable homotopy by arithmetic and geometric means.

In fact, the third point, which we discuss in Section 4.1, is a direct consequence of the fourth. This final fact is our jumping off point for chromatic homotopy theory.

Definition 3.1.2. The *moduli stack of formal groups*, $\mathcal{M}_{\mathrm{fg}}$, is the (*fpqc*-)stack

$$\mathcal{M}_{\mathrm{fg}} : \mathsf{Ring} \to \mathsf{An}$$

assigning to each ring R the groupoid of formal groups over $\mathrm{Spec}(R)$ with their isomorphisms.

By the Yoneda lemma, we can equivalently think of the R-points of $\mathcal{M}_{\mathrm{fg}}$ (i.e., the objects of $\mathcal{M}_{\mathrm{fg}}(R)$) as maps $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$. That is, a formal group over $\mathrm{Spec}(R)$ is equivalent to the data of a "classifying map" $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$.

Let $\mathcal{F}\mathrm{gl}$ be the moduli of formal group laws. By a result of Lazard, $\mathcal{F}\mathrm{gl}$ is actually an affine scheme,

$$\mathcal{F}\mathrm{gl} \cong \mathrm{Spec}(L)$$

for a ring L called the *Lazard ring*. The association of a formal group law F to the formal group it presents, G_F yields a morphism of *fpqc*-sheaves

$$\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}.$$

Since a formal group $G \to \mathrm{Spec}(R)$ is presented by a formal group law Zariski-locally on R, the induced map $\pi_0\,\mathcal{F}\mathrm{gl} \to \pi_0\mathcal{M}_{\mathrm{fg}}$ is a surjection of sheaves. It follows that that $\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$ is an effective epimorphism [Lur09, Corollary 7.2.1.15], and so the map

$$\mathrm{colim}\left(\cdots \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \rightrightarrows \mathcal{F}\mathrm{gl}\right) \to \mathcal{M}_{\mathrm{fg}}$$

is an equivalence.

We can better understand the left hand side of this equivalence through the following proposition.

Proposition 3.1.3 ([Pst21, Proposition 12.2])**.** *The morphism* $\mathcal{F}\mathrm{gl} \to \mathcal{M}_{fg}$ *discussed above is faithfully flat, and*

$$\mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{fg}} \mathcal{F}\mathrm{gl} \simeq \mathcal{F}\mathrm{gl} \times \mathbb{G}_{\mathrm{inv}}\,.$$

Here,

$$\mathbb{G}_{\mathrm{inv}} = \mathrm{Spec}(\mathbb{Z}[b_0^{\pm}, b_1, b_2, \ldots])$$

is the affine group scheme of invertible power series, i.e., the scheme with R-points corresponding to power series

$$\varphi(t) = b_0 t + b_1 t^2 + b_2 t^3 + \ldots$$

with $b_j \in R$ and $b_0 \in R^\times$ made into a group under power-series composition.

Informally, the latter part of this proposition is saying that the data of two formal group laws presenting the same formal group is equivalent to the data of one formal group law and a choice of twist (isomorphism). Letting $\mathbb{G}_{\mathrm{inv}}$ act on $\mathcal{F}\mathrm{gl}$ in the natural way and act trivially on $\mathcal{M}_{\mathrm{fg}}$, we obtain the following presentation of $\mathcal{M}_{\mathrm{fg}}$ as an *fpqc*-algebraic stack (see Definition A.5.3 and Definition A.5.6 for details):

Theorem 3.1.4 ([Pst21, Proposition 12.3])**.** $\mathcal{M}_{fg} \cong \mathcal{F}\mathrm{gl} /\!\!/ \mathbb{G}_{\mathrm{inv}}$.

We will also be interested in a certain intermediate object between $\mathcal{F}\mathrm{gl}$ and $\mathcal{M}_{\mathrm{fg}}$, obtained as follows. Let

$$\mathbb{G}^s_{\mathrm{inv}} = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \ldots])$$

be the affine group scheme of *strictly*-invertible power series—that is, the invertible power series with leading coefficient 1. We can observe that over any ring R, there is twisting action of $R^\times$ on $\mathbb{G}^s_{\mathrm{inv}}(R)$ defined by

$$a \cdot \varphi(t) = a\varphi(a^{-1}t)$$

for any unit $a \in R^\times$ and $\varphi \in \mathbb{G}^s_{\mathrm{inv}}(R)$. We can deduce

$$\mathbb{G}_{\mathrm{inv}} \cong \mathbb{G}^s_{\mathrm{inv}} \rtimes \mathbb{G}_m\,.$$

With this in mind, we make the following definition

Definition 3.1.5. The *strict moduli of formal groups* is the quotient (*fpqc*-)stack

$$\mathcal{M}^s_{\mathrm{fg}} = \mathcal{F}\mathrm{gl} /\!\!/ \mathbb{G}^s_{\mathrm{inv}}\,.$$

Remark 3.1.6. The map $\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}^s$ is naturally $\mathbb{G}_{\mathrm{inv}}$-equivariant since (strictly) isomorphic formal group laws determine the same formal group, and so $\mathcal{M}_{\mathrm{fg}}^s$ inherits a $\mathbb{G}_m$ action. By construction, this makes $\mathcal{M}_{\mathrm{fg}}^s$ into a $\mathbb{G}_m$-torsor over $\mathcal{M}_{\mathrm{fg}}$ [Pst21, Page 56]. In summary, we have the following tower of torsors over $\mathcal{M}_{\mathrm{fg}}$:

$$\begin{tikzcd} \mathcal{F}\mathrm{gl} \arrow[d, dash, "\mathbb{G}_{\mathrm{inv}}^s"'] \arrow[dd, dash, bend left, "\mathbb{G}_{\mathrm{inv}}"] \\ \mathcal{M}_{\mathrm{fg}}^s \arrow[d, dash, "\mathbb{G}_m"'] \\ \mathcal{M}_{\mathrm{fg}}. \end{tikzcd}$$

Remark 3.1.7. The adjective "strict" for $\mathcal{M}_{\mathrm{fg}}^s$ comes from that fact that $\mathcal{M}_{\mathrm{fg}}^s$ classifies formal groups up to strict isomorphism in the sense that for any formal groups G_F and $G_{F'}$ over $\mathrm{Spec}(R)$ presented by a formal group laws F and F', the set of morphisms

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{fg}}^s(R)}(G_F, G_{F'})$$

exactly corresponds to the set of *strict formal group law isomorphisms* $F \to F'$ in the sense of [Rav86, Definition A2.1.5].

In [Pst21], the stack $\mathcal{M}_{\mathrm{fg}}^s$ is obtained in a different but equivalent way as the stack with R-points consisting of pairs (G, ψ) where $G \to \mathrm{Spec}(R)$ is a formal group and $\psi : \omega_G \simeq \tilde{R}$ is a choice of global trivialization of the line bundle ω_G of invariant differentials on G. This formulation is discussed more in Section B.1.

The following foundational theorem of Quillen is the justification for point (iv) of Remark 3.1.1, and brings us back to homotopy theory.

Theorem 3.1.8 (Quillen [Rav86, Theorem 4.1.6])**.** *The map $L \to MU_*$ classifying the formal group law over MU_* induced by the identity complex orientation $MU \to MU$ is an isomorphism.*

Remark 3.1.9. The structure of L was known to Lazard and the coefficient ring of MU was originally computed by Milnor using the classical $H\mathbb{F}_p$-Adams spectral sequence. This latter computation is relatively easy: the spectral sequence collapses at the E_2-page. But if E is a complex oriented ring spectrum, just knowing that MU_* and L are abstractly isomorphic is not enough to relate the map $MU_* \to E_*$ obtained by applying π_* to the orientation $MU \to E$ and the map $L \to E_*$ classifying the associated formal group law over E_*. Relating these two maps is exactly the content of Quillen's much more difficult theorem: for any complex oriented E, the following diagram commutes:

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend left] & MU_* \arrow[r] & E_*. \end{tikzcd}$$

Quillen's theorem tells us that the algebraic information of the associated formal group law over E_* carries interesting topological information. One way to phrase this is that MU-algebras in hSp (the inputs to the Adams-Novikov spectral sequence) can be studied by understanding $\mathcal{M}_{\mathrm{fg}}$. To make this precise, we rephrase Quillens theorem in geometric language.

Corollary 3.1.10 ([Hop99, Proposition 6.5])**.** *There are natural, $\mathbb{G}_m$-equivariant isomorphisms of affine schemes*

$$\mathrm{Spec}(MU_*) = \mathcal{F}\mathrm{gl}$$
$$\mathrm{Spec}(MU_*MU) = \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{fg}^s} \mathcal{F}\mathrm{gl}\,.$$

Theorem 3.1.11 ([Pst21, Prop 12.8])**.** *There is an equivalence of symmetric monoidal abelian categories between even-graded comodules over the Hopf algebroid (MU_*, MU_*MU) and $\mathsf{QCoh}(\mathcal{M}_{fg})$.*

Proof. We trace the following sequence of equivalences of categories:

$$\begin{aligned}
\mathsf{Comod}^{\mathrm{ev}}(MU_*, MU_*MU) &\simeq \mathsf{Mod}^{\mathrm{ev}}(\pi_* MU^{\otimes[n]}) && \text{(Lemma A.4.21)}\\
&\simeq \mathsf{QCoh}_{\mathbb{G}_m}(\mathrm{Spec}(\pi_* MU^{\otimes[n]}))) && \text{(Lemma A.4.13; Lemma A.4.17)}\\
&\simeq \mathsf{QCoh}_{\mathbb{G}_m}(\mathcal{F}\mathrm{gl}^{\times[n]}) && \text{(Corollary 3.1.10)}\\
&\simeq \mathsf{QCoh}_{\mathbb{G}_m}(\mathcal{M}^s_{\mathrm{fg}}) && \text{(Definition 3.1.5)}\\
&\simeq \mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}}) && \text{(Remark 3.1.6).}
\end{aligned}$$

□

In particular, the above proof shows that the stack associated to MU in the sense of Definition 2.0.8 is

$$\mathcal{M}_{MU} = \operatorname{colim} \mathrm{Spec}(\pi_* \mathcal{D}_{MU}) = \mathcal{M}^s_{\mathrm{fg}},$$

which, after taking the quotient by $\mathbb{G}_m$, gives $\mathcal{M}_{\mathrm{fg}}$.

The E_2-page of the classical Adams-Novikov spectral sequence concerns Ext in the category of comodules over (MU_*, MU_*MU). By Theorem 3.1.11, there should be an equivalent formulation in terms of Ext in the category of quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$. This is exactly the formulation of the generalized Adams spectral sequence in Theorem 2.0.13 taking $E = MU$. So to complete our translation of classical chromatic homotopy theory into the language of stacks, we want to identify the quasi-coherent sheaves $\mathcal{M}_{MU}(\mathbb{S}^{2t})$ over $\mathcal{M}_{\mathrm{fg}}$, i.e., the sheaves corresponding to the (even-graded) (MU_*, MU_*MU)-comodules $\Sigma^{-2t}MU_*$.

Lemma 3.1.12 ([Pet19, Example 2.3.4])**.** *Under the equivalence of categories of Theorem 3.1.11, the (MU_*, MU_*MU)-comodule $\Sigma^{-2t}MU_*$ is sent to the quasi-coherent sheaf $\omega^{\otimes t}$ over $\mathcal{M}_{fg}$, where ω is the sheaf of invariant differentials of Definition B.2.7.*

Proof. The proof has to do with the fact that MU is the Thom spectrum of the universal bundle over BU. Let $\mathcal{L}$ denote the restriction of this bundle to the universal line bundle over $\mathbb{CP}^\infty$. This bundle pulls back along the map $0 : * \to \mathbb{CP}^\infty$ to give a line bundle $0^* \mathcal{L}$ over $*$. Applying the Thom space construction gives the inclusion map

$$S^2 \simeq \mathrm{Th}(0^* \mathcal{L}) \hookrightarrow \mathrm{Th}(\mathcal{L}) \simeq \Sigma^2 \mathbb{CP}^\infty .$$

where the latter equivalence follows from [Pet19, Example 1.1.3]. This inclusion induces the quotient map

$$MU^*[\![t]\!] \to MU^*[\![t]\!] / t^2 \tag{3.1.13}$$

on MU-cohomology.

Recall that for any (real) rank n vector bundle ξ over a base space B, we have the Thom isomorphism

$$MU^* B_+ \cong \tilde{MU}^{*+n}\,\mathrm{Th}(\xi),$$

where $\tilde{MU}^*$ denotes reduced MU-cohomology. We have a sequence of isomorphisms of MU_*MU-comodules:

$$\begin{aligned}
\pi_* MU &\cong MU^* S^0\\
&\cong \tilde{MU}^{*+2}\,\mathrm{Th}(0^* \mathcal{L})\\
&\cong 0^* \tilde{MU}^{*+2}\,\mathrm{Th}(\mathcal{L})\\
&\cong 0^* \tilde{MU}^{*+2} \Sigma^2 \mathbb{CP}^\infty\\
&\cong 0^* \tilde{MU}^{*} \mathbb{CP}^\infty .
\end{aligned}$$

Note that $\pi_* MU$ is a free rank-1 MU^*MU-comodule, and so determines a line bundle over $\mathcal{M}_{\mathrm{fg}}$. This line bundle pulls back along the map $\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$ classifying the universal formal group over $\mathcal{F}\mathrm{gl}$ to define a line bundle over the universal formal group, $\mathbb{G}_{MU} = \mathrm{Spf}(MU^* \mathbb{CP}^\infty)$ which we denote by

$$\widetilde{\pi_* MU}.$$

From the above sequence of isomorphisms, and by the functoriality of the correspondence between MU^*MU-comodules and quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, we have

$$\widetilde{\pi_* MU} \cong (0^* \widetilde{\tilde{MU}^* \mathbb{CP}^\infty}) \cong 0^*(\widetilde{\tilde{MU}^* \mathbb{CP}^\infty}).$$

The sheaf $\mathcal{I} = \widetilde{\tilde{MU}^* \mathbb{CP}^\infty}$ is exactly the sheaf of functions on $\mathbb{G}_{MU}$ which vanish at the identity. To see this, we can choose a coordinate for $\mathbb{G}_{MU}$ and notice that

$$\begin{aligned}\tilde{MU}^* \mathbb{CP}^\infty &= \ker(MU^* \mathbb{CP}^\infty \to MU^*)\\ &\cong \ker(MU^*[\![t]\!] \to MU^*),\end{aligned}$$

where the latter map is induced by the zero section $\mathcal{F}\mathrm{gl} \to \mathbb{G}_{MU}$ and sends the coordinate t to 0. From the computation in 3.1.13, applying 0^* to $\mathcal{I}$ adds the restriction that $t^2 = 0$. Thus, $0^* \mathcal{I}$ is the sheaf of linear functions on $\mathbb{G}_{MU}$ which vanish at 0. That is, $0^* \mathcal{I}$ is canonically identified with the cotangent space of $\mathbb{G}_{MU}$ at the identity:

$$0^* \mathcal{I} \cong T_0^* \mathbb{G}_{MU}.$$

This cotangent space at the identity is the sheaf of invariant differentials ω, or "Lie algebra," of $\mathbb{G}_{MU}$ defined in Definition B.2.7.

One can check that replacing $\pi_* MU$ with $\pi_{*-2n} MU$ in the above proof has the effect of replacing $\tilde{MU}^* \mathbb{CP}^\infty$ with $(\tilde{MU}^* \mathbb{CP}^\infty)^{\otimes n}$, essentially because the Thom construction is a monoidal functor:

$$\mathrm{Th}(\xi_1 \oplus \xi_2) \simeq \mathrm{Th}(\xi_1) \otimes \mathrm{Th}(\xi_2).$$

The result is that instead of ω in the end, we obtain $\omega^{\otimes n}$, as desired. □

Following Theorem 2.0.13, we can now make the following identification.

Proposition 3.1.14. *The E_2-page of the Adams-Novikov spectral sequence converging to $\pi_* \mathbb{S}$ is given by*

$$E_2^{s,2t} := \mathrm{Ext}^s_{MU_* MU}(MU_*, \Sigma^{-2t} MU_*) \cong \mathrm{Ext}^s_{\mathsf{QCoh}(\mathcal{M}_{fg})}(\mathcal{O}_{\mathcal{M}_{fg}}, \omega^{\otimes t}).$$

By definition, the latter Ext *term is the cohomology group $H^s(\mathcal{M}_{fg}, \omega^{\otimes t})$.*

Remark 3.1.15. It is common in chromatic homotopy theory to fix a prime p and work p-locally. There is a further simplification we can make: instead of working directly with $MU_{(p)}$, we work with a retract called the *Brown-Peterson spectrum*, BP.

This spectrum was originally constructed by Brown and Peterson from $H\mathbb{Z}$ using Postnikov systems. A more canonical construction was later given by Quillen, who defined a spectrum-level idempotent map $MU_{(p)} \to MU_{(p)}$ inducing an idempotent natural transformation $MU^*_{(p)}(-) \to MU^*_{(p)}(-)$, the image of which determines the spectrum BP by Brown-representability. See [Rav86, Chapter 4.1].

At the level of coefficient rings, the analogue of the Quillen idempotent comes from a theorem of Cartier which says that every p-local formal group is canonically (strictly) isomorphic to a p-typical one. That is, if a formal group law over a $\mathbb{Z}_{(p)}$-algebra R is classified by a map $L \otimes \mathbb{Z}_{(p)} \to R$, then there is a factorization through the ring V over which the universal p-typical formal group law is defined. It turns out that $V = BP_*$, and after identifying $L = MU_*$, the factorization becomes

$$\begin{tikzcd} (MU_{(p)})_* \arrow[r, "\cong"] \arrow[d] & L \otimes \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & R \\ BP_* \arrow[r, "\cong"] & V \arrow[ur, dashed] & \end{tikzcd}$$

where the left hand vertical map is the one induced by the Quillen idempotent. This remark is summarized by the following theorem.

Theorem 3.1.16 (Cartier [Goe08, Theorem 2.43], [Rav86, Theorem A2.1.18])**.** *Every formal group over a $\mathbb{Z}_{(p)}$-algebra is canonically (strictly) isomorphic to a p-typical one. This induces correspondence induces an equivalence of stacks*

$$\mathcal{M}_{BP} \simeq \mathcal{M}^s_{fg} \times \mathrm{Spec}(\mathbb{Z}_{(p)}).$$

3.2. **Heights, v_n, and the chromatic filtration**

In the previous section, we motivated the study of $\mathcal{M}_{\mathrm{fg}}$ as a way to understand stable homotopy theory. We have already had a look at the local structure of $\mathcal{M}_{\mathrm{fg}}$: its R-points correspond to formal groups over R. But the feature of $\mathcal{M}_{\mathrm{fg}}$ which returns the richest insights into stable homotopy theory is a certain global structure called the *height filtration*. In this section, we will introduce and study the height filtration, starting from the classical story of formal group laws and leading to a modern arithmetic picture in terms of formal groups.

Definition 3.2.1. The *p-series* of a formal group law F defined over R is the power series

$$[p]_F(x) := \underbrace{x +_F \ldots +_F x}_{p \text{ times}} \in R[\![x]\!]\,.$$

Recall that a formal group law F is called *p-typical* if its p-series is of the form

$$[p]_F(x) = px + a_1 x^p + a_2 x^{p^2} + \ldots + a_n x^{p^n} + \ldots.$$

Definition 3.2.2 ([Rav86, A2.2.7])**.** We say a p-typical formal group law F over a $\mathbb{Z}_{(p)}$-algebra R is of *height at least* n if its p-series is of the form

$$[p]_F(x) = h(x^{p^n})$$

for some power series h defined over R. We say F is of *height exactly* n if h is invertible. If F is of height at least n for all n, we say F has *height* ∞.

By the above definition, every formal group law is of height at least 0.

Example 3.2.3. This definition makes it easy to compute the heights of several formal group laws arising from familiar complex-oriented spectra:

Spectrum	Formal Group Law	p-series	Height
$H\mathbb{Q}$	$x + y \in \mathbb{Q}[\![x, y]\!]$	px	0
$KU_{/p}$	$x + y + uxy \in \mathbb{F}_p[u^{\pm}][\![x, y]\!]$	$px + pu(x^2 + \ldots + x^{p-1}) + ux^p \equiv_p ux^p$	1
$H\mathbb{F}_p$	$x + y \in \mathbb{F}_p[\![x, y]\!]$	$px \equiv_p 0$	∞

Following these computations, it is straightforward to deduce that the p-series of any formal group law F will be of the form

$$[p]_F(x) = px + \text{higher order terms},$$

and so if F is defined over a ring where p is invertible, F is of height 0. Conversely, over an $\mathbb{F}_p$-algebra, every formal group law has height at least 1.

Example 3.2.4. A formal group law need not be of any exact height. Consider the formal group law associated to $H\mathbb{Z}_{(p)}$:

$$F(x, y) = x + y \in \mathbb{Z}_{(p)}[\![x, y]\!]\,.$$

As with $H\mathbb{Q}$ and $H\mathbb{F}_p$, we compute $[p]_F(x) = px$. Note that F is not of height at least 1 since the p-series does not factor through x^p. On the other hand, F is not of height exactly 0 since p is not a unit in $\mathbb{Z}_{(p)}$.

Definition 3.2.5. Let $v_n \in V$ be the coefficient of x^{p^n} in the p-series of the universal p-typical formal group law. Note that $v_0 = p$.

Remark 3.2.6. Suppose F is a p-typical formal group law over R classified by a map $\theta : V \to R$. This map determines elements $\theta(v_n)$ which are the coefficients of x^{p^n} in $[p]_F(x)$ by the universal property of V. Note that F is of height at least n if $v_0, \ldots, v_{n-1}$ are in $\ker\theta$, and moreover of height exactly n if $\theta(v_n)$ is a unit in R.

In order to use heights to understand $\mathcal{M}_{\mathrm{fg}}$, we need to develop a notion of the height of a formal group. In some sense, this should exactly be a "coordinate free" version of the discussion of the heights of formal group laws. The process of making this idea precise is the beginning of a very elegant arithmetic story, for which we will follow [Pst21, Section 13] and [Goe08, Section 5].

To begin, recall that for any $\mathbb{F}_p$-algebra R, the *Frobenius homomorphism* is the map $\mathrm{Frob}_R : R \to R$ sending r to r^p. We can extend this definition to obtain a self map on any (*fpqc*-)sheaf X over $\mathrm{Spec}(\mathbb{F}_p)$

called the *absolute Frobenius* of X. If X is an $\mathbb{F}_p$-scheme, the absolute Frobenius is the map $\mathrm{Frob}_X : X \to X$ which restricts on any open affine-subscheme $\mathrm{Spec}(R)$ to $\mathrm{Spec}(\mathrm{Frob}_R)$. More generally, we can write any *fpqc*-sheaf X as the (homotopy) colimit of a collection of affine $\mathbb{F}_p$-schemes, $\{\mathrm{Spec}(R_\alpha)\}$ (see Lemma A.5.8). Each $\mathrm{Spec}(R_\alpha)$ has its own Frobenius map, and by the sheaf condition, these maps glue to produce a map $\mathrm{Frob}_X : X \to X$ as in the following diagram:

$$
\begin{tikzcd}
\mathrm{Spec}(R_\alpha) \arrow[d, "\mathrm{Frob}_{R_\alpha}"] & \\
\mathrm{Spec}(R_\alpha) \arrow[ddr, bend right] & \\
 & X \cong \mathrm{colim}(\mathrm{Spec}(R_\alpha)) \arrow[d, dashed, "\mathrm{Frob}_X"'] \arrow[dd, bend left] \\
 & X \arrow[d] \\
 & \mathrm{Spec}(\mathbb{F}_p)
\end{tikzcd}
$$

As in the above diagram, the absolute Frobenius always defines a self map of sheaves over $\mathrm{Spec}(\mathbb{F}_p)$, but may not respect structure maps to some other base, e.g. $\mathrm{Spec}(R)$ for general $\mathbb{F}_p$-algebras R. To make sense of this, we introduce the following.

Definition 3.2.7 ([Pst21, Definition 13.1])**.** Let Y be a sheaf over $\mathrm{Spec}(\mathbb{F}_p)$ and $X \to Y$ a sheaf over Y. The *Frobenius twist* of X over Y is the pullback

$$X^{(p)} := \mathrm{Frob}_Y^* X.$$

The *relative Frobenius* is the natural morphism $\mathrm{Frob}_{X/Y} : X \to X^{(p)}$ obtained in the following pullback diagram:

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\mathrm{Frob}_{X/Y}"] \arrow[drr, bend left, "\mathrm{Frob}_X"] \arrow[ddr, bend right] & & \\
 & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
 & Y \arrow[r, "\mathrm{Frob}_Y"'] & Y
\end{tikzcd}
$$

The *nth iterate of the relative Frobenius* is the map

$$\mathrm{Frob}^n_{X/Y} : X \to (\mathrm{Frob}^n_{X/Y})^* X$$

obtained by replacing Frob_Y with Frob_Y^n in the above diagram, and similarly we will write $X^{(p^n)}$ for $(\mathrm{Frob}_Y^n)^* X$

Remark 3.2.8. By construction, the nth relative Frobenius of X factors through the $(n-1)$st and the relative Frobenius of $X^{(p^{n-1})}$, as in the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{Frob}_X^{n-1}"] \arrow[rr, bend right, "\mathrm{Frob}_X^n"'] & X^{(p^{n-1})} \arrow[r, "\mathrm{Frob}_{X^{(p^{n-1})}}"] & X^{(p^n)}
\end{tikzcd}
$$

Our main use of the relative Frobenius will be on formal groups over an $\mathbb{F}_p$-algebra R, so we study this case in detail in the following example.

Example 3.2.9 ([Pst21, Example 13.2])**.** Let $X = \mathrm{Spf}(R[\![x]\!]) \to \mathrm{Spec}(R)$ be the formal affine line over an $\mathbb{F}_p$-algebra R. Note that

$$\mathrm{Spf}(R[\![x]\!])^{(p)} = \mathrm{Spf}(R[\![x]\!]) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R^{(p)}) = \mathrm{Spf}(R[\![x]\!] \,\hat{\otimes}_R R^{(p)})$$

where $R^{(p)}$ is the R algebra induced by $\mathrm{Frob}_R : R \to R$ (the Frobenius twist of R). The map $R \to R \otimes_R R^{(p)}$ given by $r \mapsto 1 \otimes r$ is an isomorphism of rings (but not necessarily of R-algebras), so we can compute

$$\begin{aligned}
\mathrm{Spf}(R[\![x]\!])^{(p)} &\cong \mathrm{Spf}(R[\![x]\!] \hat{\otimes}_R R^{(p)}) \\
&\cong \varinjlim \mathrm{Spec}((R[x]/x^n) \otimes_R R^{(p)}) \\
&\cong \varinjlim \mathrm{Spec}((R \otimes_R R^{(p)})[x]/x^n) \\
&\cong \varinjlim \mathrm{Spec}(R[x]/x^n) \\
&\cong \mathrm{Spf}(R[\![x]\!]),
\end{aligned}$$

where the isomorphisms are of formal schemes, but not necessarily of formal schemes over $\mathrm{Spec}(R)$. The upshot is that we can identify the relative Frobenius

$$\mathrm{Spf}(R[\![x]\!]) \longrightarrow \mathrm{Spf}(R[\![x]\!])^{(p)}$$

by the (x-adically continuous) map it induces on the rings of global sections

$$R[\![x]\!] \longleftarrow (R \otimes_R R^{(p)})[\![x]\!] \xleftarrow{\cong} R[\![x]\!]$$

which is the R-linear self-map of $R[\![x]\!]$ sending x to x^p.

By contrast, the absolute Frobenius of $\mathrm{Spf}(R[\![x]\!])$ is not necessarily R-linear on global sections. Using the fact that the induced map on global sections is continuous with respect to the x-adic topology on $R[\![x]\!]$ and so must extend maps on each quotient $R[x]/x^n$, and noting that $\mathrm{char}\, R = p$, we can compute that the absolute Frobenius is the map defined on each power series in $R[\![x]\!]$ by

$$\sum_{k \geq 0} r_k x^k \mapsto \sum_{k \geq 0} r_k^p x^{kp}.$$

This map is not generally R-linear since it extends the Frobenius of R, sending r to r^p.

The functorial construction of the Frobenius implies that for any formal group $G \to \mathrm{Spec}(R)$, the formal scheme $G^{(p)}$ has the structure of a formal group and the relative Frobenius is a morphism of formal groups. That is, for each R-algebra S, the relative Frobenius induces a group homomorphism on groups of S-points:

$$G(S) \to G^{(p)}(S).$$

A diagram chase allows us to be precise about what this homomorphism is exactly, as the following lemma records.

Lemma 3.2.10 ([Pst21, Example 13.6]). *Let $F(x, y) = \sum_{i,j \geq 0} r_{i,j} x^i y^j$ be a formal group law over an $\mathbb{F}_p$-algebra R, and let G_F be its associated formal group. The formal group $G_F^{(p)}$ is presented by the formal group law $F'(x, y) = \sum_{i,j \geq 0} r_{i,j}^p x^i y^j$. It follows that the relative Frobenius*

$$G_F \to G_F^{(p)} \cong G_{F'}$$

is induced by the morphism of formal group laws $\varphi : F \to F'$ given by $\varphi(x) = x^p$.

Proof. See [Poo17, Example 3.7.2]. ☐

Informally, the lemma is saying that the relative Frobenius of a formal group is locally induced by the Frobenius twist on formal group laws $x \mapsto x^p$. Notice that Definition 3.2.2 can be rephrased to say that a formal group law F is of height at least n if its p-series factors through the nth iterate of this Frobenius twist, that is, if $[p]_F$ factors through $x \mapsto x^{p^n}$ as in the following diagram:

$$\begin{tikzcd}
F \arrow[r, "x^{p^n}"] \arrow[dr, "{[p]_F}"'] & F' \arrow[d, dashed, "h"] \\
& F
\end{tikzcd}$$

With this setup, it is natural to extend our "local" definition of the height of a formal group law to a global definition for general formal groups in the following way.

Definition 3.2.11 ([Pst21, Definition 13.8])**.** A formal group G over a base scheme S over $\mathbb{Z}_{(p)}$ is of *height at least* n if the multiplication by p map $[p]_G : G \to G$ factors through the nth iterate of the relative Frobenius:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Frob}_{G/S}"] \arrow[dr, "{[p]_G}"'] & G^{(p)} \arrow[d, dashed, "h"] \\ & G \end{tikzcd}$$

We say G is of *height exactly* n if such an h exists and is an isomorphism of formal groups. Finally, G is of *height* ∞ if the above factorization happens for all n.

Remark 3.2.12. Taking the 0th relative Frobenius to be the identity map on G, we see that the above definition says that all formal groups are of height at least 0.

It turns out that there is a convenient geometric criterion for when this factorization through the relative Frobenius happens.

Proposition 3.2.13 ([Pst21, Proposition 13.7])**.** *Let $f : G \to H$ be a morphism of formal groups over* $\mathrm{Spec}(R)$ *for some $\mathbb{F}_p$-algebra R. Then f factors uniquely through the realtive Frobenius of G if and only if the induced map on sheaves of invariant differentials* $\mathsf{Lie}(f) : \omega_H \to \omega_G$ *is* 0.

Proof. Note that the property that f factors *uniquely* through the relative Frobenius of G and the property that $\mathsf{Lie}(f) = 0$ are both local on $\mathrm{Spec}(R)$, and so we can assume G and H are both presented by formal group laws F_G and F_H over R and f corresponds to a formal group law homomorphism $f \in R[\![x]\!]$.

The map $\mathsf{Lie}(f) : \omega_H \to \omega_G$ is the map of free rank-1 R-modules obtained by restricting the map $df : f^*\Omega^1_{H/R} \to \Omega^1_{G/R}$ of rank-1 $R[\![x]\!]$-modules along the zero section $x \mapsto 0$ of G, and the canonical generator $\eta_H(x)dx$ of ω_H generates $\Omega^1_{H/R}$ as an $R[\![x]\!]$-module, and so $\mathsf{Lie}(f)$ vanishes if and only if df vanishes (see Lemma B.2.10).

So, if $\mathsf{Lie}(f) = 0$, we have $df(x) = f'(x)dx = 0$. Writing

$$f(x) = \sum_i a_i x^i,$$

this forces $ia_i = 0$ for all i. This means $a_i = 0$ unless i is a multiple of p. This exactly means that we can write $f(x) = f'(x^p)$, i.e., f factors through the relative Frobenius.

On the other hand, if we have $f(x) = h(x^p)$ for some h, then

$$df(x) = f'(x)dx = px^{p-1}h'(x^p)dx = 0.$$

This completes the proof. □

So, we can check if a formal group G over some base scheme S over $\mathbb{Z}_{(p)}$ is of height at least 1 if the multiplication by p map $[p]_G : G \to G$ induces the zero map on invariant differentials. In this case, we get a factorization of the Frobenius twist which we call V_1:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "{[p]_G}"'] & G^{(p)} \arrow[d, dashed, "V_1"] \\ & G \end{tikzcd}$$

Remark 3.2.14. Here, V_1 is the "*Verschiebung* morphism" for G. This observation leads to rich connection to Dieudonné theory.

The map V_1 itself induces a map on differentials $\mathsf{Lie}(V_1) : \omega_G \to \omega_{G^{(p)}}$, and we can ask whether $\mathsf{Lie}(V_1) = 0$, in which case Proposition 3.2.13 gives us a map $V_2 : G^{(p^2)} \to G$. Inductively, if G is of height at least n, then each of $\mathsf{Lie}(V_0), \dots, \mathsf{Lie}(V_{n-1})$ vanish and we obtain a factorization

$$\begin{tikzcd} G \arrow[r] \arrow[drrr, "{[p]}"'] & G^{(p)} \arrow[r, dashed, "\cdots"] \arrow[drr, "V_1"] & G^{(p^{n-1})} \arrow[r] \arrow[dr, "V_{n-1}"] & G^{(p^n)} \arrow[d, dashed, "V_n"] \\ & & & G \end{tikzcd}$$

inducing a map $\mathsf{Lie}(V_n) : \omega_G \to \omega_{G^{(p^n)}}$. Note the following lemma.

Lemma 3.2.15 ([Goe08, Lemma 4.13])**.** *Let G be a formal group over a base scheme S over $\mathbb{F}_p$. Then*

$$\omega_G^{\otimes p^n} \cong \mathrm{Frob}^*_{G/S}\, \omega_{G^{(p^n)}}.$$

Using this, we can rewrite $\mathsf{Lie}(V_n)$ as a map $\mathsf{Lie}(V_n) : \omega_G \to \omega_G^{\otimes p^n}$. Since ω_G is invertible as an $\mathcal{O}_S$-module, we can multiply by its inverse ω_G^{-1} to obtain the map of sheaves

$$v_n(G) := \mathsf{Lie}(V_n) \otimes \omega_G^{-1} : \mathcal{O}_S \to \omega_G^{\otimes p^n - 1}.$$

That is, $v_n(G)$ is a global section of the line bundle $\omega_G^{\otimes p^n-1}$ over S, an element of $H^0(S, \omega_G^{\otimes p^n-1})$.

Remark 3.2.16. Note that this construction of $v_n(G)$ is functorial in the sense that if $G' \to S'$ is another formal group over a different base $\mathbb{F}_p$-scheme, and $f : S' \to S$ is a map of $\mathbb{F}_p$-schemes inducing an isomorphism $f^*G \cong G'$, then $f^*v_n(G) = v_n(G')$ as elements of $H^0(S', \omega_{G'}^{\otimes p^n-1})$. This fact follows from the functoriality of the Frobenius morphism and of Lie. See [Goe08, Proposition 5.3].

Thus, for each n, we have inductively determined a section v_n of the universal sheaf $\omega^{\otimes p^n-1}$ over $\mathcal{M}_{\mathrm{fg}}$ defined on some suitable "sub-object" of $\mathcal{M}_{\mathrm{fg}}$ where $v_0, \ldots, v_{n-1}$ are defined and vanish. Our next goal will be to specify this "sub-object" precisely. To do this, we will need to recall some machinery from algebraic geometry.

Remark 3.2.17. Given an *fpqc*-stack $\mathcal{M}$, an invertible sheaf $\mathcal{L}$ over $\mathcal{M}$, and an injective section $\sigma : \mathcal{O}_{\mathcal{M}} \to \mathcal{L}$, the substack of $\mathcal{M}$ where σ vanishes is the closed substack defined by the ideal sheaf $\mathcal{I}(\sigma) \subset \mathcal{O}_{\mathcal{M}}$ which is the image of

$$\sigma \otimes \mathcal{L}^{-1} : \mathcal{L}^{-1} \to \mathcal{O}_{\mathcal{M}}\,.$$

By [Sta25, Tag 01WS], $\mathcal{I}(\sigma)$ defines an *effective Cartier divisor* D on $\mathcal{M}$ with $\mathcal{O}_D = \mathcal{O}_{\mathcal{M}} \,/\, \mathcal{I}(\sigma)$. For us, this means we have a way of turning a section of a line bundle over $\mathcal{M}$ into a closed substack D. In particular, we want to use the sections v_n to inductively define closed substacks of $\mathcal{M}_{\mathrm{fg}}$.

The section

$$v_0 : \mathcal{O}_{\mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})} \to \mathcal{O}_{\mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})}$$

(which is multiplication by p) vanishes exactly at the $\mathbb{F}_p$-points of $\mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$ (i.e., the points at which $p = 0$), and so we define $\mathcal{M}_{\mathrm{fg}}^{\geq 1}$ to be the closed substack of formal groups over $\mathbb{F}_p$-schemes, which by the discussion in Example 3.2.3, is exactly the moduli of formal groups of height at least 1. We extend this discussion to the following definition.

Definition 3.2.18 ([Goe08, Definition 5.5])**.** Let

$$\mathcal{M}_{\mathrm{fg}}^{\geq 0} = \mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)}).$$

Inductively, suppose $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ is defined and each section $v_0, \ldots, v_{n-1}$ vanishes on $\mathcal{M}_{\mathrm{fg}}^{\geq n}$. Then the section v_n of $\omega_G^{\otimes p^n-1}$ is defined on all of $\mathcal{M}_{\mathrm{fg}}^{\geq n}$. Following Remark 3.2.17, v_n determines a closed inclusion $\mathcal{M}_{\mathrm{fg}}^{\geq n+1} \subset \mathcal{M}_{\mathrm{fg}}^{\geq n}$. We call $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ the *moduli of formal groups of height at least* n. The resulting filtration of the (p-local) moduli of formal groups by closed substacks is the *height filtration* (or, sometimes, the *chromatic filtration*):

$$\ldots \subset \mathcal{M}_{\mathrm{fg}}^{\geq n+1} \subset \mathcal{M}_{\mathrm{fg}}^{\geq n} \subset \ldots \subset \mathcal{M}_{\mathrm{fg}}^{\geq 1} \subset \mathcal{M}_{\mathrm{fg}}^{\geq 0} = \mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)}).$$

For each n, let $\mathcal{I}_n \subset \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq 0}}$ be the ideal sheaf specifying the closed inclusion $\mathcal{M}_{\mathrm{fg}}^{\geq n} \subset \mathcal{M}_{\mathrm{fg}}^{\geq 0}$.

From the discussion preceding this definition, we deduce the following fact.

Proposition 3.2.19. *There are natural identifications*

$$\mathcal{M}_{fg}^{\geq 1} = \mathcal{M}_{fg}^{\geq 0} \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p) = \mathcal{M}_{fg} \times \mathrm{Spec}(\mathbb{F}_p).$$

Remark 3.2.20. It is worth taking a moment to justify our notation. Speficially, we should check that $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ does in fact classify formal groups of height at least n and their isomorphisms. To see this consider G a formal group over a $\mathbb{Z}_{(p)}$-scheme S. By definition, G is classified by a map $\theta : S \to \mathcal{M}_{\mathrm{fg}}^{\geq 0}$. If θ factors through the inclusion $\mathcal{M}_{\mathrm{fg}}^{\geq n} \subset \mathcal{M}_{\mathrm{fg}}^{\geq 0}$, then the pullback $\theta^*\, \mathcal{I}_n \subset \mathcal{O}_S$ is zero.

Working Zariski-locally of S, assume G is presented by a formal group law F over a $\mathbb{Z}_{(p)}$-algebra R. For each $k = 0, \dots, n-1$, pulling back along θ determines a section $\theta^* v_n$ of $\omega_G^{\otimes p^k - 1}$. The choice of coordinate for G induces an identification $\omega_G^{\otimes p^k - 1} \cong R$, and so we obtain $\theta^* v_k$ in R such that

$$\theta^* \mathcal{I}_n = (\theta^* v_0, \dots, \theta^* v_{n-1}).$$

Since $\mathcal{I}_n = 0$, this forces each $\theta^* v_k = 0$. Tracing through definitions, we find that $\theta^* v_k$ is exactly the coefficient of x^{p^k} in $[p]_F(x)$. This aligns with the discussion in Remark 3.2.6, and so F (and hence G) are of height at least n.

Remark 3.2.21. It is standard in chromatic homotopy theory to fix a prime p once and for all and from then on assume all objects are p-local. We will do the same: from now on, we will usually write $\mathcal{M}_{\text{fg}}$ and say "the moduli of formal groups" to mean what we have defined as $\mathcal{M}_{\text{fg}} \times \operatorname{Spec}(\mathbb{Z}_{(p)})$, the moduli of p-typical formal groups. This convention is always implicit when dealing with heights of formal groups since there is a different notion of height for every prime p. We hope it will be clear from context when we use $\mathcal{M}_{\text{fg}}$ to mean to moduli of formal groups at all primes.

Definition 3.2.22 ([Goe08, Definition 5.5])**.** For each finite n, let $\mathcal{M}_{\text{fg}}^{\leq n}$ be the (open) complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$ inside $\mathcal{M}_{\text{fg}}$. We call $\mathcal{M}_{\text{fg}}^{\leq n}$ the *moduli of formal groups of height at most* n.

Definition 3.2.23 ([Goe08, Definition 5.5])**.** For each finite n, let $\mathcal{M}_{\text{fg}}^{=n}$ be the (open) complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$ inside $\mathcal{M}_{\text{fg}}^{\geq n}$. Let $\mathcal{M}_{\text{fg}}^{=\infty}$ be the intersection $\bigcap_n \mathcal{M}_{\text{fg}}^{\geq n}$. We call $\mathcal{M}_{\text{fg}}^{=n}$ and $\mathcal{M}_{\text{fg}}^{=\infty}$ the *moduli of formal groups of height (exactly)* n and ∞, respectively.

Remark 3.2.24. Again, we should justify our notation and argue, for example, why the S-points of $\mathcal{M}_{\text{fg}}^{=n}$ are the formal groups of height exactly n over a base scheme S. By definition, (see [Goe08, Remark 1.3]) an S is a point of $\mathcal{M}_{\text{fg}}^{\geq n}$ (i.e., a formal group over S) is in the complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$ if the intersection

$$\mathcal{M}_{\text{fg}}^{\geq n+1} \times_{\mathcal{M}_{\text{fg}}^{\geq n}} S$$

is empty as a closed subscheme of S.

Working locally on S, it is sufficient to consider a formal group G over $\operatorname{Spec}(R)$ presented by a formal group law. Supposing G is of height at least n, we can pull $\mathcal{I}_{n+1}$ back along the classifying map $\theta : \operatorname{Spec}(R) \to \mathcal{M}_{\text{fg}}$ to obtain an ideal

$$I_{n+1}(G) = \theta^* \mathcal{I}_{n+1} \subset \mathcal{O}_{\operatorname{Spec}(R)} = R.$$

By definition, the intersection of $\operatorname{Spec} R$ with $\mathcal{M}_{\text{fg}}^{\geq n+1}$ in $\mathcal{M}_{\text{fg}}^{\geq n}$ is $\operatorname{Spec}(R/I_n(G))$, i.e., the following is a pullback square:

$$\begin{tikzcd}
\operatorname{Spec}(R/I_{n+1}(G)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec}(R) \arrow[d, "\theta"] \\
\mathcal{M}_{\text{fg}}^{\geq n+1} \arrow[r, hook] & \mathcal{M}_{\text{fg}}^{\geq n}
\end{tikzcd}$$

So if G is in the complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$, then $\operatorname{Spec}(R/I_{n+1}(G))$ is empty, which forces $I_{n+1}(G) = R$. But since G is of height at least n, each $\theta^* v_0, \dots, \theta^* v_{n-1}$ vanishes, so $I_{n+1}(G) = (\theta^* v_n)$. It follows that $\theta^* v_n$ is a unit in R, and so from Remark 3.2.6, we conclude that G is of height exactly n.

Note that while $\mathcal{M}_{\text{fg}}^{=n}$ is by definition the complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$ in $\mathcal{M}_{\text{fg}}^{\geq n}$, it is not the case that $\mathcal{M}_{\text{fg}}$ is a disjoint union of the various height-n pieces. This is due to the existence of formal groups which are not of any particular height, and is the reason for the inherent "stackiness" of $\mathcal{M}_{\text{fg}}$. We illustrate this with the following example which is a stacky version of Example 3.2.4.

Example 3.2.25. Let G be the formal group over $\mathbb{Z}_{(p)}$ presented by the additive formal group law, $F(x, y) = x + y$. As a p-local formal group, G is classified by a map $\operatorname{Spec}(R) \to \mathcal{M}_{\text{fg}}$. We have seen that G is not of height at least 1 since its multiplication by p does not factor through the Frobenius twist of G. To see that G is not of height exactly 0, we compute the intersection

$$\mathcal{M}_{\text{fg}}^{\geq 1} \times_{\mathcal{M}_{\text{fg}}} \operatorname{Spec}(\mathbb{Z}_{(p)}).$$

In Proposition 3.2.19, we computed,

$$\mathcal{M}_{\mathrm{fg}}^{\geq 1} = \mathcal{M}_{\mathrm{fg}} \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p),$$

and so the above intersection is naturally isomorphic to $\mathrm{Spec}(\mathbb{F}_p)$, which is non-empty. That is, G is a $\mathbb{Z}_{(p)}$-point of $\mathcal{M}_{\mathrm{fg}}$ which is not a $\mathbb{Z}_{(p)}$-point of either $\mathcal{M}_{\mathrm{fg}}^{\geq 1}$ or its complement, $\mathcal{M}_{\mathrm{fg}}^{=0}$.

Incidentally, it is interesting to note that the natural projection map

$$\mathrm{Spec}(\mathbb{F}_p) = \mathcal{M}_{\mathrm{fg}}^{\geq 1} \times_{\mathcal{M}_{\mathrm{fg}}^{\geq 0}} \mathrm{Spec}(\mathbb{Z}_{(p)}) \to \mathcal{M}_{\mathrm{fg}}^{\geq 1}$$

classifies the additive formal group over $\mathbb{F}_p$, which has infinite height. This means G is "totally contained" in the height 0 piece of $\mathcal{M}_{\mathrm{fg}}$, except for its one-point intersection with the infinite height piece.

The takeaway is that the geometry of $\mathcal{M}_{\mathrm{fg}}$ is complicated. This is to be expected: as we saw in Proposition 3.1.14, the cohomology of $\mathcal{M}_{\mathrm{fg}}$ approximates $\pi_* \mathbb{S}$. But as we will see in Section 3.4, the geometry of each piece $\mathcal{M}_{\mathrm{fg}}^{=n}$ is relatively simple, and in Section 4.1 we will discuss the *chromatic spectral sequence*, which is the machine for assembling cohomological information about the various $\mathcal{M}_{\mathrm{fg}}^{=n}$ into information about $\mathcal{M}_{\mathrm{fg}}$.

For now, we conclude this section by formalizing some constructions which will be useful to use later. Recall the local construction of generators for the pullback of $\mathcal{I}_n$ along the classifying map for a formal group presented by a formal group law of Remark 3.2.20. We record the universal case of this construction in the following definition.

Definition 3.2.26. Let $\theta : \mathrm{Spec}(BP_*) \to \mathcal{M}_{\mathrm{fg}}$ be the map classifying the formal group associated to the universal p-typical formal group law F over BP_*. Let $I_n \subset BP_*$ be the ideal

$$I_n = \theta^* \mathcal{I}_n \subset \mathcal{O}_{\mathrm{Spec}(BP_*)} = BP_*.$$

Let F_n be the reduction of F to BP_*/I_n. The formal group associated to F_n is of height at least n, and is classified by a map $\theta_n : BP_*/I_n \to \mathcal{M}_{\mathrm{fg}}^{\geq n}$. Let $v_n \in BP_*/I_n$ be defined by pulling back the global section v_n of $\omega^{\otimes p^n - 1}$ along θ_n.

For a p-typical formal group G over R presented by a formal group law which is classified by a map $\varphi : BP_* \to R$, we define $I_n(G) = \varphi(I_n)$. This is a reformulation of of Remark 3.2.6 in new language.

Remark 3.2.27. By an abuse of notation, we will sometimes write

$$I_n = (v_0, \ldots, v_{n-1}) \subset BP_*.$$

This is slightly suspicious because each v_k is only canonically defined (up to a unit) in BP_* after the ideal I_k is modded out.

Following [Goe08, Remark 5.7], in the presence of a p-typical coordinate on a formal group G over a $\mathbb{Z}_{(p)}$-algebra R, it is possible to choose elements of R which inductively generate the ideals $I_n(G)$. That is, we can choose elements $u_k \in R$ such that $I_n(G) = (u_0, \ldots, u_n)$ for each n. We stress, however, that while the ideals $I_n(G)$ are canonically defined, the elements u_n are not. This is the reason for the distinction between the Hazewinkel and Araki generators for BP_* as discussed in [Rav86, A.2].

With this construction in mind, we can tie the filtered structure of $\mathcal{M}_{\mathrm{fg}}$ back to chromatic homotopy theory with the following definition and lemma which give familiar presentations of these stacks.

Definition 3.2.28 ([Goe08, [Definition 6.5]). We say a Hopf algebroid (A, Γ) (Definition A.4.19) is of *Adams type* if the left unit $\eta_L : A \to \Gamma$ is flat and the (A, Γ)-comodule Γ is a filtered colimit of comodules Γ_i, each of which are finitely generated and projective as A-modules.

We say a *fpqc*-algebraic stack $\mathcal{M}$ (Definition A.5.6) is of *Adams type* if there is a presentation $\mathrm{Spec}(A) \to \mathcal{M}$ and some ring Γ so that

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(\Gamma),$$

and the resulting Hopf algebroid (A, Γ) is of Adams type.

Lemma 3.2.29 ([Goe08, Proposition 6.8]). *The map*

$$\mathrm{Spec}(BP_*) = \mathrm{Spec}(\mathbb{Z}_{(p)}[u_1, u_2, \ldots]) \to \mathcal{M}_{fg} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$$

classifying the universal p-typical formal group over BP_* presents $\mathcal{M}_{fg}\times\mathrm{Spec}(\mathbb{Z}_{(p)})$ as an Adams type algebraic stack. Pulling this map back along substack inclusions into $\mathcal{M}_{fg}\times\mathrm{Spec}(\mathbb{Z}_{(p)})$ gives the following presentations of Adams type algebraic stacks:

$$\mathrm{Spec}(BP_*/I_n) = \mathrm{Spec}(\mathbb{F}_p[u_n, u_{n+1}, \ldots]) \to \mathcal{M}_{fg}^{\geq n}$$

$$\mathrm{Spec}(v_n^{-1}BP_*/I_n) = \mathrm{Spec}(\mathbb{F}_p[u_n^{\pm}, u_{n+1}, \ldots]) \to \mathcal{M}_{fg}^{=n}$$

$$\mathrm{Spec}(E(n)) = \mathrm{Spec}(\mathbb{Z}_{(p)}[u_1, \ldots, u_{n-1}, u_n^{\pm}]) \to \mathcal{M}_{fg}^{\leq n}.$$

3.3. **The Honda formal group**

The chromatic filtration of $\mathcal{M}_{\mathrm{fg}}$ into closed substacks $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ (Definition 3.2.18) organizes the category of formal groups, and therefore MU-algebra spectra, into more manageable and richly structured pieces. A natural first step to studying this filtration is to study it's "factors," i.e., the substacks of formal groups of exact height: $\mathcal{M}_{\mathrm{fg}}^{=n} = \mathcal{M}_{\mathrm{fg}}^{\geq n} - \mathcal{M}_{\mathrm{fg}}^{\geq n+1}$.

It turns out that these substacks are relatively simple: they each have only one geometric point. This geometric point is called the *Honda formal group of height n, H_n*. In this section, we will give a characterization of this formal group.

The jumping-off point is the following foundational result from Lubin-Tate local class field theory which says that if R is the ring of integers of a local field, there is essentially only one formal group over R, up to canonical isomorphism. First, we briefly recall some necessary terminology.

By *local field*, we a mean field obtained as the metric completion of some finite extension of $\mathbb{Q}$ with respect to some valuation. The prototypical example is $\mathbb{Q}_p$, which is the completion of $\mathbb{Q}$ with respect to the p-adic valuation. A local field K necessarily has $\mathbb{Z}$ as a subring, and the elements of K which are roots of monic polynomials with $\mathbb{Z}$ coefficients form the *ring of integers* in K, denoted $\mathcal{O}_K$. This ring is always a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ generated by an element π called the *uniformizer*. The residue field $\mathcal{O}_K/\mathfrak{m}$ is always finite. In the case of the local field $\mathbb{Q}_p$, the ring of integers is $\mathbb{Z}_p$, the uniformizer is p, and the residue field is $\mathbb{F}_p$.

Theorem 3.3.1 (Lubin-Tate [Pst21, Theorem 14.13])**.** *Let K be a local field with uniformizer $\pi \in \mathcal{O}_K$ and residue field of order $q = p^n$. For any $f \in \mathcal{O}_K[\![t]\!]$ of the form*

$$f(t) = \pi t + \textit{higher order terms...}$$

such that $f(t) \equiv t^q \pmod{\pi}$, there is a unique formal group law $F \in \mathcal{O}_K[\![x, y]\!]$ with f as an endomorphism.

Moreover, for any other choice of f, say f' with associated formal group law F', there is a canonical isomorphism $F \cong F'$.

We won't attempt to prove this result here since setting up the necessary machinery would take us too far afield. The motivation for including it here is the rich connection to arithmetic geometry it indicates. It also puts us in the right setting for computing the Morava stabilizer group.

Remark 3.3.2. One feature of the proof which we will highlight is that, by construction, the multiplication by π endomorphism of the formal group law F associated to f is exactly $[\pi]_F(x) = f(x)$ [Pst21, Remark 14.15].

Another feature of the theorem is that for any $a \in \mathcal{O}_K$, there is a canonical endomorphism $\varphi_a(t)$ of F with leading term at which commutes with f.

The technical fact to prove here is the existence of a unique such $\varphi_a(t)$ commuting with f (this is [Pst21, Lemma 14.10]). Once we know $f(\varphi_a(t)) = \varphi_a(f(t))$ though, we can see that for $h(x, y)$ either $F(\varphi_a(x), \varphi_a(y))$ or $\varphi(F(x, y))$, both have the same linear term, and in both cases we have that $f(h(x, y)) = h(f(x, y))$, and so by the uniqueness property of Theorem 3.3.1, we must have

$$\varphi_a(F(x, y)) = F(\varphi_a(x), \varphi_a(y)).$$

Example 3.3.3 ([Pst21, Example 14.16])**.** Consider the case where $K = \mathbb{Q}_p[\zeta_{q-1}]$ where ζ_{q-1} is a $q-1$ root of unity. The ring of itegers in K is $\mathcal{O}_K = \mathbb{Z}_p[\zeta_{q-1}]$ which is exactly the ring of Witt vectors $W(\mathbb{F}_q)$. The uniformizer of $\mathcal{O}_K$ is p, and $\mathcal{O}_K/(p) = \mathbb{F}_q$.

Thus, there is a unique formal group law F over $\mathcal{O}_K$ with p-series $[p]_F(x) = px + x^q$. Modulo p, we obtain a formal group law F/p over $\mathbb{F}_q$ with p-series $[p]_{F/p}(x) = x^q$, and so obtain a formal group law of height exactly n over $\mathbb{F}_q$. Moreover, since the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ on $\mathcal{O}_K[\![t]\!]$ fixes $px + x^q$, the uniqueness claim of Theorem 3.3.1 guarantees that the Galois action on $\mathcal{O}_K[\![x, y]\!]$ also fixes F. Thus, F must have coefficients in $(\mathcal{O}_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} = \mathbb{Z}_p$. That is, F/p in fact has coefficients in $\mathbb{F}_p$.

Definition 3.3.4. The *Honda formal group law of height* n, written H_n, is the formal group law over $\mathbb{F}_p$ which is the mod-p reduction of the unique formal group law $\tilde{H}_n$ over $W(\mathbb{F}_q)$ with p-series $[p]_{\tilde{H}_n}(x) = px + x^q$.

The *Honda formal group of height* n, also written H_n, is the associated formal group.

Remark 3.3.5. By definition, H_n is a formal group law over $\mathbb{F}_p$. But by pushing H_n forward along the unit map of any $\mathbb{F}_p$ algebra R, we obtain a formal group law over R which we call *the Honda formal group law over* R and denote by H_n/R.

The following series of results makes our claim that there is locally only one formal group of height exactly n precise.

Theorem 3.3.6 ([Pst21, Proposition 15.11])**.** *Let R be an $\mathbb{F}_p$-algebra, and F a formal group law over R of height exactly n. Then there is a faithfully flat R-algebra R' over which F is (strictly) isomorphic to the Honda formal group law of height n.*

Proof. We will give the setup for the proof and refer the reader to [Pst21, Proposition 15.11] for the explicit arithmetic construction of R'.

Since F is a formal group law of height exactly n, by definition, there is some invertible $g(t) \in R[\![t]\!]$ such that the p-series of F is given by

$$[p]_f(x) = g(x^{p^n}).$$

By the uniqueness claim of Theorem 3.3.1, it is enough to show that F is isomorphic to a formal group law G with p-series

$$[p]_G(x) = x^{p^n}.$$

If $\varphi : F \to G$ is such an isomorphism, we have

$$\varphi(x)^{p^n} = \varphi(g(x^{p^n}))$$

as power series over R. Writing $\varphi(t) = \sum_{ib_it^{i+1}}$ and $g(t) = \sum_k a_k t^{k+1}$, this equation becomes

$$\Big(\sum_k b_k x^{k+1}\Big)^{-1} \circ \Big(\sum_i b_i x^{p^n(i+1)}\Big) = g(x^{p^n}).$$

We want to show that there is a sequence of coefficients $b_0, b_1, b_2, \dots$ in some faithfully flat extension R' of R making this equation hold.

This is done by inductively constructing finite étale extensions

$$R \hookrightarrow R_1 \hookrightarrow R_2 \hookrightarrow \dots$$

where the coefficients $b_0, \dots, b_k$ are determined in R_k. The proof concludes by taking the direct limit of the R_k and arguing that the resulting formal group law isomorphism is well-defined. ☐

For later, we will also need the following technical generalization of the above fact.

Corollary 3.3.7. *Let R be an $\mathbb{F}_p$-algebra. There is a faithfully-flat extension $R \to \tilde{R}$ over which all formal group laws over R are strictly isomorphic to H_n.*

Proof. By Theorem 3.3.6, for each $F \in \mathfrak{F}\mathrm{gl}(R)$ we obtain a faithfully flat extension $R \to R_F$ over which F is strictly isomorphic to H_n. Iterating this construction, we can produce a directed system

$$\{R_J : J \subset \mathfrak{F}\mathrm{gl}(R), J \text{ finite}\}$$

of faithfully flat extensions (with an arrow $R_J \to R_{J'}$ whenever $J \subset J'$) such that every formal group law over R is strictly isomorphic to H_n over $\tilde{R} = \varinjlim R_J$.

We will check that $\tilde{R}$ is faithfully flat as an R-algebra. Flatness follows from the fact that Tor commutes with filtered colimits: for any R-module M, we have

$$\mathrm{Tor}_i^R(M, \tilde{R}) \cong \varinjlim \mathrm{Tor}_i^R(M, R_J) \cong 0$$

for each $i > 0$ since $R \to R_J$ is flat. Faithful flatness then follows from the fact that $\mathrm{Spec}(\tilde{R}) \to \mathrm{Spec}(R)$ is surjective since each $\mathrm{Spec}(R_J) \to \mathrm{Spec}(R)$ is. ☐

A related statement is the following theorem, originally due to Lazard.

Theorem 3.3.8 (Lazard [Rav86, Theorem A2.2.11])**.** *Over a separably-closed field of positive characteristic, any formal group of height n is isomorphic to H_n.*

Remark 3.3.9. Note that this theorem of Lazard is enough to show that all formal groups over an $\mathbb{F}_p$-algebra R are isomorphic to the Honda formal group after flat base-change along the composition

$$R \to \mathsf{Frac}(R) \to \mathsf{Frac}(R)^{\mathrm{sep}}$$

(localizations and field extensions are flat). Corollary 3.3.7 is slightly stronger in that it produces an extension which is *faithfully* flat.

Finally, note the following Lemma.

Lemma 3.3.10 ([Pst21, Lemma 17.1])**.** *The map* $\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{fg}^{=n}$ *classifying the Honda formal group is a faithfully flat cover.*

Proof. We need to show that for any height-n formal group G over an $\mathbb{F}_p$-algebra R classified by a map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}^{=n}$, the natural map of schemes

$$\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(R)$$

is flat and surjective. These properties are local on $\mathrm{Spec}(R)$, and so we can assume G is presented by a formal group law F. This makes R into an algebra over the Lazard ring L, and so we obtain the map $R \cong R \otimes \mathbb{F}_p \to R \otimes_L \mathbb{F}_p$. The result of [Pst21, Corollary 15.3] shows that this map makes $R \otimes_L \mathbb{F}_p$ into a faithfully flat R-algebra, which is what we needed to show. □

Remark 3.3.11. There is actually something even more special about the cover

$$\mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}_{\mathrm{fg}}^{=n} :$$

it is an (affine) *Galois cover* with respect to the automorphism group $\mathbb{G}_n$ of $H_n/\mathbb{F}_q$. One very nice result of this fact is that, in a precise sense, $\mathcal{M}_{\mathrm{fg}}^{=n}$ is (related to) the quotient of $\mathrm{Spec}(\mathbb{F}_q)$ by $\mathbb{G}_n$. Some care is necessary since the Galois group $\mathrm{Gal}(\mathbb{F}_q \,/\, \mathbb{F}_p)$ acts non-trivially on $\mathrm{Spec}(\mathbb{F}_q)$; the result is that

$$\mathrm{Spec}(\mathbb{F}_q) /\!\!/ \mathbb{S}_n \simeq \mathrm{Spec}(\mathbb{F}_q) \times \mathcal{M}_{\mathrm{fg}}^{=n}.$$

This will turn out to be the content of the Miller-Ravenel change of rings theorem (Theorem 4.2.3).

The next section is dedicated to studying this automorphism group in detail.

3.4. The Morava stabilizer group

In the previous section, we saw that $\mathcal{M}_{\mathrm{fg}}^{=n}$ has only one geometric point, corresponding to the Honda formal group. So, to understand $\mathcal{M}_{\mathrm{fg}}^{=n}$, we consider the stalk at this point. This stalk, which comprises the automorphisms of the Honda formal group H_n defined over $\bar{\mathbb{F}}_p$, is the *Morava stabilizer group*.

Our goal for this section will be to compute the Morava stabilizer group as the group of units in the endomorphism ring $\mathrm{End}(H_n/\bar{\mathbb{F}}_p)$, for which we will follow [Rav86, Section A2.2] and [Pst21, Section 16]. A first observation that will somewhat simplify this computation is the following which lets us replace $\bar{\mathbb{F}}_p$ with $\mathbb{F}_q$ where $q = p^n$.

Lemma 3.4.1 ([Pst21, Lemma 15.10])**.** *Let R be an $\mathbb{F}_p$ algebra, and let $R' = \{r \in R : r^q = r\}$. Then the coefficients of H_n/R are all in R', as are the coefficients of any endomorphism of H_n/R.*

Proof. Note that $[p]_{H_n}(t) = t^q$ is an endomorphism of H_n/R, so $H_n(x^q, y^q) = H_n(x,y)^q$, and therefore writing $H_n(x,y)$ as the R-power series $\sum_{i,j} a_{i,j}x^iy^j$ gives

$$\sum_{i,j} a_{i,j}x^{iq}y^{jq} = \sum_{i,j} a_{i,j}^q x^{iq}y^{jq}.$$

Thus, each $a_{i,j} = a_{i,j}^q$.

Also, any endomorphism of a formal group law commutes with its p-series, so in the case of H_n/R, we have $\varphi(t^q) = \varphi(t)^q$ for any endomorphism $\varphi(t)$ over R. Writing $\varphi(t) = \sum_i a_it^i$, gives

$$\sum_i a_i t^{iq} = \sum_i a_i^q t^{iq}.$$ □

Note that if k is any field of characteristic p, then the set of elements x in k such that $x^q = x$ is exactly $k \cap \mathbb{F}_q$. This fact combined with the previous lemma implies that if k is an extension of $\mathbb{F}_q$ (in particular, if $k = \bar{\mathbb{F}}_p$), then $\operatorname{End}(H_n/k) = \operatorname{End}(H_n/\mathbb{F}_q)$.

Our next step is to identify this endomorphism ring as a set.

Lemma 3.4.2 ([Pst21, Propositions 16.2, 16.3]). *For any endomorphism $\varphi(t)$ of $H_n/\mathbb{F}_q$, there is a unique sequence $\{a_i\} \subset \mathbb{F}_q$ such that*

$$\varphi(x) = \sum_i{}^{H_n} a_i x^{p^i},$$

where the sum is taken with respect to the addition defined by H_n. Conversely, any such sum defines an endomorphism of $H_n/\mathbb{F}_q$.

Proof. First we show that for $a \in \mathbb{F}_q$ we have $[a]_{H_n}(t) = at$. Let $\tilde{H}_n$ be the unique life of H_n to the Witt ring $W(\mathbb{F}_p)$, and consider $\tilde{a}$, the Teichmüller representative of a in $W(\mathbb{F}_q)$. Since $\operatorname{char} W(\mathbb{F}_p) = 0$, there is a unique power series $[\tilde{a}]_{\tilde{H}_n}(t)$ of the form

$$[\tilde{a}]_{\tilde{H}_n}(t) = \tilde{a}t + \text{higher order terms} \ldots$$

which is an endomorphism of $\tilde{H}_n$ [Rav86, Proposition A2.1.20]. It is therefore enough to show that $\tilde{a}t$ is an endomorphism of $\tilde{H}_n$, and by Remark 3.3.2, to do this we can just show that $\tilde{a}t$ commutes with $[p]_{\tilde{H}_n}$. Indeed, since $\tilde{a}^q = \tilde{a}$,

$$[p]_{\tilde{H}_n}(\tilde{a}t) = p\tilde{a}t + (\tilde{a}t)^q = p\tilde{a}t + \tilde{a}t^q = \tilde{a}[p]_{\tilde{H}_n}(t).$$

Reducing mod p proves that $[a]_{H_n}(t) = at$.

Now, let $\varphi(t)$ be any endomorphism of H_n with leading term at for $a \in \mathbb{F}_q$. Then $\varphi(t) -_{H_n} [a]_{H_n}$ is an endomorphism of H_n with leading term 0. By Proposition 3.2.13, $\varphi(t) -_{H_n} [a]_{H_n}$ must factor through the relative Frobenius endomorphism, $S(x) = x^p$. Thus, there is a unique endomorphism $\varphi'(t)$ such that $\varphi(t) -_{H_n} [a]_{H_n}(t) = \varphi'(S(t))$. We have

$$\varphi(t) = [a]_{H_n} +_{H_n} \varphi'(S(x)),$$

and we can then give a similar decompotion for $\varphi'(t)$, and so we obtain the desired decomposition for $\varphi(t)$. □

This result leads to the following identification of the automorphism group of $H_n/\mathbb{F}_q$. In fact, the result holds for a general $\mathbb{F}_p$-algebra R and H_n/R.

Corollary 3.4.3 ([Pst21, Theorem 16.4]). *The $\mathbb{F}_p$ algebra classifying automorphisms of H_n, called the* Morava stabilizer algebra, *is*

$$A_n = \mathbb{F}_p[t_0, t_1, t_2, \ldots]/(t_0^{q-1} - 1, t_i^q - t_i : i > 0).$$

That is, at the level of sets, $\operatorname{Aut}(H_n/R) \cong \operatorname{Spec}(A_n)(R)$ *for any $\mathbb{F}_p$-algebra R.*

Of course, it is not enough to understand just the elements of $\operatorname{Aut}(H_n/R)$: we are really interested in the group structure of $\operatorname{Aut}(H_n/R)$ and more generally the algebra structure on $\operatorname{End}(H_n/R)$. To achieve this, we could determine the Hopf algebra structure on A_n by a direct computation. This method is discussed in [Rav86, Theorem 6.2.3].

A different method, which works in the setting where R is a perfect field is the following. For simplicity, we will just consider the case $R = \mathbb{F}_q$, although this method works in more generality. Recall the map

$$[-] : W(\mathbb{F}_q) \to \operatorname{End}(\tilde{H}_n/W(\mathbb{F}_q)) \to \operatorname{End}(H_n/\mathbb{F}_q)$$

discussed in the proof of Lemma 3.4.2 which sends each element a of $W(\mathbb{F}_q)$ to the unique multiplication by a endomorphism of $H_n/W(\mathbb{F}_p)$, and then reduces modulo p. One way to interpret Lemma 3.4.2 is that this map becomes a surjection after adjoining a free variable S representing the Frobenius endomorphism:

$$W(\mathbb{F}_q)\langle S\rangle \to \operatorname{End}(H_n/\mathbb{F}_q).$$

Here, S is not assumed to commute with the elements of $W(\mathbb{F}_p)$. Our next step is to identify the kernel of this map by determining relations between S and the Witt vectors.

First, note that the Frobenius automorphism of $\mathbb{F}_q$ lifts to an automorphism σ of $W(\mathbb{F}_q)$ which acts coordinatewise on Teichmüller representatives:

$$\sigma(a_0, a_1, \ldots) = (a_0^q, a_1^q, \ldots)$$

[Rav86, Lemma A2.2.15(d)]. Note that for any $\tilde{a} \in W(\mathbb{F}_q)$, the multiplication by $\tilde{a}$ endomorphism on $H_n/W(\mathbb{F}_q)$ is given by $[\tilde{a}](t) = \tilde{a}t$ since $W(\mathbb{F}_q)$ has characteristic 0. Letting $a = \tilde{a} \pmod{p}$, we obtain an endomorphism $[a](t)$ of $H_n/\mathbb{F}_q$ and compute

$$S([a](t)) = (at)^p = a^p t^p = [a^p](S(t)).$$

Thus, we have determined that $Sa - \sigma(a)S$ is in the kernel of $W(\mathbb{F}_q) \to \mathrm{End}(H_n/\mathbb{F}_q)$. That is, there is a surjective ring homomorphism

$$W(\mathbb{F}_q)\langle S\rangle/(Sa - \sigma(a)S) \to \mathrm{End}(H_n/\mathbb{F}_q).$$

In the case $\tilde{a} = 0 \pmod{p}$, the above relation becomes $Sp - p^p S = 0 \pmod{p}$. But we know that $[p]_{H_n} = x^q \pmod{p}$, and so we have one extra relation in $W(\mathbb{F}_q)$, namely $p = S^n$. Since there are no other possible relations (we have already considered the commutators $[S, a]$ for all $a \in W(\mathbb{F}_q)$, and the only other defining property of H_n is its p-series), we conclude that the map

$$W(\mathbb{F}_q)\langle S\rangle/(Sa - \sigma(a)S, S^n - p) \to \mathrm{End}(H_n/\mathbb{F}_q)$$

is an isomorphism of rings. We record what we have proved in the following theorem.

Theorem 3.4.4 (Lubin, Dieudonné [Rav86, Thoerem A2.2.17], [Pst21, Theorem 16.6])**.** *The endomorphism ring of the Honda formal group over $\mathbb{F}_q$ is*

$$\mathrm{End}(H_n/\mathbb{F}_q) \cong W(\mathbb{F}_q)\langle S\rangle/(Sa - \sigma(a)S, S^n - p).$$

Since our goal is to understand $\mathcal{M}_{\mathrm{fg}}^{=n}$, we are more interested in automorphisms of $H_n/\mathbb{F}_q$. These are exactly the units in the above endomorphism ring, which we can identify in the following way.

By Lemma 3.4.2, we can represent each element f of $\mathrm{End}(H_n/\mathbb{F}_q)$ as a power series

$$f = \sum_{j \geq 0} a_j S^j \tag{3.4.5}$$

where each a_j is an element of $\mathbb{F}_q$. Addition and multiplicaton in $\mathrm{End}(H_n/\mathbb{F}_q)$ are then addition and multiplication of power series over $W(\mathbb{F}_q)$, where the coefficients a_j are lifted to $W(\mathbb{F}_q)$ via $a_j \mapsto (a_j, 0, 0, \ldots)$. Using the relations $Sa - \sigma(a)S$ and $S^n - p$, it is always possible to write the sum or product of two such power series in the form above. The units in $\mathrm{End}(H_n/\mathbb{F}_q)$ exactly correspond to the multiplicatively-invertible such power series, that is, the power series of the above form with $a_0 \in \mathbb{F}_q^\times$. (Strict automorphisms will correspond to such power series with $a_0 = 1$.) We make this into the following definition.

Definition 3.4.6. The *Morava stabilizer group*, $\mathbb{S}_n$, is the group of units in $\mathrm{End}(H_n/\mathbb{F}_q)$. Concretely, it can be identified with the multiplicative group of power series over $W(\mathbb{F}_q)$ of the form of Equation (3.4.5). The *strict Morava stabilizer group*, S_n, is the subgroup of such power series with leading coefficient 1.

Remark 3.4.7. The strict Morava stabilizer group S_n is the kernel of the group homomorphism $\mathbb{S}_n \to \mathbb{F}_q^\times$ sending a power series f of the form of Equation (3.4.5) to its leading coefficient, a_0.

Remark 3.4.8. The profinite topology on the ring of Witt vectors $W(\mathbb{F}_q)$ is determined by its identification with the ring of p-adic integers with an $(n-1)$-root of unity:

$$W(\mathbb{F}_q) \cong \mathbb{Z}_p[\zeta_{n-1}] = (\varprojlim_k \mathbb{Z}/p^k)[\zeta_{n-1}].$$

Together with the relation $S^n - p$, this defines a (linear) topology on $\mathrm{End}(H_n/\mathbb{F}_q)$ where a basis of neighborhoods at 0 is given by

$$\left\{ \sum_{i>k} a_i p^i = \sum_{i>k} a_i S^{n^i} : a_j \in \mathbb{F}_q \right\}_{k \geq 0}.$$

Translating to a basis at 1, we obtain the profinite topology on $\mathbb{S}_n$.

Remark 3.4.9. In the literature, there is a small zoo of objects with the name "Morava stabilizer —," each closely related to the rest. To avoid confusion, we organize them in the following table, along with their various characterizations and relationships. Here, R is an $\mathbb{F}_p$-algebra, H_n/R denotes the height-n Honda formal group defined over R, and $q = p^n$.

Object	Characterization	Computation
S_n	*"Strict" Morava Stabilizer Group* Profinite group scheme of *strict* auts of H_n $S_n(R) = \mathrm{Aut}^s_{\mathbb{F}_p}(H_n/R)$	Group of units in $\mathrm{End}_{\mathbb{F}_p}(H_n/R)/(S-1)$
$\mathbb{S}_n$	*Morava Stabilizer Group* Profinite group scheme of *all* auts of H_n $\mathbb{S}_n(R) = \mathrm{Aut}_{\mathbb{F}_p}(H_n/R)$	Group of units in $\mathrm{End}_{\mathbb{F}_p}(H_n/R)$ $0 \to S_n \to \mathbb{S}_n \to \mathbb{F}_q^\times \to 0$
$\mathbb{G}_n$	*"Big" Morava Stabilizer Group* Automorphisms of the pair $(H_n, \mathbb{F}_q)$	$\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_q\,/\,\mathbb{F}_p)$
$\Sigma(n)$	*Morava Stabilizer Algebra* $\mathbb{Z}$-graded Hopf algebra $K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$	$K(n)_*[t_1, t_2, \ldots]/(v_n t_i^q - v_n^{p^i} t_i : i > 0)$
$S(n)$	*(Strict) Morava Stabilizer Algebra* $\mathbb{Z}/2(p^n-1)$-graded (or ungraded) Hopf algebra $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$ $S(n)^* \otimes_{\mathbb{F}_p} \mathbb{F}_q = \mathbb{F}_q[S_n]$ $\mathrm{Spec}(S(n)) \cong S_n$	$\mathbb{F}_p[t_1, t_2, \ldots]/(t_i^q - t_i : i > 0)$
A_n	*(General) Morava Stabilizer Algebra* $\mathbb{Z}/2(p^n-1)$-graded (or ungraded) Hopf algebra $\mathrm{Spec}(A_n) \cong \mathbb{S}_n$	$\mathbb{F}_p[t_0, t_1, t_2, \ldots]/(t_0^{q-1} - 1, t_i^q - t_i : i > 0)$
$\mathrm{End}_{\mathbb{F}_p}(H_n/R)$	*Endomorphism algebra of H_n/R* Non-commutative $W(\mathbb{F}_q)$-algebra	$W(\mathbb{F}_q)\langle S\rangle/(S\zeta - \zeta^p S, S^n - p)$ where ζ primitive $(q-1)$ root of 1, and $W(\mathbb{F}_q) \cong \mathbb{Z}_p[\zeta]$

The BP_*BP-comodule algebra $\Sigma(n)$ in the above table is the version of the Morava stabilizer algebra discussed by Ravenel in [Rav86, Chapter 6] and earlier in [Rav76] and [Rav77]. In this context, it is apparent that $\Sigma(n)$ should have a strong relationship with BP_*BP. In fact, the two have isomorphic cohomologies; this is the content of the Miller-Ravenel change of rings theorem (Theorem 4.0.1).

On the other hand, base changing $\Sigma(n)$ from $K(n)_*$ to $\mathbb{F}_p$ along the $\mathbb{F}_p$-algebra map sending v_n to 1 ("flattening the v_n-periodicity"), we obtain a Hopf algebra $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$ which corepresents the strict Morava stabilizer group: $\mathrm{Spec}(S(n)) = S_n$. This fact and the following, which bring the Morava stabilizer group closer to the structure of BP will be important for our proof of the Change of Rings Theorem.

Lemma 3.4.10 ([Rav86, Proposition 6.2.1]). *The functor $- \otimes_{K(n)_*} \mathbb{F}_p$ induces an equivalence of abelian categories*

$$\mathsf{Comod}(\Sigma(n)) \to \mathsf{Comod}(S(n)).$$

In particular,

$$\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p \cong \mathrm{Ext}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p).$$

We give a proof of a more general version of this fact in Lemma 4.3.4.

4. The Miller-Ravenel change of rings theorem

In this section will study the following theorem relating the cohomology of a certain BP_*BP-comodule and the Morava stabilizer algebra $\Sigma(n)$, originally proved in [MR77]:

Theorem 4.0.1 (Miller-Ravenel Change of Rings [Rav86, Theorem 6.1.1])**.** *Let M be a BP_*BP comodule annihilated by $I_n = (p, v_1, \ldots, v_{n-1})$. There is a natural isomorphism*

$$\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}M) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*, M \otimes_{BP_*} K(n)_*).$$

As it appears in [Rav86], this theorem is proven using a formal "change of rings" theorem for general Hopf algebroids. The main effort of this proof method is in verifying the technical hypotheses of the general theorem, in particular, producing an isomorphism

$$K(n)_* \otimes_{BP_*} BP_*BP \to \Sigma(n) \otimes_{K(n)_*} v_n^{-1}BP_*/I_n.$$

This isomorphism is explicitly constructed, and a corollary of the construction is a presentation of $\Sigma(n)$ as a $K(n)_*$-Hopf algebra in terms of generators and relations [Rav86, Corollary 6.1.16] (this presentation appears in the table in Remark 3.4.9).

This computation of $\Sigma(n)$ is conceptually different from the one given in Section 3.4: it is done entirely in the language of BP_*BP-comodules, whereas our computation was done by identifying the representing Hopf algebra for the automorphism group of the Honda formal group.

One of our motivations in this section will be to unify these two methods of computation by giving an alternate, conceptual proof of Theorem 4.0.1 in terms of the moduli of formal groups. In the course of this proof, it will become clear how the Morava stabilizer algebra alternately appears as the $K(n)_*$ Hopf algebra $K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$ and as the representing algebra of $\mathrm{Aut}(H_n/\mathbb{F}_q)$.

We begin with a section outlining the original importance of the change of rings theorem, and its relation to the Adams-Novikov spectral sequence.

4.1. The chromatic spectral sequence

The *chromatic spectral sequence* is a spectral sequence converging to the E_2-page of the Adams-Novikov spectral sequence, $\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*)$, arising from a filtration of this E_2 page into "v_n-torsion" pieces called the *chromatic filtration*. In this section, we will briefly outline the construction of this spectral sequence following [Rav86, Chapter 5.1], discuss how it motivates the change of rings theorem, and indicate the connection to the chromatic filtration of $\mathcal{M}_{\mathrm{fg}}$ from Definition 3.2.18.

Remark 4.1.1. As a historical note, the chromatic spectral sequence, introduced in [MRW77], is the original source of the term "chromatic" in the literature. The motivation for the language is that the chromatic spectral sequence, on its E_1-page, organizes the Adams-Novikov E_2-page into columns with distinct periodicities. The idea is that this is analogous to passing white light through a prism, breaking it into monochromatic pieces. The chromatic spectral sequence, along with Quillen's theorem (Theorem 3.1.8), form the original basis of the field of chromatic homotopy theory.

Remark 4.1.2. Following the notation of [Rav86, Chapter 5.1], we will write $\mathrm{Ext}^*(M)$ to mean

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, M)$$

for any BP_*BP-comodule M. In other places in the literature, this may also be written as the cohomology $H^*(M)$.

First, let x be an element of $\mathrm{Ext}^*(BP_*)$ and consider the multiplication by p map $BP_* \to BP_*$ which is an element of $\mathrm{Ext}^0(BP_*)$. If x is annihilated by some power of p, say p^i, we say x is *p-torsion* and obtain an element x' of $\mathrm{Ext}^{*-1}(BP_*/p^i)$ via the long exact sequence in Ext^*:

$$\cdots \to \mathrm{Ext}^{*-1}(BP_*/p^i) \to \mathrm{Ext}^*(BP_*) \xrightarrow{p^i} \mathrm{Ext}^*(BP_*) \to \mathrm{Ext}^*(BP_*/p^i) \to \cdots.$$

Now, $\mathrm{Ext}^0(BP_*/p^i)$ has an element corresponding to multiplication by v_1 on BP_*/p^i. If x' is v_1-*torsion*, that is, x is annihilated by some power of v_1, say v_1^j, we can similarly obtain an element x'' in $\mathrm{Ext}^{*-2}(BP_*/(p^i, v_1^j))$. We can continue this process, each time asking whether we obtain an element which is v_n-*torsion*, until possibly we obtain an element which not annihilated by any power of v_n; we will call such elements v_n-*periodic*.

The *chromatic filtration* of $\mathrm{Ext}^*(BP_*)$ is the filtration into v_n-torsion pieces for the various n. We make this precise in the following way.

First, we want to consider elements of $\mathrm{Ext}^*(BP)$ which are annihilated by some p^i for all i simultaneously. To achieve this, we consider the following diagram of BP_*BP-comodules:

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r, "p^i"] \arrow[d, no head, "="'] & BP_* \arrow[r] \arrow[d, "p"'] & BP_*/p^i \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & BP_* \arrow[r, "p^{i+1}"] & BP_* \arrow[r] & BP_*/p^{i+1} \arrow[r] & 0
\end{tikzcd}$$

where the dashed arrow is the unique map making the diagram commute. We can iterate this diagram vertically for each i. Taking the colimit in each column, we obtain a short exact sequence

$$0 \to BP_* \to \mathbb{Q} \otimes BP_* \to BP_*/p^\infty \to 0$$

where BP_*/p^∞ is defined to be $\varinjlim BP_*/p^i$. We can think of BP_*/p^∞ to be the "p-torsion" piece of BP_*. An equivalent notation is $\mathbb{Q}/\mathbb{Z}_{(p)} \otimes BP_*$.

The multiplication by v_1 maps on each BP_*/p^i induce a multiplication by v_1 map on BP_*/p^∞, and so we can form a similar diagram to obtain a BP_*BP-comodule $BP_*/(p^\infty, v_1^\infty)$. Inductively, we get diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] & BP_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[r, "v_n^i"] \arrow[d, no head, "="'] & BP_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[r] \arrow[d, "v_n"'] & \big(BP_*/(p^\infty, \dots, v_{n-1}^\infty)\big)/v_n^i \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & BP_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[r, "v_n^{i+1}"] & BP_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[r] & \big(BP_*/(p^\infty, \dots, v_{n-1}^\infty)\big)/v_n^{i+1} \arrow[r] & 0,
\end{tikzcd}$$

and after taking the colimit in i, we obtain a short exact sequence:

$$0 \to BP_*/(p^\infty, \dots, v_{n-1}^\infty) \to v_n^{-1}BP_*/(p^\infty, \dots, v_{n-1}^\infty) \to BP_*/(p^\infty, \dots, v_{n-1}^\infty, v_n^\infty) \to 0 \tag{4.1.3}$$

We name the three BP_*BP-comodules in (4.1.3) N^n, M^n, and N^{n+1}, respectively. Roughly, elements of $\mathrm{Ext}^*(N^n)$ correspond to elements of $\mathrm{Ext}^*(BP_*)$ which are v_k-torsion for each $k = 0, \dots, n-1$, and elements of $\mathrm{Ext}^*(M^n)$ correspond to v_n-periodic or "monochromatic" elements of $\mathrm{Ext}^*(BP_*)$. Of course, one needs to check that these constructions do in fact yield short exact sequences of BP_*BP-comodules. This is shown in [Rav86, Lemma 5.1.6].

Remark 4.1.4. It is important that the comodules N^n are defined inductively. The notation above makes it tempting to give N^n a closed-form definition as

$$\text{“}N^n = BP_*/(p^\infty, \dots, v_{n-1}^\infty).\text{”}$$

This is morally correct, but we need to be careful because in the above colimit construction, modding out v_{n-1}-torsion is only well-defined after v_k-torsion has been killed for each $k < n-1$. That is, the notation $(p^\infty, \dots, v_{n-1}^\infty)$ doesn't represent an ideal of BP_* in the usual sense, but some related construction. In particular, the order of the elements $p, \dots, v_{n-1}$ matters.

Going forward we will remember this subtlety, but continue to use the pseudo-notation above—after all, the notation v_n^∞ is custom, too.

Note however, that we do always have a well-defined equality

$$M^n = v_n^{-1}BP_* \otimes_{BP_*} N^n.$$

Now that we have introduced some notation, let us be very concrete about the "chromatic filtration" we have defined on $\mathrm{Ext}^*(BP_*)$. Each short exact sequence as in (4.1.3) induces a long exact sequence in Ext. We can string together the connecting homomorphisms from each of these long exact sequences to get the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\delta_2"'] & & \vdots \arrow[d, "\delta_3"'] & & & \\
\cdots \arrow[r, "j_1"] & \mathrm{Ext}^{*-3}(N^2) \arrow[r, "i_2"] \arrow[d, "\delta_1"'] & \mathrm{Ext}^{*-3}(M^2) \arrow[r, "j_2"] & \mathrm{Ext}^{*-3}(N^3) \arrow[d, "\delta_2"'] & & \vdots \arrow[d, "\delta_3"'] & \\
\cdots \arrow[r, "j_0"] & \mathrm{Ext}^{*-2}(N^1) \arrow[r, "i_1"] \arrow[d, "\delta_0"'] & \mathrm{Ext}^{*-2}(M^1) \arrow[r, "j_1"] & \mathrm{Ext}^{*-2}(N^2) \arrow[r, "i_2"] \arrow[d, "\delta_1"'] & \mathrm{Ext}^{*-2}(M^2) \arrow[r, "j_2"] & \mathrm{Ext}^{*-2}(N^3) \arrow[r, "i_3"] \arrow[d, "\delta_2"'] & \cdots \\
 & \mathrm{Ext}^{*-1}(N^0) \arrow[r, "i_0"] & \mathrm{Ext}^{*-1}(M^0) \arrow[r, "j_0"] & \mathrm{Ext}^{*-1}(N^1) \arrow[r, "i_1"] \arrow[d, "\delta_0"'] & \mathrm{Ext}^{*-1}(M^1) \arrow[r, "j_1"] & \mathrm{Ext}^{*-1}(N^2) \arrow[r, "i_2"] \arrow[d, "\delta_1"'] & \cdots \\
 & & & \mathrm{Ext}^{*}(N^0) \arrow[r, "i_0"] & \mathrm{Ext}^{*}(M^0) \arrow[r, "j_0"] & \mathrm{Ext}^{*}(N^1) \arrow[r, "i_1"] \arrow[d, "\delta_0"'] & \cdots \\
 & & & & & \mathrm{Ext}^{*-1}(N^0) \arrow[r, "i_0"] & \cdots
\end{tikzcd}$$

Above, the different color arrows indicate the various long exact sequences in Ext. Recall that $\mathrm{Ext}^*(N^0) = \mathrm{Ext}^*(BP_*)$ is the Adams-Novikov E_2-page.

If an element x in $\mathrm{Ext}^*(BP_*)$ is p-torsion (in the kernel of i_0), it is in the image of δ_0 and so lifts to some element x' in $\mathrm{Ext}^{*-1}(N^1)$ which is determined "up p-periodicity", i.e., up to an element in the image of j_0. But whether x' is v_1-torsion (in the kernel of i_1) is independent of p-periodicity since $i_1 \circ j_0 = 0$. And so if x' is v_1-torsion, it lifts to an element x'' in $\mathrm{Ext}^{*-2}(N^2)$, which may or may not be v_2-periodic, and so on. Thus, we have a well-defined way of determining whether each x is v_n-torsion for each n. This defines the chromatic filtration on $\mathrm{Ext}^*(N^0)$.

Remark 4.1.5 (The Greek-letter construction)**.** For each k, the element v_n^k in $\mathrm{Ext}^0(N^n)$ determines an element

$$\alpha_k^{(n)} = \delta_0\delta_1 \dots \delta_{n-1}(v_n^k)$$

in $\mathrm{Ext}^n(N^0)$. These abstractly-defined classes, called the *Greek letter elements* of the Adams-Novikov E_2-page, are inherently interesting to attempt to identify concretely. They were understood for $n = 0$ and 1 in [MRW77], and to date are not well-understood for larger n.

Splicing together short the exact sequences of (4.1.3), we obtain a long exact sequence

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to \cdots .$$

The *chromatic spectral sequence* is the *resolution spectral sequence* [Rav86, Theorem A1.3.2] associated to this long exact sequence.

Theorem 4.1.6 (Chromatic Spectral Sequence [Rav86, Proposition 5.1.8])**.** *There is a spectral sequence converging to the Adams-Novikov E_2-page,* $\mathrm{Ext}^*(BP_*)$*, with*

$$E_1^{n,s} = \mathrm{Ext}^s(M^n).$$

That is, the nth column of the E_1-page of the chromatic spectral sequence consists of the v_n-periodic elements of $\mathrm{Ext}^*(BP_*)$, and the chromatic spectral sequence "assembles" this periodicity information into the Adams-Novikov E_2-page.

Remark 4.1.7. We can obtain smaller BP_*BP-comodules from M^n and N^n by killing the v_0 through v_m torsion parts, that is, killing the ideals I_m. Specifically, we let

$$N_m^n = N^n/I_m = BP_*/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{m+n-1}^\infty)$$

and $M^n_m = M^n/I_m = v_{n+m}^{-1}BP_*\otimes_{BP_*} N^n_m$. (Again, take care to observe the subtlety of this notation discussed in Remark 4.1.4.) The later collection of comodules fit into short exact sequences of the following form:

$$0 \to M^{n-1}_{m+1} \to M^n_m \xrightarrow{v_m} M^n_m \to 0,$$

and there are Bockstein spectral sequences converging to $\mathrm{Ext}^*(M^n_m)$ with E_1-pages related to $\mathrm{Ext}^*(M^{n-1}_{m+1})$ [Rav86, Lemma 5.1.16].

The advantage of this perspective is the following nice conceptual picture of the homotopy groups of $\mathbb{S}$. In principle, this sequence of Bockstein spectral sequences lets us obtain information about $\mathrm{Ext}^*(M^n) = \mathrm{Ext}^*(M^n_0)$, the nth column of the E_1-page of the chromatic spectral sequence, from information about $\mathrm{Ext}^*(M^0_n) = \mathrm{Ext}^*(v_n^{-1}BP_*/I_n)$. The content of the Miller-Ravenel change of rings theorem is that this latter Ext-term is relatively easy to compute: it is the cohomology of the Morava stabilizer group, $\mathbb{S}_n$. In practice, however, the process of actually obtaining information about the Adams-Novikov E_2-page from the cohomology of $\mathbb{S}_n$ is extremely difficult.

Remark 4.1.8. In Proposition 3.1.14, we identified the Adams-Novikov E_2-page with the cohomology of $\mathcal{M}_{\mathrm{fg}}$ with respect to the powers of a certain line bundle. With this in mind, we can view the chromatic spectral sequence as a tool for approximating $H^*(\mathcal{M}_{\mathrm{fg}})$. Indeed, there is a whole story which can be told here of approximating this cohomology in terms of the cohomology of the various substacks of $\mathcal{M}_{\mathrm{fg}}$ from Definition 3.2.18. We will not go into detail about this story, but refer the reader to [Pst21, Section 23].

The essential takeaway is that the nth column of the E_1-page of the chromatic spectral sequence becomes $H^*(\mathcal{M}_{\mathrm{fg}}^{\geq n})$, and that there is a series of Bockstein spectral sequences for approximating this column coming from the sequence of substack inclusions

$$\mathcal{M}_{\mathrm{fg}}^{=n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\geq n-1,\leq n} \hookrightarrow \dots \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\geq 1,\leq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\leq n},$$

where $\mathcal{M}_{\mathrm{fg}}^{\geq k,\leq n}$ is the intersection

$$\mathcal{M}_{\mathrm{fg}}^{\geq k,\leq n} = \mathcal{M}_{\mathrm{fg}}^{\geq k} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{M}_{\mathrm{fg}}^{\leq n}.$$

The result is that, at the most basic level, information about the homotopy groups of $\mathbb{S}$ can be obtained by understanding the cohomology of each $\mathcal{M}_{\mathrm{fg}}^{=n}$.

Remark 4.1.9. In Section 3.2, we constructed the elements v_n as cohomology classes in $H^0(\mathcal{M}_{\mathrm{fg}}^{\geq n}, \omega^{\otimes p^n-1})$ for each n. Under the identification of Proposition 3.1.14, we have

$$H^0(\mathcal{M}_{\mathrm{fg}}^{\geq n}, \omega^{\otimes p^n-1}) \cong \mathrm{Ext}^0((M^n)_{2(p^n-1)})$$

where the right hand Ext^0 term expands to

$$\mathrm{Hom}_{BP_*BP}(BP_*, v_n^{-1}BP_{*+2(p^n-1)}/(p^\infty, \dots, v_{n-1}^\infty)).$$

Under this identification, the section $v_n : \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \to \omega^{\otimes p^n-1}$ corresponds to the multiplication by v_n map on BP_*, where it is canonically defined modulo $(p, \dots, v_{n-1})$-torsion. This is to say that our constructions of the v_n elements in the context of $\mathcal{M}_{\mathrm{fg}}$ and in the context of the Adams-Novikov E_2-page are consistent.

4.2. The change of rings theorem and $\mathcal{M}_{\mathrm{fg}}$

In this section, we will prove the following stacky variant of Theorem 4.0.1. We will see in Section 4.3 how the latter can be obtained as a corollary of this stronger statement.

We first start with identifying the stacks associated to the relevant Hopf algebroids.

Lemma 4.2.1. *The stack associated to the Hopf algebroid $(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n)$ is $\mathcal{M}_{fg}^{=n,s}$, the strict moduli of height-n, p-typical formal groups.*

Proof. Following Definition 3.2.23, $v_n^{-1}BP_*/I_n$ corepresents the functor assigning each $\mathbb{Z}_{(p)}$-algebra R to the set of height-n formal groups over R and $v_n^{-1}BP_*BP/I_n$ represents their strict isomorphisms. An argument similar to that in Theorem 3.1.4 identifies the map

$$\mathrm{Spec}(v_n^{-1}BP_*/I_n) \to \mathcal{M}_{\mathrm{fg}}^{=n}$$

classifying the universal height-n formal group as an *fpqc*-presentation of the moduli of height-n formal groups. $\square$

Lemma 4.2.2. *The stack associated to the Hopf algebroid $(\mathbb{F}_p, S(n))$ is BS_n, the classifying stack of the strict Morava stabilizer group.*

Proof. Note that $(\mathbb{F}_p, S(n))$ is actually a Hopf *algebra* corepresenting the group scheme S_n of strict automorphisms of H_n. Following Example A.5.5, the associated stack is

$$\mathrm{Spec}(\mathbb{F}_p) /\!\!/ \mathrm{Spec}(S(n)) \cong * /\!\!/ S_n \cong BS_n,$$

the classifying stack of S_n in the category *fpqc*-sheaves over $\mathbb{F}_p$. □

Note that base changing along the map $K(n)_* \to \mathbb{F}_p$ sending v_n to 1 sends the classifying stack of $\Sigma(n)$ in $\mathsf{Sch}_{/K(n)_*}$ to BS_n in $\mathsf{Sch}_{/\mathbb{F}_p}$. That is,

$$BS_n = \big(\mathrm{Spec}(K(n)_*) /\!\!/ \mathrm{Spec}(\Sigma(n))\big) \times_{\mathrm{Spec}(K(n)_*)} \mathrm{Spec}(\mathbb{F}_p).$$

(We will prove this fact later as Lemma 4.3.4.)

We are now ready to state and prove the main theorem of this section.

Theorem 4.2.3 (Miller-Ravenel Change of Rings)**.** *Let $n \geq 1$, let $\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{fg}^{=n}$ be the map classifying the Honda formal group of height n, and let $\mathcal{M}_{fg}^{=n,s} \to \mathcal{M}_{fg}^{=n}$ be the map exhibiting $\mathcal{M}_{fg}^{=n,s}$ as a $\mathbb{G}_m$-torsor over $\mathcal{M}_{fg}^{=n}$. There is an equivalence of* fpqc*-stacks*

$$\theta : BS_n \to \mathcal{M}_{fg}^{=n,s} \times_{\mathcal{M}_{fg}^{=n}} \mathrm{Spec}(\mathbb{F}_p)$$

inducing an equivalence of symmetric monoidal abelian categories of quasi-coherent sheaves over these stacks.

Proof. We first define a map $BS_n \to \mathcal{M}_{\mathrm{fg}}^{=n,s}$ as follows. For each $\mathbb{F}_p$-algebra R, let $BS_n(R) \to \mathcal{M}_{\mathrm{fg}}^{=n,s}(R)$ be the fully-faithful embedding which sends the lone object $*$ of $BS_n(R)$ to the Honda formal group H_n in $\mathcal{M}_{\mathrm{fg}}^{=n,s}(R)$ with its strict automorphisms. This assignment is natural in R, and so defines a map of stacks.

We obtain a natural map $\theta : BS_n \to \mathcal{M}_{\mathrm{fg}}^{=n,s} \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p)$ as in the following diagram.

$$\begin{tikzcd}
BS_n \arrow[dr, dashed, "\theta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{M}_{\mathrm{fg}}^{=n,s} \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{fg}}^{=n,s} \arrow[d, "/\,\mathbb{G}_m"] \\
& \mathrm{Spec}(\mathbb{F}_p) \arrow[r, "H_n"] & \mathcal{M}_{\mathrm{fg}}^{=n}
\end{tikzcd}$$

For convenience, let $\mathcal{N}$ denote $\mathcal{M}_{\mathrm{fg}}^{=n,s} \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p)$. For an $\mathbb{F}_p$-algebra R, the R-points of $\mathcal{N}$ form the groupoid where the objects are formal groups isomorphic to H_n over R, and the morphisms are *strict* isomorphisms of formal groups over R. The functor θ_R is fully faithful and sends $*$ to the Honda formal group defined over R. We will prove that θ is an equivalence of stacks in the sense of Definition A.3.11. (Note that since we are dealing with stacks valued in An_1, we only need to check that $\pi_0\theta$ and $\pi_1\theta$ are isomorphisms.)

We first handle the case of $\pi_0\theta$. Since $BS_n(R)$ has just one object, $\pi_0 BS_n(R)$ is the one element set consisting of (the path component of) $*$. On the other hand, $\pi_0\,\mathcal{N}(R)$ consists of the strict isomorphism classes of formal groups isomorphic to H_n over R. Note that $\pi_0\theta(R)$ is injective for every R since its domain has one element. That $\pi_0\theta$ is locally surjective follows from the fact that every formal group in $\mathcal{N}(R)$ is presented by a formal group law and from Corollary 3.3.7: there is a faithfully flat extension $R \to \tilde{R}$ over which every formal group law over R is strictly isomorphic to H_n.

Now consider $\pi_1\theta$. Following Example A.5.5, we have $\pi_1 BS_n(R, *) = S_n(R)$ for every $\mathbb{F}_p$-algebra R. On the other hand,

$$(\theta^*(\pi_1\,\mathcal{N}))(R, *) = \pi_1(\mathcal{N}(R), \theta_R(*)) = \pi_1(\mathcal{N}(R), H_n)$$

is the group of strict isomorphism loops in $\mathcal{N}(R)$ based at H_n, up to equivalence. This is exactly the strict automorphism group of H_n over R, which is $S_n(R)$ by definition, and so $\pi_1\theta_R$ is an isomorphism for all R because θ_R is fully faithful. □

4.3. Recovering 4.0.1 from 4.2.3

First, we can make the identification

Lemma 4.3.1 ([MR77, Proposition 1.3])**.**

$$\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}_{v_n^{-1}BP_*BP/I_n}(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*/I_n).$$

Next, we show that a map inducing a "Morita equivalence" of stacks (as in Theorem 4.2.3) specializes to induce an isomorphism of cohomologies.

Lemma 4.3.2. *Suppose a map of stacks $f : \mathcal{M}_1 \to \mathcal{M}_2$ induces an equivalence of symmetric monoidal abelian categories $f^* : \mathsf{QCoh}(\mathcal{M}_2) \simeq \mathsf{QCoh}(\mathcal{M}_1)$. Then for any $\mathcal{F} \in \mathsf{QCoh}(\mathcal{M}_2)$,*

$$H^*(\mathcal{M}_1, f^*\mathcal{F}) \cong H^*(\mathcal{M}_2, \mathcal{F}).$$

Proof. Since f is a monoidal equivalence, it sends the monoidal unit of $\mathsf{QCoh}(\mathcal{M}_2)$ to the monoidal unit of $\mathsf{QCoh}(\mathcal{M}_1)$, i.e. $f^*\mathcal{O}_{\mathcal{M}_2} \cong \mathcal{O}_{\mathcal{M}_1}$. On the other hand, since f is fully faithful, there is a bi-natural isomorphism

$$\mathrm{Hom}_{\mathcal{M}_2}(-,-) \cong \mathrm{Hom}_{\mathcal{M}_1}(f^*-, f^*-).$$

Thus we have a natural isomorphism of functors from $\mathsf{QCoh}(\mathcal{M}_2)$ to Ab:

$$\mathrm{Hom}_{\mathcal{M}_2}(\mathcal{O}_{\mathcal{M}_2}, -) \cong \mathrm{Hom}_{\mathcal{M}_1}(\mathcal{O}_{\mathcal{M}_1}, f^*-).$$

This induces natural isomorphisms on the derived functors $H^i(\mathcal{M}_1, f^*-)$ and $H^i(\mathcal{M}_2, -)$ for each i, as desired. □

Lemma 4.3.3 ([Rav86, Theorem A1.1.3])**.** *Let (A, Γ) be a Hopf algebroid such that the left unit map $A \to \Gamma$ is flat. Then the category of left Γ-comodules is ableian.*

We will need the following generalization of [Rav86, Proposition 6.2.1].

Lemma 4.3.4. *Let (A, Γ) be a Hopf algebroid over $K(n)_*$ such that the left unit $\eta_L : A \to \Gamma$ is flat. Let $K(n)_* \to \mathbb{F}_p$ be the $\mathbb{F}_p$-algebra map sending v_n to 1. Then the functor $-\otimes_{K(n)_*}\mathbb{F}_p$ is an equivalence of abelian categories between $\mathbb{Z}$-graded (A, Γ)-comodules and $\mathbb{Z}/2(p^n-1)$-graded $(A \otimes_{K(n)_*} \mathbb{F}_p, \Gamma \otimes_{K(n)_*} \mathbb{F}_p)$-comodules.*

Proof. Since $K(n)_*$ is a graded field, any $K(n)_*$-module M splits as a sum of suspensions of $K(n)_*$. Since $K(n)_*$ is $2(p^n-1)$ periodic, without loss of generality, we can write

$$M = \bigoplus_{j \in J} \Sigma^{k_j} K(n)_*$$

with $0 \le k_j < 2(p^n - 1)$ for each j. Accordingly, after base changing to $\mathbb{F}_p$, we obtain the $2(p^n-1)$-graded $\mathbb{F}_p$ module

$$M \otimes_{K(n)_*} \mathbb{F}_p = \mathbb{F}_p[x_j : j \in J]$$

with $|x_j| = k_j$. It is clear that base changing back along $\mathbb{F}_p \to K(n)_*$ recovers M; indeed,

$$(M \otimes_{K(n)_*} \mathbb{F}_p) \otimes_{\mathbb{F}_p} K(n)_* = K(n)_*[x_j : j \in J] \cong \bigoplus_{j \in J} \Sigma^{|x_j|} K(n)_* = M,$$

and so $-\otimes_{K(n)_*}\mathbb{F}_p$ and $-\otimes_{\mathbb{F}_p} K(n)_*$ are inverse equivalences of categories. That $-\otimes_{K(n)_*}\mathbb{F}_p$ is an equivalence of abelian categories follows from the fact that $K(n)_* \to \mathbb{F}_p$ is faithfully flat. □

Remark 4.3.5. We will also need to show that the equivalence of stacks

$$\theta : BS_n \to \mathcal{M}_{\mathrm{fg}}^{=n,s} \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p)$$

comes from a map of the relevant Hopf algebroids, (BP_*, BP_*BP) and $(K(n)_*, \Sigma(n))$. Indeed, this equivalence is obtained from the BP_*-bimodule map $\beta : BP_*BP \to \Sigma(n)$.

To see this, first note that β descends to a map $\bar{\beta}$ from $v_n^{-1}BP_*BP/I_n$ since I_n annihilates $\Sigma(n)$ and v_n acts on $\Sigma(n)$ by automorphisms. Note that $BP_*BP/I_n^s \cong v_n^{-1}BP_*BP/I_n \otimes_{K(n)_*} \mathbb{F}_p$, and so after base changing the map

$$\mathrm{Spec}(\Sigma(n)) \xrightarrow{\mathrm{Spec}(\bar{\beta})} \mathrm{Spec}(v_n^{-1}BP_*BP/I_n)$$

along $K(n)_* \to \mathbb{F}_p$, we obtain a map

$$S_n \to \mathrm{Spec}(BP_*BP/I_n^s),$$

where I_n^s is the ideal generated by I_n and the relation $v_n = 1$. By construction, this map covers a map $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(BP_*/I_n^s)$, and so yields a map of groupoid schemes. Our goal is to show this map equals θ after stackification.

Finally, we are ready to prove the classical change of rings theorem as a corollary of our stacky version.

Proof of 4.0.1 using 4.2.3. Let $\mathcal{N} = \mathcal{M}_{\mathrm{fg}}^{=n,s} \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p)$. Applying Lemma 4.3.2 to the result of Theorem 4.2.3 gives

$$H^*(BS_n, \mathcal{O}_{BS_n}) \cong H^*(\mathcal{N}, \mathcal{O}_{\mathcal{N}}). \tag{4.3.6}$$

Note that BS_n and $\mathcal{N}$ are the stacks associated to the Hopf algebroids $(\mathbb{F}_p, S(n))$ and $(BP_*/I_n^s, BP_*BP/I_n^s)$ respectively. Using Corollary A.3.18, we can interpret (4.3.6) as an statement about the cohomology of the sheaves of groupoids corepresented by these Hopf algebroids. Now, using the equivalence of abelian categories of Theorem 3.1.11, we can rephrase (4.3.6) as

$$\mathrm{Ext}^*_{S(n)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}_{BP_*BP/I_n^s}(BP_*/I_n^s, BP_*/I_n^s). \tag{4.3.7}$$

Using the (faithfully-flat) $K(n)_*$-module structure on $\mathbb{F}_p$ induced by the $\mathbb{F}_p$-algebra map $K(n)_* \to \mathbb{F}_p$ sending v_n to 1, we can rewrite the left-hand side of (4.3.7) (according to Remark 3.4.9) as

$$\mathrm{Ext}^*_{\Sigma(n)\otimes_{K(n)_*}\mathbb{F}_p}(K(n)_* \otimes_{K(n)_*} \mathbb{F}_p, K(n)_* \otimes_{K(n)_*} \mathbb{F}_p).$$

Likewise, we can rewrite the right-hand side as

$$\mathrm{Ext}^*_{v_n^{-1}BP_*BP/I_n\otimes_{K(n)_*}\mathbb{F}_p}(v_n^{-1}BP_*/I_n \otimes_{K(n)_*} \mathbb{F}_p,\ v_n^{-1}BP_*/I_n \otimes_{K(n)_*} \mathbb{F}_p).$$

Since base change along $K(n)_* \to \mathbb{F}_p$ is an equivalence of abelian categories (Lemma 4.3.4), we can rewrite these Ext modules as

$$\mathrm{Ext}^*_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p$$

and

$$\mathrm{Ext}^*_{v_n^{-1}BP_*BP/I_n}(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*/I_n) \otimes_{K(n)_*} \mathbb{F}_p,$$

respectively.

In summary, we have produced a sequence of isomorphisms yielding

$$\mathrm{Ext}^*_{v_n^{-1}BP_*BP/I_n}(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*/I_n) \otimes_{K(n)_*} \mathbb{F}_p \cong \mathrm{Ext}^*_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p\,.$$

By the argument in Remark 4.3.5, this isomorphism is obtained by applying $-\otimes_{K(n)_*}\mathbb{F}_p$ to the map on Ext

$$\bar{\beta}_* : \mathrm{Ext}^*_{v_n^{-1}BP_*BP/I_n}(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*/I_n) \to \mathrm{Ext}^*_{\Sigma(n)}(K(n)_*, K(n)_*)$$

induced by $\bar{\beta} : v_n^{-1}BP_*BP/I_n \to \Sigma(n)$. Since $K(n)_* \to \mathbb{F}_p$ is faithfully-flat and $\bar{\beta}_* \otimes_{K(n)_*} \mathbb{F}_p$ is an isomorphism, so is $\bar{\beta}_*$. Finally, by Lemma 4.3.1, we obtain

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}^*_{\Sigma(n)}(K(n)_*, K(n)_*).$$

This completes the proof.

□

5. Landweber exactness

5.1. The Landweber exact functor theorem

One major reward for understanding chromatic homotopy theory via its relationship to the moduli of formal groups is the picture this perspective affords of the Landweber exact functor theorem. The story of this theorem over $\mathcal{M}_{\mathrm{fg}}$ is especially elegant, and a stacky proof of the theorem is given by Hopkins in one of the original documents on the subject [Hop99]. We will discuss this proof following [Pst21, Section 19] and [Goe08, Section 6.3] after recalling some motivation for the theorem.

In Theorem 3.1.11, we saw that given a spectrum X, its MU-homology determines a MU_*MU-comodule which yields a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$:

$$X \mapsto \widetilde{MU_*X}.$$

With this construction in mind, we might hope to lift information about quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$ back to information about spectra. A natural first question to ask is whether given an arbitrary such sheaf there is a spectrum X realizing it.

The main way to answer questions like this in homotopy theory is via the Brown representability theorem. Suppose the quasi-coherent sheaf $\mathcal{F}$ over $\mathcal{M}_{\mathrm{fg}}$ we're considering is the push-forward of some (even-graded) MU_*-module M along the *fpqc*-cover

$$p : \mathrm{Spec}(MU_*) \to \mathcal{M}_{\mathrm{fg}}$$

discussed in Proposition 3.1.3. We have a functor from Sp to (even-graded) MU_*-modules

$$X \mapsto M \otimes_{MU_*} MU_*X. \tag{5.1.1}$$

If this functor is a homology theory on hSp, then Brown representability says there is some spectrum E for which $E_*X = M \otimes_{MU_*} MU_*X$. Specifically, to be representable in hSp, this functor needs to

(i) be stable and homotopy invariant,
(ii) preserve direct sums, and
(iii) take cofiber sequences to long exact sequences.

The first two properties are immediately guaranteed because $X \mapsto MU_*X$ itself is already homology theory. The last point, however, depends strongly on the structure of M as an MU_*-module.

Certainly if M is a flat MU_*-module, then point (iv) is satisfied and (5.1.1) determines a homology theory. This is a very strong condition to place on M, however, since MU_* is infinitely-generated. It turns out that we can get away with a weaker condition.

Recall from Section 2 that given an (even) MU_*-module M with an MU_*MU-comodule structure, we can obtain a module over the descent object $\mathcal{D}_{MU_*}$ as in 2.0.1 and by faithfully-flat descent, there is a canonical quasi-coherent sheaf $\mathcal{F}_M$ over $\mathcal{M}_{\mathrm{fg}}$, such that

$$M \cong p^* \mathcal{F}_M$$

as MU_*-modules. In particular, to a spectrum X with even MU-homology, there is some canonical quasi-coherent $\mathcal{F}_X$ over $\mathcal{M}_{\mathrm{fg}}$ such that

$$MU_*X \cong p^* \mathcal{F}_X\,.$$

Thus, for M an MU_*MU-comodule, we have

$$M \otimes_{MU_*} MU_*X \cong p^* \mathcal{F}_M \otimes_{MU_*} p^* \mathcal{F}_X \cong p^*(\mathcal{F}_M \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{F}_X)$$

since p^* is a monoidal functor. We can therefore see that for (5.1.1) to be a homology theory, it is enough for $\mathcal{F}_M$ to be flat as an $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module.

Landweber's theorem is a criterion for $\mathcal{F}_M$ to be flat. A classical statement is the following.

Theorem 5.1.2 (Landweber Exact Functor Theorem [Pst21, Theorem 19.5])**.** *Let M be an even MU_*MU-comodule. Each prime p determines elements $v_0, v_1, \ldots$ in MU_*; suppose that for every p,*

(i) v_n acts injectively on M/I_n, and
(ii) $M/I_n = 0$ for n large enough.

Then $\mathcal{F}_M$ is a flat $\mathcal{O}_{\mathcal{M}_{fg}}$-module, and hence the construction in (5.1.1) determines an object of Sp by Brown representability.

In Section 3.2, we constructed the sections

$$v_n : \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \to \omega^{\otimes p^n - 1},$$

and in a precise sense, these v_n are the $\mathcal{M}_{\mathrm{fg}}$-analogue of the elements v_n in the ring MU_*. We want to translate the criteria of Theorem 5.1.2 to to make use of this alternate definition of the v_n.

Remark 5.1.3. In Remark 3.2.21, we established the convention that we will usually use $\mathcal{M}_{\mathrm{fg}}$ to implicitly mean the moduli of p-typical formal groups. For this section, we will break this convention and use $\mathcal{M}_{\mathrm{fg}}$ to mean the moduli of formal groups at all primes.

Let

$$j_n : \mathcal{M}_{\mathrm{fg}}^{\geq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

be the inclusion of the closed substack of formal groups of height at least n into $\mathcal{M}_{\mathrm{fg}}$. As we saw in Remark 3.2.17, for each n, the inclusions

$$\mathcal{M}_{\mathrm{fg}}^{\geq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\geq n-1}$$

are of effective Cartier divisors. This gives the functors j_n^* and $(j_n)_*$ nice cohomological properties, which we record in the following lemma.

Lemma 5.1.4 ([Pst21, Lemma 19.7])**.** *The inclusion j_n induces an adjoint pair $j_n^* \dashv (j_n)_*$ on categories of quasi-coherent sheaves with the properties that*

(i) $(j_n)_*$ is exact and fully-faithful, and
(ii) $R^s j_n^* = 0$ for all $s > n$.

Now, recall that $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ is, by definition, the vanishing locus of v_n on $\mathcal{M}_{\mathrm{fg}}^{\geq n}$, so we have a short exact sequence of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules:

$$0 \to \omega^{\otimes 1-p^n} \otimes (j_n)_* j_n^* \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \xrightarrow{v_n} (j_n)_* j_n^* \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \to (j_{n+1})_* j_{n+1}^* \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \to 0.$$

Since $(j_n)_*$ and $(j_{n+1})_*$ are fully-faithful embeddings into the category of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules, we might also express this short exact sequence as

$$0 \to \omega^{\otimes 1-p^n} \otimes \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \xrightarrow{v_n} \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \to \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n+1}} \to 0.$$

Given a quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module $\mathcal{F}$, we can tensor with this short exact sequence to obtain a right-exact sequence

$$\omega^{\otimes 1-p^n} \otimes \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \otimes \mathcal{F} \xrightarrow{v_n} \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \otimes \mathcal{F} \to \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n+1}} \otimes \mathcal{F} \to 0,$$

which we can simplify to

$$\omega^{\otimes 1-p^n} \otimes j_n^* \mathcal{F} \xrightarrow{v_n} j_n^* \mathcal{F} \to j_{n+1}^* \mathcal{F} \to 0. \tag{5.1.5}$$

Note that property (i) of Theorem 5.1.2 is exactly equivalent to this sequence also being exact on the right.

With this, we are ready to produce a stack-theoretic statement of Landweber's theorem.

Theorem 5.1.6 (Landweber Exact Functor Theorem [Pst21, Theorem 19.11])**.** *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{M}_{fg}}$-module. Suppose that for every p,*

(i) the right-exact sequence in (5.1.5) is also left-exact, and
(ii) $j_n^* \mathcal{F} = 0$ for n large enough.

Then $\mathcal{F}$ is flat, and hence, after pulling back along the faithfully-flat cover $\mathrm{Spec}(MU_*) \to \mathcal{M}_{fg}$*, the construction in* (5.1.1) *determines an object of* Sp *by Brown representability.*

We will prove this theorem using the language of quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$. Before giving the main proof, we will handle a couple of lemmas. The first (which we will not prove since doing so would require some additional technical constructions) has to do with the process of inverting v_n on appropriate $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules. (This result will also become relevant in Definition 6.2.3).

Let

$$i_n : \mathcal{M}_{\mathrm{fg}}^{\leq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

be the open inclusion of the moduli of formal groups of height at most n into $\mathcal{M}_{\mathrm{fg}}$.

Definition 5.1.7. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module annihilated by $\mathcal{I}_n$. Then there is some quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}$-module $\mathcal{E}$ so that $\mathcal{F} = (j_n)_* \mathcal{E}$, and we can define the *v_n-mapping telescope* of $\mathcal{F}$ to be the $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module

$$v_n^{-1} \mathcal{F} := (j_n)_* \varinjlim \big(\mathcal{E} \xrightarrow{v_n} \mathcal{E} \otimes \omega^{p^n-1} \xrightarrow{v_n} \mathcal{E} \otimes \omega^{2(p^n-1)} \xrightarrow{v_n} \cdots \big).$$

Lemma 5.1.8 ([Goe08, Proposition 6.15, Remark 6.16])**.** *There is an isomorphism*

$$v_n^{-1} \mathcal{F} \cong (i_n)_* i_n^* \mathcal{F}.$$

Remark 5.1.9. We will sometimes also want to consider the mapping telescope construction as a $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}$-module. In this case, we can simply drop the extra push-forward along j_n in the definition to obtain $v_n^{-1}\mathcal{E}$.

Typically it we will be explicit whether we mean to consider a sheaf as an $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$- or $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}$-module. In particular, we highlight that for $\mathcal{F}$ an $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module, we define

$$\mathcal{F} / \mathcal{I}_n := (j_n)_* j_n^* \mathcal{F}$$

to be another $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module.

Next, we prove two lemmas to do with the vanishing of certain Tor groups and their relation to the flatness of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules.

Lemma 5.1.10 ([Goe08, Proposition 6.24])**.** *Suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{M}_{fg}}$-module satisfying conditions (i) and (ii) of Theorem 5.1.6. Suppose further that for each $s > n$,*

$$\mathrm{Tor}_s(v_n^{-1}(\mathcal{F} / \mathcal{I}_n), \mathcal{N}) = 0$$

for any $\mathcal{N}$. Then $\mathcal{F}$ is flat.

Proof. To prove $\mathcal{F}$ is flat, it is enough to show that each

$$\mathrm{Tor}_s(\mathcal{F}, \mathcal{N}) = 0 \tag{5.1.11}$$

for all positive s and any $\mathcal{N}$. By property (ii), $\mathcal{F} / \mathcal{I}_k$ vanishes for large enough k, and so we have

$$\mathrm{Tor}_s(\mathcal{F} / \mathcal{I}_k, \mathcal{N}) = 0.$$

We will proceed by downwards induction to obtain the desired equation (5.1.11) in the case when $k = -1$. To begin, suppose that

$$\mathrm{Tor}_s(\mathcal{F} / \mathcal{I}_{n+1}, \mathcal{N}) = 0 \tag{5.1.12}$$

for some $n+1$ and for all $s > 0$.

By property (i) and the fact that $(j_n)_*$ is exact (Lemma 5.1.4), we have a short exact sequence of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq}}$-modules

$$0 \to (\mathcal{F} / \mathcal{I}_n) \otimes \omega^{\otimes 1-p^n} \xrightarrow{v_n} \mathcal{F} / \mathcal{I}_n \to \mathcal{F} / \mathcal{I}_{n+1} \to 0.$$

Applying $\mathrm{Tor}_*(-, \mathcal{N})$ to get a long exact sequence and we see that

$$\mathrm{Tor}_s((\mathcal{F} / \mathcal{I}_n) \otimes \omega^{\otimes 1-p^n}, \mathcal{N}) \xrightarrow{v_n} \mathrm{Tor}_s(\mathcal{F} / \mathcal{I}_n, \mathcal{N})$$

is injective since $\mathrm{Tor}_{s+1}(\mathcal{F} / \mathcal{I}_{n+1}, \mathcal{N}) = 0$. Since $\omega^{\otimes 1-p^n}$ is locally free, we find that

$$\mathrm{Tor}_s(\mathcal{F} / \mathcal{I}_n, \mathcal{N}) \xrightarrow{v_n} \mathrm{Tor}_s(\mathcal{F} / \mathcal{I}_n, \mathcal{N}) \otimes \omega^{\otimes p^n - 1}$$

is also injective. Finally, we have

$$\begin{aligned} 0 &= \mathrm{Tor}_s(v_n^{-1}(\mathcal{F} / \mathcal{I}_n), \mathcal{N}) \\ &\cong \mathrm{Tor}_s\left(\mathrm{colim}_{t \geq 0} (\mathcal{F} / \mathcal{I}_n) \otimes \omega^{\otimes t(1-p^n)}, \mathcal{N}\right) \\ &\cong \mathrm{colim}_{t \geq 0} \mathrm{Tor}_s((\mathcal{F} / \mathcal{I}_n) \otimes \omega^{\otimes t(1-p^n)}, \mathcal{N}) \\ &\cong \mathrm{colim}_{t \geq 0} \mathrm{Tor}_s((\mathcal{F} / \mathcal{I}_n), \mathcal{N}) \otimes \omega^{\otimes t(1-p^n)}. \end{aligned}$$

Since each of the maps along the colimit are injective, we deduce that each of the object must be 0; in particular,

$$\mathrm{Tor}_s((\mathcal{F} / \mathcal{I}_n), \mathcal{N}) = 0.$$

This completes the proof. □

In what follows, let

$$h_n : \mathcal{M}_{\mathrm{fg}}^{=n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

be the inclusion of the moduli of height exactly n formal groups inside $\mathcal{M}_{\mathrm{fg}}$.

Lemma 5.1.13 ([Goe08, Proposition 6.25], [Pst21, Proof of Theorem 19.11])**.** *Let $\mathcal{F}$ and $\mathcal{N}$ be a quasi-coherent sheaves over $\mathcal{M}_{fg}$. For all $s > n$,*

$$\mathrm{Tor}_s((h_n)_* h_n^* \mathcal{F}, \mathcal{N}) = 0.$$

Proof. Note that as $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules,

$$(h_n)_* h_n^* \mathcal{F} = \mathcal{F}(h_n)_* \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}},$$

and so

$$\begin{aligned}(h_n)_* h_n^* \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{N} &= \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{N} \\ &= \left(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}}\right) \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}}} \left(\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathcal{N}\right) \\ &= (h_n)_* h_n^* \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} (h_n)_* h_n^* \mathcal{N}.\end{aligned}$$

Also note that $(h_n)_*$ is a fully-faithful embedding, and so it is enough to show that the derived functors

$$R^s(h_n^* \mathcal{F} \otimes h_n^* \mathcal{N})$$

vanish above degree n in the category of quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}}$-modules.

Recall from Remark 3.3.11 that $\mathcal{M}_{\mathrm{fg}}^{=n}$ admits an (affine) $\mathbb{G}_n$-Galois covering by $\mathrm{Spec}(\mathbb{F}_{p^n})$, and so $\mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}}^{=n})$ is equivalent to the category of continuous $\mathbb{G}_n$-representations over $\mathbb{F}_{p^n}$. In particular, every object in this category is flat, so tensoring over $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{=n}}$ is exact. Thus,

$$R^s(h_n^* \mathcal{F} \otimes h_n^* \mathcal{N}) = h_n^* \mathcal{F} \otimes R^s h_n^* \mathcal{N}.$$

But note that $h_n^* = g_n^* j_n^*$ where g_n is the open embedding

$$g_n : \mathcal{M}_{\mathrm{fg}}^{=n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

The push-forward g_n^* is exact since g_n is an open embedding, and j_n^* has vanishing right derived functors in degrees above n by Lemma 5.1.4. This completes the proof. □

We are now ready for the proof of the main theorem.

Proof of Theorem 5.1.6. It is enough to work one prime at a time, since, after pulling back along any $f : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$, the R-module $f^* \mathcal{F}$ is flat if and only if each $f^* \mathcal{F} \otimes \mathbb{Z}_{(p)}$ is. Thus, we will fix a prime p and implicitly assume we are working with the p-localized $\mathcal{M}_{\mathrm{fg}}$.

Our goal is to show that the Tor groups

$$\mathrm{Tor}_s(v_n^{-1}(\mathcal{F} / \mathcal{I}_n), \mathcal{N})$$

vanish for all $s > n$, and then apply Lemma 5.1.10. From Definition 3.2.23, we have a pullback square

$$\begin{tikzcd} \mathcal{M}_{\mathrm{fg}}^{=n} \arrow[r, "g_n"] \arrow[d, "k_n"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{\mathrm{fg}}^{\geq n} \arrow[d, "j_n"] \\ \mathcal{M}_{\mathrm{fg}}^{\leq n} \arrow[r, "i_n"'] & \mathcal{M}_{\mathrm{fg}} \end{tikzcd}$$

and so since $\mathcal{F} / \mathcal{I}_n = (j_n)_* j_n^* \mathcal{F}$, we have

$$i_n^*(\mathcal{F} / \mathcal{I}_n) = (k_n)_* g_n^* j_n^* \mathcal{F}$$

by [Sta25, Tag 02RG]. Combining this with the result of Lemma 5.1.8, we have

$$v_n^{-1}(\mathcal{F} / \mathcal{I}_n) = (i_n)_*(k_n)_* g_n^* j_n^*(\mathcal{F} / \mathcal{I}_n).$$

But note that $j_n^*(\mathcal{F} / \mathcal{I}_n) = j_n^* \mathcal{F}$ since $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}$-modules are $\mathcal{I}_n$-torsion. It follows that

$$v_n^{-1}(\mathcal{F} / \mathcal{I}_n) = (i_n)_*(k_n)_* g_n^* j_n^* \mathcal{F} = (h_n)_* h_n^* \mathcal{F}.$$

With this substitution, we have

$$\mathrm{Tor}_s(v_n^{-1}(\mathcal{F} / \mathcal{I}_n), \mathcal{N}) = \mathrm{Tor}_s((h_n)_*(h_n^* \mathcal{F}), \mathcal{N})$$

which is 0 for all $s > n$ by Lemma 5.1.13. □

With the theorem proved, let us make the following definition which we will examine in some more detail in the next section.

Definition 5.1.14. Let F be a formal group law over a ring R. Then R is naturally an MU_*-algebra via the map $MU_* \to R$ classifying F. We say F is *Landweber exact* if R satisfies conditions (i) and (ii) of Theorem 5.1.2.

Remark 5.1.15. The condition of F being Landweber exact is equivalent to the map

$$\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$$

classifying the formal group G_F associated to F being flat [Pst21, Remark 20.3]. We make this into a definition: a formal group is *Landweber exact* if its classifying map into $\mathcal{M}_{\mathrm{fg}}$ is flat.

Similarly, we call an MU-module spectrum X *Landweber exact* if X_* is Landweber exact as an MU_*-module.

Example 5.1.16. The spectrum

$$E(n) := v_n^{-1} BP\langle n\rangle\,,$$

called the nth *Johnson-Wilson spectrum*, has coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}],$$

and is Landweber exact. In fact, it is standard to *define* $E(n)$ to be the Landweber exact spectrum with this prescribed MU_*-module structure on its coefficient ring.

5.2. **Application: Lubin-Tate spectra**

Starting with any Landweber exact formal group, we can obtain a (complex-oriented) spectrum—this is nice for multiple reasons: first, this construction gives a "partial converse" to the construction in Quillen's theorem (Theorem 3.1.8); second, Landweber's theorem provides a new source of "designer spectra" which have interesting arithmetic properties. In this brief section, we will indicate one example of this latter point. The starting point is Lubin-Tate deformation theory.

Let G be a formal group over a ring R. Let R' be a "nilpotent thickening" of R, that is, suppose we have a surjection $R' \to R$ with nilpotent kernel. Then $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ is a closed inclusion. A *deformation* of G to R' is a formal group G' over R' along with a choice of isomorphism φ as in the following diagram:

$$\begin{tikzcd}
G \arrow[r, dashed, "\cong"', "\varphi"] \arrow[dr, bend right] & \mathrm{Spec}(R) \times_{\mathrm{Spec}(R')} G' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G' \arrow[d] \\
& \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(R')
\end{tikzcd}$$

The theory of deformations of formal groups (called Lubin-Tate deformation theory) is very rich and well-studied. A key application for us is that it allows us to study the formal neighborhoods of points in $\mathcal{M}_{\mathrm{fg}}$—in particular, as we will see in Definition 6.2.6, studying the formal neighborhood of the point

$$\mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}_{\mathrm{fg}}$$

classifying the height-n Honda formal group is directly related to understanding $K(n)$-local stable homotopy.

A central result of Lubin-Tate deformation theory is the following.

Theorem 5.2.1 (Lubin-Tate [Pst21, Corollary 18.19])**.** *Let k be a perfect field and G a formal group of finite height n over k. There exists a complete local noetherian $W(k)$-algebra E_0 called the* Lubin-Tate ring, *and a deformation $\tilde{G}$ of G to E_0 which is universal in the sense that deformations of G to any other nilpotent thickening R of k are in natural correspondence with (continuous) maps $E_0 \to R$.*

Moreover, E_0 can be (non-canonically) identified with the power series ring

$$E_0 \cong W(k)[\![u_1, \dots, u_{n-1}]\!]\,.$$

When the formal group of height n in the theorem is taken to be the Honda formal group over $\mathbb{F}_q$, then we obtain a universal deformation $\tilde{H}_n$ over E_0. After identifying E_0 with a power series ring as above, we can find that $\tilde{H}_n$ is the formal group associated to the formal group law over E_0 with $v_i(\tilde{H}_n) = u_i$. One can check that this makes $\tilde{H}_n$ a Landweber exact formal group, and so we obtain a Landweber exact spectrum E_n. In fact, we can make this construction in slightly more generality:

Definition 5.2.2 ([Pst21, Definition 20.22])**.** The *Lubin-Tate spectrum* $E(G)$ associated to a formal group G of finite height over a perfect field k is the (weakly even-periodic) Landweber exact spectrum associated to the universal deformation of G to the Lubin-Tate ring $E_0(G)$.

In particular, when $k = \mathbb{F}_q$ and G is the Honda formal group, we denote the associated Lubin-Tate spectrum by E_n.

To conclude our brief discussion, we make a couple comments on the properties of $E(G)$. The first is that

$$\pi_0 E(G) = E_0(G).$$

Next, since the construction of the universal deformation is functorial, the action of the Morava stabilizer group $\mathbb{G}_n$ on $H_n/\mathbb{F}_q$ lifts to an action on $\tilde{H}_n$ and then to an action on E_n. It is a theorem of Goerss-Hopkins-Miller that E_n has a natural $\mathbb{E}_\infty$-ring structure, and in fact, this action of $\mathbb{G}_n$ can be further lifted to an action by $\mathbb{E}_\infty$-ring maps. These excellent arithmetic and homotopical properties make the Lubin-Tate spectra central objects of study in modern chromatic homotopy theory.

6. Chromatic localization and chromatic convergence

6.1. Classical chromatic localizations

In Remark 3.1.1, we discussed a few motivations for studying the Adams-Novikov spectral sequence, and the essential takeaways were that (a) it has certain intrinsically interesting structures (v_n-periodicities and a filtration by v_n-torsion pieces) and (b) it is accessible to study via other methods, namely through the theory of cobordism and Thom spectra (as Novikov did), and more importantly for us, through the relation to the cohomology of $\mathcal{M}_{\mathrm{fg}}$.

Both of these motivations lead directly to the field of chromatic homotopy theory, which unpacks these phenomena and produces new objects to help us study them such as the Morava E- and K-theories. This is all nice, but it may seem that through this series of abstractions and constructions we have ended up quite far afield from the original goal of understanding the the sphere itself and the category of spectra.

In this section, we discuss another way one might arrive at chromatic homotopy theory and see how in some sense, the Morava E- and K-theories are "inevitable" when studying the category of spectra. To begin, we introduce some terminology.

Definition 6.1.1. A *finite spectrum* is a compact object of Sp. Equivalently, it is the colimit of a finite diagram in Sp comprised of self-maps of $\mathbb{S}$.

We denote the full subcategory of Sp spanned by the finite spectra by $\mathsf{Sp}^{\mathrm{fin}}$. (It is sometimes also written as Sp^{ω}.)

Definition 6.1.2. A (finite) spectrum X is *type n* if $K(n)_*X \neq 0$, but each $K(m)_*X = 0$ for $m < n$.

Theorem 6.1.3 ([Rav84, Theorem 2.11])**.** *For any finite spectrum X, if $K(n)_*X = 0$, then $K(m)_*X = 0$ for each $m < n$.*

We denote the full subcategory finite spectra of type at least n by $\mathsf{Sp}^{\mathrm{fin}}_{\geq n}$. From the theorem, it follows that

$$\mathsf{Sp}^{\mathrm{fin}}_{\geq n} \subset \mathsf{Sp}^{\mathrm{fin}}_{\geq n-1}$$

The *thick subcategory theorem* says that the thick subcategories of $\mathsf{Sp}^{\mathrm{fin}}$—that is, the full subcategories closed under taking direct summands—are precisely these nested subcategories $\mathsf{Sp}^{\mathrm{fin}}_{\geq n}$ for each n and at each prime. These subcategories of finite $K(n)_*$-acyclic spectra (which are roughly the "kernels" of the homology theories $K(n)_*$) determine localization functors on $\mathsf{Sp}^{\mathrm{fin}}$. It is natural then to consider the Bousfield localizations of the entire category of spectra with respect to these $K(n)$:

$$L_{K(n)} : \mathsf{Sp} \to \mathsf{Sp}_{K(n)}\,.$$

It is also interesting to consider the Bousfield localizations with respect to the sum of the first n Morava K-theories:

$$L_n = L_{K(0)\oplus \ldots \oplus K(n)}.$$

As we will see in Theorem 6.2.1, it turns out that L_n coincides with the Bousfield localization with respect to *any* height-n Landweber-exact spectrum, for example, the Lubin-Tate and Johnson-Wilson spectra: $L_n = L_{E_n} = L_{E(n)}$ [Rav84, Theorem 2.1(d)].

Remark 6.1.4 (Telescopic localizations)**.** There is a natural third class of localization functors on Sp of interest. From any set $\mathcal{A}$ of spectra, [Mil92] produces a localization $L^f_{\mathcal{A}}$ which, when $\mathcal{A}$ is taken to be the set of finite E-acyclic spectra for some E, is like the Bousfield localization with respect to E with an extra finiteness condition. When E is $K(n)$ (so that $\mathcal{A}$ is the set of objects of $\mathsf{Sp}^{\mathrm{fin}}_{\geq n}$), the functor is denoted L^f_n, and has the property of sending each finite $K(n-1)$-acyclic spectrum X with a v_n-self map to its mapping telescope $v_n^{-1}X$ [Mil92, Proposition 14].

This functor is related to the Bousfield localization $L_{T(n)}$ where $T(n)$ is the mapping telescope of (any) finite type-n spectrum in the following way:

$$L^f_n = L_{T(0)\oplus\ldots\oplus T(n)}.$$

Certainly there is a natural map $L^f_n \to L_n$ since L_n is a localization of Sp with respect to *all* $E(n)$-local spectra, not just the finite ones. The question of whether the functors L^f_n and L_n (or $L_{T(n)}$ and $L_{K(n)}$) actually coincide is the content of the telescope conjecture. The answer turns out to be no.

Another perspective related to the thick subcategory theorem is the Balmer spectrum of $\mathsf{Sp}^{\mathrm{fin}}$, which, as a topological space, is illustrated by the following picture:

$$\begin{tikzcd}
H\mathbb{F}_2 & H\mathbb{F}_3 & H\mathbb{F}_5 & \cdots & H\mathbb{F}_p \\
\vdots \arrow[u, dash] & \vdots \arrow[u, dash] & \vdots \arrow[u, dash] & & \vdots \arrow[u, dash] \\
K(2) \arrow[u, dash] & K(2) \arrow[u, dash] & K(2) \arrow[u, dash] & \cdots & K(2) \arrow[u, dash] \\
K(1) \arrow[u, dash] & K(1) \arrow[u, dash] & K(1) \arrow[u, dash] & & K(1) \arrow[u, dash] \\
& & H\mathbb{Q} \arrow[ull, dash] \arrow[ul, dash] \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

Figure 6.1.5. The Balmer spectrum of the category of finite spectra, $\mathrm{Spc}(\mathsf{Sp}^{\mathrm{fin}})$. (In $\mathbb{E}_\infty$-geometry, this picture is also sometimes caled "$\mathrm{Spec}(\mathbb{S})$," $\mathbb{S}$ being the initial $\mathbb{E}_\infty$-ring.) See [Bal20, Theorem 3.3].

Here, each point of $\mathrm{Spc}(\mathsf{Sp}^{\mathrm{fin}})$ corresponds to the "kernel" (thick $\otimes$-ideal) of some Morava K-theory (where, by convention, $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$). A vertical line indicates that the higher point is in the closure of the lower point. The closed points then correspond to the mod-p Eilenberg-Maclane spectra for each prime p, and $H\mathbb{Q}$ is the generic point in the sense that its closure is all of $\mathsf{Sp}^{\mathrm{fin}}$.

We will highlight three takeaways from this picture in the following remarks.

Remark 6.1.6. Notice the resemblance between $\mathrm{Spc}(\mathsf{Sp}^{\mathrm{fin}})$ and the increasing filtration of $\mathcal{M}_{\mathrm{fg}}$ by the open substacks $\mathcal{M}^{\leq n}_{\mathrm{fg}}$ discussed in Definition 3.2.18. Indeed, as topological spaces, $\mathrm{Spc}(\mathsf{Sp}^{\mathrm{fin}})$ and the space of geometric points of $\mathcal{M}_{\mathrm{fg}}$ are homeomorphic.

Remark 6.1.7. Another perspective to have in mind is the comparison of Figure 6.1.5 to the corresponding picture of $\mathrm{Spec}(\mathbb{Z})$:

$$\begin{tikzcd}
\mathbb{F}_2 & \mathbb{F}_3 & \mathbb{F}_5 & \cdots & \mathbb{F}_p \\
& & \mathbb{Q} \arrow[ull, dash] \arrow[ul, dash] \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

Figure 6.1.8. $\mathrm{Spec}(\mathbb{Z})$.

Specifically, taking the Eilenberg-Maclane embedding $H : \mathsf{Ring} \to \mathsf{Sp}$ affords us infinitely many "new" localizations between looking at the rationalization and p-torsion pieces for each p. In some sense, this makes the category of spectra a richer place for studying rings and their geometry. (Recall the theory of faithfully-flat descent is enriched by passing to the derived setting, too—see Section 2.)

Remark 6.1.9. One more perspective to take is that this means mod-p homology is hard to understand—there are infinitely many steps to take between understanding the rational homology and mod-p homology of a space.

The point is that understanding localization with respect to Morava E- and K-theories is an essential part of understanding Sp, independent of the Adams-Novikov spectral sequence. Accordingly, we can hope for a nice picture of these localizations in terms of quasi-coherent sheaves over the moduli of formal groups. Indeed a very elegant and instructive picture exists here if we are willing to fully embrace geometric language.

6.2. Chromatic localizations of sheaves over $\mathcal{M}_{\mathrm{fg}}$

We saw in Remark 2.0.12 how the E homology of a spectrum X yields a quasi-coherent sheaf over the associated stack $\mathcal{M}_E$. In the case that $E = MU$, this construction gives a sheaf over $\mathcal{M}_{\mathrm{fg}}$. In this section, we will discuss the interactions between this sheaf and the height filtration on $\mathcal{M}_{\mathrm{fg}}$.

First, we point out the following fundamental relationship between quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ and classical homotopy-theoretic information coming from spectra.

Theorem 6.2.1 ([Pst21, Corollary 21.9])**.** *Let E be any Landweber exact homology theory of height n. Then there is an equivalence of symmetric monoidal abelian categories*

$$\mathsf{QCoh}(\mathcal{M}_{fg}^{\leq n}) \simeq \mathsf{Comod}^{\mathsf{ev}}(E_*E).$$

Proof. The key fact that we will prove is that if E is any height-n Landweber-exact homology theory, then the map

$$\mathrm{Spec}(E_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

classifying the formal group $G_E = \mathrm{Spf}(E^*\,\mathbb{CP}^\infty)$ is an *fpqc* cover by Lemma 3.2.29. From this fact, it will follow that

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Spec}(E_*) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(E_*) \rightrightarrows \mathrm{Spec}(E_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

is a colimit diagram of *fpqc*-stacks. This is equivalent to

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Spec}(E_*E) \rightrightarrows \mathrm{Spec}(E_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

being a colimit diagram. From this, we can make an argument similar to that in the proof of Theorem 3.1.11 to deduce the result.

We need to show that $\mathrm{Spec}(E_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ is faithfully flat. Flatness is immediate since E is Landweber-exact. In fact, this map is also affine by an argument similar to Proposition 3.1.3, so we just need to show that for each $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$, the flat map

$$\mathrm{Spec}(A) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(E_*) \to \mathrm{Spec}(A)$$

is in fact *faithfully* flat, i.e., surjective as a map of affine schemes. Let $\mathrm{Spec}(B)$ denote the affine scheme $\mathrm{Spec}(A) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(E_*)$.

To do this, we check that the intersection of $\mathrm{Spec}(B)$ with each closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ of $\mathrm{Spec}(A)$ is non-trivial, i.e., that $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$ is non-empty. Without loss of generality, we can suppose $\mathrm{Spec}(A)$ has just one closed point, so let $A = k$. In this case, the map $\mathrm{Spec}(k) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ classifies a formal group G of some exact height $m \leq n$ since a formal group over a field is always of some exact height.

Now, since E is Landweber exact of height n, we we know E_*/I_m is non-zero, and that v_m acts injectively on E_*/I_m, so $v_m^{-1}E_*/I_m$ is also non-zero. Pulling G_E back along the map

$$\mathrm{Spec}(v_m^{-1}E_*/I_m) \to \mathrm{Spec}(E_*)$$

yields a formal group G_E' of height exactly m.

Attaching pullback squares together, we have

$$\mathrm{Spec}(B) \times_{\mathrm{Spec}(E_*)} \mathrm{Spec}(v_m^{-1}E_*/I_m) \cong \mathrm{Spec}(k) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(v_m^{-1}E_*/I_m).$$

The right hand side is the affine scheme of formal group isomorphisms $G \to G'_E$ which is faithfully flat over $\operatorname{Spec}(k) \times \operatorname{Spec}(v_m^{-1}E_*/I_m)$ [Pst21, Theorem 15.2]. We conclude that B is faithfully flat as a k-algebra. □

Remark 6.2.2. A key aspect of the previous result is that E can be taken to be *any* height-n Landweber-exact homology theory—this gives us some freedom to choose our favorite. The two common choices are the Lubin-Tate theory E_n (Definition 5.2.2) and the Johnson-Wilson theory $E(n)$ (Example 5.1.16). The former is preferable for its excellent theoretical properties, while the later is somewhat simpler to work with in computation.

As in Section 5.1, for each n, let

$$i_n : \mathcal{M}_{\mathrm{fg}}^{\leq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$
$$j_n : \mathcal{M}_{\mathrm{fg}}^{\geq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

be the open and closed substack inclusions of height at most n and height at least n formal groups into $\mathcal{M}_{\mathrm{fg}}$. Pulling back and pushing forward gives us an adjunctions $i_n^* \dashv (i_n)_*$ and $j_n^* \dashv (j_n)_*$:

$$\mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \underset{i_n^*}{\overset{(i_n)_*}{\rightleftarrows}} \mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}}) \underset{(j_n)_*}{\overset{j_n^*}{\rightleftarrows}} \mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n}).$$

Note in particular that since i_n is an open embedding, it is an isomorphism on stalks and hence flat. It follows that i_n^* is exact [Sta25, Tag 076X].

Our first step is to give an algebraic analogue of E_n-localization. Recall from Lemma 5.1.8 that inverting v_n on a quasi-coherent $\mathcal{I}_n$-torsion $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module (i.e., forming the v_n mapping telescope) is equivalent to base changing to $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq}}$ and then pushing forward along the open inclusion i_n. We make a derived version of this construction into a definition.

Definition 6.2.3. Let $\mathcal{F}$ be a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$. Define the *algebraic $E(n)$-localization* of $\mathcal{F}$ to be

$$\mathcal{L}_n\,\mathcal{F} := \mathbb{R}((i_n)_* i_n^*)\,\mathcal{F}$$

as an object of the derived category of quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules. There is a natural map $\mathcal{F} \to \mathcal{L}_n\,\mathcal{F}$ given by

$$\mathcal{F} \to (i_n)_* i_n^*\,\mathcal{F} \to \mathbb{R}(i_n)_* i_n^*\,\mathcal{F}$$

where the first map is the unit of the adjunction $i_n^* \dashv (i_n)_*$, and the latter is an equivalence by [Goe08, Proposition 6.15].

Remark 6.2.4. Essentially, this incarnation of $E(n)$-localization is (derived) base-change of quasi-coherent sheaves from $\mathcal{M}_{\mathrm{fg}}$ to $\mathcal{M}_{\mathrm{fg}}^{\leq n}$:

$$\mathcal{L}_n(-) \cong (-) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \mathbb{R}(i_n)_*\,\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}$$

This is a reflection of the fact that at the level of spectra, $L_n : \mathsf{Sp} \to \mathsf{Sp}_{E(n)}$ is a *smashing localization*; that is, there is a natural isomorphism

$$L_n(-) \cong (-) \otimes L_n\,\mathbb{S}\,.$$

We can interpret this to mean that at the level of spectra, L_n is essentially "base-change" from $\mathbb{S}$-modules (spectra) to $L_n\,\mathbb{S}$-modules ($E(n)$-local spectra).

Lemma 6.2.5 ([Goe08, Proposition 8.19])**.** *For all quasi-coherent sheaves $\mathcal{F}$ on $\mathcal{M}_{fg}$, the natural map*

$$\mathcal{L}_{n-1}\,\mathcal{F} \to \mathcal{L}_{n-1}\,\mathcal{L}_n\,\mathcal{F}$$

obtained by applying $\mathcal{L}_{n-1}$ to the localization $\mathcal{F} \to \mathcal{L}_n\,\mathcal{F}$ is an equivalence.

Proof. As in Definition 6.2.3, the map $\mathcal{F} \to \mathcal{L}_n\,\mathcal{F} = \mathbb{R}(i_n)_* i_n^*\,\mathcal{F}$ factors as

$$\mathcal{F} \to (i_n)_* i_n^*\,\mathcal{F} \to \mathbb{R}(i_n)_* i_n^*\,\mathcal{F}$$

where the latter map is an equivalence. Thus it is enough to show that the adjunction unit map $\mathcal{F} \to (i_n)_* i_n^*\,\mathcal{F}$ becomes an equivalence after applying $\mathbb{R}(i_{n-1})_* i_{n-1}^*$. Consider the open inclusion $f : \mathcal{M}_{\mathrm{fg}}^{\leq n-1} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\leq n}$, and note that $i_{n-1} = i_n \circ f$. We have

$$\mathbb{R}(i_{n-1})_* i_{n-1}^* (i_n)_* i_n^*\,\mathcal{F} = \mathbb{R}(i_{n-1})_* f^* i_n^*\,\mathcal{F} = \mathbb{R}(i_{n-1})_* i_{n-1}^*\,\mathcal{F}\,.$$

Thus we have a natural equivalence $\mathcal{L}_{n-1}\mathcal{F} \simeq \mathcal{L}_{n-1}\mathcal{L}_n\mathcal{F}$. By the universal property of the localization map $\mathcal{F} \to \mathcal{L}_n\mathcal{F}$, the map $\mathcal{L}_{n-1}\mathcal{F} \to \mathcal{L}_{n-1}\mathcal{L}_n\mathcal{F}$ is an isomorphism. □

We are also interested in $K(n)$-localizations of spectra. Producing the right notion at the level of $\mathcal{M}_{\mathrm{fg}}$ is slightly more technical than for $E(n)$-localization: roughly, it will involve the restriction of a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$ to an infinitesimal neighborhood of $\mathcal{M}_{\mathrm{fg}}^{=n}$. There are some hurdles to making this precise, though. In particular, the process of completing a quasi-coherent sheaf at an ideal may not produce an $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module sheaf which is still quasi-coherent. The good news is that such modules still have resolutions by sums of finite vector bundles which are easier to understand. This motivates passing to the derived category of quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules.

In some sense, the algebraic notion of $K(n)$-localization being more complicated than $E(n)$-localization is to be expected since $L_{K(n)}$ is not smashing like L_n.

Definition 6.2.6 ([Goe08, Definition 6.4])**.** Let $\mathcal{J}_n = i_n^*\mathcal{I}_n$ be the quasi-coherent ideal sheaf over $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ defining the closed inclusion $\mathcal{M}_{\mathrm{fg}}^{=n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\leq n}$. Let $\mathcal{F}$ be a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$. Define the *algebraic $K(n)$-localization* to be the *derived completion* of $\mathcal{F}$ at $\mathcal{M}_{\mathrm{fg}}^{=n}$ which is the following object of the derived category of quasi-coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}$-modules:

$$\mathcal{L}_{K(n)}\mathcal{F} := \mathbb{L}(\mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{=n}} = \varprojlim_k((\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}/\mathcal{J}_n^k) \otimes^{\mathbb{L}} i_n^*\mathcal{F}).$$

By construction, there is a natural map $\mathcal{F} \to \mathcal{L}_{K(n)}\mathcal{F}$.

Example 6.2.7. To be very concrete, we can speak of how to evaluate the sheaf $\mathbb{L}(\mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{=n}}$ at an R-point of $\mathcal{M}_{\mathrm{fg}}^{\leq n}$. Let $\theta : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ be an *fpqc*-morphism classifying a formal group of height at most n, and let $\mathcal{P}_* \to \mathcal{F}$ be a resolution of $\mathcal{F}$ by sums of (finite rank) vector bundles (such a resolution exists by [Goe08, Proposition 6.1]).

Then $\mathbb{L}(\mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{=n}}(\theta)$ is the following object of the derived category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules (which we identify with the derived category of R-modules):

$$\begin{aligned}
\mathbb{L}(\mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{=n}}(\theta) &= \varprojlim_k((\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}/\mathcal{J}_n^k) \otimes^{\mathbb{L}} \mathcal{F})(\theta)\\
&= \varprojlim_k((\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}/\mathcal{J}_n^k)(\theta) \otimes \mathcal{P}_*(\theta))\\
&= \varprojlim_k(\mathcal{O}_{\mathrm{Spec}(R)}/\mathcal{J}_n(R)^k \otimes \theta^*\mathcal{P}_*)\\
&= \varprojlim_k((R/I_n(R)^k) \otimes_R \theta^*\mathcal{P}_*)\\
&= (\theta^*\mathcal{P}_*)^{\wedge}_{I_n(R)},
\end{aligned}$$

where $\theta^*\mathcal{P}_*$ is a chain complex of R-modules and the completion at $I_n(R)$ is level-wise (since at the beginning, the completion is of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}$ which is a chain complex concentrated in degree zero).

Let us test this computation on a simple example. Take $\mathcal{F}$ to be the sheaf arising from the $E(n)_*$-module $E(n)_*$ itself where $E(n)$ is the nth Johnson-Wilson theory. The projective resolution $\mathcal{P}_*$ can be taken to be just $E(n)_*$ concentrated in degree 0, and the result of evaluating at the point $\theta : \mathrm{Spec}(E(n)_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ is

$$\begin{aligned}
\mathcal{L}_{K(n)}\mathcal{F}(\theta) &= (\theta^* E(n)_*)^{\wedge}_{I_n(E(n)_*)}\\
&= (E(n)_*)^{\wedge}_{I_n}\\
&= \left(\mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm}]\right)^{\wedge}_{(p, v_1, \ldots, v_{n-1})}\\
&= \mathbb{Z}_p[\![v_1, \ldots, v_{n-1}]\!][v_n^{\pm}]
\end{aligned}$$

This is an algebraic shadow of the fact in Sp that $L_{K(n)}E(n)$, sometimes called the *completed Johnson-Wilson theory* and denoted $\hat{E}(n)$, is related to the homotopy fixed points of the Lubin-Tate theory E_n under the action of the subgroup $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ of $\mathbb{G}_n$.

At the level of spectra, there is a homotopy-cartesian square relating the $K(n)$ and $E(n)$ localizations of a spectrum X:

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

We can hope that our algebraic shadows of these spectrum-level localizations leads to a similar fracture square at the level of sheaves over $\mathcal{M}_{\mathrm{fg}}$. This is indeed the case:

Theorem 6.2.8 (Algebraic Fracture Square [Goe08, Theorem 8.17])**.** *Let $\mathcal{F}$ be a quasi-coherent sheaf over $\mathcal{M}_{fg}$. The following is a homotopy pullback square in the derived category of quasi-coherent sheaves over $\mathcal{M}_{fg}$:*

$$\begin{tikzcd} \mathcal{L}_n \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}_{K(n)} \mathcal{F} \arrow[d] \\ \mathcal{L}_{n-1} \mathcal{F} \arrow[r] & \mathcal{L}_{n-1} \mathcal{L}_{K(n)} \mathcal{F}, \end{tikzcd}$$

where the maps are obtained from those in Definition 6.2.3 and Definition 6.2.6 after identifying $\mathcal{L}_{n-1} \mathcal{F}$ with $\mathcal{L}_{n-1} \mathcal{L}_n \mathcal{F}$ via Lemma 6.2.5.

Remark 6.2.9. Written all out, the diagram above becomes

$$\begin{tikzcd} \mathbb{R}((i_n)_* i_n^*) \mathcal{F} \arrow[r] \arrow[d] & \mathbb{L}(\mathbb{R}((i_n)_* i_n^*) \mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \arrow[d] \\ \mathbb{R}((i_{n-1})_* i_{n-1}^*) \mathcal{F} \arrow[r] & \mathbb{R}((i_{n-1})_* i_{n-1}^*) \mathbb{L}(\mathbb{R}((i_n)_* i_n^*) \mathcal{F})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}, \end{tikzcd}$$

from which we can see that the fracture square is essentially recording the interactions between three of the localization functors we can apply to $\mathcal{F}$: base change to $\mathcal{M}_{\mathrm{fg}}^{\leq n}$, base change to $\mathcal{M}_{\mathrm{fg}}^{\leq n-1}$, and completion at an infinitesimal neighborhood of $\mathcal{M}_{\mathrm{fg}}^{=n}$.

To prove the theorem, it is enough to show that the map induced on the fibers of the vertical maps in the square above is an equivalence [Hov99, Remark 7.1.12]. As it turns out, these fibers can be interpreted using local cohomology.

Definition 6.2.10 ([Goe08, Definition 8.4])**.** Let $\mathcal{Z} \hookrightarrow \mathcal{M}$ be a closed substack with open complement $i : \mathcal{U} \to \mathcal{M}$. Let $\mathcal{E}$ be a quasi-coherent sheaf over $\mathcal{M}$. The *(derived) local cohomology sheaf* of $\mathcal{E}$, denoted $\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{E})$, is the homotopy fiber of the unit map $\mathcal{E} \to \mathbb{R}\, i_* i^* \,\mathcal{E}$. That is, the following is an exact triangle in the derived category of quasi-coherent sheaves over $\mathcal{M}$:

$$\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{E}) \to \mathcal{E} \to \mathbb{R}\, i_* i^* \,\mathcal{E} \to \mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{E})[1].$$

The *local cohomology* of $\mathcal{E}$ at $\mathcal{Z}$, written $H^*_{\mathcal{Z}}(\mathcal{M}, \mathcal{E})$, is the cohomology of $\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{E})$:

$$H^*_{\mathcal{Z}}(\mathcal{M}, \mathcal{E}) := H^* \,\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{E}).$$

We are interested in showing that the map

$$\mathbb{R}\,\Gamma_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}(\mathcal{L}_n \mathcal{F}) \to \mathbb{R}\,\Gamma_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}(\mathcal{L}_{K(n)} \mathcal{L}_n \mathcal{F})$$

induced on the fibers of the vertical maps in Theorem 6.2.8 is an equivalence. This will be a consequence of the following general lemma, after taking $\mathcal{E}$ to be $\mathcal{L}_n \mathcal{F}$, taking $\mathcal{Z} \subset \mathcal{M}$ to be the closed substack $\mathcal{M}_{\mathrm{fg}}^{\geq n} \subset \mathcal{M}_{\mathrm{fg}}$ defined by the ideal sheaf $\mathcal{I}_n$, and recalling by Lemma 3.2.29 that there is an *fpqc*-presentation

$$\varphi : \mathrm{Spec}(BP_*) \to \mathcal{M}_{\mathrm{fg}}$$

where by Remark 3.2.27, the ideal $\varphi^* \mathcal{I}_n$ in BP_* is generated by a (non-canonical) regular sequence $u_0, \dots, u_{n-1}$.

Lemma 6.2.11 ([Goe08, Theorem 8.16]). *Let $\mathcal{F}$ be a quasi-coherent sheaf over a stack $\mathcal{M}$ with affine* fpqc*-presentation $\varphi : \operatorname{Spec}(V) \to \mathcal{M}$, and let $\mathcal{Z} \subset \mathcal{M}$ be a closed substack defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{M}}$ such that $\varphi^* \mathcal{I} = (u_0, \ldots, u_{n-1})$ for some regular sequence $u_0, \ldots, u_{n-1}$ in V. Then the natural map*

$$\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{F}) \to \mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathbb{L}(\mathcal{F})^{\wedge}_{\mathcal{Z}})$$

is an equivalence in the derived category of quasi-coherent sheaves over $\mathcal{M}$.

Proof. Let $\mathcal{P}_* \to \mathcal{F}$ be a projective resolution. It is enough to show that the map in question is a levelwise equivalence, i.e., that

$$\mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathcal{P}_i) \to \mathbb{R}\,\Gamma_{\mathcal{Z}}(\mathbb{L}(\mathcal{P}_i)^{\wedge}_{\mathcal{Z}})$$

is an equivalence for each i. We can further reduce to proving this locally: pulling back along the *fpqc*-cover $\varphi : \operatorname{Spec}(V) \to \mathcal{M}$, the statement to prove becomes that

$$\mathbb{R}\,\Gamma_I(P) \to \mathbb{R}_I(P^{\wedge}_I)$$

is an equivalence in the derived category of V-modules, where $I = \varphi^* \mathcal{I}$ is the ideal in V generated by the regular sequence $u_0, \ldots, u_{n-1}$, and $P = \varphi^* \mathcal{P}_i$ is a projective V-module.

From here, the proof relies on a series of technical identifications. To prove each of them fully requires some machinery to do with regular sequences and Koszul resolutions. For brevity, we will simply give an outline of the steps involved and refer the reader to [Goe08, Section 8.3].

Because I is generated by a regular sequence $u_0, \ldots, u_{n-1}$, [Goe08, Lemma 8.5] gives a natural equivalence

$$\operatorname{colim}_k \mathbb{R}\mathrm{Hom}(R/(u_0^k, \ldots, u_{n-1}^k), M) \to \mathbb{R}\,\Gamma_I(M)$$

for M either P or $P^{\wedge}_I$. At each k, taking cohomology gives

$$H^*\,\mathbb{R}\mathrm{Hom}(R/(u_0^k, \ldots, u_{n-1}^k), M) = \mathrm{Ext}^*_R(R/(u_0^k, \ldots, u_{n-1}^k), M),$$

and by a computation, $\mathrm{Ext}^*_R(R/(u_0^k, \ldots, u_{n-1}^k), M) = 0$ whenever $* \neq n$ for M either P or $P^{\wedge}_I$. On the other hand, in degree n, we consider the map on Ext induced by the completion map $P \to P^{\wedge}_I$:

$$\mathrm{Ext}^n_R(R/(u_0^k, \ldots, u_{n-1}^k), P) \to \mathrm{Ext}^n_R(R/(u_0^k, \ldots, u_{n-1}^k), P^{\wedge}_I)$$

[Goe08, Equation 8.9]. By another computation, this map on Ext is identified with the R-linear completion map

$$P/(u_0^k, \ldots, u_{n-1}^k) \to P^{\wedge}_I/(u_0^k, \ldots, u_{n-1}^k)$$

which is an isomorphism. We conclude that

$$\operatorname{colim}_k \mathbb{R}\mathrm{Hom}(R/(u_0^k, \ldots, u_{n-1}^k), P) \simeq \operatorname{colim}_k \mathbb{R}\mathrm{Hom}(R/(u_0^k, \ldots, u_{n-1}^k), P^{\wedge}_I),$$

and so $\mathbb{R}\,\Gamma_I(P) \simeq \mathbb{R}\,\Gamma_I(P^{\wedge}_I)$, as desired. □

Unfortunately, there is not a good way of understanding telescopic localization of spectra in terms of a localization of quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$. This is essentially because our method of producing a quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$ from a spectrum X is to understand the BP-module $BP \otimes X$—the problem is that the localizations L_n and L_n^f coincide for BP-modules:

Theorem 6.2.12 ([Hov93, Corollary 1.10]). *For any spectrum X, the natural map*

$$L_n^f X \to L_n X$$

is a BP-equivalence.

This is to say that we cannot distinguish $T(n)$- and $K(n)$-localizations of spectra at the level of $\mathcal{M}_{\mathrm{fg}}$ (at least, not as we have defined $\mathcal{M}_{\mathrm{fg}}$ in this document—there is a more sophisticated construction of $\mathcal{M}_{\mathrm{fg}}$ as a functor on the category of $\mathbb{E}_\infty$-rings over which we might hope to distinguish $T(n)$- and $K(n)$-localizations as completion and "hypercompletion" at a sufficient $\mathbb{E}_\infty$-analogue of the ideal $\mathcal{I}_n$).

6.3. **Chromatic convergence**

Finally, we mention an analogue of the classical chromatic convergence theorem over $\mathcal{M}_{\mathrm{fg}}$. Recall that the classical theorem of Hopkins-Ravenel says that for any finite spectrum X, the natural map

$$X \to \varprojlim L_n X$$

is an equivalence. This theorem is especially strong because it is true at the "topological"-level. Our algebraic version over $\mathcal{M}_{\mathrm{fg}}$ is slightly weaker since it is a statement made at the level of BP-homology.

Theorem 6.3.1 (Algebraic Chromatic Convergence [Goe08, Theorem 8.22])**.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{M}_{fg}$. Then the natural map*

$$\mathcal{F} \to \varprojlim \mathcal{L}_n \mathcal{F}$$

is an equivalence in the derived category of $\mathcal{O}_{\mathcal{M}_{fg}}$-modules.

Remark 6.3.2. The theorem states that any coherent sheaf over $\mathcal{M}_{\mathrm{fg}}$ can be recovered from its restrictions to finite height open substacks and the maps relating these restrictions. There is indeed something to check here since the above expression does not involve any information to do with the infinite-height point

$$\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\mathrm{fg}}$$

classifying the mod-p additive formal group.

Proof. From the definition of local cohomology, for $\mathcal{F}$ a (coherent) $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-module, the homotopy fiber of the localization map $\mathcal{F} \to \mathcal{L}_n \mathcal{F}$ in the derived category of $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$-modules is

$$\mathbb{R}\,\Gamma_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \mathcal{F} \to \mathcal{F} \to \mathcal{L}_n \mathcal{F}\,.$$

Since (homotopy) limits commute with (homotopy) limits, to prove the theorem, it is enough to show that the limit of these local cohomologies vanishes:

$$\varprojlim \mathbb{R}\,\Gamma_{\mathcal{M}_{\mathrm{fg}}^{\geq n}}(\mathcal{F}) \simeq 0.$$

The key observation is that if $\mathcal{F}$ is coherent, then it is is the pullback of some coherent $\mathcal{F}_0$ along the map

$$\mathcal{M}_{\mathrm{fg}} \to \mathcal{M}_{\mathrm{fg}}\left\langle r \right\rangle,$$

where $\mathcal{M}_{\mathrm{fg}}\left\langle r \right\rangle$ is the *moduli of r-buds of formal groups* [Goe08, Theorem 3.27]. We will not discuss r-buds of (p-typical) formal groups except to say that they correspond to r-buds of formal group laws which are classified by ring maps out of truncated Brown-Peterson spectra:

$$BP\left\langle r \right\rangle_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_r],$$

and so are the finite "buds" of honest formal groups.

The upshot of this theory is that for $n > r$, the localizations $\mathcal{L}_{n+1} \mathcal{F}$ and $\mathcal{L}_n \mathcal{F}$ coincide. It follows that the map on local cohomology groups

$$H^*_{\mathcal{M}_{\mathrm{fg}}^{\leq n+1}}(\mathcal{M}_{\mathrm{fg}}, \mathcal{F}) \to H^*_{\mathcal{M}_{\mathrm{fg}}^{\leq n}}(\mathcal{M}_{\mathrm{fg}}, \mathcal{F})$$

induced by the map $\mathcal{L}_{n+1} \mathcal{F} \to \mathcal{L}_n \mathcal{F}$ is 0 [Goe08, Theorem 8.20]. This proves the result. ☐

Appendix A. Functorial geometry

A.1. **A brief picture of higher category theory**

Definition A.1.1. A *groupoid* is a (small) category in which all morphisms are isomorphisms.

Remark A.1.2. A groupoid is determined by a set O of objects and set M of morphisms and functions as in the following diagram, subject to the obvious coherence relations:

$$\begin{tikzcd} O \arrow[r, "\mathsf{id}"] & M \arrow[l, bend right, "\mathsf{codomain}"'] \arrow[l, bend left, "\mathsf{domain}"] \arrow[loop below, "(\ )^{-1}"] & O \times_M O \arrow[l, "\circ"'] \end{tikzcd}$$

Our goal in this section is not to give an introduction to ∞-categories, but rather to solidify some of the notations and conventions we will use going forward. With that in mind, we make the following approximate definition.

Definition A.1.3. An ∞-*category* is a simplicial set which satisfies a "weak horn filling property" [Lur09, Definition 1.1.2.4]. An ∞-*groupoid* is an ∞-category which is also a Kan complex, i.e., which satisfies a "strong horn filling property."

In particular, ∞-groupoids are the ∞-categories in which all morphisms of all orders are invertible.

Remark A.1.4. Definition A.1.3 might more accurately be stated as a definition of $(\infty, 1)$-categories, i.e., categories where every n-morphism with $n > 1$ is invertible. Most theory of ∞-categories is developed to handle $(\infty, 1)$-categories. Dropping the assumption that higher order morphisms are invertible quickly leads to extreme technical issues e.g. with defining coherent associativity axioms (see the discussion at [Lur09, page 5]).

Remark A.1.5 (∞-groupoids as spaces)**.** An extremely convenient heuristic in higher category theory is to attempt to understand ∞-groupoids homotopically as "spaces." Indeed, every ∞-groupoid is the fundamental ∞-groupoid of some topological space X, and the homotopy type of X can be recovered from the fundamental ∞-groupoid via geometric realization. Thus, studying ∞-groupoids is the "same" as doing homotopy theory [Lur09, Example 1.1.1.4].

This is the content of the *homotopy hypothesis*, which is the statement that the fundamental ∞-groupoid construction

$$\Pi_{\leq\infty} : \mathsf{An} \to \mathsf{Gpd}_\infty \tag{A.1.6}$$

is an equivalence of categories between the category An of *anima*, or "homotopy types," obtained from Top by simplicial localization at weak equivalence of spaces, and Gpd_∞, the ∞-category of ∞-groupoids.

The phrase "homotopy hypothesis" stems from a guiding heuristic of Grothendieck that any concrete model for the category of ∞-groupoids should satisfy this equivalence to "spaces" induced by the fundamental ∞-groupoid construction. (An interesting historical note: Grothendieck originally discussed this heuristic in his letters to Quillen which comprise the beginning of his 1983 monograph *Pursuing Stacks*, [Gro83].)

Indeed, there are specific models for "∞-groupoids" where this equivalence is a theorem, including if ∞-groupoids are taken to be Kan complexes as in Definition A.1.3. In this formulation, the homotopy hypothesis can be proven using the model category structures on Top and SSet and the adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\mathrm{Sing}"] & \mathsf{SSet} \arrow[l, shift left, "|-|"] \end{tikzcd}.$$

This is developed in Joyal's theory of *quasi-categories*.

In what follows, we will use the terms "anima," "homotopy type," "∞-groupoid," and (somewhat abusively) "space" interchangeably. Likewise, we will use both An_1 and Gpd as models of a category of "homotopy 1-types," or (again, abusively) "1-truncated spaces."

Finally, we will need the following fundamental construction which gives a natural way to "upgrade" an ordinary 1-category into an ∞-category.

Definition A.1.7 ([Lur09, Page 9])**.** Let $\mathcal{C}$ be a (small) category. The *nerve* of $\mathcal{C}$ is the ∞-category which is defined, as a simplicial set, to have

$$N(\mathcal{C})_n := \mathsf{Fun}([n], \mathcal{C})$$

with face and degeneracy maps induced by the various order-preserving maps $[n] \to [n+1]$ and $[n] \to [n-1]$, respectively.

Remark A.1.8. Note that $N(\mathcal{C})$ contains precisely "the same" information as $\mathcal{C}$ itself—higher morphisms in $N(\mathcal{C})$ witness the arithmetic of composition in $\mathcal{C}$. For example, given two composable morphisms f and g in $\mathcal{C}$, there is a unique 2-morphism $\eta : [2] \to \mathcal{C}$ as in the following picture:

$$\begin{tikzcd} & * \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ * \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & * \end{tikzcd}$$

Intuitively, the 2-morphisms in $N(\mathcal{C})$ witness composition in $\mathcal{C}$, the 3-morphisms witness the associativity property of composing three morphisms in $\mathcal{C}$, the 4-morphisms witness the associativity of composing four morphisms, and so on.

Our primary use for the nerve construction will be to embed our ordinary categories into an ∞-categorical context. The reason to do this is that some of the yoga of studying sheaves and stacks simplifies when they are equipped with "infinitely much" coherence data.

A.2. **Sites and Sheaves**

We now turn to the matter of defining sheaves on categories. Essentially, a sheaf on a category behaves like a sheaf on a topological space, after the category is equipped with the right notion of a "topology."

After outlining the classical 1-categorical notion of sheaves, we will show how the definition extends to the ∞-categorical setting. We will mostly restrict our attention to ∞-categories arising as the nerves of 1-categories, since this is the context in which most sheaves (and later, stacks) in chromatic homotopy theory arise.

Definition A.2.1 ([Ols16, Definition 2.1.2]). Let $\mathcal{C}$ be a category. A (*Grothendieck*) *topology* on $\mathcal{C}$ is an association to every object X in $\mathcal{C}$ a set of collections of morphisms $\{U_i \to X\}_{i\in I}$ called *covers* for X with the following properties:

(i) (*identity cover*) If $U \to X$ is an isomorphism, then the singleton $\{U \to X\}$ is a cover for X.
(ii) (*transitivity*) If $\{U_i \to X\}_{i\in I}$ is a cover for X and $\{V_{ij} \to U_i\}_{j\in J_i}$ is a cover for U_i for each i, then the set of compositions
$$\{V_{ij} \to X\}_{i\in I, j\in J_i}$$
is also a cover for X.
(iii) (*stability under base change*) If $\{U_i \to X\}_{i\in I}$ is a cover for X, then for any map $Y \to X$,
$$\{U_i \times_X Y \to Y\}_{i\in I}$$
is a cover for Y.

A category $\mathcal{C}$ equipped with a choice of topology is called a *site*.

The fundamental motivating example of sites is the following:

Example A.2.2. Let X be a topological space and $\mathsf{Op}(X)$ the poset category which has the open subsets of X as objects and subspace inclusions as morphisms. The collections of open covers $\{U_i \to A\}_{i\in I}$ for each open subspace $A \subset X$ make $\mathsf{Op}(X)$ into a site.

Definition A.2.3. A *presheaf* on a category $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$.

Definition A.2.4 ([Ols16, Definition 2.2.2(ii)]). A *sheaf* on a site $\mathcal{C}$ is a presheaf $\mathcal{F}$ such that for any object $X \in \mathcal{C}$ and any cover $\{U_i \to X\}$, the natural dashed arrow in the following diagram is a bijection:
$$\mathcal{F}(X) \overset{\cong}{\dashrightarrow} \varprojlim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_X U_j)\right).$$

Example A.2.5. A standard first example is the sheaf of continuous, functions on a topological space X valued in some other topological space Y,
$$C_Y(-) : \mathsf{Op}(X)^{\mathsf{op}} \to \mathsf{Set}$$
which sends each open subspace $U \subset X$ to the set of continuous functions $C_Y(U) = \mathsf{Top}(U, Y)$.

The functor $C_Y(-)$ being a presheaf encodes the fact that functions on U can be restricted to functions on a further open subspace $V \subset U$. That $C_Y(-)$ is a sheaf encodes the fact that functions on U and U' which restrict to the same function on $U \cap U'$ can be "glued" uniquely to obtain a function on $U \cup U'$.

Remark A.2.6. For any site $\mathcal{C}$, we will write $\mathsf{Shv}(\mathcal{C})$ (respectively, $\mathsf{Psh}(\mathcal{C})$) for the category of sheaves (presheaves) on $\mathcal{C}$ where the morphisms are natural transformations. If $\mathcal{C}$ is small, there is a "localization functor" $\mathsf{Psh}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$ called *sheafification* which is left adjoint to the inclusion $\mathsf{Shv}(\mathcal{C}) \hookrightarrow \mathsf{Psh}(\mathcal{C})$ [Lur09, Lemma 6.2.2.7]. A category is called a *topos* if it is equivalent to the category of sheaves on some site.

Remark A.2.7. We will want to study sheaves over $\mathcal{C}$ valued not just in Set, but in other more structured categories. Such definitions can be made internally to the topos $\mathsf{Shv}(\mathcal{C})$ in the following way. (See [Ols16, Remark 2.2.11].)

For example, a *sheaf of abelian groups* on a site $\mathcal{C}$ is an abelian group object in $\mathsf{Shv}(\mathcal{C})$, i.e. a sheaf $\mathcal{F}$ with multiplication and unit maps

$$\mathcal{F} \times \mathcal{F} \to \mathcal{F}$$
$$* \to \mathcal{F}$$

satisfying the axioms of an abelian group. (Here, $*$ denotes the constant singleton-valued sheaf on $\mathcal{C}$.) By the Yoneda lemma, this abelian group object structure is equivalent to providing a lift of $\mathcal{F}$ through the forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$:

$$\begin{tikzcd} & \mathsf{Ab} \arrow[d, "U"] \\ \mathcal{C}^{\mathsf{op}} \arrow[ur, dashed] \arrow[r, "\mathcal{F}"'] & \mathsf{Set}\,. \end{tikzcd}$$

Many more definitions of structured sheaves are possible in this vein. If $\mathsf{Shv}(\mathcal{C})$ is monoidal, we have a notion of sheaves of modules as the modules over some monoid object $\mathcal{O}$, and if $\mathsf{Shv}(\mathcal{C})$ is additive, we can speak of sheaves of rings.

Remark A.2.8. Each category $\mathcal{C}$ can be given a topology called the *canonical topology* which is the finest topology with respect to which every representable presheaf is a sheaf. That is, for every object X in $\mathcal{C}$, the presheaf $\mathrm{Hom}(-, X)$ is a sheaf with respect to the canonical topology, and any topology on $\mathcal{C}$ with "more" covering families does not have this property. (A precise definition of the canonical topology is outlined in [Ols16, Exercise 2.N].) Any topology on $\mathcal{C}$ with respect to which all representable presheaves are sheaves is called *subcanonical*.

Intuitively, topologies with "fewer" covering families have "fewer conditions" for a presheaf to be a sheaf, meaning more presheaves satisfy the sheaf condition. The weakest topology—i.e. the topology which has "the most" sheaves—is the *discrete topology* in which the covering families consist only of isomorphisms. With respect to the discrete topology, every presheaf is a sheaf.

For intuition, the following diagram records the relative strengths of some common topologies on the category of affine schemes which nest between the discrete and canonical topologies. The inclusions here denote inclusion of covering families (i.e. if $A \subset B$, then any cover in A is a cover in B).

$$\text{discrete} \subset \text{crystalline} \subset \text{Zariski} \subset \text{Nisnevich} \subset \text{étale} \subset \textit{fppf} \subset \textit{fpqc} \subset \text{canonical}$$
$$\leftarrow \text{“more presheaves are sheaves”} \qquad\qquad \text{“more covering families”} \rightarrow$$

We will discuss the *fpqc*-topology in Section A.4.

Having indicated the classical categorical constructions of sites and sheaves, we now turn briefly to their ∞-categorical generalizations. All the following machinery is a direct generalization of the above in the sense that if $\mathcal{C}$ is an ordinary category and $N(\mathcal{C})$ its nerve, there are natural correspondences between topologies and sheaves on $\mathcal{C}$ and the objects we will define below: ∞-topologies and ∞-sheaves on $N(\mathcal{C})$ [Lur09, Remark 6.2.2.3]. Since, in this document, we are primarily concerned with sheaves on the 1-category Ring, almost of of our examples will fit into this framework. With this being our motivation, we will only consider ∞-sheaves over ∞-sites arising from the nerves of ordinary categories. This will allow us to be more concrete and to avoid some of the extra subtlety which comes from defining sheaves over a general ∞-category.

One advantage of the ∞-categorical setting is (roughly) that it allows us to make certain sheaf-y constructions (e.g., Definition A.3.8) in a natural way without the need to "re-sheafify," as we might in the 1-categorical setting.

We first describe the ∞-categorical generalization of sites.

Definition A.2.9 ([Lur09, Definition 6.2.2.1])**.** For $\mathcal{C}$ an ∞-category and X an object of $\mathcal{C}$, a *sieve* on X is a full subcategory $\mathcal{C}'_{/X}$ of $\mathcal{C}_{/X}$ such that if $U \to X$ is in $\mathcal{C}'_{/X}$ and morphism $V \to U$ is any morphism over X in $\mathcal{C}$, then V is also in $\mathcal{C}'_{/X}$.

Given a morphism $f : Y \to X$ in $\mathcal{C}$, we also define the *pullback sieve* $f^* \mathcal{C}'_{/X}$ to be the subcategory of $\mathcal{C}_{/Y}$ with objects $U \times_X Y \to Y$ arising by pulling back objects $U \to X$ of $\mathcal{C}'_{/X}$ along f.

A (*Grothendieck*) *topology* on the ∞-category $\mathcal{C}$ is an association to every object X in $\mathcal{C}$ a collection of distinguished sieves on X, called *covering sieves*, with the following properties:

(i) (*identity sieve*) $\mathcal{C}_X$ is a covering sieve for X,
(ii) (*stability under base change*) if $f : Y \to X$ is a morphism in $\mathcal{C}$ and $\mathcal{C}'_{/X}$ a covering sieve for X, then $f^* \mathcal{C}'_{/X}$ is a covering sieve for Y,
(iii) (*gluing property/transitivity*) if $\mathcal{C}'_{/X}$ is a covering sieve for X and $\mathcal{C}''_{/X}$ any sieve with the property that for each $f : U \to X$ in $\mathcal{C}'_{/X}$, the pullback $f^* \mathcal{C}''_{/X}$ is a covering sieve for U, then $\mathcal{C}''_{/X}$ is a covering sieve.

An ∞-category with a topology is called an ∞-*site*.

Remark A.2.10. To gain intuition for this definition, we can think of the analogy to the site of open sets, $\mathsf{Op}(X)$, for X a topological space as in Example A.2.2. For A and open subset of X, a covering sieve on A is a collection of opens $\mathcal{U} = \{U_i\}_{i \in I}$ which is

(i) an open cover for A, i.e. $A \subset \bigcup_{i \in I} U_i$,
(ii) downwards-closed with respect to the poset structure on $\mathsf{Op}(X)$, i.e., if U is in $\mathcal{U}$ and V is an open subset of U, then V is also in $\mathcal{U}$.

Recall that in Example A.2.2, we required only (i) and not (ii). The advantage of discussing covering sieves (as opposed to ordinary covers) is that they allow us access to all levels of "intersection data" of the covering objects.

Remark A.2.11. We note that the above definition exactly coincides with the notion of a (1-categorical) Grothendieck topology on $h\mathcal{C}$. The advantage of this formulation is that it is "more canonical" and does not rely on set-theoretic notions related to constructing covering families and so is more adaptable to large ∞-categories.

With this intuition in mind, we make the following construction.

Definition A.2.12. Given a 1-categorical site $\mathcal{C}$ admitting fiber products, we can define a topology on $N(\mathcal{C})$ by taking the covering sieves of an object X to be the free sieves generated by each of the covering families $\{U_j \to X\}$ in $\mathcal{C}$. We will call the resulting ∞-site $N(\mathcal{C})$ the *nerve* of the site $\mathcal{C}$.

Let us now turn to the higher-categorical generalizations of sheaves. Thinking of sets as (small) "0-categories," it becomes natural to extend the notion of sheaf to that of a 2-*sheaf.* A 2-sheaf is a functor $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$ that now must satisfy a local coherence condition one level higher than a sheaf. Specifically, for any $X \in \mathcal{C}$, there must be a cover $\{U_i \to X\}$ so that the the natural dashed arrow in the following diagram is an equivalence of categories:

$$\mathcal{F}(X) \overset{\simeq}{\dashrightarrow} \varprojlim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_X U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \mathcal{F}(U_i \times_X U_j \times_X U_k) \right).$$

From the point of view of higher category theory, there is no reason to stop here. This leads us to the following definition.

Definition A.2.13. Let $\mathcal{C}$ be the nerve of some 1-categorical site. An ∞-*sheaf* on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ such that for any $X \in \mathcal{C}$ and any covering sieve on X generated by a covering family $\{U_i \to X\}$, the natural dashed arrow in the following diagram is an equivalence:

$$\mathcal{F}(X) \overset{\simeq}{\dashrightarrow} \varprojlim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_X U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \mathcal{F}(U_i \times_X U_j \times_X U_k) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

We will denote the limit term on the right-hand side of this diagram by

$$\mathrm{Tot}_n(\mathcal{F}(U_i^{\times[n]}))$$

and call it the *totalization* of the cosimplicial object $\mathcal{F}(U_i^{\times[n]})$ in Cat_∞.

A.3. Stacks

In the previous section, we gave a very general definition of sheaves as functors valued in Cat_∞ satisfying a sheaf condition. We now turn our attention to the particularly important case of sheaves valued in ∞-groupoids: in this section, we will treat the general theory of stacks over a site. After developing the theory of schemes in Section A.4, we will discuss the most important case for our purposes of stacks over the category of affine schemes in Section A.5.

We begin with a definition.

Definition A.3.1. A *stack* on a site $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Gpd}$ which is a 2-sheaf. An ∞-*stack* on $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Gpd}_\infty$ which is an ∞-sheaf.

An intuitive first example of a stack is the *classifying stack* of a group scheme discussed in Example A.5.5. We will postpone this example until then, when we will have defined a sufficient notion of "scheme."

Remark A.3.2. Following Remark A.1.5, we may equivalently think of an ∞-stack as a "sheaf of spaces," or a "sheaf of anima." Moreover, under the equivalence of (A.1.6), ordinary 1-groupoids are identified with homotopy 1-types, i.e., anima with vanishing homotopy groups above dimension 1. This allows us to think of ordinary stacks as sheaves of (1-truncated) spaces as well.

Note that a stack $\mathcal{X}$ over a 1-site $\mathcal{C}$ can be "upgraded" to an ∞-stack over $N(\mathcal{C})$ in a natural way, and the resulting ∞-stack $\tilde{X}$ is, up to equivalence, valued in homotopy 1-types as well. We will make this notion of equivalence precise below in Definition A.3.11.

This system of identifications provides convenient homotopical heuristics for thinking about stacks and the ways they behave like spaces themselves. In particular, it gives valuable intuition for quotient stacks, as we examine in Section A.5.

Remark A.3.3. As in the case of presheaves and sheaves discussed in Remark A.2.6, under nice conditions on the site $\mathcal{C}$, there is a localization functor called *stackification* which is left adjoint to the inclusion of the category of stacks on $\mathcal{C}$ into the category of sheaves of simplicial sets on $\mathcal{C}$. We define this functor precisely below.

Definition A.3.4. Let $\mathcal{X} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$ be a simplicial presheaf on a site $\mathcal{C}$, i.e., $\mathcal{X}$ is a presheaf on $\mathcal{C}$ valued in simplicial sets:

$$\begin{tikzcd} \mathcal{X}(U)_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{X}(U)_1 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathcal{X}(U)_2 \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

The *associated stack* of $\mathcal{X}$ is the functor $\mathcal{M}_\mathcal{X} : \mathcal{C} \to \mathsf{An}$ sending each $U \in \mathcal{C}$ to the geometric realization of $\mathcal{X}(U)$. With respect to the model structure on the category of simplicial presheaves discussed in [Hol08], $\mathcal{M}_\mathcal{X}$ is the homotopy colimit

$$\mathcal{M}_\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n \,.$$

For our purposes, the following explicit 1-categorical computation of an associated stack is useful. In particular, it highlights the advantage of thinking about homotopy-coherent ∞-stacks instead of just ordinary sheaves. For more discussion, see [Pet19, Rmk 3.1.17].

Lemma A.3.5. *Let X_0 and X_1 be sheaves of sets on a site $\mathcal{C}$ so that for each $U \in \mathcal{C}$, the pair $(X_0(U), X_1(U))$ form the objects and morphisms of a groupoid in the sense of Remark A.1.2. After identifying* $\mathsf{Gpd} \simeq \mathsf{An}_{\leq 1}$*, the stackification (or* associated stack *of the pair (X_0, X_1)) is the stack $X_0 /\!\!/ X_1$ on $\mathcal{C}$ defined on each $U \in \mathcal{C}$ to be the homotopy pushout*

$$\begin{tikzcd} X_1(U) \arrow[r, "\mathsf{range}"] \arrow[d, "\mathsf{domain}"'] & X_0(U) \arrow[d] \\ X_0(U) \arrow[r] & (X_0 /\!\!/ X_1)(U) \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

Remark A.3.6. Note that for any object U in a (∞-)site $\mathcal{C}$ with a subcanonical topology, $\operatorname{Hom}(-, U)$ is naturally a stack.

For a stack $\mathcal{X}$ over a site $\mathcal{C}$ with subcanonical topology, we can define the *slice category* $\mathcal{C}_{/\mathcal{X}}$ to be the category that has objects which are the morphisms of stacks $\operatorname{Hom}(-, U) \to \mathcal{X}$ for U an object in $\mathcal{C}$, and morphisms which are commuting triangles:

$$\begin{tikzcd} \operatorname{Hom}(-, U) \arrow[rr] \arrow[dr] & & \operatorname{Hom}(-, V) \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

We can identify $(\mathcal{C}_{/\,\mathcal{X}})^{\mathsf{op}}$ as the *category of points* of $\mathcal{X}$ in the following way. The objects of the category of points (the *points* of $\mathcal{X}$) are the pairs (U, a) with $U \in \mathcal{C}$ and $a \in \mathcal{X}(U)$ and the morphisms are those $f : U \to V$ in $\mathcal{C}$ such that $\mathcal{X}(F)(a) = b$. The Yoneda lemma, gives a correspondence between objects of $\mathcal{C}_{/\,\mathcal{X}}$ and points of $\mathcal{X}$, and similarly between morphisms in $\mathcal{C}_{/\,\mathcal{X}}$ and morphisms between points of $\mathcal{X}$.

This language gives a convenient way to interpret the following definitions.

Definition A.3.7. Let $\mathcal{X}$ be a stack over $\mathcal{C}$. A *sheaf over* $\mathcal{X}$ is a sheaf on the site $\mathcal{C}_{/\,\mathcal{X}}$.

Definition A.3.8 ([Lur09, Definition 6.5.1.1])**.** For $\mathcal{X}$ a stack over a site $\mathcal{C}$, for each $n \geq 0$, we define the *sheaf of homotopy groups* $\pi_n\,\mathcal{X} : (\mathcal{C}_{/\,\mathcal{X}})^{\mathsf{op}} \to \mathsf{Grp}$ to be the sheafification of the presheaf of groups

$$(\pi_n\,\mathcal{X})(U, a) = \pi_n(\mathcal{X}(U), a).$$

(Note that for $n = 0$, this is actually a sheaf of (pointed) sets.)

A natural first computation of homotopy sheaves is of the classifying stack of a group scheme. We discuss this in Example A.5.5.

Remark A.3.9. If we are willing to forget that π_0 is valued in *pointed* sets, not sets, then we have an easier, otherwise equivalent definition of $\pi_0\,\mathcal{X}$ as the sheaf of sets on $\mathcal{C}$ defined at each $U \in \mathcal{C}$ by

$$(\pi_0\,\mathcal{X})(U) = \pi_0(\mathcal{X}(U)_+).$$

This exactly recovers the sheaf of "path components" or "isomorphism classes" of $\mathcal{X}$, which is a useful notion in itself. We will implicitly use this simpler definition when talking about π_0 in what follows.

The reason we cannot make such a simplification for the definition of the higher homotopy sheaves has to do with the fact that the positive degree homotopy groups π_n are basepoint-dependent.

As we would hope, this assignment of sheaves of homotopy groups has functorial properties: given two stacks $\mathcal{X}_1$ and $\mathcal{X}_2$ on $\mathcal{C}$ and a map $g : \mathcal{X}_1 \to \mathcal{X}_2$, there is an induced map relating the sheaves of homotopy groups. Constructing this induced map takes a bit of care though, since $\pi_n\,\mathcal{X}_1$ and $\pi_n\,\mathcal{X}_2$ are sheaves over different sites.

To construct the desired map, first note that from g, we get a pushforward functor $g_* : \mathcal{C}_{/\,\mathcal{X}_1} \to \mathcal{C}_{/\,\mathcal{X}_2}$ which is a map of sites in the sense of [Ols16, Definition 2.2.20]. This induces a map of sheaf topoi

$$g^* : \mathsf{Shv}(\mathcal{C}_{/\,\mathcal{X}_2}) \to \mathsf{Shv}(\mathcal{C}_{/\,\mathcal{X}_1})$$

which sends a sheaf of groups $\mathcal{F}$ over $\mathcal{C}_{/\,\mathcal{X}_2}$ to the sheaf $g^*\,\mathcal{F}$ over $\mathcal{C}_{/\,\mathcal{X}_1}$ defined at each $(U, a) \in (\mathcal{C}_{/\,\mathcal{X}_1})^{\mathsf{op}}$ by the formula

$$(g^*\,\mathcal{F})(U, a) := \mathcal{F}(g_*(U, a)) = \mathcal{F}(U, g_U(a)),$$

where by g_U, we denote the functor of (∞-)groupoids $\mathcal{X}_1(U) \to \mathcal{X}_2(U)$ obtained by evaluating g at U. In fact, we can view g_U as a map of pointed spaces:

$$g_U : (\mathcal{X}_1(U), a) \to (\mathcal{X}_2(U), g_U(a)),$$

and so g_U induces a map on homotopy groups

$$\pi_n(g_U) : \pi_n(\mathcal{X}_1(U), a) \to \pi_n(\mathcal{X}_2(U), g_U(a)).$$

Thus, we have constructed a natural map of sheaves of groups over $\mathcal{C}_{/\,\mathcal{X}_1}$:

$$\pi_n(g) : \pi_n\,\mathcal{X}_1 \to g^*(\pi_n\,\mathcal{X}_2) \tag{A.3.10}$$

We summarize this construction in the following diagram. (Take care to note the distinction between "$\mapsto$" which denotes the application of a functor, and "$\to$" which denotes maps of pointed spaces or groups.)

$$\begin{tikzcd}
 & (U, a) \arrow[r, mapsto, "g_*"] \arrow[d, mapsto, "\mathcal{X}_1"'] & (U, g_U(a)) \arrow[d, mapsto, "\mathcal{X}_2"] & \\
 & (\mathcal{X}_1(U), a) \arrow[r, "g_U"] \arrow[d, mapsto, "\pi_n"'] & (\mathcal{X}_2(U), g_U(a)) \arrow[d, mapsto, "\pi_n"] & \\
(\pi_n\,\mathcal{X}_1)(U, a) \; := & \pi_n(\mathcal{X}_1(U), a) \arrow[r, "\pi_n(g_U)"] & \pi_n(\mathcal{X}_2(U), g_U(a)) & =: \; (g^*(\pi_n\,\mathcal{X}_2))(U, a)
\end{tikzcd}$$

Definition A.3.11. We say two stacks $\mathcal{X}_1, \mathcal{X}_2 : \mathcal{C}^{\mathsf{op}} \to \mathsf{An}$ are *equivalent* if there is a morphism $w : \mathcal{X}_1 \to \mathcal{X}_2$ such that the natural map $\pi_n(w) : \pi_n \mathcal{F}_1 \to w^*(\pi_n \mathcal{F}_2)$ of A.3.10 is an isomorphism of sheaves of groups over $\mathcal{C}_{/\mathcal{X}_1}$ for all $n \geq 0$ [Hol08, Definition 5.1].

In particular, two sheaves of groupoids have vanishing higher homotopy groups and so are equivalent if there is a map between them inducing isomorphisms on π_0 and π_1.

Remark A.3.12. A different definition of equivalent sheaves of groupoids is given in [Hov01, Definition 3.1] using more concrete language. There, an equivalence of sheaves of groupoids is defined to be a map $w : \mathcal{F}_1 \to \mathcal{F}_2$ such that

(i) w is *locally essentially surjective*, i.e., for each $U \in \mathcal{C}$, there is a cover $\{f_i : U_i \to U\}$ such that for any object b in $\mathcal{F}_2(U)$, there are collections of objects $\{b_i \in \mathcal{F}_2(U)\}$ with $\mathcal{F}_2(f_i)(b_i) = b$ and $\{a_i \in \mathcal{F}_1(U)\}$ with $w(U_i)(a_i) \cong b_i$;
(ii) for each $U \in \mathcal{C}$, the functor $w(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ is fully faithful.

Note that (i) is a sheaf-theoretic weakening of the condition that $w(U)$ be essentially surjective for each U: we only require that $w(U)$ becomes essentially surjective after passing to some cover.

Lemma A.3.13 ([Hol08, Corollary 5.11])**.** *The notion of equivalent sheaves of groupoids of Definition A.3.11 agrees with the one in Remark A.3.12.*

The theory of homotopy groups of sheaves and stackification are nicely packaged up in a model structure on the category of sheaves of groupoids on a site, due to [Hol08]. We summarize some key properties of this model structure in the following theorem.

Theorem A.3.14 (Model structure on $\mathsf{Shv}^{\mathsf{Gpd}}(\mathcal{C})$ [Hol08, Propositions 4.4, 5.10])**.** *There is a model structure on the category of sheaves of groupoids on a site $\mathcal{C}$ with the following properties:*

(i) weak equivalences are the maps inducing isomorphisms on sheaves of homotopy groups,
(ii) stacks are the fibrant objects, and
(iii) stackification is fibrant replacement.

Corollary A.3.15. *Given two equivalent sheaves of groupoids, there is a zigzag of equivalences between their stackifications.*

Proof. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two sheaves of groupoids on $\mathcal{C}$, and let $\tilde{\mathcal{X}}_1$ and $\tilde{\mathcal{X}}_2$ be their stackifications. Suppose $w : \mathcal{X}_1 \to \mathcal{X}_2$ is an equivalence. With respect to the model structure of Theorem A.3.14, stackification is fibrant replacement. Thus, we have a zigzag of equivalences:

$$\tilde{\mathcal{X}}_1 \leftarrow \mathcal{X}_1 \xrightarrow{w} \mathcal{X}_2 \to \tilde{\mathcal{X}}_2.$$ □

In Section 4, we will also make use of the following very powerful theorem.

Theorem A.3.16 ([Hov01, Theorem 4.5])**.** *Sheaves of groupoids equivalent on the* fpqc*-site have equivalent categories of quasi-coherent sheaves.*

Remark A.3.17. We will not prove the above theorem here, but the following is an indication of a way in which it is a very natural, homotopical fact from the perspective of higher topos theory.

The theorem is essentially a version of Whitehead's theorem in the setting of the (∞-)topos of sheaves of spaces on the *fpqc*-site (whereas the usual Whitehead theorem is true in the category An, which we might think of the ∞-topos of sheaves of spaces over a point). A theorem like this holds in great generality, and leads to the theory of *hypercomplete sheaves*, that is (roughly) sheaves which are local with respect to morphisms inducing isomorphisms on homotopy groups.

With this machinery in place, the theorem then follows from the fact that sheaves of groupoids are hypercomplete, as indeed are any n-truncated objects of an ∞-topos [Lur09, Lemma 6.5.2.9].

Corollary A.3.18 ([Hop99, Prop 11.5])**.** *The category of quasi-coherent sheaves over groupoid sheaf is equvalent to quasi-coherent sheaves over its stackification.*

Proof. From Theorem A.3.14, stackification is fibrant replacement. The result then follows from Theorem A.3.16. □

A.4. **Schemes**

We will give a brief introduction to the functorial geometry of schemes, following [Goe08, Section 1.1].

Definition A.4.1. Let R be a ring. An *affine scheme* over R is a represenatable presheaf over $\mathsf{Alg}_R^{\mathsf{op}}$. We denote the affine scheme represented by an R-algebra A by $\mathrm{Spec}(A)$, so that

$$\mathrm{Spec}(A)(B) = \mathrm{Hom}_{\mathsf{Alg}_R}(A, B).$$

We will write write Aff_R for the full subcategory of representable presheaves in $\mathsf{Psh}(\mathsf{Alg}_R^{\mathsf{op}})$. If R is unspecified, we assume $R = \mathbb{Z}$.

Remark A.4.2. The Yoneda embedding $\mathsf{Ring}^{\mathsf{op}} \to \mathsf{Aff} \subset \mathsf{Psh}(\mathsf{Ring}^{\mathsf{op}})$ sending R to $\mathrm{Hom}_{\mathsf{Ring}}(R, -)$ gives an equivalence of categories:

$$\mathrm{Spec} : \mathsf{Ring}^{\mathsf{op}} \rightleftarrows \mathsf{Aff} : \mathrm{Hom}_{\mathsf{Aff}}(-, \mathbb{A}^1),$$

where $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$. Indeed, for any ring R, by the Yoneda lemma,

$$\mathrm{Hom}_{\mathsf{Aff}}(\mathrm{Spec}(R), \mathbb{A}^1) \cong \mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], R) \cong R.$$

We can define a topology, called the *Zariski topology*, on an affine scheme $\mathrm{Spec}(A)$ in the following way. If I is an ideal of A, we let the define an "open" subfunctor of $\mathrm{Spec}(A)$ by

$$U_I(B) = \{f \in \mathrm{Spec}(A)(B) : f(I)B = B\}$$

with "closed" complement subfunctor

$$Z_I(B) = \{f \in \mathrm{Spec}(A)(B) : f(I)B = 0\}.$$

(Note that set theoretically, $U_I(B) \cup Z_I(B) = \mathrm{Spec}(A)(B)$ only if A is a field.)

For a general presheaf $X : \mathsf{Alg}_R \to \mathsf{Set}$, we say a subfunctor $U \subset X$ is "open" if for all $\mathrm{Spec}(B) \to X$, the restriction

$$U \times_X \mathrm{Spec}(B) \to \mathrm{Spec}(B)$$

is open. A collection of open subfunctors $\{U_i \to X\}$ is called an open cover if

$$\coprod_i U_i(k) \to X(k)$$

is surjective for all fields k.

Definition A.4.3. A *scheme* over R is a presheaf over $\mathsf{Alg}_R^{\mathsf{op}}$ which has an open cover by affine R-schemes and which is a sheaf with respect to the *Zariski topology* on Aff_R, i.e., for each R-algebra B and $b_1, \ldots, b_n$ generating the unit ideal, the natural dashed arrow below is a bijection:

$$X(B) \dashrightarrow \mathsf{eq}\Big(\prod_i X(B[b_i^{-1}]) \rightrightarrows \prod_{i,j} X(B[b_i^{-1}, b_j^{-1}])\Big).$$

We denote the category of schemes over R by Sch_R. As with affine schemes, when R is left unspecified, we assume $R = \mathbb{Z}$.

Remark A.4.4. The equalizer condition in Definition A.4.3 goes by the name *Zariski descent*. It's precise formulation will not be so important to think about for us. The upshot of this perspective is that we can think of schemes as sheaves over Aff or simply as functors from Ring to Set satisfying "certain conditions."

Definition A.4.5. Let X be a scheme, and R a ring. We call the elements of the set $X(R)$ the *R-points* of X. If k is an algebraically closed field, then the k-points of X are called *geometric points*. Note that each map from a field determines a geometric point.

Remark A.4.6. Of course, this construction of the category of schemes is quite different from the classical one which begins with studying the geometry of the prime ideal spectrum of a ring. While the classical story has the advantage of being more concrete and affords geometric intuition as a direct generalization of algebraic varieties, this functorial formulation better suits our needs. Indeed, as a branch of math, homotopy theory is largely concerned with universal properties and functorial constructions. Such concepts are made more accessible through the perspective on schemes we take here. (To be sure, the two perspectives are equivalent, i.e., the category of "schemes" is the same in both cases. For example, the underlying locally-ringed space of a scheme as we have defined can be obtained by putting a topology on its set of geometric points. See, e.g. [Goe08, Remark 1.3].)

Remark A.4.7. A sheaf over a scheme X is a functor $\mathcal{F} : \mathsf{Aff}^{\mathsf{op}}_{/X} \to \mathsf{Set}$. Since $\mathsf{Aff}^{\mathsf{op}} = \mathsf{Ring}$ is additive, $\mathcal{F}$ is naturally a sheaf of abelian groups.

Definition A.4.8. The *structure sheaf* of a scheme X is the sheaf of rings

$$\mathcal{O}_X : \mathsf{Aff}_{/X} \to \mathsf{Ring}$$

defined on each affine open $U : \mathrm{Spec}(B) \to X$ by $\mathcal{O}_X(U) = B$. This definition extends to a sheaf on X by the sheaf condition.

Example A.4.9. For an affine scheme $\mathrm{Spec}(A)$, evaluating $\mathcal{O}_{\mathrm{Spec}(A)}$ on the open subfunctor $\mathrm{Spec}(A)$ returns A itself. As we saw in Remark A.4.2, this is the same as applying $\mathrm{Hom}(-, \mathbb{A}^1)$ to $\mathrm{Spec}(A)$. For a general scheme X, the ring $\mathcal{O}_X(X)$ is called the *ring of global sections* of X, and is always isomorphic to $\mathrm{Hom}(X, \mathbb{A}^1)$.

Definition A.4.10. A sheaf of abelian groups $\mathcal{F}$ over a scheme X is an $\mathcal{O}_X$*-module* if for each open U in X, the abelian group $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module and this module structure is natural in the topology on X, i.e., and for any morphism of open subfunctors $U \to V$ of X, the induced map of abelian groups $\mathcal{F}(V) \to \mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module map.

The category of $\mathcal{O}_X$-modules form an abelian category, denoted $\mathsf{Mod}(\mathcal{O}_X)$.

Definition A.4.11. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module sheaf over a scheme X. We say $\mathcal{F}$ is *quasi-coherent* if there is a cover of X by open subschemes $f_i : U_i \to X$ an exact sequence of $\mathcal{O}_{U_i}$-modules:

$$\bigoplus \mathcal{O}_{U_i} \to \bigoplus \mathcal{O}_{U_i} \to f_i^* \, \mathcal{O}_X \to 0.$$

The following lemma is essential for passing between homological algebra and algebraic geometry.

Lemma A.4.12 (Serre)**.** *For a ring R, the categories of R-modules and quasi-coherent sheaves over* $\mathrm{Spec}(R)$ *are equivalent:*

$$\mathsf{Mod}(R) \simeq \mathsf{QCoh}(\mathrm{Spec}(R)).$$

This lemma extends to the setting where R is a co-simplicial ring and $\mathrm{Spec}(R)$ is a simplicial scheme:

Lemma A.4.13. *For $R_{[*]}$ a cosimplicial ring, there is an equivalence of categories:*

$$\mathsf{Mod}(R_{[*]}) \simeq \mathsf{QCoh}(\mathrm{Spec}(R_{[*]})).$$

Remark A.4.14. Later, we will be interested in discussing many of the above concepts over more general geometric objects than schemes, such as stacks. While we will not give them here, the definitions in these more general contexts are similar.

We now recall some relevant objects, facts, and constructions to help us translate objects from homotopy theory into algebraic geometry.

Definition A.4.15. A *group scheme* is a group object in the category of schemes.

Example A.4.16. The *multiplicative group* is the affine group scheme

$$\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[t^{\pm}])$$

with group structure induced by the map $\mathbb{Z}[t^{\pm}] \to \mathbb{Z}[t^{\pm}] \otimes \mathbb{Z}[t^{\pm}]$ sending t to $t \otimes t$. For any ring R, the R-points of $\mathbb{G}_m$ are naturally identified with the units in R:

$$\mathbb{G}_m(R) = R^{\times}.$$

Lemma A.4.17 ([Pst21, Proposition 6.5])**.** *For any ring R, the following pieces of data are equivalent:*

(i) An even grading on R, that is, a direct sum decomposition

$$R \cong \bigoplus_{i \in \mathbb{Z}} R_{2i}$$

which respects the multiplication: $R_{2i} \cdot R_{2j} \subset R_{2(i+j)}$.

(ii) A $\mathbb{G}_m$ action on $\mathrm{Spec}(R)$*.*

Proof. Note that if R is concentrated in even degrees, it is honestly commutative (rather than graded commutative), and so it makes sense to consider $\mathrm{Spec}(R)$.

Suppose R is an even-graded ring. Define a map $R \to R[x^{\pm}]$ by sending each r in r to $rx^{|r|/2}$. After identifying $R[x^{\pm}] \cong \mathbb{Z}[x^{\pm}] \otimes R$, we apply Spec to obtain a map $\mathbb{G}_m \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$. One can check that this map defines a group action.

On the other hand, if R is ungraded, a $\mathbb{G}_m$-action on $\mathrm{Spec}(R)$ determines a map $\mathbb{G}_m \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ which corresponds to one of the following form in the category of rings:

$$R \to R \otimes \mathbb{Z}[x] \cong R[x^{\pm}].$$

Define an even grading on R by setting the elements in degree $2n$ to be those r which are sent to rx^n by this map. One can check that this determines an even-grading on R. $\square$

Remark A.4.18. If we were content to allow our graded rings to be commutative instead of graded commutative, the previous lemma could be stated as "a grading on R is equivalent to a $\mathbb{G}_m$ action on $\mathrm{Spec}(R)$". The purpose restricting to even gradings is to make this statement while respecting the Koszul sign convention. Fortunately for us in the context of chromatic homotopy theory, many graded rings we encounter are even-graded already, so we are free to translate many of our statements into the $\mathbb{G}_m$-equivariant scheme setting.

We will occasionally want to refer to the notion of a *groupoid scheme*, by which we mean a pair of schemes (X_0, X_1) which together have the structure of a presheaf of groupoids on Aff in the sense of Remark A.1.2. That is, a groupoid scheme is a groupoid object in the category of schemes. To each groupoid scheme, Definition A.3.4 gives an associated stack. Of particular importance to us is the following affine case.

Definition A.4.19. A *Hopf algebroid* is a pair of rings (A, Γ) equipped with the necessary structure maps which make $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ into a groupoid scheme.

A *(left)* (A, Γ)*-comodule* is an A-module M with an A-linear map $M \to \Gamma \otimes_A M$ satisfying counit and coassociativity properties.

Lemma A.4.20 ([Rav86, Theorem A1.1.3])**.** *If Γ is flat as a left A-module, then the category of (A, Γ)-comodules is abelian.*

A key observation for us will be the following.

Lemma A.4.21. *Let (A, Γ) be a Hopf algebroid and M an (A, Γ)-comodule. There is a natural assignment of M to a cosimplicial module over the cosimplicial ring $\Gamma^{\otimes [n]}$, and this construction gives an equivalence of categories*

$$\mathsf{Comod}(A, \Gamma) \xrightarrow{\simeq} \mathsf{Mod}(\Gamma^{\otimes [*]}).$$

We will not prove this lemma in detail, but the intuition is that Hopf algebroids are groupoid objects forming the cosimplicial ring on the right hand side is analagous to the nerve construction on groupoids. That is, no "additional information" is added by extending a Hopf algebroid to a cosimplicial ring.

A.5. Stacks in the *fpqc* topology

In Remark A.2.8, we compared the relative strengths of several common topologies on Aff without defining them. In Definition A.4.3, we defined schemes to be a certain subcategory of sheaves on Aff with the Zariski topology. For our purposes, we will primarily be interested in the *fpqc* topology, which we define next.

Definition A.5.1 ([Goe08, Definition 1.10])**.** The *fpqc topology* on Sch is the topology with morphisms consisting of flat maps of schemes. That is, an *fpqc*-cover of an affine scheme U is a finite set of affine schemes $\{U_i\}$ and flat morphisms $\{U_i \to U\}$ such that

$$\coprod_i U_i \to U$$

is surjective. An *fpqc*-cover of an arbitrary scheme X is a finite collection of morphisms $\{V_i \to X\}$ which restrict to an *fpqc*-cover on each affine open U in X. That is, for each such U,

$$\{V_i \times_X U \to U\}$$

is an *fpqc*-cover.

The *fpqc site on* X is $\mathsf{Sch}_{/X}$ equipped with the *fpqc*-topology.

Remark A.5.2. The name *fpqc*, which stands for faithfully flat and quasi compact (in French), is actually a slight misnomer, as morphisms in covering families need not be quasi-compact. The reason for the terminology is the following theorem: any jointly-surjective collection of flat and quasicompact morphisms of schemes $\{X_i \to X\}$ form an *fpqc*-cover for X [Goe08, Proposition 1.11]. But not all *fpqc*-covers arise in this way. For this reason, the *fpqc*-topology is sometimes also refered to as the *flat topology* on Sch. However, this alternate term has its own disadvantage of being mistook for the related *fppf*-topology.

Definition A.5.3. Let X be a scheme equippied with an action of a group scheme $\mathbb{G}$. Thinking of X as a sheaf of (discrete) spaces, the *homotopy quotient* of X by $\mathbb{G}$ is the stack $X /\!\!/ \mathbb{G}$, that is, the colimit

$$\operatorname{colim}\left(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X \times \mathbb{G} \times \mathbb{G} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X \times \mathbb{G} \rightrightarrows X \right) \xrightarrow{\simeq} X /\!\!/ \mathbb{G}$$

taken in the ∞-category of *fpqc*-sheaves of spaces.

Remark A.5.4. Taking the colimit in the previous definition in the ∞-category of all spaces is essential. For an aritrary ring R, the R-points of $X /\!\!/ \mathbb{G}$ can be computed as the homotopy quotient $X(R)/\,\mathbb{G}(R)$ which may not be finite dimensional or of finite type, even if $X(R)$ and $\mathbb{G}(R)$ are. An example of this is given next.

Example A.5.5. Let $\mathbb{G}$ be a group scheme over some base scheme X. The *classifying stack* $B\mathbb{G}$ for $\mathbb{G}$ in the category $\mathsf{Sch}_{/X}$ is the quotient $X /\!\!/ \mathbb{G}$ with respect to the trivial $\mathbb{G}$-action on X. The Y-points of $B\mathbb{G}$ classify principle $\mathbb{G}$-bundles over Y.

This is reminiscent of the classifying space construction for a discrete group, G, where BG is the geometric realization of the nerve of G. Like with $B\mathbb{G}$, maps from a space Y into BG classify principle G-bundles over Y.

The analogy extends to homotopy groups, as well. For any finite group scheme $\mathbb{G}$, the classifying stack $B\mathbb{G}$ satisfies

$$\pi_n(B\mathbb{G}(R), *) = \begin{cases} * & n = 0 \\ \mathbb{G}(R) & n = 1 \\ * & n > 1. \end{cases}$$

for every ring R. Again, this resembles the classifying space construction: when G is a discrete group, BG has the homotopy type of the Eilenberg-Maclane space $K(G, 1)$.

For the purposes of doing algebraic geometry, we prefer our stacks to be relatively well behaved. One such restriction we will want to put on stacks is the following.

Definition A.5.6 ([Sta25, Tag 04TI], [Goe08, Definition 2.29])**.** Let $\mathcal{M}$ be a stack over a base scheme X. We say $\mathcal{M}$ is an *fpqc algebraic stack* if the following hold:

- (i) the diagonal morphism $\mathcal{M} \to \mathcal{M} \times_X \mathcal{M}$ is representable, separated, and quasicompact;
- (ii) there is a scheme X and a surjective, flat, and quasi-compact morphism $X \to \mathcal{M}$. This morphism is called a *presentation* of $\mathcal{M}$.

Example A.5.7. Under nice conditions, the map $X \to X /\!\!/ \mathbb{G}$ of Definition A.5.3 gives a presentation of $X /\!\!/ \mathbb{G}$ as an *fpqc*-algebraic stack. An example is seen in Theorem 3.1.4.

Lemma A.5.8 ([Pst21, Remark 3.3])**.** *An* fpqc *stack is a colimit of affine schemes in the ∞-category of* fpqc-*sheaves of spaces.*

Proof. We will not prove the lemma in detail, but the key idea is akin to the fact that a topological space is a "colimit of its points." ∎

Appendix B. Formal groups

B.1. Formal groups and their local structure

Definition B.1.1. The *formal spectrum* of a topological R-algebra A is the *fpqc*-sheaf $\operatorname{Spf}(A) : \mathsf{Alg}(R) \to \mathsf{Set}$ with T-points given by continuous R-algebra maps from A to T

$$\operatorname{Spf}(A)(T) := \operatorname{Hom}_R^{\text{cts}}(A, T)$$

for each (discrete) R-algebra T.

The case we will care the most about is when our topological ring R is obtained by completing some ring at an ideal and equipping it with the inverse limit topology. In particular, we will care about the following.

Definition B.1.2. The *formal affine line* over a ring R, written $\hat{\mathbb{A}}^1_R$, is the formal spectrum of the completion of $R[x]$ at the ideal (x); that is, the formal spectrum of $R[\![x]\!]$ equipped with the (x)-adic topology:

$$\hat{\mathbb{A}}^1_R := \mathrm{Spf}(R[\![x]\!]).$$

Remark B.1.3. Let T be a (discrete) R-algebra. An R-linear map $f : R[\![x]\!] \to T$ is determined by where it sends x, so let us examine the possible values for $f(x)$ if f is to be continuous. With respect to the (x)-adic topology on $R[\![x]\!]$, the sequence $\{x^n\}_{n\geq 0}$ converges to 0 in $R[\![x]\!]$ as $n \to \infty$. So, if f is continuous, then the sequence $\{f(x^n)\}_{n\geq 0}$ must converge to $f(0) = 0$. Since T is discrete, this forces $f(x^n) = 0$ for all n larger than some integer N. Thus, $f(x)$ is nilpotent in T since $f(x)^N = f(x^N) = 0$. We conclude that there is a natural bijection of sets

$$\mathrm{Spf}(R[\![x]\!])(T) = \mathrm{Nil}(T).$$

With this definition in place, we are prepared to give a definition of our central objects of study.

Definition B.1.4. A (1-dimensional) *formal group* over a base scheme S is an *fpqc*-sheaf G over S which is

(i) a group object, i.e., G is equipped with a multiplication map $G \times_S G \to G$ which makes $\mathrm{Hom}_S(X, G)$ into a group for any other *fpqc*-sheaf $X \to S$

(ii) isomorphic to the formal affine line Zariski-locally on S. That is, there exists an affine open cover $\{\mathrm{Spec}(R_j)\}$ of S so that

$$\mathrm{Spec}(R_j) \times_S G \cong \hat{\mathbb{A}}^1_{R_j}$$

for each j.

A *morphism of formal groups* $G \to H$ is a natural transformation of group objects.

Remark B.1.5. The choice of affine open cover of S over which G is isomorphic to $\hat{\mathbb{A}}^1$ is generally far from being unique, and even over the same affine open, the choice of isomorphism is not canonical since it can be twisted by an automorphism of $\hat{\mathbb{A}}^1$.

A useful analogy to have in mind is of 1-manifolds and their local homeomorphisms to $\mathbb{R}^1$. Local charts on a manifold let us pick local coordinates, and in the same sense, the local identification of a formal group G with $\hat{\mathbb{A}}^1_R = \mathrm{Spf}(R[\![x]\!])$ gives us a local coordinate x for G. But just as local charts for manifolds can be reparameterized, so can these "local charts" for formal groups.

The ability to choose local coordinates on a formal group $G \to S$ lets us be very explicit about the group structure. Consider the multiplication map $\mu : G \times_S G \to G$, and let $\mathrm{Spec}(R) \hookrightarrow S$ be an affine open subscheme over which G is isomorphic to $\hat{\mathbb{A}}^1_R$. Base changing μ to $\mathrm{Spec}(R)$, we obtain a multiplication map on $\hat{\mathbb{A}}^1_R$ over $\mathrm{Spec}(R)$ as in the following diagram:

$$\begin{tikzcd}
\hat{\mathbb{A}}^1_R \times_{\mathrm{Spec}(R)} \hat{\mathbb{A}}^1_R \arrow[r] \arrow[d] \arrow[dr, dashed] & \hat{\mathbb{A}}^1_R \times_S \hat{\mathbb{A}}^1_R \arrow[r, "i\times i"] & G \times_S G \arrow[d, "\mu"] \\
\mathrm{Spec}(R) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R) \arrow[dr, bend right, dash, "="'] & \hat{\mathbb{A}}^1_R \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d] \\
& \mathrm{Spec}(R) \arrow[r] & S
\end{tikzcd}$$

After choosing a coordinate on each copy of $\hat{\mathbb{A}}^1_R$, we get a sequence of identifications

$$\begin{aligned}
\hat{\mathbb{A}}^1_R \times_{\mathrm{Spec}(R)} \hat{\mathbb{A}}^1_R &\cong \mathrm{Spf}(R[\![x]\!]) \times_{\mathrm{Spec}(R)} \mathrm{Spf}(R[\![y]\!]) \\
&\cong \mathrm{Spf}(R[\![x]\!] \mathbin{\hat{\otimes}}_R R[\![y]\!]) \\
&\cong \mathrm{Spf}(R[\![x, y]\!]),
\end{aligned}$$

and the multiplication becomes a map

$$\mathrm{Spf}(R[\![x, y]\!]) \to \mathrm{Spf}(R[\![t]\!])$$

which on rings of global sections is determined by an R-linear map from $R[\![t]\!]$ to $R[\![x,y]\!]$. The image of t under this map is a power series $F(x,y) \in R[\![x,y]\!]$ called a *formal group law*. The fact that $F(x,y)$ locally induces a group structure on G puts restrictions on the coefficients of $F(x,y)$ which we make into an intrinsic definition as follows.

Definition B.1.6. A *formal group law* over a ring R is a power series $F(x,y) \in R[\![x,y]\!]$ with the following properties:

(i) (*unital*) $F(x,0) = F(0,x) = x$,
(ii) (*associative*) $F(F(x,y),z) = F(x,F(y,z))$,
(iii) (*commutative*) $F(x,y) = F(y,x)$,
(iv) (*inverse*) There is a power series $i \in R[\![t]\!]$ such that $F(x,i(x)) = 0$.

A *morphism of formal group laws* $F(x,y) \to F'(x,y)$ is a power series $h(t) \in R[\![t]\!]$ such that

$$h(F(x,y)) = F'(h(x),h(y)).$$

We say $F(x,y)$ and $F'(x,y)$ are *isomorphic* if an invertible such power series $h(t)$ exists and *strictly isomorphic* if, moreover, $h'(0) = 1$.

Remark B.1.7. If R is reduced, then properties (i) and (ii) of Definition B.1.6 imply property (iii). Regardless, we will always assume our formal groups and formal group laws are abelian.

Property (iv) is always implied by the first three [Rav86, Proposition A2.1.2].

Definition B.1.8. Let $F(x,y) \in R[\![x,y]\!]$ be a formal group law over R. The formal group *associated to* or *presented by* $F(x,y)$ is the formal group $G_F := \mathrm{Spf}(R[\![t]\!])$ over $\mathrm{Spec}(R)$ where the multiplication on $\mathrm{Spf}(R[\![t]\!])$ is induced by the R-linear map $R[\![t]\!] \to R[\![x,y]\!]$ sending t to $F(x,y)$.

Conversely, we say a formal group G over $\mathrm{Spec}(R)$ is *presented by* a formal group law if it is isomorphic to G_F for some formal group law $F(x,y) \in R[\![x,y]\!]$.

Specifically, if G is a formal group over R presented by a formal group law F, then $G \cong \mathrm{Spf}(R[\![x]\!])$ as group objects where for any R-algebra T, the group structure on $\mathrm{Spf}(R[\![x]\!])(T) = \mathrm{Nil}(T)$ is defined by

$$t_1 +_F t_2 := F(t_1, t_2)$$

for $t_1, t_2 \in \mathrm{Nil}(T)$, where the right-hand side represents evaluating the power series F with addition and multiplication in T. We remark that the nilpotents of T are in some sense "exactly the right elements" to consider since they are the elements with will lead to only finitely many nonzero terms in F, making the above power series evaluation well-defined.

Example B.1.9. We record some examples of formal group laws and their associated formal groups.

(i) The *additive formal group* over R is the formal group $\hat{\mathbb{G}}_a$ associated to the formal group law

$$F_a(x,y) = x + y.$$

(ii) The *multiplicative formal group* over R is the formal group $\hat{\mathbb{G}}_m$ associated to the formal group law

$$F_m(x,y) = x + y + xy.$$

The reason for the name "multiplicative" is the following connection to the ordinary multiplicative group $\mathbb{G}_m$ (Example A.4.16). Note that $\hat{\mathbb{G}}_m(T) = \mathrm{Nil}(T)$ can be equivalently identified as

$$\hat{\mathbb{G}}_m(T) = \{1 + t : t \in \mathrm{Nil}(T)\},$$

and under this identification the group structure is given by ordinary multiplication in T.

(iii) Given a 1-dimensional (abelian) Lie group G with a choice of coordinate in a neighborhood U of the identity given by a diffeomorphism $\psi : U \to \mathbb{R}$, we can obtain a formal group law

$$F(x,y) = \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)) \in \mathbb{R}[\![x,y]\!]$$

where $\mu : G \times G \to G$ is the multiplication map on G. Note that the formal group associated to F is not generally G itself, but some infinitesimal neighborhood of the identity in G. (Recall that any *open* neighborhood of the identity in G generates the identity component under multiplication, and the identity component may not have a global choice of coordinate.)

(iv) Similarly, to an elliptic curve E over a field k we can associate a formal group law by studying the multiplication on E in an infinitessimal neighborhood of the base point $\mathrm{Spec}(k) \to E$. A detailed construction of this formal group law is given in [Sil86, Chapter 4.1]. This large family of examples is one of the original inspirations for the connection between chromatic homotopy theory and arithmetic geometry.

Remark B.1.10. It is possible for a formal group $G \to \mathrm{Spec}(R)$ to be presented by multiple formal group laws. Indeed, for any unit u in R, the formal group law $F_u(x, y) = x + y + uxy$ presents $\hat{\mathbb{G}}_m$. The formal group laws F_u are all isomorphic to F_m under the invertible power series

$$h_u(t) : F_m \to F_u$$

sending t to $u^{-1}t$. At the level of formal schemes, we can think of this $h(t)$ as a "change of coordinate" on $\hat{\mathbb{G}}_m$:

$$\hat{\mathbb{G}}_m \cong \mathrm{Spf}(R\llbracket u^{-1}t\rrbracket) \xrightarrow{\mathrm{Spf}(h(t))} \mathrm{Spf}(R\llbracket t\rrbracket) \cong \hat{\mathbb{G}}_m.$$

This is an example of the motivation for the analogy to Lie groups given in Remark B.1.5.

Remark B.1.11. Let us take a moment to observe the difference between the additive group $\mathbb{G}_a$, which is defined to be $\mathbb{G}_a(R) = R^+$ where R^+ is the additive abelian group underlying R, and the additive *formal* group, $\hat{\mathbb{G}}_a$.

For one thing, following Remark B.1.3, we have $\hat{\mathbb{G}}_a(R) = 0$ for any reduced ring, but $\mathbb{G}_a$ is only 0 on the zero ring, so $\hat{\mathbb{G}}_a$ and $\mathbb{G}_a$ are indeed different. But certainly we have a map $\hat{\mathbb{G}}_a \to \mathbb{G}_a$ induced by the natural inclusion $\mathrm{Nil}(R) \hookrightarrow R$. We can ask for a picture of how $\hat{\mathbb{G}}_a$ sits inside $\mathbb{G}_a$.

Consider the ring $T = k[\varepsilon]/\varepsilon^3$. As an abelian group, $\mathbb{G}_a(T)$ is a 3-dimensional k-vector space with basis $\{1, \varepsilon, \varepsilon^2\}$. On the other hand, as a set $\hat{\mathbb{G}}_a(T) = \mathrm{Nil}(T)$ consists of all the k-linear combinations of the elements ε and ε^2, and under the additive formal group law, $\hat{\mathbb{G}}_a(T)$ is a 2-dimensional k-vector space with basis $\{\varepsilon, \varepsilon^2\}$. That is, $\hat{\mathbb{G}}_a(T)$ is the vector subspace of $\mathbb{G}_a(T)$ spanned by the "infinitesimal" (nilpotent) directions ε and ε^2. We illustrate this in the following picture.

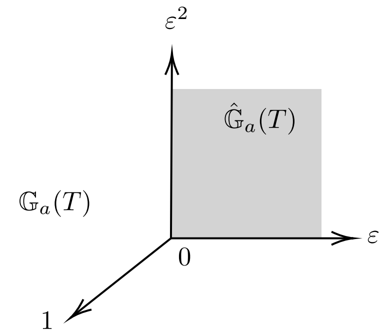

FIGURE B.1.12. The T-points of the formal group $\hat{\mathbb{G}}_a$ as a k-linear subspace of the T-points of $\mathbb{G}_a$.

In general, we can think of $\hat{\mathbb{G}}_a$ as an "infinitesimal neighborhood" of the origin inside $\mathbb{G}_a$. A similar remark can be made about the distinction between the multiplicative group, $\mathbb{G}_m$, and the multiplicative *formal* group, $\hat{\mathbb{G}}_m$.

Remark B.1.13. When $(R, \mathfrak{m})$ is a local ring, any formal group over R is equivalent to one associated to a formal group law over R. This is because property (ii) of Definition B.1.4 is Zariski-local on $\mathrm{Spec}(R)$, and $\mathrm{Spec}(R)$ itself is the only open neighborhood of the closed point corresponding to $\mathfrak{m}$.

B.2. **Invariant differentials**

In this section we will introduce the sheaf of invariant differentials associated to a formal group. Following with the analogy between formal groups and Lie groups, the sheaf of invariant differentials is like the "Lie algebra" of a formal group.

We begin by defining the sheaf of Kähler differentials.

Definition B.2.1 ([Sta25, Tag 01UR])**.** Let $f : X \to Y$ be morphism of schemes (or, e.g., of *fpqc*-sheaves). The *sheaf of Kähler differentials of* f (when it exists) is the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$ defined by the universal property that for each $\mathcal{O}_X$-module $\mathcal{F}$, the natural map

$$d^*_{X/Y} : \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \to \mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$$

is a bijection, where the right-hand object is the set of $\mathcal{O}_Y$-module derivations from $\mathcal{O}_X$ to $\mathcal{F}$, and $d_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$ is the universal $\mathcal{O}_Y$-derivation.

Another way to phrase this definition is that the functor $\mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, -)$ is (sometimes) corepresentable in the category of $\mathcal{O}_X$-modules, and the corepresenting object, when it exists, is $\Omega_{X/Y}$.

Example B.2.2. Consider the map of *fpqc*-sheaves $\hat{\mathbb{A}}^1 \to \mathrm{Spec}(R)$ (as in the case of a formal group over R presented by a formal group law). In this context, maps from $\Omega_{\hat{\mathbb{A}}^1/R}$ to another R-module M correspond to the set of R-linear derivations $R[\![x]\!] \to M$.

The universal R-linear derivation is identified with the differential dx, and so the natural bijection in the above definition identifies an R-module map $f : \Omega_{\hat{\mathbb{A}}^1/R} \to M$ with the derivation $f(x)dx : R[\![x]\!] \to M$. That is, $\Omega_{\hat{\mathbb{A}}^1/R}$ is freely-generated as an $R[\![x]\!]$-module by dx:

$$\Omega_{\hat{\mathbb{A}}^1/R} \cong R[\![x]\!]\{dx\}$$

Next, we will also give a couple alternative ways of thinking about Kähler differentials. The first is the following lemma which gives an intuition that $\Omega_{X/Y}$ somehow tracks how X "sits inside" its self-intersection over Y.

Lemma B.2.3 ([Sta25, Tag 01UR])**.** *Let $f : X \to Y$ be a morphism of schemes. The sheaf of Kähler differentials $\Omega_{X/Y}$ is canonically isomorphic to the conormal sheaf of the diagonal*

$$X \to X \times_Y X.$$

Alternatively, we can think about $\Omega_{X/Y}$ as a sheaf recording all the ways the morphism $X \to Y$ can be extended to an "infinitesimal thickening" of X. This perspective will be especially useful for us as we pass to the world of formal schemes.

To make this precise, let R be a k-algebra and M and R-module. We can put a k-algebra structure on $R \oplus M$ with multiplication defined by

$$(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

In particular, the elements $(0, m)$ in $R \oplus M$ square to 0, and so the retract

$$R \oplus M \to R$$

has nilpotent kernel. In this sense, we can think of the closed inclusion

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(R \oplus M)$$

as a "nilpotent thickening."

With this language in place, we are prepared to give an alternate definition of the sheaf of Kähler differentials.

Definition B.2.4 ([Pst21, Definition 4.15])**.** Let $X \to \mathrm{Spec}(k)$ be a morphism of *fpqc*-sheaves. The *sheaf of Kähler differentials*, when it exists is the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/k}$ with the property that for every point $f : \mathrm{Spec}(R) \to X$ and R-module M, there is a natural bijection between

$$\mathrm{Hom}_R(f^*\Omega_{X/k}, M)$$

and the set of extensions of f to a nilpotent thickening of $\mathrm{Spec}(R)$, as in the following diagram:

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\
\mathrm{Spec}(R \oplus M) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(k)
\end{tikzcd}$$

We are interested in talking about sheaves of differentials over a formal group $G \to \mathrm{Spec}(R)$. But as we see from the definition, $\Omega_{G/R}$ has nothing to do with the multiplicative structure on G; this is reminiscent of the fact that the bundle of differential forms on a Lie group depends only on its structure as a smooth manifold. But just we can define invariant vector fields on Lie groups to fix this issue, we can define invariant differentials over formal groups.

Definition B.2.5. A *differential 1-form* on (*fpqc*-)sheaf X over $\mathrm{Spec}(R)$ is a global section ω of $\Omega_{X/R}$. We denote the R-module of 1-forms by $\Omega^1_{X/R}$.

When $G \to \mathrm{Spec}(R)$ is a formal group presented by a formal group law F so that G is identified with $\hat{\mathbb{A}}^1$, and $\Omega_{G/R}$ is identified with the free $R[\![x]\!]$-module on the generator dx (as in Example B.2.2), a differential on G is equivalent to a choice of some $\omega(x) = f(x)dx$ with $f(x) \in R[\![x]\!]$. We say ω is *invariant* (with respect to the multiplication on G) if

$$\omega \circ F(x, y) = \omega(x).$$

That is, if

$$f(F(x,y))\frac{\partial F}{\partial x}(x,y) = f(x)dx$$

[Sil86, Section IV.4].

We make this property of being invariant into a coordinate free definition for general formal groups.

Definition B.2.6 ([Pst21, Definition 5.6])**.** Let $G \to \mathrm{Spec}(R)$ be a formal group with multiplication map m, and let p_1 and p_2 be the two coordinate projections $G \times_{\mathrm{Spec}(R)} G \to G$. We say a differential $\omega \in \Omega^1_{G/R}$ is *invariant* if

$$m^*\omega = p_1^*\omega$$

as elements of $\Omega^1_{G\times_{\mathrm{Spec}(R)}G/G}$, the differentials with respect to p_1.

Note that the set of invariant differentials of a formal group $G \to \mathrm{Spec}(R)$ has a natural R-module structure.

Definition B.2.7 ([Pst21, Definition 5.16])**.** Let $G \to \mathrm{Spec}(R)$ be a formal group. The *sheaf of invariant differentials* ω_G is the quasi-coherent sheaf over $\mathrm{Spec}(R)$ which associates to any $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the S-module of invariant differentials of the formal group $f^*G \to \mathrm{Spec}(S)$.

The following lemma completes the analogy between the sheaf of invariant differentials of a formal group and the (dual) Lie algebra of a Lie group by identifying ω_G with the R-module of differentials at the identity.

Lemma B.2.8 ([Pst21, Lemma 5.17])**.** *Let $G \to \mathrm{Spec}(R)$ be a formal group and $0 : \mathrm{Spec}(R) \to G$ its zero section. There is a natural isomorphism of quasi-coherent sheaves over* $\mathrm{Spec}(R)$

$$\omega_G \to 0^*\Omega_{G/R}.$$

For this reason, we will occasionally adopt the following notation. If $f : G \to H$ is a map of formal groups over $\mathrm{Spec}(R)$, there is a natural map

$$df : f^*\Omega_{H/R} \to \Omega_{G/R}.$$

Since f is a map of formal groups, we can pull back along the zero-section of G to obtain a map of R-modules

$$0^*f : \omega_H \to \omega_G.$$

With the previous lemma in mind, we will denote this map by $\mathsf{Lie}(f)$.

An important corollary of the lemma is the following.

Corollary B.2.9. *The sheaf of invariant differentials of a formal group is a line bundle.*

This means that on neighborhoods in $\mathrm{Spec}(R)$ where G is presented by a formal group law, we can choose generators of the R-module ω_G to locally identify it with R itself. In fact, we can be quite explicit about this choice of generator, as the following lemma shows.

Lemma B.2.10 ([Sil86, Proposition IV.4.2])**.** *Let $G \to \mathrm{Spec}(R)$ be a formal group presented by a formal group law F. There exists a unique invariant differential ω_0 on G with the property that $\omega_0(0) = dt$. Explicitly,*

$$\omega_0(t) = \frac{\partial F}{\partial x}(0,t)^{-1}dt.$$

Moreover, every invariant differential over G is of the form $a\omega_0$ for some a in R.

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