ON PRESENTABLE ∞-CATEGORIES AND ∞-TOPOI

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Abstract. These notes are an attempt to record some facts and definitions about presentable ∞-categories and ∞-topoi come up in practice. In the spirit of making something which functions as a "users' guide," intuitions are emphasized over proofs (though references are provided as much as possible). My main goals with this document were to solidify my understanding of certain foundational concepts like the existence of adjoint functors and Kan extensions, the monoidal structure on Pr^L, and of hypercomplete objects. I also strongly recommend [Rez22] for a very efficient introduction to the subject. Of course, any mistakes herein are my own.

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Notation

Cat_∞	∞-category of small ∞-categories
$\widehat{\mathsf{Cat}}_\infty$	∞-category of locally essentially small ∞-categories
$\mathcal{S}$	∞-category of spaces (anima, ∞-groupoids, CW-complexes, Kan complexes etc.)
$\hat{\mathcal{S}}$	∞-category of large spaces
Sp	∞-category of spectra
$h\mathcal{C}$	homotopy category of an ∞-category $\mathcal{C}$
$\mathsf{Fun}(\mathcal{C},\mathcal{D})$	∞-category of functors between $\mathcal{C}$ and $\mathcal{D}$
$\mathsf{Fun}^L(\mathcal{C},\mathcal{D})$	full subcategory of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ comprising functors which are left adjoints
$\mathcal{P}(\mathcal{C})$	∞-category of presheaves of spaces on $\mathcal{C}$, i.e., $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathcal{S})$
$\mathcal{P}(\mathcal{C},\mathcal{D})$	∞-category of $\mathcal{D}$-valued presheaves, i.e., $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathcal{D})$
$\mathsf{Maps}_{\mathcal{C}}(c,d)$	mapping space between two objects c and d of an ∞-category $\mathcal{C}$
$\mathsf{maps}_{\mathcal{C}}(c,d)$	mapping spectrum between two objects c and d of a stable ∞-category $\mathcal{C}$
$\mathcal{M}\mathsf{aps}_{\mathcal{C}}(c,d)$	internal mapping object between two objects c and d of a closed monoidal ∞-category
$[c,d]$	set of morphisms between two objects c and d of $h\mathcal{C}$; equivalently, $\pi_0\,\mathsf{Maps}_{\mathcal{C}}(c,d)$
$1_{\mathcal{C}}$	terminal object in $\mathcal{C}$ (when it exists)
$\pi_!$	forgetful functor $\mathcal{C}_{/c}\to\mathcal{C}$; postcomposition with $\pi : c\to 1_{\mathcal{C}}$.
$\mathsf{Disc}(\mathcal{C})$	discrete (0-truncated) objects of $\mathcal{C}$

1. Constructions in ∞-categories

1.1. **Size issues**

When attempting to make many ∞-categorical constructions precise, we quickly run into some set-theoretic difficulties which we call "size issues." These are not much fun to think about, but the good news is that we almost never have to. In this brief section we give an example of how to address these issues so we do not have to worry about them again. See [Lur09, 1.2.15] for more discussion.

Let κ be an uncountable *strongly inaccessible cardinal*. Let $\mathcal{U}(\kappa)$ denote the collection of sets of cardinality less than κ so that $\mathcal{U}(\kappa)$ is a *Grothendieck universe*. This means that any set theoretic construction involving sets in $\mathcal{U}(\kappa)$ will result in a set still in $\mathcal{U}(\kappa)$. We will call an object *small* if it belongs to $\mathcal{U}(\kappa)$, and *essentially small* if it is "equivalent" to a small object. We will call an object *large* if it is not essentially small.

Remark 1.1.1. In certain situations, it is necessary to speak of several nested universes, and distinguish between small, large, and "very large" objects, etc. We will not worry about making this too precise here, but will use this notion a couple times.

1.2. **Limits and colimits**

Most of the properties of presentable ∞-categories and ∞-topoi we will discuss have to do with the existence or preservation of certain kinds of limits and colimits. We record the definitions of some of these kinds in this section.

Recall that in general, lim (and dually, colim) is a (partially defined) functor

$$\lim : \mathsf{Fun}(\mathcal{D},\mathcal{C})\to\mathcal{C}$$

which, given a "diagram ∞-category" $\mathcal{D}$ and a $\mathcal{D}$-shaped diagram in the ∞-category $\mathcal{C}$ (i.e., a functor $F:\mathcal{D}\to\mathcal{C}$), assigns an object $\lim F$ in $\mathcal{C}$ which is "universal" in some appropriate sense. The different types of (co)limits we will discuss have to do with the different possibilities for the diagram category $\mathcal{D}$.

Definition 1.2.1. A *finite (co)limit* is a (co)limit indexed on a diagram category with finitely many objects.

Example 1.2.2. A nice perspective to have is that abelian categories have all finite limits and colimits and that *left exact* functors between abelian categories (as they're classically defined) are the (additive) functors preserving finite limits. Dually, *right exact* functors of abelian categories are the (additive) functors which preserve finite colimits. We will see this notion in some more generality in the next definition.

Definition 1.2.3. A *small (co)limit* is a (co)limit indexed on a diagram category which is small.

Definition 1.2.4 ([Lur09, 5.3.2.9])**.** A functor $F:\mathcal{C}\to\mathcal{D}$ of ∞-categories is

(i) *left exact* if it preserves finite limits when they exist in $\mathcal{C}$,
(ii) *right exact* if it preserves finite colimits when they exist in $\mathcal{C}$,
(iii) *exact* if it is both left exact and right exact.

Remark 1.2.5. There seems to be some occasional conflict in the literature over whether "left exact" should mean "preserves *finite* limits" or "preserves *small* limits." We will stick to the former definition since we have the following alternate terminology for the latter.

Definition 1.2.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is

(i) *continuous* if it preserves small limits, and
(ii) *cocontinuous* if it preserves small colimits.

See Warning 1.2.9.

The next definition is most important in the colimit case. (There is a dual notion of *cofiltered limits*, but these will not be important to us.)

Definition 1.2.7. A *filtered colimit* is a colimit indexed on a category D which is *filtered*, i.e., every diagram in D with a small collection of arrows admits a cocone in D.

The definition of filtered colimit can be equivalently formulated via the following characterization of filtered categories.

Lemma 1.2.8 ([Lur09, 5.3.3.3])**.** *The ∞-category $\mathcal{D}$ is filtered if and only if the colimit functor*

$$\operatorname{colim}_{\mathcal{D}} : \mathsf{Fun}(\mathcal{D}, \mathcal{S}) \to \mathcal{S}$$

preserves small limits.

Warning 1.2.9. Beware that [Lur09, 5.3.4.5] uses "continuous functor" to mean a functor which preserves filtered colimits. As far as I can tell, this is pretty nonstandard, and so we will not adopt this language.

The most important appearance of filtered colimits is the following.

Definition 1.2.10 ([Lur09, 5.3.4.5])**.** An object c of an ∞-category $\mathcal{C}$ is *compact* if filtered colimits exist in $\mathcal{C}$, and the functor corepresented by c,

$$\mathsf{Maps}_{\mathcal{C}}(c, -) : \mathcal{C} \to \hat{\mathcal{S}},$$

commutes with filtered colimits. (Here $\hat{\mathcal{S}}$ denotes large spaces.) We denote the full subcategory of compact objects of $\mathcal{C}$ by $\mathcal{C}^{\omega}$.

Warning 1.2.11. Note that a smallness hypothesis is intrinsic to our definition of filtered colimits. As a result, the definitions of continuous functors and compact objects also carry a smallness condition.

1.3. $\mathsf{Ind}(\mathcal{C}_0)$ and accessibility

Many definitions and constructions in the world of ∞-categories come with some mention of "accessibility." Essentially, the point of an accessibility hypothesis is to enforce some kind of "smallness," even when dealing with large categories. This is mainly to address set-theoretic foundational issues, and in practice we don't need to worry about it so much. For completeness, we record some basic definitions here.

We beging by defining the Ind-construction via a universal property.

Definition 1.3.1 ([Lur09, 5.3.5.10])**.** Let $\mathcal{C}_0$ be a small ∞-category. The ∞-category $\mathsf{Ind}(\mathcal{C})$ is (essentially) uniquely characterized by the following

(1) There is a functor $j : \mathcal{C}_0 \to \mathsf{Ind}(\mathcal{C}_0)$,
(2) $\mathsf{Ind}(\mathcal{C})$ admits all small colimits, and
(3) given any ∞-category $\mathcal{D}$ admitting small filtered colimits, restriction along j induces an equivalence

$$\mathsf{Fun}^{\mathsf{cts}}(\mathsf{Ind}(\mathcal{C}_0), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}_0, \mathcal{D})..$$

Essentially, this is saying $\mathsf{Ind}(\mathcal{C}_0)$ is obtained from $\mathcal{C}_0$ by freely adjoining colimits of small filtered diagrams [Lur09, 5.3.2]. In practice, the presheaf category $\mathcal{P}(\mathcal{C}_0)$ often serves as a model for $\mathsf{Ind}(\mathcal{C}_0)$, and the functor j is the Yoneda embedding. We axiomatize this as follows.

Definition 1.3.2 ([Lur09, 5.4.2.1])**.** An ∞-category $\mathcal{C}$ is *accessible* if there is some small $\mathcal{C}_0$ and an equivalence

$$\mathsf{Ind}(\mathcal{C}_0) \to \mathcal{C}\,.$$

Unwinding definitions, we get the following equivalent definition of accessibility.

Proposition 1.3.3 ([Lur09, 5.4.2.2])**.** *An ∞-category $\mathcal{C}$ is accessible if and only if*

(i) $\mathcal{C}$ admits all filtered colimits, and
(ii) there is a small collection of compact objects which generate $\mathcal{C}$ under small filtered colimits.

Definition 1.3.4 ([Lur09, 5.4.2.5])**.** A functor $F : \mathcal{C} \to \mathcal{D}$ if ∞-categories is *accessible* if $\mathcal{C}$ is accessible and F is preserves filtered colimits.

Remark 1.3.5. To summarize, $\mathcal{C}$ is accessible if it is "built from" a small collection of compact objects, and a functor $F : \mathcal{C} \to \mathcal{D}$ is accessible if it can be computed by evaluating on the compact generators of $\mathcal{C}$, and then taking filtered colimits in $\mathcal{D}$.

That is, there is a small set $C_0 = \{c_i : i \in I\}$ in $\mathcal{C}$ such that every object c in $\mathcal{C}$ is of the form

$$c \simeq \varinjlim c_j$$

(where the colimit is over some filtered system in the full subcategory spanned by C_0) and $F(c)$ can be computed by

$$F(c) = F\big(\varinjlim c_j\big) = \varinjlim F(c_j).$$

The following definition is also useful.

Definition 1.3.6 ([Lur09, 5.5.7])**.** An ∞-category $\mathcal{C}$ is *compactly generated* if the inclusion $\mathcal{C}^\omega \hookrightarrow \mathcal{C}$ induces an equivalence

$$\mathsf{Ind}(\mathcal{C}^\omega) \xrightarrow{\simeq} \mathcal{C}\,.$$

Remark 1.3.7. In other words (which will definite in Section 2), compactly generated ∞-categories are the *countably* presentable ∞-categories [Lur09, 5.5.7.1].

1.4. **Localizations**

Definition 1.4.1 ([Lur09, 5.2.7.2], [Lur12, I.6], [Lur17, 1.2.1], [Lur18, 1.5.4.2])**.** A functor $L : \mathcal{C} \to \mathcal{D}$ is a *localization* if it admits a fully faithful right adjoint.

Warning 1.4.2 ([Lur09, 5.2.7.3])**.** Beware that some authors may employ different definitions of "localization," e.g. as *any* functor which is determined by the class of morphisms it inverts (whether or not it admits any adjoints, let alone a fully faithful left adjoint). Localizations as we have defined them above are sometimes called (e.g., by nLab) "reflective localizations" as they are left adjoints to the fully faithful inclusions of reflective subcategory. As you can see by our numerous citations for Definition 1.4.1, Lurie has always been consistent with this definition of localization, so we will be, too.

We might call a functor which admits a fully faithful left adjoint a *colocalization* (though we won't have need for this terminology here).

Remark 1.4.3. It is common to abuse notation and write a localization functor $L : \mathcal{C} \to \mathcal{D}$ as $L : \mathcal{C} \to \mathcal{C}$, implicitly post-composing L with the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$, identifying $\mathcal{D}$ with its essential image in $\mathcal{C}$.

Remark 1.4.4 (Digression on sub-∞-categories [Lur09, 1.2.11])**.** This is as good a place as any to take a moment to be precise about what we mean by a sub-∞-category (from now on, just "subcategory") of an ∞-category. Such inclusions $\mathcal{C}' \hookrightarrow \mathcal{C}$ always arise by specifying a 1-categorical subcategory inclusion $h\mathcal{C}' \hookrightarrow h\mathcal{C}$ and then forming the pullback in $\widehat{\mathsf{Cat}}_\infty$:

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\ N(h\mathcal{C}') \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

We say the subcategory inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is *full* if the 1-categorical inclusion $h\mathcal{C}' \hookrightarrow h\mathcal{C}$ is full in the usual sense. In particular, to describe a subcategory of an ∞-category it is always enough to specify the objects and (homotopy equivalence classes of) morphisms.

Philosophically, a localization $L : \mathcal{C} \to \mathcal{D}$ is determined (up to equivalence in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$) by the class of morphisms it inverts, that is, the class of morphisms f in $\mathcal{C}$ such that Lf is an equivalence in $\mathcal{D}$. This is made precise through the following sequence of definitions and propositions.

Definition 1.4.5. Let $L : \mathcal{C} \to \mathcal{C}$ be a localization. A morphism f in $\mathcal{C}$ is an *L-equivalence* if Lf is an equivalence.

Definition 1.4.6. [Lur09, 5.5.4.1] Let S be an arbitrary collection of morphisms in $\mathcal{C}$. An object c of $\mathcal{C}$ is *S-local* if, for every morphism $s : x \to y$ in S, the induced map

$$s^* : \mathsf{Maps}_{\mathcal{C}}(y, c) \to \mathsf{Maps}_{\mathcal{C}}(x, c)$$

is a homotopy equivalence.

A morphism $f : c \to d$ in $\mathcal{C}$ is an *S-equivalence* if for every S-local z in $\mathcal{C}$,

$$f^* : \mathsf{Maps}_{\mathcal{C}}(d, z) \to \mathsf{Maps}_{\mathcal{C}}(c, z)$$

is a homotopy equivalence.

Example 1.4.7 (Bousfield localizations of Sp)**.** Recall that given a spectrum E, we say another spectrum X is "E-local" in the sense of Bousfield if whenever A is E-acyclic (i.e., $E \otimes A \simeq 0$) we have $\mathsf{maps}(A, X) \simeq 0$. Note that this is exactly the same as saying that X is local with respect to the E-equivalences. Indeed, if

$$A \to Y \xrightarrow{f} Z$$

is a fiber sequence where f is an E-equivalence, then A is E-acyclic. There is a fiber sequence

$$\mathsf{maps}(Z, X) \xrightarrow{f^*} \mathsf{maps}(Y, X) \to \mathsf{maps}(A, X),$$

and so we can see that X is E-local in the Bousfield sense exactly if it is local with respect to the E-equivalences in the sense of Definition 1.4.6.

Proposition 1.4.8 ([Lur09, 5.5.4.2])**.** *Let $L : \mathcal{C} \to \mathcal{C}$ be a localization; let $L\mathcal{C} \subset \mathcal{C}$ denote its essential image. Let S denote the collection of L-equivalences in $\mathcal{C}$.*

(i) An object c of $\mathcal{C}$ is S-local if and only if it is in $L\mathcal{C}$.
(ii) Every S-equivalence in $\mathcal{C}$ belongs to S.

Remark 1.4.9. Any small collection of morphisms S in $\mathcal{C}$ determines minimal "strongly saturated" class of morphisms $\bar{S}$ containing S. This class $\bar{S}$ determines a localization $L : \mathcal{C} \to \mathcal{C}$ with $\bar{S}$ exactly comprising the L-equivalences. In this sense, given such an S, we can always form a universal localization

$$L : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

"inverting S" as desired. See [Lur09, 5.5.4] for details.

In Section 2.3, we will see that when $\mathcal{C}$ is presentable, most of our intuitions about the resulting category $\mathcal{C}[S^{-1}]$ turn out to be true.

Finally, we note that the various definitions of "accessible localization" we might make coincide, as the following proposition records.

Proposition 1.4.10 ([Lur09, 5.5.4.2(iii)])**.** *Let $L : \mathcal{C} \to \mathcal{C}$ be a localization and S the collection of L equivalences. The following are equivalent:*

(i) $L\mathcal{C}$ is accessible,
(ii) $L : \mathcal{C} \to \mathcal{C}$ is accessible,
(iii) there exists a set $S_0 \subset S$ such that every S_0-local object is S-local.

1.5. **Presheaf ∞-categories**

In this section, we highlight a couple characterizations of the presheaf ∞-category of a small ∞-category $\mathcal{C}_0$ and a few useful theorems. By definition,

$$\mathcal{P}(\mathcal{C}_0) = \mathsf{Fun}(\mathcal{C}_0^{\mathsf{op}}, \mathcal{S}).$$

Abstractly, we will think of $\mathcal{P}$ as a functor

$$\mathcal{P} : \mathsf{Cat}_\infty \to \widehat{\mathsf{Cat}}_\infty .$$

Remark 1.5.1. Note that $\mathcal{P}(\mathcal{C}_0)$ is small only when $\mathcal{C}_0$ is a contractible Kan complex! For this reason, we will only ever form the presheaf categories of small ∞-categories.

The first characterization of $\mathcal{P}(\mathcal{C}_0)$ comes from the adjointness of $- \times \mathcal{D}$ and $\mathsf{Fun}(-, \mathcal{D})$ in $\widehat{\mathsf{Cat}_\infty}$.

Theorem 1.5.2 (First universal property of $\mathcal{P}$, [Lur09, 5.1.5]). *For any small ∞-category $\mathcal{C}_0$ and (possibly large) ∞-category $\mathcal{D}$, there is an equivalence of ∞-categories*

$$\mathsf{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C}_0)) \simeq \mathsf{Fun}(\mathcal{D} \times \mathcal{C}_0^{\mathsf{op}}, \mathcal{S})$$

natural in $\mathcal{C}_0$.

The second characterization uses the ∞-categorical version of the Yoneda embedding, which we introduce first.

Theorem 1.5.3 (Yoneda lemma, [Lur09, 5.1.3.1 and 5.1.3.2]). *Let $\mathcal{C}_0$ be a small ∞-category. The Yoneda functor*

$$y : \mathcal{C}_0 \to \mathcal{P}(\mathcal{C}_0)$$

sending an object c of $\mathcal{C}_0$ to $\mathsf{Maps}_{\mathcal{C}_0}(c, -)$ is fully faithful. Moreover, y preserves all small limits which exist in $\mathcal{C}_0$.

Remark 1.5.4. Constructing the Yoneda embedding ∞-categorically takes some work.

Theorem 1.5.5 (Second universal property of $\mathcal{P}$, [Lur09, 5.1.5.6]). *Let $\mathcal{C}_0$ be a small ∞-category, and let $\mathcal{D}$ be an ∞-category admitting all small colimits. Restriction along the Yoneda embedding induces an equivalence of ∞-categories*

$$\mathsf{Fun}^L(\mathcal{P}(\mathcal{C}_0), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}_0, \mathcal{D}).$$

Finally, we state a theorem which makes precise the claim that $\mathcal{P}(\mathcal{C}_0)$ is the "free cocompletion of $\mathcal{C}_0$."

Theorem 1.5.6 ([Lur09, 5.1.5.8]). *Let $\mathcal{C}_0$ be a small ∞-category. The image of the Yoneda embedding generates $\mathcal{P}(\mathcal{C}_0)$ under small colimits.*

2. Presentable ∞-categories

Presentable ∞-categories are some of the nicest (or at least most manageable) ∞-categories around. In many ways, they behave like small ∞-categories, and have the excellent properties of being complete and cocomplete. Many of our classical categorical intuitions carry through in this setting. There are also many very pleasant theorems, and the category of presentable ∞-categories, Pr^L is very nicely behaved.

2.1. Definitions and basic properties

Definition 2.1.1. A *presentable ∞-category* is an ∞-category $\mathcal{C}$ such that

(i) There exists a small ∞-category $\mathcal{C}_0$ and
(ii) an adjunction $L \dashv i$

$$L : \mathcal{P}(\mathcal{C}_0) \rightleftarrows \mathcal{C} : i$$

where the right adjoint i is fully faithful (i.e. L is a localization), and
(iii) i is accessible.

Example 2.1.2. $\mathcal{S}$ and Sp are presentable.

Warning 2.1.3. Beware that some sources (e.g., nLab) use the terminology "locally presentable" to refer to what [Lur09] and [Lur17] call "presentable." We will stick to the latter terminology.

Theorem 2.1.4 ([Lur09, 5.5.2.4]). *A presentable ∞-category $\mathcal{C}$ admits all small limits and small colimits.*

Remark 2.1.5. Note that the property of being cocomplete in the previous theorem is immediate: if $L : \mathcal{P}(\mathcal{C}_0) \to \mathcal{C}$ is a presentation of $\mathcal{C}$, then colimits in $\mathcal{C}$ can be computed in $\mathcal{P}(\mathcal{C}_0)$ (which we view as the free cocompletion of $\mathcal{C}_0$) and then pushing forward along L which preserves colimits as it is a left adjoint.

The following (somewhat surprising) fact says that $\mathcal{C}$ and $\mathcal{C}^{op}$ are rarely both presentable.

Proposition 2.1.6. *Let $\mathcal{C}$ be a presentable ∞-category. Then $\mathcal{C}^{\mathsf{op}}$ is presentable if and only if it is a poset.*

Proposition 2.1.7 ([Lur09, 5.5.3.10])**.** *Let c be an object of a presentable ∞-category $\mathcal{C}$. Then $\mathcal{C}_{/c}$ is also presentable.*

The previous proposition holds in more generality. If $p : \mathsf{D} \to \mathcal{C}$ is any small diagram in $\mathcal{C}$, then the category $\mathcal{C}_{/p}$ of objects over p is presentable.

One special case of slices over diagrams is worth highlighting for us.

Lemma 2.1.8. *Let $f : d \to c$ be a morphism in an ∞-category $\mathcal{C}$. There is a natural equivalence*

$$(\mathcal{C}_{/c})_{/f} \simeq \mathcal{C}_{/f} \simeq \mathcal{C}_{/d}\,.$$

2.2. Adjoint functor theorem and existence of Kan extensions

A central theorem that makes presentable ∞-categories nice to study is the following.

Theorem 2.2.1 (Adjoint functor theorem, [Lur09, 5.5.2.9])**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

(i) The functor F admits a right adjoint if and only if it preserves small colimits.
(ii) The functor F admits a left adjoint if and only if it is accessible and preserves small limits.

Remark 2.2.2. Point (i) of the preceding theorem actually does not require $\mathcal{D}$ to be presentable, only that is it locally small [Lur09, 5.5.2.10].

One very powerful application of the adjoint functor theorem is the following.

Theorem 2.2.3 (Representability criterion, [Lur09, 5.5.2.2])**.** *Let $\mathcal{C}$ be presentable and $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{S}$ a functor. Then F is representable by an object of $\mathcal{C}$ if and only if it preserves small limits.*

Dually, $G : \mathcal{C} \to \mathcal{S}$ is corepresentable by an object of $\mathcal{C}$ if and only if it is accessible and preserves small limits.

Another important application of the adjoint functor theorem is to show the existence of Kan extensions in presentable settings. Any functor $F : \mathcal{C} \to \mathcal{D}$ induces a restriction

$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{A}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{A}),$$

and the left and right Kan extensions along F, if they exist, form adjoint pairs

$$\mathsf{Lan}_F \dashv F^* \dashv \mathsf{Ran}_F$$

[Lur09, 4.3.3.7]. Thus, questions about the existence of Kan extensions can be rephrased into questions about the existence of adjoints which we know how to answer well in the presentable case. An example statement we can make is the following.

Proposition 2.2.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a colimit preserving functor between presentable ∞-categories, and let $\mathcal{A}$ be presentable. Then the left Kan extension*

$$\mathsf{Lan}_F : \mathsf{Fun}(\mathcal{C}, \mathcal{A}) \to \mathsf{Fun}(\mathcal{D}, \mathcal{A})$$

exists.

Proof. This can be seen using the adjoint functor theorem and the fact that F^* can be viewed as a morphism in Pr^L by Proposition 2.4.4. □

2.3. Localizations of presentable ∞-categories

Presentable ∞-categories turn out to be a very nice place to talk about localizations. Indeed, the presentable setting is where most of our intuitions about localizations tend to be true, as the theorems in this section demonstrate.

The first theorem says that when $\mathcal{C}$ is presentable and S is a set of morphisms in $\mathcal{C}$, we have a lot of control over the category of S-local objects.

Theorem 2.3.1 ([Lur09, 5.5.4.15])**.** *Let $\mathcal{C}$ be presentable, and S a small collection of morphisms in $\mathcal{C}$. Let $\bar{S}$ be the strongly saturated class of morphisms in $\mathcal{C}$ generated by S (see Remark 1.4.9). Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of S-local objects. Then*

(i) for each c in $\mathcal{C}$, there is a morphism $s : c \to c'$ such that c' is S-local and s belongs to $\bar{S}$,

(ii) *$\mathcal{C}'$ is presentable, and*
(iii) *the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left adjoint L.*

Next, a theorem stating that, in the presentable setting, S, $\bar{S}$, and $L : \mathcal{C} \to \mathcal{C}'$ are all essentially encoding the "same data."

Theorem 2.3.2 ([Lur09, 5.5.4.15])**.** *Let S, $\bar{S}$, and $L : \mathcal{C} \to \mathcal{C}'$ be as in the previous theorem. For every morphism f in $\mathcal{C}$, the following are equivalent:*
(i) *f is an S-equivalence,*
(ii) *f belongs to $\bar{S}$,*
(iii) *Lf is an equivalence in $\mathcal{C}'$.*

Remark 2.3.3. Note that in the setting of the above theorem, $\mathcal{C}'$ is frequently denoted by $\mathcal{C}[S^{-1}]$, $S^{-1}\,\mathcal{C}$, or $L\mathcal{C}$.

Remark 2.3.4. Note that the functor $L : \mathcal{C} \to \mathcal{C}'$ depends only on the strongly saturated class $\bar{S}$ generated by S. In particular, two different small collections of morphisms S and T may induce equivalent localizations.

Example 2.3.5. An important example of localizations of presentable ∞-categories are *Bousfield localizations* of Sp with respect to a homology theory. If E is a spectrum, we say a morphism of spectra $f : X \to Y$ is an *E-equivalence* if $f \otimes E : X \otimes E \to Y \otimes E$ is an equivalence. Since Sp is presentable, the class of E-equivalences determines a localization of Sp:

$$L_E : \mathsf{Sp} \to \mathsf{Sp}_E\,.$$

Finally, in the presentable setting, localizations are characterized by a familiar universal property. Roughly, that the localization $L : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is initial among functors admitting right adjoints and inverting S.

Theorem 2.3.6 ([Lur09, 5.5.4.20])**.** *Let $\mathcal{C}$ be presentable and $\mathcal{D}$ an arbitrary ∞-category. Let S be a small collection of morphisms in $\mathcal{C}$ and $L : \mathcal{C} \to \mathcal{C}[S^{-1}]$ the associated (accessible) localization. Then*

$$L^* : \mathsf{Fun}^L(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathsf{Fun}^L(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image consists of functors $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an equivalence for each s in S.

Equivalently, any functor $F : \mathcal{C} \to \mathcal{D}$ admitting a right adjoint and such that $F(s)$ is an equivalence for each s in S factors (essentially) uniquely through L, as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D}\,. \\ \mathcal{C}[S^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

2.4. **Basic properties of Pr^L**

Definition 2.4.1 ([Lur09, 5.5.3.1])**.** Let $\widehat{\mathsf{Cat}}_\infty$ denote the ∞-category of locally essentially small ∞-categories. By Pr^L (respectively, Pr^R), we denote the subcategory where the objects are the presentable ∞-categories and the morphisms are functors which are left adjoints (respectively, right adjoints).

Remark 2.4.2. Pr^L and Pr^R are (canonically) anti-equivalent. In practice, we usually make statements about Pr^L.

Remark 2.4.3. In light of Theorem 2.2.1, we can equivalently think of Pr^L as the ∞-category of presentable ∞-categories and colimit preserving functors.

Proposition 2.4.4 ([Lur09, 5.5.3.8])**.** *Pr^L is closed. That is, if $\mathcal{C}$ and $\mathcal{D}$ are presentable, then $\mathsf{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable.*

Theorem 2.4.5 ([Lur09, 5.5.3.13 and 5.5.3.18])**.** *Pr^L and Pr^R admit all small limits and small colimits. Moreover, the inclusions*

$$\mathsf{Pr}^L, \mathsf{Pr}^R \hookrightarrow \widehat{\mathsf{Cat}}_\infty$$

preserve small limits.

Remark 2.4.6. Since Pr^L and Pr^R are antiequivalent, it follows that colimits in Pr^L can be computed by first taking the opposite diagram in Pr^R, and then taking the limit in $\widehat{\mathsf{Cat}}_\infty$. Indeed, a small diagram $f : \mathcal{D} \to \mathsf{Pr}^L$ determines a diagram $f' : \mathcal{D}^{\mathsf{op}} \to \mathsf{Pr}^R$, and if $i : \mathsf{Pr}^L \hookrightarrow \widehat{\mathsf{Cat}}_\infty$ is the fully faithful inclusion, we have

$$\operatorname{colim} f \simeq \lim f' \simeq \lim i \circ f',$$

where the colimit is taken in Pr^L, the first limit in Pr^R, and the second limit in $\widehat{\mathsf{Cat}}_\infty$. A dual remark can be made about computing colimits in Pr^R as limits in Pr^L.

2.5. Symmetric monoidal structure on Pr^L

Let $\widehat{\mathsf{Cat}}_\infty(\mathcal{K})$ denote the ∞-category of cocomplete ∞-categories and colimit preserving functors. In this section, we will make use of the fact that $\widehat{\mathsf{Cat}}_\infty(\mathcal{K})$ is symmetric monoidal under the Cartesian product with unit $\mathcal{S}$. There are some technical, set-theoretic hurdles to make this precise, but we will not worry about them.

Theorem 2.5.1 ([Lur17, 4.8.1.15 and 4.8.1.17])**.** *The symmetric monoidal structure on $\widehat{\mathsf{Cat}}_\infty(\mathcal{K})$ restricts to one on Pr^L. For any $\mathcal{C}$ and $\mathcal{D}$ presentable, we have*

$$\mathcal{C} \otimes \mathcal{D} = \mathsf{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathcal{D}).$$

Moreover, $\otimes$ preserves colimits in both variables, and $\mathcal{S}$ is again the unit for Pr^L.

The following are a collection of useful constructions made with the tensor product in Pr^L. We refer the reader to the relevant section in [Lur17] for details.

Example 2.5.2 (Categories of pointed objects in Pr^L, [Lur17, 4.8.1.20])**.** For any ∞-category $\mathcal{C}$, the category of *pointed objects* in $\mathcal{C}$ is the full subcategory of $\mathsf{Fun}(\Delta^1, \mathcal{C})$ spanned by functors F such that $F(0)$ is terminal in $\mathcal{C}$. In particular, for $\mathcal{C}$ presentable, we have

$$\mathcal{C} \otimes (\mathcal{S}_*) \simeq \mathcal{C}_* .$$

Example 2.5.3 (Truncation in Pr^L, [Lur17, 4.8.1.22])**.** Let $n \geq -2$ be an integer. By $\tau_{\leq n}\,\mathcal{C}$ we denote the truncated objects in $\mathcal{C}$, i.e., the objects c in $\mathcal{C}$ for which $\mathsf{Maps}_{\mathcal{C}}(-, c)$ takes values in n-truncated spaces. When $\mathcal{C}$ is presentable, we have

$$\mathcal{C} \otimes \tau_{\leq n}\,\mathcal{S} \simeq \tau_{\leq n}\,\mathcal{C} .$$

Example 2.5.4 (Contruction of the smash product of spectra, [Lur17, 4.8.2])**.** Let $\mathsf{Pr}^L_{\mathrm{st}}$ be the full subcategory of Pr^L spanned by the stable presentable ∞-categories (see Section 2.6). It can be shown that the symmetric monoidal structure on Pr^L restricts to one on $\mathsf{Pr}^L_{\mathrm{st}}$ with Sp as the unit. The result is that Sp is a commutitive algebra object in $\mathsf{Pr}^L_{\mathrm{st}}$, were the map

$$\mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$$

is the smash product of spectra. It can be shown that this symmetric monoidal structure is essentially unique, has the sphere spectrum as its unit, and makes Sp into the *initial* algebra object in $\mathsf{Pr}^L_{\mathrm{st}}$.

2.6. Stability, stabilization, and stabilization in Pr^L

Definition 2.6.1 ([Lur17, 1.1.1.9])**.** An ∞-category $\mathcal{C}$ (not necessarily presentable) is *stable* if it

(i) is *pointed*, i.e., $\mathcal{C}$ has an object $*$ which is both initial and terminal,
(ii) admits all *fibers* and *cofibers*, i.e., the diagrams $* \to a \leftarrow b$ and $* \leftarrow a \to b$ admit limits and colimits for all objects a and b in $\mathcal{C}$, and
(iii) any commuting diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ * \arrow[r] & c \end{tikzcd}$$

in $\mathcal{C}$ is a pullback diagram if and only it is a pushout diagram.

Remark 2.6.2. We emphasize that stability is a property of an ∞-category $\mathcal{C}$, rather than a piece of extra structure. Note that for any stable $\mathcal{C}$, the homotopy category $h\mathcal{C}$ has a natural triangulated structure.

Definition 2.6.3 ([Lur17, 1.4.2.25])**.** Let $\mathcal{C}$ be an ∞-category admitting finite limits. Let $\mathcal{C}_*$ be its category of pointed objects (see Example 2.5.2). Note that $\mathcal{C}_*$ also admits finite limits. The *stabilization* of $\mathcal{C}$ is the stable ∞-category

$$\mathsf{Sp}(\mathcal{C}) := \lim\big(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\big),$$

where the limit is taken in $\widehat{\mathsf{Cat}}_\infty$ (or, equivalently, in Pr^R if $\mathcal{C}_*$ is presentable).

The stabilization of $\mathcal{C}$ is also occaisionally denoted $\mathcal{C}^{\mathsf{st}}$. To avoid confusion, we will stick to the notation above.

Remark 2.6.4. We should think of the above construction of stabilization as a process for formally inverting the loops functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$.

Remark 2.6.5 ([Lur17, 1.4.2.25])**.** The stabilization of $\mathcal{C}$ can equivalently be described as the category of *spectrum objects* in $\mathcal{C}$.

As we would hope, the following is true.

Lemma 2.6.6 ([Lur17, 1.4.2.21])**.** *A stable ∞-category is equivalent to its stabilization. That is, if $\mathcal{C}$ is stable, then $\mathcal{C} \simeq \mathsf{Sp}(\mathcal{C})$.*

Example 2.6.7. The ∞-category of spectra, Sp, is the stabilization of $\mathcal{S}$. That is, $\mathsf{Sp} = \mathsf{Sp}(\mathcal{S})$.

Remark 2.6.8. Following Remark 2.4.6, if $\mathcal{C}_*$ is presentable, since we have an adjunction $\Sigma \dashv \Omega$, the limit in Definition 2.6.3 can be computed as the following colimit in Pr^L:

$$\mathsf{Sp}(\mathcal{C}) \simeq \operatorname{colim}\big(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \cdots\big). \tag{2.6.9}$$

Expressed this way, the stabilization has the effect of formally inverting $\Sigma : \mathcal{C}_* \to \mathcal{C}_*$.

Warning 2.6.10. Beware that it matters that the colimit (2.6) is taken in Pr^L. An interesting question is what we get if we take this colimit in $\widehat{\mathsf{Cat}}_\infty$ instead of Pr^L. For example, taking $\mathcal{C}$ to be the (prestable) ∞-category $\mathsf{Sp}_{\geq 0}$ of connective spectra and evaluating the colimit in $\widehat{\mathsf{Cat}}_\infty$, we obtain a category $\mathsf{Sp}\_$ with a natural functor to the usual stabilization, $\mathsf{Sp}(\mathsf{Sp}_{\geq 0}) \simeq \mathsf{Sp}$:

$$\mathsf{Sp}_- = \operatorname{colim}\big(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\big) \hookrightarrow \mathsf{Sp}\,.$$

In fact, $\mathsf{Sp}\_$ turns out to be the ∞-category of spectra which are bounded below, and the map to Sp is the inclusion. Note that $\mathsf{Sp}\_$ is *not* presentable, though. Indeed it does not admit all colimits; for example, the direct sum over all $n \geq 0$ of $\mathbb{S}^{-n}$ does not exist in Sp_-.

The category we obtain by taking the colimit in $\widehat{\mathsf{Cat}}_\infty$ is interesting in its own right. We record it in the following definition.

Definition 2.6.11 ([Lur18, C.1.1.1])**.** Let $\mathcal{C}$ be an ∞-category admitting finite colimits. The *Spanier-Whitehead category* of $\mathcal{C}$ is

$$\mathrm{SW}(\mathcal{C}) := \operatorname{colim}\big(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \cdots\big),$$

where the colimit is taken in $\widehat{\mathsf{Cat}}_\infty$.

Example 2.6.12 ([Lur18, 0.2.3.1])**.** The Spanier-Whitehead category of the homotopy category $h\mathcal{S}^{\mathrm{fin}}$ of finite spaces,

$$\mathrm{SW}(h\mathcal{S}^{\mathrm{fin}}) \simeq h\mathrm{SW}(\mathcal{S}^{\mathrm{fin}}),$$

(sometimes just called *the* Spanier-Whitehead category) is the full subcategory of $h\mathsf{Sp}$ consisting of spectra of the form $\Sigma^n\Sigma^\infty X$ for a finite space X and (possibly negative) integer n.

Lemma 2.6.13 ([Lur18, C.1.1.7])**.** $\mathrm{SW}(\mathcal{C})$ *is stable.*

Warning 2.6.14. Warning 2.6.10 shows that $\mathrm{SW}(\mathcal{C})$ need not be presentable, even if $\mathcal{C}$ is.

Example 2.6.15 (Stabilization in Pr^L, [Lur17, 4.8.1.22])**.** Noting that Sp is presentable, for a given presentable ∞-category $\mathcal{C}$, we have the following chain of equivalences:

$$\mathcal{C} \otimes \mathsf{Sp} \simeq \mathsf{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathsf{Sp}) \simeq \lim \mathsf{Fun}^R(\mathcal{C}, \mathcal{S}_*) \simeq \lim \mathcal{C} \otimes \mathcal{S}_* \simeq \lim \mathcal{C}_* \simeq \mathsf{Sp}(\mathcal{C}).$$

That is, the stabilization of $\mathcal{C}$ is naturally identified with the product $\mathcal{C} \otimes \mathsf{Sp}$ in Pr^L.

In particular, this means $\mathsf{Sp}(\mathcal{C})$ is presentable if $\mathcal{C}$ is.

Remark 2.6.16 (Natural t-structure)**.** After identifying $\mathsf{Sp}(\mathcal{C}) \simeq \mathcal{C} \otimes \mathsf{Sp}$, the t-structure $(\mathsf{Sp}_{\geq 0}, \mathsf{Sp}_{\leq 0})$ on Sp induces a natural t-structure on $\mathsf{Sp}(\mathcal{C})$ with

$$\mathsf{Sp}(\mathcal{C})_{\geq 0} = \mathcal{C} \otimes \mathsf{Sp}_{\geq 0}\,.$$

The following theorem witnesses presentable stable ∞-categories as the result of replacing $\mathcal{S}$ with Sp in the definition of presentable ∞-categories (Definition 2.1.1).

Theorem 2.6.17 (Stable Giraud theorem [Lur17, 1.4.4.9])**.** *An ∞-category $\mathcal{C}$ is stable and presentable if and only if it is an accessible left-exact localization of $\mathcal{P}(\mathcal{C}_0, \mathsf{Sp})$ for some small ∞-category $\mathcal{C}_0$.*

Finally, let us discuss the notion of ∞-categories which are prestable.

Definition 2.6.18 ([Lur18, C.1.2.1])**.** We say an ∞-category $\mathcal{C}$ is *prestable* if it

(i) is pointed and admits finite colimits,
(ii) the functor $\Sigma : \mathcal{C}_* \to \mathcal{C}_*$ is fully faithful, and
(iii) for every morphism $f : c \to \Sigma c$ in $\mathcal{C}$, there is a bicartesian square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d, "f"] \\ * \arrow[r] & \Sigma c. \end{tikzcd}$$

Remark 2.6.19 ([Lur18, C.1.2.2])**.** Note that, given condition (i) in the previous definition, condition (ii) is equivalent to the canonical map $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ into the Spanier-Whitehead category being fully faithful. If both (i) and (ii) hold, then (iii) is equivalent to the essential image of $\mathcal{C} \to \mathrm{SW}(\mathcal{C})$ being closed under extensions.

Proposition 2.6.20 ([Lur18, C.1.2.9])**.** *An ∞-category $\mathcal{C}$ is prestable and admits all finite limits if and only if there is some stable $\mathcal{D}$ with a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ such that $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$.*

Warning 2.6.21 ([Lur18, C.1.2.10])**.** Given a prestable ∞-category $\mathcal{C}$ with finite limits, the stable ∞-category with t-structure of Proposition 2.6.20 is not canonically determined. Indeed, both $\mathrm{SW}(\mathcal{C})$ and $\mathsf{Sp}(\mathcal{C})$ are stable ∞-categories with t-structures, and

$$\mathrm{SW}(\mathcal{C})_{\geq 0} \simeq \mathcal{C} \simeq \mathsf{Sp}(\mathcal{C})_{\geq 0}.$$

In a precise sense, these two examples are the initial and terminal examples. Indeed, if $\mathcal{D}$ is an arbitrary stable ∞-category with a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$, then there are natural functors

$$\mathrm{SW}(\mathcal{D}_{\geq 0}) \hookrightarrow \mathcal{D} \to \mathsf{Sp}(\mathcal{D}_{\geq 0}).$$

This can be seen from the fact that the t-structure on $\mathrm{SW}(\mathcal{C})$ is right bounded (bounded below), and $\mathsf{Sp}(\mathcal{C})$ is the right (Postnikov) completion of the t-structure on $\mathrm{SW}(\mathcal{C})$.

The moral of the story is that a prestable ∞-category $\mathcal{C}$ with finite limits can be embedded into (potentially) many stable ∞-categories as the connective part of their t-structure, but there are two “canonical” choices for which category to embed into: $\mathrm{SW}(\mathcal{C})$ and $\mathsf{Sp}(\mathcal{C})$.

2.7. **Truncations and connectivity**

Definition 2.7.1. For $n \geq 0$ *n-truncated space* is a space X such that $\pi_k(X, b) = 0$ for all $k > n$ and points $b \in X$. By convention, we say a (-1)-truncated space is either empty or $*$, and the only (-2)-truncated space is $*$.

Definition 2.7.2 ([Lur09, 5.5.6.1 and 5.5.6.10])**.** Let $\mathcal{C}$ be a general ∞-category. An *n-truncated object* of $\mathcal{C}$ is an object c such that $\mathsf{Maps}_{\mathcal{C}}(x, c)$ is an n-truncated space for all x in $\mathcal{C}$. An *n-truncated morphism* in $\mathcal{C}$ is a morphism $f : c \to d$ in $\mathcal{C}$ which is n-truncated as an object of $\mathcal{C}_{/d}$.

Proposition 2.7.3 ([Lur09, 5.5.6.18])**.** *When $\mathcal{C}$ is a presentable ∞-category, there is an n-*truncation functor *$\tau_{\leq n}^{\mathcal{C}}$ which is left adjoint to the inclusion of the full subcategory $\tau_{\leq n}\,\mathcal{C}$ of n-truncated objects:*

$$\tau_{\leq n}^{\mathcal{C}} : \mathcal{C} \rightleftarrows \tau_{\leq n}\,\mathcal{C} : i.$$

When the category $\mathcal{C}$ is clear from context, we will simply denote $\tau_{\leq n}^{\mathcal{C}}$ by $\tau_{\leq n}$.

Remark 2.7.4. The unit of this adjunction provides a morphism

$$c \to \tau_{\leq n} c$$

in $\mathcal{C}$ which is the initial example of a morphism from c to an n-truncated object.

Definition 2.7.5 ([Lur09, 6.5.1.12])**.** An object c of a presentable ∞-category is an *n-connective object* if $\tau_{\leq n-1} c$ is terminal in $\mathcal{C}$. An *n-connective morphism* in $\mathcal{C}$ is a morphism $f : c \to d$ in $\mathcal{C}$ which is n-connective as an object of $\mathcal{C}_{/d}$.

Remark 2.7.6 (Truncation factorization)**.** When $\mathcal{C}$ is presentable, the slice $\mathcal{C}_{/c}$ is as well, so given a morphism $f : d \to c$ in $\mathcal{C}$, we can discuss its n-truncation as an object of $\mathcal{C}_{/c}$. As in Remark 2.7.4, this yields a morphism $f \to \tau_{\leq n} f$ which corresponds to a triangle

$$\begin{tikzcd} d \arrow[r, "g"] \arrow[dr, "f"'] & d' \arrow[d, "\tau_{\leq n} f"] \\ & c \end{tikzcd}$$

in $\mathcal{C}$ where $\tau_{\leq n} f$ is the initial n-truncated morphism factoring f in this way. It can be shown that g is always $(n+1)$-connective [Lur09, 5.2.8.16]. This truncation factorization arises from a *factorization system* on $\mathcal{C}$ in the sense of [Lur09, 5.2.8.8].

Warning 2.7.7 ([Lur17, 1.2.1.9])**.** When $\mathcal{C}$ is a stable presentable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, the notions of being "$n$-truncated" and of "$n$-coconnective" *do not coincide.* In fact, *a stable ∞-category has no nonzero truncated objects*! Indeed, the identity map of a (nonzero) object c of $\mathcal{C}$ produces a nontrivial element of

$$\pi_0 \operatorname{\mathsf{Maps}}_{\mathcal{C}}(c, c) \cong \pi_0 \operatorname{\mathsf{Maps}}_{\mathcal{C}}(\Sigma^n \Omega^n c, c) \cong \pi_n \operatorname{\mathsf{Maps}}_{\mathcal{C}}(\Omega^n c, c)$$

for all positive n.

Not all is lost; if we restrict to the connective piece of the t-structure on $\mathcal{C}$, then the following statement is true: an object of $\mathcal{C}_{\geq 0}$ is n-truncated (as an object of $\mathcal{C}_{\geq 0}$) if and only it is n-coconnective. Indeed, for each (nonzero) c in $\mathcal{C}_{\geq 0}$, there is some n for which the (iterated) counit map $\Sigma^n \Omega^n c \to c$ is not an equivalence, breaking the counterexample given above.

The following fact is useful when dealing with truncation functors in different categories.

Proposition 2.7.8 ([Lur09, 5.5.6.28])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a left exact, colimit preserving functor. For all n, there is a natural equivalence*

$$F \circ \tau_{\leq n}^{\mathcal{C}} \simeq \tau_{\leq n}^{\mathcal{D}} \circ F.$$

3. ∞-Topoi

We now turn to ∞-topoi, which are an especially nice class of presentable ∞-categories. Indeed, ∞-topoi *almost* behave like categories of sheaves of spaces on a small site and have rich geometric properties. There are also many ways in which ∞-topoi behave like spaces—these analogies motivate many of the definitions and theorems we'll state. For brevity, we won't go into detail about these analogies here, but I recommend [Rez22] for an efficient and well-written introduction to the subject.

3.1. **Definitions and basic properties**

Definition 3.1.1 ([Lur09, 6.1.0.4])**.** An ∞-*topos* is a presentable ∞-category $\mathcal{X}$ with a presentation

$$L : \mathcal{P}(\mathcal{X}_0) \rightleftarrows \mathcal{X} : i$$

such that L is left exact (preserves finite limits).

The property of the localization $L : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$ being left exact turns out give us a lot of extra control over $\mathcal{X}$ as we will see in the following sections. A convenient property for checking left exactness is the following.

Proposition 3.1.2 ([Lur09, 6.2.1.1])**.** *Let $L : \mathcal{C} \to \mathcal{D}$ be a localization of ∞-categories, and suppose $\mathcal{C}$ admits finite limits. Then L is left exact if and only if L-equivalences are stable under pullback. That is, for every L-equivalence $f : a \to b$ in $\mathcal{C}$, and every pullback square*

$$\begin{tikzcd} a' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[d, "f"] \\ b' \arrow[r] & b \end{tikzcd}$$

in $\mathcal{C}$, f' is also an L-equivalence.

Remark 3.1.3. ∞-Topoi are in particular presentable ∞-categories, and so are, accessible, complete, and cocomplete.

Remark 3.1.4. Another important property of ∞-topoi is the existence of a *subobject classifiers*. That is, an object Ω of an ∞-topos $\mathcal{X}$ which represents the functor $\mathrm{Sub}(-) : \mathcal{X} \to \mathcal{S}$ which assigns each object X to the space of (equivalence classes of) monomorphisms into X. This is the subject of [Lur09, 6.1.6].

Theorem 3.1.5 ([Lur09, 6.3.5.1])**.** *Every slice $\mathcal{X}_{/U}$ of an ∞-topos is an ∞-topos.*

A very useful fact about slice topoi is the following.

Theorem 3.1.6 (Colimits are universal in ∞-topoi [Lur09, 6.1.0.6])**.** *Let $f : U \to V$ be a morphism in an ∞-topos $\mathcal{X}$. Pulling back along f induces a functor*

$$f^* : \mathcal{X}_{/V} \to \mathcal{X}_{/U} \,.$$

The functor f^* preserves small colimits.

Proof. The proof proceeds by choosing a presentation of $\mathcal{X}$ and reducing to the case of $\mathcal{P}(\mathcal{C}_0)$. From here, since colimits and limits in $\mathcal{P}(\mathcal{C}_0)$ are computed levelwise, its enough to prove the result for $\mathcal{S}$. This is done in [Lur09, 6.1.3.14]. $\square$

Finally we introduce the notion of effective epimorphisms.

Definition 3.1.7 ([Lur09, 6.2.3.5])**.** Let $f : U \to X$ be a morphism in an ∞-topos $\mathcal{X}$. Then f is an *effective epimorphism* if and only if the Čech nerve $\check{C}(f)$ (an object of $\mathcal{X}_\Delta$) is a *simplicial resolution* of X. That is, if the natural map

$$\operatorname{colim}\Big(\cdots U \times_X U \times_X U \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} U \times_X U \rightrightarrows U \Big) \dashrightarrow X$$

is an equivalence.

Note that effective epimorphisms can also be defined in general ∞-categories. They are more relevant to the study of ∞-topoi, though.

Definition 3.1.8. A set $\{U_j\}$ of objects in an ∞-topos $\mathcal{X}$ is called a *cover* of $\mathcal{X}$ if the unique map

$$\coprod u_j \to 1_{\mathcal{X}}$$

is an effective epimorphism.

Lemma 3.1.9. *Suppose a set of objects $\{U_j\}$ of an ∞-topos $\mathcal{X}$ generates $\mathcal{X}$ under small colimits. Then there is some subset of $\{U_j\}$ which covers $\mathcal{X}$.*

3.2. Characterization of sheaf ∞-topoi

Unlike in the 1-categorical setting, *not every ∞-topos arises as the category of sheaves of spaces on a site* [Lur09, 6.2]. This is because ∞-categorically, there can be sheaves of spaces which are not "hypercomplete." In Remark 3.7.6 we give a characterization of all ∞-topoi which addresses this subtlety. For now, let's discuss a way to identify whether an ∞-topos is a category of sheaves.

It turns out that the left-exact localization $L : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$ presents $\mathcal{X}$ as the ∞-category of sheaves on $\mathcal{X}_0$ with respect to some Grothendieck topology exactly if L is a *topological localization*, which we define below.

Recall from Section 1.4 that every class of morphisms S in $\mathcal{C}$ determines a minimal "strongly saturated class" of morphisms $\bar{S}$ in $\mathcal{C}$ which in turn determines a localization $L : \mathcal{C} \to \mathcal{C}[S^{-1}]$.

Definition 3.2.1 ([Lur09, 6.2.1.5])**.** Let $\mathcal{C}$ be a presentable ∞-category, and let $\bar{S}$ be a strongly saturated class of morphisms in $\mathcal{C}$. We say $\bar{S}$ is *topological* if the following conditions are satisfied:

(i) There exists some $S \subset \bar{S}$ consisting of monomorphisms which generates $\bar{S}$ as a strongly saturated class of morphisms, and
(ii) $\bar{S}$ is stable under pullback, as in Proposition 3.1.2.

We say a localization $L : \mathcal{C} \to \mathcal{D}$ is *topological* if the class $\bar{S}$ of L-equivalences in $\mathcal{C}$ is topological.

Remark 3.2.2. Using Proposition 3.1.2, we can see that all topological localizations are left exact.

Remark 3.2.3. The data of a Grothendieck topology on a (small) ∞-category $\mathcal{C}$ exactly coincides with the data of a Grothendieck topology on the 1-category $h\mathcal{C}$ in the classical sense [Lur09, 6.2.2.3]. The formalism of covering sieves of [Lur09, 6.2.2.1] is useful in making universal constructions while avoiding set-theoretic issues.

Theorem 3.2.4 (Characterization of sheaf ∞-topoi, [Lur09, 6.2.2.9])**.** *Let $\mathcal{C}_0$ be a small ∞-category. There is a bijective correspondence between Grothendieck topologies on $\mathcal{C}_0$ and (equivalence classes of) topological localizations of $\mathcal{P}(\mathcal{C}_0)$.*

Corollary 3.2.5. *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is (equivalent to) the category of sheaves on some site if and only if there is a topological localization*

$$L : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$$

for some small subcategory $\mathcal{X}_0 \subset \mathcal{X}$.

Remark 3.2.6. It may seem unfortunate that our definition ∞-topoi does not exactly recover the notion of ∞-categories of sheaves over a site. This is not an accident. It turns out that ∞-categories of sheaves of spaces are much more subtle than their 1-categorical counterparts: there are sheaves of spaces which are not "hypercomplete." We discuss this more in Section 3.7

3.3. Sheaves over ∞-topoi

In the last section, we saw that ∞-topoi are not always the realizable as the ∞-category of sheaves of spaces over a site. In this section, we will introduce a different notion of sheaves as functors on an ∞-topos.

Definition 3.3.1 ([Lur09, 6.3.5.16], [Lur18, 1.3.1.4])**.** Let $\mathcal{X}$ be an ∞-topos. A *$\mathcal{D}$-valued sheaf* on $\mathcal{X}$ is a (small) limit preserving functor $F : \mathcal{X}^{\mathsf{op}} \to \mathcal{D}$. We denote the full subcategory of $\mathsf{Fun}(\mathcal{X}^{\mathsf{op}}, \mathcal{D})$ spanned by the sheaves by $\mathsf{Shv}(\mathcal{X}, \mathcal{D})$.

Warning 3.3.2. Note that the notion of sheaves over an ∞-topos defined here and the notion of sheaves on a site are *different*: we have always required that our sites are small. See [Lur18, 1.3.1.5] for more discussion. Note that in general, this means our categories of sheaves over ∞-topoi are "very large."

By Theorem 2.2.3, every limit-preserving functor $\mathcal{X}^{\mathsf{op}} \to \mathcal{S}$ is representable in $\mathcal{X}$. This yields the following remarkable fact.

Proposition 3.3.3. *Let $\mathcal{X}$ be an ∞-topos. The Yoneda embedding restricts to an equivalence*

$$\mathcal{X} \simeq \mathsf{Shv}(\mathcal{X}, \mathcal{S}).$$

A consequence of this is the following fact.

Corollary 3.3.4 ([Rez22, 3.5]). *With respect to truncation functors we define in Section 2.7, every ∞-topos satisfies*

$$\mathsf{Shv}(\mathcal{X}, \mathsf{Set}) \simeq \tau_{\leq 0}\, \mathcal{X} = \mathsf{Disc}(\mathcal{X}).$$

In fact, $\mathsf{Disc}(\mathcal{X})$ *is naturally equivalent to (the nerve of) a* 1*-topos and every* 1*-topos arises in this way. However, nonequivalent ∞-topoi can have equivalent underlying* 1*-topoi.*

Objects of $\mathsf{Disc}(\mathcal{X})$ are called *discrete objects* of $\mathcal{X}$.

The two preceding facts can be seen as consequences of the following.

Proposition 3.3.5 ([Lur18, 1.3.1.6]). *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a presentable ∞-category. Then there is a natural equivalence*

$$\mathsf{Shv}(\mathcal{X}, \mathcal{C}) \simeq \mathcal{X} \otimes \mathcal{C}$$

where $\otimes$ is the product on Pr^L *of Section 2.5. In particular,* $\mathsf{Shv}(\mathcal{X}, \mathcal{C})$ *is presentable.*

Through a theory of *descent* for ∞-topoi outlined in [Lur09, 6.1.3], we can deduce the following very beautiful fact which illustrates how the slices of $\mathcal{X}$ — which we can see as "subtopoi" — glue together to recover $\mathcal{X}$ in the same way that sheaves of sets over open subsets of a topological space glue together when they agree on intersections.

Theorem 3.3.6 ([Rez22, 4.10], [Lur09, 6.1.3.7]). *Let $\mathcal{X}$ be an ∞-topos. The functor*

$$\mathcal{X}^{\mathsf{op}} \to \widehat{\mathsf{Cat}}_\infty$$

sending an object U of $\mathcal{X}$ to $\mathcal{X}_{/U}$ is limit preserving, and so is a sheaf on $\mathcal{X}$ valued in large ∞-categories.

We will mention how a stronger theory of descent arises along geometric morphisms in the next section.

3.4. Geometric morphisms and ∞-categories of ∞-topoi

We have seen that ∞-topoi are not always ∞-categories of sheaves on a site. Nevertheless, they do have a very rich internal structure which an arbitrary functor of ∞-categories may not preserve. For this reason we introduce the notion of *geometric morphisms* which behave the way we want when an ∞-topos is in fact an ∞-category of sheaves.

Definition 3.4.1. A *geometric morphism* of ∞-topoi $f : \mathcal{X} \to \mathcal{Y}$ is an adjoint pair of functors

$$f^* : \mathcal{Y} \rightleftarrows \mathcal{X} : f_*$$

such that the left adjoint f^* is left exact (preserves finite limits). The functor f_* is called the *direct image*, and f^* is *pullback* or *preimage*.

Remark 3.4.2. By Theorem 2.2.1, to construct a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$, it is enough to produce a functor $f^* : \mathcal{Y} \to \mathcal{X}$ which preserves finite limits and small colimits. The latter property is typically easy to check given a presentation for $\mathcal{Y}$.

Example 3.4.3. A presentation of an ∞-topos $\mathcal{X}$ corresponds to a geometric morphism $\mathcal{X} \to \mathcal{P}(\mathcal{C})$.

Example 3.4.4. Any morphism $f : U \to V$ in an ∞-topos $\mathcal{X}$ induces a geometric morphism $f : \mathcal{X}_{/U} \to \mathcal{X}_{/V}$ where the left exact left adjoint is given by pullback along f.

Example 3.4.5 (Projection morphism [Lur09, 6.3.5.1]). A special case of the previous example is when the morphism is taken to be $\pi : U \to 1_{\mathcal{X}}$. In this case, the forgetful functor $\pi_! : \mathcal{X}_{/U} \to \mathcal{X}$ admits a right adjoint π^*. By Theorem 3.1.6, π^* preserves small colimits, and so admits its own right adjoint π_*. The pair

$$\pi^* : \mathcal{X} \rightleftarrows \mathcal{X}_{/U} : \pi_*$$

is a geometric morphism.

We denote the ∞-categories of geometric morphisms $f : \mathcal{X} \to \mathcal{Y}$ with natural transformations of the left adjoints f^* or right adjoints f_* by $\mathsf{Fun}^*(\mathcal{Y}, \mathcal{X})$ and $\mathsf{Fun}_*(\mathcal{X}, \mathcal{Y})$, respectively. These categories are canonically anti-equivalent [Lur09, 6.3.1.11] and generally not small, but are always accessible [Lur09, 6.3.1.13].

We define the subcategories of ∞-topoi LTop_∞ and RTop_∞ in $\widehat{\mathsf{Cat}}_\infty$ to be the ∞-categories with objects which are ∞-topoi and mapping spaces between $\mathcal{X}$ and $\mathcal{Y}$ which are the "maximal ∞-groupoid" (i.e., the *core*) inside $\mathsf{Fun}^*(\mathcal{Y}, \mathcal{X})$ or $\mathsf{Fun}_*(\mathcal{X}, \mathcal{Y})$, respectively [Lur09, 6.3.1.5]. Note that LTop_∞ and RTop_∞ are canonically anti-equivalent as well.

Remark 3.4.6. Note that RTop_∞ and LTop_∞ are more naturally considered as "∞-bicategories" since we might consider non-invertible natural transformations of geometric morphisms. Doing this leads to extreme technical difficulties which we won't get into. See the discussion at the end of [Lur09, 6.3.4].

Example 3.4.7 ([Lur09, 6.3.4.1])**.** The terminal object of RTop_∞ is $\mathcal{S}$. That is, there is an essentially unique geometric morphism $f : \mathcal{X} \to \mathcal{S}$ from any ∞-topos $\mathcal{X}$.

It is convenient to view the right adjoint f_* as a *global sections* functor, Γ. Indeed, for any X in $\mathcal{X}$, we have

$$f_*(X) \simeq \Gamma(X) \simeq \mathsf{Maps}_{\mathcal{X}}(1_{\mathcal{X}}, X).$$

Theorem 3.4.8 ([Lur09, 6.3.2.3, 6.3.3.1, and 6.3.4.7])**.** *RTop_∞ admits all small limits and small colimits. Moreover, the inclusion*

$$\mathsf{RTop}_\infty \hookrightarrow \widehat{\mathsf{Cat}}_\infty$$

preserves small colimits and small filtered limits.

Remark 3.4.9. The statements about colimits in the previous theorem are relatively formal to prove [Lur09, 6.3.2]. The statements about limits are much more subtle. For example, the inclusion into $\widehat{\mathsf{Cat}}_\infty$ does not preserve final objects since it does not send $\mathcal{S}$ to the terminal category.

Example 3.4.10. Finite products in RTop_∞ are given by the restriction of the tensor product on Pr^L [Lur09, 7.3.3].

Given any ∞-category $\mathcal{A}$ admitting small limits, a geomtric morphism $f : \mathcal{X} \to \mathcal{Y}$ induces a direct image functor $f_* : \mathsf{Shv}(\mathcal{X}, \mathcal{A}) \to \mathsf{Shv}(\mathcal{Y}, \mathcal{A})$ given by *precomposition* with f^*.

If $\mathcal{A}$ is compactly generated, we can formulate this differently. In this case, we can identify

$$\mathsf{Shv}(\mathcal{X}, \mathcal{A}) \simeq \mathsf{Fun}^{\mathrm{lex}}((\mathcal{A}^\omega)^{\mathsf{op}}, \mathcal{X})$$

where $\mathsf{Fun}^{\mathrm{lex}}$ denotes left exact functors. Now a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ induces direct image and pullback functors

via *postcomposition* with $f_* : \mathcal{X} \to \mathcal{Y}$ and $f^* : \mathcal{Y} \to \mathcal{X}$, respectively. See [Rez22, 6.16] and [Lur12, V.1.1.8].

An immediate consequence of this is the following which is a strengthening of Theorem 3.3.6.

Theorem 3.4.11 (Descent for sheaves over ∞-topoi, [Rez22, 6.17])**.** *Let $\mathcal{A}$ be a compactly generated ∞-category, and suppose $\mathcal{X} \simeq \operatorname{colim}_i \mathcal{X}_i$ in LTop_∞. Then*

$$\mathsf{Shv}(\mathcal{X}, \mathcal{A}) \simeq \lim_i \mathsf{Shv}(\mathcal{X}_i, \mathcal{A})$$

in $\widehat{\mathsf{Cat}}_\infty$, where the limit is taken over the pullback functors defined above.

In particular, if $U \simeq \operatorname{colim}_i U_i$ in $\mathcal{X}$, then

$$\mathsf{Shv}(\mathcal{X}_{/U}, \mathcal{A}) \simeq \lim_i \mathsf{Shv}(\mathcal{X}_{/U_i}, \mathcal{A}).$$

Proof. Barring some technicalities, a good way to see this is using Theorem 2.5.1. □

Definition 3.4.12 ([Lur09, 6.3.6.1 and 6.3.6.2])**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. The *image* of f, $\mathrm{Im}(f)$, is the smallest subcategory of $\mathcal{X}$ containing $f^*\mathcal{Y}$ and which is closed under small colimits and finite limits. Note that $\mathrm{Im}(f)$ is always an ∞-topos itself. We say f is *algebraic* if $\mathcal{X} \simeq \mathrm{Im}(f)$.

Proposition 3.4.13 ([Lur09, 6.3.6.2])**.** *Any geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ factors as a composition of geometric morphisms*

$$\mathcal{X} \xrightarrow{g} \mathrm{Im}(f) \xrightarrow{h} \mathcal{Y}$$

where g^* is fully faithful and h is algebraic.

There is also a nice criterion for checking whether a geometric morphism is an equivalence. To state it, we need to introduce the notion of n-localic ∞-topoi.

Definition 3.4.14 ([Lur09, 6.4.5.9]). Let $\mathcal{X}$ be an ∞-topos. We say $\mathcal{X}$ is *n-localic* if, for any ∞-topos Y, the natural map

$$\mathsf{Fun}_*(\mathcal{Y}, \mathcal{X}) \to \mathsf{Fun}_*(\tau_{\leq n-1}\,\mathcal{Y}, \tau_{\leq n-1}\,\mathcal{X})$$

is an equivalence of ∞-categories.

Proposition 3.4.15 ([Lur09, 6.4.5.9]). *Let $\mathcal{X}$ be an n-localic ∞-topos. Any topological localization of $\mathcal{X}$ is also n-localic.*

Proposition 3.4.16 ([Lur09, 6.3.6.7]). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. Suppose*

(i) f^* is fully faithful,
(ii) $\tau_{\leq 1}\,\mathcal{X}$ is contained in the essential image of f^*, and
(iii) $\mathcal{X}$ is n-localic for some n.

Then f is an equivalence of ∞-topoi.

3.5. **Étale morphisms**

Let us first illustrate the classical picture of étale morphisms of topological spaces.

Example 3.5.1 ([Lur09, 6.3.5]). Let $f : X \to Y$ be a continuous map of topological spaces. We say f is *étale* if it is a local homeomorphism. There is an equivalence of categories

$$\{f : X \to Y \text{ étale}\} \simeq \mathsf{Shv}(Y, \mathsf{Set})$$

given by sending an étale map $f : X \to Y$ to its sheaf of sections $\mathcal{F}$. The inverse functor is obtained by identifying

$$\mathsf{Shv}(Y, \mathsf{Set})_{/\mathcal{F}} \simeq \mathsf{Shv}(X, \mathsf{Set}).$$

It turns out that there is a notion of étaleness for geometric morphisms of ∞-topoi which buys us similar results.

Definition 3.5.2 ([Lur09, 6.3.5]). A geometric morphism of ∞-topoi $f : \mathcal{X} \to \mathcal{Y}$ is *étale* if f_* admits a factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "\varphi"] \arrow[dr, "f_*"'] & \mathcal{Y}_{/U} \arrow[d, "\pi_*"] \\ & \mathcal{Y} \end{tikzcd}$$

where φ is an equivalence of categories and U is an object of $\mathcal{Y}$.

Remark 3.5.3. Note that in the above definition, f^* itself admits a left adjoint, $f_! = \pi_! \circ \varphi$. In particular, f^* preserves all limits, not just finite limits.

Remark 3.5.4 ([Lur09, 6.3.5.2]). Given an étale morphism $f : \mathcal{X}_{/U} \to \mathcal{X}$, we have a double adjunction

$$f_! \dashv f^* \dashv f_*.$$

The three morphisms are characterized as follows:

(i) $f_! : \mathcal{X}_{/U} \to \mathcal{X}$ is the "forgetful functor" sending $X \to U$ to X.
(ii) $f^* : \mathcal{X} \to \mathcal{X}_{/U}$ is the "pullback functor" sending X to $X \times_{1_{\mathcal{X}}} U \to U$.
(iii) $f_* : \mathcal{X}_{/U} \to \mathcal{X}$ is the "global sections functor" sending $p : X \to U$ to the object P in $\mathcal{X}$ representing the (limit preserving) functor

$$\mathsf{Maps}_{\mathcal{X}_{/U}}(f^*(-), p) : \mathcal{X}^{\mathsf{op}} \to \hat{\mathcal{S}}.$$

One can check that for any V in $\mathcal{X}$, $\mathsf{Maps}_{\mathcal{X}_{/U}}(f^*(V), p)$ exactly corresponds to the space of sections $s : U \to X$ of p.

Remark 3.5.5. This notion of étaleness is a generalization of the classical story with schemes. To an étale morphism of schemes $f : X \to Y$, we can associate the same three functors $f_!$, f^*, and f_* on categories of sheaves of modules over X and Y. Analogously to our situation, f^* is exact since f is flat.

The collection of étale morphisms in RTop_∞ contains all equivalences and is stable under composition, so we can consider the subcategory $\mathsf{RTop}_{\infty,\text{ét}}$ consisting of all ∞-topoi and the étale morphisms.

Theorem 3.5.6 (Characterization of étale morphisms [Lur09, 6.3.5.11])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then f is étale if and only if the following conditions hold.*

(i) The functor f^* admits a left adjoint $f_!$,
(ii) The functor $f_!$ is conservative*, i.e., if α is a morphism in $\mathcal{X}$ such that $f_!\alpha$ is an equivalence in $\mathcal{Y}$, then α is an equivalence,*
(iii) for every morphism $X \to Y$ in $\mathcal{X}$, every object Z in $\mathcal{Y}$, and every morphism $f_!Z \to Y$, the following induced diagram is a pullback:

$$\begin{tikzcd} f_!(f^*X \times_{f^*Y} Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & f_!Z \arrow[d] \\ X \arrow[r] & Y. \end{tikzcd}$$

Remark 3.5.7. Condition (iii) of the previous theorem should be read as a "push-pull formula." That is, it canonically identifies

$$f_!(f^*X \times_{f^*Y} Z) \simeq X \times_Y f_!Z.$$

Theorem 3.5.8 (Colimits along étale morphisms [Lur09, 6.3.5.13])**.** $\mathsf{RTop}_{\infty,ét}$ *admits all small colimits, and the inclusion*

$$\mathsf{RTop}_{\infty,ét} \hookrightarrow \mathsf{RTop}_\infty$$

preserves small colimits.

The following (formal) fact gives a further justification to the method of studying an ∞-topos $\mathcal{X}$ by studying its slices $\mathcal{X}_{/U}$ and how they "glue together." Essentially the theorem says that $\mathcal{X}$ is equivalent to its own "small étale site." (Compare this to Theorem 3.3.6.)

Theorem 3.5.9 ([Lur09, 6.3.5.10])**.** *Let $\mathcal{X}$ be an ∞-topos. The functor $\chi : \mathcal{X} \to \mathsf{RTop}_\infty$ sending each object U to $\mathcal{X}_{/U}$ is fully faithful and its essential image consists of ∞-topoi étale over $\mathcal{X}$. That is, we have a diagram of ∞-categories*

$$\mathcal{X} \simeq (\mathsf{RTop}_{\infty,ét})_{/\mathcal{X}} \hookrightarrow \mathsf{RTop}_\infty\,.$$

Finally, here are a few more nice facts about étale morphisms.

Lemma 3.5.10 ([Lur09, 6.3.5.9])**.** *Suppose we are given a diagram*

$$\begin{tikzcd} & \mathcal{Y} \arrow[dr, "g"] & \\ \mathcal{X} \arrow[ur, "f"] \arrow[rr, "h"'] & & \mathcal{Z} \end{tikzcd}$$

in RTop_∞ with g étale. Then f is étale if and only if h is étale.

Proposition 3.5.11 ([Lur09, 6.3.6.4])**.** *Every étale morphism of ∞-topoi is algebraic.*

3.6. Homotopy sheaves and connectivity in ∞-topoi

Let $\mathcal{X}$ be an ∞-topos, U an object of $\mathcal{X}$, and K a space. Consider the functor

$$\mathsf{Maps}_{\mathcal{S}}(K, \mathsf{Maps}_{\mathcal{X}}(-, U)) : \mathcal{X}^{\mathsf{op}} \to \mathcal{S}$$

which is clearly limit preserving. Since $\mathcal{X}$ is presentable, by Theorem 2.2.3, this functor is representable in $\mathcal{X}$ by an object U^K. (See Section 3.9 for more discussion of this construction.)

Let $\eta : * \to S^n$ be a map of spaces defining a basepoint of S^n. Evaluation at this basepoint induces a restriction map

$$\mathsf{Maps}_{\mathcal{X}}(-, U^{S^n}) \simeq \mathsf{Maps}_{\mathcal{S}}(S^n, \mathsf{Maps}_{\mathcal{X}}(-, U)) \xrightarrow{\eta^*} \mathsf{Maps}_{\mathcal{S}}(*, \mathsf{Maps}_{\mathcal{X}}(-, U)) \simeq \mathsf{Maps}_{\mathcal{X}}(-, U).$$

By Proposition 3.3.3, we obtain a map

$$\mathsf{ev}_*^U : U^{S^n} \to U$$

corresponding to evaluation at the basepoint of S^n.

Definition 3.6.1 ([Lur09, 6.5.1.1]). Let $\mathcal{X}$ be an ∞-topos and U and object of $\mathcal{X}$. The *nth homotopy sheaf* of U is the object of $\mathsf{Disc}(\mathcal{X}_{/U})$ defined by

$$\pi_n(U) := \tau_{\leq 0}(\mathsf{ev}_*^U).$$

Let F_U be the source object of $\pi_n(U)$ in $\mathcal{X}$. To be explicit, the morphism $\pi_n(U)$ fits into the following triangle in $\mathcal{X}$.

$$\begin{tikzcd} U^{S^n} \arrow[r] \arrow[dr, "\mathsf{ev}_*^U"'] & F_U \arrow[d, "\pi_n U"] \\ & U. \end{tikzcd}$$

Remark 3.6.2. As we have constructed it, $\pi_n(U)$ is a sheaf of sets over $\mathcal{X}_{/U}$. The map $S^n \to *$ induces a map $U \to U^{S^n}$ which makes $\pi_n(U)$ a sheaf of pointed sets. In fact, for $n \geq 1$, $\pi_n(U)$ is naturally a sheaf of groups, and for $n \geq 2$, a sheaf of abelian groups. This can be deduced using the usual cogroup structure on S^n for $n \geq 1$ and from the fact that in any ∞-topos, the truncation functor $\tau_{\leq n}$ commutes with finite products [Lur09, 6.5.1.2].

Let $f : U \to X$ be a morphism of $\mathcal{X}$. Since $\mathcal{X}_{/X}$ is an ∞-topos itself, we can define $\pi_n(f)$ just as in Definition 3.6.1. After identifying

$$(\mathcal{X}_{/X})_{/f} \simeq \mathcal{X}_{/f} \simeq \mathcal{X}_{/U},$$

we can consider $\pi_n(f)$ as an object $\pi_n(f) : F_f \to U$ of $\mathcal{X}_{/U}$ and form the following diagram in $\mathcal{X}$.

$$F_f \xrightarrow{\pi_n(f)} U \xrightarrow{f} X.$$

The object F_f of $\mathcal{X}$ should be thought of as the "generalized homotopy fiber" of f in sense that the vanishing of $\pi_n(f)$ gives us information about the connectivity of f as we will see later in this section.

The homotopy sheaves of an object U of an ∞-topos $\mathcal{X}$ can be understood similarly by considering the (essentially) unique map η from U to the terminal object of $\mathcal{X}$. As above, we get

$$F_U \xrightarrow{\pi_n(\eta)} U \xrightarrow{\eta} 1_{\mathcal{X}}.$$

In fact, we can naturally identify $\pi_n(\eta) \simeq \pi_n(U)$ as objects of $\mathcal{X}_{/U}$. Choosing a base point $u : 1_{\mathcal{X}} \to U$, we can form the pullback

$$\begin{tikzcd} u^*\pi_n(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_U \arrow[d, "\pi_n(U)"] \\ 1_{\mathcal{X}} \arrow[r, "u"'] & U \end{tikzcd}$$

which determines a group (or set, if $n = 0$)

$$\pi_n(U, u) := u^*\pi_n(U)$$

which we can consider the "nth homotopy group of U with respect to the base point u."

Example 3.6.3 ([Lur09, 6.5.1.6]). If $\mathcal{X} = \mathcal{S}$ and $x : * \to X$ is a pointed space, then $x^*\pi_n(X)$ can naturally be identified with the (classical) homotopy group $\pi_n(X, x)$ of X, relative to the base point x.

Remark 3.6.4 ([Lur09, 6.5.1.3]). The following is an alternative recursive definition of the homotopy groups of a morphism $f : U \to X$ in an ∞-topos $\mathcal{X}$. Let $n > 0$, and $\Delta : X \to X \times_U X$. We get a natural isomorphism

$$\pi_n(f) \simeq \pi_{n-1}(\Delta)$$

in $\mathsf{Disc}(\mathcal{X}_{/U})$. Thus, it is sufficient to define π_0 of morphisms. We have

$$\pi_0(f) = U \times_X \tau_{\leq 0}(f).$$

Remark 3.6.5 ([Lur09, 6.5.1.4]). Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, and $g : Y \to Y'$ a morphism in $\mathcal{Y}$. Proposition 2.7.8 gives a canonical isomorphism

$$f^*\pi_n(g) \simeq \pi_n(f^*(g))$$

of objects in $\mathsf{Disc}(\mathcal{X}_{/f^*Y})$.

Now let us discuss what homotopy sheaves tell us about connectivity.

Remark 3.6.6 ([Lur09, 6.2.3.5])**.** Effective epimorphisms in ∞-topoi can equivalently be characterized as follows. Let $f : U \to X$ be a morphism in an ∞-topos $\mathcal{X}$. We say f is an *effective epimorphism* if $\tau_{\leq -1} f$ is a final object of $\mathcal{X}_{/X}$.

Example 3.6.7. A map of spaces $f : X \to Y$ is an effective epimorphism if and only if it is a π_0-surjection.

Proposition 3.6.8 ([Lur09, 6.5.1.10])**.** *Let $f : U \to X$ be a morphism in an ∞-topos $\mathcal{X}$ and $0 \leq n \leq \infty$. Then f is n-connective if*

(i) f is an effective epimorphism, and
(ii) $\pi_k(f) = *$ for $0 \leq k < n$.

An object X of $\mathcal{X}$ is n-connective *if the map $X \to 1_{\mathcal{X}}$ is. By convention, we say every morphism in $\mathcal{X}$ is (-1)-connective.*

This characterization of n-connective morphisms coincides with Definition 2.7.5. Another alternative characterization of n-connective morphisms is the following.

Proposition 3.6.9 ([Lur09, 6.5.1.14])**.** *Let $f : U \to X$ be a morphism in an ∞-topos $\mathcal{X}$. Then f is n-connective if and only if the natural map*

$$\mathsf{Maps}_{\mathcal{X}_{/X}}(\mathsf{id}_X, g) \to \mathsf{Maps}_{\mathcal{X}_{/X}}(f, g)$$

is an equivalence for every $(n-1)$-truncated g in $\mathcal{X}_{/X}$.

Remark 3.6.10. Taking X to be $1_{\mathcal{X}}$, the previous proposition implies all $(n-1)$-truncated objects of $\mathcal{X}$ are local with respect to n-connective morphisms in $\mathcal{X}$.

3.7. **Hypercomplete objects**

Recall from Proposition 3.6.8 that an ∞-connective morphism $f : U \to X$ in an ∞-topos $\mathcal{X}$ is exactly an effective epimorphism for which $\pi_n(f) = *$ for all n.

Regarding the discussion in Section 3.6, we should think of this as meaning the "homotopy fiber of f vanishes." Note that in $\mathcal{S}$, morphisms with vanishing homotopy fibers are precisely the weak equivalences. It turns out that a general Whitehead theorem does not hold in ∞-topoi: ∞-*connective morphisms in* $\mathcal{X}$ *are not necessarily equivalences in* $\mathcal{X}$ (however all equivalences are ∞-connective).

Hypercomplete ∞-topoi are precisely the ∞-topoi where such a Whitehead theorem does hold. This leads us to our present discussion.

Definition 3.7.1. An object U of an ∞-topos is *hypercomplete* if it is local with respect to the class of ∞-connected morphisms in $\mathcal{X}$ in the sense of Definition 1.4.6. That is, whenever $f : V \to V'$ is an ∞-connected morphism in $\mathcal{X}$, the restriction

$$f^* : \mathsf{Maps}_{\mathcal{X}}(V', U) \to \mathsf{Maps}_{\mathcal{X}}(V, U)$$

is a homotopy equivalence.

Definition 3.7.2. We say an ∞-topos $\mathcal{X}$ is *hypercomplete* if all ∞-connective morphisms in $\mathcal{X}$ are equivalences.

We denote the full subcategory of hypercomplete objects in $\mathcal{X}$ by $\mathcal{X}^\wedge$. The inclusion $\mathcal{X}^\wedge \hookrightarrow \mathcal{X}$ admits an accessible, left exact left adjoint, so $\mathcal{X}^\wedge$ is an ∞-topos. As we might hope, $\mathcal{X}^\wedge$ is itself hypercomplete [Lur09, 6.5.2.12].

Remark 3.6.10 implies the following.

Proposition 3.7.3 ([Lur09, 6.5.2.9])**.** *Truncated objects in an ∞-topos are hypercomplete.*

The next result is a universal property of hypercompletion.

Theorem 3.7.4 ([Lur09, 6.5.2.13])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and suppose $\mathcal{Y}$ is hypercomplete. The inclusion $\mathcal{X}^\wedge \to \mathcal{X}$ induces an equivalence*

$$\mathsf{Fun}_*(\mathcal{Y}, \mathcal{X}^\wedge) \to \mathsf{Fun}_*(\mathcal{Y}, \mathcal{X}).$$

Definition 3.7.5 ([Lur09, 6.5.2.17])**.** Let $\mathcal{X}$ be an ∞-topos and $L : \mathcal{X} \to L\mathcal{X}$ an accessible left-exact localization. We say L is a *cotopological* localization if for every morphism u in $\mathcal{X}$, if Lu is an equivalence then u is ∞-connective.

Remark 3.7.6 ([Lur09, 6.5.2.20])**.** Every ∞-topos $\mathcal{X}$ has a presentation

$$\mathcal{P}(\mathcal{C}_0) \xrightarrow{L_1} \mathsf{Shv}(\mathcal{C}_0) \xrightarrow{L_2} \mathcal{X}$$

where $\mathcal{C}_0$ is a (small) Grothendieck site, L_1 is a topological localization, and L_2 is a cotopological localization.

The upshot is that every ∞-topos can be built via the following recipe.

(i) Begin with a small ∞-category $\mathcal{C}_0$, and form $\mathcal{P}(\mathcal{C}_0)$.
(ii) Choose a Grothendieck topology on $\mathcal{C}_0$.
(iii) Form the associated topological localization of $\mathcal{P}(\mathcal{C}_0)$ to obtain $\mathsf{Shv}(\mathcal{C}_0)$.
(iv) Finally, form a cotopological localization of $\mathsf{Shv}(\mathcal{C}_0)$ by inverting some class of ∞-connective morphisms.

Note that this means every ∞-topos $\mathcal{X}$ fits into a sequence of cotopological localizations

$$\mathsf{Shv}(\mathcal{C}_0) \to \mathcal{X} \to \mathsf{Shv}(\mathcal{C}_0)^\wedge$$

for some small Grothendieck site $\mathcal{C}_0$.

Definition 3.7.7. Let $\mathcal{X}$ be an ∞-topos. A *point* of $\mathcal{X}$ is a geometric morphism $\mathcal{S} \to \mathcal{X}$.

Remark 3.7.8. Recall that $\mathcal{S}$ is terminal in RTop_∞, and is equivalently the ∞-topos of (hypercomplete) sheaves of spaces on $*$. Thus if $\mathcal{X}$ is an ∞-topos of sheaves on a site $\mathcal{C}_0$, a point of $\mathcal{X}$ exactly coincides with the classical notion of the adjoint pair $f^* \dashv f_*$ induced by the inclusion of a point $f : * \to \mathcal{C}_0$.

Remark 3.7.9 ([Lur09, 6.5.4.7])**.** We say an ∞-topos $\mathcal{X}$ has *enough points* if it has the following property: a morphism f in $\mathcal{X}$ is an equivalence if and only if $p^*f : \mathcal{S} \to \mathcal{S}$ is an equivalence for every point p of $\mathcal{X}$. That is, roughly, that equivalences in $\mathcal{X}$ can be checked stalk-wise.

Note that if f is ∞-connective, then p^*f is ∞-connective, and therefore an equivalence in $\mathcal{S}$. It follows that an ∞-topos has enough points if and only it is hypercomplete.

3.8. The coskeleton monad and hyperdescent

In this section, we'll give a brief overview of hyperdescent, which is some stronger version of descent for sheaves of spaces on a site. This will result in an alternate characterization of the hypercompletion $\mathcal{X}^\wedge$.

The theory of hypercovers is originally due to Verdier. The place to start is with simplicial stuff.

Definition 3.8.1. Let Δ be the (nerve of the) simplex category, and for each $n \geq 0$, let $\Delta^{\leq n}$ be the full subcategory spanned by the objects $\{[0], \dots, [n]\}$. Given a presentable ∞-category $\mathcal{C}$, let $\mathcal{C}_\Delta$ denote the ∞-category $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathcal{C})$ of *simplicial objects* in $\mathcal{C}$.

For $n \geq 0$, the *(simplicial) truncation functor* tr_n is given by restriction along the inclusion $\Delta^{\leq n} \hookrightarrow \Delta$:

$$\mathrm{tr}_n : \mathcal{C}_\Delta \to \mathsf{Fun}((\Delta^{\leq n})^{\mathsf{op}}, \mathcal{C}),$$

taking values in *n-truncated simplicial objects* in $\mathcal{C}$. By Proposition 2.2.4, each object of $\mathcal{C}_{\Delta^{\leq n}}$ admits left and right Kan extensions along $\Delta^{\leq n} \hookrightarrow \Delta$. Thus, tr_n admits (fully faithful) left and right adjoints:

$$\mathrm{tr}_n^L \dashv \mathrm{tr}_n \dashv \mathrm{tr}_n^R .$$

We define the *n-skeleton comonad*, sk_n, and *n-coskeleton monad*, cosk_n, to be the comonad and monad of the adjunctions $\mathrm{tr}_n^L \dashv \mathrm{tr}_n$ and $\mathrm{tr}_n \dashv \mathrm{tr}_n^R$, respectively. That is $\mathrm{sk}_n \simeq \mathrm{tr}_n^L \circ \mathrm{tr}_n$ and $\mathrm{cosk}_n \simeq \mathrm{tr}_n^R \circ \mathrm{tr}_n$ as objects of $\mathrm{End}(\mathcal{C}_\Delta)$.

By convention, for a given simplicial object U_* in $\mathcal{C}$, we say $\mathrm{sk}_{-1} U_*$ is the initial object of $\mathcal{C}$ and $\mathrm{cosk}_{-1} U_* \simeq 1_\mathcal{C}$ (when these objects of $\mathcal{C}$ exist).

Remark 3.8.2. The counit of sk_n and unit of cosk_n induce natural maps

$$\mathrm{sk}_n U_* \to U_* \qquad \text{and} \qquad U_* \to \mathrm{cosk}_n U_*$$

for any simplicial object U_* in $\mathcal{C}$. If $\mathcal{C}$ is pointed and admits finite limits, the fiber of the latter map may rightfully be thought of as the *n-connective cover* of U_*.

Remark 3.8.3. Intuitively, the n-skeleton functor takes a simplicial object, truncates it at degree n, and then freely attaches degenerate simplicies above degree n. The n-coskeleton functor truncates a simplicial object at degree n and then, for each $m > n$, attaches an m-simplex to every compatible family of n-faces. That is $\mathrm{sk}_n U_*$ is the "smallest" simplicial object containing $\mathrm{tr}_n U_*$ and $\mathrm{cosk}_n U_*$ is the "largest."

Remark 3.8.4 (Postnikov towers)**.** Note that the fully faithfullness property of Definition 3.8.1 guarantees that for $m \leq n$, we have

$$(\operatorname{sk}_m U_*)_n \simeq U_m \simeq (\operatorname{cosk}_m U_*)_n.$$

If X is a space (Kan complex), then the canonical map $X \to \operatorname{cosk}_n X$ is an isomorphism on π_k for $k \leq n$, while $\pi_k \operatorname{cosk}_n X = 0$ for $k > n$. Thus, the extra cells in $\operatorname{cosk}_n X$ should be thought of as "killing higher homotopy groups." Indeed, the usual *Postnikov tower* for X looks like

$$X \to \cdots \to \operatorname{cosk}_n X \to \operatorname{cosk}_{n-1} X \to \cdots \to \operatorname{cosk}_0 X \to \operatorname{cosk}_{-1} X = *.$$

This identification is made as early as [AM69]. If X is a CW complex, this is a limit diagram. And if X is pointed, the homotopy fiber of the map $\operatorname{cosk}_n X \to \operatorname{cosk}_{n-1} X$ is the Eilenberg-Maclane space $K(\pi_n X, n)$.

Definition 3.8.5 ([Lur09, 6.5.3.2])**.** Let $\mathcal{X}$ be an ∞-topos. A simplicial object U_* in $\mathcal{X}_\Delta$ is a *hypercover* of $\mathcal{X}$ if for each $n \geq 0$, the natural map

$$U_n \to (\operatorname{cosk}_{n-1} U_*)_n$$

is an effective epimorphism. Moreover, U_* is an *effective hypercover* if $\operatorname{colim} U_* \simeq 1_{\mathcal{X}}$.

Remark 3.8.6 ([Lur09, 6.5.3.3], [AM69, 8.5(b)])**.** Intuitively, a simplicial object U_* in $\mathcal{X}_\Delta$ is a hypercover of $\mathcal{X}$ if each map

$$\begin{aligned}
&U_0 \to 1_{\mathcal{X}},\\
&U_1 \to U_0 \times U_0,\\
&U_2 \to (U_1 \times_{U_0} U_1 \times_{U_0} U_1) \times_{(U_0 \times U_0)} U_0,\\
&\quad \cdots\\
&U_n \to (\operatorname{cosk}_{n-1} U_*)_n
\end{aligned}$$

is an effective epimorphism. Notice that by Remark 3.8.4, $(\operatorname{cosk}_{n-1} U_*)_n$ is totally determined by $U_0, \ldots, U_{n-1}$. Thus, we can think of building a hypercover U_* inductively by starting with an effective epimorphism $U_0 \to 1_{\mathcal{X}}$, forming $(\operatorname{cosk}_0 U_*)_1$, choosing any object U_1 and effective epimorphism $U_1 \to (\operatorname{cosk}_0 U_*)_1$, and then forming $(\operatorname{cosk}_1 U_*)_2$, and so on, as in the following diagram.

$$\begin{tikzcd}
\cdots (\operatorname{cosk}_2 U_*)_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & U_2 \arrow[d, dashed] & & & \\
& (\operatorname{cosk}_1 U_*)_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U_1 \arrow[d, dashed] & & \\
& & (\operatorname{cosk}_0 U_*)_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & U_0 \arrow[d, dashed] & \\
& & & (\operatorname{cosk}_{-1} U_*)_0 \arrow[r, "\simeq"] & 1_{\mathcal{X}}
\end{tikzcd}$$

The result is a generalization of the usual notion of cover in Definition 3.1.8; instead of letting U_n dictate the "n-fold intersections" of the cover $U_0 \to X$, we allow ourselves at each step to pass to a cover of the intersection itself.

A hypercover of an object X in an ∞-topos $\mathcal{X}$ is a hypercover of $\mathcal{X}_{/X}$ in the sense above.

Theorem 3.8.7 ([Lur09, 6.5.3.12])**.** *Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is hypercomplete if and only if for each object X of $\mathcal{X}$, every hypercovering U_* of $\mathcal{X}_{/X}$ is effective.*

Remark 3.8.8 ([Lur09, 6.5.3.13])**.** The previous theorem characterizes hypercomplete ∞-topoi as the categories of sheaves which satisfy *hyperdescent.* Let us make this precise. Taking the colimit (geometric realization) of a hypercover U_* of $\mathcal{X}_{/X}$, we obtain a morphism

$$\operatorname{colim} U_* \to X.$$

To say that U_* is effective is to say that this morphism is an equivalence. It turns out that if S is the class of all morphisms in $\mathcal{X}$ which arise in this way, then

$$\mathcal{X}[S^{-1}] \simeq \mathcal{X}^\wedge.$$

That is, hypercompletion is "localization at the geometric realizations of hypercovers."

We refer the reader to [Lur09, 6.5.4] for a discussion of the subtle differences between descent and hyperdescent. In many important ways, $\mathsf{Shv}(\mathcal{C}_0)$ tends to be more well-behaved that $\mathsf{Shv}(\mathcal{C}_0)^\wedge$.

3.9. Closed monoidal structure

Theorem 3.9.1. *Every ∞-topos is Cartesian closed. That is, for any object X of an ∞-topos $\mathcal{X}$, the functor $X \times (-) : \mathcal{X} \to \mathcal{X}$ admits a right adjoint:*

$$X \times (-) \dashv \mathcal{M}\mathsf{aps}_{\mathcal{X}}(X, -).$$

Proof. Let $\pi : X \to 1_{\mathcal{X}}$. The functor $X \times (-) : \mathcal{X} \to \mathcal{X}$ can be written

$$X \times (-) \simeq \pi_! \pi^*.$$

Note that π^* preserves colimits by Theorem 3.1.6, and $\pi_!$ is left adjoint to π^* and so also preserves colimits. It follows by Theorem 2.2.1 that $X \times (-)$ admits a right adjoint. □

We can rephrase the above argument to say that $\mathcal{M}\mathsf{aps}_{\mathcal{X}}(U, V)$ is the object of $\mathcal{X}$ representing the (limit preserving) functor

$$\mathsf{Maps}_{\mathcal{X}}(U \times (-), V) : \mathcal{X}^{\mathsf{op}} \to \mathcal{S}\,.$$

Recall from Example 3.4.7 that every ∞-topos $\mathcal{X}$ admits an essentially unique geometric morphism to $\mathcal{S}$:

$$\mathrm{const} : \mathcal{S} \rightleftarrows \mathcal{X} : \Gamma$$

The right adjoint is given by taking global sections. The left adjoint is the *constant sheaf* functor defined as follows. Let K be a space; the functor $\mathcal{X}^{\mathsf{op}} \to \mathcal{S}$ sending each object of $\mathcal{X}$ to K is limit preserving, and so is representable by an object $\mathrm{const}(K)$ of $\mathcal{X}$ by Theorem 2.2.3.

Definition 3.9.2. Let K be a space, and X an object of an ∞-topos $\mathcal{X}$. The *power object* or *cotensor*, X^K, (when it exists) is the object of $\mathcal{X}$ characterized by the universal property

$$\mathsf{Maps}_{\mathcal{X}}(U, X^K) \simeq \mathsf{Maps}_{\mathcal{S}}(K, \mathcal{M}\mathsf{aps}_{\mathcal{X}}(U, X))$$

for any other object U of $\mathcal{X}$.

By the discussion above, the cotensor X^K exists in $\mathcal{X}$ for every object X and space K. Thus, we say $\mathcal{X}$ is *cotensored* over $\mathcal{S}$ [Lur09, 5.5.2.6].

Remark 3.9.3. Tensors and cotensors in an ∞-topos $\mathcal{X}$ are also characterized in the following way. Let K be a space and X an object of $\mathcal{X}$. We have

$$\mathcal{M}\mathsf{aps}_{\mathcal{X}}(\mathrm{const}(K), X) \simeq \lim_K X$$

and

$$\mathrm{const}(K) \times X \simeq \mathrm{colim}_K\, X.$$

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