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Trend Following Trading under a Regime Switching Model*

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Abstract. This paper is concerned with the optimality of a trend following trading rule. The idea is to catch a bull market at its early stage, ride the trend, and liquidate the position at the first evidence of the subsequent bear market. We characterize the bull and bear phases of the markets mathematically using the conditional probabilities of the bull market given the up to date stock prices. The optimal buying and selling times are given in terms of a sequence of stopping times determined by two threshold curves. Numerical experiments are conducted to validate the theoretical results and demonstrate how they perform in a marketplace.

Key words. optimal stopping time, regime switching model, Wonham filter, trend following trading rule

AMS subject classifications. 91G80, 93E11, 93E20

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1. Introduction.

Trading in organized exchanges has increasingly become an integrated part of our life. Big moves of market indices of major stock exchanges all over the world are often the headlines of news media. By and large, active market participants can be classified into two groups according to their trading strategies: those who trade contra-trend and those who follow the trend. On the other hand, there are also passive market participants who simply buy and hold for a long period of time (often indirectly through mutual funds). Within each group of strategies there are numerous technical methods. Much effort has been devoted to theoretical analysis of these strategies.

Using optimal stopping time to study optimal exit strategy for stock holdings has become the standard textbook method. For example, Øksendal [24, Examples 10.2.2 and 10.4.2] considered optimal exit strategy for stocks whose price dynamics were modeled by a geometric Brownian motion. To maximize an expected return discounted by the risk-free interest rate, the analysis in [24] showed that if the drift of the geometric Brownian motion was not high enough in comparison to the discount of interest rate, then one should sell at a given threshold. Although the model of a single geometric Brownian motion with a constant drift was somewhat too simplistic, this result well illustrated the flaw of the so-called buy and hold strategy, which worked only in limited situations. Stock selling rules under more realistic models have gained increasing attention. For example, Zhang [31] considered a selling rule determined by two

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threshold levels, a target price and a stop-loss limit. Under a regime switching model, optimal threshold levels were obtained by solving a set of two-point boundary value problems. In Guo and Zhang [13], the results of Øksendal [24] were extended to incorporate a model with regime switching. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. For example, a stochastic approximation technique was used in Yin, Liu, and Zhang [29]; a linear programming approach was developed in Helmes [14]; and the fast Fourier transform was used in Liu, Yin, and Zhang [20]. See also Beibel and Lerche [1] for a different approach to stopping time problems. Furthermore, consideration of capital gain taxes and transaction costs in connection with selling can be found in Cadenillas and Pliska [2], Constantinides [3], and Dammon and Spatt [8], among others.

Recently, there has been an increasing volume of literature concerning trading rules that involve both buying and selling. For instance, Zhang and Zhang [30] studied the optimal trading strategy in a mean reverting market, which validated a well-known contra-trend trading method. In particular, they established two threshold prices (buy and sell) that maximized overall discounted return if one traded at those prices. In addition to the results obtained in [30] along this line of research, an investment capacity expansion/reduction problem was considered in Merhi and Zervos [22]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model was treated by Løkka and Zervos [21] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [16] studied an optimal timing of investment problem under a general diffusion market model. The objective was to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution was obtained in [16].

However, a theoretical justification of trend following trading methods is missing despite the fact that they are widely used among professional traders (see, e.g., [26]). It is the purpose of this paper to fill this void. We adopt a finite horizon regime switching model for the stock price dynamics. In this model the price of the stock follows a geometric Brownian motion whose drift switches between two different regimes representing the up trend (bull market) and down trend (bear market), respectively, and the exact switching times between the different trends are not directly observable as in the real markets. We model the switching as an unobservable Markov chain. Our trading decisions are based on current information represented by both the stock price and the historical information with the probability in the bull phase conditioning to all available historical price levels as a proxy. Assuming trading one share with a fixed percentage transaction cost, we show that the strategy that optimizes the discounted expected return is a simple implementable trend following system. This strategy is characterized by two time-dependent thresholds for the conditional probability in a bull regime signaling buy and sell, respectively. The main advantage of this approach is that the conditional probability in a bull market can be obtained directly using actual historical stock price data through a differential equation.

The derivation of this result involves a number of different technical tools. One of the main difficulties in handling the regime switching model is that the Markov regime switching process is unobservable. Following Rishel and Helmes [25] we use the optimal nonlinear filtering technique (see, e.g., [18, 28]) regarding the conditional probability in a bull regime as an observation process. Combining with the stock price process represented in terms of this observing process, we obtain an optimal stopping problem with complete observation. Our model involves possibly infinitely many buy and sell operations represented by sequences of stopping times, and it is not a standard stopping time problem. As in Zhang and Zhang [30] we introduce two optimal value functions that correspond to the starting net position being either flat or long. Using a dynamic programming approach, we can formally derive a system of two variational inequalities. A verification theorem justifies that the solutions to these variational inequalities are indeed the optimal value functions. It is interesting that we can show that this system of variational inequalities leads to a *double obstacle problem* satisfied by the difference of the two value functions. Since the solution and properties of double obstacle problems are well understood, this conversion simplifies the analysis of our problem considerably. An accompanying numerical procedure is also established to determine the thresholds involved in our optimal trend following strategy.

Numerical experiments have been conducted for a simple trend following trading strategy that approximates the optimal one. We test our strategy using both simulation and actual market data for the NASDAQ, SP500, and DJIA indices. Our trend following trading strategy outperforms the buy and hold strategy with a huge advantage in simulated trading. This strategy also significantly prevails over the buy and hold strategy when tested with the real historical data for the NASDAQ, SP500, and DJIA indices.

The rest of this paper is arranged as follows. We formulate our problem and present its theoretical solutions in the next section. Numerical results for optimal trading strategy are presented in section 3. We conduct extensive simulations and tests on market data in section 4 and conclude in section 5. Details of data and results related to simulations and market tests are collected in the appendix.

2. Problem formulation.

Let S_r denote the stock price at time r satisfying the equation

$$dS_r = S_r[\mu(\alpha_r)dr + \sigma dB_r], \quad S_t = S, \quad t \le r \le T < \infty,$$

where $\mu(i) = \mu_i$, $i = 1, 2$, are the expected return rates, $\alpha_r \in \{1, 2\}$ is a two-state Markov chain, $\sigma > 0$ is the volatility, B_r is a standard Brownian motion, and T is a finite time.

The process α_r represents the market mode at each time r: $\alpha_r = 1$ indicates a bull market and $\alpha_r = 2$ a bear market. Naturally, we assume $\mu_1 > \mu_2$. Let

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \quad (\lambda_1 > 0,\ \lambda_2 > 0)$$

denote the generator of α_r. We assume that $\{\alpha_r\}$ and $\{B_r\}$ are independent.

Let

$$t \le \tau_1 \le v_1 \le \tau_2 \le v_2 \le \cdots \le T \text{ a.s.}$$

denote a sequence of stopping times. Note that one may construct a sequence of stopping times satisfying the above inequalities from any monotone sequence of stopping times truncated at time T. A buying decision is made at τ_n and a selling decision at v_n, $n = 1, 2, \dots$.

We consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell the stock before acquiring any share. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should first buy a stock before selling any shares. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)$.

Let $0 < K < 1$ denote the percentage of slippage (or commission) per transaction. Given the initial stock price $S_t = S$, initial market trend $\alpha_t = \alpha \in \{1, 2\}$, and initial net position $i = 0, 1$, the reward functions of the decision sequences, Λ_0 and Λ_1, are given as follows:

$$J_i(S, \alpha, t, \Lambda_i) = \begin{cases} E_t \left\{ \displaystyle\sum_{n=1}^{\infty} \left[e^{-\rho(v_n - t)} S_{v_n}(1-K) - e^{-\rho(\tau_n - t)} S_{\tau_n}(1+K) \right] I_{\{\tau_n < T\}} \right\} & \text{if } i = 0, \\ E_t \left\{ e^{-\rho(v_1 - t)} S_{v_1}(1-K) \right. & \\ \left. \quad + \displaystyle\sum_{n=2}^{\infty} \left[e^{-\rho(v_n - t)} S_{v_n}(1-K) - e^{-\rho(\tau_n - t)} S_{\tau_n}(1+K) \right] I_{\{\tau_n < T\}} \right\} & \text{if } i = 1, \end{cases}$$

where $\rho > 0$ is the discount factor. Here, term $E \sum_{n=1}^{\infty} \xi_n$ is interpreted as $\limsup_{N\to\infty} E \sum_{n=1}^{N} \xi_n$ for random variables ξ_n. Our goal is to maximize the reward function.

To exclude trivial cases,[1] we always assume

$$\mu_2 < \rho < \mu_1.$$

Remark 1. Note that the indicator function $I_{\{\tau_n < T\}}$ is used in the definition of the reward functions J_i. This is to ensure that if the last buy order is entered at $t = \tau_n$, then the position will be sold at $v_n \le T$.

The indicator function I confines the effective part of the sum to a finite time horizon so that the reward functions are bounded above.

Note that only the stock price S_r is observable at time r in the marketplace. The market trend α_r is not directly available. Thus, it is necessary to convert the problem into a completely observable one. One way to accomplish this is to use the Wonham filter [28]; see also [18, 32] for recent development and the references therein in connection with Wonham filters.

Let $p_r = P(\alpha_r = 1 | \mathcal{S}_r)$ denote the conditional probability of $\alpha_r = 1$ (bull market) given the filtration $\mathcal{S}_r = \sigma\{S_u : 0 \le u \le r\}$. Then we can show (see Wonham [28]) that p_r satisfies the following SDE:

$$dp_r = \left[-(\lambda_1 + \lambda_2) p_r + \lambda_2 \right] dr + \frac{(\mu_1 - \mu_2) p_r (1 - p_r)}{\sigma} d\widehat{B}_r,$$

where $\widehat{B}_r$ is the innovation process (a standard Brownian motion; see, e.g., Øksendal [24]) given by

$$d\widehat{B}_r = \frac{d \log(S_r) - [(\mu_1 - \mu_2) p_r + \mu_2 - \sigma^2/2] dr}{\sigma}.$$

[1] Intuitively, one should never buy stock if $\rho \ge \mu_1$ and never sell stock if $\rho \le \mu_2$, which coincides with Lemma 2.4.

Given $S_t = S$ and $p_t = p$, the problem is to choose Λ_i to maximize the discounted return

$$J_i(S, p, t, \Lambda_i) = J_i(S, \alpha, t, \Lambda_i)$$

subject to

$$\begin{cases} dS_r = S_r\left[(\mu_1 - \mu_2)\, p_r + \mu_2\right] dr + S_r \sigma d\widehat{B}_r, & S_t = S, \\ dp_r = \left[-(\lambda_1 + \lambda_2)\, p_r + \lambda_2\right] dr + \dfrac{(\mu_1 - \mu_2)p_r(1 - p_r)}{\sigma} d\widehat{B}_r, & p_t = p. \end{cases}$$

Indeed, this new problem is a completely observable one because the conditional probability can be obtained using the stock price up to time r.

For $i = 0, 1$, let $V_i(S, p, t)$ denote the value functions with the states (S, p) and net positions $i = 0, 1$ at time t. That is,

$$V_i(S, p, t) = \sup_{\Lambda_i} J_i(S, p, t, \Lambda_i).$$

The following lemma gives the upper bounds of the value functions.

Lemma 2.1. *We have*

$$0 \le V_0(S, p, t) \le S[e^{(\mu_1 - \rho)(T - t)} - 1],$$
$$0 \le V_1(S, p, t) \le S[2e^{(\mu_1 - \rho)(T - t)} - 1].$$

Proof. It is clear that the nonnegativity of V_i follows from their definition. It remains to show their upper bounds. First, given Λ_0, we have

$$e^{-\rho v_n} S_{v_n} - e^{-\rho \tau_n} S_{\tau_n} = \int_{\tau_n}^{v_n} e^{-\rho r} S_r((\mu_1 - \mu_2)p_r + \mu_2 - \rho)dr + \int_{\tau_n}^{v_n} e^{-\rho r} S_r \sigma d\widehat{B}_r.$$

Note that

$$E\left[I_{\{\tau_n < T\}} \int_{\tau_n}^{v_n} e^{-\rho r} S_r \sigma d\widehat{B}_r\right] = E\left[I_{\{\tau_n < T\}} E\left[\int_{\tau_n}^{v_n} e^{-\rho r} S_r \sigma d\widehat{B}_r \Big| \tau_n\right]\right] = 0.$$

It follows that

$$\begin{aligned} &E\left(e^{-\rho v_n} S_{v_n} - e^{-\rho \tau_n} S_{\tau_n}\right) I_{\{\tau_n < T\}} \\ &= E\left[I_{\{\tau_n < T\}} \int_{\tau_n}^{v_n} e^{-\rho r} S_r((\mu_1 - \mu_2)p_r + \mu_2 - \rho)dr\right] \\ &\le (\mu_1 - \rho) \int_{\tau_n}^{v_n} e^{-\rho r} E S_r dr. \end{aligned}$$

Using the definition of $J_0(S, p, t, \Lambda_0)$, we have

$$\begin{aligned} J_0(S, p, t, \Lambda_0) &\le \sum_{n=1}^{\infty} E\left(e^{-\rho(v_n - t)}(S_{v_n} - e^{-\rho(\tau_n - t)} S_{\tau_n})\right) I_{\{\tau_n < T\}} \\ &\le (\mu_1 - \rho) \sum_{n=1}^{\infty} e^{\rho t} E \int_{\tau_n}^{v_n} e^{-\rho r} S_r dr \\ &\le (\mu_1 - \rho) e^{\rho t} \int_t^T e^{-\rho r} E S_r dr. \end{aligned}$$

It is easy to see using Gronwall's inequality that $ES_r \le Se^{\mu_1(r-t)}$. It follows that

$$J_0(S,p,t,\Lambda_0) \le (\mu_1-\rho)e^{\rho t}S\int_t^T e^{-\rho r+\mu_1(r-t)}dr = S[e^{(\mu_1-\rho)(T-t)}-1].$$

This implies that $0 \le V_0(x) \le S[e^{(\mu_1-\rho)(T-t)}-1]$.

Similarly, we have the inequality

$$J_1(S,p,t,\Lambda_1) \le Ee^{-\rho(v_1-t)}S_{v_1}(1-K) + S[e^{(\mu_1-\rho)(T-t)}-1].$$

Moreover, note that

$$Ee^{-\rho(v_1-t)}S_{v_1} - S \le (\mu_1-\rho)e^{\rho t}\int_t^T e^{-\rho r}S_r dr \le S[e^{(\mu_1-\rho)(T-t)}-1].$$

This implies that

$$V_1(S,p,t) \le S + 2S[e^{(\mu_1-\rho)(T-t)}-1] = S[2e^{(\mu_1-\rho)(T-t)}-1].$$

Therefore, $0 \le V_1(S,p,t) \le S[2e^{(\mu_1-\rho)(T-t)}-1]$. This completes the proof. ■

Let

$$\begin{aligned}\mathcal{A} = &\frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2\partial_{pp} + \frac{1}{2}\sigma^2S^2\partial_{SS} + S\left[(\mu_1-\mu_2)p(1-p)\right]\partial_{Sp}\\ &+\left[-(\lambda_1+\lambda_2)+\lambda_2\right]\partial_p + S\left[(\mu_1-\mu_2)p+\mu_2\right]\partial_S - \rho.\end{aligned}$$

Then the HJB equations associated with our optimal stopping time problem can be given formally as follows:

$$\begin{aligned}&\min\{-\partial_t V_0 - \mathcal{A}V_0, V_0 - V_1 + S(1+K)\} = 0,\\ &\min\{-\partial_t V_1 - \mathcal{A}V_1, V_1 - V_0 - S(1-K)\} = 0\end{aligned} \tag{2.1}$$

in $(0,+\infty)\times(0,1)\times[0,T)$, with the terminal conditions

$$\begin{aligned}&V_0(S,p,T) = 0,\\ &V_1(S,p,T) = S(1-K).\end{aligned} \tag{2.2}$$

The terminal condition implies that at the terminal time T the net position must be flat.

Remark 2. In this paper, we restrict the state space of p to $(0,1)$ because both $p=0$ and $p=1$ are entrance boundaries (see Karlin and Taylor [17] for the definition and discussions). Such boundaries cannot be reached from the interior of the state space. If the process begins there, it quickly moves to the interior and never returns. We next show that $p=0$ is indeed an entrance boundary. The case when $p=1$ is similar. It is easy to see that the speed density (see [17]) can be given by

$$m(x) = \frac{\sigma^2}{(\mu_1-\mu_2)^2x^2(1-x)^2s(x)},$$

where

$$s(x) = \exp\left\{\frac{2\sigma^2}{(\mu_1-\mu_2)^2}\left[\frac{\lambda_2}{x} + \frac{\lambda_1}{1-x} + (\lambda_2-\lambda_1)\log\frac{x}{1-x}\right]\right\}.$$

To show that $p = 0$ is the entrance boundary, it suffices [17] to show that, for any $0 < a < 1$,

$$\lim_{\delta\to 0^+}\int_\delta^a\left(\int_\delta^\xi s(\eta)d\eta\right)m(\xi)d\xi = \infty \tag{2.3}$$

and

$$\lim_{\delta\to 0^+}\int_\delta^a\left(\int_\xi^a s(\eta)d\eta\right)m(\xi)d\xi < \infty. \tag{2.4}$$

Now, (2.3) follows the fact that $\lim_{\delta\to 0^+}\int_\delta^\xi s(\eta)d\eta = \infty$ for any $\xi > 0$. Moreover, note that for any $A > 0$, after a change of variables,

$$\int_0^a\left(\int_\xi^a e^{A/\eta}d\eta\right)\frac{e^{-A/\xi}}{\xi^2}d\xi = \int_{1/a}^\infty\left(\int_{1/\xi}^{1/a} e^{A/\eta}d\eta\right)e^{-A\xi}d\xi = \int_{1/a}^\infty\left(\int_a^\xi \frac{e^{A/\eta}}{\eta^2}d\eta\right)e^{-A\xi}d\xi < \infty.$$

Using this estimate, it is not difficult to see (2.4).

It is easy to show that the value functions V_0 and V_1 are linear in S. This motivates us to adopt the following transformation: $U_0(p,t) = V_0(S,p,t)/S$ and $U_1(p,t) = V_1(S,p,t)/S$. Then the HJB equations (2.1) with the terminal condition (2.2) can be reduced to

$$\begin{aligned}&\min\{-\partial_t U_0 - \mathcal{L}U_0, U_0 - U_1 + (1+K)\} = 0,\\ &\min\{-\partial_t U_1 - \mathcal{L}U_1, U_1 - U_0 - (1-K)\} = 0\end{aligned} \tag{2.5}$$

in $(0,1)\times[0,T)$, with the terminal conditions

$$\begin{aligned}&U_0(p,T) = 0,\\ &U_1(p,T) = 1-K,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\mathcal{L} = {}&\frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2\partial_{pp}\\ &+ \left[-(\lambda_1+\lambda_2)p + \lambda_2 + (\mu_1-\mu_2)p(1-p)\right]\partial_p + (\mu_1-\mu_2)p + \mu_2 - \rho.\end{aligned}$$

Thanks to Lemma 2.1, we will focus on the bounded solutions of problem (2.5)–(2.6).

Lemma 2.2. *Problem* (2.5)–(2.6) *has a unique bounded strong solution* (U_0, U_1), *where* $U_i \in W_q^{2,1}([\varepsilon, 1-\varepsilon]\times[0,T])$ *for any* $\varepsilon \in (0,1/2)$, $q \in [1,+\infty)$. *Moreover,*

$$-\partial_t U_0 - \mathcal{L}U_0 = (-\partial_t Z - \mathcal{L}Z)^-, \tag{2.7}$$

$$-\partial_t U_1 - \mathcal{L}U_1 = (-\partial_t Z - \mathcal{L}Z)^+, \tag{2.8}$$

where $Z(p,t) \equiv U_1(p,t) - U_0(p,t)$ *is the unique strong solution to the following double obstacle problem:*

$$\min\{\max\{-\partial_t Z - \mathcal{L}Z, Z-(1+K)\}, Z-(1-K)\} = 0$$

or, equivalently,

$$-\partial_t Z - \mathcal{L}Z = 0 \quad \textit{if } 1-K < Z < 1+K, \tag{2.9}$$
$$-\partial_t Z - \mathcal{L}Z \geq 0 \quad \textit{if } Z = 1-K, \tag{2.10}$$
$$-\partial_t Z - \mathcal{L}Z \leq 0 \quad \textit{if } Z = 1+K \tag{2.11}$$

in $(0,1) \times [0,T)$*, with the terminal condition* $Z(p,T) = 1-K$*. Furthermore,*

$$\partial_p Z \geq 0 \tag{2.12}$$

and

$$\partial_t Z \leq 0. \tag{2.13}$$

Proof. For any strong solution (U_0, U_1) of problem (2.5)–(2.6), we first show that[2] $Z(p,t) \equiv U_1(p,t) - U_0(p,t)$ satisfies (2.9)–(2.11). Indeed, if $1-K < Z(p,t) < 1+K$, then

$$-\partial_t U_0 - \mathcal{L}U_0|_{(p,t)} = -\partial_t U_1 - \mathcal{L}U_1|_{(p,t)} = 0,$$

which gives $-\partial_t Z - \mathcal{L}Z|_{(p,t)} = 0$. If $Z(p,t) = 1-K$ or $U_1(p,t) - U_0(p,t) - (1-K) = 0$, then $U_0(p,t) - U_1(p,t) + 1 + K = 2K > 0$, from which we infer that $-\partial_t U_0 - \mathcal{L}U_0|_{(p,t)} = 0$. On the other hand, we always have $-\partial_t U_1 - \mathcal{L}U_1|_{(p,t)} \geq 0$, so that $-\partial_t Z - \mathcal{L}Z|_{(p,t)} \geq 0$. Similarly we can deduce that $-\partial_t Z - \mathcal{L}Z \leq 0$ if $Z = 1+K$.

By the penalization method (cf. Friedman [12]), we can show that the double obstacle problem has a unique strong solution

$$Z(p,t) \in W_q^{2,1}([\varepsilon, 1-\varepsilon] \times [0,T])$$

for any $\varepsilon \in (0, 1/2)$, $q \in [1, +\infty)$. To show the regularity and uniqueness of bounded solution to problem (2.5)–(2.6), it suffices to show that the solution (U_0, U_1) to problem (2.5)–(2.6) satisfies (2.7)–(2.8). Let us prove (2.7) first. When $U_0(p,t) - U_1(p,t) > -(1+K)$ or $Z(p,t) < 1+K$, we have

$$-\partial_t U_0 - \mathcal{L}U_0 = 0,$$
$$-\partial_t Z - \mathcal{L}Z \geq 0,$$

from which we can see that (2.7) holds. As a result, it suffices to show that (2.7) remains valid when $U_0(p,t) - U_1(p,t) = -Z(p,t) = -(1+K)$. In this case $U_1(p,t) - U_0(p,t) = Z(p,t) > 1-K$ and

$$-\partial_t U_1 - \mathcal{L}U_1 = 0,$$
$$-\partial_t Z - \mathcal{L}Z \leq 0.$$

[2] After finishing this paper, we found that the connection between a system of variational inequalities and a double obstacle problem was first revealed in Nakoulima [23].

It follows that

$$\begin{aligned}0 &\geq -\partial_t Z - \mathcal{L}Z = (-\partial_t U_1 - \mathcal{L}U_1) - (-\partial_t U_0 - \mathcal{L}U_0)\\ &= -\left(-\partial_t U_0 - \mathcal{L}U_0\right),\end{aligned}$$

which implies the desired result (2.7). Equation (2.8) can be proved in a similar way.

Now let us prove (2.12). We need only restrict our attention to

$$NT \equiv \{(p,t) \in (0,1) \times [0,T) : 1-K < Z(p,t) < 1+K\}, \tag{2.14}$$

in which $-\partial_t Z - \mathcal{L}Z = 0$. Differentiating the equation w.r.t. p, we have

$$-\partial_t\left(\partial_p Z\right) - \mathcal{T}\left[\partial_p Z\right] = \left(\mu_1 - \mu_2\right) Z,$$

where $\mathcal{T}$ is another differential operator. Owing to $(\mu_1 - \mu_2)\, Z \geq 0$ in NT and $\partial_p Z = 0$ on the boundary of $NT\backslash\{p=0,1\}$,[3] we then get (2.12) by using the maximum principle. It remains to prove (2.13). Clearly $\partial_t Z|_{t=T} \leq 0$, which yields the desired result again by the maximum principle. ■

The region NT defined in (2.14) refers to the no-trading region. In terms of the solution $Z(p,t)$ to the double obstacle problem (2.9)–(2.11), with the terminal condition $Z(p,T) = 1-K$, we can define the buying region (BR) and the selling region (SR) as follows:

$$\begin{aligned}BR &= \{(p,t) \in (0,1) \times [0,T) : Z(p,t) = 1+K\},\\ SR &= \{(p,t) \in (0,1) \times [0,T) : Z(p,t) = 1-K\}.\end{aligned}$$

We aim to characterize these regions through the study of the double obstacle problem. To begin with, we prove the connectivity of any t-section of BR or SR.

Lemma 2.3. *For any $t \in [0,T)$, the following hold:*

(i) *if $(p_1,t) \in BR$ and $p_2 \geq p_1$, then $(p_2,t) \in BR$;*

(ii) *if $(p_1,t) \in SR$ and $p_2 \leq p_1$, then $(p_2,t) \in SR$.*

Proof. We prove only part (i) since the proof of part (ii) is similar. Since $Z(p_1,t) = 1+K$, we infer by (2.12) that $Z(p_2,t) \geq Z(p_1,t) = 1+K$. On the other hand, $Z(p_2,t) \leq 1+K$. So we must have

$$Z(p_2,t) = 1+K,$$

i.e., $(p_2,t) \in BR$, as desired. ■

The lemma below implies that BR shrinks and SR expands as t approaches the terminal time T.

Lemma 2.4. *For any $p \in (0,1)$, the following hold:*

(i) *if $(p,t_1) \in BR$ and $t_2 \leq t_1$, then $(p,t_2) \in BR$;*

(ii) *if $(p,t_1) \in SR$ and $t_2 \geq t_1$, then $(p,t_2) \in SR$.*

Moreover,

$$BR \subset \left\{(p,t) \in (0,1) \times [0,T) : p \geq \frac{\rho - \mu_2}{\mu_1 - \mu_2}\right\}; \tag{2.15}$$

$$SR \subset \left\{(p,t) \in (0,1) \times [0,T) : p \leq \frac{\rho - \mu_2}{\mu_1 - \mu_2}\right\}. \tag{2.16}$$

[3] On $p=0$ or 1, no boundary conditions are required due to the degeneracy of the differential operator.

Proof. In view of (2.13), the proofs of parts (i) and (ii) are similar to those of Lemma 2.3. It remains to show (2.15) and (2.16). Due to (2.10), we infer that if $(p,t) \in BR$, then

$$0 \le (-\partial_t - \mathcal{L})(1-K) = -[(\mu_1 - \mu_2)p + \mu_2 - \rho],$$

namely,

$$p \le \frac{\rho - \mu_2}{\mu_1 - \mu_2},$$

which gives (2.15). By (2.11), we can similarly get (2.16). ∎

Combining $\overline{BR} \cap \overline{SR} = \emptyset$ with (2.15) and (2.16), we deduce that any t-section of NT is nonempty. In view of Lemma 2.3, we can define two free boundaries:

$$p_s^*(t) = \inf\{p \in (0,1) : (p,t) \in NT\}, \tag{2.17}$$

$$p_b^*(t) = \sup\{p \in (0,1) : (p,t) \in NT\} \tag{2.18}$$

for any $t \in [0,T)$. They are the thresholds for sell and buy, respectively.

Theorem 2.5. *Let $p_s^*(t)$ and $p_b^*(t)$ be as given in (2.17)–(2.18). Then the following hold:*

(i) *$p_b^*(t)$ and $p_s^*(t)$ are monotonically increasing in t, and*

$$p_b^*(t) \ge \frac{\rho - \mu_2}{\mu_1 - \mu_2} \ge p_s^*(t)$$

for all $t \in [0,T)$. Moreover, $p_b^*(t), p_s^*(t) \in C^\infty(0,T)$;

(ii) *$p_s^*(T) := \lim_{t\to T^-} p_s^*(t) = \frac{\rho-\mu_2}{\mu_1-\mu_2}$;*

(iii) *there is a $\delta > 0$ such that $(1,t) \in NT$ for all $t \in (T-\delta, T)$, namely,*

$$p_b^*(t) = 1 \quad \textit{for } t \in (T-\delta, T).$$

Moreover,

$$\delta \ge \frac{1}{\mu_1 - \rho} \log \frac{1+K}{1-K}. \tag{2.19}$$

Proof. The monotonicity in part (i) is a corollary of Lemmas 2.3 and 2.4. The proof for the smoothness of $p_b^*(t)$ and $p_s^*(t)$ is somewhat technical and is placed in the appendix. To show part (ii), we use the method of contradiction. Suppose not. Due to $p_s^*(t) \le \frac{\rho-\mu_2}{\mu_1-\mu_2}$, we would have

$$p_s^*(T) < \frac{\rho - \mu_2}{\mu_1 - \mu_2}.$$

Then, for any $p \in (p_s^*(T), \frac{\rho-\mu_2}{\mu_1-\mu_2})$,

$$\partial_t Z|_{t=T} = -\mathcal{L}Z|_{t=T} = -\mathcal{L}(1-K) = -[(\mu_1-\mu_2)p + \mu_2 - \rho] > 0,$$

which is in contradiction with (2.13).

It remains to show part (iii). Owing to $Z|_{t=T} = 1-K$ and (2.12), the existence of δ is apparent. We need only show (2.19). On $p=1$, problem (2.9)–(2.11) is reduced to

$$\begin{cases} -Z_t + (\rho - \mu_1) Z|_{p=1} = -\lambda_1 \partial_p Z & \text{if } Z < 1+K, \\ -Z_t + (\rho - \mu_1) Z|_{p=1} \le -\lambda_1 \partial_p Z & \text{if } Z = 1+K \end{cases} \tag{2.20}$$

in $t \in (0,T)$, with the terminal condition $Z(1,T) = 1-K$, where we have excluded the lower obstacle according to (2.16). Due to $-\lambda_1 \partial_p Z \le 0$, problem (2.20) has a supersolution $\overline{Z}(t)$ satisfying

$$\begin{cases} -\overline{Z}_t + (\rho - \mu_1)\,\overline{Z} = 0 & \text{if } \overline{Z}(t) < 1+K, \\ -\overline{Z}_t + (\rho - \mu_1)\,\overline{Z} \le 0 & \text{if } \overline{Z}(t) = 1+K \end{cases} \tag{2.21}$$

in $t \in (0,T)$, with $\overline{Z}(T) = 1-K$. It is easy to verify

$$\overline{Z}(t) = \begin{cases} e^{(\mu_1-\rho)(T-t)}\,(1-K) \text{ if } t > T - \dfrac{1}{\mu_1-\rho}\log\dfrac{1+K}{1-K}, \\ 1+K \text{ if } t \le T - \dfrac{1}{\mu_1-\rho}\log\dfrac{1+K}{1-K}. \end{cases}$$

We then infer that

$$Z(1,t) \le \overline{Z}(t) = e^{(\mu_1-\rho)(T-t)}\,(1-K) < 1+K \quad \text{for } t > T - \frac{1}{\mu_1-\rho}\log\frac{1+K}{1-K},$$

which implies that $(1,t) \in NT$ for any $t > T - \frac{1}{\mu_1-\rho}\log\frac{1+K}{1-K}$. Then (2.19) follows. ■

Remark 3. Part (iii) indicates that there is a critical time after which it is never optimal to buy stock. This is an important feature when transaction costs are involved in a finite horizon model. Similar results were obtained in the study of finite horizon portfolio selection with transaction costs (cf. [4, 6, 7, 19]). The intuition is that if the investor does not have a long enough time horizon to recover at least the transaction costs, then she/he should not initiate a long position (bear in mind that the terminal position must be flat).

Remark 4. Using the maximum principle, it is not hard to show that $Z(\cdot,\cdot;\lambda_1,\lambda_2,\rho)$ is a decreasing function of λ_1 and ρ and an increasing function of λ_2. As a consequence, $p_s^*(\cdot;\lambda_1,\lambda_2,\rho)$ and $p_b^*(\cdot;\lambda_1,\lambda_2,\rho)$ are also increasing functions of λ_1 and ρ and decreasing functions of λ_2.

By Lemma 2.2, the Sobolev embedding theorem (cf. [12]), and the smoothness of free boundaries, the solutions U_0 and U_1 of problem (2.5)–(2.6) belong to C^1 in $(0,1) \times [0,T)$. Furthermore, it is easy to show that the solutions are sufficiently smooth (i.e., at least C^2) except at the free boundaries $p_s^*(t)$ and $p_b^*(t)$. These enable us to establish a verification theorem to show that the solutions U_0 and U_1 of problem (2.5)–(2.6) are equal to the value functions V_0/S and V_1/S, respectively, and sequences of optimal stopping times can be constructed by using (p_s^*, p_b^*).

This theorem gives sufficient conditions for the optimality of the trading rules in terms of the stopping times $\{\tau_n, \upsilon_n\}$. The construction procedure will be used in the next sections to develop numerical solutions in various scenarios.

Theorem 2.6 (verification theorem). *Let (U_0, U_1) be the unique bounded strong solution to problem (2.5)–(2.6) and $p_b^*(t)$ and $p_s^*(t)$ be the associated free boundaries. Then, $w_0(S,p,t) \equiv SU_0(p,t)$ and $w_1(S,p,t) \equiv SU_1(p,t)$ are equal to the value functions $V_0(S,p,t)$ and $V_1(S,p,t)$, respectively.*

Moreover, let

$$\Lambda_0^* = (\tau_1^*, \upsilon_1^*, \tau_2^*, \upsilon_2^*, \ldots),$$

where the stopping times $\tau_1^* = T \wedge \inf\{r \ge t : p_r \ge p_b^*(r)\}$, $v_n^* = T \wedge \inf\{r \ge \tau_n^* : p_r \le p_s^*(r)\}$, *and* $\tau_{n+1}^* = T \wedge \inf\{r > v_n^* : p_r \ge p_b^*(r)\}$ *for* $n \ge 1$, *and let*

$$\Lambda_1^* = (v_1^*, \tau_2^*, v_2^*, \tau_3^*, \ldots),$$

where the stopping times $v_1^* = T \wedge \inf\{r \ge t : p_r^* \le p_s^*(r)\}$, $\tau_n^* = T \wedge \inf\{r > v_{n-1}^* : p_r \ge p_b^*(r)\}$, *and* $v_n^* = T \wedge \inf\{r \ge \tau_n^* : p_r \le p_s^*(r)\}$ *for* $n \ge 2$. *If* $v_n^* \to T$ *a.s., as* $n \to \infty$, *then* Λ_0^* *and* Λ_1^* *are optimal.*

Proof. The proof is divided into two steps. In the first step, we show that $w_i(S,p,t) \ge J_i(S,p,t,\Lambda_i)$ for all Λ_i. Then, in the second step, we show that $w_i(S,p,t) = J_i(S,p,t,\Lambda_i^*)$. Therefore, $w_i(S,p,t) = V_i(S,p,t)$ and Λ_i^* is optimal.

Using $(-\partial_t w_i - \mathcal{L} w_i) \ge 0$, Dynkin's formula, and Fatou's lemma as in Øksendal [24, p. 226], we have, for any stopping times $t \le \theta_1 \le \theta_2$, a.s.,

$$Ee^{-\rho(\theta_1 - t)} w_i(S_{\theta_1}, p_{\theta_1}, \theta_1) I_{\{\theta_1 < a\}} \ge Ee^{-\rho(\theta_2 - t)} w_i(S_{\theta_2}, p_{\theta_2}, \theta_2) I_{\{\theta_1 < a\}} \tag{2.22}$$

for any a and $i = 0, 1$.

Note that $w_0 \ge w_1 - S(1+K)$. Given $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)$, by (2.22), and noting that $w_0(S,p,T) = 0$, we have

$$\begin{aligned}
w_0(S,p,t) &\ge Ee^{-\rho(\tau_1 - t)} w_0(S_{\tau_1}, p_{\tau_1}, \tau_1) \\
&= Ee^{-\rho(\tau_1 - t)} w_0(S_{\tau_1}, p_{\tau_1}, \tau_1) I_{\{\tau_1 < T\}} \\
&\ge Ee^{-\rho(\tau_1 - t)} (w_1(S_{\tau_1}, p_{\tau_1}, \tau_1) - S_{\tau_1}(1+K)) I_{\{\tau_1 < T\}} \\
&= Ee^{-\rho(\tau_1 - t)} w_1(S_{\tau_1}, p_{\tau_1}, \tau_1) I_{\{\tau_1 < T\}} - Ee^{-\rho(\tau_1 - t)} S_{\tau_1}(1+K) I_{\{\tau_1 < T\}}.
\end{aligned}$$

Using (2.22) again with $i = 1$ and noticing that $v_1 \ge \tau_1$ and $w_1 \ge w_0 + S(1-K)$, we have

$$\begin{aligned}
w_0(S,p,t) &\ge Ee^{-\rho(v_1 - t)} w_1(S_{v_1}, p_{v_1}, v_1) I_{\{\tau_1 < T\}} - Ee^{-\rho(\tau_1 - t)} S_{\tau_1}(1+K) I_{\{\tau_1 < T\}} \\
&\ge Ee^{-\rho(v_1 - t)} (w_0(S_{v_1}, p_{v_1}, v_1) + S_{v_1}(1-K)) I_{\{\tau_1 < T\}} \\
&\quad - Ee^{-\rho(\tau_1 - t)} S_{\tau_1}(1+K) I_{\{\tau_1 < T\}} \\
&= Ee^{-\rho(v_1 - t)} w_0(S_{v_1}, p_{v_1}, v_1) I_{\{\tau_1 < T\}} \\
&\quad + E\Big[e^{-\rho(v_1 - t)} S_{v_1}(1-K) - e^{-\rho(\tau_1 - t)} S_{\tau_1}(1+K)\Big] I_{\{\tau_1 < T\}} \\
&= Ee^{-\rho(v_1 - t)} w_0(S_{v_1}, p_{v_1}, v_1) \\
&\quad + E\Big[e^{-\rho(v_1 - t)} S_{v_1}(1-K) - e^{-\rho(\tau_1 - t)} S_{\tau_1}(1+K)\Big] I_{\{\tau_1 < T\}}.
\end{aligned}$$

Continue this way and recall that $w_i(S,p,t) \ge 0$ to obtain

$$w_0(S,p,t) \ge E \sum_{n=1}^{N} \Big[e^{-\rho(v_n - t)} S_{v_n}(1-K) - e^{-\rho(\tau_n - t)} S_{\tau_n}(1+K)\Big] I_{\{\tau_n < T\}}.$$

Sending $N \to \infty$, we have $w_0(S,p,t) \ge J_0(S,p,t,\Lambda_0)$ for all Λ_0. This implies that $w_0(S,p,t) \ge V_0(S,p,t)$. Similarly, we can show that $w_1(S,p,t) \ge V_1(S,p,t)$.

We next establish the equalities. For given t, define

$$\tau_1^* = \begin{cases} t & \text{if } p \geq p_b^*(t), \\ T \wedge \inf\{r \geq t : p_r = p_b^*(r)\} & \text{if } p < p_b^*(t). \end{cases}$$

Using Dynkin's formula, we have

$$\begin{aligned} w_0(S,p,t) &= Ee^{-\rho(\tau_1^*-t)} w_0(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) \\ &= Ee^{-\rho(\tau_1^*-t)} w_0(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{\{\tau_1^*<T\}} \\ &= Ee^{-\rho(\tau_1^*-t)} (w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) - S_{\tau_1^*}(1+K)) I_{\{\tau_1^*<T\}} \\ &= Ee^{-\rho(\tau_1^*-t)} w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{\{\tau_1^*<T\}} - Ee^{-\rho(\tau_1^*-t)} S_{\tau_1^*}(1+K) I_{\{\tau_1^*<T\}}. \end{aligned}$$

Let $v_1^* = T \wedge \inf\{r \geq \tau_1^* : p_r = p_s^*(r)\}$. Noticing that $w_1(S,p,T) = S(1-K)$, we have

$$\begin{aligned} &Ee^{-\rho(\tau_1^*-t)} w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{\{\tau_1^*<T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} w_1(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{\{\tau_1^*<T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} w_1(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{\{\tau_1^*<T\}} I_{\{v_1^*<T\}} + Ee^{-\rho T} S_{T-t}(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*=T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} (w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + S_{v_1^*}(1-K)) I_{\{\tau_1^*<T\}} I_{\{v_1^*<T\}} \\ &\qquad + Ee^{-\rho T} S_{T-t}(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*=T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{\{\tau_1^*<T\}} I_{\{v_1^*<T\}} + Ee^{-\rho(v_1^*-t)} S_{v_1^*}(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*<T\}} \\ &\qquad + Ee^{-\rho(T-t)} S_T(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*=T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + Ee^{-\rho(v_1^*-T)} S_{v_1^*}(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*<T\}} \\ &\qquad + Ee^{-\rho(T-t)} S_T(1-K) I_{\{\tau_1^*<T\}} I_{\{v_1^*=T\}} \\ &\quad = Ee^{-\rho(v_1^*-t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + Ee^{-\rho(v_1^*-t)} S_{v_1^*}(1-K) I_{\{\tau_1^*<T\}}. \end{aligned}$$

It follows that

$$w_0(S,p,t) = Ee^{-\rho(v_1^*-t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + E\Big[e^{-\rho(v_1^*-t)} S_{v_1^*}(1-K) - e^{-\rho(\tau_1^*-t)} S_{\tau_1^*}(1+K)\Big] I_{\{\tau_1^*<T\}}.$$

Continue the procedure to obtain

$$\begin{aligned} w_0(S,p,t) &= Ee^{-\rho(v_n^*-t)} w_0(S_{v_n^*}, p_{v_n^*}, v_n^*) \\ &\quad + E\sum_{k=1}^{n} \Big[e^{-\rho(v_k^*-t)} S_{v_k^*}(1-K) - e^{-\rho(\tau_k^*-t)} S_{\tau_k^*}(1+K)\Big] I_{\{\tau_k^*<T\}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} w_1(S,p,t) &= Ee^{-\rho(v_n^*-t)} w_0(S_{v_n^*}, p_{v_n^*}, v_n^*) \\ &\quad + Ee^{-\rho(v_1^*-t)} S_{v_1^*}(1-K) \\ &\quad + E\sum_{k=2}^{n} \Big[e^{-\rho(v_k^*-t)} S_{v_k^*}(1-K) - e^{-\rho(\tau_k^*-t)} S_{\tau_k^*}(1+K)\Big] I_{\{\tau_k^*<T\}}. \end{aligned}$$

Recall that $v_n^* \to T$. Sending $n \to \infty$ and noticing $w_0(S,p,T) = 0$, we obtain the equalities. This completes the proof. ∎

Table 1
Parameter values.

λ_1	λ_2	μ_1	μ_2	σ	K	ρ
0.36	2.53	0.18	−0.77	0.184	0.001	0.0679

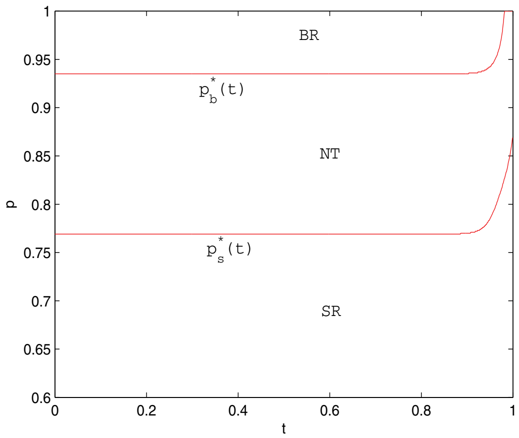

Figure 1. *Optimal trading strategy.*

3. Numerical results for optimal trading strategy.

The theoretical analysis in section 2 shows that $p_s^*(t)$ and $p_b^*(t)$ are thresholds for the optimal trend following trading strategy: buy the stock when p_t crosses $p_b^*(t)$ from below and sell the stock when p_t crosses $p_s^*(t)$ from above. Knowing the parameters of our regime switching model, we can numerically solve the double obstacle problem (2.9)–(2.11) to derive approximations of those thresholds. To do that, we employ the penalization method with a finite difference discretization (see Dai, Kwok, and You [5] and Forsyth and Vetzal [11]). The penalized approximation to the double obstacle problem is

$$-\partial_t Z - \mathcal{L}Z = \beta\,(1-K-Z)^+ - \beta\,(Z-1-K)^+\,,$$

where β is the penalty parameter. In our numerical examples, we choose $\beta = 10^7$. The right-hand side of the approximation is linearized by using a nonsmooth version of the Newton iteration. Then the resulting equations are discretized by the standard finite difference method.

We take $T = 1$ and use the model parameters in Table 1 based on the statistics for DJIA. Figure 1 represents $p_s^*(\cdot)$ and $p_b^*(\cdot)$ as functions of time t. We see that $p_s^*(t)$ approaches the theoretical value $(\rho-\mu_2)/(\mu_1-\mu_2) = (0.0679+0.77)/(0.18+0.77) = 0.882$ as $t \to T = 1$. Also, we observe that there is a $\delta > 0$ such that $p_b^*(t) = 1$ for $t \in [T-\delta, T]$, which indicates that it is never optimal to buy stock when t is very close to T. Using Theorem 2.5, the lower bound of δ is estimated as $\frac{1}{\mu_1-\rho}\log\frac{1+K}{1-K} = \frac{1}{0.18-0.0679}\log\frac{1.001}{0.999} = 0.0178$, which is consistent with the numerical result.

Figure 2 illustrates the impact of parameters λ and ρ on the optimal thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$. We can see that they are increasing functions of ρ and decreasing functions of λ_2 as indicated in Remark 2. Moreover, properties (i)–(iii) of $p_s^*(\cdot)$ and $p_b^*(\cdot)$ stated in Theorem 2.5 are also visible.

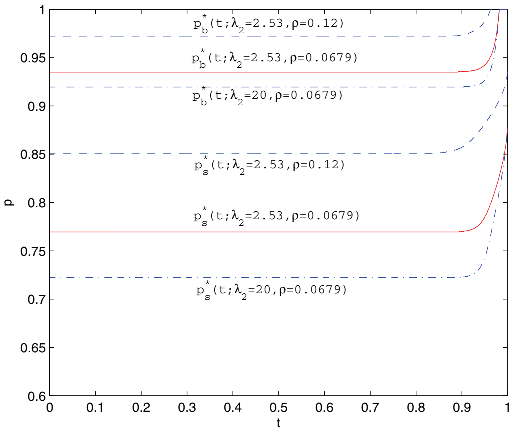

Figure 2. *Effects of parameters λ and ρ on optimal trading strategy.*

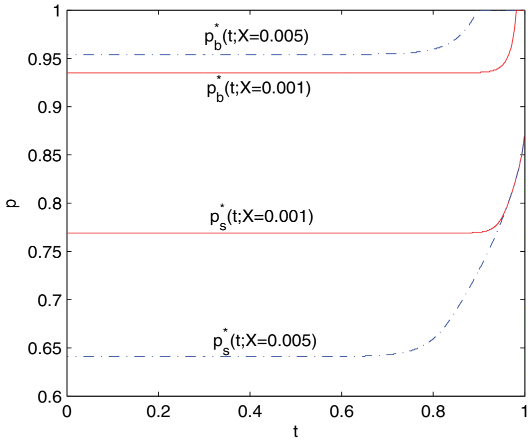

Figure 3. *Effects of transaction cost K on optimal trading strategy.*

In Figure 3, we examine the impact of transaction cost K on the optimal thresholds. It is observed that the no-trading regions expands as we increase the transaction cost from 0.001 to 0.005. This is consistent with our intuition that increasing transaction costs can decrease trading frequency.

4. Simulations and market tests. To evaluate how well our trend following trading strategy would work in practice, we conduct experiments using both simulations and market historical data. The tests are oriented towards the practicality of using p_t to detect the regime switches. Thus, we focus on an implementable trend following strategy that approximates the optimal one and test its robustness.

4.1. Method. From Figures 1–3 we can see that the thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$ are almost constant except when t gets very close to the terminal time T. In our experiments, T is always more than 10 years, which is relatively large. As a result, the contribution of the trading near T to the reward function is small. Thus, we will approximate $p_s^*(\cdot)$ and $p_b^*(\cdot)$ with two constant threshold values $p_s^* = \lim_{t\to 0+} p_s^*(t)$ and $p_b^* = \lim_{t\to 0+} p_b^*(t)$. Then we estimate p_t, the conditional probability in a bull market at time t, and check it against the thresholds to determine whether to buy, hold, or sell. The trend following trading strategy we will use is to buy the stock when p_t crosses p_b^* from below for the first time and convert to the bond when p_t crosses p_s^* from above for the first time. Moreover, we always liquidate our holdings of stock or bond at T.

Some qualitative analysis of p_t is helpful before experiments. Using the observation SDE equations, we see that p_t is related to the stock price S_t by

$$dp_t = f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} d\log(S_t), \tag{4.1}$$

where f is a third-order polynomial of p_t given by

$$f(p_t) = -(\lambda_1 + \lambda_2)\, p_t + \lambda_2 - \frac{(\mu_1 - \mu_2)p_t(1-p_t)\left((\mu_1 - \mu_2)p_t + \mu_2 - \sigma^2/2\right)}{\sigma^2}.$$

It is easy to check that $f(0) = \lambda_2 > 0$ and $f(1) = -\lambda_1 < 0$. Moreover, as p_t approaches $\pm\infty$, so does $f(p_t)$. Thus, f has exactly one root ξ in $[0,1]$. When the stock prices stay constant, p_t is attracted to ξ. This attractor is an unbiased choice for p_0. Since $(\mu_1 - \mu_2)p_t(1-p_t)/\sigma^2 \geq 0$, p_t moves in the same direction as the stock prices. This is also intuitive since the stock price movements indicate trends. The magnitude of the impact varies, though. Fixing all the parameters, the impact of stock price movement becomes relatively small when p_t is getting close to 0 or 1. Among the parameters μ_1, μ_2, and σ the latter has a larger impact since it appears in a square in the formula. A smaller σ magnifies the impact of stock movement and tends to cause more frequent trading, and a larger σ does the opposite. We will estimate p_t simply by replacing the differential in (4.1) with a difference using the trading day as the step size on a finite time horizon $[0, Ndt]$:

$$p_{t+1} = p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} \log\left(\frac{S_{t+1}}{S_t}\right),$$

where $dt = 1/250$ and $t = 0, dt, 2dt, \ldots, Ndt$.

If everything changes continuously, then $p_t \in [0,1]$. Indeed, simply changing the differential equation to a difference equation works extremely well for simulated paths. However, this could be violated in the approximation when S_t jumps. This would happen, for example, in testing the SP500: during the 1987 crash on the downside and recently on the upside. We have to restrict $p_t \in [0,1]$ in implementing the approximation. Thus, in simulation and testing, we calculate p_t iteratively with

$$p_{t+1} = \min\left(\max\left(p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} \log\left(\frac{S_{t+1}}{S_t}\right), 0\right), 1\right).$$

4.2. Simulation. In our analysis the choice of the objective function is partly due to tractability. Nevertheless, this gives us a way of recognizing trends by relatively high probabilities in the bull or bear regimes, respectively. How effective is this trend recognizing method? We check it by simulation. We use the same parameter values as in Table 1.

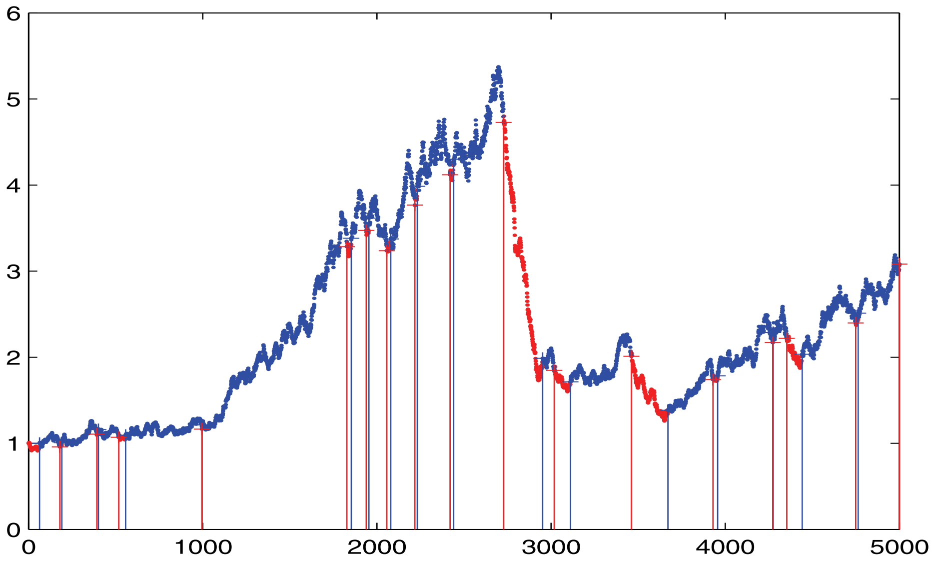

Figure 4. *Sample path.*

Table 2
Simulation results.

No. of simulations	TF	BH	TF/BH	No. of trades
1000	77.28	6.04	12.79	37.56
2000	73.97	6.00	12.33	37.66
5000	75.48	5.64	13.38	37.56
10000	73.6	5.45	13.50	37.64
50000	74.5	5.59	13.33	37.56

Numerically solving the obstacle problem for $t \to 0+$ yields thresholds $p_s^* = 0.768$ and $p_b^* = 0.934$ for down and up trends, respectively. Assuming no prior knowledge on the trend of the market, we set $p_0 = 0.8$ roughly in the middle of the two thresholds. A typical 20-year sample path is given in Figure 4, in which for the prices S_r when we are long the stocks according to our strategy ($p(t) \in BR \cup NT$ after crossing p_b^* from below) are marked in blue and those when we are flat ($p(t) \in SR \cup NT$ after crossing p_s^* from above) in red. We can see that on this typical path the signals are quite effective in detecting the regime switching.

A natural trading strategy of using these signals is to buy stock in the beginning of an up trend as signaled by p_t crossing the upper threshold p_b^* from below and switch to bond when p_t crosses the lower threshold p_s^* from above. We simulate this trend following strategy against the buy and hold strategy by using a large number of simulated paths. Again we use $p_0 = 0.8$. The average returns of the trend following (TF) strategy on one unit invested on simulation paths are listed in Table 2 along with the average number of trades on each path. We also list the average return of the buy and hold (BH) strategy for comparison.

Judging from the ratio TF/BH, the simulation tends to stabilize around 5000 rounds. We run the 5000-round simulation 10 times and summarize the mean and standard deviation in

Table 3
Statistics of 10 5000*-round simulations.*

	TF	BH	No. of trades
Mean	74.6	5.72	37.55
Stdev	1.64	0.31	0.15

Table 4
Thresholds corresponding to different parameters.

λ_1	λ_2	μ_1	μ_2	σ	ρ	p_s^*	p_b^*
0.36	*2.53*	*0.18*	*−0.77*	*0.184*	*0.067*	*0.768*	*0.934*
0.36	2.53	0.18	−0.77	0.184	**0.062**	0.761	0.931
0.36	2.53	0.18	−0.77	0.184	**0.072**	0.775	0.938
0.36	2.53	0.18	−0.77	**0.174**	0.067	0.762	0.936
0.36	2.53	0.18	−0.77	**0.194**	0.067	0.773	0.933
0.36	2.53	**0.17**	**−0.71**	0.184	0.067	0.774	0.935
0.36	2.53	**0.19**	**−0.83**	0.184	0.067	0.762	0.934
0.36	**2**	0.18	−0.77	0.184	0.067	0.769	0.935
0.36	**3**	0.18	−0.77	0.184	0.067	0.767	0.935
0.3	2.53	0.18	−0.77	0.184	0.067	0.767	0.934
0.42	2.53	0.18	−0.77	0.184	0.067	0.769	0.935

Table 5
Shifting the thresholds.

p_s^*	p_b^*	TF	BH	TF/BH	No. of trades
0.75900	0.92500	74.407	5.6478	13.174	37.039
0.76300	0.92900	76.523	5.9231	12.919	37.139
0.76700	0.93300	75.467	5.8328	12.938	37.371
0.77100	0.93700	74.232	5.6367	13.169	37.648
0.77500	0.94100	75.986	5.5975	13.575	37.766
0.77900	0.94500	77.029	6.0613	12.708	37.923

Table 3, which confirms our observation.

These simulations show that the trend following strategy has a distinctive advantage over the buy and hold strategy and is quite stable in both return and trading frequency. What if the parameters are perturbed? It turns out that the thresholds are not sensitive to the parameters as summarized in Table 4.

Next we test the robustness of the trend following trading strategy against the perturbation of the thresholds. This is important because we are using the limits of the optimal thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$ as $t \to 0+$ to approximate them. We perturb the constant thresholds p_s^* and p_b^* both by shifting and by altering the spreads. The results are summarized in Tables 5 and 6.

We can see from Table 5 that shifting the thresholds has little impact. Table 6 shows that the average number of trades in the trend following trading strategy is inversely correlated to the spreads of the thresholds. However, the relative advantage of the trend following strategy over the buy and hold strategy is not sensitive to the perturbation of the thresholds.

In the tests discussed above, the trading cost $K = 0.001$ is fixed. Increasing K, the spread of the optimal thresholds will also increase as indicated in Figure 3. Simulations summarized in Table 6 indicate that this will reduce the trading frequency, which is consistent with intuition.

Table 6
Changing the spreads of the thresholds.

p_s^*	p_b^*	TF	BH	TF/BH	No. of trades
0.75900	0.94500	74.892	6.1736	12.131	34.148
0.76300	0.94100	73.561	5.4182	13.577	35.591
0.76700	0.93700	75.480	5.7035	13.234	36.743
0.77100	0.93300	77.059	6.0427	12.753	38.122
0.77500	0.92900	74.010	5.5380	13.364	39.861
0.77900	0.92500	76.442	5.7994	13.181	41.375

Table 7
Changing the trading cost.

K	p_s^*	p_b^*	TF	BH	No. of trades
0.001	0.768	0.934	75.8	5.9	37.3
0.005	0.64	0.954	55.77	5.75	20.93
0.01	0.545	0.962	43.37	5.9	15.88
0.02	0.422	0.969	30.6	5.25	11.92

Table 8
Statistics of bull and bear markets.

Index	λ_1	λ_2	μ_1	μ_2	σ_1	σ_2	σ
SP500 (62–08)	0.353	2.208	0.196	−0.616	0.135	0.211	0.173
DJIA (62–08)	0.36	2.53	0.18	−0.77	0.144	0.223	0.184
NASDAQ (91–08)	2.158	2.3	0.875	−1.028	0.273	0.35	0.31

Simulating 5000 rounds for each of the trading cost levels $K = 0.005, 0.01, 0.02$ shows that the advantage of the trend following methods decreases as K increases as expected. However, even at $K = 0.02$, the advantage is still quite obvious. The testing results are summarized in Table 7.

The simulations convince us that the trend following trading system of using p_t crossing the constant thresholds p_s^* and p_b^* to detect the trend of the stock price movement in a regime switching model is effective and robust.

4.3. Testing in stock markets. Does this trend following strategy work in real markets? We test it on the historical data of the SP500, DJIA, and NASDAQ indices. The SP500 index started active trading in 1962 and NASDAQ in 1991, and we test them up to the end of 2008. We also test DJIA from 1962–2008.

First we need to determine the parameters. We regard a decline of more than 19% as a bear market and a rally of 24% or more as a bull market. Statistics of bull and bear markets for SP500 index and DJIA in the 47 years from 1962–2008 and NASDAQ from 1991–2008 (see Tables 11, 12, and 13 in the appendix) are shown in Table 8. Here σ_1 and σ_2 are the average annualized standard deviation corresponding to the bull and bear markets, respectively, and $\sigma = (\sigma_1 + \sigma_2)/2$. Currently, retail discount brokers usually charge \$2.5–10 per trade for unlimited number of shares (e.g., Just2Trade \$2.5 per trade, ScotTrade \$7 per trade, ETrade \$10 per trade, and TDAmeritrade \$10 per trade). For professional traders who deal with clearing houses directly, trading a \$100000 standard lot usually cost less than \$1. Assuming

Table 9
Thresholds for indices.

Index	Lower thresholds	Upper thresholds
SP500 σ_1	0.69	**0.91**
SP500 σ_2	**0.74**	0.90
DJIA σ_1	0.74	**0.94**
DJIA σ_2	**0.78**	0.93
NASDAQ σ_1	0.43	**0.69**
NASDAQ σ_2	**0.45**	0.67

\$10 per trade for an account of size \$10000, we choose $K = 0.001$ to simulate our strategy. This is close to the actual cost for a typical individual investor now or for a professional trader before the emergence of online discount brokers. We choose 10-year treasury bonds as the alternative risk-free investment instrument and use the annual yield released on the Federal Reserve Statistical Release web site [10]. This gives us an average yield of 6.7% per year from 1962–2008 and 5.4% from 1991–2008. However, to be more realistic we use the actual yield (see Table 14 in the appendix) when holding the bonds. We also take advantage of knowing σ_1 and σ_2 for the bull and bear markets. Solving the obstacle problems we derive for each index two sets of thresholds corresponding to σ_1 and σ_2, respectively, as listed in Table 9. Since when holding bonds and looking for a signal to switch to a stock position, we anticipate entering a bull market whose volatility is better represented by σ_1. Therefore, it is reasonable to choose the upper threshold related to σ_1. Similarly, for a signal to exit a stock position, we should use the lower threshold corresponding to σ_2. Thus, in conducting our test, we use the boldfaced thresholds in Table 9.

The discussions so far are based on the raw index values. This is appropriate since when pursuing market trend the raw price is what market agents can directly observe and react upon. However, for a long investment horizon, we also need to consider the effect of dividend. In this aspect the three indices are different. DJIA already includes dividend in its computation while NASDAQ and SP500 do not. The dividend paid by companies listed on NASDAQ is small; for example, the estimated 2009 average dividend for the 100 largest companies (who usually pay more dividend compared to smaller ones) listed in NASDAQ is only 0.68% (see [15]). Moreover, NASDAQ is tested for a shorter period of time. Thus, omitting dividend in the test for the NASDAQ index does not skew the comparison much. This is not the case for the SP500 index, which averages an annual dividend of about 2%. We use the annual dividend compiled in [9] to compensate the raw gains for using the trend following strategy to trade the SP500 index.

The testing results for trading the NASDAQ, SP500, and DJIA indices are summarized in Table 10, and the trading details for the trend following strategy are contained in Tables 15, 16, and 17 in the appendix, respectively. Taking NASDAQ as an example, using our trend following trading strategy, one dollar invested in the beginning of 1991 returns 8.82 at the end of 2008. By comparison, one dollar invested in the NASDAQ index using the buy and hold strategy in the same period returns only 4.24 while when invested in 10-year bonds it returns 2.63. The stories for the SP500 and DJIA are similar.

The average percentage gains per trade listed in Table 10 indicate that there is room for

Table 10

Testing results for trend following trading strategies. Legend: TF—trend following, BH—buy and hold, and G—average % gain per trade.

Index (time frame)	TF	BH	10y bonds	No. trades	G
NASDAQ (1991–2008)	8.82	4.24	2.63	66	3.35
SP500 (1962–2008)	64.98	33.5	23.44	80	5.36
DJIA (1962–2008)	26.03	12.11	23.44	80	4.16

the trend following method to absorb a higher trading cost and still outperform the buy and hold method. Using the NASDAQ test as an example, the same 66 trades with a trading cost of 1% per trade, the total return of the trend following method will be 4.64, still higher than the 4.24 from the buy and hold method. In theory, changing K to recalculate the buy and sell thresholds will yield even better returns, and this is confirmed also by the simulation reported in Table 7. However, this not the case here. In fact, using thresholds corresponding to $K = 0.01$ to test the NASDAQ data, we get a total return of only 4.04 with 22 trades. On the other hand, the same test with $K = 0.0001$ yields a total return of 10.8 with 130 trades. This corresponds to an average gain of 1.847% per trade. Had a 0.001 trading cost been charged on these same trades, we would have ended up with a total return of 9.5, which is higher than the return of 8.82 tested with the "optimized" thresholds corresponding to $K = 0.001$. Moreover, the relative advantage of the trend following method over the buy and hold method in the testing results for the stock indices is not as good as those from the simulations in the previous subsection. These indicate that the regime switching geometric Brownian motion model with those parameter values is only an approximation of the real markets.

5. Conclusion.

We show that under a regime switching model a trend following trading system can be justified as an optimal trading strategy with a discounted reward function of trading one share of stock with a fixed percentage transaction cost. The optimal trading strategy has a simple implementable approximation. Extensive simulations and tests on historical stock market data show that this "optimal" trend following trading strategy (using constant thresholds $p_b^*(t)$ and $p_s^*(t)$) significantly outperforms the buy and hold strategy and is robust when parameters are perturbed. This investigation provides a useful theoretical framework for the widely used trend following trading methods.

Appendix.

A.1. The smoothness of the free boundaries $p_b^*(t)$ and $p_s^*(t)$.

By changing variables

$$y = \log\left(\frac{p}{p-1}\right), \quad w(y,t) = Z(p,t),$$

the double obstacle problem (2.9)–(2.11) becomes

$$\min\left\{\max\left\{-\partial_t w - \mathcal{L}_1 w, w - (1+K)\right\}, w - (1-K)\right\} = 0$$

in $(-\infty, +\infty) \times [0, T)$, with the terminal condition $w(y, T) = (1 - K)$, where

$$\mathcal{L}_1 = \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}\partial_{yy} + \left[\left(\mu_1 - \mu_2 - \lambda_2 + \lambda_2 - \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}\right) + \frac{(\mu_1 - \mu_2)^2}{\sigma^2}\frac{e^y}{e^y + 1}\right]\partial_y$$
$$+ \left(\mu_1 \frac{e^y}{e^y + 1} + \mu_2 \frac{1}{e^y + 1} - \rho\right).$$

To show the smoothness of the free boundaries, it suffices to verify the so-called cone property (cf. [6, 27]):

$$(T - t)\,\partial_t w + C\partial_y w \geq 0,$$

with some constant $C > 0$, locally uniformly for $y \in (-\infty, +\infty)$. For illustration,[4] let us restrict our attention to the region $\{(y, t) : 1-K < w(y, t) < 1+K\}$, in which $-\partial_t w - \mathcal{L}_1 w = 0$. Differentiating the above equation w.r.t. y, we have

$$(-\partial_t - \mathcal{L}_2)\,(\partial_y w) = \frac{e^y}{(e^y + 1)^2}\left(\frac{(\mu_1 - \mu_2)^2}{\sigma^2}\partial_y w + (\mu_1 - \mu_2)\,w\right)$$
$$\geq \frac{e^y}{(e^y + 1)^2}(\mu_1 - \mu_2)(1 - K),$$

where $\mathcal{L}_2 = \mathcal{L}_1 - (\lambda_1 e^y + \lambda_2 e^{-y})$. On the other hand,

$$(-\partial_t - \mathcal{L}_2)\,((T - t)\,\partial_t w) = \left[\left(\lambda_1 e^y + \lambda_2 e^{-y}\right)(T - t) + 1\right]\partial_t w.$$

It is easy to see that $\partial_t w$ is uniformly bounded and $\frac{e^y}{(e^y+1)^2}$ has a local positive lower bound. By using the auxiliary function $\psi(y, t; y_0) = e^{C_1 t}(y - y_0)^2$, with some positive constant C_1, as adopted in [6, 27], we can infer from the maximum principle that

$$(T - t)\,\partial_t w + C\partial_y w + \psi(y, t; y_0) \geq 0$$

for an appropriate $C > 0$. The desired result follows by taking $y = y_0$.

[4] A rigorous proof needs the use of a penalization method (cf. [12]).

A.2. Statistics for bull and bear markets of NASDAQ, SP500, and DJIA indices.

Table 11

Statistics of NASDAQ bull and bear markets (1991–2009).

Top/bottom	Index	% Move	Mean	Stdev	Duration
08/22/1990	374.84				
07/20/1998	2014.25	4.37362608	0.000841593	0.009228481	1998
10/08/1998	1419.12	−0.295459849	−0.005991046	0.024869575	57
03/10/2000	5048.62	2.557570889	0.003450954	0.018083041	358
04/04/2001	1638.8	−0.675396445	−0.004165933	0.032639469	269
05/22/2001	2313.85	0.411917257	0.00953807	0.031383788	33
09/21/2001	1423.19	−0.384925557	−0.005883363	0.020987512	81
01/04/2002	2059.38	0.447016913	0.004609728	0.020812769	72
10/09/2002	1114.11	−0.45900708	−0.003144999	0.021500913	192
01/26/2004	2153.83	0.933229214	0.001980534	0.015351658	325
08/12/2004	1752.49	−0.186337826	−0.00138275	0.011873996	138
10/31/2007	2859.12	0.63146152	0.000581948	0.008861377	811
03/09/2009	1268.64	−0.556283052	−0.002345953	0.024687197	339

Table 12

Statistics of SP500 bull and bear markets (1962–2009).

Top/bottom	Index	% Move	Mean	Stdev	Duration
01/03/1962	71.13				
06/26/1962	52.32	−0.264445382	−0.002538268	0.011710656	121
02/09/1966	94.06	0.797782875	0.000639029	0.005328904	913
10/07/1966	73.2	−0.221773336	−0.001460123	0.007778645	167
11/29/1968	108.37	0.480464481	0.000736578	0.005908178	516
05/26/1970	69.29	−0.360616407	−0.00119353	0.007244905	369
01/11/1973	120.24	0.735315341	0.000806951	0.00682808	665
10/03/1974	62.28	−0.482035928	−0.001489909	0.011230379	436
09/21/1976	107.83	0.731374438	0.001067083	0.010019283	497
03/06/1978	86.9	−0.194101827	−0.000549579	0.00600745	366
11/28/1980	140.52	0.61703107	0.00068538	0.008509688	691
08/12/1982	102.6	−0.269854825	−0.000728014	0.008772591	430
08/25/1987	336.77	2.282358674	0.000932206	0.00905962	1274
12/04/1987	223.92	−0.335095169	−0.005525613	0.035700189	71
07/16/1990	368.95	0.647686674	0.000747921	0.009679323	659
10/11/1990	295.46	−0.199186882	−0.003455122	0.012748722	62
07/17/1998	1186.75	3.016618155	0.000699894	0.007671028	1962
08/31/1998	957.28	−0.193360017	−0.006642962	0.017942563	31
03/24/2000	1527.46	0.595625104	0.001002091	0.013475437	395
10/09/2002	776.76	−0.491469498	−0.001059809	0.014432882	637
10/05/2007	1557.59	1.005239714	0.000531502	0.008593452	1256
03/09/2009	676.53	−0.565655917	−0.001833476	0.023888376	357

Table 13
Statistics of DJIA bull and bear markets (1962–2009).

Top/bottom	Index	% Move	Mean	Stdev	Duration
01/02/1962	724.7				
06/26/1962	535.7	−0.260797571	−0.002476914	0.010625788	122
02/09/1966	995.2	0.857756207	0.000674868	0.005654829	914
10/07/1966	744.3	−0.252110129	−0.00170399	0.007878357	167
12/03/1968	985.2	0.323659815	0.000525586	0.006114526	519
05/26/1970	631.2	−0.359317905	−0.001204596	0.007520823	367
01/11/1973	1051.7	0.666191381	0.000742507	0.007302779	665
12/06/1974	577.6	−0.450793953	−0.001389029	0.012603452	481
09/21/1976	1014.8	0.756925208	0.0012054	0.009654537	453
02/28/1978	742.1	−0.268722901	−0.000804275	0.006918213	363
04/27/1981	1024	0.379867942	0.0003929	0.008882315	797
08/12/1982	776.9	−0.241308594	−0.00082839	0.008056537	328
08/25/1987	2722.42	2.504209036	0.000983971	0.009436921	1273
10/19/1987	1738.74	−0.361325585	−0.011256546	0.043065972	38
07/17/1990	2999.75	0.725243567	0.000416497	0.015688021	693
10/11/1990	2365.1	−0.211567631	−0.00383401	0.013142105	61
07/17/1998	9337.97	2.948234747	0.000690436	0.007984466	1962
08/31/1998	7539.07	−0.192643583	−0.006654443	0.016969817	31
01/14/2000	11722.98	0.554963676	0.001079512	0.012092007	347
10/09/2002	7286.27	−0.378462643	−0.000675652	0.014062882	685
10/09/2007	14164.53	0.944002899	0.000504876	0.008380232	1258
03/09/2009	6547.05	−0.537785581	−0.002149751	0.021943976	355

A.3. Yield for 10-year US treasury bonds. Table 14 summarizes the annual yield of 10-year US treasury bonds as released on the Federal Reserve Statistical Release web site [10].

Table 14
Yield of 10-year bonds (1962–2008).

Year	Yield	Year	Yield	Year	Yield
1962	3.95	1978	8.41	1994	7.09
1963	4	1979	9.43	1995	6.57
1964	4.19	1980	11.43	1996	6.44
1965	4.28	1981	13.92	1997	6.35
1966	4.93	1982	13.01	1998	5.26
1967	5.07	1983	11.1	1999	5.65
1968	5.64	1984	12.46	2000	6.03
1969	6.67	1985	10.62	2001	5.02
1970	7.35	1986	7.67	2002	4.61
1971	6.16	1987	8.39	2003	4.01
1972	6.21	1988	8.85	2004	4.27
1973	6.85	1989	8.49	2005	4.29
1974	7.56	1990	8.55	2006	4.8
1975	7.99	1991	7.86	2007	4.63
1976	7.61	1992	7.01	2008	3.66
1977	7.42	1993	5.87		

A.4. Transactions for the trend following strategy on NASDAQ, SP500, and DJIA indices.

Table 15
NASDAQ investment test (1991–2008).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain
Bond	01/02/1991	1	01/23/1991	1.004522	1.004522
COMPQX	01/23/1991	383.91	06/24/1991	475.23	1.235393
Bond	06/24/1991	1	08/13/1991	1.010767	1.010767
COMPQX	08/13/1991	514.4	03/27/1992	604.67	1.173135
Bond	03/27/1992	1	09/14/1992	1.032841	1.032841
COMPQX	09/14/1992	594.21	10/05/1992	565.21	0.9492933
Bond	10/05/1992	1	11/05/1992	1.005954	1.005954
COMPQX	11/05/1992	614.08	02/16/1993	665.39	1.081389
Bond	02/16/1993	1	05/26/1993	1.015921	1.015921
COMPQX	05/26/1993	704.09	11/22/1993	738.13	1.046249
Bond	11/22/1993	1	08/24/1994	1.053418	1.053418
COMPQX	08/24/1994	751.72	11/22/1994	741.21	0.9840468
Bond	11/22/1994	1	02/09/1995	1.01422	1.01422
COMPQX	02/09/1995	785.44	10/04/1995	1002.27	1.27351
Bond	10/04/1995	1	02/01/1996	1.021173	1.021173
COMPQX	02/01/1996	1069.46	06/18/1996	1183.08	1.104028
Bond	06/18/1996	1	09/13/1996	1.01535	1.01535
COMPQX	09/13/1996	1188.67	02/27/1997	1312.66	1.102101
Bond	02/27/1997	1	05/02/1997	1.011134	1.011134
COMPQX	05/02/1997	1305.33	10/27/1997	1535.09	1.173665
Bond	10/27/1997	1	02/02/1998	1.014123	1.014123
COMPQX	02/02/1998	1652.89	05/26/1998	1778.09	1.073595
Bond	05/26/1998	1	06/24/1998	1.004179	1.004179
COMPQX	06/24/1998	1877.76	08/03/1998	1851.1	0.9838306
Bond	08/03/1998	1	11/04/1998	1.013402	1.013402
COMPQX	11/04/1998	1823.57	05/25/1999	2380.9	1.303015
Bond	05/25/1999	1	06/18/1999	1.003715	1.003715
COMPQX	06/18/1999	2563.44	08/04/1999	2540	0.9888744
Bond	08/04/1999	1	08/24/1999	1.003096	1.003096
COMPQX	08/24/1999	2752.37	10/18/1999	2689.15	0.9750766
Bond	10/18/1999	1	10/28/1999	1.001548	1.001548
COMPQX	10/28/1999	2875.22	04/03/2000	4223.68	1.466056
Bond	04/03/2000	1	07/12/2000	1.016521	1.016521
COMPQX	07/12/2000	4099.59	07/27/2000	3842.23	0.9353486
Bond	07/27/2000	1	08/31/2000	1.005782	1.005782
COMPQX	08/31/2000	4206.35	09/08/2000	3978.41	0.9439188
Bond	09/08/2000	1	05/02/2001	1.032458	1.032458
COMPQX	05/02/2001	2220.6	05/11/2001	2107.43	0.9471381
Bond	05/11/2001	1	05/21/2001	1.001375	1.001375
COMPQX	05/21/2001	2305.59	05/30/2001	2084.5	0.9022987
Bond	05/30/2001	1	10/25/2001	1.020355	1.020355
COMPQX	10/25/2001	1775.47	10/29/2001	1699.52	0.9553082
Bond	10/29/2001	1	11/06/2001	1.0011	1.0011
COMPQX	11/06/2001	1835.08	02/04/2002	1855.53	1.009122

NASDAQ investment test (1991–2008) (continued).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain
Bond	02/04/2002	1	11/01/2002	1.034101	1.034101
COMPQX	11/01/2002	1360.7	01/27/2003	1325.27	0.9720141
Bond	01/27/2003	1	03/17/2003	1.005383	1.005383
COMPQX	03/17/2003	1392.27	02/23/2004	2007.52	1.43902
Bond	02/23/2004	1	04/05/2004	1.004913	1.004913
COMPQX	04/05/2004	2079.12	04/16/2004	1995.74	0.9579766
Bond	04/16/2004	1	10/01/2004	1.019654	1.019654
COMPQX	10/01/2004	1942.2	01/20/2005	2045.88	1.051276
Bond	01/20/2005	1	05/23/2005	1.014457	1.014457
COMPQX	05/23/2005	2056.65	09/21/2005	2106.64	1.022258
Bond	09/21/2005	1	11/11/2005	1.005994	1.005994
COMPQX	11/11/2005	2202.47	05/11/2006	2272.7	1.029823
Bond	05/11/2006	1	09/05/2006	1.015386	1.015386
COMPQX	09/05/2006	2205.7	02/27/2007	2407.86	1.08947
Bond	02/27/2007	1	07/12/2007	1.017125	1.017125
COMPQX	07/12/2007	2701.73	07/26/2007	2599.34	0.9601779
Bond	07/26/2007	1	09/26/2007	1.007865	1.007865
COMPQX	09/26/2007	2699.03	11/09/2007	2627.94	0.9717135
Bond	11/09/2007	1	04/24/2008	1.016746	1.016746
COMPQX	04/24/2008	2428.92	06/11/2008	2394.01	0.9836561
Bond	06/11/2008	1	08/11/2008	1.006117	1.006117
COMPQX	08/11/2008	2439.95	09/03/2008	2333.73	0.9545534
Bond	09/03/2008	1	12/31/2008	1.011933	1.011933

Table 16
SP500 investment test (1962–2008).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain	Dividend
Bond	01/03/1962	1	11/12/1962	1.033873	1.033873	
SP500	11/12/1962	59.59	06/09/1965	85.04	1.424231	0.085
Bond	06/09/1965	1	09/10/1965	1.010905	1.010905	
SP500	09/10/1965	89.12	03/02/1966	89.15	0.998336	0.021
Bond	03/02/1966	1	11/16/1966	1.034983	1.034983	
SP500	11/16/1966	82.37	11/03/1967	91.78	1.112012	0.033
Bond	11/03/1967	1	04/03/1968	1.023487	1.023487	
SP500	04/03/1968	93.47	01/07/1969	101.22	1.080748	0.023
Bond	01/07/1969	1	05/05/1969	1.021563	1.021563	
SP500	05/05/1969	104.37	06/09/1969	101.2	0.967688	0.003
Bond	06/09/1969	1	08/24/1970	1.088804	1.088804	
SP500	08/24/1970	80.99	06/22/1971	97.59	1.202554	0.028
Bond	06/22/1971	1	09/03/1971	1.01232	1.01232	
SP500	09/03/1971	100.69	10/20/1971	95.65	0.9480455	0.004
Bond	10/20/1971	1	12/17/1971	1.009789	1.009789	
SP500	12/17/1971	100.26	02/27/1973	110.9	1.103912	0.034
Bond	02/27/1973	1	07/24/1973	1.027588	1.027588	
SP500	07/24/1973	108.14	08/13/1973	103.71	0.9571165	0.002
Bond	08/13/1973	1	10/05/1973	1.009947	1.009947	
SP500	10/05/1973	109.85	11/14/1973	102.45	0.9307701	0.004
Bond	11/14/1973	1	03/13/1974	1.024648	1.024648	
SP500	03/13/1974	99.74	03/28/1974	94.82	0.9487704	0.002

SP500 investment test (1962–2008) (continued).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain	Dividend
Bond	03/28/1974	1	01/10/1975	1.063044	1.063044	
SP500	01/10/1975	72.61	08/04/1975	87.15	1.197847	0.023
Bond	08/04/1975	1	10/13/1975	1.015323	1.015323	
SP500	10/13/1975	89.46	04/06/1977	97.91	1.092267	0.063
Bond	04/06/1977	1	04/14/1978	1.085943	1.085943	
SP500	04/14/1978	92.92	10/20/1978	97.95	1.052024	0.028
Bond	10/20/1978	1	01/12/1979	1.021702	1.021702	
SP500	01/12/1979	99.93	10/22/1979	100.71	1.00579	0.043
Bond	10/22/1979	1	11/26/1979	1.009043	1.009043	
SP500	11/26/1979	106.8	03/17/1980	102.26	0.9555756	0.015
Bond	03/17/1980	1	05/23/1980	1.020981	1.020981	
SP500	05/23/1980	110.62	08/27/1981	123.51	1.114292	0.067
Bond	08/27/1981	1	11/03/1981	1.025933	1.025933	
SP500	11/03/1981	124.8	11/16/1981	120.24	0.9615346	0.002
Bond	11/16/1981	1	04/23/1982	1.056317	1.056317	
SP500	04/23/1982	118.64	05/26/1982	113.11	0.9514816	0.005
Bond	05/26/1982	1	08/20/1982	1.030654	1.030654	
SP500	08/20/1982	113.02	02/08/1984	155.85	1.376202	0.067
Bond	02/08/1984	1	08/02/1984	1.060081	1.060081	
SP500	08/02/1984	157.99	09/17/1985	181.36	1.145625	0.047
Bond	09/17/1985	1	11/05/1985	1.014257	1.014257	
SP500	11/05/1985	192.37	10/16/1987	282.7	1.466625	0.071
Bond	10/16/1987	1	02/29/1988	1.032975	1.032975	
SP500	02/29/1988	267.82	08/06/1990	334.43	1.246214	0.089
Bond	08/06/1990	1	12/03/1990	1.027875	1.027875	
SP500	12/03/1990	324.1	01/09/1991	311.49	0.95917	0.004
Bond	01/09/1991	1	01/18/1991	1.001938	1.001938	
SP500	01/18/1991	332.23	03/30/1994	445.55	1.338407	0.096
Bond	03/30/1994	1	08/24/1994	1.028554	1.028554	
SP500	08/24/1994	469.03	07/15/1996	629.8	1.340086	0.045
Bond	07/15/1996	1	08/22/1996	1.006705	1.006705	
SP500	08/22/1996	670.68	08/31/1998	957.28	1.424473	0.032
Bond	08/31/1998	1	09/23/1998	1.003314	1.003314	
SP500	09/23/1998	1066.09	10/01/1998	986.39	0.9233904	0.000
Bond	10/01/1998	1	10/22/1998	1.003026	1.003026	
SP500	10/22/1998	1078.48	08/10/1999	1281.43	1.185805	0.009
Bond	08/10/1999	1	08/25/1999	1.002322	1.002322	
SP500	08/25/1999	1381.79	08/30/1999	1324.02	0.9562755	0.000
Bond	08/30/1999	1	10/28/1999	1.009133	1.009133	
SP500	10/28/1999	1342.44	10/11/2000	1364.59	1.014467	0.012
Bond	10/11/2000	1	10/31/2000	1.003304	1.003304	
SP500	10/31/2000	1429.4	11/10/2000	1365.98	0.9537205	0.000
Bond	11/10/2000	1	01/23/2001	1.010177	1.010177	
SP500	01/23/2001	1360.4	02/20/2001	1278.94	0.9382402	0.001
Bond	02/20/2001	1	05/16/2001	1.01169	1.01169	
SP500	05/16/2001	1284.99	06/18/2001	1208.43	0.938539	0.001
Bond	06/18/2001	1	11/19/2001	1.02118	1.02118	
SP500	11/19/2001	1151.06	07/22/2002	819.85	0.7108319	0.012
Bond	07/22/2002	1	07/29/2002	1.000884	1.000884	
SP500	07/29/2002	898.96	09/24/2002	819.29	0.9095526	0.003

SP500 investment test (1962–2008) (continued).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain	Dividend
Bond	09/24/2002	1	10/11/2002	1.002147	1.002147	
SP500	10/11/2002	835.32	01/18/2008	1325.19	1.583273	0.096
Bond	01/18/2008	1	03/18/2008	1.006016	1.006016	
SP500	03/18/2008	1330.74	07/07/2008	1252.31	0.9391808	0.009
Bond	07/07/2008	1	08/08/2008	1.003209	1.003209	
SP500	08/08/2008	1296.31	10/09/2008	909.92	0.700527	0.005
Bond	10/09/2008	1	10/30/2008	1.002106	1.002106	
SP500	10/30/2008	954.09	12/31/2008	903.25	0.9467136	0.005

Table 17

DJIA investment test (1962–2008).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain
DJ-30	11/02/1962	604.6	07/22/1963	688.7	1.136822
Bond	07/22/1963	1	09/24/1963	1.007014	1.007014
DJ-30	09/24/1963	746	11/22/1963	711.5	0.9518458
Bond	11/22/1963	1	12/05/1963	1.001425	1.001425
DJ-30	12/05/1963	763.9	06/09/1965	879.8	1.149418
Bond	06/09/1965	1	09/16/1965	1.011609	1.011609
DJ-30	09/16/1965	931.2	03/01/1966	938.2	1.005502
Bond	03/01/1966	1	01/12/1967	1.044033	1.044033
DJ-30	01/12/1967	830	10/31/1967	879.7	1.05776
Bond	10/31/1967	1	04/08/1968	1.024723	1.024723
DJ-30	04/08/1968	884.4	01/08/1969	921.3	1.03964
Bond	01/08/1969	1	05/06/1969	1.021563	1.021563
DJ-30	05/06/1969	962.1	06/06/1969	924.8	0.9593082
Bond	06/06/1969	1	07/17/1970	1.081756	1.081756
DJ-30	07/17/1970	735.1	06/21/1971	876.5	1.18997
Bond	06/21/1971	1	09/07/1971	1.013164	1.013164
DJ-30	09/07/1971	916.5	10/18/1971	872.4	0.9499784
Bond	10/18/1971	1	12/29/1971	1.012151	1.012151
DJ-30	12/29/1971	893.7	02/07/1973	968.3	1.081306
Bond	02/07/1973	1	07/25/1973	1.031529	1.031529
DJ-30	07/25/1973	933	08/10/1973	892.4	0.9545715
Bond	08/10/1973	1	09/24/1973	1.008445	1.008445
DJ-30	09/24/1973	936.7	11/14/1973	869.9	0.9268284
Bond	11/14/1973	1	02/27/1974	1.021748	1.021748
DJ-30	02/27/1974	863.4	07/08/1974	770.6	0.8907329
Bond	07/08/1974	1	01/09/1975	1.040497	1.040497
DJ-30	01/09/1975	645.3	08/05/1975	810.2	1.253029
Bond	08/05/1975	1	10/13/1975	1.015104	1.015104
DJ-30	10/13/1975	837.8	10/12/1976	932.4	1.110689
Bond	10/12/1976	1	12/24/1976	1.01522	1.01522
DJ-30	12/24/1976	996.1	02/08/1977	942.2	0.9439972
Bond	02/08/1977	1	04/14/1978	1.099077	1.099077
DJ-30	04/14/1978	795.1	10/26/1978	821.1	1.030635
Bond	10/26/1978	1	01/15/1979	1.020927	1.020927
DJ-30	01/15/1979	848.7	10/19/1979	814.7	0.9580188
Bond	10/19/1979	1	01/10/1980	1.025992	1.025992

DJIA investment test (1962–2008) (continued).

Symbol	Buy date	Buy price	Sell date	Sell price	Gain
DJ-30	01/10/1980	858.9	03/10/1980	818.9	0.9515219
Bond	03/10/1980	1	05/20/1980	1.022234	1.022234
DJ-30	05/20/1980	832.5	12/11/1980	908.5	1.089109
Bond	12/11/1980	1	01/05/1981	1.009534	1.009534
DJ-30	01/05/1981	992.7	07/21/1981	934.5	0.9394892
Bond	07/21/1981	1	04/23/1982	1.098377	1.098377
DJ-30	04/23/1982	862.2	05/28/1982	819.5	0.9485745
Bond	05/28/1982	1	08/17/1982	1.028872	1.028872
DJ-30	08/17/1982	831.2	02/03/1984	1197	1.437206
Bond	02/03/1984	1	08/02/1984	1.061788	1.061788
DJ-30	08/02/1984	1166.1	10/16/1987	2246.74	1.92286
Bond	10/16/1987	1	02/29/1988	1.032975	1.032975
DJ-30	02/29/1988	2071.62	08/06/1990	2716.34	1.308593
Bond	08/06/1990	1	12/05/1990	1.028344	1.028344
DJ-30	12/05/1990	2610.4	01/09/1991	2470.3	0.9444374
Bond	01/09/1991	1	01/18/1991	1.001938	1.001938
DJ-30	01/18/1991	2646.78	03/30/1994	3626.75	1.367509
Bond	03/30/1994	1	08/24/1994	1.028554	1.028554
DJ-30	08/24/1994	3846.73	08/31/1998	7539.07	1.955945
Bond	08/31/1998	1	10/15/1998	1.006485	1.006485
DJ-30	10/15/1998	8299.36	02/25/2000	9862.12	1.185922
Bond	02/25/2000	1	03/16/2000	1.003304	1.003304
DJ-30	03/16/2000	10630.6	10/12/2000	10034.58	0.9420457
Bond	10/12/2000	1	12/05/2000	1.008921	1.008921
DJ-30	12/05/2000	10898.72	03/22/2001	9389.48	0.8597984
Bond	03/22/2001	1	04/05/2001	1.001925	1.001925
DJ-30	04/05/2001	9918.05	07/10/2001	10175.64	1.02392
Bond	07/10/2001	1	11/14/2001	1.017467	1.017467
DJ-30	11/14/2001	9823.61	07/22/2002	7784.58	0.7908509
Bond	07/22/2002	1	07/29/2002	1.000884	1.000884
DJ-30	07/29/2002	8711.88	09/24/2002	7683.13	0.8801503
Bond	09/24/2002	1	10/11/2002	1.002147	1.002147
DJ-30	10/11/2002	7850.29	02/12/2003	7758.17	0.9862888
Bond	02/12/2003	1	03/17/2003	1.003626	1.003626
DJ-30	03/17/2003	8141.92	04/15/2005	10087.51	1.236482
Bond	04/15/2005	1	11/11/2005	1.024682	1.024682
DJ-30	11/11/2005	10686.04	01/22/2008	11971.19	1.118024
Bond	01/22/2008	1	03/18/2008	1.005615	1.005615
DJ-30	03/18/2008	12392.66	06/26/2008	11453.42	0.9223616
Bond	06/26/2008	1	08/28/2008	1.006317	1.006317
DJ-30	08/28/2008	11715.18	09/04/2008	11188.23	0.9531099
Bond	09/04/2008	1	09/30/2008	1.002607	1.002607
DJ-30	09/30/2008	10850.66	10/06/2008	9955.5	0.9156668
Bond	10/06/2008	1	12/31/2008	1.008624	1.008624

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