Continuous-time Martingale and Brownian Motion

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1 Continuous-time Martingale

1.1 Introduction

Recall that we have introduced the discrete-time martingale, which can be interpreted as a fair game. Not surprisingly, there exists a counterpart in continuous-time framework. Let's first give the definition of *martingale* in general. The setting is usual, we have $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space equipped with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$:

Definition 1.1 A stochastic process $M = \{M_t\}_{t\geq 0}$ defined on the probability space above is said to be a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\mathbb{P}$ if

1. for every $t \geq 0$, $\mathbb{E}[|M_t|] < \infty$;

2. for every t, M_t is measurable with respect to the filtration and for all $0 \leq s \leq t$,

$$\mathbb{E}[M_t|\mathcal{F}_s] = M(s) \tag{1}$$

Equation (1) is called the *martingale property*. Its validity depends on the filtration $\mathcal{F}_t$ and the probability $\mathbb{P}$. However, when the probability measure is fixed and not subject to change, we shall just say that M is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ or an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale. Observe that if M is a martingale, it is constant in expectation: $\mathbb{E}[M(t)] = \mathbb{E}[M(0)]$ for all $t \geq 0$. As we have seen in the discrete-time case, the definition of *sup-(sub-)martingale* just needs to turn the equality to inequality, i.e., $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$ (sup-martingale), $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$(sub-martingale).

Suppose that M represents your fortune at time t is an ongoing game of chance. If s is the current time and $t > s$ a future time, (1) says that your expected fortune at time t, conditioned on everything that has happened up to time s, is just what you presently have. Thus, the game is fair, at least as measured by expected value.

1.2 Examples of Martingale

Let's first check out two martingales that are just analogue of discrete-time phenomena:

Example 1.1 We proved earlier that symmetric simple random walk is a martingale. As you know, *Brownian motion* is the limiting behaviour of certain simple random walk (with appropriate rescaling), you may expect that it is also a martingale. Let's carry out the program: since $\mathbb{E}[|W(t)|] < +\infty$ (check by yourself), (1) is satisfied; let's take $0 \leq s \leq t$,

$$\begin{aligned}\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[(W_t - W_s) + W_s|\mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] \\ &= W_s\end{aligned}$$

Indeed, we convince ourselves that W is a $\{\mathcal{F}_t\}_{t\geq 0}$-martingale. Note the standard trick here is to decompose the process into "known part", given the knowledge of the past, the value is known, and "independent part", independent of the history.

Example 1.2 You may still remember, in the discrete-time setting, if we have $\{M_t\}_{t\geq 0}$ a martingale, then betting on martingale is again a martingale. This is still true in the continuous-time setting. Let's illustrate in this context, consider a game in which you are allowed to place adapted bets on the increment of a $\{\mathcal{F}_t\}_{t\geq 0}$-martingale M. This means that if you place a bet of α at time s and hold it until time t, you will earn the amount $\alpha(M_t - M_s)$. To say the best is adapted means that if α is bet at time s, it must be $\mathcal{F}_s$-measurable; this is a way of saying you are not able to look into the future when deciding how much to bet. The expected gain of this bet is $\mathbb{E}[\alpha(M_t - M_s)|\mathcal{F}_s] = \alpha\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$. Thus, the game is fair no matter how you bet. This observation leads to an important heuristic principle; let X_t be the gain at t from betting on the increments of a martingale using adapted bets, or let X_t be a limit of such gains for a sequence of betting schemes. Then X is also a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Of course, to make this into a theorem requires a more precise formulation and further technical conditions. Stochastic integrals with respect to Brownian motion provide one example of such a precise formulation, which we will learn soon.

Now, let's given a family of sub-martingale:

Proposition 1.3 Suppose that X is a martingale, $f : \mathbb{R} \mapsto \mathbb{R}$ is a convex function such that $f(X)$ is integrable, then $f(X_t)$ is a sub-martingale.

Proof. This is a direct consequence of *Jensen's inequality*:

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] \geq f\left(\mathbb{E}[X_t|\mathcal{F}_s]\right) = f(X_s)$$

$\square$

In particular, if X has finite variance, X^2 is a sub-martingale. However, with a small modification we can recover the martingale property. We claim: $W_t^2 - t$ is a martingale:

$$
\begin{aligned}
\mathbb{E}[W_t^2|\mathcal{F}_s] &= \mathbb{E}[W_t^2 - W_s^2 + W_s^2|\mathcal{F}_s] \\
&= \mathbb{E}[W_t^2 - W_s^2|\mathcal{F}_s] + W_s^2 \\
&= \mathbb{E}[(W_t - W_s)(W_t + W_s)|\mathcal{F}_s] + W_s^2 \\
&= \mathbb{E}[(W_t - W_s)(W_t - W_s + 2W_s)|\mathcal{F}_s] + W_s^2 \\
&= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2\mathbb{E}[(W_t - W_s)W_s|\mathcal{F}_s] + W_s^2 \\
&= \mathrm{Var}(W_t - W_s) + 2W_s\mathbb{E}[W_t - W_s] + W_s^2 \\
&= t - s + W_s^2
\end{aligned}
$$

Thus, $\mathbb{E}[W_t^2 - t|\mathcal{F}_s] = W_s^2 - s$, this validates the martingale property. The calculation is standardized, but the importance is significant, it basically says that a martingale can be decomposed to a sub-martingale and an increasing predictable process. We already saw this argument in discrete setting ,that is *Doob's decomposition*, the continuous-time counterpart is called *Doob-Meyer decomposition theorem*. Let's state the theorem:

Theorem 1.4 (*Doob-Meyer decomposition*) Let Z be a cádlág sub-martingale with $Z_0 = 0$, then there exists a unique, increasing, predictable process A with $A_0 = 0$ such that $M_t = Z_t - A_t$ is a uniformly integrable martingale.

In our example, $Z_t = W_t^2$ which has value 0 at time 0 and t is trivial increasing predictable process.

One more example regarding the transformation of Brownian motion, which we shall use later:

Example 1.5 $\{W_t\}_{t\geq 0}$ is a Brownian motion adapted the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, define $X_t = e^{\sigma W_t}$ and $Y_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$, for $\sigma \geq 0$. Take $0 \leq s < t$,

$$\mathbb{E}[e^{\sigma W_t}|\mathcal{F}_s] = e^{\sigma W_s}\mathbb{E}[e^{\sigma(W_t - W_s)}|\mathcal{F}_s] = \mathbb{E}[e^{\sigma(W_t - W_s)}]$$

Set $Z = W_t - W_s \sim N(0, t-s)$, from the knowledge of M.G.F for normal random variable, we have

$$\psi_Z(\sigma) = \mathbb{E}[e^{\sigma Z}] = e^{\frac{1}{2}\sigma^2(t-s)}$$

Thus,

$$\mathbb{E}[e^{\sigma W_t}|\mathcal{F}_s] = e^{\sigma W_s}e^{\frac{1}{2}\sigma^2(t-s)} \geq e^{\sigma W_s}$$

X_t is a sub-martingale. Multiplying both sides by $e^{-\frac{\sigma^2}{2}t}$, we have

$$\mathbb{E}[e^{\sigma W_t - \frac{\sigma^2}{2}t}|\mathcal{F}_s] = e^{\sigma W_s - \frac{\sigma^2}{2}s}$$

Thus, Y_t is a martingale.

2 Right-continuous Filtration and Miscellanea

2.1 Definition of right-continuous filtration

Given a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, let

$$\mathcal{F}_{t_+} := \bigcap_{s>t} \mathcal{F}_s$$

for every $t \geq 0$. The family $\{\mathcal{F}_{t_+}\}_{t\geq 0}$ defines a new filtration that augments $\mathbb{F}$. One may think of $\mathcal{F}_{t_+}$ as the past plus an infinitesimal step into the future. Let's give an example to illustrate this type of event that is in $\mathcal{F}_{t_+}$ but not in $\mathcal{F}_t$. Let X be a continuous stochastic process for which $X_0(\omega) = 1$ for all ω, and let $\mathbb{Q}$ be the rational numbers. Then

$$\bigcap_{n=1}^{\infty} \bigcap_{t\in[0,\frac{1}{n}]\cap\mathbb{Q}} \{X_t \geq 1\}$$

is the event, that, starting from time $t = 0$, X_t stays above 1 for a positive amount of time. Assuming that neither this event nor its complement is empty, it is in $\mathcal{F}^X_{0_+}$ but not $\mathcal{F}^X_0$.

Definition 2.1 A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is *right-continuous* if $\mathcal{F}_t = \mathcal{F}_{t_+}$ for all t.

Clearly, $\{\mathcal{F}_{t_+}\}_{t\geq 0}$ is always right-continuous filtration. The example of the previous paragraph shows that, in general, the filtration generated by a process will not be right-continuous.

2.2 Remarkable Fact about Brownian motion*

In general, even a right continuous process can not have a right-continuous filtration. But we can always make the filtration right-continuous by appending the infinitesimal step filtration. Let's introduce some terminology for convenience of illustration. Suppose $\{X_t\}_{t\geq 0}$ is a continuous-time stochastic process, define

$$\mathcal{F}^0_t := \sigma(X_t : 0 \leq s \leq t)$$

which is the σ-algebra generated by the process X upto time t. When X is a Brownian motion, we call $\mathcal{F}^\circ_t$ the *raw filtration of Brownian motion*. Let's from now on restrict our attention only to Brownian motion. As the way we defined in the previous subsection, we enlarge the raw filtration by

$$\mathcal{F}_{t+} = \bigcap_{r>t} \mathcal{F}^\circ_r$$

This filtration is called *augmented filtration of Brownian motion*. Obviously, the raw filtration $\mathcal{F}^\circ_t$ is not right-continuous but after we construct $\mathcal{F}_{t+}$, it is certainly right-continuous.

Why we want to do this to make the filtration right continuous? Let's give some explanations. Recall the *strong Markov Property* of Brownian motion

Theorem 2.1 (*Markov Property*) Suppose $\{W_t : t \geq 0\}$ is a Brownian motion started in $x \in \mathbb{R}$ (not necessary standard). Let $s > 0$, then the process $\{W_{t+s} - W_s : t \geq 0\}$ is a standard Brownian motion that is independent of $\mathcal{F}_s^0$.

The theorem can be improved as we replace raw filtration by the augmented filtration.

Theorem 2.2 For every $s \geq 0$ the process $\{W_{t+s} - W_s : t \geq 0\}$ is independent of $\mathcal{F}_{s+}$.

Proof. By continuity $W_{t+s} - W_s = \lim_{n\to\infty} (W_{s_n+t} - W_{s_n})$ for a strictly decreasing sequence $\{s_n : n \in \mathbb{N}\}$ converging to s from above. By *theorem* 2.1, for any $t_1, ..., t_m \geq 0$, the vector $(W_{t_1+s} - W_s, ..., W_{t_m+s} - W_s) = \lim_{j\uparrow\infty} \left(W_{t_1+s_j} - W_{s_j}, ..., W_{t_m+s_j} - W_{s_j}\right)$ is independent of $\mathcal{F}_{s+}$, and so is the process $\{W_{t+s} - W_s : t \geq 0\}$. $\square$

As a particular case, we have the following important result:

Theorem 2.3 (*Blumenthal's* $0-1$ *law*) For any $A \in \mathcal{F}_{0+}$, $\mathbb{P}(A) \in \{0, 1\}$.

Proof. By using *theorem* 2.2, for $s = 0$, any $A \in \sigma(W_t : t \geq 0)$ is independent of $\mathcal{F}_{0+}$. This applies in particular to $A \in \mathcal{F}_{0+}$, which is therefore independent of itself,

$$\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$$

Thus, $\mathbb{P}(A)$ can either be 0 or 1. $\square$

As a small application, let's define

$$\tau = \int \{t > 0 : W_t > 0\}$$

The event

$$\{\tau = 0\} = \bigcap_{n=1}^{\infty} \{\exists 0 < \epsilon < \frac{1}{n} \text{ s.t. } W_\epsilon > 0\}$$

which is clearly in $\mathcal{F}_{0+}$. We want to show that this event has positive probability, then it is automatically 1. This follows, because

$$\mathbb{P}\{\tau = 0\} \geq \mathbb{P}\{W_t > 0\} = \frac{1}{2}$$

for $t > 0$.

If you still remember the example of last recitation, we state that, in general, $\tau_b = \inf\{t : X_t > b\}$ is not a $\{\mathcal{F}_t^X\}_{t\geq 0}$ stopping-time. But if we augment *Brownian motion* by make the filtration right-continuous, τ_b is again a stopping time but w.r.t $\{\mathcal{F}_{t+}\}_{t\geq 0}$(Why? by strong Markov property, verify by yourself!). Thus, to have nice property, such as the first entrance time of an open set is a stopping time, we always complete the raw filtration of Brownian motion by adding infinitesimal step filtration and all p-null sets.

3 Miscelleaneous

Regularity properties of sample paths are very significant to stochastic analysis. A process is said to be *continuous*, or *right-continuous*, or *left-continuous*, if each sample path is, respectively, continuous, or right-continuous, or left-continuous. Processes whose sample paths are right-continuous and also admit a finite limit, $\lim_{s\uparrow t} X_s$ at each t, $t > 0$, are particularly important. This was first realized by the French school of probabilists centered at Strasbourg under the leadership of Paul-André Meyer, and they called such process "cádlág", which is an abbreviation of "continu á droite, limites á gauche." This terminology made its way into English literature and is still in common use. Many important process, such as Brownian motion, Poisson process, are all of this type.

Many theorems remains true in continuous-time setting as long as the process has cádlág paths. Especially, *Doob's martingale and up-crossing inequality* is valid as well as *martingale convergence theorem*:

Theorem 3.1 Let $\{X\}_{t\geq 0}$ be a sub-martingale w.r.t $\{\mathcal{F}_t\}_{t\geq 0}$, which has *cádlág* sample path, let $[t_1, t_2]$ be a sub-interval of $[0, +\infty)$ and let $\alpha < \beta, \lambda > 0$ be real numbers, we have,

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t_1 \leq t \leq t_2} X_t(\omega) \geq \lambda\right) \leq \frac{\mathbb{E}[X_{t_2}^+]}{\lambda} \text{ (martingale inequality)}$$

and

$$\mathbb{E}[U_{[t_1,t_2]}(\alpha, \beta, X(\omega))] \leq \frac{\mathbb{E}[X_{t_2}^+] + |\alpha|}{\beta - \alpha} \text{ (upcrossing inequality)}$$

where $U_{[t_1,t_2]}(\alpha, \beta, X(\omega))$ denotes number of times the process up-cross the interval (α, β) during $[t_1, t_2]$.

Theorem 3.2 (*Martingale Convergence*)Let $\{X\}_{t\geq 0}$ be a sub-martingale w.r.t $\{\mathcal{F}_t\}_{t\geq 0}$, which has *cádlág* sample path, if $\sup_{t\geq 0} \mathbb{E}[X_t^+] < +\infty$, then $X_\infty(\omega) := \lim_{t\to\infty} X_t(\omega)$ exists almost surely and X_∞ is integrable (in $\mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{P})$).

Also, the *optional-stopping theorem* is true:

Theorem 3.3 (*Optional-Stopping*)Let $\{X\}_{t\geq 0}$ be a sub-martingale w.r.t $\{\mathcal{F}_t\}_{t\geq 0}$, which has *cádlág* sample path, and let $\tau_1 \leq \tau_2$ be two bounded stopping time (with respect to the same filtration $\{\mathcal{F}_t\}_{t\geq 0}$), then

$$\mathbb{E}[S_{\tau_2}|\mathcal{F}_{\tau_1}] \geq X_{\tau_1}, \text{ a.s.}$$

In particular, $\mathbb{E}[X_{\tau_2}] \geq \mathbb{E}[X_0]$.

Remark 3.4 Although all theorems are stated in terms of sub-martingale, they hold for martingale and sup-martingale as well.

Again $\{W_t\}_{t\geq 0}$ is a Brownian motion, let's recall the definition of *first hitting time*,

$$T_a = \inf\{t \geq 0, W(t)(\omega) = a\}$$

We want to compute $\mathbb{E}[e^{-\theta}T_a]$ given $\theta \geq 0$.

Notice, from previous section, that $Y_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$ is a martingale, let's define, for large N,

$$T_N = \begin{cases} T_a, & \text{if } T_a \leq N, \\ N, & \text{otherwise.} \end{cases}$$

Obviously, T_N is a bounded stopping time. By optional stopping theorem,

$$\mathbb{E}[Y_{T_N}] = Y_0 = 1$$

That is, $\mathbb{E}[e^{\sigma W_{T_N} - \frac{\sigma^2}{2}T_N}] = 1$. If $T_N = T_a$, then $Y_{T_N} = e^{-\sigma a - \frac{\sigma^2}{2}T_a}$, if $T_N = N$, $Y_{T_N} = e^{\sigma W_T} - \frac{\sigma^2}{2}N \leq e^{\sigma a - \frac{\sigma^2}{2}N}$. As a result, Y_{T_N} is bounded, i.e.,

$$0 \leq Y_{T_N} \leq e^{\sigma N}$$

As $N \to \infty$, $Y_{T_N} \to Y_{T_a}$ with probability 1 (the composition of continuous functions are continuous, which preserve the limit). By *dominant convergence theorem*:

$$1 = \lim_{N\to\infty} \mathbb{E}[Y_{T_N}] = \mathbb{E}[\lim_{N\to\infty} Y_{T_N}] = \mathbb{E}[Y_{T_a}]$$

Eventually, we have

$$\mathbb{E}[e^{\sigma W_{T_a} - \frac{\sigma^2}{2}T_a}] = 1$$

which is equivalent to:

$$\mathbb{E}[e^{-\frac{\sigma^2}{2}T_N}] = e^{-\sigma a}$$

Define $\theta = \frac{\sigma^2}{2} > 0$, we have

$$\mathbb{E}[e^{-\theta T_N}] = e^{-\sqrt{2\theta}a}$$

4 Reference

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