Mean Value Theorem and Taylor Series

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January 1, 2015

1 Differentiation

Let's first formally describe the differentiability of function:

Definition 1.1 A function $f : [a, b] \mapsto \mathbb{R}$ is *differentiable* at $x \in [a, b]$ if the following limit, which is called *derivative*, exists:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}, \ t \in (a, b), \ t \neq x$$

In calculus, usually, we visualize or interpret the derivative as the slope of the secant line at point of interest. If function f is continuous on $[a, b]$, must it be differentiable? The answer is of course no, e.g., $f(x) = |x|$ is continuous but not differentiable at point 0. But the converse is true:

Proposition 1.1 If f is differentiable on $[a, b]$, then f is continuous.

Proof. (*Idea*) To verify f is continuous, it is sufficient to have $t \to x$, $f(t) \to f(x)$. We want to have,

$$\lim_{t \to x} f(t) - f(x) = 0$$

Since the limit $f'(x)$,

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}(t - x) = 0$$

□

Next, we want to ask: if f is differentiable on $[a, b]$, must f' be continuous? The answer is NO! Let's recall the most useful function as a counter example:

$$f(x) = \begin{cases} x^{\frac{4}{3}} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The Graph is in *Figure 1*. It is differentiable, maybe the most controversial point is 0. But as you observe these two asymptotic function, we can, at least, intuitively, know that the derivative around 0 is 0. That is actually correct. Let's draw the graph of f' (in *Figure 2*), it is certainly not continuous at 0. It satisfies the intermediate value property but has 2^{ed} kind discontinuity.

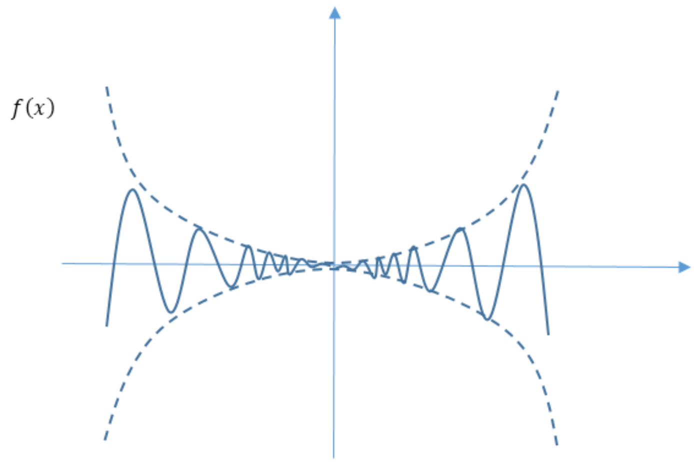

Figure 1: Topological Sine Curve

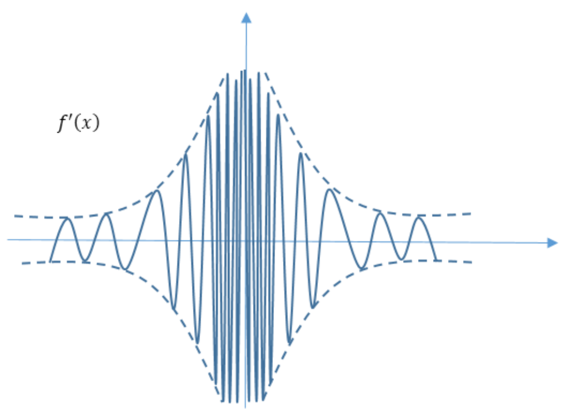

Figure 2: Derivative of Topological Sine Curve

We denote the class of functions that has first derivative and f' is continuous $\mathscr{C}^1$. Similarly, we can generalize the notion and notation to k–th differentiable and $f^{(k)}$ is continuous, $\mathscr{C}^k$, in particular, the function that is infinitely differentiable and continuous is in the class $\mathscr{C}^\infty$, we also call it *smooth function*.

One remark, also can be considered as a theorem:

Theorem 1.2 There exists functions $f : \mathbb{R} \mapsto \mathbb{R}$ that a re continuous everywhere, but differentiable nowhere.

Example 1.3

$$f(x) = \sum_{i=1}^{\infty} b^n \cos(a^n \pi x)$$

where $0 < b < 1$, a is odd integer and $ab > 1 + \frac{3\pi}{2}$.

2 Mean Value Theorem

The *mean value theorem* initiated by *Cauchy* (although the proof of *Cauchy* is problematic) is the fundamental tool to connect the value of a function with the value of f' without using limits. Let's first state the theorem:

Theorem 2.1 If f is continuous on $[a, b]$, differentiable on (a, b), $\exists$ some point $c \in (a, b)$, such that

$$f(b) - f(a) = f'(c)(b - a)$$

Example 2.2 If $f'(x) > 0$ for all $x \in (a, b)$, then $f(b) > f(a)$.

Proof.

$$f(b) - f(a) = (b - a)f'(c) > 0$$

as desired. □

Proof. Step 1 (Rolle's Theorem): If a function h on $[a, b]$ has a maximum on $c \in [a, b]$ and $f'(c)$ exists, then $h'(c) = 0$. Because

$$\frac{h(t) - h(c)}{t - c}$$

It is negative on the right hand side ($t > c$) and positive on the left hand side ($t < c$). Since the limit exists, then the left limits and right limits must be equal, thus $h'(c) = 0$.

Step 2 (Generalized Mean Value Theorem): If $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c) \tag{1}$$

Notice, if $g(x) = x$, we get the *classical mean value theorem.* Let's prove it, but firstly build the notion by the aid of graph (*Figure 3*) Observe the left hand side of 1 is the rate that L sweeps out the area, right hand side of 1 is the rate that L sweeps out the area.

Let's define

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

This can be interpreted as the difference of the area swept by time x. Clearly, $\exists c$ such that $h'(c) = 0$. But $h'(c) = (\text{LHS} - \text{RHS})$ of 1. □

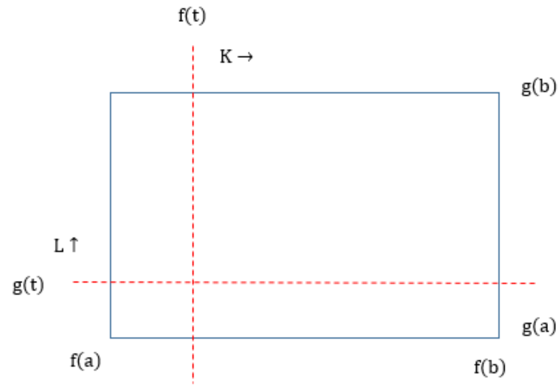

Figure 3: Graph for Proof

3 Taylor Theorem

Suppose we know $f(a)$, we want to approximate $f(b)$, by MVT:

$$f(b) = f(a) + f'(c)(b-a), \text{for some } c \in (a,b)$$

The second term can be considered as the error term. Let's push this further, if we choose this particular point to be a, $c = a$, we can certainly do that, but it will introduce another error,

$$f(b) = f(a) + f'(a)(b-a) + \frac{f"(c)}{2}(b-a)^2, \text{for some } c \in (a,b)$$

This motivates the so called *Taylor expansion*. Let's define the $n-1$ degree polynomial:

$$P(x) = f(a) + f'(a)(x-a) + \frac{f"(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

Theorem 3.1 If $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b), the $P_{n-1}(x)$ approximate $f(x)$ in the following sense:

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-a)^n, \text{for some } c \in (a,b)$$

Remark 3.2
- When $n = 1$, it is the MVP;
- P_n is the *"best"* polynomial-approximation of order n at a, i.e., it has the same value of $f, f', ..., f^{(n)}$ as $P, P', ..., P^{(n)}$ at a.

Proof. Clearly, for some number M, we can make

$$f(b) = P_{n-1}(b) + M(b-a)^n$$

Let

$$g(x) = f(x) - P_{n-1}(x) - M(x-a)^n$$

Then,

$$g^{(n)}(x) = f^{(n)}(x) - P_{n-1}^{(n)}(x) - n!M$$

It is enough to show that $g^{(n)}(c) = 0$ for some $c \in (a, b)$. Observe $g(a) = 0$, $g'(a) = 0$, ..., $g^{n-1}(a) = 0$ (since $f^k(a) = P^{(k)}(a)$) and $g(b) = 0$ by the way we define M to be. It follows from MVP by bootstrapping that, $\exists c_1 \in [a, b]$, $g'(c) = 0$; $\exists c_2 \in [a, b]$, $g"(c_2) = 0$, ..., $\exists c \in [a, b]$, $g^{(n)}(c) = 0$. This shows:

$$M = \frac{f^n(c)}{n!}$$

The assertion follows. $\square$

4 Sequence of Functions

The question we want to address (very basic treatment, more on *Functional analysis*) is that what does it mean for a sequence of functions to converge:

$$f_1(x), f_2(x), \cdots,$$

The simplest version of convergence is so called *pointwise convergence*. Fix x, does $\{f_n(x)\}$ converge? We say it has a pointwise limit if

$$f(x) = \lim_{n \to \infty} f_n(x), \ \forall x$$

Example 4.1 As in *Figure 4*

$$f_n(x) = \frac{x}{n} \to_{p.t.w} f(x) = 0$$

Example 4.2 As in *Figure 5*, The function f is defined on $[0, 1]$

$$f_n(x) = x^n$$

which converges pointwise to:

$$f(n) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{else} \end{cases}$$

Example 4.3 As in *Figure 6*

$$f_n(x) \to_{p.t.w} f(x) = 0$$

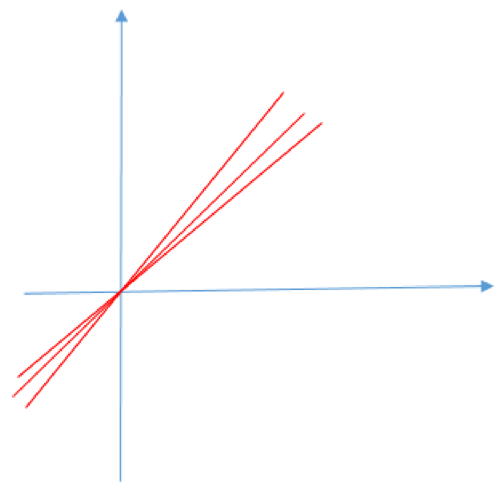

Figure 4: Example 1

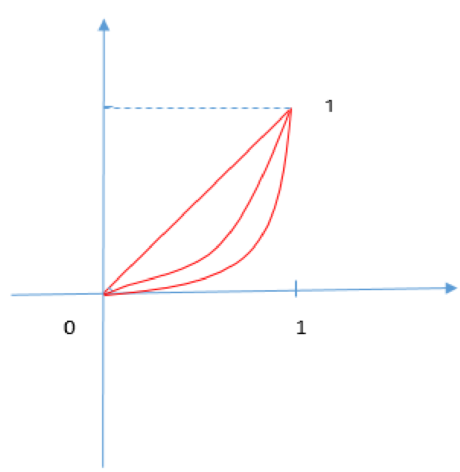

Figure 5: Example 2

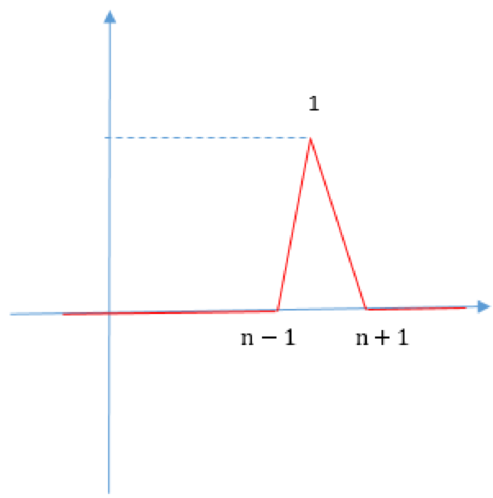

Figure 6: Example 3

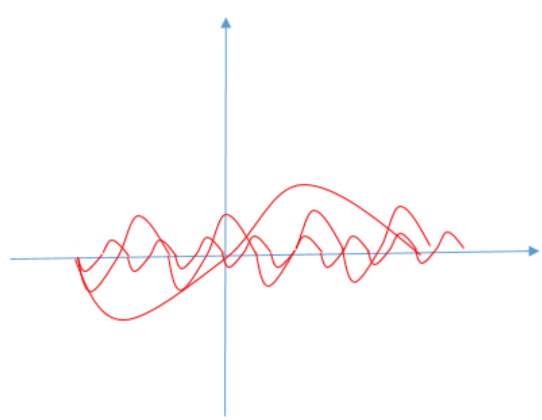

Figure 7: Example 4

Example 4.4 As in *Figure 7*

$$f_n(x) = \frac{1}{n}\sin(n^2 x) \to_{p.t.w} f(x) = 0$$

Here comes the question, what property is preserved by pointwise limit? The continuity is not preserved as shown in *Example 4.2*; the derivative is not preserved as shown in *Example 4.4*; integral is not preserved as shown in *Example 4.3*. Such convergence misses too many important properties of the sequence of the functions. This motivates another notions of convergence.

Let's define the following norm:

$$\|f\| = \sup_{x\in E} |f(x)|$$

Definition 4.1 (*Uniform Convergence*) Given $f_n \in \mathscr{C}_b(E)$[1], we say $f_n \to_u f$, if f_n converges uniformly to f on E, i.e., $\forall \epsilon > 0$, $\exists N$ such that for $n > N$,

$$\|f_n - f\| < \epsilon$$

The interpretation is such that you can draw ϵ–ribbon about limit function f and f_n eventually stays in the ribbon, besides, the same n works for all x.

Indeed, if we define the metric is induced by the norm, i.e.,

$$\|f - g\| = \sup_{x\in E} |f(x) - g(x)|$$

then, the result of *functional analysis* says:

Theorem 4.5 $\mathscr{C}_b(E)$ is complete.

Thus we can use *Cauchy Criterion*:

Theorem 4.6 f_n converges to f uniformly if and only if $\forall\epsilon > 0$, $\exists N$, $\forall m, n > N$, $\|f_n(x) - f_m(x)\| < \epsilon$.

Under such notion of convergence, at least, we have the continuity preserved after taking limit:

Theorem 4.7 If $f_n \to f$ uniformly, f_n is continuous, then f is continuous.

Proof. Observe,

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

For fixed x, $\forall\epsilon > 0$, choose f_n so that $\|f_n - f\| < \frac{\epsilon}{3}$, thus the first absolute term and the third absolute term are both $< \frac{\epsilon}{3}$. On the other hand, f_n is continuous, $\exists\delta > 0$, such that, $d(x, y) < \delta$, then the second absolute term is less than $\frac{\epsilon}{3}$ as well. So $\forall\epsilon > 0$, we can find $\delta > 0$, such that,

$$|f(x) - f(y)| < \frac{\epsilon}{3} \times 3 = \epsilon$$

as desired. □

[1] $\mathscr{C}_b(E)$: class of continuous function that is bounded on E