Network Flow Optimization

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By popular demands, I write this 'short' notes on network- flow model, the first section is devoted to the analysis of multi-period production model, while the second section is for transportation problem. They are both given in the basic (typical) format, which is the foundation of more complicated analysis. You should be very familiar with this part, read section 3, you find out how you should analyze more challenging problem.

1 Multi-Period Production Model

The first step is always to identify the model, and the characteristic of multi-period production model is so obvious that you cannot miss it. Now, let's carry out so called *kernel analysis*, and you know what I am talking about, here comes the picture:

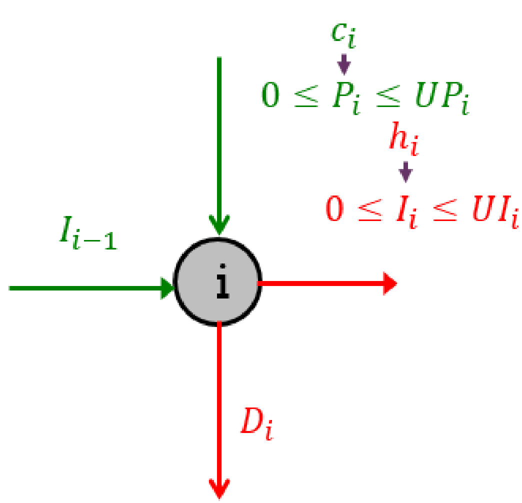

Figure 1: Stage i

You shall know what those parameters are standing for, P_i is the production in stage i, D_i is the demand, I_{i-1} is the inventory left over from last stage, I_i is the new inventory that will go in to the next period. **Usually but not necessary**, we have upper limit of the production, UP_i and the upper limit of the inventory UI_i (in real life, you can not produce infinitely many, or have a warehouse having infinite capacity !!!). In the meanwhile, we shall expect some costs associated with each P_i and I_i, they are production costs c_i and holding cost h_i. Why this kernel is enough, as I explained in the class, this is a perfect representation, the whole problem is just repeating it self as many as the number of the stage you have (e.g., if you have number of periods 5, then the index i goes from 1 to 5). The initial inventory is always known, which means I_0 is deterministic (some numerical value). Also, notice that c_i and h_i can vary for different stage, there is no need to insist on universal costs.

The decision variable is quite simple to figure out, that is:

$$P_i = \text{the amount of products produced in stage } i, \quad i = 1, ..., T$$

where T is the number of stage you have. Now the objective function, in such a simple model, we only have costs, so our goal is to minimize the cost.

- *Production cost:*

$$\sum_{i=1}^{T} c_i P_i = c_1 P_1 + \cdots + c_5 P_5$$

- *Holding cost:*

$$\sum_{i=1}^{T} h_i I_i = h_1 I_1 + \cdots + h_5 I_5$$

The objective function is:

$$\min. \sum_{i=1}^{T} c_i P_i + \sum_{i=1}^{T} h_i I_i$$

or, alternatively,

$$\min. \; c_1 P_1 + \cdots + c_5 P_5 + h_1 I_1 + \cdots + h_5 I_5$$

Those of you who feel comfortable with Σ notation, please go for that, it's time-saving ! But if you just hate this notation, keep it as the way you like, never hurts.

Let's talk about the constraints, there are two category of constraints: inequality constraints and equality constraints. The latter one, I prefer to call it dynamic constraint, because it is so clear that in such model it is the "mysterious" linkage between each stage.

- *Inequality constraints:*

$$0 \le P_i \le UP_i, \; i = 1, ..., T,$$
$$0 \le I_i \le UI_i, \; i = 1, ..., T$$

- *Equality constraints:*

$$I_i = I_{i-1} + P_i - D_i, i = 1, ..., T$$

  This is basically the right-arrow = left-arrow + upper-arrow - lower-arrow, you have inventory carried over, produce the product (addition), satisfy the demand (subtraction), this results the inventory to be carried over to the next period.

Remark 1.1 The lower limit of P_i and I_i, I always write 0, but it can be other values, just not that natural. Thus, I will keep it as 0, however, you shall be smart enough that when you encounter some artificial lower bound, for example, in stage 2, the factory should at least produce 5 products, then substitute 0 by that lower bound, 5 in this case. The same for the inventory I_i.

Let's write down the whole model again:

$$\begin{aligned} \min. \ & \sum_{i=1}^{T} c_i P_i + \sum_{i=1}^{T} h_i I_i \\ \text{subject to}: \ & 0 \leq P_i \leq UP_i, \ i = 1, ..., T, \\ & 0 \leq I_i \leq UI_i, \ i = 1, ..., T, \\ & I_i = I_{i-1} + P_i - D_i, \ i = 1, ..., T. \end{aligned} \tag{1}$$

Remark 1.2 Why P_i decision variable, I_i help variable ? Look at the *dynamic constraint* of problem (1), in the first period, I_0 is known, D_1 is known as well, then if you decide on P_1, I_1 is determined, you have no freedom to "choose" I_1. At the second period, I_1 is known, D is always known, you make a decision on P_2, do we have freedom to choose I_2, **NO!!!**. This will continue on, but you get the idea, why I_i is not decision variable, simply because you cannot decide what it is, it is determined by P_i.

Now, look at problem (1), after you identify this is a multi-period production model, you can immediately draw the kernel (Figure 1) and write down (1). Since time is your enemy, please practice this at home and in your dream.

2 Transportation Problem

For the transportation problem, it is also to identify, right? Typically, you have supply side, you have demand side, the transportation is going on between, the graphs explains (See Figure 2).

You have m factories (plants, e.t.c.), and n customers (regions, e.t.c.), m and n is not necessary identical, actually, I should say, usually not the same. Here s_i is the production capacity for each factories, d_i is the demand for different customers. p_{ij} is the selling price of a product, and q_{ij} is the cost of shipping, sometimes, you may even have the production cost, which I define as c_i which is associated with left-hand side. All those parameters are data (that is given) in the table. Note, everything here can be universal, that is, $s_1 = s_2 = \cdots = s_m$, $d_1 = d_2 = \cdots = d_n$, p_{ij} the shipping cost is typically different, provided by the table 1. Before discussing the p_{ij}, let's first define the decision variables:

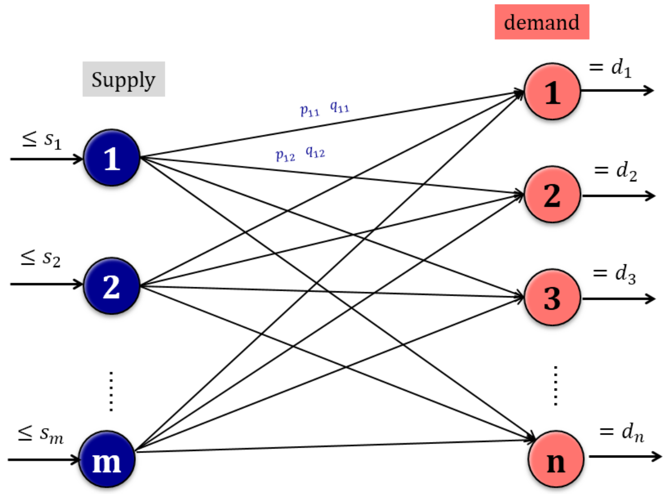

Figure 2: Transportation Problem

Table 1: Transportation Cost

S/D	1	2	3	$\cdots$
1	q_{11}	q_{12}	q_{13}	$\cdots$
2	q_{21}	q_{22}	q_{23}	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$$x_{ij} = \text{amount of products transported from } i \text{ to } j, \quad i = 1, ..., m; j = 1, ..., n.$$

Now, here it comes the important comment on the p_{ij}, that is the selling price, it can appears in so many different forms, that's why we need to talk a little bit. How to interpret this ? First of all, when you make a decision on each arc, those products must be sold, so what is the revenue on each arc, that is $p_{ij}x_{ij}$. Sometimes, you have a universal price $p_{11} = p_{12} = \cdots = p_{mn}$, for example, in your homework, $p_{ij} = 10$, but sometimes, maybe each factory has different pricing, say it can be the situation that, whenever factory 1 sells something, it should be sold at a price 12, then those arc going out of node 1 shall have a unit price 12 per shipment, $x_{11}, x_{12}, x_{13}, ...,$, however, factory insists to sell at price 20, then the arc going out of the node 2 shall have a unit price 20 per shipment. But, as discussed, the most general situation will be on each arc the pricing is different(If you are lucky, maybe there is no selling price, then just ignore this paragraph).

Time to consider the objective function, there are two cases, actually just one case, but maybe you prefer to say two:

- *No selling price:* in this case, you only minimize the cost, from production and transportation

$$
\begin{aligned}
\min. \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}q_{ij} + \sum_{i=1}^{m} c_i \sum_{j=1}^{n} x_{ij} = \min.\ & x_{11}q_{11} + x_{12}q_{12} \cdots x_{mn}q_{mn} \\
& + c_1(x_{11} + x_{12} + \cdots x_{1n}) \\
& + c_2(x_{21} + x_{22} + \cdots x_{2n}) \\
& + \cdots + c_m(x_{m1} + x_{m2} + \cdots x_{mn})
\end{aligned}
$$

Here, you know why I prefer $\sum$, clear and air-tight.

- *Selling price:* in this case, you need to maximize the profit, that is the difference between the transportation cost and the revenue you earn from each arc (but it never affect the production cost part !!!)

$$
\begin{aligned}
\max. \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}(p_{ij} - q_{ij}) - \sum_{i=1}^{m} c_i \sum_{j=1}^{n} x_{ij} = \max.\ & x_{11}(p_{11} - q_{11}) + x_{12}(p_{12} - q_{12}) \\
& + \cdots + x_{mn}(p_{mn} - q_{mn}) \\
& - c_1(x_{11} + x_{12} + \cdots x_{1n}) \\
& - c_2(x_{21} + x_{22} + \cdots x_{2n}) \\
& - \cdots - c_m(x_{m1} + x_{m2} + \cdots x_{mn})
\end{aligned}
$$

Remark 2.1 Justify why I said these two are identical, set $p_{ij} = 0$ in the second case, and change max to min while switching negative sign to positive sign, you recover the first case.

Constraints ? The easiest constraints will be non-negative constraint, i.e.,

$$x_{ij} \geq 0, i = 1, 2, ..., m; j = 1, ..., n.$$

Besides, there are two category of the constraints.

- *From the perspective of a factory:* Visualize it in Figure 3 that's what happens, factory i is a representative, it transport different amount of products to those customers, $x_{i1}, x_{i2}, ..., x_{in}$, they are not allowed to exceed s_i.

$$\sum_{j=1}^{n} x_{ij} \leq s_i,\ i = 1, ..., m$$

or,

$$x_{i1} + x_{i2} + \cdots + x_{in} \leq s_i,\ i = 1, ..., m$$

- *From the perspective of a customer:* Visualize it in Figure 4 customer j is a representative, he receive different amount of products from those factories, $x_{1j}, x_{2j}, ..., x_{mj}$, the demand needs to be satisfied, that is adding those quantity up equal to d_j,

$$\sum_{i=1}^{m} x_{ij} = d_j,\ j = 1, ..., n$$

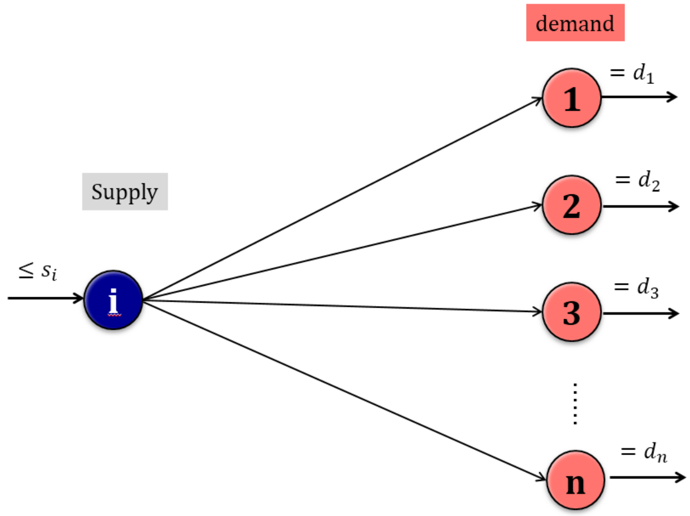

Figure 3: Factory Perspective

or,

$$x_{1j} + x_{2j} + \cdots + x_{mj} = d_j, \;\; j = 1, ..., n$$

That's the end of it, what will be the optimization problem (maximize if there is a profit, minimize if only cost, I will do the maximization)

$$\begin{aligned}
\max. \;\; & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(p_{ij} - q_{ij}) - \sum_{i=1}^{m} c_i \sum_{j=1}^{n} x_{ij} \\
\text{subject to:} \;\; & \sum_{j=1}^{n} x_{ij} \leq s_i, \;\; i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} = d_j, \;\; j = 1, ..., n, \\
& x_{ij} \geq 0, i = 1, 2, ..., m; j = 1, ..., n.
\end{aligned} \tag{2}$$

3 How to analyze the problem

As we discussed in the class, our scheme should be the following:

- Identify what problem it is;
- Draw the representative in the case of multi-period model, draw the whole picture and the representative in the transportation problem case;

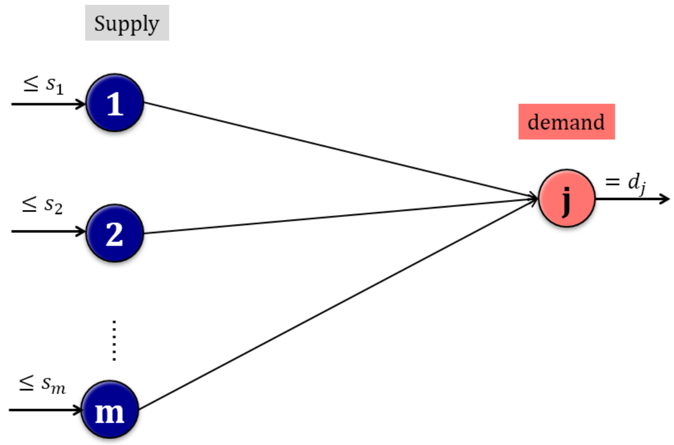

Figure 4: Customer Perspective

- Read the whole problem statement, find out what those parameters, $s_i, c_i, p_{ij}, D_i, ...$, stands for, skip the strange information, the stuff you never saw before;
- Write down the corresponding model quickly, either (1) or (2), given the simplified version (because you skipped, but the rest of the thing you can write down 'without thinking');
- Reckoning, incorporate new information, you may need to define the new decision variable like what we did in the homework, maybe we don't need to introduce new decision variable, just we have more constraints, because the customer is picky, e.t.c, like the transportation+blending model, given in the recitation.

Remember, not every problem is new, the foundation part is here, if you understand, you can write down very quickly, just (1) and (2). Then, you have done more than half of the job, now you won't be panic, just kill the new arrival information. At least, you have certain credits for the foundation part, another advantage is you save the time !!! Those foundation stuff, I wrote to you here, you know in advance. Good luck !!!