Interest Rate Basic
Under Multi-Curve Framework

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We will continue the introduction to basic interest rate instruments but under multi-curve framework. To be consistenet, we consider a market on a compact time interval $[0, T]$*, to model the uncertainty, let's introduce a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\{\mathcal{F}_t\}_{t\geq 0}$ *generated by* d*-dimensional Brownian motion* $\{W_t\}_{t\geq 0}$*. We call it Brownian market to be distinguished with the one has partial information from the Poisson measure.*

1 Motivation for Multi-curve Framework

As we have seen in the last lecture notes, the interest rate models are constructed by eliminating arbitrage opportunities. The most fundamental forward rate agreement can be hedged by zero-coupon bonds. However, the non-arbitrage relationships might not hold in practice. The existence of basis swap is an evident of arbitrage free violation. Also, the tenor swap that exchanges the two LIBORs with different tenors requires non-zero basis spread. After credit crunch in 2007, those spread basis become more visible, which indicates the counterparty risk of those financial institute(big banks) used to be reliable. As a result, it is not appropriate to use LIBOR as discounting rates. On the other hand, increasing number of financial contracts are being made with collateral agreements. Under collateral agreement, the firm receives the collateral from the counterparty when the present value of the contract is postivie, and needs to pay the margin called "collateral rate" on the outstanding collateral to payer(one can think of the counterparty deposits his capital in your collateral account). The most commonly used collateral is a currency of developed country such as USD, EUR and JPY. Similar to trading futures, the MTM of the contract is to be made quite frequently.

By modeling collateral account as a stochastic process with collateral rate process $c(t)$, we can derive, in a single currency situation, the risk-free rate can be identified by the collateral rate. In other words, in current market, most of trades are collateralised, it's more proper to set the collateral rate as discounting rate, which is usually lower than the LIBOR rate. In terms of funding cost, we can have the following interpretation: when there is a receipt of a future cashflow, we will receive a collateral from counterparty, in return, we have to pay the collateral rate and pay back the collateral at the end. It is equivalent to a loan for which we fund the position at the expense of the collateral rate; when there is a paymnent of a future cashflow, we have to post a collateral, which can be thought as a loan provided by counterparty with the same rate. The *overnight index swap(OIS)* that exchanges the fixed coupon against the daily compounded overnight rate is the proxy of the collateral rate. Indeed, since collateralised trade requires MTM very frequently, most likely daily, the funding rate is essentially the overnight rate.

Remark 1.1. *There is a short note on pricing collateralised swaps,* "Swap Pricing with Collateral"*, in which, the discounting rate under both single currency and foreign currency are derived.*

2 Multi-Curve Framework Overview

2.1 Notation

In a single currency economy, let's assume that we have N different interest rate lengths δ_1, δ_2, ..., δ_N, from which corresponding yield curves can be constructed. We will call it yield curve i if it is associated to a length δ_i interest rate. The notation $P_i(t,T)$ stands for the zero-coupon price at time t for maturity T. We also assume we are given a curve D for discounting future cashflows that is denoted by $P_D(t,T)$. The index set $\mathcal{T}^i = \{T_0^i, T_1^i, ...\}$ with subscript i denotes the curve it is associated with, and $\mathcal{T}^S = \{T_0^S, T_1^S, ...\}$ is for the fixed leg payment schedule of a swap. Typically, example of $P_i(t,T)$ can be the price of δ_i-LIBOR related zero-coupon and $P_D(t,T)$ is the discount factor stripped from the OIS curve (bootstrapped by OIS index, OIS swaps).

Now, let's define the forward rate for different curves. That is, for curve $x \in \{1, 2, ..., N, D\}$, the forward rate observed at time t and applied to future time interval $[T, S]$ is defined by

$$F_x(t; T, S) := \frac{1}{\tau_x(T, S)}\left[\frac{P_x(t, T)}{P_x(T, S)} - 1\right]$$

and, in particular,

$$L_x(T, S) := F_x(T; T, S) = \frac{1}{\tau_x(T, S)}\left[\frac{1}{P_x(T, S)} - 1\right],$$

where $\tau_x(T, S)$ is the day count function counts the year fraction for the interval $[T, S]$ under the convention of curve x. Notice, there is no difference between the classical definition and current definition of forward rates, except now the rates are not consistent so that subscript has to be present.

2.2 The valuation of FRA and Swaps

The floating leg of FRA, associated with curve i, pays at time T_1^i[1]:

$$\tau_1^i L_x(T_0^i, T_1^i) = \frac{1}{P_i(T_0, T_1)} - 1 \tag{1}$$

The fixed leg, with fixed rate K, is straightforward, the payment at time T_1 is $\tau_1^i K$. Since FRA cost nothing to enter, thus the fair time-t rate K of a FRA with notional N must satisfies:

$$\begin{aligned} & N\mathbb{E}_t^{\mathbb{Q}_D}\Big[D(t, T_1)\tau_1^i\big(L_x(T_0^i, T_1^i) - K\big)\Big] = 0 \\ \Rightarrow \qquad & N\tau_1^i P(t, T_1)\mathbb{E}_t^{\mathbb{Q}^{T_1}}\Big[\big(L_x(T_0^i, T_1^i) - K\big)\Big] = 0 \end{aligned}$$

Thus,

$$FRA(t; T_0, T_1) = \mathbb{E}_t^{\mathbb{Q}_D^{T_1}}\big[L_x(T_0^i, T_1^i)\big] \tag{2}$$

Remark 2.1. *Here, also in the following, the measure $\mathbb{Q}_D$ is associated with the discounting curve and $\mathbb{Q}_D^T$ is the T-forward measure linked with zero coupon bond $P_D(t, T)$.*

[1] For notation convenience, we write τ_k^i instead of $\tau^i(T_{k-1}, T_k)$.

If $i = D$, we are back to the case of single curve, for which we have proved in the previous notes that,

$$FRA(t; T_0^i, T_1^i) = F_i(t; T_0^i, T_1^i) \tag{3}$$

However, in the current market, we won't have above identity.

For swap with notional $N = 1$, the present value of floating legs is:

$$FL(t; T_0^i, ..., T_n^i) = \sum_{k=1}^{n} \tau_k^i P_D(t, T_k^i) \mathbb{E}_t^{\mathbb{Q}_D^{T_k}} \left[L_x(T_{k-1}^i, T_k^i) \right] \tag{4}$$

On the other hand, the fixed leg, rate K, has present value(to be general, we assume the fixed rate payment schedule is different from floating rate payment schedule $\mathcal{T}^S$):

$$Fix(t; T_0^i, ..., T_m^i) = K \sum_{j=1}^{m} \tau_j^S P_D(t, T_j^S) \tag{5}$$

where we suppress $\tau_D(T_{j-1}^S, T_j^S)$ to be τ_j^S. To combine, for a payer swap, we have

$$IRS(t; K, \mathcal{T}^i, \mathcal{T}^S) = \sum_{i=1}^{n} P_D(t, T_k^i) \mathbb{E}_t^{\mathbb{Q}_D^{T_k}} \left[L_x(T_{k-1}^i, T_k^i) \right] - K \sum_{j=1}^{m} \tau_j^S P_D(t, T_j^S) \tag{6}$$

As a result, the forward swap rate, i.ee., the fixed rate K makes forward IRS value equal to zero at time t is,

$$S^i(t) = \frac{\sum_{i=1}^{n} \tau_k^i P_D(t, T_k^i) \mathbb{E}_t^{\mathbb{Q}_D^{T_k}} \left[L_x(T_{k-1}^i, T_k^i) \right]}{\sum_{j=1}^{m} \tau_j^S P_D(t, T_j^S)} \tag{7}$$

In particular, at time 0,

$$S^i(0) = \frac{\sum_{i=1}^{n} \tau_k^i P_D(0, T_k^i) \mathbb{E}^{\mathbb{Q}_D^{T_k}} \left[L_x(T_{k-1}^i, T_k^i) \right]}{\sum_{j=1}^{m} \tau_j^S P_D(0, T_j^S)} \tag{8}$$

2.3 Basis Swap Spreads

As mentioned in the beginning, the crisis widened the spread between rates of different tenors. The tenor swap exchanges two floating legs related to, say, LIBOR 3m and LIBOR $6m$, respectively. Assume notional is one unit, one floating leg with tenor δ_i has present value:

$$FL^{\delta_i}(t) = \sum_{k=1}^{n} \tau_k^i P_D(t, T_k^i)\big(\mathbb{E}_t^{\mathbb{Q}_D^{T_k}}\left[L_x(T_{k-1}^i, T_k^i)\right] + X\big) \tag{9}$$

The other floating leg with tenor δ_j time t value:

$$FL^{\delta_i}(t) = \sum_{l=1}^{n} \tau_l^i P_D(t, T_l^i)\mathbb{E}_t^{\mathbb{Q}_D^{T_l}}\left[L_x(T_{l-1}^i, T_l^i)\right] \tag{10}$$

The zero value at initial requires:

$$X := \frac{\sum_{l=1}^{n} \tau_l^i P_D(t, T_l^i)\mathbb{E}_t^{\mathbb{Q}_D^{T_l}}\left[L_x(T_{l-1}^i, T_l^i)\right] - \sum_{k=1}^{n} \tau_k^i P_D(t, T_k^i)\mathbb{E}_t^{\mathbb{Q}_D^{T_k}}\left[L_x(T_{k-1}^i, T_k^i)\right]}{\sum_{k=1}^{n} \tau_k^i P_D(t, T_k^i)} \tag{11}$$

This defines X the spread which is quoted on the market.

Remark 2.2. *The basis swap will be used for curve construction in the next section.*

2.4 Bootstrapping the Initial Yield Curve

Unlike in the single curve case, we have only one yield curve that is bootstrapped from cash rate, FRA, Euro-dollar future and swaps, now, the are handful of curves that can be constructed. Namely, we have one for discounting, $T \mapsto P_D(0, T)$ and many forwarding curves for projection, i.e, $T \mapsto FRA(0; T^i, S^i)$ for different tenors, i.e., $\delta_1, \delta_2, ..., \delta_n$. For example, quoted LIBOR rate tenors, 1m, 3m, 6m, 12m.

There are multiple ways for bootstrapping the term structures. We start by considering the spread between forward rates of different tenors defined in (2), as well as the spread with respect to discounting curve. We bootstrap curve i, for all i, by interpolating on such spreads. All these are based on the availability of discounting curve, which we can extract by standard procedure as illustrated in the single curve framework. For example, in the case of USD, the instruments used for bootstrapping are:

- Federal funds with tenor 1BD;
- OIS swap rates with tenor 1W, 2W, 3W, 1M, 2M, 3M, 4M, 5M, 6M, 9M, 1Y, 18M, 2Y, 3Y, 4Y, 5Y, 10Y;

- FF/LIBOR 3M basis swap spread quotes as well as vanilla swap rates with tenor 15Y, 20Y.

Once the discounting curve is known, we derive from it a curve of 1d forward rates obtained by using the standard no-arbitrage arguments:

$$FRA(0; t-1d, t) = \frac{1}{1d}\Big(\frac{P(0, t-1d)}{P(0,t)} - 1\Big). \tag{12}$$

Because we assume that one day curve D coincide with one day curve i. Although its artificial, the difference between them should still be negligible. We start from δ_1 tenor, then define the difference:

$$\Delta_{\delta_i/1d}(t) := FRA(0; t-\delta_i, t) - FRA(0; t-1d, t) \tag{13}$$

The time $1d$ indicates the second term is from the 1d discounting forward curve. For the first term, we can bootstrap according to (8) from corresponding instrument, for example, a forward starting swap with floating tenor δ_i. After deriving those spreads on grids, interpolation can fits us a smooth spread. As a result, we get the δ_i-tenor forward curve by inverting the definition of $\Delta_{\delta_i/1d}(t)$.

Remark 2.3. *Although it is at one's discretion, interpolation on spread instead of forward rate can produce a much smoother basis between the δ_i and 1d curve.*

Once we know the δ_i, a similar procedure can produce curve j for other j's. While sometimes the liquidity can be an issue for certain tenors, as a result, we can use tenor swap to build other curves according to formula (11). Those, bootstrapped forward rate

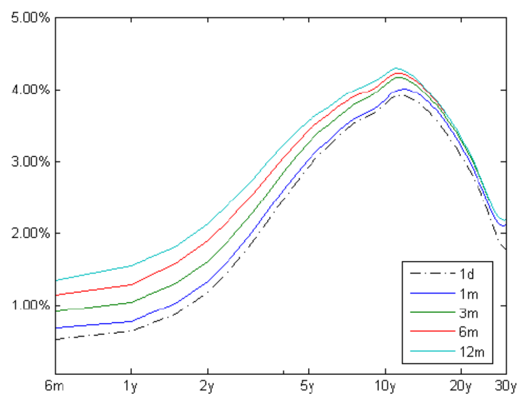

Figure 1: Multi-Curve Construction

curve i can be used to price other swaps based on the curve i.

3 Pricing of Basic Instruments

In this section, we will repeat what we did in the previous notes, pricing model semi-free products, namely, caps/floors and Swaptions but under a dual-curve framework, i.e., one discounting(curve D), one projection(index curve i).

3.1 Interest Cap

To price the cap, we can focus on pricing the caplets tied to curve i. Recall the final payoff of a *caplet* maturing at time T_k^i[2],

$$\tau_k^i \big(F_i(t; T_{k-1}^i, T_k^i) - K\big)^+ \tag{14}$$

To price such *caplets*, we working under discounting measure $\mathbb{Q}_D$ and its equivalence measure, e.g., $\mathbb{Q}_D^{T_k}$,

$$Cplt(t; T_{k-1}^i, T_k^i) = \tau_k^i P_D(t, T_k^i) \mathbb{E}_t^{\mathbb{Q}_D^{T_k^i}} \left[\big(F_i(T_{k-1}^i; T_{k-1}^i, T_k^i) - K\big)^+\right] \tag{15}$$

As already encountered in the IRS case, the problem with the $\mathbb{Q}_D^{T_k^i}$ is that the forward rate is not, in general, a martingale. To address this issue, we can assume a model for dynamics of forward rate $FRA(t; T_{k-1}^i, T_k^i)$ under its own measure $\mathbb{Q}_i^{T_k^i}$ and then use change of measure technique to evaluate the original expectation. This is very similar to the convexity adjustment technique we did in the Eurodollar future as well as LIBOR pay in arrear. Just in this case, it is called a *quanto correction*. Here, let's, instead, consider another approach of *LIBOR market model(LMM) flavor*. The forward rate, under new setting, reads:

$$FRA(t; T_{k-1}^i, T_k^i) = \mathbb{E}_t^{T_k^i}\left[F_i(T_{k-1}^i; T_{k-1}^i, T_k^i)\right] \tag{16}$$

In particular, at time T_{k-1}^i,

$$FRA(T_{k-1}^i; T_{k-1}^i, T_k^i) = F_i(T_{k-1}^i; T_{k-1}^i, T_k^i). \tag{17}$$

Thus, we can rewrite (15) as:

$$Cplt(t; T_{k-1}^i, T_k^i) = \tau_k^i P_D(t, T_k^i) \mathbb{E}_t^{\mathbb{Q}_D^{T_k^i}} \left[\big(FRA(t; T_{k-1}^i, T_k^i) - K\big)^+\right] \tag{18}$$

where FRA is by definition a martingale under the measure $\mathbb{Q}_D^{T_k^i}$. If we can choose the dynamics of such a rate, we can value the last expectation analytically and, as before,

[2] We will assume $N = 1$ most of the time, except otherwise state it explicitly.

derive a closed-formula for the caplet price, thus caps/floors. For instance, we can use log-normal model:

$$dFRA(t; T_{k-1}^i, T_k^i) = \nu_k FRA(t; T_{k-1}^i, T_k^i) dW_t^{i,k}, \quad t \le T_{k-1}^i \tag{19}$$

The $W_t^{i,k}$ term is a Brownian motion under probability measure $\mathbb{Q}_D^{T_k^i}$ and ν_k is a constant volatility. According to BS formula, we have [3].

$$Cplt(t; T_{k-1}^i, T_k^i) = \tau_k^i PD(t, T_k^i) Bl\Big(FRA(t; T_{k-1}^i, T_k^i), \nu_k \sqrt{T_{k-1}^i - t}, K, 1\Big) \tag{20}$$

Remark 3.1. *We will extensively explore the LLM model later on and give a justification of the underlying model for $FRA(t; T_{k-1}^i, T_k^i)$.*

3.2 Swaptions

The introduction of swaption is already done previously, we won't repeat here. Let's just focusing the pricing aspect of a payer swaption. Suppose the $\mathcal{T}^i = \mathcal{T}^S$(for the sake of simpler notation) and the fixed rate for the forward swap is K. Then the swaption payoff at time T_0^i is:

$$\begin{aligned}
&Payoff(T_0^i) \\
&= \sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i) \mathbb{E}_{T_0^i}^{\mathbb{Q}_D^{T_k^i}} \Big[L_i(T_{k-1}^i, T_k^i) - K \Big] \\
&= \sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i) \frac{\sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i) \mathbb{E}_{T_0^i}^{\mathbb{Q}_D^{T_k^i}} \Big[L_i(T_{k-1}^i, T_k^i) \Big] - K \sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i)}{\sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i)} \\
&= \big(S^i(T_0^i) - K\big) \sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i)
\end{aligned} \tag{21}$$

As before, it is convenient to price under the swap measure $\mathbb{Q}_D^{swap}$, whose associted numérairé is $\sum_{k=1}^n \tau_k^i P_D(T_0^i, T_k^i)$. We have

$$PS\big(t; \mathcal{T}, K\big) = \sum_{j=1}^n \tau_j^i P_D(t, T_j^i) \mathbb{E}_t^{\mathbb{Q}_D^{swap}} \Big[\big(S^i(T_0^i) - K\big) + \Big] \tag{22}$$

[3]The notation is consistent, ω decides put or call, Bl stands for BS formula for forward rate.

As in the single curve case, the forward swap rate $S^i(t)$ is a martingale under swap measure $\mathbb{Q}_D^{swap}$. If we assume the rates evovels according to driftless geometric Brownian motion:

$$dS^i(t) = \nu_n S^i(t) dW_t^{swap}, \quad t \leq T_0^i. \tag{23}$$

where ν_n is a constant volatility. As a result,

$$PS(t; \mathcal{T}^i) = Bl\big(S^i(t), \sigma_n \sqrt{T_0^i - t}, K, 1\big) \sum_{i=1}^{n} \tau_j^i P_D(t, T_j^i). \tag{24}$$