Stochastic Programming

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Lecture 2 - Probabilistic Constraints

In this chapter, we will discuss stochastic optimization problems with *probabilistic constraints*. We emphasize that imposing constraints on probability of events is particularly appropriate whenever high uncertainty is involved and reliability is a central issue. In order to study such constraint, we need to introduce generalized concavity and explore the property of generalized concave functions and measures. In the end, we will see how this properties can help us to get convexity of the feasible set.

2.1 Introduction to Probabilistic Constraint

Given probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathscr{F}$ is the σ-algebra, we discuss stochastic optimization problems with probabilistic constraints of the form:

$$\begin{aligned} &\min\ c(x) \\ &\text{subject to: } \mathbb{P}\{g_j(x, Z) \geq 0,\ j \in \mathscr{J}\} \geq p \\ &x \in \mathscr{X} \end{aligned}$$

Here, $\mathscr{X} \in \mathbb{R}^n$ is a nonempty set, $c : \mathbb{R}^n \mapsto \mathbb{R}$ and $g_j : \mathbb{R}^n \times \mathbb{R}^s \mapsto \mathbb{R}$ are both *Borel measurable*, $j \in \mathscr{J}$, where $\mathscr{J}$ is an index set, Z is an s-dimensional random vector, and p is a modeling parameter. Here, $\mathbb{P}_Z$ is the probability measure induced by the random vector Z on $\mathbb{R}^s$. The event,

$$A(x) = \{g_j(x, Z) \geq 0,\ j \in \mathscr{J}\}$$

depends on the decision vector x, and its probability $\mathbb{P}\{A(x)\}$ is calculated with respect to the probability distribution $\mathbb{P}_Z$.

Our analysis will be carried out on the constraints, for convenience of the presentation and notation, some times we may only consider $card(\mathscr{J}) = 1$. From nonlinear optimization

or convex analysis, we usually want to have feasible set to be convex (or concave), but, unfortunately, this is not the case in this probabilistic setting. Why ? If we set $G(x) = \mathbb{P}\{g(x, Z) \geq 0\}$, it is bounded in the interval $[0, 1]$ so that it can't be a concave function (it maybe a constant). Under this circumstance, we can't get a convex feasible set. We need to generalize the concept of concavity.

2.2 Generalized Concavity of Functions and Measures

2.2.1 On Function

Let's consider a Borel measurable function $f : D \mapsto \mathbb{R}_+$, where $D \subset \mathbb{R}^n$ is convex.

Definition 2.2.1 A non-negative function $f(x)$ defined on a convex set $D \subset \mathbb{R}^n$ is said to be α-*concave*, where $\alpha \in [-\infty, +\infty]$, if for all $x, y \in D$ and all $\lambda \in [0, 1]$ the following inequality holds true:

$$f(\lambda x + (1 - \lambda)y) \geq m_\alpha \left(f(x), f(y), \lambda\right)$$

where Borel measurable function $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1]$ is defined as follows:

$$m_\alpha(a, b, \lambda) = 0 \text{ if } ab = 0$$

and if $a, b > 0$, $0 \leq \lambda \leq 1$, then

$$m_\alpha = \begin{cases} a^\lambda b^{1-\lambda} & \textit{if } \alpha = 0, \\ \max\{a, b\} & \textit{if } \alpha = +\infty, \\ \min\{a, b\} & \textit{if } \alpha = -\infty, \\ (\lambda a^\alpha + (1 - \lambda)b^\alpha)^{\frac{1}{\alpha}} & \textit{otherwise.} \end{cases}$$

The generic form of α-concavity is to have f satisfies:

$$[f(\lambda x + (1 - \lambda)y)]^\alpha \geq \lambda[f(x)]^\alpha + (1 - \lambda)[f(y)]^\alpha$$

Let's observe several special and useful α. When $\alpha = 1$, we have f *concave*

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

When $\alpha = 0$, we have f *logarithmically concave* or *log-concave*, i.e., $\ln f(\cdot)$ is a concave function,

$$(\ln f)(\lambda x + (1 - \lambda)y) \geq \lambda \ln f(x) + (1 - \lambda) \ln f(y)$$

For $\alpha = -\infty$, we have f *quasi-concave*,

$$f(\lambda x + (1 - \lambda)y) \geq \min\left(f(x), f(y)\right)$$

It is important to note that if f and g are two measurable functions, then the function $m_\alpha(f(\cdot), g(\cdot), \lambda)$ is a measurable function for all α and $\lambda \in (0,1)$. And m_α has following nice property.

Lemma 2.2.1 The mapping $\alpha \mapsto m_\alpha(a,b,\lambda)$ is non-decreasing and continuous.

Proof. We first show the continuity of the mapping at $\alpha = 0$. We have

$$\begin{aligned}
\ln m_\alpha(a,b,\lambda) &= \ln\left(\lambda a^\alpha + (1-\lambda)b^\alpha\right)^{1/\alpha} \\
&= \frac{1}{\alpha}\ln\left(\lambda e^{\alpha \ln a} + (1-\lambda)e^{\alpha \ln b}\right) \\
&= \frac{1}{\alpha}\left(1 + \alpha(\lambda \ln a + (1-\lambda)\ln b) + o(\alpha^2)\right) \ \ (e^x \approx 1 + x \text{ as } x \to 0)
\end{aligned}$$

Applying the *L'Hospital rule* to the right-hand-side in order to calculate tis limit $\alpha \to 0$,

$$\begin{aligned}
\lim_{\alpha\to 0} m_\alpha(a,b,\lambda) &= \lim_{\alpha \to 0} \frac{\lambda \ln a + (1-\lambda)\ln b + o(\alpha)}{1 + \alpha(\lambda \ln a + (1-\lambda)\ln b) + o(\alpha^2)} \\
&= \lim_{\alpha \to 0} \frac{\ln(a^\lambda b^{(1-\lambda)}) + o(\alpha)}{1 + \alpha \ln(a^\lambda b^{(1-\lambda)}) + o(\alpha^2)} \\
&= \ln(a^\lambda b^{(1-\lambda)})
\end{aligned}$$

This proves the continuity at $\alpha = 0$. Let's now justify the monotonicity of the mapping. Let's firstly consider the case $0 < \alpha < \beta$, we set

$$h(\alpha) = m_\alpha(a,b,\lambda) = \exp\left(\frac{1}{\alpha}\ln[\lambda a^\alpha + (1-\lambda)b^\alpha]\right)$$

Calculating its derivative, we get

$$h'(\alpha) = h(\alpha)\left(\frac{1}{\alpha}\frac{\lambda a^\alpha \ln a + (1-\lambda)b^\alpha \ln b}{\lambda a^\alpha + (1-\lambda)b^\alpha} - \frac{1}{\alpha^2}\ln[\lambda a^\alpha + (1-\lambda)b^\alpha]\right)$$

We need to demonstrate that the expression on the RHS is non-negative. Substituting $x = a^\alpha$ and $y = b^\alpha$, we obtain

$$h'(\alpha) = \frac{1}{\alpha^2}h(\alpha)\left(\frac{\lambda x \ln x + (1-\lambda)y \ln y}{\lambda x + (1-\lambda)y} - \ln[\lambda x + (1-\lambda)y]\right)$$

Using the fact that the function $z \mapsto z \ln z$ is convex for $z > 0$ and that both $x, y > 0$, we have that

$$\frac{\lambda x \ln x + (1-\lambda)y \ln y}{\lambda x + (1-\lambda)y} - \ln[\lambda x + (1-\lambda)y] \geq 0$$

As $h(\alpha) > 0$, we conclude that $h(\cdot)$ is non-decreasing in this case. If $\alpha < \beta < 0$, we have the following chain of relations:

$$m_\alpha(a,b,\lambda) = [m_{-\alpha}\left(\frac{1}{a}, \frac{1}{b}, \lambda\right)]^{-1} \leq [m_{-\beta}\left(\frac{1}{a}, \frac{1}{b}, \lambda\right)]^{-1} = m_\beta(a,b,\lambda)$$

In the case of $0 \leq \alpha < \beta$, we can select a sequence $\{\alpha_k\}$ such that $\alpha_k > 0$ and $\lim_{k\to\infty} \alpha_k = 0$. We use the monotonicity of $h(\cdot)$ for positive arguments and the continuity at 0 to obtain the desired assertion. In the case $\alpha < \beta = 0$, we proceed in the same way, choosing appropriate sequence approaching 0. If $\alpha < 0 < \beta$, then the inequality

$$m_\alpha(a, b, \lambda) \leq m_0(a, b, \lambda) \leq m_\beta(a, b, \lambda)$$

follows from the previous two cases. It remains to investigate how the mapping behaves when $\alpha \to \infty$ or $\alpha \mapsto -\infty$. We observe that

$$\max\{\lambda^{1/\alpha} a, (1-\lambda)^{1/\alpha} b\} \leq m_\alpha(a, b, \lambda) \leq \max\{a, b\}$$

Passing to the limit, we obtain that

$$\lim_{\alpha\to\infty} m_\alpha(a, b, \lambda) = \max\{a, b\}$$

We also conclude that

$$\lim_{\alpha\to-\infty} m_\alpha(a, b, \lambda) = \lim_{\alpha\to-\infty} [m_{-\alpha}(1/a, 1/b, \lambda)]^{-1} = [\max\{1/a, 1/b\}]^{-1} = \min\{a, b\}$$

This completes the proof. □

Remark 2.2.2 This is a very important result, because it implies that α-concavity entails β-concavity, for $\alpha \geq \beta$. In particular, all α-concave functions are $-\infty$-concave, or *quasi-concave*.

Before proceeding further, let's give some examples.

Example 2.2.3 Consider the density function of a *non-degenerate multivariate normal distribution* on $\mathbb{R}^s$:

$$f(x) = \frac{1}{\sqrt{(2\Pi)^s \det(\Sigma)}} \exp\{-\frac{1}{2}(x-\mu^T)\Sigma^{-1}(x-\mu)\}$$

where Σ is a positive definite symmetric matrix of dimension $s \times s$, $\det(\Sigma)$ denotes the determinant of the matrix Σ, and $\mu \in \mathbb{R}^s$. We observe that

$$\ln f(x) = -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) - C$$

where C is some constant, is a concave function. Therefore, we conclude that f is 0-concave, or *log-concave*.

Example 2.2.4 Consider a convex body (a convex compact set with non-empty interior) $D \subset \mathbb{R}^s$. The uniform distribution on this set has density defined as follows:

$$f(x) = \begin{cases} \frac{1}{V_s(D)} & \text{if } x \in D, \\ 0 & \text{otherwise.} \end{cases}$$

where $V_s(D)$ denotes the *Lebesgue measure* of D. The function $f(x)$ is *quasi-concave* on $\mathbb{R}^s$ and $+\infty$-concave on D.

2.2.2 On Measure

We can actually extend the notion of α-concavity to set functions, in particular, for probability measure P (that is defined on $(\mathbb{R}^s, \mathscr{L}, P)$, where $\mathscr{L}$ is the Lebesgue set). Before giving the definition, we shall notice that for two Borel measurable sets $A, B \in \mathbb{R}^s$, the *Minkowski sum*

$$A + B = \{x + y; x \in A, y \in B\}$$

is *Lebesgue measurable* in $\mathbb{R}^s$. This fact will legitimate our definition of α-concavity for $\mathbb{P}$.

Definition 2.2.2 Given $(D, \mathscr{L}, P)$, D is a convex set of $\mathbb{R}^s$, P is defined on the *Lebesgue measurable* subsets of D. It is said to be α-concave if for any Borel measurable sets $A, B \in D$ and for all $\lambda \in [0, 1]$, we have the inequality,

$$P(\lambda A + (1-\lambda)B) \geq m_\alpha(P(A), P(B), \lambda)$$

where $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y : x \in A, y \in B\}$.

We say that a random vector Z with values in $\mathbb{R}^s$ has an α-concave distribution if the probability measure $\mathbb{P}_Z$[1] induced by Z on $\mathbb{R}^s$ is α-concave.

Lemma 2.2.5 If a random vector Z induces an α-concave probability measure on $\mathbb{R}^s$, then its cumulative distribution function F_Z is an α-concave function.

Proof. Let's fix $x, y \in \mathbb{R}^s$ and form a convex combination $\lambda x + (1-\lambda)y$, so

$$F_Z(\lambda x + (1-\lambda)y) = \mathbb{P}_Z\left(Z \leq \lambda x + (1-\lambda)y\right)$$

On the other hand, we define two half space by the threshold x, y,

$$A = \{Z : Z \leq x\}$$
$$B = \{Z : Z \leq y\}$$

and also

$$C = \lambda A + (1-\lambda)B = \{\lambda z_1 + (1-\lambda)z_2,\ z_1 \leq x, z_2 \leq y\}$$

Thus, $z \in C$ implies $z \leq \lambda x + (1-\lambda)y$, which leads

$$\begin{aligned}\mathbb{P}_Z\{Z \leq \lambda x + (1-\lambda)y\} &\geq \mathbb{P}_Z(C) \\ &\geq m_\alpha(\mathbb{P}_Z(A), \mathbb{P}_Z(B), \lambda)\end{aligned}$$

That is,

$$F_Z(\lambda x + (1-\lambda)y) \geq m_\alpha(F_Z(x), F_Z(y), \alpha)$$

□

[1] In previous session, we denote it as $\mathbb{P}$, but from this point onward, we denote by $\mathbb{P}_Z$ to denote its association to Z

As an application of above theorem, we have

Proposition 2.2.6 Consider the probabilistic constraint,

$$\mathbb{P}_Z(Z \leq x) \geq p$$

Equivlently,

$$F_Z(x) \geq q$$

The claim is that the feasible set $\mathbb{X} = \{x;\ F_Z(x) \geq p\}$ is convex.

Proof. To prove the assertion, we take $x, y \in \mathbb{X}$, and form the convex combination, $\lambda x + (1-\lambda)y$, $0 \leq \lambda \leq 1$. Then, by *Theorem* 2.2.5,

$$\begin{aligned} F_Z\left(\lambda x + (1-\lambda)y\right) &\geq m_\alpha(F_Z(x), F_Z(y), \lambda) \\ &\geq m_{-\infty}(F_Z(x), F_Z(y), -\lambda) \\ &= \min\{F_Z(x), F_Z(y)\} \\ &\geq p \end{aligned}$$

The assertion follows immediately. □

Lemma 2.2.7

We can also generalize the *Minkowski inequality* to the set functions, that is so called *Brunn - Minkowski inequality.*

Theorem 2.2.8 (*Brunn - Minkowski*) Let $n \geq 1$ and ν denote the Lebesgue measure on $\mathbb{R}^n$, A and B are two non-empty compact subsets of $\mathbb{R}^n$, then

$$[\mu(A+B)]^{\frac{1}{n}} \geq [\mu(A)]^{\frac{1}{n}} + [\mu(B)]^{\frac{1}{n}}$$

where $A + B$ denotes the *Minkowski sum.*

Now, we relate the α-concavity property of a measure to generalized concavity of its density:

Theorem 2.2.9 Let D be a convex subset of $\mathbb{R}^s$ and let $m > 0$ be the dimension of the smallest affine subspace L containing D. The probability measure P on D is γ-concave with $\gamma \in [-\infty, 1/m]$ if and only if its probability density function with respect to the *Lebesgue measure* on L is α-concave with

$$\alpha = \begin{cases} \lambda/(1-m\gamma) & \text{if } \gamma \in (-\infty, 1/m), \\ 1/m & \text{if } \gamma = -\infty, \\ +\infty & \text{if } \gamma = 1/m. \end{cases}$$

Corollary 2.2.10 Let an integrable function $f(x)$ be defined on a non-degenerated convex set $D \in \mathbb{R}^s$. Denote

$$c = \int_D f(x)dx$$

If $f(x)$ is α-concave with $-1/s \leq \alpha \leq \infty$ and positive on the interior of D, then the measure P on D defined by setting that

$$P(A) = \frac{1}{c}\int_A f(x)dx, \quad A \subset D$$

is γ-concave with

$$\alpha = \begin{cases} \alpha/(1+s\alpha) & \text{if } \alpha \in (-1/s, +\infty), \\ 1/s & \text{if } \alpha = +\infty, \\ -\infty & \text{if } \gamma = -1/s. \end{cases}$$

In particular, if a measure P on $\mathbb{R}^s$ has a density function $f(x)$ such that $x^{-\frac{1}{s}}$ is convex, then P is quasi-concave.

Example 2.2.11 In *Example* 2.2.4 that the density of the uniform distribution on a convex body D is $+\infty$-concave function. Hence, it generates a $1/s$-concave measure on D. On the other hand, the density of the normal distribution (*Example* 2.2.3) is log-concave, and therefore, it generates a log-concave probability measure.

2.2.3 Calculus for α-concave function

Let's provide some calculus result for α-concave function.

Theorem 2.2.12 If the function $f : \mathbb{R}^n \mapsto \mathbb{R}_+$ is α-concave and the function $g : \mathbb{R}^n \mapsto \mathbb{R}_+$ is β-concave, where $\alpha, \beta \geq 1$, then the function $h : \mathbb{R}^n \mapsto \mathbb{R}_+$, defined as $h(x) = f(x) + g(x)$ is γ-concave with $\gamma = \min\{\alpha,\ \beta\}$.

Proof. Given points $x_1, x_2 \in \mathbb{R}^n$ and a scalar $\lambda \in (0, 1)$, we form the convex combination $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Since both functions f and g are γ-concave by virtue of *Lemma* 2.2.1. Using the *Minkowski inequality*, which holds true for $\gamma \geq 1$, we obtain:

$$\begin{aligned} h(x_\lambda) &= f(x_\lambda) + g(x_\lambda) \\ &\geq [\lambda(f(x_1))^\gamma + (1-\lambda)(f(x_2))^\gamma]^{\frac{1}{\gamma}} + [\lambda(g(x_1))^\gamma + (1-\lambda)(g(x_2))^\gamma]^{\frac{1}{\gamma}} \\ &\geq [\lambda\,(f(x_1) + g(x_1))^\gamma + (1-\lambda)\,(f(x_2) + g(x_2))^\gamma]^{\frac{1}{\gamma}} \end{aligned}$$

The case when $\alpha = \beta = +\infty$, the proof is trivial. □

Theorem 2.2.13 Let f be a concave function defined on a convex set $C \in \mathbb{R}^s$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be a non-negative nondecreasing α-concave function, $\alpha \in [-\infty, +\infty]$. Then the function $g \circ f$ is α-concave.

Proof. Given $x, y \in \mathbb{R}^s$ and a scalar $\lambda \in (0,1)$, we consider $z = \lambda x + (1-\lambda)y$. We have $f(z) \geq \lambda f(x) + (1-\lambda)f(y)$. By monotonicity, we obtain the following chain of inequalities:

$$[g \circ f](z) \geq g(\lambda f(x) + (1-\lambda)f(y)) \geq m_\alpha \left(g(f(x)), g(f(y)), \lambda\right)$$

This proves the assertion. □

Theorem 2.2.14 Let the function $f : \mathbb{R}^m \times \mathbb{R}^s \mapsto \mathbb{R}_+$ be such that for all $y \in Y \subset \mathbb{R}^s$ the function $f(\cdot, y)$ is α-concave ($\alpha \in [-\infty, +\infty]$) on the convex set $X \subset \mathbb{R}^m$. Then the function $\psi(x) = \inf_{y \in Y} f(x, y)$ is α-concave on X.

Proof. Let $x_1, x_2 \in X$ and a scalar $\lambda \in (0,1)$ be given. We set $z = \lambda x_1 + (1-\lambda)x_2$, then we can find a sequence of points $y_k \in Y$ such that

$$\psi(z) = \inf_{y \in Y} f(z, y) = \lim_{k \to \infty} f(z, y_k)$$

Using the α-concavity of the function $f(\cdot, y)$, we can conclude that

$$f(z, y_k) \geq m_\alpha \left(f(x_1, y_k),\ f(x_2, y_k),\ \lambda\right)$$

The mapping $(a, b) \mapsto m_\alpha(a, b, \lambda)$ is monotone for non-negative a and b and $\lambda \in (0,1)$ (easy to check). Therefore, we can have that

$$f(z, y_k) \geq m_\alpha(\psi(x_1), \psi(x_2), \lambda)$$

Passing to the limit, we obtain the assertion. □

Theorem 2.2.15 If $\alpha_i > 0$, $i = 1, ..., m$, and $\Sigma_{i=1}^m \alpha_i = 1$, then the function $f : \mathbb{R}_+^m \mapsto \mathbb{R}$ defined as $f(x) = \prod_{i=1}^m x_i^m$ is concave.

Proof. We shall show the statement for the case of $m = 2$. For points $x, y \in \mathbb{R}_+^2$ and a scalar $\lambda \in (0,1)$, we consider $\lambda x + (1-\lambda)y$. Define the quantities

$$a_1 = (\lambda x_1)^{\alpha_1}, \quad a_2 = ((1-\lambda)y_1)^{\alpha}, \quad b_1 = (\lambda x_2)^{\alpha}, \quad b_2 = ((1-\lambda)y_2)^{\alpha_2}$$

Using Hölder's inequality, we obtain the following:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \left(a_1^{1/\alpha_1} + a_2^{1/\alpha_1}\right)^{\alpha_1} \left(b_1^{1/\alpha_2} + b_2^{1/\alpha_2}\right)^{\alpha_2} \\ &\geq a_1 b_1 + a_2 b_2 \\ &= \lambda x_1^{\alpha_1} x_2^{\alpha_2} + (1-\lambda) y_1^{\alpha_1} y_2^{\alpha_2} \end{aligned}$$

The assertion in the general case follows by induction. □

Theorem 2.2.16 If the functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}_+$, $i = 1, ..., m$, are α_i-concave and α_i are such that $\Sigma_{i=1}^m \alpha_i^{-1} > 0$, then the function $g : \mathbb{R}^n \mapsto \mathbb{R}_+$, defined as $g(x) = \prod_{i=1}^m f_i(x_i)$ is γ-concave with $\gamma = \left(\Sigma_{i=1}^m \alpha_i^{-1}\right)^{-1}$.

Proof. Fix points $x_1, x_2 \in \mathbb{R}_+^n$, a scalar $\lambda \in (0, 1)$ and set $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$. By the generzlied concavity of the function f_i, $i = 1, ..., m$, we have the following inequality:

$$\prod_{i=1}^m f_i(x_\lambda) \geq \prod_{i=1}^m (\lambda f_i(x_1)^{\alpha_i} + (1 - \lambda) f_i(x_2)^{\alpha_i})^{\frac{1}{\alpha_i}}$$

We denote $y_{ij} = f_i(x_j)^{\alpha_i}$, $j = 1, 2.$ substituting into the last displayed inequality and raising both sides to power γ, we obtain

$$\left(\prod_{i=1}^m f_i(x_\lambda)\right)^\gamma \geq \prod_{i=1}^m (\lambda y_{i1} + (1 - \lambda) y_{i2})^{\gamma/\alpha_i}$$

We continue the chain of inequality using above theorem,

$$\prod_{i=1}^m (\lambda y_{i1} + (1 - \lambda) y_{i2})^{\gamma/\alpha_i} \geq \lambda \prod_{i=1}^m [y_{i1}]^{\gamma/\alpha_i} + (1 - \lambda) \prod_{i=1}^m [y_{i2}]^{\gamma/\alpha_i}$$

Putting the inequalities together and using the substitutions at the right-hand side of the last inequality, we conclude that

$$\prod_{i=1}^m [f_1(x_\lambda)]^\gamma \geq \lambda \prod_{i=1}^m [f_i(x_1)]^\gamma + (1 - \lambda) \prod_{i=1}^m [f_i(x_2)]^\gamma$$

as required. □

Remark 2.2.17 In the special case, when the function $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, ..., k$, are concave, we can apply above theorem consecutively to conclude that $f_1 f_2$ is $\frac{1}{2}$-concave and $f_1 \cdots f_k$ is $\frac{1}{k}$-concave.

2.3 Convexity of Probability Constrained Sets

We will present one of the most general results in the convexity theory of probabilistic optimization. Recall

$$G(x) = \mathbb{P}_Z\{g(x, Z) \geq 0\} \tag{2.1}$$

so that the feasible set is:

$$\mathbb{X} = \{x : \ G(x) \geq p\}$$

Theorem 2.3.1 Let the function $g_j : \mathbb{R}^n \times \mathbb{R}^s \mapsto \mathbb{R}$, $j \in \mathscr{J}$, be *quasi-concave*. If $Z \in \mathbb{R}^s$ is a random vector that has an α-concave distribution, then the function $G(x)$ is α-concave on the set

$$D = \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^s \text{ such that } g_j(x, z) \geq 0,\ j \in \mathscr{J}\}$$

Proof. Given the pints $x_1, x_2 \in D$ and $\lambda \in (0, 1)$, we define the sets

$$A_i = \{z \in \mathbb{R}^s \ :\ g_j(x_i, z) \geq 0,\ j \in \mathscr{J}\},\ i = 1, 2,$$

and $B = \lambda A_1 + (1 - \lambda)A_2$. We consider

$$G(\lambda x_1 + (1 - \lambda)x_2) = \mathbb{P}_Z\{g_j(\lambda x_1 + (1 - \lambda)x_2,\ Z) \geq 0,\ j \in \mathscr{J}\}$$

If $z \in B$, then there exists points $z_i \in A_i$ such that $z = \lambda z_1 + (1 - \lambda)z_2$. By virtue of the *quasi-concavity* of g_j, we obtain that:

$$g_j(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2) \geq \min\{g_j(x_1, z_1),\ g_j(x_2, z_2)\} \geq 0, \quad \forall j \in \mathscr{J}$$

This implies that $z \in \{z \in \mathbb{R}^s : g_j(\lambda x_1 + (1 - \lambda)x_2, z) \geq 0,\ j \in \mathscr{J}\}$, which entails that $\lambda x_1 + (1 - \lambda)x_2 \in D$ and that

$$G(\lambda x_1 + (1 - \lambda)x_2) = \mathbb{P}_Z\{g_j(\lambda x_1 + (1 - \lambda)x_2, z) \geq 0,\ j \in \mathscr{J}\} \geq \mathbb{P}_Z\{B\}$$

Using the α-concavity of the measure, we conclude that

$$G(\lambda x_1 + (1 - \lambda)x_2) \geq \mathbb{P}_Z\{B\} \geq m_\alpha\{\mathbb{P}_Z\{A_1\},\ \mathbb{P}_Z\{A_2\},\ \lambda\} = m_\alpha\{G(x_1),\ G(x_2),\ \lambda\}$$

as desired. □

Example 2.3.2 (*The Log-Normal Distribution*) The probability density function of the one-dimensional log-normal distribution with parameters μ and σ is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This density is neither log-concave nor log-convex. However, we can show that the cumulative distribution function is log-concave. We demonstrate it for the multidimensional case.

The s-dimensional random vector Y has the log-normal distribution if the vector $Z = (\ln Y_1, ..., \ln Y_s)^T$ has a multivariate normal distribution. Recall that the normal distribution is log-concave (*Example* 2.2.3). The distribution function of Y at a point $x \in \mathbb{R}^s$, $x > 0$, can be written as

$$F_Y(x) = \mathbb{P}_Y\{Y_1 \leq x_1, ..., Y_s \leq x_s\} = \mathbb{P}_Z\{x_1 - e_1^Z \geq 0, ..., x_s - e_s^Z \geq 0\}$$

If $g_j(x, Z) = x_j - e^{Z_j}$ is quasi-concave, $j = 1, ..., s$, then by *Theorem* 2.3.1, we get F_Y is log-concave function. We only need to justify the quasi-concavity of g_j. Take (x_1, z_1),

$(x_2, z_2) \in \mathbb{R} \times \mathbb{R}$, and the convex combination $(\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2)$. Assume that $g(x_2, z_2) \geq g(x_1, z_1)$,

$$(1-\lambda)x_2 - (1-\lambda)e^{z_2} \geq (1-\lambda)x_1 - (1-\lambda)e^{z_1}$$

then

$$\begin{aligned}
& g(\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2) \\
=& \lambda x_1 + (1-\lambda)x_2 - e^{\lambda z_1} e^{(1-\lambda)z_2} \\
\geq& \lambda x_1 + (1-\lambda)x_1 - (1-\lambda)e^{z_1} + (1-\lambda)e^{z_2} - e^{\lambda z_1} e^{(1-\lambda)z_2} \\
=& x_1 - e^{z_1} + \lambda e^{z_1} + (1-\lambda)e^{z_2} - e^{\lambda z_1} e^{(1-\lambda)z_2} \quad \text{(by convexity of } e^x\text{)} \\
\geq& x_1 - e^{z_1} = g(x_1, z_1)
\end{aligned}$$

As a consequence, under the assumption of *Theorem* 2.3.1, we obtain convexity statements for sets described by probabilistic constraints.

Corollary 2.3.3 Assume that $g_j(\cdot, \cdot)$, $j \in \mathscr{J}$, are *quasi-concave* jointly in both arguments and that $Z \in \mathbb{R}^s$ is a random variable that has an α-concave distribution. Then the following set is convex and closed:

$$X_0 = \{x \in \mathbb{R}^n \ : \ \mathbb{P}_Z\{g_i(x, Z) \geq 0, i = 1, ..., m\} \geq p\}$$

Proof. Let $G(x)$ be defined as in (2.1), and let $x_1, x_2 \in X_0$, $\lambda \in [0, 1]$. We have

$$G(\lambda x_1 + (1-\lambda)x_2) \geq m_\alpha\{G(x_1),\ G(x_2),\ ,\lambda\} \geq \min\{G(x_1),\ G(x_2)\} \geq p$$

The closedness of the set follows from the continuity of α-concave functions (little bit uncomfortable about the openness of D). $\square$

Chapter 3

Lecture 3 - Separable Probability Constraints

In this chapter, we will continue our discussion about the optimization problem with *probabilistic constraints*. We will mainly investigate a special type of probabilistic constraints, *separable probability constraints*. But before proceeding to that, we will talk a little bit about *floating body*.

3.1 Floating Body

Some α-concave measures have a very nice property, they enjoy so called *floating body* for all probability levels $p \in (\frac{1}{2}, 1)$. To illustrate the notion, let's firstly give the definition of a support function of the a convex body:

Definition 3.1.1 The support function $s_A : \mathbb{R}^n \mapsto \mathbb{R}$ of a convex body $A \subset \mathbb{R}^n$ is given by

$$s_A(x) = \sup\{x^T a \ : \ a \in A\}$$

where $x \in \mathbb{R}^n$.

To give an geometric interpretation, we assume $s = 2$, i.e., $\mathbb{R}^2$. We can visualize it,

A is contained in the closed half space

$$\{y \in \mathbb{R}^2 \ : \ y \cdot x \leq s_A(x)\}$$

Floating body is highly related to this interpretation:

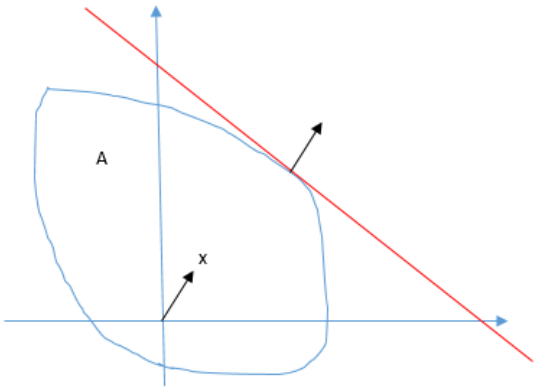

Figure 3.1: Floating Body

Definition 3.1.2 A measure P on $\mathbb{R}^s$ has a floating body at level $p > 0$ if there exists a convex body $C_p \subset \mathbb{R}^s$ such for all vectors $z \in \mathbb{R}^s$,

$$P\{x \in \mathbb{R}^s \ : \ z^T x \geq s_{C_p}(z)\} = 1 - p$$

where $s_{C_p}(\cdot)$ is the support function of the set C_p. The set C_p is called the floating body of P at level p.

Assume P has floating body, then C_p exists and the area below support function at certain point z has measure p. Without a proof, we give a deep result associated with it:

Theorem 3.1.1 Any non-degenerate probability measure with symmetric log-concave density function has a floating body C_p at all levels $p \in (\frac{1}{2}, 1)$.

How does floating body benefit us, let's give an application of it. We know from *Corollary* 2.3.3, for probabilistic constraint,

$$X_0 = \{x \in \mathbb{R}^n : \mathbb{P}_Z\{g_i(x, Z) \geq 0, \ i = 1, ..., m\} \geq p\}$$

if g is quasi-concave in both argument and Z has a α-concave distribution, then X_0 is convex and closed. But if the probability measure has a floating body, we can sometimes have a convex set even g that is not necessary quasi-concave:

Proposition 3.1.2 If Z has a log-concave distribution, symmetric around $\mu \in \mathbb{R}^s$, then the set

$$X_p = \{X : \ \mathbb{P}_Z\{\langle Z, x\rangle \leq b\} \geq p\}$$

is convex.

Proof. Set $x_1, x_2 \in X_p$, $\lambda \in (0,1)$, we need to prove that $x = \lambda x_1 + (1-\lambda)x_2 \in X_p$. We know that

$$\mathbb{P}_Z\{Z: \ \langle Z, x_1\rangle \leq b\} \geq p$$

Let $Y = Z - \mu$, so $\langle Z, x_1\rangle \leq b$ implies $\langle Y, x_1\rangle \leq b - \langle \mu, x_1\rangle$, thus

$$\mathbb{P}_Y\{Y: \ \langle Y, x_1\rangle \leq b - \langle \mu, x_1\rangle\} \geq p$$

From the definition of floating body, we have

$$\mathbb{P}_Y\{Y : \langle Y, x_1\rangle \leq S_{C_p}(x_1)\} = p$$

This implies

$$S_{C_p}(x_1) \leq b - \langle \mu, x_1\rangle$$

Similarly,

$$S_{C_p} \leq b - \langle \mu, x_2\rangle$$

Multiply by λ and $(1-\lambda)$ respectively and add them up, we have

$$\lambda S_{C_p}(x_1) + (1-\lambda) S_{C_p}(x_2) \leq b - [\lambda\langle \mu, x_1\rangle + (1-\lambda)\langle \mu, x_2\rangle]$$

Since support function is convex (i.e., $\sup_i k_i(x)$ where $k_i(x)$ are convex functions is convex), we have

$$S_{C_p}[\lambda x_1 + (1-\lambda)x_2] \leq \lambda S_{C_p}(x_1) + (1-\lambda) S_{C_p}(x_2)$$

Equivalently,

$$\mathbb{P}_Y\{Y: \ \langle Y, \lambda x_1 + (1-\lambda)x_2\rangle \leq S_{C_p}(\lambda x_1 + (1-\lambda)x_2\} \leq \mathbb{P}_Y\{Y: \ \langle Y, \lambda X_1 + (1-\lambda)x_2\rangle \leq b - \langle \mu, \lambda x_1 + (1 -$$

Thus, the assertion follows that $\mathbb{P}_Z\{Z: \ \langle Z, \lambda x_1 + (1-\lambda)x_2\rangle \leq b\} \geq p$. □

3.2 Structure of Separable Probability Constraints

We begin with the following example (which is problem 4, HW1),

$$\begin{aligned} &\min \ c(x) \\ &\text{subject to: } \ \mathbb{P}\{g(x) \geq Z\} \geq p \\ &\qquad x \in \mathscr{D} \end{aligned} \tag{3.1}$$

where $c : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \mapsto \mathbb{R}^m$ is such that each component $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ is a concave function, also $\mathscr{D}$ is a deterministic convex and closed set, Z is a random variable in $\mathbb{R}^s$. Then this problem can be converted to

$$\begin{aligned} \min\ & f(x) \\ \text{subject to: } & g(x) \geq z \\ & F(z) = \mathbb{P}\{Z \leq z\} \geq p \\ & x \in \mathscr{D} \end{aligned} \tag{3.2}$$

Under certain regularity conditions, from the knowledge of nonlinear optimization, we can find *Lagrangian multipliers* to form necessary (sometimes sufficient) conditions for optimality. But in the following, we will concentrate on deriving an equivalent algebraic description for the feasible set of problem above.

3.3 p-efficient Point

Given formulation below,

$$\begin{aligned} \min\ & f(x) \\ \text{subject to: } & g(x) \in \mathscr{Z}_p \\ & x \in \mathscr{D} \end{aligned} \tag{3.3}$$

where we defined p-level set as $\mathscr{Z}_p = \{z : \ F(z) \geq p\}$.

Lemma 3.3.1 For every $p \in (0, 1)$ the level set $\mathscr{Z}_p$ is nonempty and closed.

Proof. The statement follows from the monotonicity and right continuity of the distribution function. □

Let's visualize the constraint (drop the last deterministic constraint temporarily) in $\mathbb{R}^2$ as follows:

Thus, x should be such that $f(x)$ is in the area defined by the set Z_p (not necessary convex). To go further, let's introduce the key concept of a p-efficient point.

Definition 3.3.1 Let $p \in (0, 1)$. A point $v \in \mathbb{R}^m$ is called a p-efficient point of the probability cumulative distribution function F if $F(z) \geq p$ and there is no $z \leq v$, $z \neq v$ such that $F(z) \geq p$.

The p-efficient points are minimal points (*Pareto Optimal*) of the level set $\mathscr{Z}_p$ with respect to the partial order in $\mathbb{R}^m$ generated by the non-negative cone $\mathbb{R}^m_+$. In *Figure* 3.2 - 3.3, the

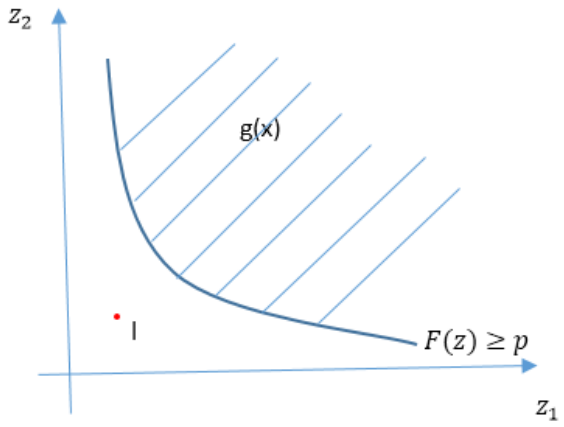

Figure 3.2: Continuous Distribution: p-level set

boundary (not necessary convex or concave) is the set of p-efficient points for continuous distributed random variable and the kinks is the p-efficient points for discrete distributed random variable. Obviously, for a scalar random variable Z, the p-efficient point is the p-quantile. And in fact, p-efficient points are bounded below.

Lemma 3.3.2 Let $p \in (0,1)$ and let

$$l = \left(F_{Z_1}^{-1}(p), ..., F_{Z_m}^{(-1)}(p)\right)$$

Then every $v \in \mathbb{R}^m$ such that $F_Z(v) \geq p$ must satisfy the inequality $v \geq l$.

Proof. Pick any $v \in \mathbb{R}^m$ satisfies $F_{\mathbb{Z}}(v) \geq p$ ($v \in \mathscr{Z}_p$), we have

$$p \leq F_Z(v) \leq F_{Z_i}(v_i)$$

where v_i, $i = 1, ..., m$, are coordinates of v. Since $F_{Z_i}^{-1}$ is monotone, then

$$l_i = F_{Z_i}^{-1}(p) \leq F_{Z_i}^{-1}\left(F_{Z_i}(v_i)\right) = v_i$$

The assertion follows. □

We can also observe the location of l in *Figure* (3.2)-(3.3), it should be always true that $l + \mathbb{R}^m$ contains $\mathscr{Z}_p$.

Let $p \in (0,1)$ and let v^j, $j \in \mathscr{E}$, be all p-efficient points of Z, here $\mathscr{E}$ is arbitrary index set. We define the cones:

$$K_j = v^j + \mathbb{R}_+^m, \quad j \in \mathscr{E}$$

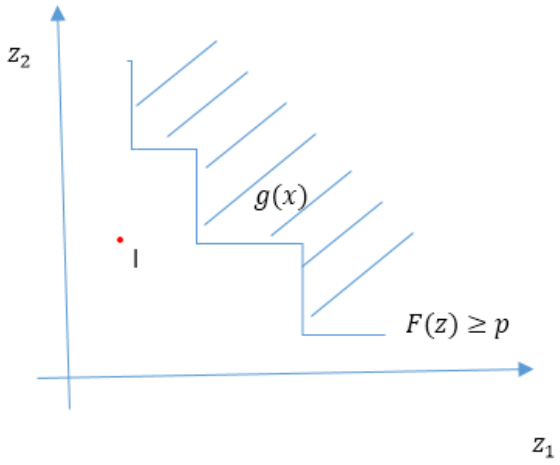

Figure 3.3: Discrete Distribution: p-level set

By Phelps Theorem[1].

Theorem 3.3.3 It holds that

$$\mathscr{Z}_p = \cup_{j \in \mathscr{E}} K_j$$

In the case of discrete random variable, visualization tells the story explicitly, see *Figure* 3.4. By the virtue of theorem above, we can get the following *disjunctive semi-infinite* formulation

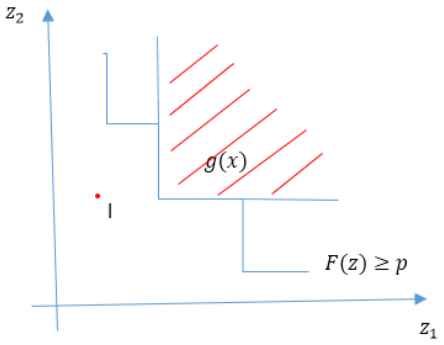

Figure 3.4: Cones generated by p-efficient point

[1]Lemma 3.12, "Convex functions, monotone operators, and differentiability, Lecture Notes in Mathematics" by R.R.Phelps.

of (3.3)

$$\begin{array}{c} \min f(x) \\ \text{subject to: } g(x) \in \cup_{j\in\mathscr{E}} K_j \\ x \in \mathscr{D} \end{array} \tag{3.4}$$

This formulation provides insight into the structure of the feasible set and the nature of its non-convexity. The main difficulty here is the implicit character of the disjunctive constraint.

Let S stand for the simplex in $\mathbb{R}^{m+1}$,

$$S = \{\alpha \in \mathbb{R}^{m+1} : \ \Sigma_{i=1}^{m+1}\alpha_i = 1, \alpha_i \geq 0\}$$

Denote the convex hull of the p-efficient points by E, i.e., $E = \text{conv}\{v^j, \ j \in \mathscr{E}\}$. Then

$$\cup_{j\in\mathscr{E}} K_j \subseteq E + \mathbb{R}^m_+$$

Lemma 3.3.4 It holds that

$$\text{conv}(\mathscr{Z}_p) = E + \mathbb{R}^m$$

Proof. We know that every point $y \in \mathscr{Z}_p$ can be represented as a convex combination of points in the cones K_j, $j \in \mathscr{E}$. By the theorem of *Caratheodory* the number of these points is no more than $m+1$. Thus, we can write $y = \Sigma_{i=1}^{m+1}\alpha_i(v^{j_i} + w^i)$, where $w^i \in \mathbb{R}^m_+$, $\alpha \in S$ and $j_i \in \mathscr{E}$. The vector $w = \Sigma_{i=1}^{m+1}\alpha_i w^i$ belongs to $\mathbb{R}^+_m$. Therefore, $y \in \Sigma_{i=1}^{m+1}\alpha_i v^{j_i} + \mathbb{R}^m_+$. □

Remark 3.3.5 Actually, we can prove that conv $(\mathscr{Z}_p)$ is closed [2].

3.4 Formulation with Discrete Distribution

Let's, from now on, restriction our attentions to the random variable Z with discrete distribution on grid. The *convex hull relaxation* of problem (3.4)

$$\begin{array}{c} \min f(x) \\ \text{subject to: } g(x) \in \text{conv}(\mathscr{Z}_p) \\ x \in \mathscr{D} \end{array} \tag{3.5}$$

or, alternatively

$$g(x) \geq \Sigma_{j\in\mathscr{E}}\alpha_j v^j$$

Here, the set v^j, $j \in \mathscr{E}$ is the collection of p-efficient points. Since Z has a discrete distribution (on a grid), then the number of p-efficient point is finite.

[2] see *Theorem 4.62*, p.117, "*Lectures on Stochastic Programming: Theory and Application*", A.Shaprio, D.Darinka, A, Ruszczyński

Theorem 3.4.1 For each $p \in (0,1)$ the set of p-efficient points of an integer random vector is nonempty and finite.

Proof. First we shall show that at least one p-efficient point exists. Since $p < 1$, there exists a point y such that $F_z(y) \geq p$. We know that the level set $\mathscr{Z}_p$ is bounded from below by the vector l of p-efficient points of one-dimensional marginals. Therefore, if y is not p-efficient, one of finitely many integer points v such that $l \leq v \leq y$ must be p-efficient.

For the finiteness of the set of p-efficient points. Suppose that there exists an infinite sequence of different p-efficient points v^j, $j = 1, 2, ...$ Since they are integer, and the first coordinate v_1^j is bounded from below by l_1, with no loss of generality we may select a subsequence which is non-decreasing in the first coordinate. By a similar token, we can select further subsequences which are nondecreasing in the first k coordinates ($k = 1, ..., m$). Since the dimension m is finite, we obtain a subsequence of different p-efficient points which is nondecreasing in all coordinates. This contradicts the definition of a p-efficient point. □

In general, the relaxation accommodates more integer points that is not p-efficient but $g(x)$ can lie on. We shall make certain assumptions under which the relaxation is identical to the original problem (3.4). Let's generalize the concept of α-concavity to discrete random variables.

Definition 3.4.1 A distribution function F is called α-concave on the set $\mathscr{A} \subset \mathbb{R}^s$ with $\alpha \in [-\infty, +\infty]$ if

$$F(z) \geq m_\alpha \left(F(x), F(y), \lambda\right)$$

for all $z, x, y \in \mathscr{A}$, and $\lambda \in (0,1)$ such that $z \geq \lambda x + (1-\lambda)y$.

The following theorem is crucial to derive the equivalence relationship between convex relaxation problem and original problem.

Theorem 3.4.2 Let $\mathscr{A}$ be the set of all possible values of an integer random vector Z. If the distribution function F_Z of Z is α-concave on $\mathscr{A} + \mathbb{Z}_+^m$ for some $\alpha \in [-\infty, +\infty]$, then for every $p \in (0,1)$ one has

$$\mathscr{Z}_p = \{y \in \mathbb{R}^m : \ y \geq z \geq \Sigma_{j \in \mathscr{E}} \alpha_j v^j, \Sigma_{j \in \mathscr{E}} \alpha_j = 1, \alpha_j \geq 0, z \in \mathbb{Z}^m\}$$

where v^j, $j \in \mathscr{E}$, are the p-efficient points of F.

The proof is not hard, one can refer to the book[3]. But what the theorem tells is significant. In *Figure* 3.5, we saw that if the distribution function is α-concave, then the convex hull

[3]See *Theorem 4.65*, page 119, "*Lectures on Stochastic Programming: Theory and Application*", A.Shaprio, D.Darinka, A, Ruszczyński

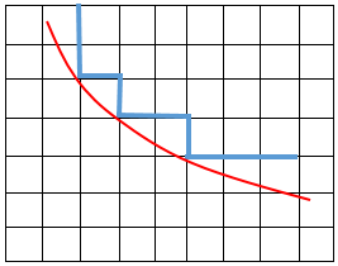

Figure 3.5: Convex hull of p-efficient point generated by α-concave distribution function

generated by p-efficient point will not include more integer point that is not p-efficient. Thus, the relaxation is identical to the following optimization problem:

$$\begin{aligned} \min\ & f(x) \\ \text{subject to: } & g(x) \geq z, \\ & z \geq \Sigma_{j\in\mathscr{E}}\alpha_j v^j, \\ & z \in \mathbb{Z}^m, \\ & \Sigma_{j\in\mathscr{E}}\alpha_j = 1, \\ & \alpha_j \geq 0, j \in \mathscr{E} \\ & x \in D \end{aligned} \tag{3.6}$$

In this way, we have replaced the probabilistic constraint by algebraic equations and inequalities. Note that, in general, we can not drop the integrality requirement. But if other conditions of the problem imply that $g(x)$ is integer, then we may remove z entirely from the problem formulation. This happens, for example, if the definition of $\mathscr{D}$ contains the constraints $x \in \mathbb{Z}^m$, and, in addition, $g(x) = Tx$, where T is a matrix with integer elements, then we can dispose of the variable z.

3.5 Optimality Condition and Algorithm

Consider a subset $\mathscr{J}$ of $\mathscr{Z}_p$, associating *Lagrangian Multipliers* μ to the first two constraints, we get

$$\begin{aligned} L(x,z,\alpha,\mu) &= f(x) + \langle \mu, z - g(x)\rangle + \langle \mu, \Sigma_{j\in\mathscr{J}}\alpha_j v^j - z\rangle \\ &= f(x) + \langle \mu, z - g(x)\rangle + \langle -\mu, z\rangle + \langle \mu, \Sigma_{j\in\mathscr{J}}\alpha_j v^j\rangle \end{aligned} \tag{3.7}$$

The *dual function* is:

$$L_D(\mu) = \inf_{\{(x,z)\in\mathscr{D}\times\mathbb{Z}^m_+, \Sigma_{j\in\mathscr{J}}\alpha_j=1, \alpha_j\geq 0\}} f(x) + \langle \mu, z - g(x)\rangle + \langle -\mu, z\rangle + \langle \mu, \Sigma_{j\in\mathscr{J}}\alpha_j v^j\rangle$$

Notice, for any $\mu \in \mathbb{R}^m_+$ the value of $\Psi(\mu)$ is a lower bound on the optimal value F^* of the original problem (3.6) with $\mathscr{Z}_p$ replaced by $\mathscr{J}$. Let's first focus our attention on the part involving p-efficient point.

$$\begin{aligned} \min\ & \Sigma_{j\in\mathscr{J}}\alpha_j\langle\mu, v^j\rangle \\ \text{subject to: } & \Sigma_{j\in\mathscr{J}}\alpha_j = 1 \\ & \alpha_j \geq 0,\ j = 1, ..., m \end{aligned}$$

If we know that $\langle\mu, v^j\rangle$ gives the smallest value, then we can put all weight on it. Thus, the optimization problem is equivalent to

$$\min_{j\in\mathscr{J}} \langle\mu, v^j\rangle \tag{3.8}$$

By a similar argument, in the original problem (3.6), we actually want to solve

$$\min_{j\in\mathscr{Z}_p} \langle\mu, v^j\rangle \tag{3.9}$$

Thus, we can use the *column generation approach*. That is,

- If $\min_{v\in\mathscr{Z}_p} = \min_{j\in\mathscr{J}}\langle\mu, v^j\rangle$, then stop;
- If $v^* = \arg\min_{v\in\mathscr{Z}_p}\langle\mu, v\rangle < \min_{j\in\mathscr{J}}\langle\mu, v^j\rangle$ then add v^* to the set of p-efficient point set $\mathscr{J}$.

3.6 Methods for Linear Models with Probabilistic Constraints and Discrete Distribution

Consider

$$\begin{aligned} \min\ & \langle c, x\rangle \\ & \mathbb{P}\{Tx \geq h\} \geq 1 - \alpha \\ & x \in \mathscr{X}_0 \end{aligned} \tag{3.10}$$

If we set $c = \mu$, $h = Z$, $T = I$ and remove the deterministic constraint $\mathscr{X}_0$. We get back problem (3.9). In the following, we will discuss how problem of this kind can be solved. There are two cases: (1) with only the right-hand side h random; and (2) with both T and h random.

3.6.1 Random Right Hand Sides

Assume that T is fixed, and h has N realizations $h_1, ..., h_N$ with probabilities $p_1, ..., p_N$. With no loss of generality we may assume that $h^k \geq 0$ for all k, thus it is necessary that $Tx \geq 0$.

We introduce binary variables $u_k \in \{0,1\}$, $k = 1, ..., N$. We can formulate the mixed-integer linear programming problem:

$$\begin{aligned} &\min \langle c, x\rangle \\ &\text{subject to: } Tx + u_k h^k \geq h^k, \quad k = 1, ..., N, \\ &\Sigma_{k=1}^N p_k u_k \leq \alpha, \\ &u_k \in \{0,1\}, \quad k = 1, ..., N, \\ &x \in X_0 \end{aligned} \tag{3.11}$$

That is identical to the problem (3.10), because if the x is chosen such that $\langle T, x\rangle < h^k$, then we add by h^k to ensure $Tx + u_k h^k \geq h^k$ valid, in this case, u_k needs to be 1, and if x is chosen such that $\langle T, x\rangle \geq h^k$, u_k can be 0 and 1. The knapsack constraint ensure that the probability of violating the constraint $Tx \geq h$ is no more than α.

Let us focus on row i together with knapsack constraint:

$$\begin{aligned} &\langle T_i, x\rangle + u_k h_i^k \geq h_i^k, \quad k = 1, ..., N, \\ &\Sigma_{k=1}^N p_k u_k \leq \alpha, \\ &u_k \in \{0,1\}, \quad k = 1, ..., N \end{aligned} \tag{3.12}$$

with T_i denoting the i-th row of T, and h_i^k - i-th entry of h^k. We find a permutation $\{\sigma_1, \sigma_2, ..., \sigma_N\}$ of $\{1, ..., N\}$ such that

$$h_i^{\sigma_1} \geq h_i^{\sigma_2} \geq \cdots \geq h_i^{\sigma_N}$$

Clearly, if $\langle T_i, x\rangle \geq h_i^{\sigma_k}$, then obviously $\langle T_i, x\rangle \geq h_i^{\sigma_l}$ for all $l \geq k$. We choose q so that

$$\Sigma_{k=1}^q p_{\sigma_k} \leq \alpha \text{ and } \Sigma_{k=1}^{q+1} p_{\sigma_k} > \alpha$$

Then (3.12) can be reformulated as:

$$\begin{aligned} &\langle T_i, x\rangle + u_{\sigma_k} h_i^{\sigma_k} \geq h_i^{\sigma_k}, \quad k = 1, ..., q+1, \\ &\Sigma_{k=1}^N p_{\sigma_k} u_{\sigma_k} \leq \alpha, \\ &u_{\sigma_j} \in \{0,1\}, \quad k = 1, ..., N, \end{aligned} \tag{3.13}$$

It is equivalent to (3.11), because we must have $\langle T_i, x\rangle \geq h_i^{\sigma_k}$ for at least one of the indices $k = 1, ..., q+1$, otherwise the original knapsack constraint would be violated. This implies, in the worst case, the solution must satisfy $\langle T_i, x\rangle \geq h_i^{\sigma_{q+1}}$. Consequently, we can further strengthen the coefficients in the constraints, by replace the first system with

$$\langle T_i, x\rangle + u_{\sigma_k}[h_i^{\sigma_k} - h_i^{\sigma_{q+1}}] \geq h_i^{\sigma_k}, \quad k = 1, ..., q+1 \tag{3.14}$$

Let's check that. If $\langle T_i, x\rangle < h_i^{\sigma_k}$, then $u_{\sigma_k} = 1$, and when $\langle T_i, x\rangle \geq h_i^{\sigma_k}$, u_{σ_k} can be 0 or 1, thus it is identical to the problem (3.13). Observe that in this last formulation, we do not rely on the fact that $\langle T_i, x\rangle \geq 0$, thus the assumption $h^k \geq 0$ is not needed. Finally,

the system of mixed-integer inequalities (3.14) yields a family of *star inequalities*, which are valid and facet defining:

$$\langle T_i, x\rangle + \Sigma_{k=1}^{J} u_{\sigma_{k_j}} [h_i^{\sigma_{k_j}} - h_i^{\sigma_{k_{j+1}}}] \geq h_i^{\sigma_{k_1}}, \quad \forall 1 \leq k_1 < k_2 < \cdots < k_J \leq N \tag{3.15}$$

This is a topic in integer optimization [4]. Similar derivations can be carried out for every row i of the system, yielding a very strong mixed-integer formulation of the original problem (3.11).

3.6.2 Random Matrix

We now generalize the idea of the previous section to the case of both T and h are random in (3.10), with realization (T^k, h^k) attained with probabilities p_k, $k = 1, ..., N$. We define convex polyhedron

$$Y^k = \{x \in \mathbb{R}^n : T^k x \geq h^k\}, \quad k = 1, ..., N$$

For simplicity, we assume that they are bounded. Again, we introduce the binary variables $u_k \in \{0, 1\}$ representing the events that $x \in Y^k$, $k = 1, ..., N$. The original problem (3.10) can be reformulated as follows:

$$\begin{aligned} &\min \langle c, x\rangle \\ &\text{subject to: } [x \notin Y^k] \Rightarrow [u_k = 1], \quad k = 1, ..., N, \\ &\qquad \Sigma_{k=1}^{N} p_k u_k \leq \alpha, \\ &\qquad u_k \in \{0, 1\}, \quad k = 1, ..., N, \\ &\qquad x \in X_0 \end{aligned} \tag{3.16}$$

We want to represent the logical constraint $[x \notin Y^k] \Rightarrow [u_k = 1]$ by linear mixed-integer inequalities of the form $\langle a_j, x\rangle + \langle \pi_j, u\rangle \geq \beta_j$, $j = 1, 2, ...$, where $a_j \in \mathbb{R}^n$, $\pi_j \in \mathbb{R}^n$, and $\beta_j \in \mathbb{R}$.

Suppose we have a collection of such valid inequalities for $j = 1, ..., J$, and we solved the problem:

$$\begin{aligned} &\min \langle c, x\rangle \\ &\text{subject to: } , \langle a_j, x\rangle + \langle \pi_j, u\rangle \geq \beta_j \quad j = 1, ..., J, \\ &\qquad \Sigma_{k=1}^{N} p_k u_k \leq \alpha, \\ &\qquad u_k \in \{0, 1\}, \quad k = 1, ..., N, \\ &\qquad x \in X_0 \end{aligned} \tag{3.17}$$

[4] See "An integer programming approach for linear programs with probabilistic constraints", Luedtke, James, Shabbir Ahmed and George L. Nemhauser.

Let (x^J, u^J) denote the solution obtained. If the logical constraints are satisfied, we have found the solution of (3.16). Otherwise, we find a scenario $\bar{k}$ such that $x^j \in Y^{\bar{k}}$ but $u^J_{\bar{k}} = 0$. Then we separate x^J from the polyhedron $Y^{\bar{k}}$ by finding a vector $a \in \mathbb{R}^n$ such that

$$\langle a, x^J \rangle < \langle a, x \rangle, \quad \forall x \in Y^{\bar{k}}$$

This can be done by projecting x^J on $Y^{\bar{k}}$, but also by special linear programming techniques. Having found a, we calculate the numbers:

$$\gamma_k = \min\{\langle a, x \rangle : x \in Y^k \cap X_0\}, \quad k = 1, ..., N$$

By construction, if $x \in Y^k \cap X_0$ then $\langle a, x \rangle \geq \gamma_k$. Temporarily assuming for simplicity that $\langle a, x \rangle \geq 0$, we can write an additional system of valid inequalities:

$$\begin{aligned} &\langle a, x \rangle + u_k \gamma_k \geq \gamma_k, \quad k = 1, ..., N, \\ &\Sigma_{k=1}^N p_k u_k \leq \alpha, \\ &u_k \in [0, 1], k = 1, ..., N. \end{aligned}$$

The current solution violates $\langle a, x \rangle + u_k \gamma_k \geq \gamma_k$ for $k = \bar{k}$. The rest is the same as in the previous section. We order the scenarios, skip many constraints, strengthen, deduce the star inequalities. The temporary assumption can be skipped. All theses valid inequalities can be now added to get a new solution. The iteration continuous, until the logical constraints are satisfied.

3.7 Normal Approximation

Let's wrap up the discussion on probability constraint by an approximation method. Again, we have the following optimization model:

$$\begin{aligned} \max . \ & c(x) \\ \text{subject to: } & \mathbb{P}_Z\{x^\top Z \geq \eta\} \geq p \\ & x \in \mathscr{X} \end{aligned} \tag{3.18}$$

Here, n-dimensional random vector Z has independent components[5] and the dimension n is relatively large, we may invoke the *central limit theorem*[6]. Under mild additional assumptions, we can conclude that the distribution of $x^\top Z$ is approximately normal and convert the probabilistic constraint into an algebraic constraint. Let's define

$$W(x) = Z^\top x$$

[5]In theory, sometimes we can even have weak dependence of random variables, this will be a study in statistics so that we will skip.

[6]A short discussion of Central limit theorem can be found in Appendix B

where $Z_1, ..., Z_n$ are independent. Apply CLT, for large number n, $W(x)$ can be asymptotically normal $W(x) \sim N(\mu^T x, x^T C x)$, where $\mathbb{E}[Z] = \mu$, $Cov[Z] = C$. In the multivariate case, assume we need to satisfies

$$\Sigma_{j=1}^{n} Z_{ij} x_j < b_i, \quad i = 1, ..., m$$

$$Z_1 = [Z_{11}, ..., Z_{n1}]^\top, ..., Z_m = [Z_{1n}, ..., Z_{mn}]^\top$$

Then, $\mathbb{E}[W(x)] = \Sigma_{j=1}^{n} \mu_j x_j$, where $\mu_j = [\mu_{ij}, ..., \mu_{mj}]^\top = \mathbb{E}[Z_j]$. For the variance

$$\begin{aligned} C &= \mathbb{E}[(W(x) - \mathbb{E}(W(x)))(W(x) - \mathbb{E}(W(x)))^\top] \\ &= \Sigma_{j=1}^{n} \Sigma_{l=1}^{n} x_j x_l \mathbb{E}[(Z_j - \mu_j)(Z_l - \mu_l)^\top] \end{aligned}$$

Then we have

$$\mathbb{P}\{N\left(\Sigma_{j=1}^{n} \mu_j x_j, \Sigma_{j=1}^{n} \Sigma_{l=1}^{n} x_j x_l C_{jl}\right) \leq b\} \geq 1 - \alpha$$

which can be easily converted to an algebraic form.[7]

[7] Check Prof. Alan Genz, one can find various of ways to solve such problem efficiently. http://www.math.wsu.edu/faculty/genz/homepage

Chapter 4

Lecture 4

4.1 Two-Stage Problem Introduction

Given probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let's consider the following two-stage models of the standard form:

$$\begin{aligned} \min_x . \ & \langle c, x\rangle + \mathbb{E}[Q(x, \xi(\omega))] \\ \text{subject to:} \ & Ax = b \\ & x \geq 0 \end{aligned} \tag{4.1}$$

Here, $\xi(\omega) = (q, h, T, W)(\omega)$ are the data of the second-stage problem. We view some or all elements of the vector ξ as random. The expectation operator is taken with respect to the probability distribution of ξ. And $Q(x, \xi(\omega))$ denotes the optimal value of the *second-stage* problem (omitting the dependence on ω):

$$\begin{aligned} \min . \ & \langle q, y\rangle \\ \text{subject to} : \ & Wy + Tx = h \\ & y \geq 0 \end{aligned} \tag{4.2}$$

This type of model is ubiquitous, let's just give one example to illustrate the above formulation. Consider a graph $\mathscr{G} = (\mathscr{N}, \mathscr{A})$, for each arc $a \in \mathscr{A}$, x_a is the capacity of the arc, q_a denotes the cost of the flow on arc a and c_a stands for the cost of increasing the capacity on a. For $n \in \mathscr{N}$, h_n denotes the demand at node n, if it is a supply node, $h_n < 0$, if it is a demand node $h_n > 0$, and if it is a transshipment node, $h_n = 0$. Overall, we shall have $\Sigma_n h_n = 0$. We have the following situation, a network flow is given, the cost of flow on each arc is prefixed. We need to make a decision on the capacity of arc at first stage, and then after we observe the demands, we shall optimally distribute the flow in the sense

of minimizing the cost. In mathematical language, the *second-stage* is:

$$\begin{aligned} \min . \ & \Sigma_{a\in\mathscr{A}} q_s y_a \\ & Wy = h \\ & 0 \le y_a \le x_a, \quad a \in \mathscr{A} \end{aligned}$$

where W is the *incidence matrix*. We can actually combine the two constraint and form a big matrix $\bar{W}y = h$ (with either $y \ge 0$ separate or involved). This is a special case of (5.1)-(5.2), where $Q(x, \xi(\omega)) = Q(x, h(\omega))$, and the second stage constraint has no association with x.

This is a difficult problem in general, we will set out by calculating the subdifferential of the recourse function $Q(\cdot, \xi)$[1] and of its expected value.

4.2 The subdifferential and directional derivatives of the recourse function

Let's focus on the *second-stage* problem:

$$\begin{aligned} \min . \ & \langle q, y\rangle \\ \text{subject to} : \ & Wy = h - Tx \\ & y \ge 0 \end{aligned} \tag{4.3}$$

The *dual problem* of (4.3) is

$$\max_{\pi\in\Pi(q)} \langle h - Tx, \pi\rangle \tag{4.4}$$

with

$$\Pi(q) = \{\pi : W^T\pi \le q\} \tag{4.5}$$

We assume that $\Pi(q) = \emptyset$. Then by *duality* in linear programming, the optimal values of (4.3) and (4.4) are identical[2].

We are interested in the case when $Q(x, \xi)$ is finite, then

Proposition 4.2.1 Suppose that for given $x = x_0$ and ξ, the value $Q(x_0, \xi)$ is finite. Then $Q(\cdot, \xi)$ is subdifferentiable[3] at x_0 and

$$\partial Q(x_0, \xi) = -T^\top \mathscr{D}(x_0, \xi) \tag{4.6}$$

where $\mathscr{D}(x_0, \xi)$ is the optimal solution of *dual problem*, i.e.,

$$\mathscr{D}(x_0, \xi) := \arg\max_{\pi\in\Pi(q)} \pi^\top(h - Tx) \tag{4.7}$$

[1] the dependence on ω will always be omitted

[2] See Appendix C - Optimization, sect 1.

[3] See Appendix C - optimization, section 2

Proof. ("$\supseteq$") If $\hat{\pi} \in \mathscr{D}(x_0, \xi)$, then for arbitrary x we have

$$\begin{aligned} Q(x,\xi) &= \max_{\pi \in \Pi(q)} \langle h - Tx, \pi \rangle \\ &\geq \langle h - Tx, \hat{\pi} \rangle \\ &= \langle h - Tx_0, \hat{\pi} \rangle - \langle Tx - Tx_0, \hat{\pi} \rangle \\ &= Q(x_0, \xi) + \langle x - x_0, -T^\top \hat{\pi} \rangle \end{aligned}$$

This proves that $-T^\top \hat{\pi}$ is a subgradient of $Q(\cdot, \xi)$ at x_0.

To prove the equivalence relationship is very involved. Suppose that $g \in \partial Q(x_0, \xi)$, but $g \notin -T^\top \mathscr{D}(x_0, \xi)$. The set $\mathscr{D}(x_0, \xi)$ is polyhedral (defined by a finite number of equations and inequalities) and so is the right-hand-side of (4.6). It is therefore a convex and closed set. We can separate g from it. By *separation theorem*, a vector d exists and also ϵ such that

$$\langle g, d \rangle \geq \langle -T^\top \pi, d \rangle + \epsilon, \quad \forall \pi \in \mathscr{D}(x_0, \xi)$$

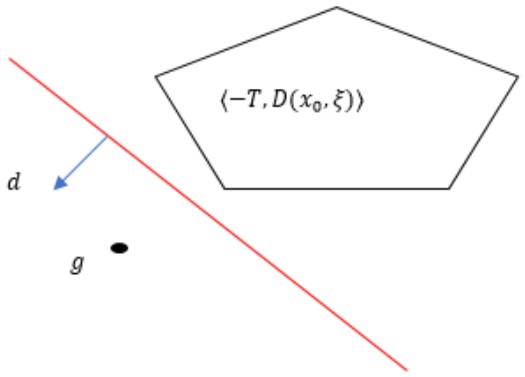

Figure 4.1: Separation

Consider the point $x_0 + \tau d$ for a small $\tau > 0$. As g is a sub-gradient, we obtain the inequality:

$$\begin{aligned} Q(x_0 + \tau d, \xi) &\geq Q(x_0, \xi) + \tau \langle g, d \rangle \\ &\geq \max_{\pi \in \Pi(q)} \langle h - Tx_0, \pi \rangle + \sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle -T^\top \pi, \tau d \rangle + \tau \epsilon \\ &\geq \sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle h - Tx_0 - \tau T d, \pi \rangle + \tau \epsilon \end{aligned}$$

Thus,

$$\sup_{\pi \in \Pi(q)} \langle h - Tx_0 - \tau T d, \pi \rangle \geq \sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle h - Tx_0 - \tau T d, \pi \rangle + \tau \epsilon \tag{4.8}$$

The key issue is whether for small $\tau > 0$ the supremum on the left hand side can be achieved at a point different than at the right hand side.

The set $\Pi(q)$ can be represented as convex hull of a finite number of points plus a conic hull of a finite number of rays (extreme rays of its recession cone):

$$\Pi(q) = \mathrm{conv}\{e^j,\ j \in \mathscr{J}\} + cone\{r^l,\ l \in \mathscr{L}\}$$

For sufficiently small $\tau > 0$ the perturbation $\tau T d$ cannot make the points e^j that are inferior in terms of the value $\langle h - Tx_0, e^j\rangle$, superior in terms of $\langle h - Tx_0 - \tau T d, e^j\rangle$. Consequently, if a supremum on the left hand side of (4.8) is achieved at some e^j, this point must be an element of $\mathscr{D}(x_0, \xi)$. But then (4.8) is a contradiction.

It remains to consider the case when one of the recession rays r^l plays a role on the left side of (4.8), which means that the supremum is $+\infty$ and is achieved along a line $\pi = e^j + \alpha r^l$, $\alpha \to \infty$, with some $e^j \in \mathscr{D}(x_0, \xi)$. This means that $\langle -\tau T d, r^l\rangle > 0$. We also must have $\langle h - Tx_0, r^l\rangle = 0$, because if it was positive, the value of $Q(x_0, \xi) = \max_{\pi\in\Pi(q)}\langle h - Tx_0, \pi\rangle$ could be driven to $+\infty$ along the line $\pi = e^j + \alpha r^l$, $\alpha \to \infty$, which contradicts its finiteness. If $\langle h - Tx_0, r^l\rangle < 0$, then for sufficiently small $\tau > 0$, the value of $\langle h - Tx_0 - \tau T d, \pi$ would not grow along r^l, in other words, $\langle h - Tx_0, r^l\rangle$ will dominate. Therefore, $\langle h - Tx_0, r^l\rangle = 0$. But then we conclude that moving along r^l is indifferent for $\langle h - Tx_0, \pi\rangle$. Consequently, the ray $e^j + \alpha r^l$, $\alpha \geq 0$, is included in $\mathscr{D}(x_0, \xi)$. This again leads to contradiction in (4.8). □

We can also prove another important fact:

Proposition 4.2.2 The directional derivative $Q(x_0, \xi)$ on d is defined as

$$Q'(x_0, d; \xi) = \lim_{\tau\downarrow 0} \frac{1}{\tau}[Q(x_0 + \tau d, \xi) - Q(x_0, \xi)]$$

If $Q(x_0, \xi)$ is finite, then

$$Q'(x_0, d; \xi) = \sup_{\pi\in\mathscr{D}(x_0,\xi)} \langle -T^\top \pi, d\rangle \tag{4.9}$$

Remark 4.2.3 This property is true in the interior of the domain of general convex functions. In our case, it is true in the entire domain, owing to the polyhedral character of the function.

Proof. We have

$$Q(x_0 + \tau d, \xi) = \sup_{\pi\in\Pi(q)} \langle h - Tx_0 - \tau T d, \pi\rangle$$
$$Q(x_0, \xi) = \sup_{\pi\in\Pi(q)} \langle h - Tx_0, \pi\rangle$$

If the "sup" in the first expression is achieved at a point $\hat{\pi} \in \mathscr{D}(x_0, \xi)$. then

$$Q(x_0 + \tau d, \xi) = \sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle h - Tx_0 - \tau Td, \pi \rangle = Q(x_0, \xi) + \sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle -\tau Td, \pi \rangle$$

Dividing by τ and send it to 0, we obtain (4.9).

If the "sup" in the first expression is not achieved at a point in $\mathscr{D}(x_0, \xi)$, arguing as before we deduce that $Q(x_0 + \tau d, \xi) = +\infty$ and that the supremum is achieved along a line $\pi = e^j + \alpha r^l$, $\alpha \to \infty$. In exactly the same way, we deduce that $\langle h - Tx_0, r^l \rangle = 0$ and that again the entire line $\pi = e^j + \alpha r^l$, $\alpha \geq 0$, is included in $\mathscr{D}(x_0, \xi)$. Thus $\sup_{\pi \in \mathscr{D}(x_0, \xi)} \langle -T^\top \pi, d \rangle = +\infty$, achieved along this line. □

Chapter 5

Lecture 5

We will continue our discussion on two-stage problem, mainly focus on the optimality conditions and algorithm, namely, solving huge linear program which is a brutal force method and solving by cutting plane.

5.1 Optimality Condition

Recall our setting:

$$\begin{aligned} \min . \ & \{\langle c, x\rangle + \mathbb{E}[Q(x,\xi)]\} \\ \text{subject to: } & Ax = b \\ & x \geq 0 \end{aligned} \tag{5.1}$$

where $\xi(\omega) = (T, W, q, h)(\omega)$ is random vector, and $Q(x, \xi)$ is the optimal value of the second stage problem, that is:

$$\begin{aligned} \min . \ & \langle q, y\rangle \\ \text{subject to: } & Tx + Wy = h \\ & y \geq 0 \end{aligned} \tag{5.2}$$

We can express $Q(x, \xi)$ by its dual form:

$$Q(x,\xi) = \max_{W^\top \pi \leq q} \langle h - Tx, \pi\rangle \tag{5.3}$$

which allows us to describe the sub-differential,

$$\partial Q(x,\xi) = -T^\top \mathscr{D}(x,\xi) \tag{5.4}$$

where $\mathscr{D}(x,\xi)$ is the set of optimal solution for the dual problem (5.3). If the space of Ω is finite, then we have finite realization of ξ, i.e., $\xi = (\xi^1, ..., \xi^K)$ with probability $p_1, ..., p_K$ attached to them respectively. By the analysis of last lecture (*Rockfellar-Moreau Theorem*), we can derive

$$\partial\mathbb{E}[Q(x,\xi)] = \Sigma_{k=1}^K p_k \partial Q(x,\xi^k)$$

Next, let's consider the optimality condition. From the knowledge of convex optimization, we know that the *Lagrangian optimality condition* for the original problem is identical to the *Lagrangian optimality condition* for the linearized version, that is

$$\begin{aligned} \min . \ & \langle c, x\rangle + \langle \hat{g}, x\rangle \\ \text{subject to: } & Ax = b \\ & x \geq 0 \end{aligned} \tag{5.5}$$

where $\hat{g} = \Sigma_{k=1}^K p_k(-T_k^\top \hat{\pi}_k)$ ($\hat{\pi}_k$ is the solution of the dual problem for scenario k). Thus, we have the following optimality condition (necessary), if $\hat{x}$ is the optimal solution, then there exists one $\hat{g} \in \partial\mathbb{E}[Q(x,\xi)]$ such that $\hat{x}$ solves the above LP (5.5).

Remark 5.1.1 If we interchange the expectation inside the optimal value function of the second-stage, i.e., $Q(x, \mathbb{E}(\xi))$, it will leads huge difference of the optimality condition and also the solution, because now we only need to have one g of the following form:

$$\bar{g} = -\bar{T}\bar{\tau} \tag{5.6}$$

where $\bar{\tau}$ is in the solution set $\mathscr{D}(x, \mathbb{E}(\xi))$.

5.2 Example: Network Optimization

Let's recall the network model. Assume that we have a graph $\mathscr{G} = (\mathscr{N}, \mathscr{A})$. Then we can use *incidence matrix* to represent the directed graph. The incidence matrix of a tree has the property that: $\amalg^\top A = 0$, and also the following important theorem:

Theorem 5.2.1 Let $T = (\mathscr{N}, \mathscr{A})$ be a directed tree with $n = \text{card}(\mathscr{N})$ nodes, then the $(n-1) \times (n-1)$ truncated matrix T_1 has a permutation to lower triangle matrix, it is irreducible and the absolute value of determinant is 1.

Corollary 5.2.2 The rank of the incidence matrix of a tree is $n-1$.

Another important concept of network model is *spanning tree*:

Definition 5.2.1 Suppose $\mathscr{G}(\mathscr{N}, \mathscr{A})$ is an undirected graph with $\text{card}(\mathscr{N}) = n$. Suppose $\mathscr{A}_1 \subseteq \mathscr{A}$ is a subset of arcs with $\text{card}(r_1) = n-1$ such that the sub-graph $T = (\mathscr{N}, \mathscr{A})$ is a tree. Then T is called a *spanning tree*. (r_1 is the rank of the incidence matrix associated with sub-graph $\mathscr{A}_1$)

Lemma 5.2.3 Let $\mathscr{G}(\mathscr{N}, \mathscr{A})$ be a *weakly undirected graph* with incidence matrix $\mathscr{A}$. Suppose we remove one node, then a subset of $n-1$ rows along with the remaining $n-1$ columns of A forms a non-singular matrix if and only if the $n-1$ rows corresponds to the arcs of a *spanning tree*.

Usually we have the following elements for network optimization model:

- Each supply node i has a maximum production capacity s_i;
- Each demand node i has a minimum demand ξ_i;
- Each arc (i, j) has a cost q_{ij};
- In some case, there are upper bound or capacity on each arc (i, j), i.e., x_{ij};
- Objective is to find how much flow y_{ij} should be distributed on each arc to minimize the cost;
- Balance constraint: total in-flow of i - total out-flow of i = demand i, after we consider supply as negative demand.

Thus, the formulation is the following:

$$\begin{aligned} \min . \ & \Sigma_{(i,j)\in\mathscr{A}} q_{ij} y_{ij} \\ \text{subject to} : \ & \Sigma_{j;(j,i)\in\mathscr{A}} y_{ji} - \Sigma_{j;(i,j)\in\mathscr{A}} y_{ij} = \xi_i, \quad \forall i \\ & 0 \le y_{ij} \le x_{ij}, \quad \forall (i,j) \in \mathscr{A} \end{aligned} \tag{5.7}$$

Algorithmically, in the network model, the spanning tree is the basic solution for the linear program, we can derive the reduced cost through very simple calculation by the virtue of the special structure of incidence matrix and decrease the flow until optimal.

Let's go back to the stage problem (with $\text{card}(\mathscr{N}) = n$), which is the following:

$$\min_{x \ge 0} \ \Sigma_{(i,j)} c_{ij} x_{ij} + \mathbb{E}[Q(x, \xi)] \tag{5.8}$$

where the optimal value of the second stage $Q(x, \xi)$ is obtained from above static network flow model (5.7) **where ξ is the realization**. We derive its *dual problem* through *Lagrangian*, attaching μ_i to the *balance constraint* and λ_{ij} to the capacity constraint $y_{ij} \le x_{ij}$,

$$\begin{aligned} L(y, \mu, \lambda) = & \Sigma_{(i,j)\in\mathscr{A}} q_{ij} y_{ij} + \Sigma_{i=1}^{n} \mu_i (\Sigma_{j;(j,i)\in\mathscr{A}} y_{ji} - \Sigma_{j;(i,j)\in\mathscr{A}} y_{ij} - \xi_i) \\ & + \Sigma_{(i,j)\in\mathscr{A}} \lambda_{ij} (y_{ij} - x_{ij}) \end{aligned} \tag{5.9}$$

Thus, the *dual function* is

$$\begin{aligned} L_D(\mu,\lambda) &= \min_{y\geq 0} L(y,\mu,\lambda) \\ &= \min_{y\geq 0} \Sigma_{(i,j\in\mathscr{A})}(q_{ij}+\lambda_{ij}+\mu_j-\mu_i)y_{ij} \\ &- \Sigma_{i=1}^n \mu_i\xi_i - \Sigma_{(i,j)\in\mathscr{A}}\lambda_{ij}x_{ij} \end{aligned}$$

The *dual problem* is obtained by taking maximum of dual function over λ,μ, thus we need to have the the coefficients in front of y_{ij} to be positive (otherwise goes to $-\infty$), i.e.,

$$\begin{aligned} \max. \ & -\Sigma_{i=1}^n \xi_i\mu_i - \Sigma_{(i,j)\in\mathscr{A}} x_{ij}\lambda_{ij} \\ \text{subject to: } & q_{ij}+\lambda_{ij}+\mu_j-\mu_i \geq 0 \\ & \lambda_{ij}\geq 0, \quad \forall (i,j)\in\mathscr{A} \end{aligned} \tag{5.10}$$

In fact, we can write it more compactly,

$$\max. \ -\Sigma_{i=1}^n \xi_i\mu_i - \Sigma_{(i,j)\in\mathscr{A}} x_{ij}\max(0,\mu_i-\mu_j-q_{ij}) \tag{5.11}$$

Thus we can express

$$\hat{g}_{ij} = \Sigma_{k=1}^K p_k\lambda^k$$

for some λ[1].

5.3 Non-anticipativity

We can relax our original problem by allowing x depending on k, that is to say, clairvoyance is permitted, one can look into the future to observe the ξ. Let's formulate such relaxation problem:

$$\begin{aligned} \min_{x_1,...,x_k, Ax_k=b, x_k=0,\forall k=1,...,K} \ & \Sigma_{k=1}^K p_k\{\langle c,x_k\rangle + \langle q,y_k\rangle\} \\ \text{subject to: } & T_kx_k+W_ky_k = h_k, \quad k=1,...,K \\ & y_k\geq 0, \quad k=1,...,K \end{aligned} \tag{5.12}$$

Note the problem is *separable*, i.e.,

$$\begin{aligned} \min_{x_k\geq 0, y_k\geq 0, k=1,...,K} . \ & \langle c,x\rangle + \langle q_k,y_k\rangle \\ \text{subject to: } & Ax_k = b \\ & T_kx_k+W_ky_k = h_k \end{aligned} \tag{5.13}$$

We can fix it by enforcing

$$x_1 = x_2 = \cdots = x_K$$

[1]See page.71, "Nonlinear Optimization", Andrezj Ruszczynski

It is equivalent to

$$x_k = \Sigma_{k=1}^{K} p_j x_k \tag{5.14}$$

(5.14) is quite important for *multi-stage modelling* and is therefore called *non-anticipative constraint*.

Let's carry out the analysis on (5.14). Define

$$\mathscr{L} = \{x_1, ..., x_K \ : \ x_1 = x_2 = \cdots = x_K\}$$

is a linear subspace of $\mathbb{X} = \mathbb{R}^n \times \mathbb{R}^n \cdots \mathbb{R}^n$ (K times). Consider $\mathbb{X}$ equipped with scalar product:

$$\langle x, y \rangle_* := \Sigma_{i=1}^{K} p_i \langle x_i, y_i \rangle$$

A linear operator $P : \mathbb{X} \mapsto \mathbb{X}$ is defined as:

$$Px := [\Sigma_{i=1}^{K} p_i x_i \cdots \Sigma_{i=1}^{K} p_i x_i]^\top = [\bar{x} \cdots \bar{x}] \tag{5.15}$$

Then the constraint (5.14) can be rewritten as:

$$x = [x_1 \ x2 \cdots x_K]^\top = [\bar{x} \ \bar{x} \cdots \bar{x}]^\top = Px \tag{5.16}$$

Moreover, we can shall that P is actually an *orthogonal projection* on $\mathbb{X}$, equipped with scalar product. Indeed $P(P(x)) = Px$ and since we have inner product for the space, the concept of orthogonality can be used.

$$\langle Px, (y - Py) \rangle_* = \langle (x - Px), Py \rangle_* = 0$$

That is equivalent to,

$$\langle x, Py \rangle_* = \langle Px, Py \rangle_* = \langle Px, y \rangle_*$$

We can verify above relation easily,

$$\begin{aligned} \langle Px, y \rangle_* &= \Sigma_{k=1}^{K} p_k \langle (Px)_k, y_k \rangle \\ &= \Sigma_{k=1}^{K} p_k \langle \Sigma_{j=1}^{K} p_j x_j, y_k \rangle \\ &= \langle \Sigma_{j=1}^{K} p_j x_j, \Sigma_{k=1}^{K} y_k p_k \rangle \\ &= \langle x, Py \rangle_* \end{aligned}$$

Before we talk about the dualization of non-anticipative constrains, let's first observe the operator $(I - P)$ is an *orthogonal operator* as well. Every vector $x \in \mathbb{X}$ can be decomposed uniquely as:

$$x = u + v$$

where $u \in Px$ and $v \in (I - P)x$. This says, $(I - P)$ is a projection to the space orthogonal to $\mathscr{L}$.

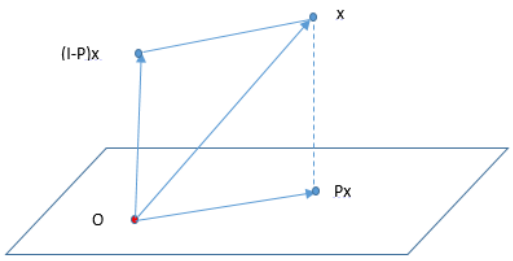

Figure 5.1: Orthogonal Decomposition

Set

$$F_k(x) = \langle c, x\rangle + Q(x, \xi^k) \tag{5.17}$$

Then our problem is:

$$\begin{aligned} \min_{x=(x_1,\ldots x_k), Ax_k=b, x_k\geq 0, k=1,\ldots,K} \quad & \Sigma_{k=1}^K p_k F_k(x) \\ \text{subject to:} \quad & x - Px = 0 \end{aligned} \tag{5.18}$$

The *Larangian* is (attaching λ to *non-anticipative constraint*),

$$\begin{aligned} L(x,\lambda) &= \Sigma_{k=1}^K p_k F_k(x_k) + \langle \lambda, x - Px\rangle_* \\ &= \Sigma_{k=1}^K p_k F_k(x_k) + \langle (I-P)\lambda, x\rangle \\ &= \Sigma_{k=1}^K p_k F_k(x_k) + \Sigma_{k=1}^K p_k \left(\lambda_k - \Sigma_{j=1}^K p_j \lambda_j\right) x_k \end{aligned}$$

Since λ is translation invariant, we can, without loss of generality, write

$$L(x,\lambda)_{P\lambda=0} = \Sigma_{k=1}^K p_k \{F_k(x), \langle \lambda_k, x_k\rangle\}$$

Then, the dual function is

$$L_D(\lambda) = \Sigma_{k=1}^K p_k \{ \min_{Ax_k=b, x_k\geq 0} [F_k(x_k) + \langle \lambda_k, x_k\rangle]\}$$

The dual problem is $\max_{P\lambda=0} L_D(\lambda)$, the optimal solution $\hat{\lambda}$ should satisfy $\Sigma_{k=1}^K p_k \lambda_k = 0$. In other words, if we input $\lambda_k = \hat{\lambda}_k$ in

$$\begin{aligned} \min_{Ax_k=b, x_k\geq 0} \quad & \langle c, x_k\rangle + \langle q_k, y_k\rangle + \langle \lambda_k, x_k\rangle \\ \text{subject to:} \quad & T_k x_k + W_k y_k = b_k \\ & y_k \geq 0 \end{aligned}$$

we shall get the optimal solution $x_k = \hat{x}_k$.

5.4 Method - Cutting Plane

A brutal force method will be solving a huge linear program:

$$\begin{aligned} \min\ & \langle c, x\rangle + \Sigma_{k=1}^{K} p_k \langle q_k, y_k\rangle \\ \text{subject to: } & Ax = b \\ & T_i x + W_i y_i = h_i \quad i = 1, ..., K \\ & x \geq 0, y_i \geq 0 \forall k \end{aligned}$$

But as the size of realization increase, this method will get very slow.

We can use instead the *cutting plane method*, set

$$F(x) = \langle c, x\rangle + \Sigma_{k=1}^{K} p_k Q(x, \xi^k)$$

and

$$\mathscr{D} = \{x;\ Ax = b, x \geq 0\}$$

which is a compact set. Suppose that for any $x \in \mathscr{D}$, $Q(x, \xi^k) < +\infty$, $k = 1, ..., K$, the terminology for this is *relative complete recourse*. We describe this method by illustration iterations.

For the first iteration, we can get a x^1 by solving the feasibility problem

$$\begin{aligned} \min .\ & \mathbf{1}^\top x \\ \text{subject to: } & Ax = b \\ & x \geq 0 \end{aligned}$$

This will finds us a point in $\mathscr{D}$. Now, we start regular j-th iterations, we solve the second stage problem for each scenario, $k = 1, ..., K$,

$$\begin{aligned} \min\ & \langle q_k, y\rangle \\ \text{subject to: } & T_k x^j + W_k y = h_k \Leftarrow \pi_k^j \\ & y \geq 0 \end{aligned}$$

From which we can get *Lagrangian multiplier* π_k^j, for $k = 1, ..., K$. If we want to $\min_{x\in\mathscr{D}} F(x)$, we can first get its sub-gradient by evaluating

$$g^j = -\Sigma_{k=1}^{K} p_k T_k \pi_k^j$$

By definition,

$$F(x) - F(x^j) \geq \langle g^j, x - x^j\rangle$$

Set

$$F^j(x) = \max_{1 \le i \le j} \{F(x^i) + \langle g^i, x - x^i \rangle\}$$

Then we can solve $\min_{x \in \mathscr{D}} F^j(x)$ by the following linear program:

$$\begin{aligned} &\min\ v^{j+1} \\ &\text{subject to: } v^{j+1} \ge F(x^i) + \langle g^i, x - x^i \rangle, \quad i = 1, ..., j \\ &x \in \mathscr{D} \end{aligned}$$

Then we can enter next iteration.

This algorithm will converge in finite steps. Let's observe that

$$v^j \le \min_{x \in \mathscr{D}} F(x)$$

and

$$F^{j+1} = v^{j+1} \ge v^j = F^j, \quad j = 1, 2, ...$$

Assume $F^* = \min_{x \in \mathscr{D}} F(x)$, then for arbitrary $\epsilon > 0$,

$$J_\epsilon = \{j \ : \ F(x^j) \ge F^* + \epsilon\}$$

Suppose $i, j \in J_\epsilon$ where $j > i$, then

$$\begin{aligned} F^* &\ge v^j \\ &\ge F(x^i) + \langle g^i, x^j - x^i \rangle \\ &\ge F^* + \epsilon + \langle g^i, x^j - x^i \rangle \end{aligned}$$

This gives us

$$C||x^i - x^j|| \ge \langle g^i, x^j - x^i \rangle \ge \epsilon$$

By the compactness of the interval, we have

$$\lim_{j \to \infty} F(x^j) = F^*$$

Chapter 6

Lecture 6

6.1 Cutting Plane Method (c.t.d)

Recall our *two-stage problem*:

$$\begin{aligned} \min . \ & \langle c, x \rangle + \sum_{k=1}^{K} p_k Q(x, \xi^k) \\ \text{subject to: } & x \in \mathcal{X} \end{aligned} \tag{6.1}$$

where $\mathcal{X} = \{x : Ax = b, x \geq 0\}$ and $Q(x, \xi^k)$ is the optimal value of the *second-stage problem*:

$$\begin{aligned} \min . \ & \langle q_k, y \rangle \\ \text{subject to: } & T_k x + W_k y = h_k \\ & y \geq 0 \end{aligned} \tag{6.2}$$

We assume that for each scenario k, the sub-problem has relative complete recourse, i.e., in finite number of realizations case, it is, $\forall x \in \mathcal{X}$, $Q(x, \xi^k) < +\infty$ for all k. Set $F(x) = \langle c, x \rangle + \sum_{k=1}^{K} p_k Q(x, \xi^k)$, the main idea of *cutting plane method* is the following: we don't know each point of the function $F(x)$, but evaluation of $F(x)$ at a particular point x can be obtained by solving sub-problems, furthermore, in each particular point, the sub-gradient $g(x)$ can be derived also via the calculation of the sub-problems, we run our iteration, in each iteration, we get an additional linearized version of $F(x)$, which is a cut, denote it as, $F^{(j)}$, and by solving the *master problem* in each iteration, we can get new point which serves in the linearization process and, more importantly, the updated minimum.

The key step is to use inequality to cut the set $\mathcal{X}$, i.e.,

$$F^{(j)}(x) = \max_{1 \leq i \leq j} \{F(x^i) + \langle g^i, x - x^i \rangle\} \tag{6.3}$$

That is equivalent to solve the *master problem* in j-th iteration,

$$\min_{x\in\mathcal{X}} F^{(j)}(x) \tag{6.4}$$

That is equivalent to:

$$\begin{aligned} \min.\ & v^{(j+1)} \\ \text{subject to:}\ & v^{(j+1)} \geq F(x^i) + \langle g^i, x - x^i\rangle,\ i = 1, ..., j \\ & x \in \mathcal{X} \end{aligned} \tag{6.5}$$

This will give us the minimizer and the current optimal value $(x^{(j+1)}, v^{(j+1)})$. Then we shall update the sub-gradient and current value of $F(x)$ by appealing the sub-problem and its dualization:

$$\begin{aligned} \max.\ & \langle h_k - T_k x^{(j+1)}, \pi_k^{(j+1)}\rangle \\ \text{subject to:}\ & W_k^\top \pi_k^{(j+1)} \leq q_k \end{aligned} \tag{6.6}$$

we can get $\mathbb{E}[Q(x^{(j+1)}, \xi^k)]$ and therefore $F(x^{(j+1)})$, also the sub-gradient

$$g^{(j+1)} = c + \sum_{k=1}^{K} p_k \left(-T_k^\top \pi_k^{j+1}\right)$$

Now, we involve the cut,

$$F^{(j+1)}(x) = \max_{1\leq i\leq j+1} \{F(x^i) + \langle g^i, x - x^i\rangle\} \tag{6.7}$$

The rationality behind (6.7) is the definition of the sub-gradient, i.e.,

$$F(x) \geq F(x^i) + \langle g^i, x - x^i\rangle \text{ for all } i \tag{6.8}$$

Thus, if we set the maximum of all cuts up to iteration j as v^j, it is also smaller than $F(x)$. In other words, the collection of supporting hyperplane serves as a lower bound of $F(x)$. In fact, in each iteration, we cut the current optimal solution. For example, in $j+1$ iteration,

$$F^{(j+1)}(x^{(j+2)}) = v^{(j+2)} \geq F(x^{(j+1)}) + \langle g^{(j+1)}, x - x^{(j+1)}\rangle \tag{6.9}$$

and

$$F(x^{(j+1)}) \geq v^{(j+1)} = F^j(x^{(j+1)}) \tag{6.10}$$

As long as the inequality in (6.10) is sharp, we cut off the current solution. If they are identical, it means that we achieved optimality. This method will stop in finite number of steps as proved in the last lecture.

Remember, we make an assumption that we have relative complete recourse, that means, our algorithm is valid only if $x \in \mathcal{X}$ such that $Q(x, \xi^k) < +\infty$, $k = 1, ..., K$. Otherwise, the algorithm will fail, we shall remedy by involving so called *induced-constraint*,

$$\mathcal{X}^{ind} = \{x \in \mathbb{R}^n : Q(x, \xi^k) < +\infty, \text{ for all } k = 1, ..., K\} \tag{6.11}$$

Recall the *Phase 1* of linear programming, let's apply it on the second stage problem, i.e.,

$$\begin{aligned} \min . \ & ||s||_1 \\ \text{subject to: } & T_k x + W_k y + s = h_k \\ & y_k \geq 0, \ k = 1, ..., K \end{aligned} \tag{6.12}$$

Let's denote the optimal value of (6.12) by $\Psi(x)$. We will show its convexity and calculate its sub-differentials. Attaching η to the first constraint, we have *Lagrangian*,

$$L(y, s, \eta) = ||s||_1 + \eta^\top (h - Tx - Wy - s) \tag{6.13}$$

$$= ||s||_1 + \eta^\top (h - Tx) - \eta^\top (Wy) - \eta^\top s \tag{6.14}$$

The dual function is:

$$L_D(\eta) = \min_{s,y \geq 0} L(y, s, \eta) \tag{6.15}$$

We investigate y and s separately. For the part involving y, we shall have $W^\top y \leq 0$, otherwise, $L_D(\eta) \to -\infty$. In terms of s, it is actually the conjugate function of $f(\eta) = ||s||_1$, i.e.,

$$f^*(\eta) = \sup_{s \in \mathbb{R}^n} \{\langle \eta, s \rangle - ||s||_1\} \tag{6.16}$$

We know that $||\cdots||_\infty$ is the *dual norm*[1] of $||\cdot||_1$. If $||\eta||_\infty > 1$, we can find $\bar{s}$ such that $||s||_1 = 1$ and $\langle \eta, \bar{s} \rangle > 1$. Since scalar product is *homogeneous*, the right hand side of (6.16) will go to $+\infty$. On the other hand, if $||\eta||_\infty \leq 1$, then $\langle \eta, s \rangle \leq ||s||_1$. Thus the supremum on the right hand side is 0. Now, we can obtain the *dual problem*

$$\begin{aligned} \max . \ & \eta^\top (h - Tx) \\ \text{subject to: } & W^\top \eta \leq 0, \\ & ||\eta||_\infty \leq 1 \end{aligned} \tag{6.17}$$

The sub-differential of $\Psi(x)$ can be obtained easily[2],

$$\partial \Psi(x) = -T^\top \mathscr{Z}(x)$$

where $\mathscr{Z}(x)$ is the set of the optimal solution of dual problem at x.

Let's get back to the *induced constraint*. If $x \in \mathcal{X}^{ind}$, $\Psi(x) \leq 0$, $k = 1, ..., K$ (Actually, $\Psi(x)$ should equal 0, but we do this for a purpose). If $\Psi(x^i) > 0$ for some k, then the second-stage is infeasible, we need to have a *feasible cut*. Suppose $\sigma_k^i \in \partial \Psi_k(x^i)$, then,

$$\Psi_k(x) \geq \Psi_k(x^i) + \langle \sigma_k^i, x - x^i \rangle$$

[1] Dual norm of $||g||$ is defined as: $||g||_* = \sup_{||d||=1} \langle g, d \rangle$

[2] See page. 71, "*Nonlinear Optimization*", Andrezj Ruszczyński

Thus, we can have the following inequality to ensure the feasibleness of second-stage,

$$\Psi_k(x^i) + \langle \sigma_k^i, x - x^i \rangle \leq 0 \tag{6.18}$$

where $\sigma_k^i = -T_k^\top \eta_k^i$ and η_k^i is the optimal solution of (6.17).

Thus, in the master program (6.4),

$v \geq F(x^i) + \langle g^i, x - x^i \rangle$, for all $i = 1, ..., j$ and also all sub - problems are solvable
$0 \geq \Psi_k(x^i) + \langle \sigma_k^i, x - x^i \rangle$, for those (i, k) whose sub - problem is infeasible at $x = x^i$

This method will stop in finite steps, because the polyhedral structure of the second-stage. Moreover, the convergence can be improved by using multi-cuts, which is a slight modification of the original method.

For each scenario k, we know that

$$Q(x, \xi^k) \geq Q(x^i, \xi^k) + \langle g_k^i, x - x^i \rangle \tag{6.19}$$

Thus, the *master problem*, at j-th iteration, becomes

$$\begin{aligned} \min . \ & \langle c, x \rangle + \sum_{k=1}^{K} p_k w_k \\ \text{subject to: } & w_k \geq Q(x^i, \xi^k) + \langle g_k^i, x - x^i \rangle \ k = 1, ..., K, \ i = 1, ..., j \\ & x \in \mathcal{X} \end{aligned} \tag{6.20}$$

For the first constraint, objective cuts are possible for scenario k. We now involve the feasible cut, then the *master problem* becomes:

$$\begin{aligned} \min . \ & \langle c, x \rangle + \sum_{k=1}^{K} p_k w_k \\ \text{subject to: } & w_k \geq Q(x^i, \xi^k) + \langle g_k^i, x - x^i \rangle \ k = 1, ..., K, \ i = 1, ..., j \\ & 0 \geq \Psi_k(x^i) + \langle \sigma_k^i, x - x^i \rangle, \text{ for all } i, k \\ & x \in \mathcal{X} \end{aligned} \tag{6.21}$$

The multi-cut method requires more memory, but it uses the previously-collected data more efficiently, because cuts from different scenarios can be combined in various ways.

6.2 Regularization

The principal difficulty associated with *cutting plane methods* is the growth of the number of cuts that need to be stored in the *master problem*. Also, there is no easy way to make use of a good starting solution. To mitigate these difficulties, we shall implement regularization.

6.2.1 Proximal Point

Suppose we have the following constrained optimization problem:

$$\min_{x\in\mathcal{X}} .\ f(x) \tag{6.22}$$

where f is a *convex, proper* function and

$$f(x) = \begin{cases} +\infty & \text{if } x \notin \mathcal{X} \\ f(x) & \text{otherwise} \end{cases} \tag{6.23}$$

Suppose x^* is the optimal solution. Let's introduce quadratic penalty term to (6.22), i.e.,

$$f_\rho(v) = \min_{x\in\mathcal{X}} \ \{f(x) + \frac{rho}{2}||x-v||^2\} \quad (\rho > 0) \tag{6.24}$$

Given arbitrary v, the above optimization problem will ensure that the minimizer of (6.24) will not go too far beyond v. Indeed, above formulation compromises between minimizing f and being near v, for this reason, it is called *proximal point* of v w.r.t f. Assume that $v = w^j$, since x^* is the minimizer for f, thus $f_\rho(w^j) > f(x^*)$. If $f_\rho(w^j) = f(w^j)$, then w^j is optimal.

Theorem 6.2.1 The sequence w^j is convergent to a solution $\hat{x}$ of the problem.

Proof. The optimality condition of (6.24) is:

$$-\rho(w^{j+1} - w^j) \in \partial f(w^{j+1}) \tag{6.25}$$

where $w^{j+1} = \arg\min f_\rho(w^j)$. Thus,

$$f(x^*) \geq f(w^{j+1}) - \rho\langle w^{j+1} - w^j, x^* - w^{j+1}\rangle \tag{6.26}$$

On the other hand,

$$\begin{aligned} &||w^{j+1} - x^*||^2 \\ =&||(w^{j+1} - w^j) + (w^j - x^*)||^2 \\ =&||w^j - x^*||^2 + 2\langle w^{j+1} - w^j, (w^j - w^{j+1} + w^{j+1}) - x^*\rangle + ||w^{j+1} - w^j||^2 \\ =&||w^j - x^*||^2 + w\langle w^{j+1} - w^j, w^j - w^{j+1}\rangle + 2\langle w^{j+1} - w^j, w^{j+1} - x^*\rangle + ||w^{j+1} - w^j||^2 \\ =&||w^j - x^*||^2 + 2\langle w^{j+1} - w^j, w^{j+1} - x^*\rangle - ||w^{j+1} - w^j||^2 \end{aligned} \tag{6.27}$$

Substitute (6.26) in,

$$||w^{j+1} - x^*||^2 \leq ||w^j - x^*||^2 + \frac{2}{\rho}[f(x^*) - f(w^{j+1})] \tag{6.28}$$

Several conclusions follows from this inequality. First, the sequence $\{w^k\}$ is bounded, because the distance to x^* is non-increasing. Secondly, summing up (6.28) from $k = 1$ to $k = \infty$,

$$\sum_{k=1}^{\infty} \big(f(w^k) - f(x^*)\big) \leq \frac{\rho}{2}||w^1 - x^*||^2 \tag{6.29}$$

So $f(w^k) \to f(x^*)$ as $k \to \infty$. Take $\hat{x}$ accumulation point of w^j, then $f(\hat{x}) = f(x^*)$. We choose one such $\hat{x}$, substitute it for x^* in (6.28) and conclude that the entire sequence $\{w^k\}$ is convergent to $\hat{x}$. □

6.2.2 Regularization for Cutting Plane Method (Bundle Method)

Recall in the cutting plane method, the *master problem* is the following:

$$f^{(j)}(x) = \max_{1 \leq i \leq j} \{f(x^i) + \langle g^i, x - x^i \rangle\}, \ g^i \in \partial f(x^i) \tag{6.30}$$

Suppose w^j is the best point known so far, then the *regularized master problem* is

$$\min_{x \in \mathcal{X}} f^j(x) + \frac{\rho}{2}||x - w^j||^2 \tag{6.31}$$

Suppose the minimizer is x^{j+1}, the following condition will be useful in the derivation later on,

$$f(x^{j+1}) \leq f(w^j) - \gamma[f^{(j)}(w^j) - f^{(j)}(x^{j+1})], \ \gamma \in (0, 1) \tag{6.32}$$

That is the observed improvement $f(w^j) - f(x^{(j+1)})$ is larger than the predicted improvement $\gamma[f^j(w^j) - f^j(x^{j+1})]$. Under this circumstance, we shall update $w^{j+1} = x^{j+1}$, which is called a *serious step*, otherwise, we remain $w^{k+1} = w^j$, that is *null step*. Now let's prove a similar convergent result as in the last sub-section.

Proof. From the optimality condition of (6.31),

$$\rho(w^j - w^{j+1}) \in \partial f^j(x^{j+1}) \tag{6.33}$$

Then

$$f(x^*) \geq f^{(j)}(x^*) \geq f^{(j)}(x^{j+1}) + \rho\langle w^j - x^{j+1}, x^* - x^{j+1} \rangle \tag{6.34}$$

As before,

$$||x^{j+1} - x^*||^2 = ||x^j - x^*||^2 + 2\langle x^{j+1} - x^j, x^j - x^* \rangle + ||x^{j+1} - x^j||^2 \tag{6.35}$$

$$\leq ||x^j - x^*||^2 + 2\langle x^{j+1} - x^j, x^{j+1} - x^* \rangle \tag{6.36}$$

At serious step, $w^{j+1} = x^{j+1}$, and we know (6.32), thus

$$||x^{j+1} - x^*||^2 \leq ||w^j - x^*||^2 + \frac{2}{\rho}[f(x^*) - f^{(j)}(w^{j+1})] \tag{6.37}$$

$$\leq ||w^j - x^*||^2 + \frac{2}{\rho}[f(x^*) - \frac{1}{\gamma}f(w^{j+1}) + \frac{1-\gamma}{\gamma}f(w^j)] \tag{6.38}$$

$$= ||w^j - x^*||^2 + \frac{2}{\rho}[f(x^*) - f(w^j)] + \frac{2}{\rho\gamma}[f(w^j) - f(w^{j+1})] \tag{6.39}$$

Thus,

$$||w^{j+1} - x^*||^2 \leq ||w^1 - x^*||^2 + \frac{2}{\rho}\sum_{\text{serious steps}}[f(x^*) - f(w^i)] + \frac{2}{\rho\gamma}\sum_{\text{serious step}}[f(w^i) - f(w^{i+1})]$$

$\frac{2}{\rho}\sum_{\text{serious steps}}[f(x^*) - f(w^i)]$ is less than 0, and the last term is bounded by $f(w^1) - f(x^*)$, thus $f(w^1) \to f(x^*)$. As before, $w^j \to \hat{x}$, where $\hat{x}$ is an accumulation point. Substitute $\hat{x}$ for x^* and choose $j_1 \in \mathscr{J}$ such that $||w^{j_1} - \hat{x}|| \leq \epsilon$, then

$$||w^{j+1} - x^*||^2 \leq ||w^1 - x^*||^2 + \frac{2}{\rho}\sum_{\text{serious steps after } j_1}[f(w^i) - f(w^{i+1})]$$

Both term can be made very small, so the convergence result follows. □

Chapter 7

Lecture 7-8

Multi-stage problem is a natural extension of second-stage problem, in which the uncertain data xi_1,..., ξ_T is revealed gradually over time, in T periods, and our decision should be adapted to this process, we can express by the flow chart below:

$$\text{decision}(x_1) \sim \text{observation}(\xi_2) \sim \text{decision}(x_2) \sim \cdots \text{observation}(\xi_T) \sim \text{decision}(x_T)$$

We view the sequence $\xi_t \in \mathbb{R}^{d_t}$, $t = 1, ..., T$, of data vectors as a stochastic process. We will use notation:

$$\xi_{[t]} := (\xi_1, ..., \xi_t)$$

to denote the history of the process up to time t.

The values of the decision vector x_t, chosen at stage t, may depend on the information $\xi_{[t]}$ available up to time t, but not on the results of the future observations. This is the basic requirement of *nonanticipativity*.

In a generic form a T-stage stochastic programming problem can be written in the nested formulation:

$$\min_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\int_{x_2 \in \mathcal{X}_2(x_1,\xi_2)} f_2(x_2,\xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}\left[\inf_{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)} f_T(x_T,\xi^T)\right]\right]\right] \tag{7.1}$$

driven by the random data process ξ_1, ξ_2, ..., ξ_T. Here $x_t \in \mathbb{R}^{n_t}$, $t = 1, ..., T$, are decision variables, $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \mapsto \mathbb{R}$ are continuous functions and $\mathcal{X}_t : \mathbb{R}^{n_t - 1} \times \mathbb{R}^{d_t} \Rightarrow \mathbb{R}^{n_t}$, $t = 1, ..., T$, are measurable closed valued multi-functions. The first-stage data, i.e., the vector ξ_1, the function $f_1 : \mathbb{R}^{n_1} \mapsto \mathbb{R}$, and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ are deterministic. It is said that the multi-stage problem is *linear* if the objective function $f_t(\cdot)$ and the constraints function consisting of the set $\mathcal{X}_t(\cdot, \cdot)$ are linear. In a typical formulation,

$$f_t(x_t, \xi_t) := c_t^\top x_t, \ \mathcal{X}_1 = \{x_1 : A_1 x_1 = b_1, \ x_1 \geq 0\}, \tag{7.2}$$

$$\mathcal{X}_t(x_{t-1}, \xi_t) := \{x_t : B_t x_{t-1} + A_t x_t = b_t, \ x_t \geq 0\}, \ t = 2, ..., T \tag{7.3}$$

Here, $\xi_1 := (c_1, A_1, b_1)$ is known at the first-stage, and $\xi_t := (c_t, B_t, A_t, b_t) \in \mathbb{R}^{d_t}$, $t = 2, ..., T$ are vectors, some (or all) elements of which can be random.

There are several equivalent ways to make this formulation precise. One approach is to consider decision variables $x_t = \mathbf{x}_t(\xi_{[t]})$, $t = 1, ..., T$, as functions of the data process $\xi_{[t]}$ up to time t. Such a sequence of measurable mappings $x_t : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t} \mapsto \mathbb{R}^{n_t}$, $t = 1, ..., T$, is called an *implementable policy*. An *implementable policy* is said to be *feasible* if it satisfies the feasibility constraints, i.e.,

$$\mathbf{x}_t(\xi_{[t]}) \in \mathscr{X}_t(\mathbf{x}_{t-1}(\xi_{[t-1]}), \xi_t), \ t - 2, ..., T, \ w.p.1. \tag{7.4}$$

We can formulate the multi-stage problem in the form:

$$\begin{array}{l} \min\limits_{x_1, \mathbf{x}_2, ..., \mathbf{x}_T} \quad \mathbb{E}[f_1(x_1) + f_2(\mathbf{x}_2(\xi_{[2]}, \xi_2)) + \cdots + f_T(\mathbf{x}_T(\xi_{[T]}), \xi_T)] \\ \text{subject to: } x_1 \in \mathscr{X}_1, \mathbf{x}_t(\xi_{[t]}) \in \mathscr{X}_t(\mathbf{x}_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, ..., T \end{array} \tag{7.5}$$

Another possible way is to write the corresponding *dynamic programming equation*. That is, consider the last stage problem:

$$\min_{x_T \in \mathscr{X}_T(x_{T-1}, \xi_T)} \tag{7.6}$$

The optimal value of this problem, denoted $Q_T(x_{T-1}, \xi_T)$, depends on the decision vector x_{T-1} and data ξ_T. At stage $t = 2, ..., T - 1$, we formulate the problem:

$$\begin{array}{rl} \min\limits_{x_t} & f_t(x_t, \xi_t) + \mathbb{E}\{Q_{t+1}(x_t, \xi_{[t+1]}) | \xi_{[t]}\} \\ \text{subject to:} & x_t \in \mathscr{X}_t(x_{t-1}, \xi_t) \end{array} \tag{7.7}$$

where $\mathbb{E}[\cdot|\xi_{[t]}]$ denotes conditional expectation. Its optimal value depends on the decision x_{t-1} at previous stage and realization of the data process $\xi_{[t]}$, and denoted $Q_t(x_{t-1}, \xi_{[t]})$. The idea is to calculate the *cost-to-go* functions $Q_t((x_{t-1}, \xi_{[t]})$, recursively, going backward in time. At the first stage we finally need to solve the problem:

$$\min_{x \in \mathscr{X}_1} \ f_1(x_1) + \mathbb{E}[Q_2(x_1, \xi_2)] \tag{7.8}$$

The corresponding dynamic programming equations are

$$Q_t(x_{t-1}, \xi_{[t]}) = \inf_{x_t \in \mathscr{X}_t(x_{t-1}, \xi_t)} \{f_t(x_t, \xi_t) + \mathscr{Q}_{t+1}(x_t, \xi_{[t]})\}, \tag{7.9}$$

where

$$\mathscr{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}\{Q_{t+1}(x_t, \xi_{[t+1]}) | \xi_{[t]}\}.$$

An implementable policy $\mathbf{x}_t^*(\xi_{[t]})$ is *optimal* if and only if for $t = 1, ..., T$,

$$\mathbf{x}_t^*(\xi_{[t]}) \in \operatorname*{arg\,min}_{x_t \in \mathscr{X}_t(\mathbf{x}_{t-1}^*(\xi_{[t-1]}), \xi_t)} \{f_t(x_t, \xi_t) + \mathscr{Q}_{t+1}(x_t, \xi_{[t]})\}, \ w.p.1$$

where for $t = T$ the term $\mathscr{X}_{T+1}$ is omitted and for $t = 1$ the set $\mathscr{X}_1$ only depends on ξ_1.

If the process $\xi_1, ..., \xi_T$ is *Markovian*, then conditional distributions in the above equations, given $\xi_{[t]}$, are the same as the respective conditional distributions given ξ_t and we can write it as $Q_t(x_{t-1}, \xi_t)$.

Let's check out an examples see how those definition features in the real world problem.

7.1 Inventory Model

Consider now the situation when the manufacturer has a planning horizon of T periods. The demand is modelled as a stochastic process D_t, $t = 1, .., T$, where each $D_t = (D_t^1, D_t^2, ..., D_t^n)$ is a random vector of demands for the products at time t. The unused parts can be stored from one period to the next, and holding one unit of part j in inventory cost h_j. For simplicity, we assume that all costs and prices are the same in all periods.

It would not be reasonable to plan specific order quantities for the entire planning horizon T. Instead, one has to make orders and production decisions at successive stages, depending on the information available at the current stage. We use symbol $D_{[t]} := (D_1, ..., D_t)$ to denote the history of the demand process in periods $1, ..., t$.

Let us denote by $x_{t-1} = (x_{t-1}^1, ..., x_{t-1}^n)$ the vector of quantities ordered at the beginning of stage t, before the demand vector D_t becomes known. The numbers of units produced in stage t will be denoted by z_t and the inventory level of parts at the end of the stage t by y_t for $t = 1, ..., T$. We use the subscript $t - 1$ for the order quantity to stress that it may depend on the past demand realizations $D_{[t-1]}$ but not on D_t, while production and storage variables at stage t may depend on $D_[t]$, which includes D_t.

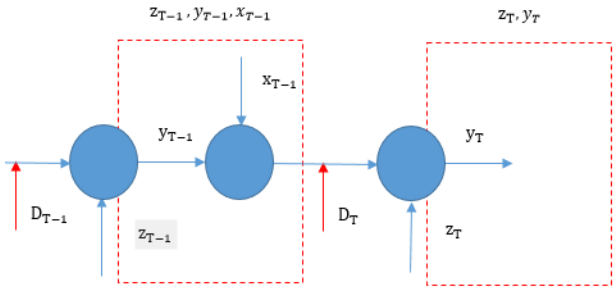

Figure 7.1: Multi-Stage Problem - Final Stage T

Suppose $T > 1$ and consider the last stage $t = T$, after the demand D_T has been observed. At this time, all inventory levels y_{T-1} of the parts, as well as the last order quantities x_{T-1} are known. The problem at stage T is therefore identical to the second stage problem of the

two-stage formulation:

$$\begin{aligned} \min_{z_T, y_T} \quad & (l-q)^\top z_T - s^\top y_T \\ \text{subject to: } & y_T = y_{T-1} + x_{T-1} - A^\top z_T, \\ & 0 \le z_T \le d_T,\ y_T \ge 0 \end{aligned} \tag{7.10}$$

where l is the price to satisfy a unit of demand (maybe transportation cost), q is the selling price and s is the salvage value, the number of parts left in inventory is y and number of product produced by z. Note, d_T here is the observed realization of D_T. Denote by $Q_T(x_{T-1}, y_{T-1}, d_T)$ the optimal value of (7.10). This optimal value depends on the latest inventory levels (y_{T-1}), order quantities (z_{T-1}) and the present demand (d_T).

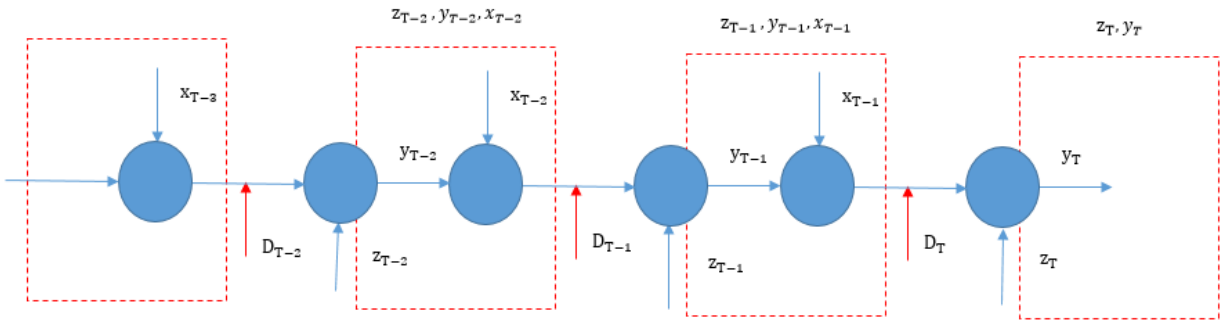

Figure 7.2: Multi-Stage Problem - $(T-1)$-Stage

At stage $T-1$, we know realization $d_{[T-1]}$ of $D_{[T-1]}$ and thus we are concerned with the conditional expectation of the last stage cost, that is the function

$$\mathcal{Q}(x_{T-1}, y_{T-1}, d_{[T-1]}) := \mathbb{E}\{Q_T(x_{T-1}, y_{T-1}, d_T) | D_{[T-1]} = d_{[T-1]}\} \tag{7.11}$$

At stage $T-1$, we solve the following problem:

$$\begin{aligned} \min_{z_{T-1}, y_{T-1}, x_{T-1}} \quad & (l-q)^\top z_{T-1} + h^\top y_{T-1} + c^\top x_{T-1} + \mathcal{Q}_T(x_{T-1}, y_{T-1}, d_{[T-1]}) \\ \text{subject to: } & y_{T-1} = y_{T-2} + x_{T-2} - A^\top z_{T-1}, \\ & 0 \le z_{T-1} \le d_{T-1},\ y_{T-1} \ge 0 \end{aligned} \tag{7.12}$$

Its optimal value is denoted by $Q_{T-1}(x_{T-2}, y_{T-2}, d_{[T-1]})$. Generally, the problem at stage $t = T-1, ..., 1$ has the form:

$$\begin{aligned} \min_{z_t, y_t, x_t} \quad & (l-q)^\top z_t + h^\top y_t + c^\top x_t + \mathcal{Q}_{t+1}(x_t, y_t, d_{[t]}) \\ \text{subject to: } & y_t = y_{t-1} + x_{t-1} - A^\top z_t, \\ & 0 \le z_t \le d_t,\ y_t \ge 0 \end{aligned} \tag{7.13}$$

with

$$\mathcal{Q}_{t+1}(x_t, y_t d_{[t]}) := \mathbb{E}[Q_{t+1}(x_t, y_t, D_{[t+1]}) | D_{[t]} = d_{[t]}] \tag{7.14}$$

The optimal value of problem (7.13) is denoted by $Q_t(x_{t-1}, y_{t-1}, d_{[t]})$, and the backward recursion continues. At stage $t = 1$, the symbol y_0 represents the initial inventory level of the parts, and the optimal value function $Q_1(x_0, d_1)$ depends only on the initial order x_0 and realization d_1 of the first demand D_1.

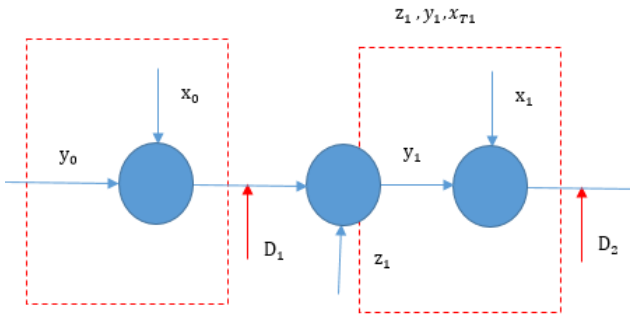

Figure 7.3: Multi-Stage Problem - Stage 1

The initial problem is to determine the first order quantities x_0. It can be written as

$$\min_{x_0 \geq 0} \quad c^\top x_0 + \mathbb{E}[Q_t(x_0, D_1)] \tag{7.15}$$

Although the first-stage problem (7.15) looks similar to the first-stage problem of the two-stage formulation, it is essentially different since the function $Q_1(x_0, d)$ is not given in a computationally accessible form but in itself is a result of recursive optimization.

7.2 The Linear case

As in the above example, we have linear model for each period, such *linear multi-stage model* is ubiquitous. Thus, from now on, we can focus our attention on the linear case. Consider the following linear programming problem:

$$\begin{aligned} \min \quad & c_1^\top x_1 + c_2^\top x_2 + c_3^\top x_3 + \cdots + c_T^\top x_T \\ \text{subject to:} \quad & A_1 x_1 = b_1, \\ & B_2 x_1 + A_2 x_2 = b_2, \\ & B_3 x_2 + A_3 x_3 = b_3, \\ & \cdots\cdots\cdots\cdots\cdots \\ & B_T x_{T-1} + A_T x_T = b_T, \\ & x_t \geq 0, \ t = 1, ..., T \end{aligned} \tag{7.16}$$

We can view this problem as a multi-period stochastic programming problem where c_1, A_1 and b_1 are known, but some (or all) the entries of the cost vectors c_t, matrices B_t and A_t, and

right-hand-side vectors b_t, $t = 2, ..., T$, are random. In the multi-stage setting, the values of the random data become known in the respective time period, and we have following sequence of actions:

$$\begin{gathered}\text{decision}(x_1)\\ \text{observation } \xi_2 := (c_2, B_2, A_2, b_2)\\ \text{decision}(x_2)\\ \cdots\cdots\\ \text{observation } \xi_T := (c_T, B_T, A_T, b_T)\\ \text{decision}(x_T)\end{gathered}$$

Our objective is to design the decision process in such a way that the expected value of the total cost is minimized while optimal decisions are allowed to be made at every time period $t = 1, ..., T$. As we already discussed, there are generally two ways of formulation. In one of such formulation $x_t = \mathbf{x}_t(\xi_{[t]})$, $t = 2, ..., T$, is viewed as a function of $\xi_{[t]}$, and the minimization is performed over appropriate functional spaces of such functions. If the number of scenarios is finite, this leads to a formulation of the linear multi-stage stochastic program as one large (deterministic) linear programming problem. We will discuss this later. Another possibility is to write dynamic programming equations.

Let's look at our problem from the perspective of the last stage T. At that time the values of all problem data, $\xi_{[T]}$, are already known, and the values of the earlier decision vectors $x_1, ..., x_{T-1}$, have been chosen. Our problem is, therefore, a simple linear programming problem

$$\begin{aligned}\min_{x_T}\ & c_T^\top x_T\\ \text{subject to:}\ & B_T x_{T-1} + A_T x_T = b_T\\ & x_T \geq 0\end{aligned} \tag{7.17}$$

The optimal value of this problem depends on the earlier decision $X_{T-1} \in \mathbb{R}^{n_{T-1}}$ and data $\xi_T = (c_T, B_T, A_T, b_T)$ and is denoted by $Q_T(X_{T-1}, \xi_T)$. At stage $T-1$ we know x_{T-2} and $\xi_{[T-1]}$. We face, therefore, the following two-stage stochastic programming problem:

$$\begin{aligned}\min_{x_T}\ & c_{T-1}^\top x_{T-1} + \mathbb{E}[Q_T(x_{T-1}, \xi_T)|\xi_{T-1}]\\ \text{subject to:}\ & B_{T-1} x_{T-2} + A_{T-1} x_{T-1} = b_{T-1}\\ & x_{T-1} \geq 0\end{aligned} \tag{7.18}$$

The optimal value of the above problem depends on $x_{T-2} \in \mathbb{R}^{n_{T-2}}$ and data $\xi_{[T-1]}$ and is denoted $Q_T(x_{T-2}, \xi_{[T-1]})$.

Generally, at stage $t = 2, ..., T-1$, we have the problem:

$$\begin{aligned}\min_{x_T}\ & c_t^\top x_t + \mathbb{E}[Q_{t+1}(x_t, \xi_{[t+1]})|\xi_t]\\ \text{subject to:}\ & B_t x_{t-1} + A_t x_t = b_t\\ & x_t \geq 0\end{aligned} \tag{7.19}$$

Its optimal value, called *cost-to-go* function, is denoted $Q_t(x_{t-1}, \xi_{[t]})$. On the top of all these problems is the problem to find the first decision, $x_1 \in \mathbb{R}^{n_1}$,

$$\begin{aligned} \min_{x_T} \; & c_1^\top x_1 + \mathbb{E}[Q_2(x_1, \xi_2)] \\ \text{subject to:} \; & A_1 x_1 = b_1 \\ & x_1 \geq 0 \end{aligned} \tag{7.20}$$

Note that all subsequent stage $t = 2, ..., T$ are absorbed in the above problem into the function $Q_2(x_1, \xi_2)$ through the corresponding expected values. Note also that since ξ_1 is not random, the optimal value $Q_2(x_1, \xi_2)$ does not depend on ξ_1.

The dynamic programming equations here take the form:

$$Q_t(x_{t-1}, \xi_{[t]}) = \inf_{x_t} \{c_t^\top x_t + \mathscr{Q}_{t+1}(x_t, \xi_{[t]}) : B_t x_{t-1} + A_t x_t = b_t, \; x_t \geq 0\} \tag{7.21}$$

where

$$\mathscr{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}\{Q_{t+1}(x_t, \xi_{[t-1]}) | \xi_{[t]}\} \tag{7.22}$$

Also an *implementable policy* $\mathbf{x}_t^*(\xi_{[t]})$ is *optimal* if for $t = 1, ..., T$, the condition

$$\mathbf{x}_t^*(\xi_{[t]}) \in \arg\min_{x_t} \{x_t^\top x_t + \mathscr{Q}_{t+1}(x_t, \xi_{[t]}) : A_t x_t = b_t - B_t \mathbf{x}_t^*(\xi_{[t-1]}), x_t \geq 0\}$$

holds for almost every realization of the random process. If the process ξ_t is *Markovian*, then each cost-to-go function depends on ξ_t rather than $\xi_{[t]}$, and we can simply write $Q_t(x_{t-1}, \xi_t)$, $t = 2, ..., T$. If, moreover, the stagewise independence condition holds, then each expectation function $\mathscr{Q}_t$ does not depend on realizations of the random process, and we can write it as $\mathscr{Q}_t(x_{t-1})$, $t = 2, ..., T$.

Someone may ask what's the difference between dynamic programming and multi-stage stochastic programming. In dynamic programming (discrete), we always have *Markov structure*, but in multi-stage model, it can be *path-dependent*. However, in terms of the value function, $Q_t(X_{t-1}, \xi_{[t]})$, it is always a *convex function* of x. This follows from the following result:

Lemma 7.2.1 If we have

$$\begin{aligned} F(b) = \min \; & f(x) \\ \text{subject to:} \; & g(x) \leq b \\ & x \in \mathscr{X} \end{aligned}$$

where $f(\cdot)$ and $g(\cdot)$ are both convex and $\mathscr{X}$ is convex and closed. Then $F(b)$ is a convex function.

7.3 Scenario Tree

In finite realizations case, we can actually visualize the process as a scenario tree. With out loss of generality, let's set $T = 3$, Black ξ is the random variable, red x is the decision

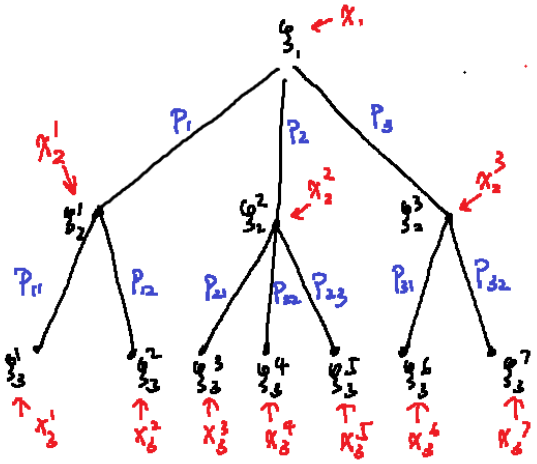

Figure 7.4: Scenario Tree $T = 3$

variable in different scenario, blue p reflects the distribution of probability measure. We can decompose the above graph scenario by scenario: Observe that we can now think of each

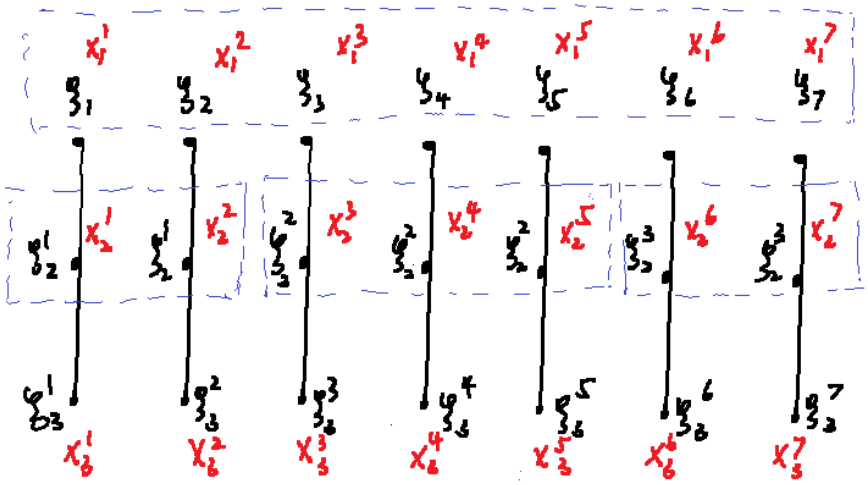

Figure 7.5: Scenario Tree - Enumeration

scenario as a path to the leaves node. Note those in the same blue dashed box means that they are in fact the same realization up to that stage. How can this helps us in numerically solving this problem ?

Suppose that in problem (7.16) there are K different scenarios. Each scenario can be considered as a path of the respective scenario tree. With each scenario, numbered k, is associated probability p_k and the corresponding sequence of decisions $\mathbf{x}^k = (x_1^k, ..., x_T^k)$. That is, with each possible scenario $k = 1, ..., K$, we can associate a sequence of decisions $\mathbf{x}^k$. Of course, it would not be appropriate to try to find the optimal values of these decision by solving the *relaxed version* of (7.16):

$$\begin{aligned} \min \ & \sum_{k=1}^{K} p_k [c_1^\top x_1^k + (c_2^k)^\top x_2^k + (c_3^k)^\top x_3^k + \cdots + (c_T^k)^\top x_T^k] \\ \text{subject to :} \ & A_1 x_1^k = b_1, \\ & B_2^k x_1^k + A_2^k x_2^k = b_2^k, \\ & \cdots\cdots \\ & B_T^k x_{T-1}^k + A_T^k x_T^k = b_T^k, \\ & x_i^k \geq 0, \ i = 1, ..., T, \ k = 1, ..., K \end{aligned} \tag{7.23}$$

The reason is the same as in the two-stage case. That is, in problem (7.23) all parts of the decision vector are allowed to depend on all parts of the random data, while each part x_t should be allowed to depend only on the data known up to stage t. In particular problem (7.23) may suggest different values of x_1, one for each scenario k, while our first-stage decision should be independent of possible realizations of the data process. In order to correct this issue we enforce the constraints:

$$x_1^k = x_1^l, \ \forall k, l \in \{1, , , K\} \tag{7.24}$$

But this is not sufficient, in general. Consider the second part of the decision vector x_2, it should be allowed to depend only on $\xi_{[2]} = (\xi_1, \xi_2)$, so it has to have the same value for all scenarios k for which $\xi_{[2]}^k$ are identical. We must, therefore, enforce the constraints:

$$x_2^k = x_2^l, \text{ for which } \xi_{[2]}^k = \xi_{[2]}^l \tag{7.25}$$

In general, at stage $t = 1, ..., T$, the scenarios that have the same history $\xi_{[t]}$ cannot be distinguished, so we need to enforce the *non-anticipativity constraints*:

$$x_t^k = x_t^l, \text{ if } l \in \mathscr{G}(k, t) \tag{7.26}$$

where

$$\mathscr{G}(k, t) = \{l : \xi_{[t]}^{(k)} = \xi_{[t]}^{(l)}\} \tag{7.27}$$

Problem (7.23) together with *nonanticipativity constraints* (7.26) becomes equivalent to the original formulation. As we did in the second-stage problem, we can also rewrite it as:

$$x_t^{(k)} = \frac{\sum_{l \in \mathscr{G}(k,t)} p_l x_t^{(l)}}{\sum_{l \in \mathscr{G}(k,t)} p_l}, \ k = 1, ..., K, \ t = 1, ..., T-1 \tag{7.28}$$

If we equip it with *discrete time filtration* $\mathscr{F}_t$ the σ-algebra of events observed up to time t, then *non-anticipative constraints* says that x_t is $\mathscr{F}_t$-measurable, that is

$$\{x_t \in \mathscr{S}\} \in \mathscr{F}_t, \ \forall \text{ Borel set } \mathscr{S} \text{ of } \mathbb{R}^n \tag{7.29}$$

Of course, an alternative way is to state

$$x_t = \mathbb{E}[x_t \mid \mathscr{F}_t], \ t = 1, ..., T \tag{7.30}$$

We have a probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$, where $\Omega = \{1, ..., K\}$. Then, Here $x : \Omega \mapsto \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (in total T times) and we have for different scenario different process x,

$$x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, ..., x_T^{(k)}\right) \tag{7.31}$$

We can define the linear operator $\mathcal{P} : \mathbb{L}_2 \mapsto \mathbb{L}_2$, such that

$$(\mathcal{P}x)_t = \mathbb{E}[x_t \mid \mathscr{F}_t], \quad t = 1, ..., T \tag{7.32}$$

We can define the inner product:

$$\langle x, y \rangle_* = \mathbb{E} \sum_{t=1}^{T} \langle x_t, y_t \rangle \tag{7.33}$$

We can claim that $\mathcal{P}$ is a *projection* on $\mathscr{L} = \{x : x = \mathcal{P}x\}$ with inner product $\langle \cdot, \cdot \rangle_*$. Because it is easy to check that $\mathcal{P}(\mathcal{P}x) = \mathcal{P}x$, moreover, $\mathcal{P}$ is *self-adjoint*, i.e., $\mathcal{P}^* = \mathcal{P}$.

Proof.

$$\begin{aligned}
\langle \mathcal{P}x, y \rangle_* &= \mathbb{E} \sum_{t=1}^{T} \langle (\mathcal{P}x)_t, y_t \rangle \\
&= \mathbb{E}\{\sum_{t=1}^{T} \langle \mathbb{E}[x_t \mid \mathscr{F}_t], y_t \rangle\} \\
&= \mathbb{E}\{\sum_{t=1}^{T} \mathbb{E}[\langle \mathbb{E}[x_t \mid \mathscr{F}_t], y_t \rangle | \mathscr{F}_t]\} \\
&= \mathbb{E}\{\sum_{t=1}^{T} \langle \mathbb{E}[x_t | \mathscr{F}_t], \mathbb{E}[y_t | \mathscr{F}_t] \rangle\} \\
&= \langle \mathcal{P}x, \mathcal{P}y \rangle_*
\end{aligned}$$

Thus, we can establish

$$\langle \mathcal{P}x, y \rangle = \langle \mathcal{P}x, \mathcal{P}y \rangle = \langle x, \mathcal{P}y \rangle \Rightarrow \mathcal{P} = \mathcal{P}^* \tag{7.34}$$

□

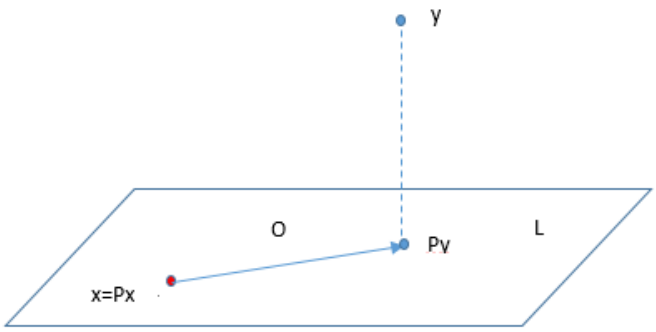

Figure 7.6: Conditional Expectation Projection

Thus, we also have $\langle x, y - \mathcal{P}y\rangle = \langle \mathcal{P}x, y - \mathcal{P}y\rangle = 0$ for $x \in \mathscr{L}$ (See Figure 7.6).

Let's recap the optimization problem:

$$\begin{aligned} \min . \ & \mathbb{E}\sum_{t=1}^{T}\langle c_t, x_t\rangle = \langle c, x\rangle_* \\ \text{subject to: } & x \in \mathcal{X} \text{ (those are all constraints describing the dynamics, } x^k \in \mathcal{X}^{(k)}) \\ & x = \mathcal{P}x \end{aligned} \tag{7.35}$$

Attaching λ to those *nonanticipative constraint*, then the *Lagrangian* is:

$$L(x, \lambda)_{x\in\mathcal{X}} = \langle c, x\rangle_* + \langle \lambda, x - \mathcal{P}x\rangle_* \tag{7.36}$$

If we have the optimal $\hat{\lambda}$, then the solution of (7.35) is the solution of the following optimization problem:

$$\min_{x\in\mathcal{X}} L(x, \hat{\lambda}) \tag{7.37}$$

Observe that if $\hat{\lambda}$ works, then $\hat{\lambda} - \mathcal{P}\hat{\lambda}$ also works, because of orthogonality, i.e.,

$$\langle \mathcal{P}\lambda, x - \mathcal{P}x\rangle_* = 0 \tag{7.38}$$

So we have freedom to choose $\mathcal{P}\hat{\lambda} = 0$. Therefore, the *Lagrangian* can be simplified:

$$L(x, \lambda)_{\{\mathcal{P}\lambda=0, x\in\mathcal{X}\}} = \langle c, x\rangle_* + \langle \lambda, x\rangle_* \tag{7.39}$$

Finally, minimizing $\langle c, x\rangle_* + \langle \hat{\lambda}, x\rangle_*$ with $x \in \mathcal{X}$ is equivalent to the original formulation. This implies, for every scenario l, we can solve

$$\min . \{\sum_{t=1}^{T} \langle c_t^{(k)}, x_t^{(k)}\rangle + \langle \hat{\lambda}_t^{(k)}, x_t^{(k)}\rangle\} \tag{7.40}$$

$$x^{(k)} \in \mathcal{X}^{(k)} \tag{7.41}$$

Remark 7.3.1 Above derivation only works for the finite scenarios case, however, in the infinite dimensional space, the existence of *Lagrangian Multipliers* are not always guaranteed. Suppose we have the only feasible constraint $\mathbf{x} \geq 0$, then we can define the set

$$K = \{x \in \mathcal{X} : \mathbf{x} \geq 0\} \tag{7.42}$$

To have *Lagrangian Multipliers*, we need to have an interior point of K to satisfy the *Qualification Constraints*. If we have *int* $K \neq \emptyset$, for $\mathcal{X} = \mathscr{L}_\infty$, one can prove that there exists $\lambda^* \in (\mathscr{L}_\infty)^*$ such that

$$\langle \lambda, x\rangle_* = \int_\Omega x(\omega)\lambda(\omega)\mathbb{P}(d\omega) \tag{7.43}$$

Chapter 8

Lecture 9 - Lecture 10

We have probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thinking of Z_x as a random outcome of cost depend on x, where x is a random variable, what we have so far is always the model based on *average performance*, i.e.,

$$\min_x \quad \mathbb{E}[Z_x]$$

The question arises that is this really a 'good' criterion or 'universal' criterion. Consider that if you run a insurance company, then the probability of certain customer having an accident is very low, they you can calculate the average relying on *Law of Large number*, but what if you are just an individual, then having a car accident is really a big thing for you ! So, your criterion is not the average but some other preferential mechanism.

One way of resolving this issue is discovered by economist, they put forward the *utility model*, then instead of minimizing the expectation, we minimize the expectation of the utility function, which depends on different people of different attitude towards risk:

$$\min_x \mathbb{E}[u(Z_x)]$$

Note that $u : \mathbb{R} \mapsto \mathbb{R}$ is non-decreasing and convex (if it is convex, then u is actually *dis-utility function*, but never minds, this is the problem for economist, not mathematician!).

Another popular approach invented by people in financial industry is the *mean-variance risk model*. We have the expectation of the cost, $\mathbb{E}[Z_x]$, and also, variability, $var[Z_x]$ (or in the finance domain, it is usually $\sigma[Z_x]$ to indicate the volatility). Now the criterion is the following:

$$\begin{aligned} &\min \ \sigma[Z_x] \\ &\text{subject to} : \mathbb{E}[Z_x] = m \end{aligned}$$

This means, you want to make certain profit but also control the risk. Or, sometimes, this can be rewritten as,

$$\min \mathbb{E}[Z_x] + \kappa\sigma[Z_x]$$

However, the controversial part of this model is it does not have *monotonicity*, which we pretty much want to have If you didn't choose κ carefully, then it violates the monotonicity

Table 8.1: Example – No Monotonicity

	0.5	0.5	$\mathbb{E}[Z_x]$	$\sigma[Z_x]$
Z	1	-1	0	1
V	1	2	1	0.5

!

8.1 Introduction to the Risk Measure

If this is the way people create preference model, then everybody can come up with their own model. That is fine, but at least, we should have certain agreement on the properties of those preference model, this motivates the birth of *risk measure*.

Definition 8.1.1 $\mathscr{Z}$ is the vector space of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, a functional $\rho : \mathscr{Z} \mapsto \mathbb{R}$ is a *convex risk measure* if it satisfies the following properties:

- **Convexity** for all $Z, V \in \mathscr{Z}$ and all $0 \leq \lambda \leq 1$,

$$\rho(\lambda Z + (1-\lambda)V) \leq \lambda\rho(Z) + (1-\lambda)\rho(V)$$

- **Monotonicity** for all $Z, V \in \mathscr{Z}$,

$$Z \leq V a.s., \Rightarrow \rho(Z) \leq \rho(V)$$

- **Translation Invariance** for all $z \in \mathscr{Z}$, for all constant $c \in \mathbb{R}$,

$$\rho(Z + c\mathbf{1}) = \rho(Z) + c$$

If additionally, we have **Positive homogeneity**, i.e., for all $z \in \mathscr{Z}$ and $\gamma \geq 0$,

$$\rho(\gamma Z) = \gamma\rho(Z)$$

we call it *coherent risk measure*.

8.2 Example of Coherent Risk Measure

The simplest example is the expectation model, which is indeed coherent risk measure. So we can relieve now, because all we did before is correct. The reader can verify by yourself. But this is just one of many coherent risk measures,

Example 8.2.1 The following risk measure is coherent risk measure:

$$\rho(Z) = \mathbb{E}[Z] + \mathbb{E}[(Z - \mathbb{E}[Z])_+]$$

To prove the convexity, let's observe that $Z \mapsto Z - \mathbb{E}[Z]$ is linear in Z, and also $t \mapsto (t)_+$ is convex on $\mathbb{R}$, we only need to show that for $V : \Omega \mapsto \mathbb{R}$,

$$F(v) = \int \psi(V(\omega))\mathbb{P}(d\omega)$$

is convex in V, given ψ is convex. It is indeed, because if $Z, W \in \mathscr{Z}$, then

$$\begin{aligned} F(\lambda V + (1-\lambda)W) &= \int \psi\left(\lambda V + (1-\lambda)W\right)\mathbb{P}(d\omega) \\ &\leq \int \left(\lambda\psi(V(\omega)) + (1-\lambda)\psi(W(\omega))\right)\mathbb{P}(d\omega) \end{aligned}$$

For monotonicity,

$$\begin{aligned} \rho(Z) &= \mathbb{E}[\mathbb{E}[Z] + \max(0, Z - \mathbb{E}[Z])] \\ &= \mathbb{E}[\max(\mathbb{E}[Z], Z)] \end{aligned}$$

similarly,

$$\rho(V) = \mathbb{E}[\max(\mathbb{E}[V], V)]$$

Monotonicity follows automatically. Translation property is also easy,

$$\begin{aligned} \rho(Z + c\mathbf{1}) &= \mathbb{E}[Z + c\mathbf{1}] + \mathbb{E}[(Z + c\mathbf{1} - \mathbb{E}[Z + c\mathbf{1}])_+] \\ &= \mathbb{E}[Z] + c + \mathbb{E}[(Z - \mathbb{E}(Z))_+] \end{aligned}$$

Lastly, the positive homogeneity,

$$\begin{aligned} \rho(\gamma Z) &= \mathbb{E}[\gamma Z] + \mathbb{E}[(\gamma Z - \mathbb{E}[\gamma Z])_+] \\ &= \gamma\mathbb{E}[Z] + \gamma\mathbb{E}[(Z - \mathbb{E}[Z])_+] \end{aligned}$$

Note, this can only be done for $\gamma \geq 0$, otherwise, it is impossible to take γ out of $(\cdot)_+$.

Example 8.2.2 By the virtue of the above example, we have a family of coherent risk measure of the following form:

$$\rho_\kappa(Z) = \mathbb{E}[Z] + \kappa\mathbb{E}[(Z - \mathbb{E}[Z])_+]$$

provided that $0 \leq \kappa \leq 1$, because

$$\rho_\kappa(Z) = (1-\kappa)\mathbb{E}[Z] + \kappa\rho_1(Z)$$

Example 8.2.3 We can generalize the $\rho_1(\cdot)$ further by putting p-norm, instead of just 1-norm, that is:

$$\rho(Z) = \mathbb{E}[Z] + ||(Z - \mathbb{E}[Z])_+||_p$$

For the convexity, we only need to show $V \mapsto ||V_+||_p$ is convex,

$$F(\lambda V + (1-\lambda)W) = ||(\lambda V + (1-\lambda)W)_+||_p$$

Then,

$$\begin{aligned} 0 \leq (\lambda V + (1-\lambda)W)_+ &\leq \lambda V_+ + (1-\lambda)W_+ \\ ||(\lambda V + (1-\lambda)W)_+||_p &\leq ||\lambda V_+ + (1-\lambda)W_+||_p \end{aligned}$$

By triangle in equality of the norm we achieve our goal, then

$$\rho(Z) = \mathbb{E}[Z] + F(AZ)$$

where $AZ = Z - \mathbb{E}[Z]$. Now for monotonicity,

$$\begin{aligned} Z - \mathbb{E}[Z] &= Z - V + V - \mathbb{E}(V) + \mathbb{E}[V] - \mathbb{E}[Z] \\ (Z - \mathbb{E}[Z])_+ &= (Z - V)_+ + (V - \mathbb{E}(V))_+ + (\mathbb{E}[V] - \mathbb{E}[Z])_+ \end{aligned}$$

Since $Z \leq V$ a.s., we have

$$(Z - \mathbb{E}[Z])_+ \leq (V - \mathbb{E}[V])_+ + \mathbb{E}[V] - \mathbb{E}[Z]$$

Now,

$$\begin{aligned} ||(Z - \mathbb{E}[Z])_+||_p &\leq ||(V - \mathbb{E}[V])_+||_p + ||\mathbb{E}[V] - \mathbb{E}[Z]||_p \\ &= ||(V - \mathbb{E}[V])_+||_p + \mathbb{E}[V] - \mathbb{E}[Z] \end{aligned}$$

Monotonicity is proved, the rest of the properties are easy !

Example 8.2.4 By the same reasoning we will again have a family of coherent risk measure, for $\kappa \in (0, 1)$,

$$\rho_{\kappa'}[Z] = \mathbb{E}[Z] + \kappa||(Z - \mathbb{E}[Z])_+||_p$$

Example 8.2.5 Let's check out one more coherent risk measure:

$$\rho(Z) = \min_{\eta}\{\eta + \frac{1}{\alpha}||(Z-\eta)_+||_p\} \quad \alpha \in (0,1)$$

First of all, the convexity. Define the function:

$$\Psi(Z, \eta) = \eta + \frac{1}{\alpha}||(Z-\eta)_+||_p$$

is convex in (Z, η), thus from convex analysis, we know that taking minimum with respect to one of the variable, η, it will be a convex function of another variable. Monotonicity follows from the monotonicity of the norm, when the input is non-negative. How about translation invariance,

$$\begin{aligned}\rho(Z+c) &= \min_{\eta\in\mathbb{R}}\{\eta + \frac{1}{\alpha}||(Z+c-\eta)_+||_p\} \quad \text{set } t = \eta - c \\ &= \min_{t}\{t + c + \frac{1}{\alpha}||(Z-t)_+||_p\} \\ &= \min_{t}\{t + \frac{1}{\alpha}||(Z-t)_+||_p\} + c \\ &= \rho(Z) + c\end{aligned}$$

Next thing is homogeneity, for the case $\gamma = 0$,

$$\rho(0) = \min_{\eta}\{\eta + \frac{1}{\alpha}||(-\eta)_+||_p\}$$

draw the picture you will find out it is a convex function that attain its minimum at 0. For $\gamma > 0$, set $\eta = \gamma t$

$$\begin{aligned}\rho(\gamma Z) &= \min_{t\in\mathbb{R}}\{\gamma t + \frac{1}{\alpha}||(\gamma Z - \gamma t)_+||_p\} \\ &= \gamma \min_{t\in\mathbb{R}}\{t + \frac{1}{\alpha}||(Z-t)_+||_p\}\end{aligned}$$

Remark 8.2.6 For $p = 1$, it is the *average value at risk at level* α, which we will learn soon.

8.3 Value-at-Risk and Average Value-at-Risk

Let's give the definition of value at risk,

$$VaR_\alpha[Z] = \min_{\eta}\{F_Z(\eta) \geq 1-\alpha\} = F_Z(\eta)^{-1}(1-\alpha)$$

It is widely adopted in finance industry, considered as the capital adequacy criterion. Because what it says is that the probability of the event that loss is larger than $VaR_\alpha[Z]$ is less than a threshold α (See Figure 8.1). It has monotonicity, but it does not have convexity, as we can easily give a counter-example. To have a coherent risk measure we can define the so called *average-value-at-risk* via *value-at-risk*, that is

$$\text{AVaR}_\alpha(Z) = \frac{1}{\alpha}\int_0^\alpha VaR_\beta(Z)d\beta = \frac{1}{\alpha}\int_0^\alpha F_Z^{-1}(1-\beta)d\beta \tag{8.1}$$

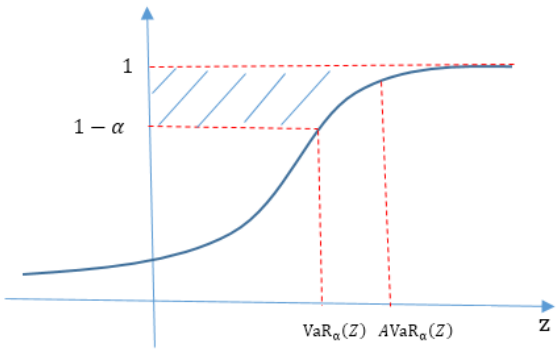

Figure 8.1: Value-at-Risk and Average Value-at-Risk

The integral represents the shaded area, thus the AVaR is certain mean value, as we can easily observe that $\text{AVaR}_\alpha(Z) \geq \text{VaR}_\alpha(Z)$.

The representation 8.1 is not good for optimization purpose, fortunately, we have a very nice transform that is illustrated by following theorem:

Theorem 8.3.1 For all $Z \in \mathscr{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ and all $\alpha \in (0, 1)$,

$$\text{AVaR}_\alpha(Z) = \min_{\eta \in \mathbb{R}} \{\eta + \frac{1}{\alpha}\mathbb{E}[(Z - \eta)_+]\} \tag{8.2}$$

We first prove this theorem in the case of (1) Z has a density function, and (2) Z has a discrete distribution. Later on, we will show that it is true for almost all distributions.

Simplification 1: Z has probability density $f(z) = \frac{dF_Z(z)}{dz}$;

Proof. We know that

$$G(\eta) = \eta + \frac{1}{\alpha}\int_{-\infty}^{\infty} \max\{0, Z - \eta\} f(z)dz = \eta + \frac{1}{\alpha}\int_{\eta}^{+\infty} (Z - \eta) f(z)dz \tag{8.3}$$

Differentiate with respect to η [1],

$$\frac{dG(\eta)}{d\eta} = 1 + \frac{1}{\alpha}\int_{\eta}^{+\infty} -f(z)dz = 1 - \frac{1}{\alpha}\mathbb{P}(Z \geq \eta) \tag{8.4}$$

[1]The validity of the calculation is ensured by *Leibniz Integral Rule*

Set it to be zero, then we get one of the optimizer is $\hat{\eta} = \text{VaR}_\alpha(Z)$, plug this optimizer back into $\min_\eta G(\eta)$, we have

$$\min_{\eta\in\mathbb{R}} G(\eta) = \hat{\eta} + \frac{1}{\alpha}\int_{\hat{\eta}}^{+\infty} (Z-\hat{\eta})\, f(z)dz = \hat{\eta} + \frac{1}{\alpha}\int_{\hat{\eta}}^{+\infty} zf(z)dz - \frac{1}{\alpha}\int_{\eta}^{+\infty} \hat{\eta} f(z)dz \tag{8.5}$$

$$= \frac{1}{\alpha}\int_{\hat{\eta}}^{+\infty} zf(z)dz \tag{8.6}$$

The following is just a one of the thing in calculus that is under the name of 'dirt trick', set $F(z) = 1-\beta$, then $f(z)dz = -d\beta$,

$$\min_{\eta\in\mathbb{R}} G(\eta) = \frac{1}{\alpha}\int_{\alpha}^{0} F^{-1}(1-\beta)(-d\beta) = \frac{1}{\alpha}\int_{0}^{\alpha} F^{-1}(1-\beta)d\beta \tag{8.7}$$

□

Simplification 2: Z has a discrete distribution z_k's are realizations and p_k's are corresponding probabilities, and we order them in the following way: $z_1 < z_2 < z_3 < \cdots < z_K$, if we have in total K realizations;

Proof. Again, we have

$$G(\eta) = \eta + \frac{1}{\alpha}\sum_{z_k\geq\eta} p_k(z_k - \eta) \tag{8.8}$$

Thus,

$$\frac{dG(\eta)^+}{d\eta^+} = 1 + \frac{1}{\alpha}\sum_{z_k>\eta} -p_k = 1 - \frac{1}{\alpha}\sum_{z_k>\eta} p_k \tag{8.9}$$

Observe that $G(\eta)$ is both piece-wise linear and convex function, and if $\eta < z_1$,

$$\frac{dG(\eta)^+}{d\eta^+} = 1 - \frac{1}{\alpha} < 0 \tag{8.10}$$

if $\eta > z_k$, then

$$\frac{dG(\eta)^+}{d\eta^+} = 1 > 0 \tag{8.11}$$

We have if $1 - \frac{1}{\alpha}\sum_{k>k^*} p_k \geq 0$, $\mathbb{P}(z > z_{k^*}) \leq \alpha$, and if $1 - \frac{1}{\alpha}\sum_{k>k^*} p_k \leq 0$, then $\mathbb{P}(z \geq z_{k^*}) \leq \alpha$. Thus,

$$\hat{\eta} = z_{k^*} \tag{8.12}$$

which is the value at risk at level α. Then

$$\min_{\eta\in\mathbb{R}} G(\eta) = z_{k^*} + \frac{1}{\alpha}\sum_{k>k^*} p_k(z_k - z_{k^*}) \tag{8.13}$$

in the meanwhile, the integral version is:

$$\frac{1}{\alpha}[\sum_{k>k^*} p_k z_k + (1-\alpha-\sum_{k>k^*} p_k) z_{k^*}] = \frac{1}{\alpha}\sum_{k>k^*} p_k(z_k - z_{k^*}) + z_{k^*} \tag{8.14}$$

□

The representation (8.2) is one special case in the previous example, thus a *coherent risk measure*. Since for practical purpose, we usually deal with discrete distribution, we can actually rewrite it as a linear programming formulation:

$$\begin{aligned} &\min_{\eta,v\in\mathbb{R}} \{\eta + \frac{1}{\alpha}\sum_{k=1}^{K} p_k v_k\} \\ \text{subject to: } &v_k \geq z_k - \eta,\ k = 1, ..., K \\ &v_k \geq 0 \end{aligned} \tag{8.15}$$

8.4 Dual Representation of Coherent Risk Measure

Again we have vector space $\mathscr{Z} = \mathscr{L}_p(\Omega, \mathcal{F}, \mathbb{P})$. The *linear continuous functional* (or sometimes, *linear operator*) mapping from $\mathscr{Z}$ to $\mathbb{R}$ that satisfies:

- *Linearity:* $l(\alpha Z + \beta V) = \alpha l(Z) + \beta l(Z)$, where $Z, V \in \mathscr{Z}$ and $\alpha, \beta \in \mathbb{R}$;
- *Continuity:* For $Z_k \in \mathscr{Z}$ that converges to Z, $l(Z_k) \to l(Z)$ as $k \to \infty$.

And in $\mathscr{L}_p$ space the continuous linear functional has a unique representation that is given by the *Riesz Representation Theorem*:

Theorem 8.4.1 For every $l : \mathscr{Z} \mapsto \mathbb{R}$ that is linear and continuous, there exists a $\mu \in \mathscr{L}_q(\Omega, \mathcal{F}, \mathbb{P})$ for which $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$l(Z) = \int_\Omega Z(\omega)\mu(\omega)\mathbb{P}(d\omega) = \langle Z, \mu\rangle \tag{8.16}$$

We shall notice that the risk functional $\rho(\cdot)$ is NOT linear but convex and continuous. We can visualize it in Figure 8.2.

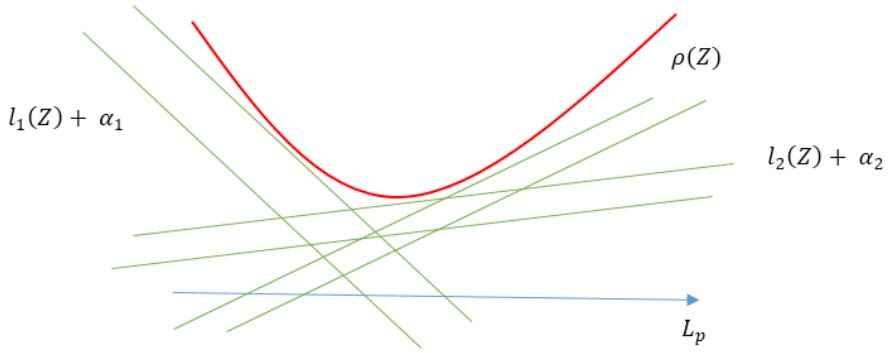

Figure 8.2: Minorant

For any arbitrary affine plane, we can push it further until it touches the convex function ρ (by *separation theorem*). The collections of such affine plane is called the *Minorant* of ρ. Let's look at each shifted linear continuous functional,

$$\begin{aligned} l(Z) - \alpha = \langle Z, \mu \rangle - \alpha &\leq \rho(Z) \\ \Rightarrow \alpha &\geq \langle Z, \mu \rangle - \rho(Z) \text{ for all } Z \\ \Rightarrow \alpha &\geq \sup_Z \{\langle Z, \mu \rangle - \rho(Z)\} \end{aligned}$$

We set the optimal value

$$\alpha^* = \rho^*(\mu) = \sup_Z \{\langle Z, \mu \rangle - \rho(Z)\} \tag{8.17}$$

$\rho^*(\mu)$ is called the *conjugate function* of ρ that is convex and lower-semi-continuous. We observe that

$$\rho^{**}(Z) = \sup_\mu \{\langle Z, \mu \rangle - \alpha^*\} \tag{8.18}$$

If ρ is convex and lower-semi-continuous, by *Fenchel-Moreau theorem*,

$$\rho(Z) = \rho^{**}(Z) = \sup_\mu \{\langle Z, \mu \rangle - \alpha^*\} \tag{8.19}$$

Remark 8.4.2 Observe in Figure 8.3, no matter how you choose μ, $l(Z) - \alpha$ will intersect $\rho(Z)$, thus $\rho^*(\mu) = +\infty$. In this situation, $\rho(Z) = -\infty$ which is not interesting, we shall look for other μ. Actually, to be interesting, we shall restrict ρ to be proper so that ρ^* is proper as well.

Theorem 8.4.3 Suppose that $\rho : \mathscr{Z} \mapsto \bar{\mathbb{R}}$ is convex, proper and lower semi-continuous. Then the representation

$$\rho(Z) = \sup_{\mu \in \mathscr{A}} \{\langle \mu, Z \rangle - \rho^*(\mu)\}, \ \forall Z \in \mathscr{Z} \tag{8.20}$$

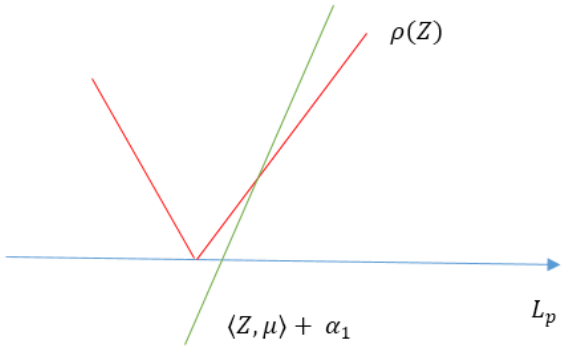

Figure 8.3: Extreme case

where $\mathscr{A} := dom(\rho^*)$ (without loss of generality, we can assume that $\rho(\mu) < +\infty$), holds. Moreover, we have that: (i) monotonicity holds if and only if every $\mu \in \mathscr{A}$ is non-negative, i.e., $\mu(\omega) \geq 0$, a.e. $\omega \in \Omega$; (ii) translation invariance hold if and ony if $\int_\Omega \mu d\mathbb{P} = 1$ for every $\mu \in \mathscr{A}$; and (iii) positive homogeneity holds if and only if ρ is the support function of the set $\mathscr{A}$, i.e., can be represented in the form:

$$\rho(Z) = \sup_{\mu \in \mathscr{A}} \langle \mu, Z \rangle, \ Z \in \mathscr{Z} \tag{8.21}$$

Proof. (Sketched) We will firstly use the monotonicity property. Define the event:

$$A = \{\omega : \mu(\omega) < 0\} \tag{8.22}$$

Suppose that $\mathbb{P}(A) > 0$. Consider the random variable $H = \mathbf{1}_A$, set $Z = tH$ for $t > 0$. Therefore,

$$\rho^*(\mu) \geq \sup_{t \geq 0} \{\langle -tH, \mu \rangle - \rho(-tH)\}$$

Since $-tH \leq 0$, $\rho(-tH) \leq \rho(0)$, thus

$$\rho^*(\mu) \geq \sup_{t \geq 0} \{-t\langle H, \mu \rangle - \rho(0)\}$$

Since $\langle H, \mu \rangle = \int_A \mu(\omega) \mathbb{P}(d\omega) < 0$, thus

$$\rho^*(\mu) = +\infty$$

Contradiction ! Thus, $\mu > 0$ almost surely.

Then, we use the translation invariance property.

$$\begin{aligned}
\rho^*(\mu) &= \sup_{Z \in \mathscr{Z}, c \in \mathbb{R}} \{\langle Z + c\mathbf{1}, \mu \rangle - \rho(Z + c\mathbf{1})\} \\
&= \sup_{Z \in \mathscr{Z}, c \in \mathbb{R}} \{\langle Z, \mu \rangle + c\langle \mathbf{1}, \mu \rangle - \rho(Z) - c\} \\
&= \sup_{Z \in \mathscr{Z}, c \in \mathbb{R}} \{\langle Z, \mu \rangle - \rho(Z)\} + \sup_{c \in \mathbb{R}} \{c\,(\langle \mathbf{1}, \mu \rangle - 1)\}
\end{aligned}$$

This proves that $\int_\Omega \mu(\omega)\mathbb{P}(d\omega) = 1$.

Lastly, we shall use positive homogeneity,

$$\begin{aligned}\rho^*(\mu) &= \sup_{Z\in\mathscr{Z}}\{\langle tZ, \mu\rangle - \rho(tZ)\} \\ &= \sup_{z\in\mathscr{Z}} t\{\langle Z, \mu\rangle - \rho(Z)\} \\ &= t\rho^*(\mu)\end{aligned}$$

It gives us the simplified representation:

$$\rho(Z) = \sup_{\mu\in\mathscr{A}}\langle Z, \mu\rangle$$

□

Remark 8.4.4 From above analysis, we actually proves that $\mathscr{A}$ is convex closed set of probability density function.

Next, let's explore more on how $\mathscr{A}$ looks like. In general,

$$\mathscr{A} = \partial\rho(0)$$

Because when $Z = 0$, the optimization has no effect, we solve a feasibility problem.

Example 8.4.5 For $\rho(Z) = \mathbb{E}[Z]$, then

$$\rho(Z) = \int_\Omega Z(\omega)\mu(\omega)\mathbb{P}(d\omega)$$

where $\mu(\omega) = 1$. Thus $\mathscr{A} = \{\mathbf{1}\}$.

Example 8.4.6 For *average-value-at-risk* in the discrete case, we have he linear programming formulation

$$\begin{aligned}\min_{\eta,v\in\mathbb{R}}\ & \eta + \frac{1}{\alpha}\sum_{k=1}^K p_k v_k \\ \text{subject to:}\ & v_k \geq z_k - \eta,\ k = 1, ..., K \\ & v_k \geq 0\end{aligned} \tag{8.23}$$

The dual problem is very easy to write down that is:

$$\begin{aligned}\max\ & \sum_{k=1}^K Z_k\lambda_k \\ \text{subject to:}\ & \lambda_k \leq \frac{p_k}{\alpha},\ k = 1, ..., K \\ & \sum_{k=1}^K \lambda_k = 1 \\ & \lambda \geq 0\end{aligned} \tag{8.24}$$

Define $\mu_k = \frac{\lambda_k}{p_k}$, then

$$\begin{aligned} &\max \sum_{k=1}^{K} p_k Z_k \mu_k \\ &\text{subject to: } 0 \le \mu_k \le \frac{1}{\alpha},\ k = 1, ..., K \\ &\sum_{k=1}^{K} p_k \mu_k = 1 \end{aligned} \tag{8.25}$$

Example 8.4.7 For $\rho(Z) = \mathbb{E} + \mathbb{E}\{(Z - \mathbb{E}[Z])_+\}$, in the discrete case, we have linear programming formulation:

$$\begin{aligned} &\min_{v \in \mathbb{R}} \sum_{k=1}^{K} p_k z_k + \sum_{k=1}^{K} p_k v_k \\ &\text{subject to: } v_k \ge z_k - \sum_{l=1}^{K} p_l z_l,\ k = 1, ..., K \\ &v_k \ge 0 \end{aligned} \tag{8.26}$$

The dual problem is

$$\begin{aligned} &\max \sum_{k=1}^{K} (Z_k - \sum_{l=1}^{K} p_l Z_k l) \lambda_k + \sum_{l=1}^{K} p_l z_l \\ &\text{subject to: } 0 \le \lambda_k \le p_l, \quad k = 1, ..., K \end{aligned} \tag{8.27}$$

Set $\xi_k = \lambda_k - p_k \sum_{j=1}^{K} \lambda_j + p_k$, observe that

$$\sum_{k=1}^{K} \xi_k = 1 \text{ and } \xi \ge 0 \tag{8.28}$$

Set

$$\xi = \{\lambda - \left(\sum_{j=1}^{K} \lambda_j \right) p + p :\ 0 \le \lambda \le p\}$$

and define $\mu_k = \xi_k / p_k$, then

$$\mu \in \{s - \langle s, \mathbf{1} \rangle \cdot \mathbf{1} + \mathbf{1} :\ 0 \le s \le 1\}$$

where $s_j = \frac{\lambda_j}{p_j}$ and $\sum_{j=1}^{K} \lambda_j = \sum_{j=1}^{K} s_j p_j = \langle s, \mathbf{1} \rangle$.

Chapter 9

Lecture 11-Lecture 12

In many context, one may encounter risk measure defined in another way but exactly identical. Remember the random variable Z is chosen to represent the loss, if it is a gain instead, how will it change the definition of risk measure:

- *Convexity:* it stays the same as before;
- *Monotonicity:* if $Z \geq V$ a.s., it implies $\rho(Z) \geq \rho(V)$;
- *Translation invariance:* $\rho(Z + c) = \rho(Z) - c$, $c \in \mathbb{R}$;
- *Homogeneity:* it stays the same as well.

Now, Z stands for the gain, then the *dual representation* is in the following foramt:

$$\rho(Z) = \max_{\mu \in \mathscr{A}} \langle -Z, \mu \rangle = -\mathbb{E}_{\mu}[Z]$$

How will *average-value-at-risk* look like in this case?

$$\mathrm{AVaR}_{\alpha}^{-}(Z) = \frac{1}{\alpha} \int_0^{\alpha} \mathrm{VaR}_{\beta}(Z) d\beta$$

where

$$\mathrm{VaR}_{\beta}(Z) = -F_Z^{-1}(\beta) \tag{9.1}$$

On the other hand,

$$\mathrm{AVaR}_{\alpha}^{-}(Z) = \min_{\eta \in \mathbb{R}} \{-\eta + \frac{1}{\alpha} \mathbb{E}[(\eta - Z)_+]\}$$

Why? In the case that Y is the loss, we have:

$$\begin{aligned}\mathrm{AVaR}_\alpha^+(Y) &= \min_{\eta\in\mathbb{R}}\{\eta + \frac{1}{\alpha}\mathbb{E}[(Y-\eta)_+]\} \\ &= \min_{\eta\in\mathbb{R}}\{\eta + \frac{1}{\alpha}\mathbb{E}[(-Z-\eta)_+]\} \\ &= \min_{t\in\mathbb{R}}\{-t + \frac{1}{\alpha}\mathbb{E}[(t-Z)_+]\} \quad (t=-\eta) \\ &= \mathrm{AVaR}_\alpha^-(Z)\end{aligned}$$

Similarly, for *mean-semi-deviation risk measure*:

$$\rho(Z) = -\mathbb{E}[Z] + \gamma\mathbb{E}\{(\mathbb{E}[Z]-Z)_+\}, \quad \gamma\in[0,1]$$

9.1 Optimization Under Risk Measure

Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have Z_x is a measurable function defined on x, $Z_x \in \mathscr{Z}$, which defines the random cost. Suppose $x \in \mathcal{X}$, we want to solve the following optimization problem:

$$\begin{aligned}&\min\ \rho(Z_x) \\ \text{subject to:}\ \ &x\in\mathcal{X}\end{aligned}$$

For example, if ρ is AVaR_α, then actually we are solving the following optimization problem:

$$\min_{x\in\mathcal{X},\eta\in\mathbb{R}}\{\eta + \frac{1}{\alpha}\mathbb{E}[(Z_x-\eta)_+]\}$$

9.1.1 Dual Representation & Min-Max Optimization

We know that the dual representation of coherent risk measure is:

$$\rho(Z_x) = \max_{\mu\in\mathscr{A}}\mathbb{E}_\mu(Z_x)$$

The optimization then becomes:

$$\min_{x\in\mathcal{X}}\max_{\mu\in\mathscr{A}}\mathbb{E}_\mu(Z_x)$$

In the continuous case $\mathbb{E}_\mu(Z_x) = \int_\Omega Z_x(\omega)\mu(\omega)\mathbb{P}(d\omega)$, or, in the discrete case, $\mathbb{E}_\mu(Z_x) = \sum_{k=1}^K p_k Z_x^{(k)}\mu_k$, where $Z_x^{(k)}$ is the cost under k-th scenario. Observe that $\mathbb{E}_\mu(Z_x)$ is linear and also continuous in μ.

Assumption 9.1.1 $x \mapsto Z_x$ is convex a.s., $\mathcal{X}$ is convex, solution $\hat{x}$ exists.

Theorem 9.1.2 Under above assumption, we have the following chains of equality,

$$\min_{x\in\mathcal{X}} \rho(Z_x) = \min_{x\in\mathcal{X}} \max_{\mu\in\mathscr{A}} \mathbb{E}_\mu[Z_x] = \min_{x\in\mathcal{X}} \mathbb{E}_{\hat{\mu}}[Z_x] = \max_{\mu\in\mathscr{A}} \inf_{x\in\mathcal{X}} \mathbb{E}_\mu(Z_x)$$

where

$$\hat{\mu} = \arg\min_{\mu\in\mathscr{A}} \mathbb{E}_\mu[Z_x]$$

The result follows from the old theorem of *Von Neumann*, it basically says that if a function $K(x, y)$ is concave in x and convex in y and $\mathcal{X}$ and $\mathcal{Y}$ are convex and compact, then $\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} K(x, y) = \max_{y\in\mathcal{Y}} \min_{x\in\mathcal{X}} K(x, y)$.

We know that

$$\mathscr{A} = \partial\rho(0)$$

Furthermore,

$$\partial\rho(Z) = \{\mu \in \mathscr{A} : \rho(Z) = \mathbb{E}_\mu(Z)\}$$

The conclusion is that there exists $\hat{\mu} \in \partial\rho(Z_{\hat{x}})$, such that $\hat{x}$ also minimizes

$$\rho(Z_{\hat{x}}) + \langle Z_x - Z_{\hat{x}}, \hat{\mu}\rangle$$

The essence of convex optimization lies on the equivalent optimality condition between linearized version and itself.

9.1.2 Connection to Game Theory & Cutting Plane Method

We have n assets, R_{kj} is the return rate of asset j in scenario k, $j = 1, ..., n$, $k = 1, ..., K$. We denote the fraction of capital invested in asset j by x_j. Obviously, $\sum_{j=1}^n x_j = 1$, $x \geq 0$. We form the portfolio, the return rate of portfolio x in scenario k is:

$$Z_k = \sum_{j=1}^n R_{kj} x_j$$

The probability of each scenario is given as p_k, $k = 1, ..., K$. Now we try to solve the following optimization problem:

$$\begin{aligned}\min_{x\in\mathcal{X}} \rho(-Z) &= \min_{x\in\mathcal{X}} \max_{\mu\in\mathscr{A}} -\sum_{k=1}^K p_k \mu_k Z_k \\ &= \min_{x\in\mathcal{X}} \max_{\nu\in\mathscr{A}'} -\sum_{k=1}^K \nu_k Z_k\end{aligned}$$

where $\nu = p_k \mu_k \in \mathscr{A}'$. We can visualize $Z = [Z_1, ..., Z_K]^\top = Rx$, $\nu = [\nu_1, ..., \nu_K]$. Thus,

$$\min_{x \in \mathcal{X}} \rho(-Z_x) = \min_{x \in \mathcal{X}} \max_{\nu \in \mathscr{A}'} -\nu^\top Rx = -\max_{x \in \mathcal{X}} \min_{\nu \in \mathscr{A}'} \nu^\top Rx$$

If $R \in \mathbb{R}^{k \times n}$, then ν can be thought as the probability distribution over rows, while x is the probability distribution over columns. The real world gives a distribution ν, you try to optimize your portfolio x, i.e., distribution over columns, to compete against ν. This is a good interpretation from the perspective of Game theory.

We can use the *cutting plane method* to solve such an optimization. Suppose we are now at iteration k, we already have $\mu^{(1)}, ..., \mu^{(k)} \in \mathscr{A}$, the *master problem* is

$$\begin{aligned} \min \quad & \nu \\ \text{subject to:} \quad & \nu \geq \text{linearized} \left(\mathbb{E}_{\mu^{(j)}}[Z_x] \right), \quad j = 1, ..., k, \\ & x \in \mathcal{X} \end{aligned}$$

Then we get the current best solution $\hat{x}^{(k)}$. The new sub-gradient can be obtained by

$$\mu^{(k+1)} \in \partial \rho(Z_{\hat{x}^{((k)}})$$

For the details, one should refer back to the *two-stage problem* discussed in previous chapters.

9.2 Law-invariant Risk Measure

We define the distribution function

$$F_z(\eta) = \mathbb{P}\{Z \leq \eta\}, \ \eta \in \mathbb{R}$$

We write $Z \sim^{\mathscr{D}} Z'$, if $F_Z(\cdot) = F_{Z'}(\cdot)$.

Definition 9.2.1 $\rho(\cdot)$ is *law-invariant* if $Z \sim^{\mathscr{D}} Z'$ implies $\rho(Z) = \rho(Z')$.

Special Case 1: Assume $\Omega = \{\omega_1, \omega_2, ..., \omega_K\}$, $p_k = \frac{1}{K}$, $k = 1, ..., K$. Let abuse the notation a little bit. Imagine $Z = [z_1, z_2, ..., z_K]$, where $z_k = Z_{\omega_k}$. $Z \sim^{\mathscr{D}} Z'$ implies that there exists a permutation Π of $\{1, 2, ..., K\}$ such that $Z_k = Z'_{\Pi(k)}$, or $Z'_k = Z_{\Pi^{-1}(k)}$, $k = 1, ..., K$.

Here is the argument:

$$\mathbb{P}\{Z \leq \eta\} = \frac{1}{K} \sum_{k: Z_k \leq \eta} 1 = \frac{1}{K} \text{card}(k : Z_k \leq \eta)$$

Observe that $j = \Pi(k)$ implies $k = \Pi^{-1}(j)$. Then

$$\begin{aligned}\mathbb{P}\{Z \le \eta\} &= \frac{1}{K}\text{card}(k : Z_k \le \eta) \\ &= \frac{1}{K}\text{card}(\Pi^{-1}(j) : Z_{\Pi^{-1}(j)} \le \eta) \\ &= \frac{1}{K}\text{card}(\Pi^{-1}(j) : Z'_j \le \eta) \\ &= \frac{1}{K}\text{card}(j : Z'_j \le \eta)\end{aligned}$$

Special Case 2: $(\Omega, \mathcal{F}, \mathbb{P})$ is atom-less standard probability space, where a uniform $U[0,1]$ distributed random variable exists.

Definition 9.2.2 $T : \Omega \mapsto \Omega$ is a measure preserving transformation if:

- *one-to-one:* if $\omega \neq \omega'$, $T(\omega) \neq T(\omega')$;
- *one-to:* $T(\Omega) = \Omega$;
- *measurable:* $\{\omega : T(\omega) \in A\} \in \mathcal{F}$, where $A \in \mathcal{F}$;
- *preserving:* $\mathbb{P}\{\omega : T(\omega) \in A\} = \mathbb{P}\{A\}$.

Proposition 9.2.1 Suppose $Z : \Omega \mapsto \mathbb{R}$, set $Z'(\omega) = Z(T(\omega))$, $\forall \omega \in \Omega$, then $Z' \sim^{\mathscr{D}} Z$.

Proof. For $\eta \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\{\omega : Z'(\omega) \le \eta\} &= \mathbb{P}\{\omega : Z(T(\omega)) \le \eta\} \qquad (\xi = T(\omega)) \\ &= \mathbb{P}\{T^{-1}(\xi) : Z(\xi) \le \eta\} \qquad \text{define } A = \{\xi : Z(\xi) \le \eta\} \\ &= \mathbb{P}\{T^{-1}(\xi) : \xi \in A\} \\ &= \mathbb{P}\{\omega : T(\omega) \in A\} \\ &= \mathbb{P}\{A\} \\ &= \mathbb{P}\{\omega : Z(\omega) \le \eta\}\end{aligned}$$

□

Remark 9.2.2 Actually, the converse is also true: if $Z \sim^{\mathscr{D}} Z$, then $\exists T$ such that $Z'(\omega) = Z(T(\omega))$. We won't prove it here, it is a little bit involved.

We write $Z'(\omega) = Z(T(\omega))$ as $Z' = Z \circ T$, thus a coherent risk measure that is law-invariant means, if $Z \sim^{\mathscr{D}} Z'$, then $\rho(Z') = \rho(Z)$, i.e., $\rho(Z \circ T) = \rho(Z)$ for all measure preserving mapping T.

Let's talk about the stability of the probability density set $\mathscr{A}$. Since ρ is coherent, it has the following dual representation:

$$\rho(Z) = \max_{\mu \in \mathscr{A}} \int_\Omega Z(\omega)\mu(\omega)\mathbb{P}(d\omega)$$

We know that the conjugate is:

$$\begin{aligned}
\rho^*(\mu) &= \sup_Z \{\int_\Omega Z(\omega)\mu(\omega)\mathbb{P}(d\omega) - \rho(Z)\} \qquad (\xi = T(\omega)) \\
&= \sup_Z \{\int_\Omega Z(T^{-1}(\xi))\mu(T^{-1}(\xi))\mathbb{P}(d\xi) - \rho(Z)\} \\
&= \sup_Z \{\int_\Omega Z'(\xi)\mu(T^{-1}(\xi))\mathbb{P}(d\xi) - \rho(Z')\} \qquad (Z' = Z \circ T^{-1}) \\
&= \rho^*(\mu \circ T^{-1}) = 0
\end{aligned}$$

We know that $\rho^*(\mu) = 0, \mu \in \mathscr{A}$, otherwise, it is infinity. This implies: if $\mu \in \mathscr{A}$, then $\mu \circ T^{-1} \in \mathscr{A}$. In other words, $\mathscr{A}$ is stable with measure preserving translation.

9.2.1 Kusuoka Representation

In this section, we will introduce an important theorem regarding law-invariant coherent risk measure discovered by *Kusuoka*. Let's firstly play around the discrete case to understand the general picture. Suppose $\mu = [\mu_1, \mu_2, \cdots, \mu_n]^\top$ and we have finite number of realizations of z, then the dual representation of ρ can be written as:

$$\rho(Z) = \max_{\mu \in \mathscr{A}} \sum_{k=1}^{n} \frac{1}{n} z_k \mu_k$$

From previous discussion, we know that $\mu \in \mathscr{A} \Rightarrow \mu \circ \Pi \in \mathscr{A}$ (or, sometimes, $\mu_{\Pi(\cdot)} \in \mathscr{A}$, what really happened is $\Pi\mu$), where Π is a permutation matrix (although we proved in the continuous case, it certainly works in the discrete case). That is, $\left(\mu_{\Pi(k)}\right)_{k=1}^n \in \mathscr{A}$. For example, if

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then,

$$\Pi\mu = \begin{pmatrix} \mu_2 \\ \mu_3 \\ \mu_1 \end{pmatrix}$$

Without loss of generality, we can define permutation matrix such that $Z'_k = Z_{\Pi(k)}$ is increasingly ordered, i.e., $Z'_1 \leq Z'_2 \leq \cdots \leq Z'_n$. We know that

$$\begin{aligned}\rho(Z) = \rho(Z') &= \max_{\mu\in\mathscr{A}} \sum_{k=1}^{n} \frac{1}{n} Z'_k \mu_k \\ &= \max_{\mu\in\mathscr{A}} \frac{1}{n} \sum_{k=1}^{n} Z_{[k]}\mu_{[k]}\end{aligned}$$

where $Z_{[1]} \leq Z_{[2]} \leq \cdots \leq Z_{[n]}$ and $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[n]}$. Thus,

$$\begin{aligned}\rho(Z) &= \max_{\mu\in\mathscr{A}} \frac{1}{n}\left[\mu_{[1]}\left(Z_{[1]} + \cdots + Z_{[n]}\right) + (\mu_{[2]} - \mu_{[1]})(Z_{[2]} + \cdots + Z_{[n]}) + \cdots + (\mu_{[n]} - \mu_{[n-1]})Z_{[n]}\right] \\ &= \max_{\mu\in\mathscr{A}} \sum_{i=1}^{n-1} (1 - \frac{i}{n})\mathrm{AVaR}_{1-\frac{i}{n}}(Z)(\mu_{[i+1]} - \mu_{[i]})\end{aligned}$$

Set $\nu_{[i+1]} = \mu_{[i+1]} - \mu_{[i]}$, then

$$\rho(Z) = \max_{\mu\in\mathscr{A}} \sum_{i=1}^{n-1} (1 - \frac{i}{n})\mathrm{AVaR}_{1-\frac{i}{n}}(Z)\nu_{[i+1]}$$

where

$$\sum_{i=0}^{n-1} (1 - \frac{i}{n})\nu_{[i+1]} = 1$$

This is the discrete version of *Kusuoka Representation of coherent law-invariant risk measure.*

In general, we assume that we work on a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define $\mathscr{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, \mathbb{P})$, where $1 \leq p < +\infty$. The dual representation of coherent risk measure gives us that:

$$\rho(Z) = \max_{\mu\in\mathscr{A}} \int_\Omega Z(\omega)\mu(\omega)\mathbb{P}(d\omega)$$

where $Z \in \mathscr{Z}$ and $\int_\Omega \mu(\omega)\mathbb{P}(d\omega) = 1$, $\mu \geq 0$. We choose U the uniform random variable on Ω, i.e., $\mathbb{P}[\mu \in [0, \alpha]] = \alpha$. If we define

$$Z' = F_Z^{-1}(U)$$

then $Z' \sim^{\mathscr{D}} Z$. Why? Observe, for $t \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{P}\{Z' \leq t\} \\ =&\mathbb{P}\{F_Z^{-1}(U) \leq t\} \\ =&\mathbb{P}\{U \leq F_Z(t)\} \\ =&F_Z(t)\end{aligned}$$

By the same token, $\mu'(\omega) = F_\mu^{-1}(U)$ has the same distribution as μ. Thus, we can re-express the risk measure as:

$$\begin{aligned}\rho(Z) &= \max_{\mu \in \mathscr{A}} \int_\Omega F_Z^{-1}(U(\omega))\mu(\omega)\mathbb{P}(d\omega) \\ &= \max_{\mu \in \mathscr{A}} \int_\Omega F_Z^{-1}(U(\omega))F_\mu^{-1}(U(\omega))\mathbb{P}(d\omega) \\ &= \max_{\mu \in \mathscr{A}} \int_0^1 F_Z^{-1}(\alpha)F_\mu^{-1}(\alpha)d\alpha\end{aligned}$$

Set $q_\mu(\alpha) = F_\mu^{-1}(\alpha)$ (subscript μ indicates its dependence on μ), it is a non-decreasing function. For some positive measure $\lambda : \mathbb{R} \mapsto \mathbb{R}_+$, we can define

$$dq_\mu(\alpha) = \frac{1}{1-\alpha}\lambda_\mu(d\alpha)$$

Now,

$$\begin{aligned}&\int_0^1 F_Z^{-1}(\alpha)F_\mu^{-1}(\alpha)d\alpha \\ =&\int_0^1 F_Z^{-1}(\alpha)\int_0^\alpha \frac{1}{1-\beta}\lambda_\mu(d\beta)d\alpha \\ =&\int_0^1 \frac{1}{1-\beta}\int_\beta^1 F_Z^{-1}(\alpha)d\alpha\lambda_\mu(d\beta) \\ =&\int_0^1 \mathrm{AVaR}_\beta(Z)\lambda_\mu(d\beta)\end{aligned}$$

Let's analyze measure $\lambda_\mu(\cdot)$ a little bit,

$$\begin{aligned}\int_0^1 \lambda_\mu(d\beta) &= \int_0^1 (1-\beta)dq_\mu(\beta) \\ &= [(1-\beta)q(\beta)]_0^1 - \int_0^1 q(\beta)d(1-\beta) \\ &= \int_0^1 q(\beta)d\beta \\ &= \int_0^1 F_\mu^{-1}d\beta \\ &= \mathbb{E}[\mu] = 1\end{aligned}$$

If we define set Λ as probability measures on $[0,1]$, then the coherent risk measure can be written as:

$$\rho(Z) = \max_{\lambda \in \Lambda} \int_0^1 \mathrm{AVaR}_\beta(Z)\lambda(d\beta)$$

This is the *Kuosoka representation* for continuous distribution. Since $|\Lambda| = 1$, we sometimes call it the spectral risk measure.

9.3 Stochastic Dominance

9.3.1 Introduction

It's very hard to get universal agreement on the risk of certain financial instrumental and other things. But at least, we have some common idea about what is riskier. We define a partial-pre-order on $\mathscr{Z}$, i.e., $Z \leq V$ if several conditions are satisfied. For the historical reason, now let's switch Z to be the gains. The classical way is to: 1) choose a family of $\mathscr{U}$ of utility functions $u : \mathbb{R} \mapsto \mathbb{R}$; and 2) define the equivalence relationship between $Z \geq V$ and $\int_\Omega u(Z(\omega))\mathbb{P}(d\omega) \geq \int u(V(\omega))\mathbb{P}(d\omega)$, $\forall u \in \mathscr{U}$, equivalently,

$$\mathbb{E}[u(Z)] \geq \mathbb{E}[u(V)]$$

9.3.2 First Order and Second order Stochastic Dominance

There are two cases that are extremely important, actually, they are sufficient for almost all applications. Let's first give the definition of them and then analyze them in details.

Definition 9.3.1 If $u : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing and bounded function, then $Z \geq_{(1)} V$ or $Z \geq_{\text{f.s.d}} V$ is called the first order stochastic dominance.

Definition 9.3.2 If $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ and $u : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing concave function with the property that $lim_{Z\to-\infty} \frac{|u(Z)|}{|Z|} < \infty$, then $Z \geq_{(2)} V$ or $Z \geq_{\text{s.s.d}} V$ is called the second order stochastic dominance.

Remark 9.3.1 For second order stochastic dominance, we can further infer, by *Jensen's inequality*,

$$u(\mathbb{E}[Z]) \geq \mathbb{E}[u(Z)]$$

Also,

$$\mathbb{E}[Z] \geq_{(2)} Z$$

For the first order dominance, we have an equivalent characterization that is

Proposition 9.3.2 $Z \geq_{(1)} V$ if and only if $F_Z(\eta) \leq F_V(\eta)$, $\forall \eta \in \mathbb{R}$.

Proof. ("⇒") Set

$$\bar{u}(Z) = \begin{cases} 1, & \text{if } Z > \eta \\ 0, & \text{if } Z \leq \eta \end{cases}$$

This is obviously a non-decreasing and bounded function. Now,

$$\mathbb{E}[\bar{u}(Z)] \geq \mathbb{E}[\bar{u}(V)] \Rightarrow \mathbb{P}[Z > \eta] \geq \mathbb{P}[V > \eta]$$

For ("$\Leftarrow$"), we have

$$\begin{aligned}\mathbb{E}[u(Z)] &= \int_{-\infty}^{+\infty} u(\eta) dF_Z(\eta) \\ &= uF_Z|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} F_Z(\eta) du(\eta) \\ &= \lim_{\eta \to \infty} u(Z) - \int_{-\infty}^{+\infty} F_Z(\eta) du(\eta)\end{aligned}$$

Similarly,

$$\mathbb{E}[u(V)] = \lim_{Z \to \infty} u(Z) - \int_{-\infty}^{\infty} F_Z(\eta) du(\eta)$$

Thus, we have the result, or equivalently,

$$F_Z^{-1}(\alpha) \geq F_V^{-1}(\alpha), \quad \forall \alpha \in (0,1)$$

□

For the second order stochastic dominance, we also have an equivalent characterization

Proposition 9.3.3 $Z \geq_{(2)} V$ if and only if $\mathbb{E}[(\eta - Z)_+] \leq \mathbb{E}[(\eta - V)_+]$, $\eta \in \mathbb{R}$.

Proof. For ("$\Rightarrow$"), we choose $\bar{u}(Z) = -(\eta - Z)_+$, then the result follows immediately. Conversely, we define $u(\cdot)$ concave, non-decreasing and piecewise linear function (Figure 9.1):

$$u(Z) = c - s_1(\eta_1 - Z)_+ - (s_2 - s_1)(\eta_2 - Z)_+ - \cdots - (s_k - s_{k-1})(\eta_k - Z)_+ + \cdots$$

$$\begin{aligned}\mathbb{E}[u(Z)] &= c - \sum (s_k - s_{k-1}) \mathbb{E}[(\eta_k - Z)_+] \\ &\geq c - \sum (s_k - s_{k-1}) \mathbb{E}[(\eta_k - V)_+] \geq \mathbb{E}[u(V)]\end{aligned}$$

The rest of the proof goes to the approximation theory, basically, it says that every non-decreasing concave function $u(\cdot)$ can be approximated by the piecewise linear one, $\tilde{u}(\cdot)$, defined as above, i.e., they differ in ϵ. Thus, if $u \in \mathscr{U}$, $Z, V, \in \mathscr{Z}$, $\tilde{u}$ is the piecewise linear function in $\mathscr{U}$ such that $\tilde{u} \leq u$ almost everywhere, then $\mathbb{E}[\tilde{u}(Z)] \geq \mathbb{E}[u(Z)] - \epsilon$ and $\mathbb{E}[\tilde{u}(V)] \geq \mathbb{E}[u(V)] - \epsilon$ for any ϵ. Therefore,

$$\mathbb{E}[u(Z)] \geq \mathbb{E}[\tilde{u}(Z)] \geq \mathbb{E}[\tilde{u}(V)] \geq \mathbb{E}[u(V)] - \epsilon$$

□

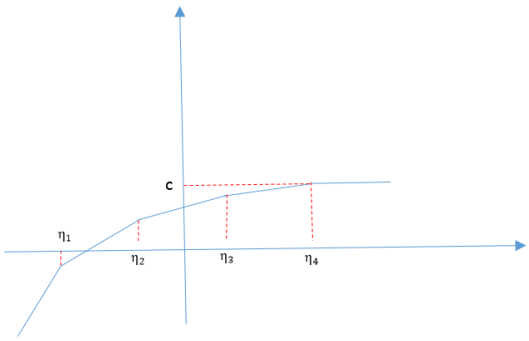

Figure 9.1: Piecewise Linear $u(\cdot)$

Remark 9.3.4 If we define

$$F^{(2)}(\eta) = \int_{-\infty}^{\eta} F_Z(t)dt, \ \eta \in \mathbb{R}$$

This is a convex function and we can claim:

$$Z \geq_{(2)} V \text{ if and only if } F_Z^{(2)}(\cdot) \leq F_V^{(2)}(\cdot)$$

Because

$$\begin{aligned} F_Z^{(2)}(\eta) = \int_{-\infty}^{\eta}\int_{-\infty}^{t} dF_Z(s)dt &= \int_{-\infty}^{\eta}\int_{-\infty}^{+\infty} \mathbf{1}_{\{s\leq t\}} dF_Z(s)dt \\ &= \int_{-\infty}^{\eta} (\eta - s)dF_Z(s) \\ &= -\int_{-\infty}^{+\infty} (\eta - s)_+ dF_Z(s) \end{aligned}$$

and

$$F_Z^{(2)}(\eta) = -\int_{-\infty}^{+\infty} (\eta - s)_+ dF_V(s)$$

From above proposition, the assertion follows.

9.4 Conjugate Duality and Second Order Stochastic Dominance

As we defined above $F_Z^{(2)}(\cdot)$ is a convex function. Thus, we can have their conjugate as follows:

$$*[F_Z^{(2)}]^*(\alpha) = \sup_{\eta\in\mathbb{R}}[\alpha\eta - F_Z^{(2)}(\eta)]$$

$$*[F_V^{(2)}]^*(\alpha) = \sup_{\eta\in\mathbb{R}}[\alpha\eta - F_V^{(2)}(\eta)]$$

By *Fenchel-Moreau Theorem*, $F_Z^{(2)} = ([F_Z^{(2)}]^*)^*$,

$$F_Z^{(2)}(\eta) = \sup_\alpha\{\alpha\eta - (F_Z^{(2)})^*(\alpha)\}$$
$$F_V^{(2)}(\eta) = \sup_\alpha\{\alpha\eta - (F_V^{(2)})^*(\alpha)\}$$

This gives us another characterization:

$$Z \geq_{(2)} V \text{ if and only if } (F_Z^{(2)})^*(\alpha) \geq (F_V^{(2)})^*(\alpha), \quad \alpha \in (0,1]$$

This looks like an even more convoluted representation but it is actually very important one, connecting risk measure and *stochastic dominance*. Notice that

$$\begin{aligned}(F_Z^{(2)})^*(\alpha) &= \sup_\eta\{\alpha\eta - \mathbb{E}[(\eta - Z)_+]\} \\ &= \alpha \sup_{\eta\in\mathbb{R}}\{\eta - \frac{1}{\alpha}\mathbb{E}[(\eta - Z)_+]\}\end{aligned}$$

If $\alpha = 0$,

$$\begin{aligned}(F_Z^{(2)})^*(0) &= \sup_\eta\{-\mathbb{E}[(\eta - Z)_+]\} \\ &= \sup_\eta \int_{-\infty}^{\eta} F_Z(t)dt = 0\end{aligned}$$

It is achieved when $\eta \to -\infty$. When $\alpha \neq 0$, we want to solve the following unconstrained optimization problem:

$$\min_\eta \ \mathbb{E}[(\eta - Z)_+] - \alpha\eta$$

The necessary condition for optimality says that there exists $g \in D_\eta\mathbb{E}[(\eta - Z)_+]$ such that

$$g - \alpha = 0$$

Let's calculate the sub-gradient, since it is in $\mathbb{R}^2$, it is very easy:

$$\frac{d^+}{d\eta}\mathbb{E}[(\eta - Z)_+] = \mathbb{P}[Z \leq \eta],$$
$$\frac{d^-}{d\eta}\mathbb{E}[(\eta - Z)_+] = \mathbb{P}[Z < \eta].$$

Thus, there exists an optimal $\tilde{g}$

$$\tilde{g} \in [\mathbb{P}\{Z < \eta\}, \mathbb{P}\{Z \leq \eta\}]$$

Equivalently, the optimal choice of η should be

$$\eta = \min\{\eta : \alpha \leq F_Z(\eta)\} = F_Z^{-1}(\alpha)$$

the α-quantile.

We have several facts about the conjugate duality, that is true for general convex function satisfying very mild assumptions ($F_Z^{(2)}$ indeed satisfies):

$$\begin{aligned}&\text{(i). } F_Z^{(2)}(\eta) + [F^{(2)_Z}]^*(\alpha) = \alpha\eta;\\ &\text{(ii). } \alpha \in \partial F_Z^{(2)}(\eta);\\ &\text{(iii). } \eta \in \partial [F_Z^{(2)}]^*(\alpha).\end{aligned}$$

Additionally, we know that

$$(F_Z^{(2)})^*(\alpha) = (F_Z^{(2)})^*(0) + \int_0^\alpha F_Z^{-1}(\beta)d\beta$$

the second term is the *Lorenz curve* in economics. So another characterization comes that: if the lorenze curve of Z lies above lorenze curve of V, then $Z \geq_{(2)} V$.

On the other hand,

$$(F_Z^{(2)})^*(\alpha) = \int_0^\alpha F_Z^{(-1)}(\beta)d\beta = \alpha \cdot \frac{1}{\alpha}\int_0^\alpha F_Z^{-1}(\beta)d\beta = -\alpha \mathrm{AVaR}_\alpha(Z)$$

Thus,

$$Z \geq_{(2)} V \text{ if and only if } \mathrm{AVaR}_\alpha(Z) \leq \mathrm{AVaR}_\alpha(V),\ \alpha \in (0,1]$$

Since any law-invariant coherent risk measure can be expressed as a convex combination of $\mathrm{AVaR}_\alpha(\cdot)$, we can further have

$$Z \geq_{(2)} V \text{ if and only if } \rho(Z) \leq \rho(V)$$

Thus, we discovered the relationship between stochastic dominance and law-invariant coherent risk measure.

Chapter 10

Lecture 13

Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have random variables X and Y. As a convention, we treat as the gain of a random event. From previous chapter, we know that $X \geq_{(2)} Y$ if and only if $\mathbb{E}[(\eta - X)_+] \leq \mathbb{E}[(\eta - Y)_+]$ for all $\eta \in \mathbb{R}$. Obviously, it leads to infinitely many inequalities, actually continuum of inequalities. However, if Y is a discrete random variable, we can actually reduce it to finitely many inequalities.

With out loss of generality, we order y ascendingly, i.e., $y_1 \leq y_2 \leq \cdots \leq y_n$. Each realization has probability p_i correspondingly, where $i = 1, ..., n$. Since it has finite number of scenarios, we can express the expectation explicitly,

$$\mathbb{E}[(\eta - Y)_+] = \sum_{k=1}^{n} p_k(\eta - y_k)_+$$

Consider this is a function of η, it is piecewise linear as in Figure 10.1. To say $\mathbb{E}[(\eta - X)_+] \leq$

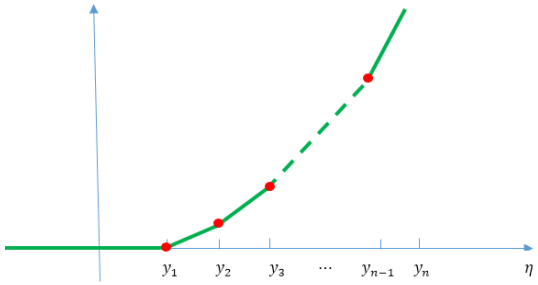

Figure 10.1: As a function of η

$\mathbb{E}[(\eta - Y)_+]$ is equivalent to say that

$$\mathbb{E}[(y_k - X)_+] \leq \mathbb{E}[(y_k - Y)_+], \quad k = 1, ..., n$$

This is because that $\mathbb{E}[(\eta - X)_+]$ is a convex function of η, thus if at each node point the function lies below $\mathbb{E}[(y_k - Y)_+]$, all intermediate point will also lie below. Notice, it doesn't matter whether X has discrete distribution or continuous, we can always reduce it to finitely many inequalities. We will later on use this observation.

10.1 Strassen Theorem

Assume the space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then $X \geq_{(1)} Y$ if and only if $\exists X' \sim^{\mathscr{D}} X$, $Y \sim^{\mathscr{D}} Y$, $X' \geq Y'$ for all $\omega \in \Omega$. This is very easy to verify, we can find a uniform distributed random variable U, then define $X' = F_X^{-1}(U)$, $Y' = F_X^{-1}(U)$. Because $F_X(\cdot) \leq F_Y(\cdot)$, $F_X^{-1}(\cdot) \geq F_Y^{-1}(\cdot)$. Thus, the result follows.

Now, we can characterize the second order stochastic dominance by conditional expectation, this goes under the name of *Strassen Theorem*.

Theorem 10.1.1 $X \geq_{(2)} Y$ if and only if $\exists X' \sim^{\mathscr{D}} X, Y' \sim^{\mathscr{D}} Y$ and $X' \geq \mathbb{E}[Y'|X']$ *a.s.*.

The proof in " $\Leftarrow$ " is easy. Pick any $u : \mathbb{R} \mapsto \mathbb{R}$ that is concave and non-decreasing, we have

$$u(X') \geq u(\mathbb{E}[Y'|X']), \ a.s.,$$

Thus,

$$\mathbb{E}[u(X')] \geq \mathbb{E}[u(\mathbb{E}[Y'|X'])]$$

From *Jensen's inequality*,

$$u(\mathbb{E}[Y'|X']) \geq \mathbb{E}[u(Y')|X']$$

Thus,

$$\begin{aligned} \mathbb{E}[u(X')] &\geq \mathbb{E}[u(\mathbb{E}[Y'|X'])] \\ &\geq \mathbb{E}[\mathbb{E}[u(Y')|X']] \\ &= \mathbb{E}[u(Y')] \end{aligned}$$

Thus, $X' \geq_{(2)} Y'$.

For the converse direction, it is very involved in general. We shall prove it in the case of finite realizations, that is, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$, $p_k = \frac{1}{n}$ for $k = 1, ..., n$. We can think of

X and Y as vectors $[x_1, \cdots, x_n]^\top$ and $[y_1, \cdots, y_n]^\top$. Then the *Strassen's Theorem* reads as follows: $X \geq_{(2)} Y$ if and only if $\exists$ doubly stochastic matrix Q of dimension $n \times n$ such that $X \geq QY$.

Without of loss of generality, we can assume that $x_1 < x_2 < \cdots x_n$. And q_{ij} is defined as $\mathbb{P}\{Y = y_i | X = x_i\}$, thus

$$(QY)_i = \sum_{j=1}^{n} \mathbb{P}(Y = y_j | X = x_i) \cdot y_j = \mathbb{E}[Y | X = x_i]$$

The direction " $\Leftarrow$ " is again easy. Take a non-decreasing concave function $u(\cdot)$, then

$$\begin{aligned}
\mathbb{E}[u(X)] &= \frac{1}{n} \sum_{i=1}^{n} u(x_i) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} u(\sum_{j=1}^{n} q_{ij} y_j) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} u(y_j) \\
&= \frac{1}{n} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} q_{ij} \right) u(y_j) \\
&= \mathbb{E}[u(Y)]
\end{aligned}$$

" $\Rightarrow$ ", we assumed that $x_1 \leq x_2 \leq \cdots \leq x_n$, $y_1 \leq y_2 \leq \cdots \leq y_n$. We know that $X \geq_{(2)} Y$ if and only if $\mathrm{AVaR}_\alpha(X) \leq \mathrm{AVaR}_\alpha(Y)$, $\forall 0 < \alpha \leq 1$. On the other hand,

$$\mathrm{AVaR}_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha F_X^{-1}(\beta) d\beta$$

Denote the *Lorenze Curve*:

$$L_X(\alpha) = \int_0^\alpha F_X^{-1}(\beta) d\beta$$

Then, $X \geq_{(2)} Y$ if and only if $L_X(\alpha) \geq L_Y(\alpha)$, $\forall 0 < \alpha \leq 1$. Since $L_X(\cdot)$ and $L_Y(\cdot)$ are certainly convex functions, thus we shall have $L_X(\frac{k}{n}) \geq L_Y(\frac{k}{n})$ for $k = 1, ..., n$. Observe that

$$L_X(\frac{k}{n}) = \frac{1}{n} (x_1 + x_2 + \cdots + x_k)$$

Thus, to have $L_X(\alpha) \geq L_Y(\alpha)$, we actually need

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j, \quad k = 1, ..., n$$

Thus, to prove the existence of Q such that $X \geq QY$ is equivalent to prove $X \geq Y'$, where $Y' = QY$, because $Y' \geq_{(2)} Y$. We can run the following algorithmic construction: find the first k such that $y_k > x_k$, if we can't find it the theorem is proved. Observe that

$$x_i \geq y_i, \ i = 1, ..., k-1$$
$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$$

We also find the first i such that $x_i > y_i$, define $Y' = MY$, where the doubly stochastic matrix is defined as follows, the diagonal are all 1, except $M_{ii} = 1 - m_{ik}$, $m_{kk} = 1 - m_{ik}$. In the meanwhile, $M_{ik} = M_{ki} = m_{ik}$. m_{ik} needs to satisfy the following relationships:

$$y_i' = (1 - m_{ik})y_i + m_{ik}y_k \leq x_i,$$
$$y_k' = m_{ik}y_i + (1 - m_{ik})y_k \geq x_k,$$
$$y_j' = y_j \text{ for all other } j.$$

We choose $m_{ik} = \min\left(\frac{x_i - y_i}{y_k - y_i}, \frac{y_k - x_k}{y_k - y_i}\right)$. This ensures that either we have $x_k = y_k$ or $x_i = y_i$, since there are only finite number of x and y. We will finally manufacture :

$$Y' = M_{i_l k_l} \cdots M_{i_2 k_2} M_{i_1 k_1} Y$$

where $M_{\cdot}$ are doubly stochastic matrix as defined above. Indeed, we find Q to make $X \geq Y'$.

Now, we have the *Strassen Theorem* that $X \geq_{(2)} Y$ is equivalent to $X \geq QY$ for some doubly stochastic matrix Q. Now if $\bar{Y}$ has fixed distribution, we can easily verify that

$$\mathscr{X}_{(2)} = \{X : X \geq_{(2)} Y\}$$

is a convex set. It is not true for the first order dominance, i.e.,

$$\mathscr{X}_{(1)} = \{X : X \geq_{(1)} Y\}$$

is not convex in general. For example, if $Y = 1, -1$ with equal probability, and X is the same but reversing the realization. Then $X \geq_{(1)} Y$ and $Y \geq_{(1)} Y$, but $\frac{1}{2}X + \frac{1}{2}Y = 0$, if you draw the distribution, you can easily find that part of the latter distribution lie above the former one, which violates the characterization of the first order stochastic dominance. However, interestingly,

$$\mathscr{X}_{(2)} = \text{conv}(\mathscr{X}_{(1)})$$

Why? If Q is doubly stochastic matrix, then there exists permutation matrix $P_1, P_2, \cdots, P_l$ and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, ..., $\lambda_l \geq 0$ such that $\sum_{j=1}^{l} \lambda_j = 1$ and

$$Q = \sum_{j=1}^{l} \lambda_j P_j$$

Thus,

$$X \geq \sum_{j=1}^{l} \lambda_j P_j Y$$

If we set $Z_j = P_j Y$, it is for sure that $Z_j \geq_{(1)} Y$, or $P_j Y \in \mathscr{X}_{(1)}$. Since $\mathscr{X}_{(1)} \subseteq \mathscr{X}_{(2)}$, the assertion follows. In general case, if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then

$$\mathscr{X}_{(2)} = \text{cl}\left(\text{conv}(\mathscr{X}_{(1)})\right)$$

10.2 Stochastic Dominance and Optimization

Suppose Y has fixed distribution, for example, it is the distribution of the SP 500 index, Z_x is a random variable depending on decision variable $x \in \mathcal{X}$, for example, return rate of portfolio x. We can formulate the following optimization problem:

$$\begin{aligned} &\max \mathbb{E}[Z_x] \\ \text{subject to: } &Z_x \geq_{(2)} Y \\ &x \in \mathcal{X} \end{aligned}$$

Assume that $Z_x = g(x, \omega)$, where g is concave for all ω and $\Omega = \{\omega_1, ..., \omega_n\}$ with probability $p_1, ..., p_n$. We can rewrite the optimization problem as:

$$\begin{aligned} &\max \sum_{i=1}^{n} p_i g(x, \omega_i) \\ \text{subject to: } &\mathbb{E}[(\eta - Z_x)_+] \leq \mathbb{E}[(\eta - Y)_+], \text{ for all } y \in \mathbb{R} \\ &x \in \mathcal{X} \end{aligned}$$

We shall focus on the analysis of the first constraint. That is,

$$\sum_{i=1}^{n} p_i (\eta - g(x, \omega_i))_+ \leq \sum_{i=1}^{n} p_i (\eta - y_i)_+, \ \forall \eta \in \mathbb{R}$$

But from the discussion of the very beginning of the chapter, we know we don't need to consider all η but only $\eta = y_k$, $k = 1, ..., n$. This will leads to the following optimization problem:

$$\begin{aligned} &\max \sum_{i=1}^{n} p_i g(x, \omega_i) \\ \text{subject to: } &\sum_{i=1}^{n} p_i (y_k - g(x, \omega_i))_+ \leq \sum_{i=1}^{n} p_i (y_k - y_i)_+, \ \forall \eta \in \mathbb{R}, k = 1, ..., n \\ &x \in \mathcal{X} \end{aligned}$$

If the qualification constraints are satisfied, then there exists Lagrangian multipliers λ, we can form the *Lagrangian*

$$L(x,\lambda) = \sum_{i=1}^{n} p_i g(x,\omega_i) - \sum_{k=1}^{n} \lambda_k [\sum_{i=1}^{n} p_i (y_k - g(x,\omega_i))_+ - \sum_{i=1}^{n} p_i (y_k - y_i)_+]$$

there exists $\hat{\lambda}$ such that $\hat{x}$ the solution of the original problem also solves:

$$\max_{x \in \mathcal{X}} L(x, \hat{\lambda})$$

Let's rewrite the Lagrangian by switching the summation:

$$L(x,\lambda) = \sum_{i=1}^{n} p_i g(x,\omega_i) - \sum_{i=1}^{n} p_i \sum_{k=1}^{n} \lambda_k (y_k - g(x,\omega_i))_+ + \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{n} p_i (y_k - y_i)_+$$

If we define $u_\lambda : \mathbb{R} \mapsto \mathbb{R}$ as

$$u_\lambda(Z) = -\sum_{k=1}^{n} \lambda_k (y_k - Z)_+$$

Then, we can write

$$L(x,\lambda) = \mathbb{E}[Z_x] + \mathbb{E}[u_\lambda(Z_x)] - \mathbb{E}[u_\lambda(Y)]$$

where u_λ is clearly concave and non-decreasing. Therefore, the optimality condition reads as follows: $\exists \hat{u}_\lambda(\cdot)$ such that

$$\max_{x \in \mathcal{X}} \mathbb{E}[Z_x + \hat{u}(Z_x)] - \mathbb{E}[\hat{u}(Y)]$$

If we denote it as $\tilde{L}(x, \hat{u})$, by Lagrangian duality, we actually choose a utility function $\hat{u}$ minimize

$$\min_{u \in \mathscr{U}} \tilde{L}(x, u)$$

Chapter 11

Dynamic Risk Measure

11.1 Initial Setup

Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we equip it with a discrete filtration: $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\{1,2,\dots,T\}}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$. Again, we define the vector space $\mathscr{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}, \mathbb{P})$. For the sequence of random variables, $Z_t \in \mathscr{Z}_t$ if and only if Z_t is $\mathcal{F}_t$–measurable and $\mathbb{E}[|Z_t|^p] < +\infty$, for $1 \leq p < +\infty$. The objective is to measure the risk of a random sequence, that is $\rho(Z_1, Z_2, \cdots, Z_T) \mapsto \mathbb{R}$. For this sake, we define the conditional risk mapping:

Definition 11.1.1 $\rho_t : \mathscr{Z}_{t+1} \mapsto \mathscr{Z}_t$ is called a *conditional risk mapping* if the following axioms are satisfied:

- *Convexity:* for all Z_{t+1}, V_{t+1} in $\mathscr{Z}_{t+1}$ and any $\lambda \in [0, 1]$,

$$\rho_t(\lambda Z_{t+1} + (1-\lambda)V_{t+1}) \leq \lambda\rho_t(Z_{t+1}) + (1-\lambda)\rho_t(V_{t+1}),$$

- *Monotonicity:* for all Z_{t+1}, V_{t+1} in $\mathscr{Z}_{t+1}$, if $Z_{t+1} \leq V_{t+1}$ a.s., then

$$\rho_t(Z_{t+1}) \leq \rho_t(V_{t+1}), a.s.$$

- *Translation Property:* for all $Z_{t+1} \in \mathscr{Z}_{t+1}$ and $V_{t+1} \in \mathscr{Z}_t$,

$$\rho_t(Z_{t+1} + V_t) = \rho_t(Z_{t+1} + V_t) \ a.s.$$

- *Positive Homogeneity:* for $\gamma > 0$ and $Z_{t+1} \in \mathscr{Z}_{t+1}$,

$$\rho_t(\gamma Z_{t+1}) = \gamma\rho_t(Z_{t+1}) \ a.s.$$

Let's give several examples of such conditional risk mapping:

Example 11.1.1 For $Z_{t+1} \in \mathscr{Z}_{t+1}$,

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t]$$

$$\rho_t(Z_{t+1})(\omega) = \min_{\eta\in\mathbb{R}}\{\eta + \frac{1}{\alpha}\mathbb{E}[(Z_{t+1}-\eta)_+|\mathcal{F}_t](\omega)\}$$

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa\mathbb{E}\{(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t])_+|\mathcal{F}_t\},\ 0 \le \kappa \le 1$$

One can easily verify that they are all $\mathcal{F}_t$–measurable random variables.

11.2 Local Property

From the axioms in the previous section about the conditional risk mapping, we can derive a very important property, so called *local property*.

Definition 11.2.1 (Local Property) For any $B \in \mathcal{F}_t$,

$$\rho_t(\mathbf{1}_B Z_{t+1}) = \mathbf{1}_B \rho_t(Z_{t+1})$$

We claim that conditional risk mapping indeed enjoy this property. Here is the sketch of the proof. Consider the dual space of $\mathscr{Z}_t$, denoted as $\mathscr{Z}_t^*$, it is the space of the linear functionals defined on $\mathscr{Z}_t$ such that the following pair returning a scalar

$$\begin{aligned} r_\eta(Z_{t+1}) &= \langle \eta, \rho_t(Z_{t+1})\rangle \\ &= \int_\Omega \eta(\omega)\rho_t(Z_{t+1})(\omega)\mathbb{P}(d\omega) \end{aligned}$$

Note, the dual space of $\mathcal{L}_p$ is $\mathcal{L}_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. We now pick special class of η such that $\eta \ge 0$ almost surely and

$$\int_\Omega \eta(\omega)\mathbb{P}(d\omega) = 1$$

We can prove that r_η is a coherent risk measure. The verification of convexity, monotonicity and homogeneity is trivial, readers should exercise on that, we only prove the translation property. Take $a \ge 0$,

$$\begin{aligned} r_\eta(Z_{t+1} + a) &= \int_\Omega \eta(\omega)\rho_t(Z_{t+1} + a)\mathbb{P}(d\omega) \\ &= \int_\Omega \eta(\omega)\rho_t(Z_{t+1})\mathbb{P}(d\omega) + \int_\Omega a\eta(\omega)\mathbb{P}(d\omega) \\ &= r_\eta(Z_{t+1}) + a \end{aligned}$$

Thus, $r_\eta : \mathscr{Z}_{t+1} \mapsto \mathbb{R}$ is coherent risk measure of risk.

Chapter 12

Appendix A - Analysis

12.1 Modes of continuity

Continuous functions is an important class of function in advanced analysis. We will introduce various concept of continuity, but let's firstly recall the definition of continuous function on metric spaces:

Definition 12.1.1 (*Version 1*) Given two metric space (X, d_X) and (Y, d_Y) and a function $f : X \mapsto Y$ then f is continuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that all x in X satisfying $d_X(x, x_0) < \delta$ will also satisfy $d_Y(f(x), f(x_0)) < \epsilon$.

Definition 12.1.2 (*Version 2*) Given two metric space (X, d_X) and (Y, d_Y), a function $f : X \mapsto Y$ is continuous if and only if for all open set $U \in Y$, $f^{-1}(U)$ is also open in X.

In the case of real function,

Definition 12.1.3 (*Version 3*) For every convergent sequence $\{x_n\}_{n\geq 1}$ in X, where X is a metric space (X, d_X), a real function f is continuous implies

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

The concept of continuity for functions between metric space can be strengthened in various ways by liming the way δ depends on ϵ and the point x_0.

12.1.1 Lipschitz Continuity & Hölder continuity

A *Lipschitz function* is limited in how fast it can change: there exists a definite real number such that, for every pair of points on the graph of the function, the absolute value of the

slope of the line connecting them is not greater than this real number. Let's give the formal definition:

Definition 12.1.4 Given two metric space (X, d_X) and (Y, d_Y), a function $f : X \mapsto Y$ is *Lipschitz Continuous* if there exists a real constant $L \geq 0$ such that, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$

L is called the *Lipschitz constant* for the function f.

In the case of real function,

Proposition 12.1.1 Continuously differentiability implies Lipschitz continuous over a compact set C.

Proof. From differentiability of $x_0 \in C$, we have

$$|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)| < \epsilon$$

for an appropriate chosen $\delta > 0$. Multiply both sides by $|x - a|$ to get

$$|f(x) - f(x_0) - (x - x_0)f'(x_0)| < |x - x_0|\epsilon$$

Triangle inequality gives

$$|f(x) - f(x_0)| - |x - x_0||f'(x_0)| < |x - x_0|\epsilon$$

or, equivalently,

$$|f(x) - f(x_0)| < (|f'(x_0)| + \epsilon)|x - x_0|$$

set $L_1 = |f'(x_0)| + \epsilon|$, it implies for $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| < L_1|x - x_0|$$

On the other hand, since f is continuous on a closed and bounded set, by *mean value theorem*, for any $|x - x_0| > \delta$

$$|f(x) - f(x_0)| \leq |f'(c)||x - x_0| \quad c \in [x, x_0]$$

Set $L_2 = |f'(c)|$, we have for any $|x - x_0| > \delta$, $|f(x) - f(x_0)| \leq L_2|x - x_0|$. We take $L = \max\{L_1, L_2\}$ to have the desired result. □

We can also have Lipschitz property locally:

Definition 12.1.5 A real function is called *locally Lipschitz continuous* if for every $\hat{x}$ in X there exists a neighborhood U of x such that f restricted to U is *Lipschitz continuous*.

And we have a corresponding result for $f \in C^1$, i.e., continuously differentiable function.

Proposition 12.1.2 Continuously differentiable real function is locally Lipschitz.

Proof. Choose some open ball $B(\hat{x}, \epsilon)$, the closure is compact so the derivative $\frac{\partial f}{\partial x}$ is bounded by some L on the ball. Now suppose $x, y \in B(\hat{x}, \epsilon)$ then using *Newton-Lebniz Rule*[1], we have

$$f(x) - f(y) = \int_0^1 \frac{\partial f(y + t(x-y))}{\partial x}(x-y)dt$$

Hence we can get the bound:

$$f(x) - f(y) \leq \int_0^1 ||\frac{\partial f(y + t(x-y))}{\partial x}||||x-y||dt \leq L||x-y||$$

The assertion follows immediately. □

Remark 12.1.3 A function f is said to be continuously differentiable if the derivative $f'(x)$ exists, and is itself a continuous function. Though the derivative of a differentiable function never has a jump discontinuity, it is possible for the derivative to have an essential discontinuity. For example, the function

$$f(x) = x^2 sin(1/x) \text{ if } x \neq 0, \text{ otherwise } 0$$

is differentiable at 0, since

$$f'(0) = \lim_{\epsilon \to \infty} \left(\frac{\epsilon^2 sin(1/\epsilon) - 0}{\epsilon} \right) = 0$$

exists. However, for $x \neq 0$,

$$f'(x) = 2xsin(1/x) - cos(1/x)$$

which has no limit as $x \to 0$.

We list several properties of Lipschitz function:

- An everywhere differentiable function $g : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous (with $L = \sup |g'(x)|$) if and only if it has bounded first derivative;
- If $\{f_n\}_{n \geq 1}$ is a sequence of Lipschitz continuous mappings between two metric spaces, and that all f_n have Lipschitz constant bounded by some L, if f_n converges to a mapping f uniformly, then f is also Lipschitz, with Lipschitz constant bounded by the same L; (The proof will be postponed after defining *uniform convergence*)

[1]Set $g(t) = f(x + t(y-x))$, then $g(1) - g(0) = \int_0^1 g'(t)dt$

- Every Lipschitz continuous map f is uniformly continuous.(The proof will be postponed after defining the *uniform continuity*)

As a generalization of Lipschitz continuous function, we also have Hölder continuous function:

Definition 12.1.6 A function f defined on X is said to be Hölder continuous or satisfy a Hölder condition of order $\alpha > 0$ on X if there exists a constant $M > 0$ such that

$$d_Y(f(x), f(y)) \leq M d_X(x, y)^\alpha$$

for all $x \in X$

Obviously, if $\alpha = 1$, then the function f satisfies Lipschitz continuity and if $\alpha = 0$, then the function is simply bounded. The notion of Hölder continuity often arises in functional analysis relevant to solving partial differential equations. We will not go into the details.

12.1.2 Uniform Continuity

Let's firstly give the definition of *uniform continuity*,

Definition 12.1.7 Given two metric space (X, d_X) and (Y, d_Y), a function $f : X \mapsto Y$ is called *uniformly continuous* if for every real number $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $d_X(x, y) < \delta$, we have

$$d_Y(f(x), f(y)) < \epsilon$$

The difference between being uniformly continuous and being simply continuous at every point, is that in uniform continuity the value of δ depends only on ϵ and not on the point in the domain.

Proposition 12.1.4 Every Lipschitz condition map between two metric space is uniformly continuous.

Proof. Given two metric space (X, d_X) and (Y, d_Y).Let $\epsilon > 0$, $x, y \in X$, let L be a Lipschitz constant for f. Take $\delta = \epsilon/L$, then if $d_X(x, y) < \delta$, we have

$$L d_X(x, y) < \epsilon$$

By the Lipschitz condition on f, we know that

$$d_Y(f(x), f(y)) \leq L d_X(x, y)$$

These last two statements together imply $d_Y(f(x), f(y)) < \epsilon$, thus f is *uniformly continuous* on (X, d_X). □

Proposition 12.1.5 Every function which is differentiable and has bounded derivative is uniformly continuous.

Proof. From differentiability of $x_0 \in C$, we have

$$|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)| < \epsilon$$

for an appropriate chosen $\delta > 0$. Multiply both sides by $|x - a|$ to get

$$|f(x) - f(x_0) - (x - x_0)f'(x_0)| < |x - x_0|\epsilon$$

Triangle inequality gives

$$|f(x) - f(x_0)| - |x - x_0||f'(x_0)| < |x - x_0|\epsilon$$

or, equivalently,

$$|f(x) - f(x_0)| < (|f'(x_0)| + \epsilon)|x - x_0|$$

Pick any $\hat{\epsilon}$ (different from ϵ fixed at the beginning and used with differentiation definition). Pick $\hat{\delta} = \min\left(\delta, \frac{\hat{\epsilon}}{|f'(x_0)+\epsilon|}\right)$. Clearly,

$$|x - x_0| < \hat{\delta} \Rightarrow |f(x) - f(x_0)| < \hat{\epsilon}$$

□

Remark 12.1.6 The exponential function $x \mapsto e^x$ is continuous everywhere on the real line but is not uniformly continuous on the line, because its derivative is not bounded.

Proposition 12.1.7 A Hölder continuous function is uniformly continuous.

Proof. Suppose that $f : X \mapsto Y$ is Hölder continuous with exponent α and constant γ. Then, given $\epsilon > 0$, choose $\delta = (\epsilon/\gamma)^{1/\alpha}$, then for any $x_0 \in X$ such that $d_X(x, x_0) < \delta$, then

$$d_Y(f(x), f(x_0)) \leq \gamma d_X(x, x_0)^\alpha < \gamma\delta^\alpha = \epsilon$$

□

Proposition 12.1.8 (*Heine-Cantor Theorem*) If $f : X \mapsto Y$ is a continuous function between two metric spaces, and C is compact, then f is uniformly continuous.

Proof. We prove by contradiction. We assume that f is continuous on the compact metric space C but not uniformly continuous, that means $\exists\epsilon_0 > 0$, $\forall\delta > 0$, $\exists x, y \in C$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) \geq \epsilon_0$. Setting $\delta = \frac{1}{n}$, for $n = 1, 2, 3, ...$ gives two sequences $\{x_n\}$, $\{y_n\}$ such that $d_X(x_n, y_n) < \frac{1}{n}$ implies

$$d_X(f(x_n), f(y_n)) \geq 0$$

Since C is compact, by *Bolzano-Weierstrass Theorem*, there exists two converging subsequence x_{n_k} to x_0 and y_{n_k} to y_0 of these two sequences. It follows that

$$d_X(x_{n_k}, y_{n_k}) < \frac{1}{n_k} \text{ and } d_Y(f(x_{n_k}), f(y_{n_k})) \geq \epsilon_0$$

As n increases, x_{n_k} and y_{n_k} will converge to the same point, that is $x_0 = y_0$. Since f is continuous, we hence conclude

$$d_Y(f(x_{n_k}), f(y_{n_k})) \to 0$$

which is a contradiction being $d_Y(f(x_n), f(y_n)) \geq \epsilon_0$. □

Proposition 12.1.9 (*Generalized Heine-Cantor Theorem*) Compactly supported continuous function is uniformly continuous.

Corollary 12.1.10 Continuous functions on $\mathbb{R}$ that vanish at infinity is uniformly continuous.

12.2 Modes of Convergence

Given two metric spaces (X, d_X) and (Y, d_Y), a sequence of function $\{f_n\}_{n\geq 1} : X \mapsto Y$, we can have different mods of convergence.

Definition 12.2.1 (*Point-wise Convergence*) $\forall x \in X$ and $\forall \epsilon > 0$, $\exists n_0(x, \epsilon)$ such that

$$d_Y(f_n(x) - f(x)) < \epsilon \quad \forall n \geq n_0$$

We say $f_n(x)$ converges to $f(x)$ point-wise.

Here, we notice that the threshold n_0 depends on both x and ϵ. Like the case of continuity, we can have another strong version of convergence, that is the *Uniform Convergence*.

Definition 12.2.2 (*Uniform Convergence*) $\forall x \in X$ and $\forall \epsilon > 0$, $\exists n_0(\epsilon)$ such that

$$d_Y(f_n(x) - f(x)) < \epsilon \quad \forall n \geq n_0$$

We say $f_n(x)$ converges to $f(x)$ uniformly.

In the uniform case, whenever a ϵ is given, we can have a threshold n_0 such that for all $n \geq n_0$, no matter what x is taken, the sequence converges to the limit. We can also consider a sequence $a_n = \sup_x\{|f_n(x) - f(x)|\}$ where the supremum is taken over all $s \in X$, Then, f_n converges to f uniformly if and only if $a_n \to 0$. Obviously, uniform convergence implies convergence, the converse is not true. Here is an example:

Example 12.2.1 Take $f_n : [0,1] \mapsto \mathbb{R}$ as $f_n(x) = x^n$ for all $n \geq 1$. Then $\{f_n\}_{n\geq 1}$ converges point-wise to the function $f(x) = 0$ if $x \leq 1$ and $f(1) = 1$. However, this convergence is not uniform, because solving for n gives $n > \frac{\log \epsilon}{\log x}$, this depends on x as well as on ϵ. It is impossible to find a suitable bound for n that does not depend on x because for any non-zero value of ϵ, the ratio $\frac{\log \epsilon}{\log x}$ grows without bounds as x tends to 1.

Also, uniform convergent can occur locally, which leads to the notion of *locally uniform convergent*.

Definition 12.2.3 (*Locally Uniform Convergence*) The sequence $\{f_n\}_{n\geq 1}$ is said to be *locally uniformly convergent* with limit f if for every $x \in X$, there exists an $r > 0$ such that $\{f_n\}_{n\geq 1}$ converges uniformly on $B(x.r) \cap X$.

Then, it immediately follows that

Proposition 12.2.2 Every uniformly convergent sequence is locally uniformly convergent.

Another straightforward result is:

Proposition 12.2.3 A sequence of function on metric space, with the image metric space being complete, is uniformly convergent if and only if it is *uniformly Cauchy*.

Proof. (" $\Rightarrow$ ") If $\{f_n\}_{n\geq 1}$ is uniformly convergent to the limit f, then for all $x \in X$ and $\epsilon > 0$, there exists $n_0(\epsilon)$, such that

$$\begin{aligned} d_Y(f_n(x), f(x)) < \epsilon/2 \quad \forall n \geq n_0 \\ d_Y(f_m(x), f(x)) < \epsilon/2 \quad \forall m \geq n_0 \end{aligned}$$

Then, by triangle inequality, we have for $n, m > n_0(\epsilon)$,

$$d_Y(f_m(x), f_n(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

(" $\Leftarrow$ ") Given *Uniformly Cauchy* sequence $\{f_n\}_{n\geq 1}$, then for all $\epsilon > 0$, $\exists N > 0$ such that for all $x \in X$:

$$d(f_n(x), f_m(x)) < \epsilon \quad m, n > N$$

since the image space Y is complete, it has a limit, we call it $f_\infty(x)$. Thus,

$$d(f_n(x), f_\infty(x)) < \epsilon, \quad \forall n \geq N$$

Then the assertion follows. □

Remark 12.2.4 As we already saw, a sequence of functions $\{f_n\}$ from a set X to a metric space Y is said to be *uniformly Cauchy* if, for all $\epsilon > 0$, $\exists N > 0$ such that for all $x \in X$:

$$d(f_n(x), f_m(x)) < \epsilon \quad m, n > N$$

It is a stronger version of being merely *Cauchy sequence*, because here the threshold n does not depend on x.

Let's give the proof for the following *proposition* that arises in *Section 1.1*:

Proposition 12.2.5 If $\{f_n\}_{n\geq 1}$ is a sequence of Lipschitz continuous mappings between two metric spaces, and that all f_n have Lipschitz constant bounded by some L, if f_n converges to a mapping f uniformly, then f is also Lipschitz, with Lipschitz constant bounded by the same L

Proof. By Lipschitz continuity, we have, for any $x, y \in X$, there exists $L_n > 0$ such that

$$d_Y(f_n(x), f_n(y)) \leq L_n d_X(x, y) \quad \forall n$$

Also, by *Uniform Continuity*, $\forall \epsilon$,

$$\begin{aligned} d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \geq n_0(\epsilon) \\ d_Y(f_n(y), f(y)) < \epsilon \quad \forall n \geq n_0(\epsilon) \end{aligned}$$

for arbitrary $x, y \in X$. Then

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + L_n d(x, y) + d(f_n(y), f(y))$$

Send $n \to \infty$, we have that

$$d(f(x), f(y)) \leq L d(x, y)$$

where $L = \sup_n\{L_n\}$. □

12.2.1 Application of Uniform Continuity

If (X, d_X) is Euclidean metric space, in particular, X is an interval. We can talk about the continuity of the functions f_n and f. The following is the more important result about uniform convergence

Theorem 12.2.6 (*Uniform Convergence Theorem*) If $\{f_n\}_{n\geq 1}$ is a sequence of continuous function which converges uniformly towards the function f on an interval X, then f is continuous on X as well.

The proof is very similar to Proposition 12.2.5, we leave to the readers to prove. This Theorem is important, since point-wise convergence of continuous function is not enough to guarantee continuity of the limit function. For a locally compact space, continuity is equivalent to local uniform continuity, and thus the uniform limit of continuous function is continuous.

For differentiability, if X is an interval and all the functions f_n are differentiable and converge to a limit f, it is often desirable to differentiate the limit function f by taking the limit of the derivatives of f_n. This is however in general not possible, even if the convergence is uniform, the limit function need not be differentiable, and even if it is differentiable, the derivative of the limit function need not be equal to the limit of the derivatives. The precise statement is the following:

Proposition 12.2.7 Let $I := [a, b]$ be an interval and $f_n : I \mapsto \mathbb{R}$ differentiable function with f_n' converges uniformly to a function $g : I \mapsto \mathbb{R}$. Suppose $\exists x_0 \in I : \lim_{n\to\infty} f_n(x_0) = L \in \mathbb{R}$. Then the f_n converge uniformly to a differentiable function $f : I \mapsto \mathbb{R}$ with $f' = g$.

Proof. Since $\{f_n(x_0)\}$ converges, for each $\epsilon > 0$ and n, m large enough we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &\leq \epsilon|x - x_0| + \epsilon \quad (\textit{By uniform convergence of } f_n') \\ &\leq \epsilon(b - a) + \epsilon \end{aligned}$$

Hence f_n converges uniformly on I to a function f, moreover for each ϵ and m, n large enough, the inequality

$$|\frac{f_n(y) - f_n(x)}{y - x} - \frac{f_m(y) - f_m(x)}{y - x}| \leq \epsilon$$

holds for each $x \neq y \in I$. The above relation implies that $\frac{f_n(y)-f_n(x)}{y-x}$ converge uniformly to $\frac{f(y)-f(x)}{y-x}$. Now we can write

$$\begin{aligned} |\frac{f(y) - f(x)}{y - x} - g(x)| \leq |&\frac{f(y) - f(x)}{y - x} - \frac{f_n(y) - f_n(x)}{y - x}| \\ &+ |\frac{f_n(y) - f_n(x)}{y - x} - f_n'(x)| + |f_n'(x) - g(x)| \end{aligned}$$

For each $\epsilon > 0$ and n large enough we get

$$|\frac{f(y) - f(x)}{y - x} - g(x)| \leq 2\frac{\epsilon}{3} + |\frac{f_n(y) - f_n(x)}{y - x} - f_n'(x)|$$

and for y close enough to x,

$$|\frac{f(y) - f(x)}{y - x} - g(x)| \leq \epsilon$$

so $f'(x)$ exists and is equal to $g(x)$. □

Similarly, one often wants to exchange integrals and limit processes. For the Riemann integral, this can be done if uniform convergence is assumed:

Theorem 12.2.8 If $f_n : [a, b] \mapsto \mathbb{R}$ is a sequence of continuous functions which uniformly converge with limit f on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

(*actually the theorem is still true for f_n just Riemman integrable, but we use this more restrictive version because the proof has fewer techinical details in it, and the continuous version will be general engouh for our purpose.*)

Proof. We know that f is continuous on a closed and bounded interval [2], hence integrable. Notice that

$$|\int_a^b f(x) - f_n(x)dx| \leq \int_a^b |f(x) - f_n(x)|dx$$

However, we also know that given $\epsilon > 0$, there exists N such that $|f(x) - f_n(x)| < \epsilon$ for $n > N$. Therefore, when $n > N$,

$$|\int_a^b f(x) - f_n(x)dx| \leq \int_a^b |f(x) - f_n(x)|dx \leq (b - a)\epsilon$$

Since $b - a$ is finite, this implies that $\lim_{n\to\infty} \int_a^b f(x) - f_n(x)dx = 0$, which is what we want to prove. □

[2] $https://proofwiki.org/wiki/Continuous\_Function\_is_R iemann\_Integrable$

Chapter 13

Appendix B - Probability

13.1 Central Limit Theorem

We will state the most general CLT of sums of independent variables. This result was established by *Lindeberg* and effectively has all the other CLTs for sums of independent random variables as special case.

Theorem 13.1.1 Let $\{X_{ni} \ : \ n \geq 1; i = 1, ..., k_n\}$ be a triangular array of random variables with $X_{n1}, ..., X_{nk_n}$ independent for each n, $\mathbb{E}(X_{ni}) = 0$ and

$$\Sigma_{i=1}^{n} \mathbb{E}\left(X_{ni}^2\right) = 1$$

If for each fixed $\epsilon > 0$,

$$\lim_{n\to\infty} \Sigma_{i=1}^{k_n} \mathbb{E}[X_{ni}^2 \mathscr{X}_{(|X_{ni}|>\epsilon)}] = 0$$

then,

$$Z_n = \Sigma_{i=1}^{k_n} X_{ni} \to Z \sim N(0,1) \quad \text{(in distribution)}$$

Note that the *CLT* for i.i.d. random variables is a special case of the *Lindeberg CLT*. More precisely, if $X_1, X_2, ...$ is a sequence of i.i.d. random variables with mean 0 and variance σ^2 then the asymptotic normality of

$$Z = \frac{\Sigma_{i=1}^{n} X_i}{\sqrt{n}}$$

follows by setting $k_n = n$ and $X_{ni} = \frac{X_i}{\sqrt{n}}$, note that

$$\begin{aligned} \Sigma_{i=1}^{k_n} \mathbb{E}[X_{ni}^2 \mathscr{X}_{|x_{ni}|>\epsilon}] &= \frac{1}{n} \Sigma_{i=1}^{n} \mathbb{E}[X_i^2 \mathscr{X}_{|X_i|>\epsilon\sqrt{n}}] \\ &= \mathbb{E}[X_1^2 \mathscr{X}_{|X_1|>\epsilon\sqrt{n}}] \to 0 \quad \text{as } n \to \infty \end{aligned}$$

By *Hölder's inequality* and the fact $X_i \in L_2$.

The condition

$$\Sigma_{n\to\infty}\Sigma_{i=1}^{k_n}\mathbb{E}[X_{ni}^2\mathscr{X}_{|X_{ni}|>\epsilon}] = 0$$

is called the *Lindeberg Condition*. Next, we will consider weighted sums of i.i.d. random variables. Suppose that $X_1, X_2, ...$ are i.i.d. random variables (with mean 0 and variance 1), $\{c_{ni}; n \geq 1, i = 1, ..., n\}$ are constants and define

$$Z_n = \frac{1}{\sigma_n}\Sigma_{i=1}^{n} c_{ni}X_i$$

where $\sigma_n^2 = c_{n1}^2 + \cdots + c_{nn}^2$; note that $\mathbb{E}(Z_n) = 0$ and $Var(Z_n) = 1$. To determine if Z_n converges to a normal distribution, we need to check the Lindeberg condition with $X_{ni} = c_{ni}X_i/\sigma_n$; we have

$$\Sigma_{i=1}^{k_n}\mathbb{E}[X_{ni}^2\mathscr{X}_{|X_{ni}|>\epsilon}] = \frac{1}{\sigma_n^2}\Sigma_{i=1}^{n} c_{ni}^2\mathbb{E}[X_i^2\mathscr{X}_{|X_i|>\epsilon\sigma_n/|c_{ni}|}]$$

We must determine conditions on the c_{ni}'s to guarantee that the right hand side above tends to 0 as $n \to \infty$.

A somewhat weaker CLT was proved by *Lyapounov* who used the condition

$$\Sigma_{i=1}^{k_n}\mathbb{E}[|X_{ni}^{2+\delta}|] \to 0 \text{ as } n \to \infty$$

where as before, we assume that

$$\Sigma_{i=1}^{k_n}\mathbb{E}\left(X_{ni}^2\right) = 1$$

It is easy to show that *Lyapounov's condition* implies *Lindeberg's condition*; however, in many applications, *Lyapounov's condition* is easier to verify and so is quite uuseful in practice.

Chapter 14

Appendix C - Optimization

14.1 Duality in Linear Programming

We have linear programming *primal problem* defined as:

$$\begin{aligned} \min . \ & \langle c, x \rangle \\ \text{subject to: } & Ax = b \\ & x \geq 0 \end{aligned} \tag{14.1}$$

then the *dual problem* associated with (14.1),

$$\begin{aligned} \min . \ & \langle b, y \rangle \\ \text{subject to: } & A^\top y \leq c \quad \text{or } A^\top y + s = c \\ & y \text{ is unrestricted or } (s \geq 0) \end{aligned} \tag{14.2}$$

Then we have:

Theorem 14.1.1 (*Weak - Duality*) Suppose x is feasible for primal problem and let (y, s) be feasible for the maximization problem. Then $\langle c, x \rangle \geq \langle b, y \rangle$.

Remark 14.1.2 The difference between $\langle c, x \rangle$ and $\langle b, y \rangle$ is called *duality gap*, if the duality gap is 0 then its optimal for both dual and primal.

Theorem 14.1.3 (*Strong - Duality*) Consider the primal and dual of LP:

- if the primal is unbounded, the dual is feasible;
- if the dual is unbounded, the primal is infeasible;
- if both primal and dual are feasible, then both have optimal solutions.

Remark 14.1.4 There are exceptions that both dual and primal can be infeasible. For example, the primal is

$$\begin{aligned} &\min . \ x \\ &\text{subjecto to:} \ \ 0 \cdot x \leq -1 \end{aligned}$$

and the dual problem:

$$\begin{aligned} &\max . \ z \\ &\text{subject to:} \ \ 0 \cdot z + 1 = 0 \\ &\qquad\qquad z \geq 0 \end{aligned}$$

14.2 Convex Analysis Element

14.2.1 Convex Sets

A set $X \in \mathbb{R}^n$ is *convex* if for any two points $a_1, a_2 \in X$, the line segment connecting a_1 and a_2 is also contained in X. Let's list several examples of convex set:

(1) Trivial sets, empty set and the whole space, are both convex;
(2) A *hyperplane*, especially an affine plane is $L = \{x \mid a^T x \leq b\}$, is convex;
(3) A *half-space* $\{x \mid a^\top x \leq b\}$ is convex;
(4) Intersection of convex sets is convex (up to countable many);
(5) A *polyhedron* is intersection of finite number of half-spaces: $a_1^T x \leq b, ..., a_n^T x \leq b_n$ or in matrix notation $Ax \leq b$, we can easily observe that feasible set of an LP is a polyhedron;
(6) *convex hull* is the convexification of a set $X \in \mathbb{R}^n$ in the following sense:

$$conv(X) = \bigcap_{X \subseteq C_i} C_i$$

where C_i is convex, $\forall i$. The convex hull is obviously convex by (4);
(7) The convex hull of a finite set of points is called a *polytope*, i.e., $conv(a_1, a_2, ..., a_k)$, where $a_i \in \mathbb{R}^n$.

Now let's introduce *Cone*:

Definition 14.2.1 A set $K \in \mathbb{R}^n$ is a *cone* if whenever $x \in K$, then $\alpha x \in K$ for all $\alpha \geq 0$.

In general, cone is not convex, but we can define the convex cone if it is both a cone and a convex set.

Definition 14.2.2 A cone K is convex if for any $x_1, x_2 \in K$, $x_1 + x_2 \in K$, $\alpha_1, \alpha_2 \geq 0$.

Next, let's discuss the faces, facets, extreme points and extreme rays of convex set. In a convex set $C \subseteq \mathbb{R}^n$, a subset $F \subseteq C$ is called a *face* if for any two points $a_1, a_2 \in C$, a point strictly between them $a = ta_1 + (1-t)a_2$, $0 < t < 1$, $a \in F$ implies the entire line segment $a_1 - a_2$ is in F. The set C itself is a face of dimension n. Faces of dimension $n-1$ are called *facets* and faces of dimension 0 is called the *extreme point*. Another characterization of extreme point is the following: if $a \in C$ cannot be expressed as strict convex combination of other points of C: $a = \alpha b + (1-\alpha)c$, then $b = a$ or $c = a$.

Let $H = \{x \in \mathbb{R}^n \ : \ v^\top x \geq h\}$ and $P = \{x \in \mathbb{R}^n \ : \ v^\top x = h\}$. If C is a convex set such that $C \subseteq H$, then P is called an *isolating hyperplane* for C. If $C \subset H$, then P is a *strictly isolating hyperplane*.

Theorem 14.2.1 If $P = \{x \in \mathbb{R}^n \ : \ v^\top x = u\}$ is an isolating hyperplane for C and $F = P \cap C$, then F is a face of C.

Theorem 14.2.2 If F_1 and F_2 are faces of the closed convex set $C \in \mathbb{R}^n$, then $F_1 \cap F_2$ is a face of C.

Theorem 14.2.3 (*Minkowski Theorem*) Let C be a closed and bounded convex set then C is the convex hull of its extreme points.

In case of a convex cone K, the only possible extreme point is the tip of the cone O. In this case, we are interested in the one dimensional faces called *extreme rays*. Let $X \in \mathbb{R}^n$, the *conic hull* of X is the intersection of all convex cones containing X. It can be shown that cone(X) is the set of all finite non-negative linear combinations of elements of X, i.e.,

$$cone(x) = \{t_1 a_1 + \cdots + t_n a_n \mid a_i \in x_i, t_i \geq 0, k \text{ is an integer}\}$$

An important category of cone is called *recession cone*.

Definition 14.2.3 Let $C \in \mathbb{R}^m$ be a convex set. The set

$$C_\infty := \{d : C + d \subset C\}$$

is called the *recession cone* of X.

Let C be a convex set and $a \in C$. If there is a vector y such that $a + \alpha y \in C$ for all $\alpha \geq 0$, then y is called a *recession vector* of C.

Lemma 14.2.4 If y is a recession vector from a point $a \in C$, then it is recession vector for any point $b \in C$

Theorem 14.2.5 The set of all recession vectors of a convex set forms a convex cone.

Definition 14.2.4 A polyhedron of the form $P = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$ is called a *polyhedral cone*.

Theorem 14.2.6 Let $C \subset \mathbb{R}^n$ be the polyhedral cone defined by the matrix A. Then the following are equivalent:
(1) The zero vector is an extreme point of C;
(2) The cone C does not contain a line;
(3) The row of A span $\mathbb{R}^n$.

Consider a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and fix a point $y \in P$. The recession cone at y is the set of all directions along which we can move indefinitely from y and still be in P, i.e.,

$$\{d \in \mathbb{R}^n \mid A(y + \lambda d) \geq b, \forall \lambda \geq 0\}$$

This turns out to be

$$\{d \in \mathbb{R}^n \mid Ad \geq 0\}$$

and is hence a polyhedral cone independent of y. The non-zero elements of the recession cone are called the *rays* of P. For a polyhedron in standard form, the ray must satisfy: $Ad = 0$, $d \geq 0$. Polyhedral cone has a finite number of extreme directions $y_1, y_2, ..., y_L$

Theorem 14.2.7 Every convex polyhedral set C can be represented as:

$$conv(a_1, ..., a_l) + cone(y_1, y_2, ..., y_L)$$

14.2.2 Subdifferential

Convex functions are not necessary differentiable, although it is differentiable almost everywhere. Thus, we need to generalize the notion of differentiable and gradient. Before give the definition of subdifferential, let's first look at the following definition:

Definition 14.2.5 A function is called proper if $f(x) > -\infty$ for all x and $f(x) < +\infty$ for at least one x.

Definition 14.2.6 Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ be a propert convex function and let $x \in \text{dom} f$. A vector $g \in \mathbb{R}^n$ such that

$$f(y) \geq f(x) + \langle g, y - x \rangle \text{ for all } y \in \mathbb{R}^n$$

is called a *subgradient* of f at x. We denote it as $\partial f(x)$.

The relationship between the subgradients and subdifferential is given by the following theorem:

Theorem 14.2.8 Let $f : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ be a convex function. Assume that $x \in \text{int dom } f$. Then $\partial f(x)$ is a non-empty, convex, closed and bounded set. Furthermore, for every direction $d \in \mathbb{R}^n$ one has

$$f'(x; d) = \max_{g \in \partial f(x)} \langle g, d \rangle$$

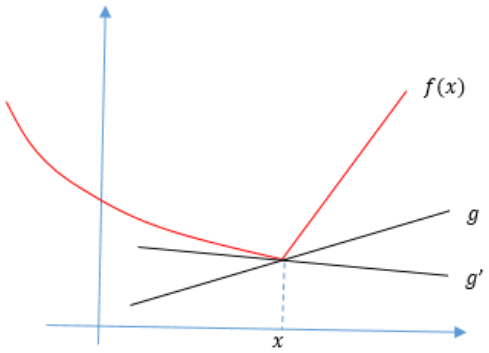

Figure 14.1: subdifferential