Credit Risk Modeling – Bascis

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In this notes, we discuss the basics of credit risk modeling, for which we start from the Poisson random variable and its inter-arrival distribution. Then, the we build its connection to the first arrival of default time. As building blocks of derivatives, we introduce two fundamental cashflows, this leads to the pricing of credit default swap(CDS) and other related topics. We focus on the reduced model instead of structural model, although ther is a brief discussion of structural model in appendix.

1 Poisson Process and Related

In this section, we give a quick review of *Poisson process*, including *standard Poisson process*, *non-time-homogeneous Poisson process* and *Cox process*. The setting for the discussion will be a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Generally speaking, all those processes are based on the notion of *counting process*, denoted by $\{N(t)\}_{t\geq 0}$, a integer-valued process satisfies the following assumptions:

(i) *Non-Negativity:* $N(t) \geq 0$ for all $t \geq 0$;
(ii) *Monotonicity:* If $0 \leq s < t$, then $N(s) \leq N(t)$;
(iii) *Regularity:* The paths $t \mapsto N(t, \omega)$[1] are increasing, piecewise constant, right continuous and increase only by unit jumps.

The further characterization of $N(t)$ gives the corresponding processes to be introduced below.

[1]We unhide the dependence on ω of N when it is necessary.

1.1 A Standard Poisson Process

We start by the most basic counting process:

Definition 1.1. *A counting process is said to be a* time-homogeneous Poisson process*, or* standard Possion process*, with rate* $\lambda > 0$ *if:*

(i) *Normalized:* $N(0, \omega) = 0$ *for all* ω*;*

(ii) *Independence Increment: for any* $0 < t_1 < t_2 < \cdots < t_n$*, the random variable* $N(t_1)$*,* $N(t_2)-N(t_1)$*,* $N(t_3)-N(t_2)$*, ...,* $N(t_n)-N(t_{n-1})$*, are independent;*

(iii) *Stationary Increment: for any* $t, s \geq 0$ *and any* $h > 0$*, the increments* $N(t+h)-N(t)$ *and* $N(s+h)-N(s)$ *are both Poisson random variables with rate* λh*, i.e.,*

$$\mathbb{P}\big(N(r+h) - N(r) = n\big) = \frac{e^{-\lambda h (\lambda h)^n}}{n!}, \quad n \in \mathbb{Z}^+ \cup \{0\}, \quad r \geq 0.$$

The parameter λ given above is also called the *intensity of Poisson process*. Indeed, the last property says the average occurrence of events during a fixed time is the time scaled intensity, namely,

$$\mathbb{E}\big[N(r+h) - N(r)\big] = \lambda h. \tag{1}$$

Notice the variance of the counts for a fixed interval is equal to the mean, thus, it is also λh. Moreover, in a infinitesimal level,

$$\lim_{h \to 0} \frac{1}{h} \mathbb{P}\bigg(N(h) = 1\bigg) = \lambda. \tag{2}$$

Besides Poisson distribution, another one that is closely related to standard Poisson process is the exponential distribution, which is used to model the inter-arrival time of a Poisson process. The probability density function(PDF) and cumulative distribution function(CDF) of an exponential distribution with arrival rate λ are:

$$f^\lambda(t) = \lambda e^{-\lambda t}, \quad F_\tau^\lambda(t) = \mathbb{P}(\tau \leq t) = 1 - e^{-\lambda t} \quad , t \geq 0. \tag{3}$$

We should be aware of that the exponential distribution has support on $\mathbb{R}^+$. The defining feature of exponential distribution is the *memoryless property*, i.e.,

$$\mathbb{P}(\tau > s + t \,|\, \tau > s) = \mathbb{P}(\tau > t) \tag{4}$$

In fact, the exponential distribution is the only memoryless continuous probability distribution. To sample from exponential distribution, we can use the inverse of CDF at a sampled uniform random variable, i.e., $(F^{\lambda})^{-1}(u) = t$. With this being said, the standard Poisson process can be defined in terms of inter-arrival time, namely,

$$N(t) := \sum_{n \geq 1} \mathbf{1}_{(0,t]}(T_n), \quad t \geq 0, \tag{5}$$

for a sequence $\{T_n\}_{n\geq 1}$ having i.i.d. increments $T_1, T_2 - T_1, ...$ with an exponential distribution of rate λ, i.e., $T_i - T_{i-1} \sim exp(\lambda)$, for $i = 1, 2, ...$ with $T_0 = 0$. Such constructive definition addresses the existence of standard Poisson process.

Based on the discussion above, we can simulate Poisson process in the following way:

(i) Generate $u_i \sim Unif(0, 1)$, $i = 1, 2, ...,$ using any decent random number generator;

(ii) Set $\tau_i = -\frac{1}{\lambda} \log(1 - u_i)$;

(iii) Set $S_i = \sum_{j=1}^{u} \tau_j$, where S_i denotes the time at which $N(t)$ increases by 1 for i-th time.

An example for above simulation with $\lambda = 0.5$ is shown in *Figure 1*.

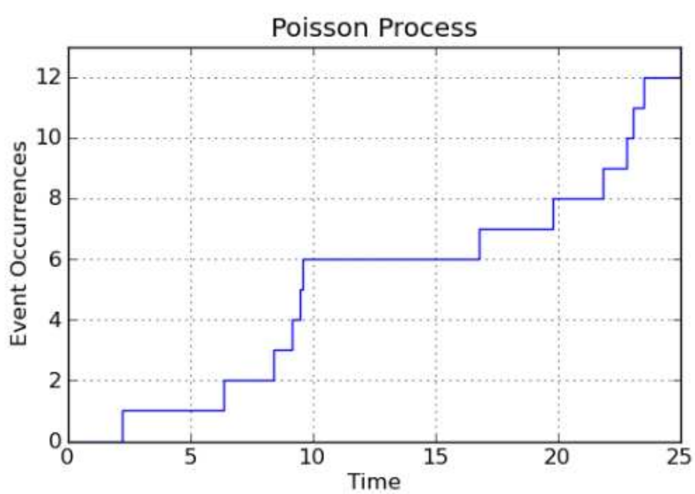

Figure 1: A standard Poisson process with $\lambda = 0.5$ for time $0 \leq 25$

1.2 Generalization of Standard Poisson Process

Let's generalize process $N(t)$ in above section to allow the arrival rate to be time-dependent.

Definition 1.2. *A counting process $N(t)$ is said to be a* non-time-homogeneous Poisson process*, if the rate function $\lambda(t)$ is bounded above by λ^*, for all $t \geq 0$, if*

(i) *Normalized: $N(0, \omega) = 0$ for all ω;*

(ii) *Independence Increment: for any $0 < t_1 < t_2 < \cdots < t_n$, the random variable $N(t_1)$, $N(t_2) - N(t_1)$, $N(t_3) - N(t_2)$, ..., $N(t_n) - N(t_{n-1})$, are independent;*

(iii) *Time-dependent Increment: for any $t, s \geq 0$ and any $h > 0$, the increments $N(r+h) - N(r)$ is exponential distributed with rate $m(r+h) - m(r)$ with* cumulative rate function *defined as $m(t) := \int_0^t \lambda(x)dx$,*

$$\mathbb{P}\big(N(t) = n\big) = \frac{e^{-m(t)}m(t)^n}{n!}, \quad n \geq 0. \tag{6}$$

As we can see, the only difference from the standard Poisson process is (iii). If we set $\lambda(\cdot) = \lambda$, then we recover the previous case. Similar to (2), we have

$$\lim_{h \to 0} \frac{1}{h}\mathbb{P}\big(N(r+h) - N(r) = 1\big) = \lambda(r). \tag{7}$$

It follows the cumulative distribution function in this case is:

$$f^{\lambda}(t) = \lambda(t)e^{-\int_0^t \lambda(x)dx}, \quad F_{\tau}^{\lambda}(t) = \mathbb{P}(\tau \leq t) = 1 - e^{-\int_0^t \lambda(x)dx} \quad, t \geq 0. \tag{8}$$

Remark 1.1. *The derivation is, from 0 to t, we make an equal partition, i.e., $dt = t/N$. In each interval, the survival probability can be calculated, $(1 - \lambda(idt)dt$, for $i = 1, ..., N$, then the probability of surviving to time t is, according to (7),*

$$\mathbb{P}(\tau > t) = (1 - \lambda(dt)dt)(1 - \lambda(2dt)dt)\cdots(1 - \lambda(t)dt) \tag{9}$$

As $dt \to 0$, we obtain:

$$\mathbb{P}(\tau > t) = \exp\left\{-\int_0^t \lambda(x)dx\right\}. \tag{10}$$

Thus, the CDF and PDF follow immediately.

The simulation of non-homogeneous Poisson process is accomplished by a *"thinning method"*. Specifically, we firstly repeat the simulation of standard Poisson process, that is sequentially generating i.i.d. exponential rate λ^* inter-arrival times and using the recursion $S_{i+1} = S_i + \big(-\frac{1}{\lambda^*}\log(u_{i+1})\big)$. As we were using a larger intensity than actual process, a acceptance-rejection algorithm is used to modify the original process. To be precise, for each arrival time S_i, it will be accepted with probability $p_i = \lambda(S_n)/\lambda^*$, and reject it with probability $1 - p_i$.

A further generalization is called *Cox process*, which has two fold stochasticities: one from the non-homogeneous Poisson process and one from the random evolution of $\lambda(\cdot)$. Mathematically, we have to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ large enough to support a standard Poisson process with rate 1 and a non-negative stochastic process $\lambda(t)$ that is independent of N that has cádlág path and integrable on any finite interval,

$$\int_0^t \lambda(s)ds < +\infty, \quad t \in [0, T]. \tag{11}$$

The Cox process is defined as:

$$\widetilde{N}(t) := N(\int_0^t \lambda(s)ds) \tag{12}$$

It is a counting process, where the rate function $\{\lambda(t)\}_{t\geq 0}$ is defined above, $\widetilde{N}(t)$ satisfies:

$$N(r+h) - N(r) \sim e^{\int_0^{r+h}\lambda(x)dx - \int_0^r \lambda(x)dx} \textit{ almost surely.} \tag{13}$$

We will not explore the technical details here, but give some intuitive understanding of the properties of Cox process. Usually, we use a diffusion process to model the randomness of hazard rate, that is,

$$d\lambda(t) = b(t, \lambda(t))dt + \sigma(t, \lambda(t))dW_t, \quad \lambda(0) = \lambda_0. \tag{14}$$

We can interpret *Cox process* as a non-time-homogenoeous Poisson process for each realization of rate process. Then, pathwisely, $N(t)$ satisfies (8). For simulation of Cox process, we adopt the following algorithm:

(i) Generate a realization of $\lambda(\cdot)$ for a sufficiently large time according to the SDE model chosen;
(ii) Set $\lambda^* = \max\{\lambda(t)\}$;
(iii) Follow the steps of simulating non-time-homogeneous Poisson process.

Remark 1.2. *The common model for the rate processes are short rate models, CIR, Vasicek, e.t.c..*

1.3 Connection to Hazard Rate Model

The specialty of credit risk modeling is to model default time. This is generally done by using an indicator function,

$$\mathbf{1}_{\{\tau>T\}} = \begin{cases} 1, & \tau > T \text{ survives before } T, \\ 0, & \tau \leq T \text{ defaults before } T. \end{cases} \tag{15}$$

where $\tau(\omega)$ is the default time which is a random variable. For a random time, we are interested in its statistical properties under appropriate measure.

It is *Jarrow*, *Turnbull* and *Lando*'s idea to model τ as the first arrival time of a Poisson process. As discussed earlier, the inter-arrival time is exponential distributed determined by the rate function $\lambda(t)$[2]. In particular, the first arrival of default time τ has CDF,

$$F_\tau^\lambda(t) = 1 - e^{-\int_0^t \lambda(x)dx}, \tag{16}$$

The corresponding survival function is defined as:

$$S_\tau^\lambda(t) := 1 - F_\tau^\lambda(t) = e^{-\int_0^t \lambda(x)dx}. \tag{17}$$

Both of them are controlled by rate function. To have more insights of $\lambda(t)$, let's introduce the concept of *hazard rate* defined in terms of CDF and survival function, i.e.,

$$h(t) := \lim_{dt\to 0} \frac{\mathbb{P}(t \leq \tau < t + dt)}{dt\mathbb{P}(\tau > t)} = \lim_{dt\to 0} \frac{F(t+dt) - F(t)}{dt \cdot S(t)}. \tag{18}$$

We claim that if τ is the first arrival time of time-homogeneous Poisson process, $h(t) = \lambda(t)$(this can be verified by direct calculation). The definition of hazard rate tells the probability of defaulting in an infinitesimal step further is the product of current hazard rate and the length of the step.

The definition of hazard rate enables us to simulate paths of default process to determine the time of default, conditional on it being before time T. Specifically, we can run the following algorithm:

(i) Set $t = dt$;
(ii) At time t, draw $u \sim Unif(0, 1)$;
(iii) If $u \leq \lambda(t)dt$, default has occurred in $[t - dt, t]$, the algorithm terminates, otherwise, we continue;

[2] In the case of Cox process, we fix a path, thus rate function is just a function of time.

(iv) Set $t = t + dt$;

(v) If $t > T$, terminates, else go back to (ii).

Notice, in the case of Cox process, one has to firstly simulate the path of $\lambda(t)$ up to time T according to certain diffusion process.

Let's conclude the discussion by mentioning the advantage of using Cox process. The hazard rate describe the intensity of default, it, however, could changes over time as the arrival of new information. As the market changes its view about the default risk of the reference entity, the hazard rate should reflect such innovation. A stochastic model is exactly for this purpose. If one assumes a Gaussian model for the reference entity, then the evolution of its risk will incorporate the information updating of the Brownian market.

2 Fundamental Cashflows

In interest rate modeling, the building block for derivatives is zero-coupon bond. Similarly, the important step in credit risk modeling is to understand the cashflow of zero-coupon with credit risk(RZCB) and a fixed payment upon default. Let's first discuss RZCB, the embedded risk cannot guarantee the pay back of face value at maturity. In other words, if default happens before the maturity, then the owner of the instrument loses the notional(see *Figure 2*). To price the RZCB, let's invoke

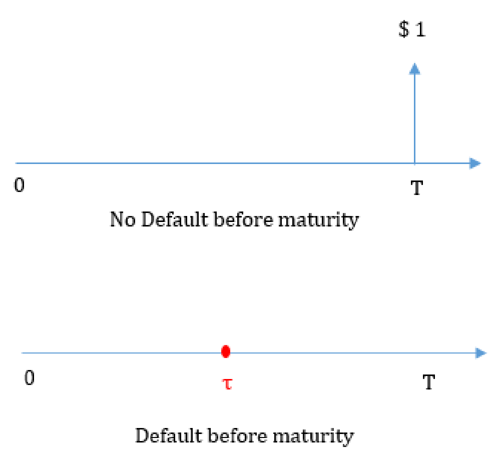

Figure 2: Payoff of Risky Bond

the equivalent risk-neutral measure $\mathbb{Q}$ and the indicator function to model default in (15),

$$\widetilde{P}(0,T) = \mathbb{E}^{\mathbb{Q}}\Big[D(0,T)\mathbf{1}_{\{\tau>T\}}\Big] \tag{19}$$

The $D(0,T)$ is the discount factor $D(0,T) = e^{-\int_0^T r(s)ds}$. We shall appreciate our discussion in Poisson process, which gives us a explicit solution for the above evaluation,

$$\widetilde{P}(0,T) = D(0,T)\mathbb{Q}(\tau > T) = D(0,T)S(0,T) = P(0,T)S(0,T). \tag{20}$$

where $\lambda(s)$ is the hazard function for default time[3] and $P(0,T)$ is the risk-less zero-coupon price. The above calculation is valid if both short rate and hazard rate are deterministic function. We next consider the cox process and stochastic short rates, that is, both hazard rate and interest rate are stochastic, i.e., driven by corresponding SDEs.

The formal setup is the following, we construct a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ large enough to support a unit exponential random variable E and and two non-negative stochastic process $\lambda(t)$ and $r(t)$ driven by Brownian motion W_t that is independent of E. The first default time can be defined as:

$$\tau = \inf\Big\{t : \int_0^t \lambda(s, \{W_u\}_{0\le u\le s})ds \ge E\Big\} \tag{21}$$

This default time can be thought of as the first jump time of a Cos process with intensity process $\lambda(s, \{W_u\}_{0\le u\le s})$. If we fix a path of Brownian motion up to some time t, then it follows,

$$\mathbb{Q}\big(\tau > t \,\big|\, \{\mathcal{F}_s^W\}_{0\le s\le t}\big) = \exp\Big(-\int_0^t \lambda(s, \{W_u\}_{0\le u\le s})ds\Big) \tag{22}$$

where $\{\mathcal{F}_s^W\}_{0\le s\le t}$ is the filtration generated by Brownian motion. By iterated conditioning,

$$\mathbb{Q}(\tau > t) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big\{-\int_0^t \lambda(s, \{W_s\}_{0\le u\le s}ds\Big\}\Big] \tag{23}$$

[3] If there is no confusion, we will write the survival function $S(\cdot) = S_\tau^\lambda(\cdot)$.

The specific functional form of λ and r are usually Itô processes that can be correlated in general, e.g., two correlated short rate models, e.g.,

$$\begin{aligned} dr_t &= \alpha_t^r(\theta_t^r - r_t)dt + \sigma_t^r dW_t^1, \\ d\lambda_t &= \alpha_t^\lambda(\theta_t^\lambda - \lambda_t)dt + \sigma_t^\lambda dW_t^2, \\ dW_t^1 dW_t^2 &= \rho dt. \end{aligned}$$

The complement of Brownian sub-sigma-algebra is the filtration below,

$$\mathcal{H}_t = \sigma\{\mathbf{1}_{\{\tau \le s\}} : 0 \le s \le t\}, \tag{24}$$

which holds the information of whether there has been a default at time t. As a result, the whole filtration is the sigma algebra generated by the union of two sub-sigma algebras, $\mathcal{F}_t^W$ and $\mathcal{H}_t$,i.e.,

$$\mathcal{F}_t = \mathcal{F}_t^W \cup \mathcal{H}_t, \tag{25}$$

which corresponding to knowing the evolution of state variables up to time t and whether default has occurred or not. To price risky bond, we have

$$\begin{aligned} \widetilde{P}(0,T) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T)\mathbf{1}_{\{\tau>T\}}\Bigg|\mathcal{F}_t^W\Bigg]\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T)\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbf{1}_{\{\tau>T\}}\Bigg|\mathcal{F}_t^W\Bigg]\Bigg] \\ &= \mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T)\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbf{1}_{\{\tau>T\}}\Bigg]\Bigg] \quad \textit{(Because } \mathcal{F}_t^W \perp \mathcal{H}_t\textit{)} \\ &= \mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T)e^{-\int_0^T \lambda(s)ds}\Bigg] \end{aligned} \tag{26}$$

To evaluate (26), one can run the following simulation:

(i) Initialize the simulation path index to be $i = 1$;
(ii) Set $t = dt$;
(iii) Compute $\lambda^i(t)$ and $r^i(t)$ according to their stochastic processes;
(iv) Set $t = t + dt$. If $t < T$, return to the last step;
(v) Calculate the stochastic discount factor $D^i(0,T)$ and the survival function $S^i(0,T)$, where

$$D^i(0,T) = \exp\big\{-\int_0^T r^i(t)dt\big\}, \quad S^i(0,T) = \exp\big\{-\int_0^T \lambda^i(t)dt\big\}.$$

(vi) Set $i = i + 1$, if $i < M$, M is the total number of simulation required, we return to step (iii);

(vii) Average out calculations above, i.e., $\widetilde{P}(0,T) = \sum_{i=1}^{M} D^i(0,T)S^i(0,T)$.

In this fully stochastic case, practitioners usually makes an assumption of independence between short rate and hazard rate, which separate the expectations in (26), namely,

$$\widetilde{P}(0,T) = \mathbb{E}^{\mathbb{Q}}\big[D(0,T)\big]\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^T \lambda(s)ds}\big] = P(0,T)S(0,T) \tag{27}$$

Under this assumption, (27) and (19) has the same form. The advantage of such expression is to reduce pricing to a simple multiplication, given the term structure of $P(0,T)$ and $S(0,T)$.

Next, let's consider an uncertain cashflow at a default time $\tau \leq T$, which pays \$1. Notice payment of this type occurs frequently in credit risk modeling, such as the protection leg of CDS contract and risky bond with recovery. The mechanism is simple and visualized in the graph below(see *Figure 3*). We can derive a closed

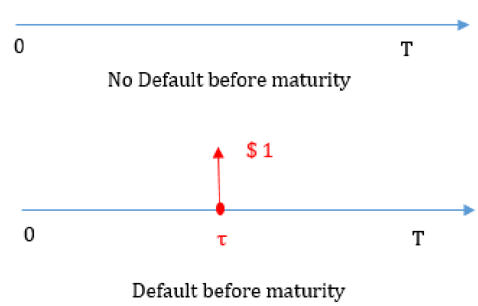

Figure 3: Fixed Payment at Default

form for the present value of the cash-flow, if deterministic hazard rate and short rate,

$$
\begin{aligned}
\hat{P}(0,T) =& \mathbb{E}^{\mathbb{Q}}\Big[D(0,\tau)\mathbf{1}_{\{\tau\le T\}}\Big] \\
=& \int_0^T P(0,\tau)\lambda(\tau)S(0,\tau)d\tau \\
=& \int_0^T P(0,\tau)d(-S(0,\tau)).
\end{aligned}
\tag{28}
$$

In the fully stochastic case, we can repeat the argument in the risky bond, by iterated conditioning, we have

$$
\hat{P}(0,T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T P(0,\tau)\lambda(\tau)e^{-\int_0^\tau \lambda(s)ds}d\tau\Big] \tag{29}
$$

The double integral inside expectation in generally can not give an explicit formula. Thus, one has to resort to Monte Carlo simulation introduced in the Poisson process section, there we are able to simulate one path, now, one should also simulate the $r(t)$ along the path up to the first arrival of default. Then, by exponentiation, the discount factor $D(0,\tau)$ can be obtained for this path. The rest of it is just repetition and averaging out.

As in the case of risky bond, the independence of short rate and hazard rate assumption is always made as a convention, which simplifies (29) further,

$$
\begin{aligned}
\hat{P}(0,T) =& \int_0^T \mathbb{E}^{\mathbb{Q}}\big[D(0,\tau)\big]\mathbb{E}^{\mathbb{Q}}\Big[\lambda(\tau)\exp\Big\{-\int_0^\tau \lambda(s)ds\Big\}\Big]d\tau \\
=& \int_0^T P(0,\tau)\big(-dS(0,\tau)\big)
\end{aligned}
\tag{30}
$$

We can use numerical integration to evaluate summation instead. Then, term structure of two rates again makes evaluation extremely efficient.

3 Fixed Rate Bond

3.1 Mechanism

As the name suggests, *fixed rate bond* pays fixed coupons on the payment dates, with notional paid back at the maturity. They're typically issued into the primary market. Then, those owners can sell the bonds on the secondary market. To buy a previously issued fixed rate bond, an investor has to execute a trade which is settled on a date, usually $1 \sim 3$ days after the trade date. On the settlement date, the purchaser pays the full bond to the seller, in return, will get periodic coupon payments(last one is coupled with notional) on the scheduled payment dates. Although market quotes the clean price(the price resulted from risk-neutral pricing formula), the full price, or, dirty price, is actually the amount being paid. The relationship is,

$$\textit{Full Price} = \textit{Clean Price} + \textit{Accrued Interest}. \tag{31}$$

Here, the accrued interest is the amount of money paid to the seller of the bond as compensation for his or her share of upcoming interest rate payment.

For exmaple, a corporate bond has a coupon rate 7.2% and pays 4 times a year, 15th of January, April, July and October uses $30/360$ US day count convention. A trade for $1,000$ par value of the bond settles on January 25. The prior coupon date was January 15, thus the accrued interest reflects ten day's interest, $7.2\% \times 1,000 \times 10/360 = \$2$. The bonds are purchased from the market at 985.5, given that the \$2 pays the accrued interest rate, the remainder represents the underlying value of the bonds.

Compared to risk-free coupon-bearing-bond, the future cash-flows of risky bond are not guaranteed, they're subject to the credit status of the issuer. A quantitative description is the (annualized) yield to maturity, y, that satisfies the following identity

$$P = \sum_{i=1}^{n} \frac{c/f}{(1 + y/f)^n} + \frac{1}{(1 + y/f)^n} \tag{32}$$

where c is annualized coupon rate and f is number of payments per year. Notice P is the full price thus different from $P(0, T)$ in the amount of accrued interest. Because of the exposure, there must exists a spread between risk free YTM and risky bond YTM(both bonds share the same payments schedule and coupon rate), i.e.,

$$y^{\textit{risky}} = y^{\textit{risk-free}} + s \tag{33}$$

In addition, from (32), we can justify why risky bond is more expensive compared to risk-free bond.

On the other hand, bond yield is a more meaningful quantity to compare bonds of different maturities and coupon rate. Because bond price can differ a lot because of the future cashflows, being converted to YTM makes all bonds comparable. Thus, in the market, it is very common that traders communicate with YTMs.

3.2 Sensitivities and Hedging

As we have discussed above, for fixed coupon rate, schedule and maturity, the fixed rate bond price is determined by the risk free yield and spread, i.e.,

$$P = \sum_{i=1}^{n} \frac{c/f}{(1 + (y^{\textit{risk-free}} + s)/f)^n} + \frac{1}{(1 + (y^{\textit{risk-free}} + s)/f)^n} \tag{34}$$

The *Taylor's expansion* gives:

$$dP = \frac{\partial P}{\partial Y^{\textit{risk-free}}}(dy^{\textit{risk-free}}) + \frac{1}{2}\frac{\partial^2 P}{\partial (y^{\textit{risk-free}})^2}(dy^{\textit{risk-free}})^2 + \frac{\partial P}{\partial s}(ds) + \frac{1}{2}\frac{\partial^2 P}{\partial s^2} + \cdots \tag{35}$$

The first-order sensitivity to changes in the risk-free yield and spread is known as the *modified duration* D, while the second-order sensitivity is known as the convexity, C, with these new notations, we can rewrite

$$\frac{dP}{P} = -D_{y^{\textit{risk-free}}} dy^{\textit{risk-free}} - D_s ds + \frac{1}{2} C_{y^{\textit{risk-free}}} (dy^{\textit{risk-free}})^2 + \frac{1}{2} C_s (ds)^2 + \cdots \tag{36}$$

Notice the modified duration, the proportional change in price movement in yield, is defined with a negative sign in front,

$$D_{y^{\textit{risk-free}}} = -\frac{1}{P}\frac{\partial P}{\partial y^{\textit{risk-free}}}$$

so that the it can be positive. On the other hand, we denote the approximated negative of the first term appear in (32) as *DV01*, namely,

$$\textit{DV01} = -\big(P(y + 1bp) - P(y)\big) \approx -\frac{\partial P}{\partial y}\Delta y. \tag{37}$$

It measures the absolute change in the full price of a bond per 1bp increase in the yield. Unlike duration, DV01 is an additive measure across bonds. That is, suppose we have a portfolio of K bonds, P^k, with face value F^k, for $k = 1, 2, ..., K$, then

$$\textit{DV01}_{\textit{portfolio}} = \sum_{k=1}^{K} F_k \textit{DV01}_k. \tag{38}$$

The full price of both risky coupon-bearing bond and treasury(risk-less) coupon-bearing bond both have dependence on the risk-free treasury yield. Thus, if we have a long position of a unit risky bond, we can hedge out the interest rate risk by shorting F amount of treasury bond. The F is set in such a way that the duration is cancelled out,

$$V = P - FP^{treasury}, \quad \textit{such that}, \quad \frac{\partial V}{\partial y^{free}} = 0 \tag{39}$$

which can be solved as:

$$F = \left(\frac{\partial P^{treasury}}{\partial y^{risk\text{-}free}}\right)^{-1}\left(\frac{\partial P}{\partial y^{risk\text{-}free}}\right). \tag{40}$$

Remark 3.1. *Notice, as observed from* (34)*, by the symmetry of spread and yield , the yield duration is equal to spread duration,* $D_s = D_{y^{risk\text{-}free}}$*, as well as convexity,* $C_s = C_{y^{risk\text{-}free}}$*.*

4 Credit Default Swaps

4.1 Terminology and Mechansim

The *credit default swap(CDS)* is a single-name credit derivative contract that is dominant in terms of notional in the credit market. It is a bilateral over-the-counter product that enables the owner to protect themselves when the counterparty defaults. There are a large number of legal terms associated with CDS contract to avoid legal risk most specified by *international swaps and derivatives association(ISDA)*. The main contribution of ISDA is the standardization of the contract, which makes it very liquid.

Technically speaking, there are three parities need to be considered when discussing CDS contract, the *protection buyer*, *protection seller* and the *issuer of the bonds or loans*. After trade date, the contract starts on the *effective date*, the lag in between is determined by convention according to different market. It is until the *termination date* the protection becomes invalid. CDS is linked to the insured asset that is issued by *reference entity*. Such asset is known as *deliverable obligation*. As in the IRS case, there is no initial cost to enter into the contract because of no exchange of money in the beginning. After effective date, there are *two legs*:

- *The protection leg:* if there is a credit event, counter-party defaults, before the maturity, the protection seller has to compensate the buyer in one of the two ways: either making a face value pay to the buyer and getting back the deliverable obligations, or paying buyer a portion of the face value, namely, $(1 - R)$ is the ratio, where R is is called *recovery rate*[4];

[4]The first one is called physical settlement, the other one is called cash settlement.

Figure 4: CDS Convention

- *The premium leg:* It refers to the series of payments made from protection buy to the seller, the size of which is the *spread*. Such payments continue until the credit event happens.

One may think it as a special type of insurance, where the insurance company charges premium periodically and makes the compensation when there is a claim. The buyer of the insurance shorts the risk on reference entity, just like the buyer of the CDS essentially short the credit risk.

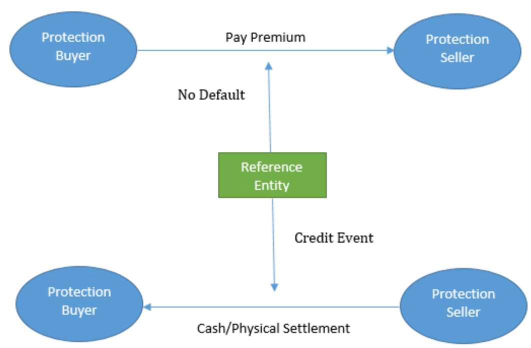

Figure 5: CDS Mechanism

As we realized that CDS is a contract to hedge credit risk, as a hedger, we now analyze the cashflows of such hedging. Suppose the investor buys a face value \$100 of a T-maturity floating rate notes(FRN) with a full price at par, the coupon of which is LIBOR plus F. The purchase is funded at a cost of LIBOR plus B. Then, the investor can hedge the default risk of the FRN by purchasing face value \$100 of CDS with spread s maturing at T. If we assume the same payment schedule, there are two scenarios:

- *No credit event:* the CDS protection ends at time T, when the hedger receives the par from the FRN and pay back it to the borrowed par amount. In this case, all cashflows at maturity net out to be zero;

- *Credit event:* The investor delivers the floating rate asset to the protection seller in return for par of \$100. They the use the \$100 to repay the funding. Again, all cashflow nets out.

According to arbitrage free condition, the trading strategy start with 0 value should also end with 0 value, equivalently, we have

$$s = F - B \tag{41}$$

under some reasonable assumptions. According to arbitrage

4.2 Valuation of CDS

We will discuss the valuation of CDS, including premium leg, default leg and MTM. We makes the assumption that both interest rate and hazard rate are stochastic and independent with each other.

4.2.1 Valuation of Premium Leg

Suppose the schedule associated to CDS is $\mathcal{T} = \{T_0, T_1, ..., T_N\}$, where $T_0 = 0$ and $\tau(T_{n-1}, T_n)$ counts the days between T_{n-1} and T_n. There are two types of cash-flows contributes to premium: one is the scheduled payments that has to be paid until the first arrival of the default at the end of each sub-interval, namely, T_1, T_2, ..., T_n, for $T_n \leq \tau$. Let's pick any T_i from the date list just mentioned, as discussed previously, the expected present value of the \$1 payment at time T_i conditioning on survival is

$$\widetilde{P}(0, T_i) = \mathbb{E}\left[\exp\left\{-\int_0^{T_i} r(s)ds\right\}\mathbf{1}_{\{\tau > T_i\}}\right] = P(0, T_i)Q(0, T_i) \tag{42}$$

Notice, we assume the effective date and trade date coincides for simplicity. Therefore, the premium leg turns out to be:

$$S_0 \sum_{i=1}^{N} \tau(T_{i-1}, T_i)P(0, T_i)Q(0, T_i) \tag{43}$$

where S_0 is the spread determined by the contract at inception, which is annualized.

The other type of payment is contingent on the default. Again, recall the discussion in fundamental cash-flow, in particular, formula (30). The present value of \$1 paid at default which occurs in the short interval $[s, s + ds]$ is given by

$$P(0, s)\big(-dQ(0, s)\big) \tag{44}$$

If the credit event happens at the start of a premium period, no premium needs to paid, if it happens at the end of the period, then a full premium payment will have to be paid. Otherwise, the default time s falls into the interval (T_{i-1}, T_i), the the extra amount to be paid at s discounted back to time 0 in this interval is,

$$S_0 \tau(T_{i-1}, s) P(0, s)\big(-dS(0, s)\big) \tag{45}$$

Such payment can occur any time in (T_{i-1}, T_i), in other words, $s \sim Unif(T_{i-1}, T_i)$, thus the premium accrued is

$$S_0 \int_{T_{i-1}}^{T_i} \tau(T_{i-1}, s) P(0, s)\big(-dQ(0, s)\big). \tag{46}$$

By summing over all periods, we have

$$S_0 \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} \tau(0, T_i) P(0, s)\big(-dS(0, s)\big) \tag{47}$$

Therefore, the present value of the premium leg is:

$$Premium\ Leg\ PV = S_0 RPV01(0, T), \tag{48}$$

where

$$RPV01(0, T) = \sum_{i=1}^{N} \left(\tau(T_{i-1}, T_i) P(0, T_i) Q(0, T_i) + \int_{T_{i-1}}^{T_i} \tau(0, T_i) P(0, s)\big(-dS(0, s)\big) \right). \tag{49}$$

In practice, integration are replaced by summation with the following approximation:

$$\int_{T_{i-1}}^{T_i} \tau(0, T_i) P(0, s)\big(-dS(0, s) \approx \frac{1}{2} \tau(T_{n-1}, T_n) P(0, T_n)\big(S(0, T_{n-1}) - S(0, T_n)\big). \tag{50}$$

which enables us write $RPV01$ in a more compact form:

$$RPV01(0, T) = \frac{1}{2} \sum_{i=1}^{N} \tau(T_{i-1}, T_i) P(0, T_i)\big(S(0, T_{i-1}) + S(0, T_i)\big). \tag{51}$$

The approximation is based on the observation that default occurs in the middle of each period on average, which results in a accrued premium $S_0 \tau(T_{n-1}, T_n)/2$. The discounting from the end of interval instead of mid-point is on the purpose to obtaina compact form.

The above calculation enables us to calculate the premium leg when the valuation date coincide with one of the payment dates. In general, this may not be the case. Suppose $t \in [T^{j-1}, T_j]$, then we have to add an extra term, which is the payment of the part of the coupon which has accrued in the time period between the previous coupon payment date and the valuation date. We write

$$\begin{aligned} Premium\,PV = S_0 \int_t^{T_j} & \tau(T_{j-1}, s) P(t,s)\big(-dS(t,s)\big) \\ & + S_0 \sum_{i=j+1}^{N} \int_{T_{i-1}}^{T_i} \tau(T_{i-1}, s) P(t,s)\big(-dQ(t,s)\big) \\ & + S_0 \sum_{i=j}^{N} \tau(T_{i-1}, T_i) P(t, T_i) S(t, T_i) \end{aligned} \tag{52}$$

The first term appears in (52) ensure if the reference credit defaults in between t to T_j, we are paid the fraction of the next coupon which has accrued since the previous coupon date. In addition, the first term can be approximated by

$$S_0 \tau(T_{j-1}, t) P(t, T_j)\big(1 - S(t, T_j)\big). \tag{53}$$

4.3 Valuation of Protection Leg

Compared to premium leg, the protection leg of CDS contract is much easier. The protection seller only pays the $(1-R)$ portion of the notional, where R is recovery rate. Thus, the present value of protection is,

$$Protection\ Leg\ PV = (1-R) \int_t^T P(0,s)\big(-dS(0,s)\big) \tag{54}$$

As usual, the integral can be numerically computed by taking summation of integrands at discrete point, which yields the following formula:

$$Protection\,PV = (1-R) \sum_{i=1}^{K} P(0, \delta_k)\Bigg(S(0, \delta_{k-1}) - S(0, \delta_k)\Bigg) \tag{55}$$

Here, the whole interval $[0, T]$ is partitioned into equal length interval of size T/K, the grids are thus $0, \delta_1, \delta_2, ..., \delta_{K-1}, \delta_K$. However, the numerical experiments showed that such approximation requires too many steps to have decent accuracy. To address the issue, we recall that $P(0,s)$ and $S(0,s)$ are both non-increasing function, which gives us the upper bound and lower bound for protection leg PV below:

$$upper = (1 - R) \sum_{k=1}^{K} P(t, T_k) \Big(S(0, T_{k-1}) - S(0, T_k) \Big)$$
$$lower = (1 - R) \sum_{k=1}^{K} P(t, T_{k-1}) \Big(S(0, T_{k-1}) - S(0, T_k) \Big)$$

The refined approximation for protection PV is:

$$\begin{aligned} Protection\ PV =& \frac{1}{2}(upper + lower) \\ =&(1 - R) \sum_{k=1}^{K} \big(P(0, T_{k-1}) + P(0, T_k) \big) \big(S(0, T_{k-1}) - S(0, T_k) \big) \end{aligned} \tag{56}$$

4.4 Spread & Mark-to-Market

For a new CDS contract, it should specify the annuity for the premium leg, which is the CDS spread, or *break-even spread*. By definition of CDS, the spread should be determined in such a way that the protection leg equals to the premium leg,

$$S_0 \times RPV01 = \frac{1 - R}{2} \sum_{k=1}^{K} \big(P(0, T_{k-1}) + P(0, T_k) \big) \big(S(0, T_{k-1}) - S(0, T_k) \big) \tag{57}$$

which implies:

$$S_0 = \frac{1 - R}{2} \frac{\frac{1-R}{2} \sum_{k=1}^{K} \big(P(0, T_{k-1}) + P(0, T_k) \big) \big(S(0, T_{k-1}) - S(0, T_k) \big)}{RPV01} \tag{58}$$

Notice, for the premium leg, we are not considering the accrued premium. That is,

$$RPV01 = \frac{1}{2} \sum_{i=1}^{N} \tau(T_{i-1}, T_i) P(0, T_i) \big(S(0, T_{i-1}) + S(0, T_i) \big). \tag{59}$$

Under the assumption of constant hazard rate, we consider a continuous approximation of protection leg and premium leg, which will give us an important. The approximating premium leg is

$$\int_0^T P(0, s) S(0, s) ds, \tag{60}$$

which can be thought as a continuous version of (43). The protection leg is this case is straightforward:

$$\lambda(1-R)\int_0^T P(0,t)S(0,t)dt \tag{61}$$

Then, (58) can be simplified to:

$$S_0 = \lambda(1-R). \tag{62}$$

We call this *credit triangle*, it is a identity relates three important quantity in CDS pricing, spread, recovery rate and hazard rate.

The *mark-to-market value* reflects how much the contract worths if unwound. To be explicit, one has to subtract the future cash outflow which is determined by initial spread from the future cash inflow (the protection leg) and discount back to current time t. In formula,

$$\begin{aligned} V^{MTM}(t) := {} & \frac{1-R}{2}\sum_{k=1}^{K}\big(P(t,T_{k-1})+P(t,T_k)\big)\big(S(t,T_{k-1})-S(t,T_k)\big) \\ & - S_0 RPV01(t,T) = (S_t - S_0)RPV01(t,T) \end{aligned} \tag{63}$$

with

$$RPV01(t,T) = \tau(T_{j-1},t)P(t,T_j)\big(1-S(t,T_j)\big) + \frac{1}{2}\tau)(t,T_j)P(t,T_j)\big(1-S(t,T_j)\big)$$
$$+\tau(T_{j-1},T_j)P(t,T_j)S(t,T_j) + \frac{1}{2}\sum_{i=j+1}^{N}\tau(T_{i-1},T_i)P(t,T_i)\big(S(t,T_{i-1})+S(t,T_i)\big)$$

where S_t is the break-even spread for the new CDS contract starting at t ending at T. The second equality in (63) follows from the fact that the new protection leg equals to new premium leg.

5 CDS Sensitivity Analysis

In this section, we focus our attentions on the sensitivity analysis of CDS. Recall the MTM value $V(\cdot)$ of a CDS position at time t, e.g., (63), is a function of initial spread S, the term structure of the yield curve Y, the passage of time t and recovery rate R. A *Taylor expansion* up to second order gives:

$$dV(t) \approx \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial Y}(dY) + \frac{\partial V}{\partial R}dR + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}(dY)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial R^2}(dR)^2 \tag{64}$$

The main risk of the position comes from the first-order sensitivities. The second order are quadratic, that only captures the magnitude not direction, thus has relatively smaller effects in the change of value.

We have various formulas for RPV01, thus, MTM value, to gain intuition of sensitivities, however, it is more preferred to have a further simplified formula that assumes both hazard rate and yield curve term structure observed at time t is flat, at a value equal to $\lambda(t)$ and $r(t)$ respectively. Then, both protection leg and premium leg calculation get simplified, namely, we have

$$\begin{aligned} \textit{Protection Leg PV}(t,T) &= (1-R)\int_t^T \lambda(t)\exp\big\{-(r(t)+\lambda(t))s\big\}ds \\ &= \frac{\lambda(t)(1-R)\big(1-\exp\big\{-(r(t)+\lambda(t)(T-t)\big\}\big)}{r(t)+\lambda(t)}. \end{aligned} \tag{65}$$

When considering premium leg, we ignore the effect of accrued premium on default as well as the accrued coupon, as a result,

$$\textit{Premium Leg PV}(t,T) = S_0\varDelta\sum_{i=1}^{N}\exp\big\{-(r(t)+\lambda(t))(T_i-t)\big\} \tag{66}$$

where we set $\varDelta = 0.25$ since premium is usually paid quarterly and assume there are N premium payments between t and T, namely, $T_1 = t$, $T_N = T$. The time t MTM value of CDS contract matures at T is:

$$V(t) = (S_0 - S_t)\varDelta\sum_{i=1}^{N}\exp\big\{-(r(t)+\lambda(t))\tau_i\big\} \tag{67}$$

where S_t is the spread for a CDS contract start from t ends at time T and $\tau_i = T_i - t$. From the formula, the MTM is defined for a short protection position. Recall the credit triangle identity:

$$\lambda(t) \approx \frac{S_t}{1-R}, \tag{68}$$

the MTM value can be re-written as:

$$V(t) = (S_0 - S_t)\varDelta\sum_{i=1}^{N}\exp\left\{-(r(t)+\frac{S_t}{1-R})\tau_i\right\}. \tag{69}$$

The equation (69) is very informative, because it represents the MTM as a simple function of S, Y, R and τ_i's.

5.1 Credit DV01 and Spread Convexity

The *Credit DV01* is defined as the change in value of MTM for a unit basis point increases in CDS curve,

$$\textit{Credit DV01} = -\big(V(S_0 + 1bp) - V(S_0)\big) \tag{70}$$

Notice the negative sign is to ensure the Credit DV01 of a short position turns out to be positive. The quantity can be calculated by the analytical formula below:

$$\textit{Credit DV01} \approx -\frac{\partial V}{\partial S} bp. \tag{71}$$

Immediately follows from (69),

$$\frac{\partial V(t)}{\partial S_t} = -\textit{RPV01}(t,T) - \frac{S_0 - S_t}{1-R} \Delta \sum_{i=1}^{N} \tau_i \exp\left\{ -(r(t) + \frac{S_t}{1-R})\tau_i \right\} \tag{72}$$

Thus,

$$\textit{Credit DV01} = \left(\textit{RPV01}(t,T) + \frac{S_0 - S_t}{1-R} \Delta \sum_{i=1}^{N} \tau_i \exp\left\{ -(r(t) + \frac{S_t}{1-R})\tau_i \right\} \right) \times 1bp. \tag{73}$$

The above result implies that the credit DV01 increase or decreases depending on the difference of S_t and S_0, and it is when $S_t = S_0$, credit DV01 becomes *RPV01* times 1 basis point.

The spread convexity, also called *gamma*, is the second-order sensitivity to changes in the market CDS spread. The similar argument as above yields:

$$\begin{aligned} \frac{\partial^2 V(t)}{\partial S_t^2} &= \frac{2\Delta}{1-R} \sum_{i=1}^{N} \tau_i \exp\left\{ -(r(t) + \frac{S_t}{1-R})\tau_i \right\} \\ &\quad + \frac{S_0 - S_t}{(1-R)^2} \Delta \sum_{i=1}^{N} \tau_i^2 \exp\left\{ -(r(t) + \frac{S_t}{1-R})\tau_i \right\}. \end{aligned} \tag{74}$$

As mentioned previously, the effect of spread convexity is "negligible" due to the square of the difference between $S_t - S_0$. It's worth to mention, if $S_0 = S_t$, the convexity becomes 0.

5.2 DV01, Theta and Recovery Rate Sensitivity

The *interest rate DV01* as expected is the sensitivity to the change of interest rate,

$$\textit{Interest Rate DV01}(t,T) = -\big(V(Y + \textit{1bp}) - V(Y)\big). \tag{75}$$

The analytical formula gives:

$$\textit{Interest Rate DV01}(t,T) \approx -\frac{\partial V(t)}{\partial r(t)} \times \textit{1bp}, \tag{76}$$

where

$$\frac{\partial V(t)}{\partial r(t)} = -(S_0 - S_t)\Delta \sum_{i=1}^{N} \exp\left\{ -(r(t) + \frac{S_t}{1-R})\tau_i \right\}. \tag{77}$$

The conclustion of the above expression is the following:
(i) The DV01 becomes 0 if $S_t = S_0$;
(ii) The sign of DV01 in general depends on the difference between S_t and S_0;
(iii) The sign of DV01 is opposite to MTM.
(iv) The interest DV01 is much smaller than credit DV01 in general.
The (iv) implies that CDS is almost a pure credit derivative.

The *Theta* of a CDS is the sensitivity to passage of time, i.e., t, for the MTM of a CDS contract. Set $\tau = T - t$, theta is calculated as t moves forward in time. Again, we have

$$\Theta = -\frac{\partial V}{\partial} dt \tag{78}$$

where dt is usually set to be 1 day. This time (69) can not give an explicit formula because S_t also changes as t increases. But, it still implies if the swap rate curve is upward sloping, then the MTM value will fall according to (69). We have to step back to the identity below:

$$\begin{aligned} V(t) = {} & S_0 \Delta \sum_{i=1}^{N} \exp\big\{ -(r(t) + \lambda(t))\tau_i \big\} \\ & - \frac{(1-R)\lambda_t\big(1 - \exp\big\{-(r(t)+\lambda(t))(T-t)\big\}\big)}{r(t) + \lambda(t)}. \end{aligned} \tag{79}$$

From this, it follows

$$\frac{\partial V(t)}{\partial t} = S_0(r(t) + \frac{S_t}{1-R})\textit{RPV01}(t,T) + S_t \exp\left\{ -(r(t) + \frac{S(t)}{1-R})(T-t) \right\} \tag{80}$$

Notice Θ does not take into account what happens across a premium payment date, that is, we only differentiate τ_1 with respect to t. The lesson taken from (80) is the value of the premium increases with t while the protection leg falls. To explain, as approaching to maturity, t moves closer to a payment, thus, it results in a greater value, on the other hand, the shortening of maturity gives less chance for a credit event, therefore, the protection leg decreases. Consequently, the short protection position will always increase as t moves forward.

Lastly, the sensitivity to R can be calculated by:

$$\frac{\partial V(t)}{\partial R} = \frac{S_t(S_0 - S_t)}{(1-R)^2}\Delta \sum_{i=1}^{N} \tau_i \exp\left\{-(r(t) + \frac{S_t}{1-R}\tau_i\right\}, \tag{81}$$

which leads to the following observations:

(i) If $S_t = S_0$, the recovery rate has to impact;

(ii) The sensitivity depends again on the difference of S_t and S_0;

(iii) The sensitivity appears to be quadratic in S_t.

The (iii) property suggests that when S_t is in a lower level, the change of R will not bring too much effect, while the impact increases as S_t becomes greater, and explode when S_t is on a high level.

5.3 Hedging

After developing intuitions fo those sensitivities, let's now move to the hedging part. We denote the value of a CDS position at time t with maturity T as V, which is a function of spread and interest rate, let's assume the following instruments for hedging purpose:

- A collection of M CDS contracts with times to maturities $T_1, ..., T_M$, spread $S_1, ..., S_M$ and notional $N_1, ..., N_M$. The MTM values are denoted by $V_1, ..., V_M$;
- A collection of M spot-starting IRS with maturities $\hat{T}_1, ..1., \hat{T}_M$, spread $Y_1, ..., Y_M$ and notional $N_1^I, .., .N_M^I$. The MTM values are denoted by $W_1, ..., V_M$.

Notice the tenors of CDS and IRS are chosen to coincide. Suppose we have a portoflio of a CDS with MTM value V, M CDS and IRS as specified above. The total portfolio worths, at time t,

$$\Pi = V + \sum_{m=1}^{M} \big(N_M V_M + N_m^I W_m\big). \tag{82}$$

Since time t is the inception of all hedging CDS's and IRS's, $\Pi = V$. The linearity of portfolio allows to represents the change of portfolio in terms of changes in all components of portfolio, ie.,

$$d\Pi = dV + \sum_{i=1}^{M} \left(N_m dV_m + N_m^I dW_m \right) = d\Pi_S + d\Pi_Y. \tag{83}$$

The change due to spread is then

$$d\Pi_S = \sum_{i=1}^{M} \left(\frac{\partial V}{\partial S_m} dS_m + N_m \sum_{n=1}^{M} \frac{\partial V_m}{\partial S_n} dS_n \right) \tag{84}$$

on the hand,

$$d\Pi_Y = \sum_{i=1}^{M} \left(\frac{\partial V}{\partial Y_m} dY_m + N_m \sum_{n=1}^{M} \frac{\partial V_m}{\partial Y_n} dY_n + N_m^I \sum_{n=1}^{M} \frac{\partial W_m}{\partial Y_n} dH_n \right) \tag{85}$$

The objective is to determine the notionals of CDS and IRS for hedging. Since the CDS hedges have an interest rate sensitivity but no credit spread sensitivity, the hedging procedure is to firstly calculate the credit spread hedging then switch to interest rate hedging.

For credit spread hedging, we require $d\Pi_S = 0$. That is,

$$-\frac{\partial V}{\partial S_i} = \sum_{j=1}^{M} N_j \frac{\partial V_j}{\partial S_i}, \quad i = 1, ..., M,$$

Notice the cross sensitivity does not contribute anything, because the bumping in credit spread that does not corresponds to the CDS will not have impact, i.e., $\frac{\partial V_j}{\partial S_i} = 0$ for $i \neq j$. By observing

$$\frac{\partial V_i}{\partial S_i} = -RPV01(t, T_i). \tag{86}$$

After some algebraic manipulation, we have

$$N_i = \big(-\frac{\partial V}{\partial S_i}\big)\big(\frac{\partial V_i}{\partial S_i}\big)^{-1}, \quad i = 1, ..., M. \tag{87}$$

The next step is to consider the hedging of interest rate. We require

$$d\Pi_Y = \sum_{i=1}^{M} \left(\frac{\partial V}{\partial Y_m} dY_m + N_m \sum_{n=1}^{M} \frac{\partial V_m}{\partial Y_n} dY_n + N_m^I \sum_{n=1}^{M} \frac{\partial W_m}{\partial Y_n} dH_n \right) = 0 \tag{88}$$

Notice $\partial W_k/\partial H_i = 0$ for $i \neq j$ for the same reasoning as in spread curve, as well as $\partial V_m/\partial Y_n = 0$. Therefore,

$$\sum_{m=1}^{M}\left(\frac{\partial V}{\partial Y_m} + \frac{\partial W_m}{\partial Y_m}\right)dY_m = 0. \tag{89}$$

Since both partial derivatives are non-positive and dY_m is positive, we enforces the terms inside brackets to be 0, which gives

$$N_m^I = -\left(\frac{\partial V}{\partial Y_m}\right)\left(\frac{\partial W_m}{\partial Y_m}\right)^{-1}, \quad m = 1, ..., M. \tag{90}$$

6 Forward Starting CDS

6.1 Mechanism

In last section, we have discussed spot starting CDS, for which the effective day follows trade date immediately. We now discuss the *forward-starting CDS*, as the name suggests, the effective date of the contract can starts in the future.

Recall the forward starting IRS, the mechanism of forward starting CDS is very similar. The protection buyer agrees to enter into a contract to buy protection at a forward date T_F and a contractual spread $S(t; T_F, t)$ which is determined at time t, the protection ends in terminal time $T > T_F$. There is no payment between t and T_F. If there is a credit event before T_F, the contact cancels at no cost to either party, otherwise, the CDS starts at time T_F with premium $S(t; T_F, T)$. A credit event happens after T_F ceases the premium payment and triggers the $(1 - R)$ recovery.

Remark 6.1. *In the forward-starting IRS, no credit event is considered, thus only plain exchange of fixed leg and floating leg should be taken care of, the fixed leg is a reference rate K which is determined in time t, one can view it as the $S(t; T_F, T)$ in the credit risk world.*

6.2 Forward Starting CDS Valuation

Let's first analyze the premium leg and protection leg of forward starting CDS. The schedule for the contract is $T_0 := T_F$, $T_N := T$, and $T_1, ..., T_{N_1}$ are in between. We denote $\tau(T_{i-1}, T_i)$ as the number of days in the interval $[T_{i-1}, T_i]$, for $i = 1, ..., N$. The present value of premium leg is

$$\textit{Premium Leg PV}(t) = S(t l T_F, T)\textit{RPV01}(t; T_F, T), \tag{91}$$

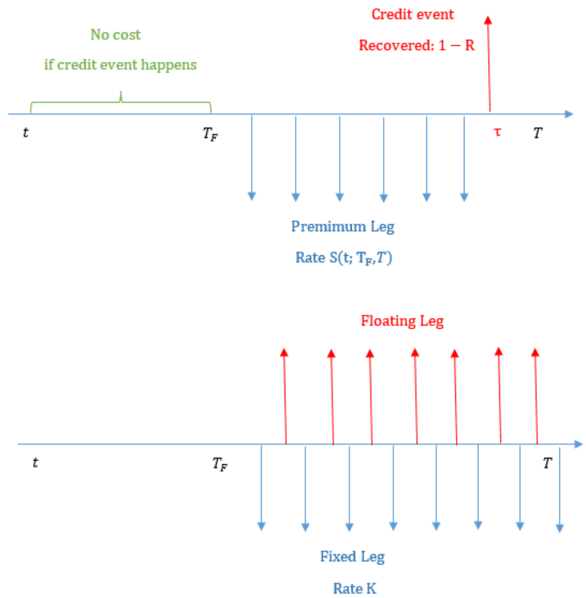

Figure 6: Upper: Forward Starting CDS, Lower: Forward Starting IRS

where

$$RPV01(t; T_F, T) = \frac{1}{2}\sum_{i=1}^{N}\tau(T_{i-1}, T_i)\big(S(t, T_{i-1}) + S(t, T_i)\big) \tag{92}$$

Remember the above formula takes into account the effect of accrued coupon.

The protection leg, we shall only consider the credit event after T_F, thus,

$$Protection\ Leg\ PV(t; T_F, T) = (1 - R)\int_{T_F}^{T} P(t, s)\big(-dS(t, s)\big) \tag{93}$$

We denote the time t MTM value of the forward starting CDS as $V(t; T_F, T)$,

$$V(t; T_F, T) = (1 - R)\int_{T_F} P(t, s)\big(-dS(t, s)\big) - S(t; T_F, T)RPV01(t; T_F, T), \tag{94}$$

which is required to be 0 at inception. This gives the break-even forward spread:

$$S(t; T_F, T) = \frac{(1 - R)\int_{T_F}^{T} P(t, s)\big(-dS(t, s)\big)}{RPV01(t; T_F, T)}. \tag{95}$$

Observe, due to the linear structure of RPV01, the forward RPV01 can be expressed by the difference of two spot starting RPV01, namely,

$$\mathit{RPV01}(t; T_F, T) = \mathit{RPV01}(t, T) - \mathit{RPV01}(t, T_F). \tag{96}$$

In addition, since integral is also linear, the forward starting protection leg can be rewritten as:

$$\begin{aligned}&(1-R)\int_{T_F}^{T} P(t,s)\big(-dS(t,s)\big)\\ =&(1-R)\bigg(\int_{t}^{T} P(t,s)\big(-dS(t,s) - \int_{T_F}^{T} P(t,s)\big(-dS(t,s)\bigg)\end{aligned}$$

Equivalently,

$$\textit{Protection Leg PV}(t; T_F, T) = \textit{Protection Leg PV}(t, T) - \textit{Protection Leg PV}(t, T_F) \tag{97}$$

To combine, since the present value of a spot starting protection leg must equal to the value of the premium leg at the present credit spread, the numerator of (95) can be expressed as the spot premium multiplied by corresponding RPV01. Therefore, we have an alternative formula for the forward spread in terms of purely spot starting CDS spread and corresponding RPV01,

$$S(t; T_F, T) = \frac{S(t,T)\mathit{RPV01}(t,T) - S(t,T_F)\mathit{RPV01}(t,T_F)}{\mathit{RPV01}(t,T) - \mathit{RPV01}(t,T_F)} \tag{98}$$

An interesting observation is that if the spot startgin CDS spread curveis flat, i.e., $S(t, T) = S(t, T_F)$, then (98) gives

$$S(t; T_F, T) = S(t, T) = S(t, T_F), \tag{99}$$

which implies the flat forward spread curve.

The unwind value of forward starting CDS at sometimes $t^* > t$ is

$$V(t) = \big(S(t^*; T_F, T) - S(t; T_F, T)\big)\mathit{RPV01}(t^*; T_F, T). \tag{100}$$

If we set $T_F = t$, the we recover the CDS pricing formula. It says that the gain and loss of CDS value is mainly due to the change of the forward credit spread curve.

6.3 The Shape of Forward CDS Curve

To build link between shapes of forward spread curve and spot spread curve, let's take a look at the following approximation formula for calculating break-even forward spread:

$$S(t;T+\alpha,T+\alpha)=\frac{(1-R)\int_{t+\alpha}^{T+\alpha}h(s)\exp\left\{\int_{t+\alpha}^{s}(r(u)+\lambda(u))du\right\}ds}{\int_{t+\alpha}^{T+\alpha}\exp\left\{-\int_{t+\alpha}^{s}(r(u)+h(u))du\right\}ds}, \qquad (101)$$

If we assume

$$\int_{t+\alpha}^{T+\alpha}\big(r(u)+h(u)\big)du << 1, \qquad (102)$$

then, we have

$$S(t;t+\alpha,T+\alpha)\approx\frac{1-R}{T-t}\int_{t+\alpha}^{T+\alpha}h(s)ds. \qquad (103)$$

Here, $h(\cdot)$ is the deterministic continuous forward default rate. Rewrite above identity as:

$$S(t;t+\alpha,T+\alpha)\approx S(t,T)+\frac{1-R}{T-t}\left(\int_{T}^{T+\alpha}h(s)ds-\int_{t}^{t+\alpha}h(s)ds\right), \qquad (104)$$

if the CDS curve is upward sloping, it is more likely to default in the future rather than now, thus

$$\int_{T}^{T+\alpha}h(s)ds-\int_{t}^{t+\alpha}h(s)ds>0. \qquad (105)$$

It makes the forward curve sits above 1the spot curve, with difference increasing with forward time α, according to (104). On the contrary, if the CDS curve spread curve is inverted, then the forward curve will sit below the spot CDS curve.

7 Survival Curve Construction

As discussed in interest rate modeling, the yield curve plays a significant role in derivative pricing. The credit risk modeling requires not only yield curve but also the survival curve, i.e., $S(0,\cdot)$. To valuate exotic derivatives in credit market, we have to extract a full term structure of survival probabilities from liquid instruments. The credit derivative we have introduced so far are risky bond and CDS, however, the bonds are less liquid. The market practice is to extract default probability information from CDS of various maturities. Notice, the frequently traded CDS contracts are of maturities $3Y$, $5Y$, $7Y$ and $10Y$, which is a very sparse collection of instruments. Therefore, the construction of survival curve also heavily relies on the interpolation method, which will be covered in this section as well.

To have a desirable curve, the following criterion are important for interpolation:

(i) The construction should exact fits the existing CDS market quotes;
(ii) The construction should be local, that is, local perturbation should not have dramatic impact on the whole curve. It can be interpreted as *stability* of construction;
(iii) The efficiency of construction should be guaranteed, e.g., faster algorithm.

Remark 7.1. *One may wonder why smoothness is not paid attention to. That's because its conflict with localness, which we prefer to have.*

7.1 Bootstrap

Suppose the CDS market quotes S_1, S_2, ..., S_N are available on the market with maturities T_1, T_2,..., T_N. The bootstrap of survival curve starts from the shortest dated contracts to longest, while assuming the recover rate remains the same for all maturities. The cornerstone is the break-even spread, namely, the protection leg has to be equal to premium leg associated with the market observed spread. In formula, we write

$$\begin{aligned} & S_0 \times \frac{1}{2}\sum_{i=1}^{N} \tau(T_{i-1}, T_i) P(0, T_i)\big(S(0, T_{i-1}) + S(0, T_i)\big) \\ =& \frac{1-R}{2}\sum_{k=1}^{K} \big(P(0, T_{k-1}) + P(0, T_k)\big)\big(S(0, T_{k-1}) - S(0, T_k)\big) \end{aligned} \tag{106}$$

The objective is to produce a vector of $S(0, T_i)$ at $N + 1$ times that re-price the CDS contract with chosen interpolation. Usually we set $K = N$, although in general they can be different. The boostrap algorithm works as follows:
(i) Initialize $S(0, 0) = 1$;
(ii) Set the $i = 1$;
(iii) Solve for $S(0, T_i)$ according to (106) with all $S(0, T_1)$, $S(0, T_2)$,..., $S(0, T_{i-1})$ already determined and interpolated. To avoid arbitrage, we enforce also $0 < S(0, T_i) \leq S(0, T_{i-1})$;
(iv) Add $(T_i, S(0, T_i))$ to the curve and set $i = i + 1$;
(v) The algorithm ends when we have $N + 1$ points at time $0, T_1, ..., T_N$.
The most important step is (iii), which requires a root-finding algorithm, bisection, Newton-Raphson, e.t.c.., are all common choices. In addition, it requires an interpolated surival curves before current maturity, because the most reliable quotes of CDS are of maturity $3Y$, $5Y$, $7Y$, $10Y$, while CDS typically has a quartely coupon. We will discuss interpolation in details next.

7.2 Interpolation

Because the exponential distribution of first arrival time, survival curve has same structure as yield curve, i.e., exponentiation of an integral with integrand being the instantaneous rate(interest rate in yield curve, hazard rate in credit curve). To be more concrete, we make a map between them,

(i) The no arbitrage condition requires $r(\cdot) \geq 0$, so does $\lambda(\cdot)$, i.e., $\lambda(\cdot) \geq 0$;

(ii) The discounting factor and survival function, in fully stochastic case, takes the same form:

$$P(0,t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^t r(s)ds}\big], \quad S(0,t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^t \lambda(s)ds}\big]. \tag{107}$$

(iii) The zero rate and zero default rate are also of the same format:

$$R(0,t) := -\frac{\ln P(0,t)}{t}, \quad \mathcal{S}(0,t) = -\frac{\ln S(0,t)}{t}. \tag{108}$$

(iv) The forward rate and forward default rate are:

$$f(0,t) = -\frac{1}{P(0,t)}\frac{\partial P(0,t)}{\partial t}, \quad f^S(0,t) = -\frac{1}{S(0,t)}\frac{\partial S(0,t)}{\partial t}. \tag{109}$$

The interpolation is based on those quantities above, for which linear interpolation is considered along the time horizon. Mathematically, a *linear interpolation* of function $l(t)$ is, $\forall t^* \in [T_{i-1}, T_i]$(see also *Figure 7*,

$$l(t^*) = \frac{(T_i - t^*)l(T_{i-1}) + (t^* - T_{i-1})l(T_i)}{T_i - T_{i-1}}. \tag{110}$$

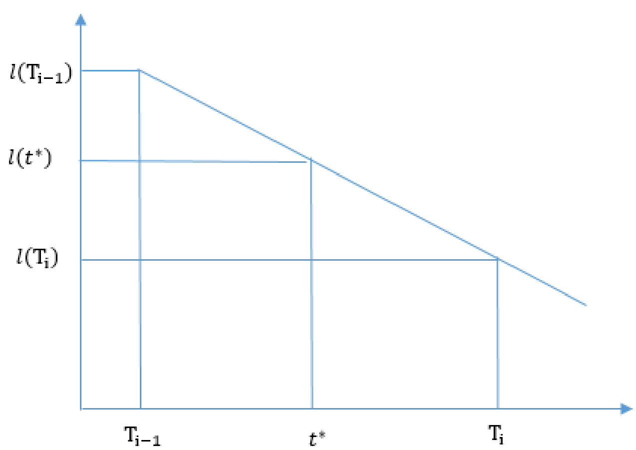

Figure 7: Linear Interpolation

1. Interpolation of log-survival function.

The first interpolation method is to interpolate the logarithm of the survival function, i.e.,

$$l(t) = -\ln S(0,t). \tag{111}$$

Since hazard rate is deterministic, it equals to the instantaneous forward default rate Lebesgue almost surely, thus,

$$-\ln S(0,t) = \int_0^t \lambda(s)ds = l(t). \tag{112}$$

which implies:

$$\lambda(t^*) = \frac{\partial l(t^*)}{\partial t^*} = \frac{l(T_i) - l(T_{i-1})}{T_i - T_{i-1}} = \frac{\ln S(0,T_{i-1})/S(0,T_i)}{T_i - T_{i-1}}. \tag{113}$$

Observe that the hazard rate is constant in $[T_{i-1}, T_i]$ by its independence of t^*. As a result, we can recover,

$$S(0,t^*) = S(0,T_{i-1}) \exp\big\{-(T_i - T_{i-1})\lambda(T_{i-1})\big\}. \tag{114}$$

Such interpolation is automatically arbitrage free because of the piecewise constant nature of hazard rate function.

2. Interpolation of Zero Default Rate.

In this case, choose to linear interpolate zero default rate. To recover survival probability from zero curve,

$$S(0,t) = \exp\{-t\mathcal{S}(0,t))\} \tag{115}$$

The interpolation function is:

$$l(t^*) = \mathcal{S}(0,t^*) = -\frac{\ln S(0,t^*)}{t^*} \tag{116}$$

for $t^* \in [T_{i-1}, T_i]$. The very right hand side can be computed by (110).

Let's make the following observation,

$$\mathcal{S}(0,t^*) = -\ln S(0,t^*) = \int_0^t f^S(0,s)ds = \int_0^t \lambda(s)ds \tag{117}$$

Thus,

$$f^S(0,t^*) = \lambda(t^*) = \frac{\partial t^* \mathcal{S}(0,t^*)}{\partial t^*} = \mathcal{S}(t^*) + t^* \left(\frac{\partial \mathcal{S}(0,t^*)}{\partial t^*} \right) \tag{118}$$

It implies such linear interpolation scheme can cause sudden jump in the forward default rate(as well as hazard rate) which is not a desirable property. In addition, it can not guarantee arbitrage free becasue of the possibility of sudden jump in the zero default rate.

3. Interpolation of the Instantaneous Forward Default Rate.

For this interpolation,

$$l(t) = -\frac{1}{S(0,t)} \frac{\partial S(0,t)}{\partial t}. \tag{119}$$

Use (110), we can easily derive $l(t^*)$ for $t^* \in [T_{i-1}, T_i]$. Then, by definition:

$$S(0,t^*) = S(0,T_{i-1}) \exp\left\{ \int_{T_{i-1}}^{t^*} l(s) ds \right\}. \tag{120}$$

This interpolation method also suffers from stability problem, it will often results a oscillating forward curve shape. Arbitrage free is not ensured.

7.3 Calibration Hazard Rate Process

The pricing of credit derivatives sometimes needs a nonlinear transformation of credit curve, thus, we may model the hazard rate as a stochastic process directly instead of assuming deterministic hazard rate. This is an analogy of interest rate modeling. Remember, there, we also construct a yield curve, but in the meanwhile, we specify a short rate model to be used in pricing more complicated derivatives. The calibration of short rate model is done by considering vanilla derivatives, such as zero coupon bond options, cap/floors and Swaptions. To calibrate credit derivatives, we choose CDS of different maturities.

To illustrate, let's pick a short rate model for hazard rate, for example, Vasicek model,

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda_t} dW_t^{\mathbb{Q}}, \quad \lambda(0) = \lambda_0. \tag{121}$$

Vasicek model has a closed solution for zero-coupon bond, thus also survival probability function $S(0,\cdot)$, i.e.,

$$S(0,t) = A(0,t)e^{-B(0,t)\lambda(t)}, \tag{122}$$

where $A(0,t)$ and $B(0,t)$ are both deterministic functions of parameters, $\Theta = \{\kappa, \theta, \sigma, \lambda_0\}$. The CDS market quotes, the credit spread, can be expressed by (58), given the yield curve. Thus, we set up the following optimization problem:

$$\min_{\Theta} \sum_{i=1}^{M} \left(S_i^{Market} - S_i^{theory}(\Theta)\right)^2. \tag{123}$$

where we have M CDS of different maturities available. The optimal solution gives the best choice of Θ.

Sometimes, an analytical solution for survival function may not exist, then the calculation of spread relies on Monte Carlo simulation. Also, the endogenous short rate model suffers from the bad performance of term structure of survival probability. In order to fit the term structure of survival probability, we can use Hull-White or other exogenous models as in interest rate modeling.

8 Appendix – Overview of Structural Model

For historical reason, we will briefly go over the structural models for modeling default. As a contrast to reduced model methodology, it models the total asset of a company as a stochastic process. According to accounting equation, the company's asset consists of two components: debt and equity, and

$$A(t) = D(t,T) + E(t) \tag{124}$$

where $A(\cdot)$, $D(\cdot, T)$, $E(\cdot)$ stands for asset, debt matures at time T and equity, respectively. Suppose the debt is represented by a face value F of T-maturity zero-coupon bonds with total present value D, and the equity of present value E does not pay dividend. The model defines the default as the asset falls below the debt only at time T.

At terminal time T, one of the following two scenarios can happen:

(i) The total asset values $A(T)$ that exceeds the debt F. In this case, all bond holders are repaid, and equity holders get $A(T) - F$;

(ii) The total asset values $A(T)$ is less than debt F. This implies the default of the bond, also the equity holders get nothing.

It is straightforward to define the survival probability (only at time T) by

$$S(t,T) = \mathbb{Q}\big(A(T) \geq F\big), \tag{125}$$

To compute such probability, we have to specify the concrete form of stochasticity. Let's firstly write the terminal payoff in such model,

$$D(T,T) = F - \max\{F - A(T), 0\} = \min\{F, A(T)\}, \quad E(T) = \max\{A(T) - F, 0\}. \tag{126}$$

It turns out that the equity terminal payoff can be thought as a payoff of a call option, while the debt as a short position in put option and long position in cash of value F. If the asset of the company follows geometric Brownian motion,

$$\frac{dA(t)}{dt} = \mu dt + \sigma_A dW_t \tag{127}$$

where σ_A is the percentage volatility of the asset value process. To calculate the present value of debt and equity, we have to work under risk-neutral measure, which amounts to replace the drift coefficient by risk-free rate r. Then *Black Scholes formula* yields:

$$\begin{aligned} E(t) &= A(t)N(d_+) - F \exp\big\{-r(T-t)\big\}N(d_-), \\ D(t,T) &= F \exp\big\{-r(T-t)\big\}N(d_-) + A(t)N(-d_-1). \end{aligned} \tag{128}$$

where

$$d_+ = \frac{\ln(A(t)/F) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}}, \quad d_- = d_+ - \sigma_A\sqrt{T-t}. \tag{129}$$

Now, we can calculate (125) easily,

$$S(t,T) = N(d_-). \tag{130}$$

We can also calculate the expected recovery rate in the structural model, that is the asset value conditioning on default at time T, denoted as $R := \mathbb{E}_t^{\mathbb{Q}}[A(T)|A(T) < F]$. We can express the present value of the debt in the following form

$$\begin{aligned} D(t,T) &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\big[\mathbf{1}_{\{A(T)\geq F\}} + \mathbb{E}_t^{\mathbb{Q}}[A(T)|A(T) < F]\big] \\ &= e^{-r(T-t)}\big(S(t,T) + (1 - S(t,T))R\big) \\ &= F \exp\big\{-r(T-t)\big\}\big(N(d_-) + N(d_-)R\big). \end{aligned}$$

By equating to the debt pricing in (128), we obtain:

$$R = \frac{A(t)N(-d_+)}{F\exp\{-r(T-t)\}N(-d_-)}. \tag{131}$$

On the other hand, we can also extract credit spread from the model. Due to the uncertainty of debt, its present value can also be expressed as:

$$D(t,T) = F \exp\big\{-(r + s(t))(T-t)\big\}, \tag{132}$$

where $s(t)$ is the time-t credit spread. Equivalently,

$$s(t) = \frac{1}{T-t}\ln\left(\frac{D(t,T)}{F}\right) - r. \tag{133}$$

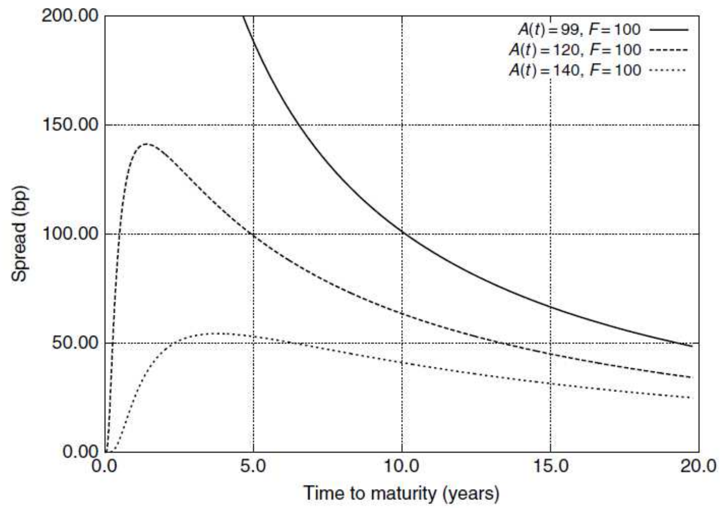

Figure 8: Credit Spread Term Structure

By (128), we can calculate $s(t)$ for $t \in [0, T]$,

$$s(t) = \frac{1}{T-t} \ln \left(\frac{F \exp\big\{-r(T-t)\big\} N(d_-) + A(t) N(-d_-1)}{F} \right) - r. \tag{134}$$

This allows characterizing the term-structure of credit spread(see *Figure 8*).

One more observation to make in structural model is the volatility. The equity preset value par to (128) implies it as a function of $A(t)$, since $A(t)$ follows geometric Brownian motion, apply Itô's formula to $E(t)$ gives us

$$\sigma_E = \sigma_A N(d_+) \frac{A(t)}{E(t)}, \quad \Leftrightarrow \quad \sigma_A = \frac{\sigma_E}{N(d_+)} \frac{E(t)}{A(t)}. \tag{135}$$

The identity above allows us to calibrate the asset volatility by the information of stock volatility.

The drawbacks of structural model is obvious, we list main concerns here:

- It only allows the defualt happening at a specified time T;
- The bond issued by company is restricted to zero coupon, this eliminates the possibility of issuing coupon-bearing bonds;
- As shown in figure 8, the credit spread $s(t^*)$ goes to 0 as $t^* \to t$, this is un-realistic, because in the market, even a very highly rated company regarded as a risk-free one.

As these being said, it is not that popular to use structural modeling method nowadays.