Topology - Limit Point / Accumulation Point

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June 30, 2014

1 Limit Point and Accumulation Point

The concept of limit point and accumulation point have subtle difference, which always leads to confusion and misunderstanding. We will clarify the issue by studying them in the context of set and sequence.

1.1 Set

Let's firstly give the formal definition of a **limit point**:

Definition 1.1 A is a subset of a topological space X, point p is a limit point of a set A if every open set containing p (neighborhood of p) contains at least one point of A distinct from p.

Note that the distinction from p is essential, otherwise call it p is an **adherent point**. And there are two particular kind of limit points:

- **ω - accumulation points:** every open set containing p (neighborhood of p) must contain infinitely many points of A;
- **condensation points:** every open set containing p must contain uncountably many points of A.

Obviously, not all limit point is ω - accumulation point, although it is true for most topology (e.g., the Euclidean Topology). The counter-example is finite particular point topology (it is not a metric space). Also, not all limit point are condensation points. Counter-example: $\mathbb{R}$ is a real line with Euclidean topology, we consider the following subset, $A \cup [2, 3]$ where A be the set of all points in the form $\frac{1}{n}$ for $n = 1, 2, 3, ...$ Then 0 is a limit point and accumulation point but not condensation point. Actually, we can observe in most case, limit point is equivalent to accumulation point in most space with suitable neighborhood topology function.

1.2 Sequence

The definition of limit point and accumulation point of a sequence is different from the definition for the set which consists of the sequence element, it does not require they are distinct.

Definition 1.2 A point p is said to be a limit point (*usually called the limit*) of a sequence $\{x_n\}_{n=1}^{\infty}$, if every open set containing p (neighborhood of p) contains all but finitely many terms of the sequence. (The sequence is then said to converge to the point p.)

In other words, the limit point (or the limit of the sequence) means that for every neighborhood of p, there are only finitely many indices of such that the corresponding elements of the sequence do not belong to the open set (neighborhood). A weaker condition on p is that every open set containing p contains infinitely many terms of the sequence, in this case, p is called an **accumulation point** of $\{x_n\}_{n=1}^{\infty}$. We consider the following example: if $\{a_n\}_{n=1}^{\infty}$ is the sequence, $1, 1, 1, 2, 1, 3, ...,$, then 1 is the only accumulation point of the sequence, since every neighborhood containing 1 contains infinitely many terms of the sequence(actually they are infinitely many 1's), but **not** a limit point (because the indices corresponding to the neighborhood of 1 is not finite but countable). An important observation is that limit point (limit) of a sequence is unique, it is the point where a infinite sequence converge to, while the accumulation point can be distinct (e.g., $-1, 1, -1, 1-1, 1, ...$ has 1 and -1 as limit point). We now discuss a little more about the **accumulation point** of a sequence. Firstly, we give the formal definition :

Definition 1.3 A point p is an **accumulation point** of the sequence $\{x_n\}_{n=1}^{\infty}$ if, for every open set containing p, there are infinitely many indices such that the corresponding elements of the sequence belong to the open set.

The other common way to characterize the accumulation point is given by a necessary condition

Definition 1.4 Given a sequence $\{x\}_{n=1}^{\infty}$, if, for any $\epsilon > 0$ and any $N \geq 1$, there exists an $n \geq N$ such that $|x_n - p| < \epsilon$. then p is a **cluster point** or **accumulation point** of $\{x\}_{n=1}^{\infty}$.

In the second definition, it essentially says the limit point of a convergent sequence is an accumulation point. But the converse is **not true**, that is, if p is an accumulation point of a sequence $\{x_n\}_{n=1}^{\infty}$ (illustrate by the oscillating sequence above). However, it is true that p is the limit of the subsequence of it. In fact, in metric space, we have another way to define the accumulation point:

Proposition 1.1 p is an accumulation point of $\{x_n\}_{n=1}^{\infty}$ **if and only if** we can find a subsequence $\{x_k\}_{k \in \mathscr{K}}$(where $\mathscr{K} \in \mathbb{N}$) converging to p.

2 General Theories

In this section, we will introduce the general set theory and provide important results from real analysis and topology that is crucial in advanced topic.

2.1 Open set and Closed Set

Recall the definition of the limit point of a set, in metric space, we have following result:

Theorem 2.1 If x is a limit point of A in X, then every neighborhood of x contains infinitely many points of A.

This actually states the equivalence between the limit point and the accumulation point (*in metric space*). Another special point of a set is called the **interior point**:

Definition 2.1 x is an interior point of A in topological space X if tthere exists a neighborhood N of x such that $N \subset A$

Based on the definitions of limit point and interior point, we can now define the openness and closedness of a set.

Definition 2.2 Set $A \subset X$ is open if every point of A is an interior point of A; set $A \subset X$ is closed if A contains all its limit points.

If we denote the limit point of $A \subset X$ as A', then we can define the **closure of** A by including all its limit points, i.e., $\bar{A} = A + A'$. Here is the list of several important fact related to the closure of a set:

Theorem 2.2 $\bar{A}$ is a closed set.

Proof. Consider p a limit point of $\bar{A}$, we want to show that p is in $\bar{A}$. That is to say, think of a neighborhood N of p, assume p is not in A, we want to give the proof that N contains a point of A.
Since p is a limit point of $\bar{A}$, N contains a point of $\bar{A}$ call it q. If $q \in A$ we are done, if not, q is limit point of A. Consider $N-1$ neighborhood of q such that $N-1$ is contained in N □

Theorem 2.3 A is closed if and only if $A = \bar{A}$.

Proof. Since A is closed, it contains all its limit point, i.e., $A' \subset A$. Thus, $A \cap A' \subset A$, then $\bar{A} \subset A$. But $A \subset \bar{A}$, combine of above inclusions, we have $A = \bar{A}$.If $A = \bar{A}$, then A contains all its limit points, by definition, A is closed. □

Theorem 2.4 If $A \subset B$, where B is closed, then $\bar{A} \subset B$. ($\bar{A}$ is the smallest closed set that contains A)

Proof. Assume p is a limit point of A, then p is a limit point of B. But B continas all its limit point, thus B contains A and $\bar{A}$. □

Another definition to describe the subset of a set is the concept of *denseness*:

Definition 2.3 A is **dense** in X if every point of X is a limit point of A or in A; or if $\bar{A} = X$; or every open set of X contains a point of A.

Obviously, rational number $\mathbb{Q}$ is dense in real number $\mathbb{R}$. Now we establish the relationships between closed set and open set:

Theorem 2.5 A is open if and only if A^C is closed.

Proof. A is open if and only if any points $x \in A$ is an interior point. It is equivalent to $\forall x \in A$, there exists a neighborhood N of x, where N is disjoint from A^C. That is $\forall x \in A$, x is not a limit point of A^C. Equivalently, A^C contains all its limit point. $\square$

Theorem 2.6 Union and intersection of open/closed sets

- finite intersection of open sets are open, arbitrary intersection of open sets are not open(e.g., $\cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$);
- finite union of closed sets are closed, arbitrary union of closed set is not closed(e.g., $\cup_{n\in\mathbb{N}}[0, 1 - \frac{1}{n}] = [0, 1)$);
- arbitrary union of open sets are open;
- arbitrary intersection of closed set is closed.

2.2 Compactness

The notion of **compactness of a set** is quite significant, intuitively, it is the *"next best thing to be finite"*. In order to give the definition of a compact set, we need to define:

Definition 2.4 An **open cover** of A in X is a collection of open sets $\{G_\alpha\}$, whose union contains A; a sub-cover is a sub-collection $\{G_{\alpha_\gamma}\}$ that still cover A.

Thus, the definition of the compactness of a set is :

Definition 2.5 A set K is **compact** (in X) if every open cover of K contains a finite sub-cover.

But what is an example of a compact set.

Theorem 2.7 Any finite sets are in fact compact.

Then we may ask what are the properties of a compact set:

Theorem 2.8 A closed subset B of a compact set K is compact.

Theorem 2.9 Compact set is bounded.

Theorem 2.10 Compact set is closed.

In general, the converse is not true. We will give the condition under which the converse is true. But before that let us talk about the space $\mathbb{R}^n$,

Theorem 2.11 Nested closed intervals in $\mathbb{R}^n$ are not empty. (Nested: if $m \geq n$, then $a_n \leq a_m \leq b_m \leq b_n$)

Based on that we can claim:

Theorem 2.12 Any closed intervals $[a, b]$ is compact in $\mathbb{R}^n$.

Now we can have the *Heine-Borel Theorem*:

Theorem 2.13 In $\mathbb{R}^n$, K is compact if and only if K is closed and bounded.

In more general metric space, the *Heine-Borel Theorem* takes the form:

Theorem 2.14 K is compact if and only if it is complete and totally bounded.

An equivalent way to say that K is compact is every infinite subset of K has a limit point in K. Then an immediate corollary reads:

Corollary 2.15 Every bounded subsets A of $\mathbb{R}$ has a limit point.

Because if the subset is bounded, A must be in some compact $K - cell$, then apply above definition, we can have for any infinite subset, it has a limit point in the $K - cell$(not necessary in the infinite subset itself).

2.3 Sequence and Subsequence

Again, after the discussion of general concept of a set, we can continue our study along the sequence. We want to inherit results from the set. What is the concept of converging sequence:

Definition 2.6 $\{p_n\}_{n=1}^{\infty}$ converges if there exists a $p \in X$ such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, for $n \geq N$, $d(p_n, p) < \epsilon$.

Here is a list of important facts:

Theorem 2.16 if $p_n \to p$ and $p_n \to p'$, then $p = p'$, in other words, the limit of a convergent sequence is unique

Theorem 2.17 If $\{p_n\}_{n=1}^{\infty}$ converges then it is bounded, $d(p_n, 0) \leq M(M > 0)$, $\forall n \in \mathbb{N}$.

Theorem 2.18 If p is a limit point of $A \in X$, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in A such that $p_n \to p$.

Theorem 2.19 $p_n \to p$ if and only if every neighborhood of p contains all but finitely many p_n.

Recall the definition of subsequence, we want to ask the following questions:

- If $p_n \to p$, must any subsequence converges to p ? Answer: Yes, because every neighborhood of p contains all but finitely many many points of p_n.
- Must every sequence contains a convergent subsequence ? Answer: No (e.g., natural numbers).
- If a sequence is bounded, must it have a convergent subsequence ? Answer: No (e.g., $\{3, 3.1, 3.14, 3.141, 3.1415, ...\}$).

After answering these questions, let us give a new definition about compactness:

Definition 2.7 A metric space is sequentially compact if every sequence has a convergent subsequence.

Thus it is obvious that:

Theorem 2.20 If X is compact, then X is sequentially compact. That is to say in compact space, every sequence has a subsequence converging to a point in the space.

On the other hand, we get the *Bolzano-Weierstrass Theorem* and its corollary:

Theorem 2.21 Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

Corollary 2.22 Every bounded sequence in $\mathbb{R}^n$ has at least one accumulation point.

Now we ask ourself another question, how can I tell $\{p_n\}_{n=1}^{\infty}$ converges, if we don't know what the limit it has ? The strategy is the following: if they do converge, then they must becomes closer to each other. So we need another important concept *Cauchy Sequence* to proceed further:

Definition 2.8 The sequence $\{p_n\}_{n=1}^{\infty}$ is **Cauchy** if and only if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for $m, n \geq \mathbb{N}$, $d(p_n, p_m) < \epsilon$.

Theorem 2.23 If $\{p_n\}_{n=1}^{\infty}$ converges, it is **Cauchy**.

But the converse is not true, it is true for the **complete metric space**:

Theorem 2.24 A metric space is complete, if every Cauchy sequence converges to some x in X.

$\mathbb{R}^n$ is an example of complete space. More generally,

Theorem 2.25 Compact metric space are in fact complete.

Although there are some spaces that are not compact, however, we can complete it:

Theorem 2.26 Every metric space (X, d) has a completion (X^*, d)

One more thing about the sequence:

Theorem 2.27 Bounded monotone sequences converge.