Interest Model Review
Under Multi-Curve Framework

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In the lecture notes "Interest Rate Basic – Under Multi-curve Framework", we have shown how to value swaps, caps and Swaptions when the OIS discounting curve is taken into consideration. As for exotics, we have to use other interest rate models. This leads us to the question: when switching from single curves to multi-curves, what kind of interest rate models are adopted? Indeed, since the projection curve is different from discounting curve, we have to distinguish, for example, between short rate model for risk-free rate as well as short rate "implied" by LIBOR. In this notes, we will discuss those issues. The standard setting for the discussion is – a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\{\mathcal{F}_t\}_{t\geq 0}$ *generated by* d*-dimensional Brownian motion* $\{W_t\}_{t\geq 0}$.

1 Direct Modeling Risky Rate

In this section, we consider directly modeling the rates implied by LIBOR of different tenors. We will first consider HJM approach, which can be specified to short rate model and stochastic model, then we will discuss the LIBOR market model.

1.1 HJM Framework

HJM is a very general way for interest rate modeling, almost all interest rate models can be derived within such framework. Once we specify the volatility structure, we can obtain, for instance, a short rate model.

Recall, the HJM model, under OIS discounting risk-neutral measure $\mathbb{Q}_D$ follows,

$$df(t,T) = \sigma(t,T)\bigg(\int_t^T \sigma(t,u)du\bigg)dt + \sigma(t,T)dW_t^{\mathbb{Q}_D}, \quad f(t,t) = f. \tag{1}$$

This is essentially now the instantaneous forward rate model for discounting rate (OIS). As a result, the bond, regardless of its existence, with both projection curve and discounting curve as OIS has dynamics,

$$dP(t,T) = P(t,T)\bigg(r_t d_t - \int_t^T \bigg(\int_t^T \sigma(t,u)du\bigg)dW_t^{\mathbb{Q}_D}\bigg) \tag{2}$$

The projection curve, since the existence of risk, can not have such dynamics for f. However, since they're still of similar nature, it is proposed that

$$df^{\Delta}(t,T) = \sigma^{\Delta}(t,T)\big(\int_t^T \sigma^{\Delta}(t,u)du\big)dt + \sigma^{\Delta}(t,T)dW_t^{\Delta,\mathbb{Q}_D}, \quad f^{\Delta}(t,t) = f. \tag{3}$$

is the dynamics of instantaneous forward rate corresponding to projection rate, i.e., LIBOR rate. Here, $\sigma^{\Delta}(t,T)$ is its volatility and $dW_t^{\Delta,\mathbb{Q}_D}$ is a $\mathbb{Q}_D$-Brownian motion correlated to $W_t^{\mathbb{Q}_D}$ by ρ^{Δ}.

Remark 1.1. *Notice, this is not a rigorous way of modeling. The* (3) *has no theoretical foundation but is based one's intuition about the dynamics of rates. For more mathematical sounding treatments, one can refer to "a general HJM framework for multiple yield curve modeling". Nevertheless, in practice, assuming* (3) *will not deviate from the reality too much.*

Having the definition of f^{Δ}, the corresponding bond price can be defined as:

$$P^{\Delta}(t,T) := \exp\bigg\{-\int_t^T f^{\Delta}(t,u)du\bigg\}, \tag{4}$$

from which it follows the dynamic of the bond by Itô's formula:

$$dP^{\Delta}(t,T) = P^{\Delta}(t,T)\bigg(r_t^{\Delta}dt - \int_t^T \bigg(\sigma^{\Delta}(t,u)du\bigg)dW_t^{\Delta,\mathbb{Q}_D}\bigg) \tag{5}$$

where $r_t^{\Delta} = f^{\Delta}(t,t)$ is generally different from r_t appeared in (2) to reflect the risk embedded. We define the *modified forward rate* by mimicking the definition under single framework:

$$\widetilde{F}(t; T-\Delta, T) := \frac{1}{\Delta}\left(\frac{P^{\Delta}(t, T-\Delta)}{P^{\Delta}(t,T)} - 1\right) \tag{6}$$

Notice this is just a definition not how the market quotes it. In previous notes, we have shown the market quotes of FRA is:

$$FRA(t; T-\Delta, T) := \mathbb{E}_t^{\mathbb{Q}_D^T}\big[L(T-\Delta, T)\big] \tag{7}$$

with

$$L(T-\Delta, T) := \frac{1}{\Delta}\left(\frac{1}{P^{\Delta}(t, T-\Delta)} - 1\right) \neq \frac{1}{\Delta}\left(\frac{1}{P(t,T)} - 1\right) \tag{8}$$

where $\mathbb{Q}_D^T$ is the T-forward measure that is equivalent to OIS discounting risk-neutral measure $\mathbb{Q}$. This quantity is used almost in all interest derivatives pricing. We can bridge the modified forward rate $\widetilde{F}$ and FRA:

$$FRA(t; T-\Delta, T) = \widetilde{F}(t; T-\Delta, T)\left(1 + \frac{1 + \Delta F(t; T-\Delta, T)}{\Delta F_t(T-\Delta, T)}\big(\Theta^{\Delta}(t,T) - 1\big)\right) \tag{9}$$

where

$$\Theta^{\Delta}(t,T) := \exp\left\{\int_t^{T-\Delta}\int_{T-\Delta}^{T} \sigma^{\Delta}(u,v)\phi^{\Delta}(u,T)du dv\right\},$$
$$\phi(t,T) := \int_t^T \big(\sigma^{\Delta}(t,T) - \rho^{\Delta}\sigma(t,T)\big)du.$$

Remark 1.2. *To derive* (9)*, apply Itô's lemma to modified forward rate $\widetilde{F}($ as defined in* (6)*) and then switch measure from $\mathbb{Q}_D$ to discounting forward measure, $\mathbb{Q}_D^T$, by using numéraire $P(t,T)$ characterized by* (5)*, which results in:*

$$d\widetilde{F}(t; T-\Delta, T) = \big(\widetilde{F}(t; T-\Delta, T) + \Delta^{-1}\big)\left(\int_{T-\Delta}^{T} \sigma^{\Delta}(t,u)du\right)\big(\phi^{\Delta}(t,T) - \rho^{\Delta}\sigma(t,u)\big) \tag{10}$$

Then, by direct integration, i.e., (7)*, we can obtain* (9)*.*

To obtain a short rate model that corresponds to (3), thus (5), the volatility needs to be specified. For example, if one assume the short rate corresponding to r^{Δ} implied by LIBOR is affine term-structure class, one can derive bond price dynamic by applying Itô's formula to (11)

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{11}$$

then, compare it with (5) to determine the short rate model coefficient structure. Introducing time-dependent coefficients to fit the initial structure, calibrating the rest coefficients from vanilla derivatives all remain the same as before. In this way, we actually have done nothing, the detour to HJM seems to be unnecessary, because one can start with a short rate model for the implied LIBOR and follow the usual routine to fix the coefficients. This is absolutely correct, HJM model here is more like a motivation and philosophy to maintain consistency. But HJM is more than short rate model with deterministic coefficients, the stochasticity can be added to the volatility function appears in SDE for instantaneous forward rate which results a Gaussian models with uncertain parameters.

1.2 LIBOR Market Model – Under Multiple Curve

We discussed extensively in previous notes the mechanism of LMM. On contrast to short rate model, it directly models the fundamental quantity – forward LIBOR rate. More precisely, in single curve LMM, it models the joint evolution of a family of consecutive forward LIBOR rates under appropriate forward measure (or, spot measure). By specifying tractable diffusion coefficients structure, we can price vanilla instruments, cap/floors, Swaption, by exact formula or efficient approximation, therefore calibration can also be achieved.

In a multi-curve setting, the forward rates associated to OIS and LIBOR has to be distinguished. Recall, the essential quantity in pricing under multi-curve framework is

$$FRA^{\tau}(t) := \mathbb{E}_t^{\mathbb{Q}_D^S}\big[L_\tau(T,S)\big],\ S \geq T,$$

and $\tau = S - T$, which is a $\mathbb{Q}_D^S$-martingale. We will use previous notation, to repeat, denoting by $\mathcal{T}^x = \{T_0^x, T_1^x, .., T_M^x\}$ the schedule for LMM associated to LIBOR with tenor x. The martingale property of FRA suggests the following dynamics:

$$dFRA_k^x(t) = \sigma_k(t) FRA_k^x(t) dW_k(t),\ t \leq T_{k-1}^x, \tag{12}$$

where the instantaneous volatility $\sigma_k(t)$ is deterministic and W_k is the k-th component of an M-dimensional $\mathbb{Q}_D^{T_k^x}$-Brownian motion W with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M}$.

On the other hand, the forward rate related to OIS curve,

$$F_k^D(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^D}\left(\frac{P_D(t, T_{k-1}^i)}{P_D(t, T_k^i)} - 1\right), \text{ where } \tau_k^D = \tau_D(T_{k-1}^i, T_k^i) \tag{13}$$

The dynamics of each rate $F_k^D(t)$ under $\mathbb{Q}_D^{T_h^x}$ is given by:

$$dF_h^D(t) = \sigma_h^D(t) F_h^D(t) dW_h^D(t), \quad t \le T_{h-1}^x, \tag{14}$$

where the instantaneous volatility $\sigma_h^D(t)$ is deterministic and W_h^D is the h-th component of an M-dimensional $\mathbb{Q}_D^{T_h^x}$-Brownian motion W^D with instantaneous correlation matrix $(\rho_{k,h}^{D,D})_{k,h=1,\dots,M}$. Two process are correlated by

$$dW_k(t) dW_h^D(t) = \rho_{k,h}^{i,D} dt. \tag{15}$$

In matrix form, we have

$$R = \begin{bmatrix} \rho & \rho^{i,D} \\ (\rho_{i,D})' & \rho^{D,D} \end{bmatrix} \tag{16}$$

where $\rho = (\rho_{k,j})_{k,j=1,\dots,M}$, $\rho^{D,D} = (\rho_{k,h}^{D,D})_{k,h=1,\dots,M}$ and $(\rho_{k,h}^{i,D})_{k,h=1,\dots,M}$ must be chosen in such a way that R is positive semi-definite.

The next step is to derive the dynamics of $FRA_k^x(t)$ under the forward measure $\mathbb{Q}_D^{T_j^x}$, where $j < k$ and $j > k$. By change of numéraire technique, the drift term of $FRA_k^x(t)$ under $\mathbb{Q}_D^{T_j^i}$ is equal to[1]:

$$\textit{new drift} = -\left\langle FRA_k^x(t), \ln \frac{P_D(t, T_k^x)}{P_D(t, T_j^x)} \right\rangle \tag{17}$$

In the case $j < k$,

$$\begin{aligned} \ln \frac{P_D(t, T_k^x)}{P_D(t, T_j^x)} &= \ln\left(\frac{1}{\prod_{h=j+1}^k \left(1 + \tau_h^D F_h^D(t)\right)}\right) = -\sum_{h=j+1}^k \ln\left(1 + \tau_h^D F_h^D(t)\right) \\ &= \sum_{h=j+1}^k \frac{\tau_h^D}{1 + \tau_h^D F_h^D(t)} \langle FRA_k^x(t), F_h^D(t) \rangle \end{aligned}$$

[1] $\langle \cdot, \cdot \rangle$ represents the quadratic variation operator.

Unlike in the single curve situation, we need to involve both forward rate $FRA_k^x(t)$ and OIS associated forward rate $F_h^D(t)$. We end up with

$$new\ drift = \sigma_k(t) FRA_k^x(t) \sum_{h=j+1}^{k} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1+\tau_h^D F_h^D(t)} \tag{18}$$

The same derivation works for $j > k$. We summarize the result as the following: the dynamics of FRA_k^x and F_k^D under forward measure $\mathbb{Q}_D^{T_j^x}$ in the three cases, $j < k$, $j = k$ and $j > k$ are,

$$j < k, t \leq T_j^x : \begin{cases} dFRA_k^x(t) = \sigma_k(t) L_k^x(t) \left[\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t0}{1+\tau_h^D F_h^D(t)} dt + dW_k^j(t) \right] \\ dF_k^D(t) = \sigma_k^D(t) F_k^D(t) \left[\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{D,D} \tau_h^D(t) F_h^D(t0}{1+\tau_h^D F_h^D(t)} dt + dW_k^{j,D}(t) \right] \end{cases}$$

$$j = k,\ t \leq T_{k-1}^x : \begin{cases} dFRA_k^x(t) = \sigma_k(t) FRA_k^x(t) dW_k^j(t) \\ dF_k^D(t) = \sigma_k^D(t) F_k^D(t) dW^{j,D}(t) \end{cases}$$

$$j < k, t \leq T_j^x : \begin{cases} dFRA_k^x(t) = \sigma_k(t) L_k^x(t) \left[-\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t0}{1+\tau_h^D F_h^D(t)} dt + dW_k^j(t) \right] \\ dF_k^D(t) = \sigma_k^D(t) F_k^D(t) \left[-\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{D,D} \tau_h^D(t) F_h^D(t0}{1+\tau_h^D F_h^D(t)} dt + dW_k^{j,D}(t) \right] \end{cases} \tag{19}$$

where W_j^j and $W_k^{j,D}$ are the k-th components of M-dimensional $\mathbb{Q}_D^{T_j^x}$-Brownian motions W^j and $W^{j,D}$ with correlation matrix R.

As we can see, the comprehensive LMM dynamics (19) raise concerns on numerical issues. The doubled number of rates to simulate adds burden on computations. In addition, the parameters in the drift terms of both models makes calculations very time consuming. Thus, the calibration for such LMM is not easy.

Remark 1.3. *The dynamics above can be simplified if we switched to spot measure as the same in single curve case. Under spot measure $\mathbb{Q}_D^d$,*

$$dFRA_k^x(t) = \sigma_k(t) FRA_k^x(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{i,D} \tau_h^D(t) \sigma_h^D(t) F_h^D(t)}{1+\tau_h^D F_h^D(t)} dt + \sigma_k(t) FRA_k^x(t) dW_k^d(t),$$

$$dF_k^D(t) = \sigma_k^D(t) F_k^D(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{D,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1+\tau_h^D F_h^D(t)} dt + \sigma_k^D(t) F_k^D(t) dW_k^{d,D}(t). \tag{20}$$

where $\beta(t) = m$ *if* $T^x_{m-2} < t \leq T^x_{m-1}$, $m \geq 1$ *so that* $t \in (T^]x_{\beta(t)-2}, T^x_{\beta(t)-1}]$, $W^d = \{W^d_1, ..., W^d_M\}$ *and* $W^{d,D} = \{W^{d,D}_1, .., W^{d,D}_M\}$ *are* M*-dimensional* $\mathbb{Q}^d_D$*-Brownian motions with correlation matrix* R*.*

Let's briefly mention calibration in the end. For caps/floors, as in single curve framework, it doesn't require correlation matrix. The Black formula can be used with volatility specified by $\sigma_k(\cdot)$ for computing caplets, namely,

$$Cplt(t, K; T^x_{k-1}, T^x_k) = \tau^x_k P_D(t, T^x_k) Bl(t, FRA^x_k(t), \sqrt{T_{k-1} - t}\,\nu_{T_k}), \tag{21}$$

where

$$\nu_k(t) = \sqrt{\int_t^{T_{k-1}} \sigma^2_k(u) du}. \tag{22}$$

As for Swaption, because of the joint distribution of $FRA^x_k(t)$'s, correlation can be taken into consideration. In this case, analytical formula is almost impossible, thus approximation method has to be used. One can firstly work with LIBOR swap rate model and then relates it back to LMM to represents the Swaption price as a function of LMM volatilities.

2 Indirect Modeling – Spread Method

2.1 Deterministic Shift

The HJM framework essentially requires modeling the joint evolution of a discount OIS curve and multiple forward curves. It is manageable but most computation for pricing therefore calibration require numerical methods, i.e., Monte Carlo simulation, numerical solution to PDE. What most banks are currently implementing is to use a *deterministic basis method*. The idea is to build the forward curves at a deterministic spread over the OIS curve.

We consider a tenor x and an associated time schedule $\mathcal{T}^x = \{0 < T_0, ..., T_M\}$, with $T_k - T_{k-1} = x$, $k = 1, ..., M$. Denote S^x_k the additive spreads, i.e.,

$$S^x_k(t) := FRA(t; T_{k-1}, T_k) - F(t; T_{k-1}, T_k), \ \forall k. \tag{23}$$

here, F is the OIS forward rates defined as in the classic single-curve paradigm,

$$F(t; T_{k-1}, t_k) = \frac{1}{\tau(T_{k-1}, T_k)} \left[\frac{P(t, T_{k-1}}{P(t, T_k)} - 1 \right]$$

A deterministic spread means,

$$S^x_k(t) \equiv S^x_k(0), \tag{24}$$

that is the initial spread term-structure remains the same for the future. This is a reasonable assumption when the market is abscent of shocks. Then, the basis spreads tend to be relatively stable(see *Figure 1*). In *deterministic basis method*, we start by modeling OIS

Figure 1: Stable Market

rates, by a short rate model, general HJM model, LIBOR market model. Then LIBIOR rates can be defined by setting:

$$FRA(t; T_{k-1}, T_k) = F(t; T_{k-1}, T_k) + S_k^x(0). \tag{25}$$

The pricing of caps/floors, Swaption are fairly simple, because the deterministic shift does not change martingale property.

2.2 Stochastic Shift

In general, (23) suggests modeling the spread as a stochastic process. The necessity become more visible if we try to price exotic derivatives, for example, a derivative whose underlying asset is the spread between OIS and LIBOR. In addition, stochastic basis is needed when we want to apply risk measure, e.g., CVA, that's where the basis risk lurk in. In the following, we will discuss modeling of basis as a stochastic process. In particular, we will focus on stochastic LIBOR market model. Under $\mathbb{Q}_D^{T_k^x}$, we assume the dynamic of $F_k^x(\cdot)$ follows,

$$
\begin{aligned}
dF_k^x(t) &= \phi_k^F(t, F_k^x(t))\psi_k^F(t, V_k^F(t))dW_k^F(t) \\
dV_k^F(t) &= \alpha_k^F(t, V_k^F(t))dt + b_k^F(t, V_k^F(t))dZ_k^F(t).
\end{aligned}
\tag{26}
$$

and dynamic for the stochastic spread as:

$$
\begin{aligned}
dS_k^x(t) &= \phi_k^S(t, S_k^x(t))\psi_k^S(t, V_k^S(t))dW_k^S(t) \\
dV_k^S(t) &= \alpha_k^S(t, V_k^S(t))dt + b_k^S(t, V_k^S(t))dZ_k^S(t).
\end{aligned}
\tag{27}
$$

where all coefficients functions are deterministic and Z_k^F, W_k^F, Z_k^S, W_k^S are $\mathbb{Q}_D^{T_k^x}$-Brownian motions.

Usually, the independence assumption is made such that the Brownian motion driving $F_D^x(t)$ is independent of those driving the spread $S_k^x(t)$. Now, as long as the marginal density of the stochastic process $F_k^x(\cdot)$ and $S_k^x(\cdot)$ are known, pricing cap/floors and Swaptions are tractable. For example, we consider a strike K-caplet, which pays out at time T_k^x,

$$
\tau_k^x \Big(FRA_k^x(T_{k-1}) - K \Big)^+ \tag{28}
$$

The caplet price at time t is thus,

$$
Cplt(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \mathbb{E}_t^{\mathbb{Q}_D^{T_k^x}} \left[\Big(FRA_k^x(T_{k-1}) - K \Big)^+ \right] \tag{29}
$$

Since $FRA_k^x(T_{k-1}^x) = F_k^x(T_{k-1}^x) + S_k^x(T_{k-1}^x)$, by independence assumption, the conditional density $f_{T_{k-1}^x}(\cdot)$ is equal to the convolution of conditional density densities $f_{F_k^x(T_{k-1}^x)}(\cdot)$ and $f_{S_k^x(T_{k-1}^x)}(\cdot)$, the pricing can be calculated by:

$$
Cplt(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \int_{-\infty}^{+\infty} (l - K)^+ f_{T_{k-1}^x}(l) dl. \tag{30}
$$

The Swaption pricing is a little bit involved, approximation method has to be adopted.