Interest Rate Basic
Under Single-curve Framework

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In these two notes, we will review some basics about interest rate based product under both single curve and multi-curve. In the meanwhile, we will introduce notations that will be used throughout the series of notes. For mathematical precision, we consider a market on a compact time interval $[0, T]$, to model the uncertainty, let's introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\{\mathcal{F}_t\}_{t\geq 0}$ generated by d-dimensional Brownian motion $\{W_t\}_{t\geq 0}$. We call it Brownian market to be distinguished with the one has partial information from the Poisson measure.

1 Introduction to Various Rates

The first concept to be considered is the *bank account*, or *money-market account*. It is a risk-free investment where profit is accrual continuously at risk-free rate. A formal mathematical definition is:

Definition 1.1. *(Bank Account) Set $B(t)$ to be the time-t value of a bank account for $t \geq 0$ and assume $B(0) = 1$. The bank account satisfies:*

$$dB(t) = r_t B(t) dt, \quad B(0) = 1,$$

where r_t is $\mathcal{F}_t$-adapted process. Solving ODE with boundary condition specified yileds,

$$B(t) = \exp\left\{\int_0^t r_s ds\right\}. \tag{1}$$

r_t *is called a* short rate *or* instantaneous rate.

A directly related notion is the *discount factor*. When the risk-free rate is deterministic, the discount factor can be interpreted as the units of currency to hold at time t in order to get 1 unit currency back at T, here $t \geq T$. In general, a stochastic risk-free rate is often encountered, the previous interpretation gives a good intuition to make the following definition:

Definition 1.2. *(Stochastic Discount Factor) For any* $0 \leq t \leq T$*, the discount factor* $D(t,T)$*,* $t \in [0,T]$*, is the amount at time t that is "equivalent" to one unit of currency payable at time* T*. It is a* $\mathcal{F}_T$*-measurable random variable defined by:*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left\{-\int_0^t r_s ds\right\}. \tag{2}$$

The equivalence is understood as a summarization of the stochasticity in between t and T. In other words, it will not be known until time T that the equivalent units of currency to be hold at inception.

A risk-free investment requires absolutely no risk, that is the counter party (providing such investment) has not default risk. The proxy of risk free rate is used to be short-dated government bond, treasury bills as well LIBOR rate. However, after credit crisis, the *overnight index*, e.g., *fed funds* in US, and *overnight index swaps rate* become the proxy of risk-free rate. We will discuss more later on. But, since we are disucssing now under single curve framework, we assume all those rates mentioned above are of no risk, then a *zero coupon bond* is defined based those rates:

Definition 1.3. *(Zero-coupon bond) A* T-maturity zero-coupon bond *is a contract that guarantees its holder* 1 *unit currency at time* T*, without intermediate cash-flow. The contract value at time* $t \in [0,T]$ *is denoted by* $P(t,T)$*. In particular,* $P(T,T) = 1$.

The definition of $P(t,T)$ is closely related to $D(t,T)$, as we illustrated previously, if the risk-free rate process is known in advance, then $P(t,T) = D(t,T)$. However, if not, there are completely different. One is a $\mathcal{F}_t$-measurable random variable, one is $\mathcal{F}_T$-measurable. Nevertheless, since zero coupon bond is a derivative with payoff 1 at maturity, according to fundamental theory of pricing, we have, under risk neutral measure $\mathbb{Q}$,

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\,|\,\mathcal{F}_t]$$

Based on zero-coupon bond, different compounding method defines corresponding rates:

Definition 1.4. *(Continuously-compounded Spot Interest Rate) The continuously-compounded spot interest rate prevailing at time t for the maturity T is denoted by $R(t,T)$, then*

$$R(t,T) := -\frac{\ln P(t,T)}{\tau(t,T)}, \tag{3}$$

where $\tau(t,T)$ is the date counts between t and T[1].

The rate $R(t,T)$ is annualized rate, because it is constant on t to T, we use R instead. If we divided it n periods, then for one unit of currency at time T, the single compounding gives

$$P(t,T)\left(1+\frac{R}{n}\right)^{n\tau(t,T)} = 1.$$

The continuously compounding requires $n \to \infty$, which yields:

$$P(t,T)\lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{n\tau(t,T)} = P(t,T)\exp\{R(t,T)\tau(t,T)\} = 1.$$

Thus,

$$P(t,T) = \exp\{-R(t,T)\tau(t,T)\} \quad \Rightarrow \quad R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)}.$$

On the other hand, we can define the simply compounded spot interest rate.

Definition 1.5. *(Simply-compounded Spot Interest Rate) The simply compounded spot interest rate between t and T is denoted by $L(t,T)$. It is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity. In formula,*

$$L(t,T) := \frac{1-P(t,T)}{\tau(t,T)P(t,T)}. \tag{4}$$

[1] The date count convention is very important in practice, but we will not pursue here to obscure the main idea.

The reason for denoting as L is because the market LIBOR rates are simply-compounded rates. Alternatively, we can write the following identities:

$$P(t,T)(1+L(t,T)\tau(t,T)) = 1, \quad L(t,T) = \frac{1}{1+L(t,T)\tau(t,T)}. \tag{5}$$

Both $R(t,T)$ and $P(t,T)$ are based on short rate r, in the limit, we have the following relationship:

$$r(t) = \lim_{T \to t^+} R(t,T) = \lim_{T \to t^+} L(t,T).$$

When pricing interest rate derivatives, the *zero-coupon curve* plays a pivot role. It is used for projection(for index curve), thus implied from LIBOR related products. On the other hand, the *zero-bond curve* is used for discounting, it is built from $P(t,T)$. We can switch back and forth by using formula (4).

Remark 1.1. *In single curve framework, the index curve and discounting curve are the same. Essentailly, we only needs to bootstrap the curve from LIBOR based basic interst rate instruments, e.g., LIBOR fixings, FRAs, swaps.*

2 Model Free Instruments & Curve Construction

2.1 Forward Rate Agreement

Forward rate, F, is the interest rate that can be locked in today for an investment in a future period. Such rate is based on the instrument called *forward rate agreement(FRA)*. The mechanism is the following (for $0 < t \leq T_1 \leq T_2$),

- at the trade date of FRA, there is no exchange of cashflow, thus the value of FRA is zero;
- at forward starting time T_1, the interest rate starts to accrual;
- at maturiy T_2, the effective payment is equal to $(F - L(T_1,T_2))\tau(T_1,T_2)$.

Essentially, such agreement enable one to pay a fixed rate F between T_1 and T_2 instead of an unknown $L(T_1,T_2)$ to hedge the forward interest rate. We can replicate the FRA by long and short zero-coupons. Without loss of generality, let's assume the notional $N = 1$. To have final payoff $(F - L(T_1,T_2))\tau(T_1,T_2)$, we can write equivalently as $(1 + F\tau(T_1,T_2)) - (1 + L(T_1)\tau(T_1,T_2))$. To have the former part at maturity, we can simply buy a zero-coupon with face value $1 + F\tau(T_1,T_2)$, while 1 unit currency invested at T_1 results $(1 - L(T_1,T_2)\tau(T_1,T_2))$ at time T_2 which worth $P(t,T_1)$ in the beginning. To conclude above, at initial, one can buy a zero-coupon bond with face value $(1+F\tau(T_1,T_2))$(worth $P(t,T_2)(1+F\tau(T_1,T_2))$) and sell a one with maturity T_1(see *figure 1*). Since FRA is entered

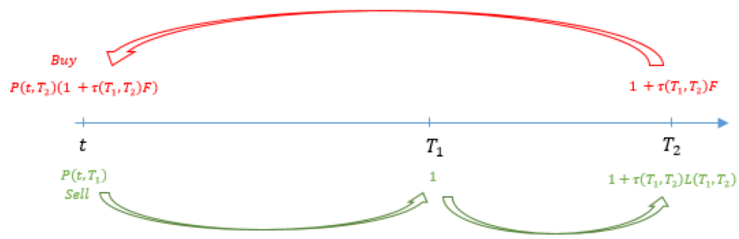


Figure 1: FRA Replication

with no cost, the following identity is valid:

$$P(t,T_2)(1+F\tau(T_1,T_2),N,K) - P(t,T_1) = 0,$$

which gives the fair FRA rate:

$$FRA(t;T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left(\frac{P(t,T_1)}{P(t,T_2)} - 1\right) \tag{6}$$

Alternatively, one can derive the forward rate by risk-neutral pricing theory. The fixed leg of FRA is $NK\tau(T_1,T_2)$ while the floating leg is $N\tau(T_1,T_2)L(T_1,T_2)$. The payoff is at time T_2, thus, the time t price of the contract is:

$$\begin{aligned}
FRA(t;T_1,T_2,N,K) &= N\tau(T_1,T_2)\mathbb{E}_t^{\mathbb{Q}}\Big[D(t,T_2)\big(F - L(T_1,T_2)\big)\Big] \\
=& N\tau(T_1,T_2)\mathbb{E}^{\mathbb{Q}}\Big[D(t,T_2)F - D(t,T_2)\frac{1}{\tau(T_1,T_2)}\Big(\frac{1}{P(T_1,T_2)} - 1\Big)\Big] \\
=& N\tau(T_1,T_2)P(t,T_2)F - N\mathbb{E}_t^{\mathbb{Q}}\big[D(t,T_1)\mathbb{E}_{T_1}^{\mathbb{Q}}\big[\frac{B(t,T_1)P(T_2,T_2)}{B(t,T_1)P(T_1,T_2)}\big]\big] + NP(t,T_2) \\
=& N\tau(T_1,T_2)P(t,T_2)F - NP(t,T_1) + NP(t,T_2)
\end{aligned} \tag{7}$$

Remark 2.1. *To make notation clear,* $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ *is the* t*-conditional expectation under risk-neutral measure, while* $\mathbb{E}_t^{\mathbb{T}_{\not\vDash}}[\cdot]$ *is* t*-conditional expectation under* T_2*-forward measure. When* $t=0$*, they're interpreted as expectation.*

Set $FRA(t; T_1, T_2)$ to zero yields the same forward rate as in (6). Let's turn it into definition:

Definition 2.1. *(Simply-compounded Forward Interest Rate) The simply-compounded forward interest rate prevailing at time t for the expiry $T_1 > t$ and maturity $T_1 > T_1$ is denoted by $F(t; T_1, T_2)$ and defined as in* (6)*. It is the value of the fixed rate renders FRA a fair contract at inception. In particular,*

$$F(T_1; T_1, T_2) = L(T_1, T_2). \tag{8}$$

One may wonder if forward rate is an estimate of future LIBOR rate. Indeed, if we rewrite (6) as:

$$F(t; T_1, T_2) P(t, T_2) = \frac{P(t, T_1) - P(t, T_2)}{\tau(T_1, T_2)}$$

The right hand side is the rescaled difference of two bonds thus can be traded on market. According to equivalent martingale measure theory, under T_2-forward measure, the zero-coupon discounted payoff is a martingale, i.e., for $u \geq t$,

$$\begin{aligned} \mathbb{E}_u^{T_2}\left[\frac{F(t; T_1, T_2) P(t, T_2)}{P(t, T_2)}\right] &= \mathbb{E}_u^{T_2}\left[\frac{P(t, T_1) - P(t, T_2)}{\tau(T_1, T_2) P(t, T_2)}\right] \\ = \frac{P(u, T_1) - P(u, T_2)}{\tau(T_1, T_2) P(u, T_2)} &= \frac{F(u; T_1, T_2) P(u, T_2)}{P(u, T_2)} \end{aligned}$$

Equivalently,

$$F(u; T_1, T_2) = \mathbb{E}_u^{T_2}\Big[F(t; T_1, T_2)\Big]. \tag{9}$$

In particular,

$$F(t; T_1, T_2) = \mathbb{E}_t^{T_2}\big[F(T_1; T_1, T_2)\big] = \mathbb{E}_t^{T_2}\big[L(T_1, T_2)\big] \tag{10}$$

As expected, forward rate $F(t; T_1, T_2)$ is essentially the expected LIBOR rate, where the expectation is taken under an appropriate measure, namely T_2-forward measure.

When the maturity of forward rate collapse towards its expiry, $T_2 \to T_1$, we have

$$
\begin{aligned}
\lim_{T_2 \to T_1} F(t; T_1, T_2) &= -\frac{1}{P(t,T_1)} \lim_{T_2 \to T_1} \frac{P(t,T_2) - P(t,T_1)}{\tau(T_1,T_2)} \\
&= -\frac{1}{P(t,T_1)} \frac{\partial P(t,T_1)}{\partial T_1} \\
&= -\frac{\partial \ln P(t,T_1)}{\partial T_1} := f(t,T_1).
\end{aligned}
$$

We call $f(t,T_1)$ *instantaneous forward interest rate*. As a result, we obtain a characterization of zero-counpon bond via $f(t,T_1)$,

$$
P(t,T_1) = \exp\left\{ -\int_t^{T_1} f(t,u)du \right\} \tag{11}
$$

Instantaneous forward rates is vital for interest rate theory. It is directly related to the notion of "fairness", which refers to absence of arbitrage opportunities. We will discuss more when exploring HJM model.

2.2 Interest-Rate Swaps

As a generalization of FRA, *(forward-start)interest rate swap(IR Swap)* is a contract exchanges two different indexed legs periodically, starting at future. Specifically, at every $T_i^f \in \{T_{1,...,M}^f\} := \mathcal{T}^f$, the fixed leg pays out the amount, $N\tau_i K$, corresponding to a fixed rate $K(\tau_i^f$ is the fixed leg year fraction between T_{i-1}^f and $T_i^f)$. On the other hand, the floating let has schedule payment on $T_j^l \in \{T_1^l, ..., T_N^l\} := \mathcal{T}^l$, with year fraction between T_{j-1}^l and T_j^l denoted by τ_j^l, it pays $N\tau_j^l L(T_{j-1}, T_j)$.

When the fixed leg is paid and the floating leg is received, the IRS is termed *payer IRS(PFS)*, while in the other case, it is termed *receiver IRS(RFS)*. Since they only differ by the sign, let's focus on *RFS*,

$$
\begin{aligned}
&RFS(t, \mathcal{T}^f, \mathcal{T}^l, N, K) \\
=&NK \sum_{i=1}^{M} \tau_i^f \mathbb{E}_{T_i^f}^{\mathbb{Q}} \Big[D(t, T_i) \Big] - N \sum_{j-1}^{N} \tau_j^l \mathbb{E}_t^{\mathbb{Q}} \Big[D(t, T_j) L(T_{j-1}^l, T_j^l) \Big] \\
=&N \Big(K \sum_{i=1}^{M} \tau_i^f P(t, T_i) - \sum_{j=1}^{N} \tau_j^l P(t, T_j) F(t; T_{j-1}^l, T_j^l) \Big) \\
=&N \Big(K \sum_{i=1}^{M} \tau_i^f P(t, T_i) - \sum_{j=1}^{N} P(t, T_{j-1}) - P(t, T_j) \Big) \\
=&N \Big(K \sum_{i=1}^{M} \tau_i^f P(t, T_i) - P(t, T_0) + P(t, T_N) \Big)
\end{aligned}
$$

As in the case of FRA, the IRS contract is entered with zero cost, which leads to a fair swap rate making:

Definition 2.2. *The* forward swap rate $S^{f,l}(t)$ *at time t for IRS schedules* $\mathcal{T}^f$ *and* $\mathcal{T}^l$ *is the rate in the fixed leg makes IRS a fair contract at present time, i.e.,*

$$
S^{f,l}(t) = \frac{P(t, T_0^l) - P(t, T_N^l)}{\sum_{i=1}^{M} \tau_i^f P(t, T_i^f)}. \tag{12}
$$

We can also express the swap rate in terms of forward rate. To ease the notation, we assume $\mathcal{T}^f = \mathcal{T}^l$. Divide both numerator and denominator in (12) by $P(t, T_0)$ and noticing $P(t, T_i)/P(t, T_{i-1}) = 1/(1 + \tau_i F(t; T_{i-1}, T_i)$[2], we obtain:

$$
S(t; T_0, T_n) := S(t) = \frac{1 - \prod_{j=1}^{N} \frac{1}{1+\tau_j F(t; T_{j-1}, T_j)}}{\sum_{i=1}^{N} \tau_i \prod_{j=1}^{i} \frac{1}{1+\tau_j F(t; T_{j-1}, T_j)}}. \tag{13}
$$

2.3 FRA, IR Swap – LIBOR in Arrears

Firstly, we briefly go over the market convention for FRA and IR swap. Using forward IR swap as an example, the *Figure 2* describes the convention for IR swap mentioned above. The *trade date(or, reference date)*, is the date when the

[2] Here, we assumed $\tau_\cdot^f = \tau_\cdot^l$, $M = N$, $T_\cdot^f = T_\cdot^l$.

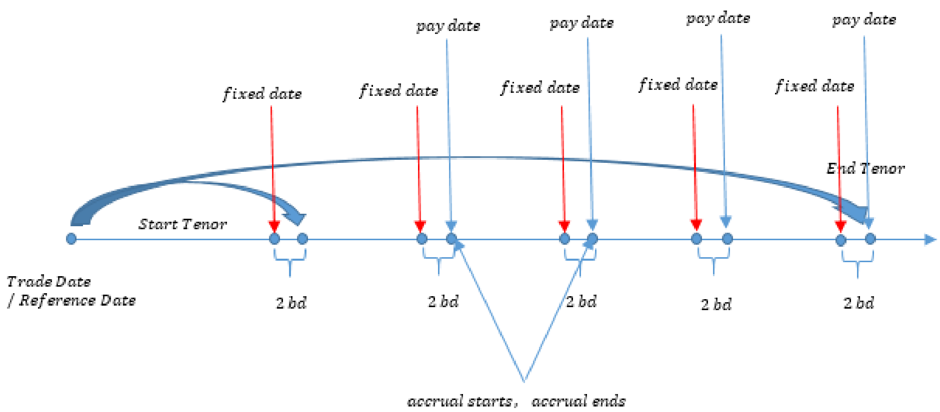


Figure 2: Forward IR Swap Convention

trade is entered. For forward IR swap, the *start tenor* determines the date when exchange of legs start, while the *end tenor* tells the expiry of the contract. Floating leg, LIBOR rate, for exchange during $[T_{i-1}, T_i]$ is prefixed at *fixed date*, usually two business day before the accrual starts. The calculations in previous sections are based on the fact that exchange floating leg and fixed leg happens at pay date in *figure 2*. In some contracts, the pay date can be set as the date accrual starts, which is legitimated because floating leg is fixed already(see *Figure 3*). To address this

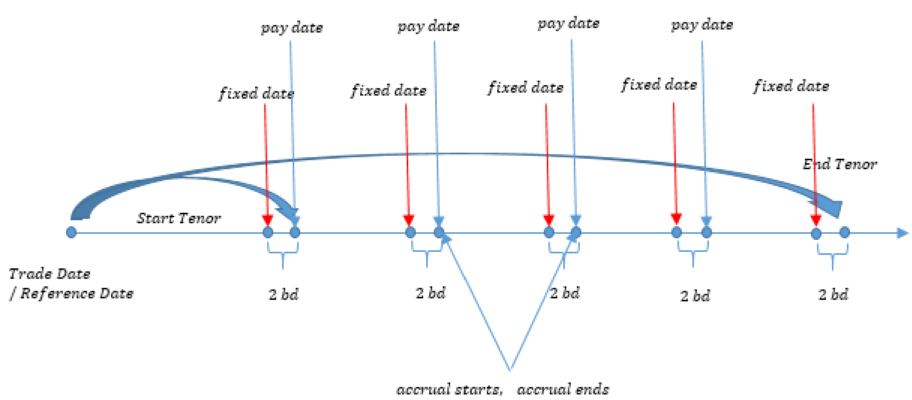


Figure 3: Forward IR Swap Convention Pay In Arrears

difference, the key is to value future LIBOR payment. Without loss of generality, assume we are concerned with time t value of $L(T_1, T_2)$. Again, by risk-neutral pricing theory:

$$V_{lib}(t) = \mathbb{E}_t^{\mathbb{Q}}\Big[D(t, T_1)L(T_1, T_2)\Big] = P(t, T_1)\mathbb{E}_t^{T_1}\Big[L(T_1, T_2)\Big] \tag{14}$$

Notice, now, discounting is from T_1, therefore, we can only switch to T_1-forward measure, under which $L(T_1, T_2)$ is not a martingale. Such situation arises frequently in interest rate derivatives pricing, the expectation is taken in a wrong martingale measure. The usual way to overcome the hurdle is to apply *convexity adjustment*. By *Radon-Nikodym theorem*, the transition between T_1-forward and T_2-forward measure can be facilitated by *Radon-Nikodym derivative*, which after conditioning on t becomes

$$Z_t = \frac{P(T_1, T_2)P(t, T_1)}{P(t, T_2)}$$

Then, since $\mathbb{E}_t^{T_1}[X] = \mathbb{E}_t^{T_2}[X\frac{1}{Z_t}]$ for $\mathcal{F}_{T_1}$-measurable random variable, we have

$$\begin{aligned}
&\mathbb{E}_t^{T_1}\Big[L(T_1, T_2)\Big] = \mathbb{E}_t^{T_2}\Big[L(T_1, T_2)\frac{P(t, T_2)}{P(T_1, T_2)P(t, T_1)}\Big] \\
=&\frac{1}{1 + \tau(T_1, T_2)F(t; T_1, T_2)}\mathbb{E}_t^{T_2}\Big[L(T_1, T_2)\Big(1 + \tau(T_1, T_2)L(T_1, T_2)\Big)\Big] \\
=&\frac{F(t; T_1, T_2) + \tau(T_1, T_2)Var\big[L(T_1, T_2)\big] + \tau(T_1, T_2)F^2(t; T_1, T_2)}{1 + \tau F(t; T_1, T_2)} \\
=&F(t; T_1, T_2)\Big(1 + \frac{\tau(T_1, T_2)Var\big[L(T_1, T_2)\big]}{F(t; T_1, T_2)\big(1 + \tau(T_1, T_2)F(t; T_1, T_2)\big)}\Big) \\
=:&F(t; T_1, T_2) + \Delta(T_1, T_2)
\end{aligned} \tag{15}$$

To give an explicit formula for the convexity correction term, a model of $F(t; T_1, T_2)$ has to be established under T^2-forward measure. For example, let's use the simplest, but widely used, BS model with constant volatility σ,

$$dF(s; T_1, T_2) = \sigma F(s; T_1, T_2)dW_s^{T_2}, \quad s \in [t, T_1]. \tag{16}$$

which has solution:

$$L(T_1, T_2) = F(T_1; T_1, T_2) = F(t; T_1, T_2)\exp\Big\{\sigma W^{T_2}(T_1 - t) - \frac{1}{2}\sigma^2)(T_1 - t)\Big\}$$

Plug into (15),

$$
\begin{aligned}
\mathbb{E}_t^{T_1}\left[L(T_1,T_2)\right] &= F(t;T_1,T_2) + \Delta(T_1,T_2) \\
&= F(t;T_1,T_2) + F(t;T_1,T_2)\frac{\tau(T_1,T_2)F(t;T_1,T_2)}{1+\tau(T_1,T_2)F(t;T_1,T_2)}\exp\left\{\sigma^2(T_1-t)-1\right\}
\end{aligned}
$$

Now, we can using these building blocks to evaluate FRA and IR Swaps. Notice that the convexity adjustment is model dependent, thus one should not expect a universal solution. In the meanwhile, to apply convexity adjustment, one has to calibrate the model in the first place. In our case, the implied volatility of forward rate process has to be determined in order to make things work.

2.4 Euro-dollar Future

Firstly, a quick introduction to *Euro-dollar future(EDF)*, $3M$-EDF is the most popular interest rate futures contract in the US traded by CME Group. Investors purchase the derivative to lock in the future interest rate. While an *Eurodollar* is a dollar deposited in a U.S. or foreign bank outside the US, EDF interest rate is the rate of interest earned on Eurodollars deposited by one bank with another bank, which is essentially the same as the LIBOR.

To understand the mechanism of $3M$-EDF. The Eurodollar future quote at time t is denoted by $Fut_L(t,T)$, corresponding to implied LIBOR rate:

$$
Implied_L(t) = \frac{(100 - Fut_L(t,T))}{100}.
$$

At maturity, the EDF payoff is,

$$
Payoff(T) = N \times \Delta \times \frac{100 - Fut_L(T,T)}{100} \tag{17}
$$

where N is notional, $\Delta = 1/4$ and $L(T, T+\Delta)$ is LIBOR $3M$ rate from T to $T + 3M$. As in the case of equity future, it costs zero to enter the contract and requires daily mark-to-market. That is, increase 1 basis point on EDF quotes corresponds to a gain of \$25. In formula,

$$
margin(t+1d) = 25 \times 100 \times \left(Fut_L(t+1d,T) - Fut_L(t,T)\right)
$$

For example, if today's quote of EDF is 99.6, the second day, the quote goes up to 99.7, the EDF owner has to post a margin of:

$$
(99.7 - 99.6) \times 100 \times \$25 = \$250
$$

Compared to forward rate agreement, EDF is standard and needs MTM. In addition, FRA usually expires within one year, EDF contracts expire up to three years are very liquid.

Let's repeat the above mechanism in a well-defined math framework. Let $0 = T_0 < T_1 < \cdots < T_n = T$ partition the life of the EDF into days. At each end of the day, buyer and seller settle any difference due to the change in future quotes so that the current value of the contract is always 0. Suppose we go long one futures contract quotes $F_L(T_k, T)$ at time T_k. One day later, the quote changes to $Fut_L(T_k, T)$. Two scenario can happen:

- $Fut_L(T_k, T) > Fut_L(T_{k-1}, T)$, we pay $2,500\big(Fut_L(T_k, T) - Fut_L(T_{k-1}, T)\big)$ to the clearing house;
- $Fut_L(T_{k-1}, T) > Fut_L(T_k, T)$, we receive $2,500\big(Fut_L(T_{k-1}, T) - Fut_L(T_k, T)\big)$ from the clearing house.

Now, effectively, we have a future contract purchased at T_{k+1} and the value of the contract at T_{k+1} to both parties become 0 again. Following this procedure, we can calculate the total of all payments to the long party entering a contract at time T_k and holding it until T us:

$$\sum_{j=k+1}^{n} 2,500\big(Fut_L(T_j, T) - Fut_L(T_{j-1}, T)\big) = 2,500\big(Fut_L(T, T) - F_L(T_k, T)\big)$$

At the end, the long party will get $N \times 0.25 \times \frac{100 - Fut_L(T,T)}{100}$. Therefore, the total netincome change for the long side is:

$$\begin{aligned} &2,500\big(Fut_L(T, T) - F_L(T_k, T)\big) + N \times 0.25 \times \frac{100 - Fut_L(T, T)}{100} \\ =&N \times 0.25 \times \frac{100 - Fut_L(T_k, T)}{100} \end{aligned}$$

This means that the long side essentially locked in LIBOR rate with $(100 - Fut_L(T_k, T))/100$ for the future 3-month. In term of pricing, we have the following proposition:

Proposition 2.2. *$Fut_L(T_k, T)$ is a martingale with respect to* spot measure*,*

$$Fut_L(t_k, T) = \mathbb{E}^{spot}_{T_k}\Big[Fut_L(T, T)\Big] = \mathbb{E}^{spot}_{T_k}\Big[g(L(T, T + \Delta))\Big]. \tag{18}$$

where g is a linear function of $L(T, T + \Delta)$.

Remark 2.3. Spot measure *is defined as:*

$$\begin{aligned} B^d(t) &:= \frac{P(t, T_{\beta(t)-1})}{\prod_{j=1}^{\beta(t)-1} P(T_{j-1}, T_j)} \\ &= \prod_{j=1}^{\beta(t)-1} \Big(1 + \tau_j F(T_{j-1}; T_{j-1}, T_j)\Big) P(t, T_{\beta(t)-1}). \end{aligned} \tag{19}$$

where $\beta(t) - 1$ is the end point of the last tenor preceeding t. One can think of it as bank account re-balanced only on the times in our discrete-tenor structure. The interpretation of $B^d(t)$ is that of the value at time t of a portfolio defined as follows. It starts with one unit of currency, exactly as in the continuous bank account case, but this unit amount is now invested in a quantity of X_1 of T_1-zero-coupon bonds, Such X_1 is readily found by noticing that $X_1 P(T_0, T_1) = 1$, hence $X_1 = 1/P(T_0, T_1)$. At time T_1, X_1 can be cashed out from the bonds and invested in a quantity $X_2 = X_1/P(T_1, T_2) = \frac{1}{P(T_0,T_1)P(T_1,T_2)}$ of T_2-zero-coupon bonds. Continuing this procedure until we reach the last tenor start date $T_{\beta(t)-2}$, where we invested:

$$X_{\beta(t)-1} = \frac{1}{\prod_{j=1}^{\beta(t)-1} P(T_{j-1}, T_j)}. \tag{20}$$

Then, the present value at the current time t of this investment is $X_{\beta(t)-1} P(t, T_{\beta(t)-1})$.

Proof. Enter into the contract at time t, this promises a payoff of $Fut_L(T_{k+1}, T) - Fut_L(T_k, T)$ at time T_{k+1}. The value of this at T_k is:

$$\begin{aligned} &\mathbb{E}_{T_k}^{spot}\left[\frac{B^d(T_k)}{B^d(T_{k+1})} 2,500 \Big(Fut_L(T_{k+1}, T) - Fut_L(T_k, T)\Big)\right] \\ =&2,500 \times \frac{B^d(T_k)}{B^d(T_{k+1})}\left[\mathbb{E}_{T_l}^{spot}\big[Fut_L(T_{k+1}, T)\big] - Fut_L(T_k, T)\right] \end{aligned}$$

This must equal to 0, since the cost of entering a futures contract is zero. This requires:

$$Fut_L(T_k, T) = \mathbb{E}_{T_k}^{spot}\big[Fut_L(T, T)\big] = \mathbb{E}_{T_k}^{spot}\big[g\big(L(T, T + \Delta)\big)\big] \tag{21}$$

$\square$

To compare the forward rate and Eurodollar future price as $k = 0$, i.e., at time $t = T_0 = 0$, observe that

$$\mathbb{E}^{spot}\big[L(T, T + \Delta)\big] = \mathbb{E}^{T+\Delta}\left[L(T, T + \Delta)\frac{P(0, T + \Delta)}{\prod_{j=1}^{\beta(t)} P(T_{j-1}, T_j)}\right] \tag{22}$$

Thus,

$$\Delta_{EDF/FRA} = \mathbb{E}^{T+\Delta}\left[L(T, T + \Delta)\left(\frac{P(0, T + \Delta)}{\prod_{j=1}^{\beta(t)} P(T_{j-1}, T_j)} - 1\right)\right]$$

The convexity adjustment term $\Delta_{EDF/FRA}$ is tractable if we can use a short rate term structure model, which we will discuss in the interest rate models notes.

Remark 2.4. *Since $g(\cdot)$ is a linear function of LIBOR, we only focus on $L(\cdot, \cdot)$ to not obscure the essence.*

2.5 Zero Curve Construction

Cash deposit, forward rate agreement, interest rate swap, Euro-dollar future are fundamental LIBOR related instruments, from which zero-curve can be bootstrapped and interpolated[3]. By zero-curve, we mean the graph corresponding to the mapping $T \mapsto P(0, T)$. Let's use US market to illustrate the idea. The cash rates are of short range, e.g., 1BD, 1M and $3M$, we can directly observe on the market. After that, we need mid range instruments, both FRA and EDF are used, although EDF are usually longer than FRA. Finally, interest rate swap rate up to 30Y is sufficient for curve construction. To summarize, we need following quotes:

- *short range:* 1BD, 1M and $3M$ cash rates;
- *middle range:* ED1, ED2, ED3, ED4, ED5, ED6, ED7 and ED8 Euro-dollar future contracts, or FRA 1×7, FRA 2×8, FRA 3×9, FRA 4×10, e.t.c.;
- *long range:* 3Y, 4Y, 5Y, 6Y, 7Y, 10Y, 15Y, 20Y, 25Y and 30Y swap rates.

The short range product is equivalent to the $P(0, T)$ itself, thus trivial to get. The convexity-adjusted Euro-dollar futures is treated as a forward rate agreement, which we have worked out the formula:

$$FRA(0; T_1, T_2) = \frac{1}{\tau(T_1, T_2)}\left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right) \tag{23}$$

[3] We will not go to the details of interpolation, market practice is to use cubic spline to interpolate continuous compounded rate – instantaneous forward rate.

Since the left hand side is observable from market, we can work from short maturity to maturity to obtain $P(0, T_i)$, for different T_i's. As for interest rate swaps, the market observed swap rate is expressed as a non-linear function of $P(0, T)$'s, i.e.,

$$S^{f,l}(0) = \frac{P(0, T_0^l) - P(0, T_N^l)}{\sum_{i=1}^{M} \tau_i^f P(0, T_i^f)}, \tag{24}$$

therefore, those $P(0, T)$'s can be stripped from the curve as well. Figure 4 shows a typical zero curve, remember, in single curve framework, we do not distinguish discounting curve and projection curve.

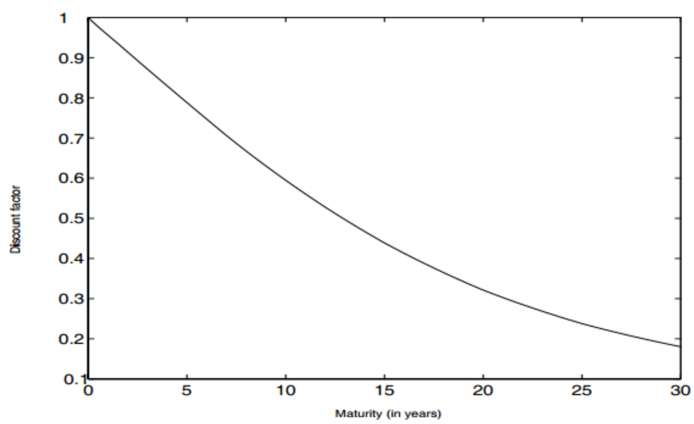


Figure 4: Zero Curve

3 Model Semi-Free Instruments

In this section, we will introduce two main derivatives of the interest-rate market, interest rate cap/floor and Swaptions. We assumed classical BS log-normal model for the underlying, later on, after discussing other interest rate model, we will return to valuation of them again.

3.1 Interest-Rate Caps/Floors

A *cap* is a rate limited tool, which provide the long party an interest rate ceiling on floating rate debts. The cap itself provides a periodic payment conditional on the positive amount by which the reference index rate exceeds the strike rate. At certain period, $[T_{i-1}, T_i]$, $L(T_{i-1}, T_i)$ is greater than K, the owner will be compensated with the difference, otherwise, no cashflow will be received. This financial instrument is primarily used by a borrower of floating rate debts, ensuring that rate never exceeds the cap specified. As shown in *figure 5*, only green part needs to be paid. On contrast, a *floor* guarantees a lower bound for the rate of interest

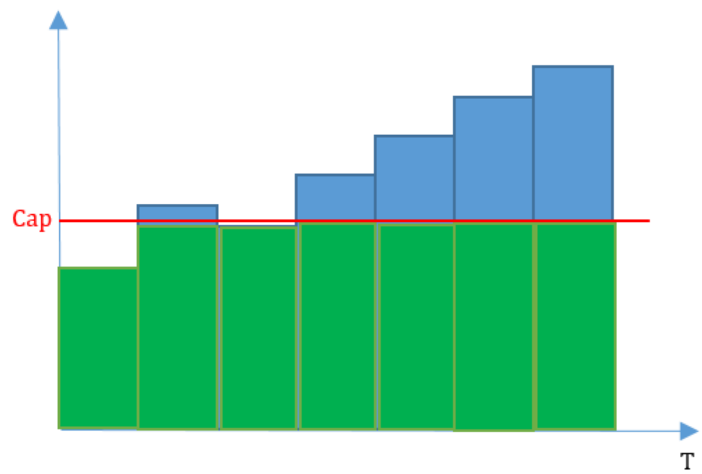


Figure 5: Interest Rate Cap

received on an investment. It is often used when one holds a long position of a floating rate note, which pays the owner periodic LIBOR payments. When the LIBOR rate is lower (than floor), the difference will be compensated by a long position on floor. To be formal, let's again create a schedule for cap(or floor), i.e., $\mathcal{T} = \{T_0, T_1, ..., T_n\}$, and day count factor τ_i counts between T_{i-1} to T_i. A strike rate for cap and floor is denoted by K and notional is again N. The discounted payoff of cap and floor at observation time $t < T_0$ is thus,

$$\sum_{i=1}^{n} D(t, T_i) N \tau_i \big(L(T_{i-1}, T_i) - K\big)^+, \quad \sum_{i=1}^{n} D(t, T_i) N \tau_i \big(K - L(T_{i-1}, T_i)\big)^+ \tag{25}$$

Since the pricing is similar, let's use cap for illustration. The fair price at time 0, according to risk-neutral pricing theory is:

$$
\begin{aligned}
Cap\big(0, \mathcal{T}, N, K\big) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n} D(0,T_i) N \tau_i \big(L(T_{i-1},T_i) - K\big)^{+}\Bigg] \\
&= N\sum_{i=1}^{n} \tau_i \mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T_i)\big(L(T_{i-1},T_i) - K\big)^{+}\Bigg] \\
&= N\sum_{i=1}^{n} P(0,T_i)\tau_i \mathbb{E}^{T_i}\Bigg[\big(F(T_{i-1};T_{i-1},T_i) - K\big)^{+}\Bigg]
\end{aligned}
\tag{26}
$$

Since $F(T_{i-1}; T_{i-1}, T_i)$ is a martingale under T_i-forward measure, if we use BS model under T_i-forward measure, for $i = 1, 2, ..., n$, then

$$
dF(t; T_{i-1}, T_i) = \sigma^{1,n} F(t; T_{i-1}, T_i) dW_t^{T_i}. \tag{27}
$$

The expectation in the end of (26) can be decomposed to caplets[4], each of which can computed by BS formula,

$$
\begin{aligned}
Cpl^{black}(0; T_{i-1}, T_i) &= P(0,T_i)\mathbb{E}^{T}\Bigg[\big(F(T_{i-1};T_{i-1},T_i) - K\big)^{+}\Bigg] \\
&= P(0,T_{i-1}) Bl(F(0;T_{i-1},T_i), \nu_i, K, 1)
\end{aligned}
\tag{28}
$$

where

$$
Bl(F, \nu_i, K, \omega) = F\omega N(\omega d_+) - K\omega N(\omega d_-)
$$

with

$$
\begin{aligned}
\omega &= 1, \quad \textit{for call} \\
d_{\pm} &= \frac{\ln\frac{F}{K} \pm \frac{\nu_i^2}{2}}{\nu_i} \\
\nu_i &= \sigma^{1,n}\sqrt{T_{i-1}}.
\end{aligned}
$$

It is plausible to put a constant volatility $\sigma^{1,n}$ for the whole lifetime of a cap, nevertheless, it is market convention to quote by $\sigma^{1,n}$. Similarly, we can derive the fair price for floor at time 0 as:

$$
Flr\big(0, \mathcal{T}, N, K\big) = N\sum_{i=1}^{n} P(0,T_i) Bl(F(0;T_{i-1},T_i), \nu_i, K, -1) \tag{29}
$$

[4]Caplet can be think of as the simplest cap that has one tenor.

Intuitively speaking, the comparable quantity of strike rate K is the forward swap rate, because they both are compared to the floating rate. The following definition gives a formal statement:

Definition 3.1. *The cap (floor) is said to be* at-the-money(ATM) *if and only if*

$$K = K_{ATM} := S(0) = \frac{P(0,T_0) - P(0,T_n)}{\sum_{i=1}^{n} \tau_i P(0,T_i)} \tag{30}$$

The cap is ITM if $K < K_{ATM}$ and OTM if $K > K_{ATM}$, with the converse holding for a floor.

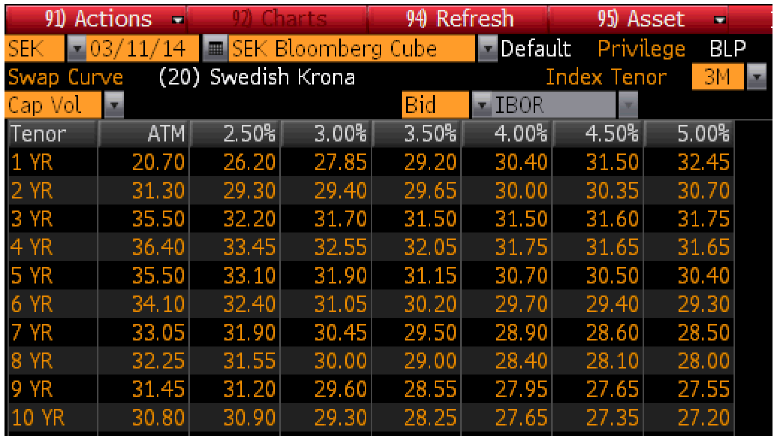
One may find that the optionalities of cap and floor are very similar to equity option. Thus, the put-call parity should remain true in for cap and floor. Indeed, using $(L-K)^+ - (K-L)^+ = L - K$, we have

$$\begin{aligned}
Cap\big(0,\mathcal{T},N,K\big) - Flr\big(0,\mathcal{T},N,K\big) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n} D(0,T_i) N \tau_i \big(L(T_{i-1},T_i) - K\big)^+\Bigg] \\
&- \mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n} D(0,T_i) N \tau_i \big(K - L(T_{i-1},T_i)\big)^+\Bigg] \\
&= N \sum_{i=1}^{N} \tau_i \mathbb{E}^{\mathbb{Q}}\Big[D(0,T_i)\big(L(T_{i-1},T_i) - K\big)^+\Big] = PFS(0,\mathcal{T},N,K),
\end{aligned}$$

the difference between a cap and the corresponding floor is equivalent to a forward start IR swap with fixed rate K. Therefore, if $K = S(0)$, that is $K = K_{ATM}$, the cap price is equals to floor price.

Let's conclude this section by illustrating a useful technique – *stripping cap volatility*, in particular, we choose the *term structure stripping method*. Since cap can be decomposed to a series of caplets, we can determine the BS log-normal volatility for caplets in a consistent way such that it produce the original price for the cap. *Figure 6* shows a typical cap log-normal volatility quotes. It is quoted according to different maturities, i.e., 1YR, 2YR, ..., 10YR, and different strike, i.e., ATM, 2.5%, ..., 5.5%. The stripping method deals with each column of quotes separately. That is, for each strike K, the caplet volatility is a function of maturity only, denoted by $\sigma_K(T)$. The algorithm works as follows, for a fixed strike K,

- order the caps in ascending order of maturity;
- compute the sum of the prices of the caplets whose volatility has already determined;


91) Actions 92) Charts 94) Refresh 95) Asset
SEK 03/11/14 SEK Bloomberg Cube Default Privilege BLP
Swap Curve (20) Swedish Krona Index Tenor 3M
Cap Vol Bid IBOR

Tenor	ATM	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%
1 YR	20.70	26.20	27.85	29.20	30.40	31.50	32.45
2 YR	31.30	29.30	29.40	29.65	30.00	30.35	30.70
3 YR	35.50	32.20	31.70	31.50	31.50	31.60	31.75
4 YR	36.40	33.45	32.55	32.05	31.75	31.65	31.65
5 YR	35.50	33.10	31.90	31.15	30.70	30.50	30.40
6 YR	34.10	32.40	31.05	30.20	29.70	29.40	29.30
7 YR	33.05	31.90	30.45	29.50	28.90	28.60	28.50
8 YR	32.25	31.55	30.00	29.00	28.40	28.10	28.00
9 YR	31.45	31.20	29.60	28.55	27.95	27.65	27.55
10 YR	30.80	30.90	29.30	28.25	27.65	27.35	27.20

Figure 6: Cap Log-Normal Vol Quotes

- calculate the price of the current cap according to market quote. The difference between these two prices must be matched by the remaining caplets. Notice, if the difference happens to be smaller than 0, this signals an arbitrage opportunity in the input data;
- for the remaining caplets, construct a single objective function that interpolates starting from the last caplet of the current.
- work out the volatility quotes by matching the difference found above.

Ideally, we shall be able to get, for different maturity, volatility skews. The caplets volatility surface will be used for construction of volatility cube, which will be explained shortly.

3.2 Swaptions

Swaption is an option on IR swap. An *European payer swaption* is a derivative give the owner right to enter a payer IR swap at a given swaption maturity where he becomes the fixed rate payer, thus floating rate receiver. A receiver swaption is the opposite. The participants in the swaption market are mostly large corporations, banks, financial institutions and hedge funds. They typically use swaptions to manage interest rate risk arising from their core business or from their financing arrangements.

For example, a corporation wants protection from rising interest rates might buy a payer swaption. Then in the future, the corp can be financed at the fixed rate if the floating rate at that time has increasing trend. A bank that holds a mortgage portfolio might buy a receiver swaption to protect against lower interest rates that might lead to early prepayment of the mortgages. A hedge fund believing that interest rates will not rise by more than a certain amount can sell a payer swaption, aiming to make money by collecting the premium.

Let's assume swaption maturity coincides with starting date of the underlying IR swap, which has schedule $\mathcal{T} := \{T_0, ..., T_n\}$ as defined before, the IR swap length $(T_n - T_0)$ is called *swaption tenor*. For a fixed rate K, i.e., strike of swaption, the value of the swap a time T_0 is:

$$\begin{aligned} N \sum_{i=1}^{n} \tau_i \mathbb{E}_{T_0}^{\mathbb{Q}} \Big[D(T_0, T_i) \big(L(T_{i-1}, T_i) - K \big) \Big] &= N \sum_{i=1}^{n} \tau_i P(T_0, T_i) \mathbb{E}_{T_0}^{T_i} \Big[L(T_{i-1}, T_i) - K \Big] \\ &= N \sum_{i=1}^{n} \tau_i P(T_0, T_i) \Big(F(T_0; T_{i-1}, T_i) - K \Big) \end{aligned}$$

The option will be exercised only if this value is positive, as a result, to obtain the swaption payoff at time T_0, we take the positive part of it, just as in the equity option case. As a result, the discounted final payoff of swaption is:

$$N D(t, T_0) \Bigg(\sum_{i=1}^{n} \tau_i P(T_0, T_i) \Big(F(T_0; T_{i-1}, T_i) - K \Big) \Bigg)^+ \tag{31}$$

Since $(\cdot)^+$ is a convex function, it holds that:

$$\begin{aligned} &\Bigg(\sum_{i=1}^{n} P(T_0, T_i) \tau_i \big(F(T_0; T_{i-1}, T_i) - K \big) \Bigg)^+ \\ \leq &\sum_{i=1}^{n} P(T_0, T_i) \tau_i \big(F(T_0; T_{i-1}, T_i) - K \big)^+ \\ = &\sum_{i=1}^{n} P(T_0, T_i) \tau_i \big(\mathbb{E}_{T_0}^{T_i} [L(T_{i-1}, T_i) - K] \big)^+ \\ = &\sum_{i=1}^{n} P(T_0, T_i) \tau_i \mathbb{E}_{T_0}^{T_i} \big[\big(L(T_{i-1}, T_i) - K \big)^+ \big] \end{aligned}$$

Notice the left hand side is the payoff of cap at time T_0. This chains of inequalities imply that a payer swaption has a vlaue that is always smaller than the value of the corresponding cap contract.

To obtain time 0 value of swaption, we take expectation of (31) under risk neutral measure,

$$
\begin{aligned}
& N\mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T_0)\Bigg(\sum_{i=1}^{n}\tau_i P(T_0,T_i)\Big(F(T_0;T_{i-1},T_i)-K\Big)\Bigg)^{+}\Bigg] \\
=& N\mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T_0)\Bigg(\sum_{i=1}^{n}\tau_i P(T_0,T_i)\Big(\frac{1}{\tau_i}\frac{P(T_0,T_{i-1})-P(T_0,T_i)}{P(T_0,T_i)}-K\Big)\Bigg)^{+}\Bigg] \\
=& N\mathbb{E}^{\mathbb{Q}}\Bigg[D(0,T_0)\sum_{i=1}^{n}\tau_i P(T_0,T_i)\Big(S(T_0)-K\Big)^{+}\Bigg] \\
=& N\mathbb{E}^{SWP}\Bigg[\Big(S(T_0;T_0,T_n)-K\Big)^{+}\Bigg]\sum_{i=1}^{n}\tau_i P(0,T_i)
\end{aligned}
$$

where the last expectation is taken under *swap measure* defined by $SWP(t) = \sum_{i=1}^{n}\tau_i P(t,T_i)$. Recall the forward rate is a martingale under corresponding forward measure, it is not surprising that forward swap rate is a martingale under swap measure, i.e.,

$$
\mathbb{E}_u^{SWP}\Big[S(t;T_0,T_n)\Big] = S(u;T_0,T_n), \quad u \le t \le T_0. \tag{32}
$$

To prove, notice

$$
\begin{aligned}
S(t;T_0,T_n) &= \frac{P(t,T_0)-P(t,T_n)}{SWP(t)} \\
\Rightarrow S(t;T_0,T_n)SWP(t) &= P(t,T_0)-P(t,T_n)
\end{aligned}
$$

The right hand side of the last identity can be traded on the market, thus we can price it under t-forward measure, which yields:

$$
\mathbb{E}_u^{t}\Big[P(t,T_0)-P(t,T_n)\Big] = \frac{P(u,T_0)-P(u,T_n)}{P(u,t)} \tag{33}
$$

On the other hand, by change of numéraire,

$$
\begin{aligned}
& \mathbb{E}_u^{t}\Big[S(t;T_0,T_n)SWAP(t)\Big] = \mathbb{E}_u^{SWP}\Big[S(t;T_0,T_n)SWP(t)\frac{SWP(u)P(t,t)}{SWP(t)P(u,t)}\Big] \\
=& \frac{SWP(u)}{P(u,t)}\mathbb{E}_u^{SWP}\Big[S(t;T_0,T_n)\Big]
\end{aligned} \tag{34}
$$

Equating (33) and (34) proves the (32).

By martingale representation, forward swap rate process is drift-less under swap measure. If log-normal dynamic is assumed for forward rate process, i.e.,

$$dS(t;T_0,T_n) = \sigma^{0,n} S(t;T_0,T_n)dW_t^{SWP}, \quad 0 \le t \le T_0,$$

then, we can use BS formula to obtain:

$$PS(0,\mathcal{T},N,K) = NBl(S(0;T_0,T_n),\sigma^{0,n}\sqrt{T_0},1)\sum_{i=1}^{n} P(0,T_i). \tag{35}$$

For receiver swaption,

$$RS(0,\mathcal{T},N,K) = NBl(S(0;T_0,T_n),\sigma^{0,n}\sqrt{T_0},-1)\sum_{i=1}^{n} P(0,T_i). \tag{36}$$

Assuming BS model greatly simplify the calculation and that's how market quotes swaptions. Similar to cap/floor, we have the following definition:

Definition 3.2. *The swaption is said to be ATM if and only if:*

$$K = K_{ATM} := S(0;T_0,T_n) = \frac{P(0,T_0) - P(0,T_n)}{\sum_{i=1}^{n} \tau_i P(0,T_i)}.$$

The payer swaption is instead said to be ITM if $K < K_{ATM}$ and OTM if $K > K_{ATM}$. The opposite hold for receiver swaption.

The put-call-parity is identical to the cap/floor's, that is, the difference between a payer swaption and receiver swaption is a forward-start swap.

Unlike cap/floor that has only maturity and strike, swaption is three dimensional, i.e., start tenor, end tenor and strike. Thus, the volatility quotes are called volatility cube(see *Figure 7*).

3.3 Volatility Cube Construction by SABR

3.3.1 Based on Swaption

Stochastic-Alpha-Beta-Rho model, which we introduce in the equity model notes, are advantageous in construction of swaption volatility cube. Let's recall the classical SABR model:

$$\begin{cases} dF_t = \alpha_t F_f^{\beta} dW_t^1, & F_0 = f, \\ d\alpha_t = \nu \alpha_t dW_t^2, & \alpha_0 = \alpha. \end{cases} \tag{37}$$

Expiry		1 YR	2 YR	3 YR	4 YR	5 YR	6 YR	7 YR	8 YR	9 YR	10 YR
1 MO	Vol	50.60	45.75	49.90	49.15	47.90	42.55	41.40	38.60	37.00	36.20
	Strike	1.0805	0.5764	0.7039	0.9160	1.1493	1.3760	1.6113	1.7936	1.9395	2.0898
3 MO	Vol	48.30	48.70	49.15	49.10	48.45	44.50	42.35	39.70	38.80	38.30
	Strike	1.0466	0.5992	0.7513	0.9723	1.2092	1.4420	1.6704	1.8489	1.9962	2.1367
6 MO	Vol	56.30	54.05	54.25	50.30	47.70	44.20	42.50	39.70	39.20	39.05
	Strike	0.8399	0.5850	0.7860	1.0336	1.2787	1.5230	1.7426	1.9114	2.0619	2.1933
9 MO	Vol	59.75	55.70	52.90	50.45	46.60	42.70	40.70	39.40	38.90	40.30
	Strike	0.5174	0.5520	0.8144	1.0898	1.3454	1.6023	1.8127	1.9742	2.1301	2.2491
1 YR	Vol	61.40	57.65	53.95	49.25	46.05	42.80	41.20	38.90	38.30	37.90
	Strike	0.0477	0.4835	0.8253	1.1308	1.3981	1.6694	1.8726	2.0273	2.1860	2.2954
2 YR	Vol	67.00	56.50	50.55	44.80	42.30	39.85	38.30	36.40	35.50	34.95
	Strike	0.9180	1.2132	1.4934	1.7420	2.0039	2.1907	2.3259	2.4723	2.5667	2.6785
3 YR	Vol	55.30	48.85	43.80	39.60	37.75	36.20	35.00	33.20	32.80	32.10
	Strike	1.5133	1.7882	2.0257	2.2881	2.4608	2.5786	2.7152	2.7956	2.8990	2.9579
4 YR	Vol	46.75	40.90	38.20	35.60	34.35	33.15	32.10	31.40	30.80	30.15
	Strike	2.0804	2.3001	2.5662	2.7192	2.8140	2.9402	3.0053	3.1007	3.1482	3.1943

Figure 7: Swaption Volatility Cube

where the two processes are correlated by $dW_t^1 dW_t^2 = \rho dt$. The underlying process depends on the instrument. For example, for caplets, the underlying asset is forward rate, thus the dynamic is under T_i-forward measure. While for swaption, the underlying dynamic is the forward swap rate, in this case, the probability measure is *swap measure*.

The significance of SABR is the approximation formula of BS implied volatiltiy. Namely, for a given maturity T, the equivalent BS implied volatiltiy is interpolated by:

$$\sigma_{impl}(K, f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K + \cdots\right\}}$$
$$\times \left(\frac{z}{\eta(z)}\right) \times \left\{1 + \left[\frac{1-\beta^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right\} \tag{38}$$

where

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \quad \eta(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

In particular, for at the money options(caplet, cap, floor, swaption), formula (38) is simplified to:

$$
\begin{aligned}
&\sigma_{impl,\, atm} = \sigma_{impl}(f, f) \\
=&\frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{4f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right\}
\end{aligned} \tag{39}
$$

Remark 3.1. *It is worth to mention that the approximation formula* (38) *breaks down for small strike and long maturity. The Anotnov, et al, proposes a new approximation method that is both computationally efficient and accurate in wings.*

Our objective is to calibrate SABR model to the volatility cube quote from the market (e.g. figure 7). Since SABR model is maturity time dependent, in addition, swaption has not only option tenor but also swap tenor, the SABR model has to adjusted to each combination of tenors. In other words, we fix a option tenor m-year and a swap tenor n-year, calibrating SABR along strike directions. The optimal parameter can be extracted by solving the following optimization problem:

$$
\min_{\Theta} \sum_{K} \left(\sigma_{mkt}^{m,n}(K) - \sigma_{impl}^{m,n}(K, \Theta)\right)
$$

where $\Theta = (\nu, \rho)$(given α and β are pre-fixed). Figure 8 is a visualization of calibration.

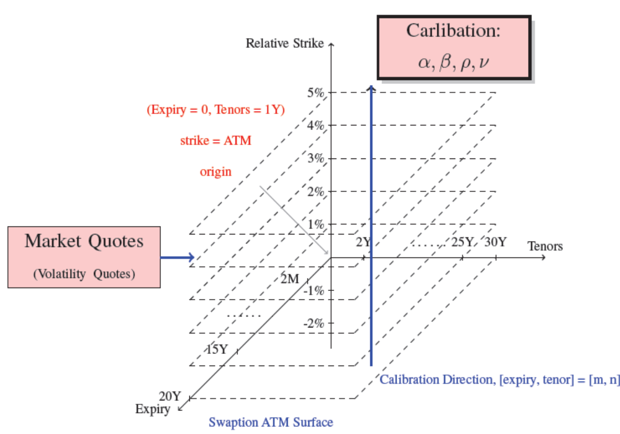


Figure 8: Vol Cube Construction based SWPT

3.3.2 Based on Swaption & Cap

Constructing a volatility cube requires volatility quotes across a range of strikes, from ATM to ATM$\pm$300-basis points. However, the OTM swaption are less traded so that it maybe neither achievable nor informative. The alternative to construct volatility cube is to use an ATM swaption surface and a surface of cap volatility quote together.

Specifically, we firstly calibrate SABR on caplets along each strike directions. The caplets can be stripped off from the cap volatility surface as explained in previous sections. Notice, we match the option tenor of the caplets with the swaption tenor, hoping the caplets share some information of the swaptions. Now, we may better understand the advantage of stripping the caplets, otherwise, the universal volatility of cap is less compatible with swaptions. To continue, the parameter for the SABRs are obtained for different option tenors. In order to get SABR for each pair of option tenor and swap tenor, (39) can be used to matching the swaption ATM market quotes. Consequently, we have SABR models for each [tenor, expiry] pair (see *Figure 9*).

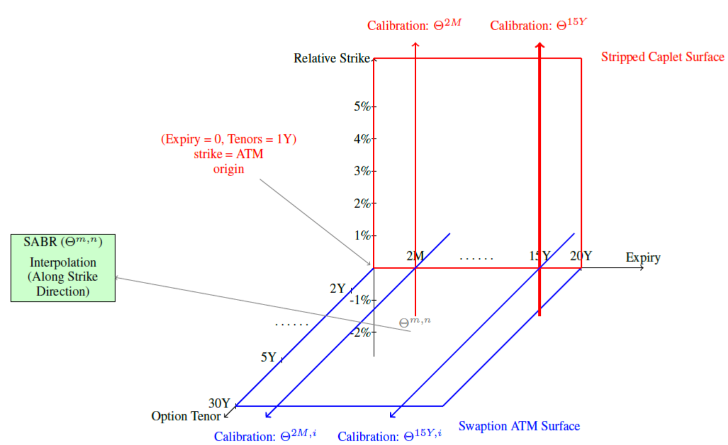


Figure 9: Vol Cube Construction based SWPT + CAP

Remark 3.2. *Such construction is controversial since the mixture information of cap/floor and swaption. Althuogh they are similar in structure, essentially the underlying process are different, one forward rate process, one forward swap rate process. However, since caplet only provide kind of initial guess for the parame-*

ters, as long as Swaption ATM volatility quotes are matched, the method is more or less OK.