Risk-neutral Pricing and Feynman-Kac PDE

Jianing Yao
Department of MSIS-RUTCOR
Rutgers University, the State University of New Jersey
Piscataway, NJ 08854 USA

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1 Markov Process

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a general filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{X(t)\}_{t\geq 0}$ be a stochastic process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

Definition 1.1 $\{X(t)\}_{t\geq 0}$ is a *Markov process* (with respect to $\{\mathcal{F}_t\}_{t\geq 0}$), if, for any $0 \leq s < t$ and any bounded function f:

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \mathbb{E}[f(X(t))|X(s)] \tag{1}$$

To check (1), it suffices to show that there is a function g such that

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = g(X(s))$$

The validity of (1) for all $0 \leq s < t$ and bounded f is called the *Markov property.* Why we are bothered to realize a process has Markov property or not? Because, as you will see shortly, the Markov property simplifies the problem of computing a conditional expectation of the form $\mathbb{E}[f(X(t))|\mathcal{F}_s]$.

Let's introduce a very important result for financial mathematics:

Lemma 1.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1, ..., X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1, ..., Y_L$ are independent of $\mathcal{G}$. Let $f(x_1, ..., x_K, y_1, ..., y_L)$ be a function of the dummy variables $x_1, ..., x_K$ and $y_1, ..., y_L$, and define

$$g(x_1, ..., x_K) = \mathbb{E}[f(x_1, ..., x_K, Y_1, ..., Y_L)] \tag{2}$$

Then,

$$\mathbb{E}[f(X_1, ..., X_k, Y_1, ..., Y_L)|\mathcal{G}] = g(X_1, ..., X_K) \tag{3}$$

Instead of giving the rigorous proof, let's just give an intuition. The idea here is that since the information in $\mathcal{G}$ is sufficient to determine the values of $X_1, ..., X_K$, we should hold these random variables constant when estimating $f(X_1, ..., X_k, Y_1, ..., Y_K)$. The other random variables, $Y_1, ..., Y_L$, are independent of $\mathcal{G}$, and so we should integrate them out without regard to the information in $\mathcal{G}$. These two steps, holding $X_1, ..., X_K$ constant and integrating out $Y_1, ..., Y_L$, are accomplished by (2). We get an estimate that depends on the values of $X_1, ..., X_K$ and, to capture this fact, we replace the dummy (non-random) variables $x_1, ..., x_k$ by the random variables $X_1, ..., X_K$ at the last step.

Example 1.2 Given $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration of Brownian motion W, let $X(t)$ solves

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ S(0) = x \tag{4}$$

That is, for any $s \leq t$,

$$S(t) = S(s)\exp\{\sigma(W(t) - W(s)) + (\mu - \frac{1}{2}\sigma^2)(t - s)\} \tag{5}$$

Thus,

$$\mathbb{E}[f(S(t))|\mathcal{F}_s] = \mathbb{E}[f(S(s)e^{\sigma(W(t)-W(s))+(\mu-\frac{1}{2}\sigma^2)(t-s)})|\mathcal{F}_s] = g(S(s))$$

where

$$\begin{aligned} g(x) &= \mathbb{E}[f(xe^{\sigma(W(t)-W(s))+(\mu-\frac{1}{2}\sigma^2)(t-s)})] \\ &= \int_{-\infty}^{\infty} f(xe^{\sigma(t-s)y+(\mu-\frac{1}{2}\sigma^2)(t-s)})e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \end{aligned}$$

Thus, we have shown that $\{S(t)\}_{t\geq 0}$ is a Markov process.

Remark 1.3 The definition of *Markov process* can be easily extended to the multi-dimensional process. That is, $X(t) = (X_1(t), ..., X_m(t))$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be *Markov* if for any $0 \leq s < t$ and any bounded $f(x_1, ..., x_m)$,

$$\mathbb{E}[f(X_1(t), ..., X_m(t))|\mathcal{F}_s] = \mathbb{E}[f(X_1(t), ..., X_m(t))|X_1(s), ..., X_m(s)]$$

which is true if and only if there is a function $g(x_1, ..., x_m)$ such that

$$\mathbb{E}[f(X_1(t), ..., X_m(t))|\mathcal{F}_s] = g(X_1(s), ..., X_m(s))$$

We very much want to have, when possible, price models are Markov processes. Stochastic differential equations provide such models. Suppose $X(t) = (X_1(t), ..., X_m(t))$ satisfies a stochastic differential equation if there are functions $\mu_i(r, x_1, ..., x_m)$ and $\sigma_{ij}(r, x_1, ..., x_m)$ and Brownian motion $W_1(r), ..., W_d(r)$ such that

$$dX_i(r) = \mu_i(r, X_1(r), ..., X_m(r))dr + \sum_{j=1}^{d} \sigma_{ij}(r, X_1(r), ..., X_m(r))dW_j(r),\ 1 \leq i \leq m \tag{6}$$

when the coefficient function μ_i and σ_{ij} satisfy certain conditions (Lipschitz continuous and linear growth), a solution to (6) will be Markov. The fact we showed above, *example 1.2*, is a special case of the general principle.

2 Stochastic Representation of PDE (Feynman-Kac) – Application to Pricing

Let $X(t) = (X_1(t), ..., X_m(t))$, $0 \leq t \leq T$, be a Markov processes that satisfies the system of equations (6). Let $H(x_1, ..., x_m, T)$ be a function and consider

$$V(t) = \mathbb{E}[H(X_1(T), ..., X_m(T), T)|\mathcal{F}_t]$$

Because $\{X(t)\}_{t\geq 0}$ is a Markov process,

$$V(t) = F(t, X_1(t), ..., X_m(t)) \tag{7}$$

for some function $F(t, x_1, ..., x_m)$, $t < T$. But $V(t)$, $t \leq T$, is also a martingale (any conditional expectation is a martingale !). If we assume that $F(t, x_1, ..., x_m)$ has enough continuous derivatives that It$\bar{o}$'s formula can be applied, we can compute the It$\bar{o}$'s differential $dV(t, X_1(t), ..., X_m(t))$. For the convenience of calculation, let's suppose we only have one dimensional It$\bar{o}$ process, i.e.,

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

And we want to compute

$$V(t) = \mathbb{E}[H(X(T), T)|\mathcal{F}_t] = F(t, X(t))$$

Since F is sufficiently smooth (by assumption), let's apply It$\bar{o}$'s formula:

$$\begin{aligned}
dF(t, X(t)) &= \frac{\partial F(t, X(t))}{\partial t}dt + \frac{\partial F(t, X(t))}{\partial x}\big[\mu(t, X(t))dt + \sigma(t, X(t))dW(t)\big] \\
&\qquad + \frac{1}{2}\frac{\partial^2 F(t, X(t))}{\partial X^2}\sigma^2(t, X(t))dt \\
&= \big[\frac{\partial F(t, X(t))}{\partial t} + \frac{\partial F(t, X(t))}{\partial X}\mu(t, X(t)) + \frac{1}{2}\frac{\partial^2 F(t, X(t))}{\partial X^2}\sigma^2(t, X(t))\big]dt \\
&\qquad + \frac{\partial F(t, X(t))}{\partial X}\sigma(t, X(t))dW(t)
\end{aligned}$$

Since $V(t)$ is a martingale, the $'dt'$ term of $dF(t, X(t))$ must be equal to 0. Thus, we get the following partial differential equation for F,

$$\begin{cases} F_t(t, x) + [F_x\mu](t, x) + \dfrac{1}{2}[F_{xx}\sigma^2](t, x) = 0, \text{ for all } 0 \leq t \leq T \text{ and } x \in \mathbb{R}, \\ F(T, x) = H(x) \text{ for all } x \in \mathbb{R} \end{cases} \tag{8}$$

This is so-called *Feynman-Kac equation.* This is very useful because it turns the probabilistic problem of computing the conditional expectation $V(t)$ for $t < T$ into a problem of solving a PDE. Even if this PDE cannot be solved explicitly, one may be able to apply one of the many algorithms available to obtain an approximate solution numerically.

Remark 2.1 Advanced stochastic analysis reveals a very deep result: every conditional expectation (even conditioning on filtration) can be represented as a solution of PDE (possibly path-dependent PDE).

3 Risk-neutral Pricing and Black-Scholes Formula

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Brownian motion W and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for W. r is constant risk-free rate, that is to say, each dollar invested in risk-free bond B at time 0 will produce $B(t) = e^{rt}$ at time t, in differential form: it is $dB(t) = rB(t)dt$. In addition, $S(t)$ is price processes of risky assets satisfying:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{9}$$

where, μ is the mean rate of growth and σ represents the volatility.

Let's consider the portfolio process $[\Delta_0(t)\ \Delta_1(t)]$, which is a vector-valued adapted process, for $0 \leq t \leq T$. Here $\Delta_0(t)$ stands for the amount of money invested in risk-free bond and $\Delta_1(t)$ represents the number of shares to be held in risky asset. Let $V(t)$ be the dollar amount of wealth resulting from using above portfolio process when the initial endowment is $V(0)$ and the self-financing condition is imposed. The self-financing condition means that, at each time t,

$$V(t) - \Delta_1(t)S(t)$$

which is the amount of wealth not invested in risky assets, is invested in the risk-free bond, and there are no additional income sources. It follows that

$$\begin{align} dV(t) &= r[V(t) - \Delta_1(t)S(t)]dt + \Delta_1(t)dS(t) \tag{10} \\ &= r\Delta_0 B(t)dt + \Delta_1(t)(\mu S(t) + \sigma S(t)dW(t)) \tag{11} \\ &= (\Delta_1 S(t)\mu + \Delta_0 e^{rt} r)dt + \Delta_1 S(t)\sigma dW(t) \tag{12} \end{align}$$

On the other hand, if we apply Itō's formula on discounted stock price,

$$\begin{aligned} d(e^{-rt}S(t)) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= e^{-rt}[-rS(t)dt + dS(t)] \end{aligned}$$

and

$$\begin{aligned} d[e^{-rt}V(t)] =& -re^{-rt}V(t)dt + e^{-rt}dV(t) \\ =&\Delta_1(t)[e^{-rt}\,(-rS(t)dt + dS(t))] \end{aligned}$$

Using the result above,

$$d[e^{-rt}V(t)] = \Delta_1(t)d[e^{-rt}S(t)] \tag{13}$$

As in the discrete-time, we want to price under risk neutral measure, let's firstly give the definition:

Definition 3.1 An equivalent *risk-neutral measure* $\mathbb{Q}$ for the model above is a probability measure on $(\Omega, \mathcal{F})$ satisfying:

- $\mathbb{Q} \sim \mathbb{P}$ ($\mathbb{P}$ is the original measure for the model);

- $\{e^{-rt}S(t)\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale under $\mathbb{Q}$, i.e.,

$$\mathbb{E}_{\mathbb{Q}}[e^{-rT}S(T)|\mathcal{F}_t] = e^{-rt}S(t), \text{ for } t \leq T. \tag{14}$$

For the simplicity of notation, let's use $\widetilde{\mathbb{E}}[Y] = \mathbb{E}_{\mathbb{Q}}[Y]$.

Theorem 3.1 (*First Fundamental Theorem of Asset Pricing*) The model is arbitrage-free if there is a risk-neutral measure $\mathbb{Q}$.

As before, we won't give a proof but just realize the essential insight here:

under $\mathbb{Q}$*,* $\{e^{-rt}V(t)\}_{t\geq 0}$ *is a martingale for any portfolio process* $[\Delta_0(t)\ \Delta_1(t)]$. (**)

The reason that this is true is due to (13). Since $e^{-rt}S(t)$ is a martingale, we have (very loosely expressed):

$$\widetilde{\mathbb{E}}[d[e^{-rt}S(t)]|\mathcal{F}_t] = \widetilde{\mathbb{E}}[e^{-r(t+dt)}S(t+dt) - e^{-rt}S(t)|\mathcal{F}_t] = 0$$

Thus, from (13),

$$\begin{aligned}\widetilde{\mathbb{E}}[d(e^{-rt}V(t))] &= \widetilde{\mathbb{E}}[\Delta_1(t)d\left(e^{-rt}S(t)\right)|\mathcal{F}_t] \\ &= \Delta_1(t)\widetilde{\mathbb{E}}[d\left(e^{-rt}S(t)\right)|\mathcal{F}_t] \\ &= 0\end{aligned}$$

Thus, at least at the formal, infinitesimal level,

$$\widetilde{\mathbb{E}}[e^{-r(t+dt)}V(t+dt)|\mathcal{F}_t] = e^{-rt}V(t)$$

Now, if (**) is true, there can be no arbitrage. For example, if $\mathbb{P}(V(T) \geq 0) = 1$, then $\mathbb{Q}(V(T) \geq 0) = 1$. Also, since $\mathbb{Q} \sim \mathbb{P}$ and

$$V(0) = e^{-r\times 0}V(0) = \widetilde{\mathbb{E}}[e^{-rT}V(T)] \geq 0$$

Since $\mathbb{Q}(D(T) > 0) = 1$. Or, if $\mathbb{P}(V(T) \geq 0) = 1$ and $\mathbb{P}(V(T) > 0) > 0$, then $\mathbb{Q}(V(T) \geq 0) = 1$ and $\mathbb{Q}(V(T) > 0) > 0$ and so

$$V(0) = \widetilde{\mathbb{E}}[e^{-rT}V(T)] > 0$$

In either case, arbitrage is not possible.

Now, let's assume such $\mathbb{Q}$ do exist, we can use it for pricing replicable claims. A contingent claim H, considered as a pay-off at time T is just some $\mathcal{F}_T$-measurable random variable. We say H can be *replicated* if:

there exists a portfolio process Δ *such that* $V(T) = H$ *with probability 1.*

- In this case, $[\Delta_0\ \Delta_1](t)$ is called a replicating portfolio. Since an investment of $V(t)$ at time t can lead to a terminal value of H using a self-financing trading strategy there will be arbitrage unless the price, call it $C(t)$, of the contingent claim at t equals $V(t)$; to repeat:

$$C(t) = V(t) \tag{15}$$

is the price of the contingent claim at t. But $\{e^{-rt}V(t)\}_{0\leq t\leq T}$ is a martingale under $\mathbb{Q}$ so

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}_t] = \widetilde{\mathbb{E}}[e^{-rT}H|\mathcal{F}_t]$$

Hence, if $\mathbb{Q}$ is a risk-neutral measure and if H is a replicable contingent claim, its no-arbitrage price process is:

$$C(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[H|\mathcal{F}_t] \tag{16}$$

This is the pricing formula in a risk-neutral model that we propose.

Let's next address the question of existence of risk-neutral measure $\mathbb{Q}$. Recall *Girsanov's theorem* (a simplified version, for the most general case, read notes from last time),

Theorem 3.2 Let $\theta(t)$ be an adapted process and define:

$$Z(t) = \exp\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\} \tag{17}$$

define:

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z(T)] \tag{18}$$

Then, under $\mathbb{Q}$, $\bar{W}(t)$, where

$$\bar{W}(t) := W(t) + \int_0^t \theta(u)du, \tag{19}$$

is a Brownian motion.

Define

$$\widetilde{W}_i(t) = W(t) + \int_0^t \theta(u)du \tag{20}$$

for some processes θ to be specified. Then

$$dW(t) = d\widetilde{W}(t) - \theta(t)dt \tag{21}$$

Plug into the stock price process (9)

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)[d\widetilde{W}(t) - \theta(t)dt] \\ &= [\mu - \sigma\theta(t)]S(t)dt + \sigma S(t)d\widetilde{W}(t) \end{aligned}$$

What we want is that, under $\mathbb{Q}$, discounted stock price process is a martingale. Suppose θ is defined by the following equation

$$\mu - \sigma\theta = r, \tag{22}$$

then,

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t). \tag{23}$$

It follows that

$$d[e^{-rt}S(t)] = e^{-rt}\sigma S(t)d\widetilde{W}(t) \tag{24}$$

Therefore, in this case, the risk-neutral measure is found by defining

$$Z(t) = \exp\{-(\frac{\mu - r}{\sigma})(W(t) - W(0)) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\} \tag{25}$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T) \tag{26}$$

Since we want to eventually compute the conditional expectation:

$$C(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[H(S(T))|\mathcal{F}_t] \tag{27}$$

where H is the final payoff function, in vanilla option, only depending on the final state, e.g., $H(S(T)) = (S(T) - K)^+$. If we rearrange the term and define

$$F(t, S(t)) := e^{-rt}C(t) = \widetilde{\mathbb{E}}[e^{-rT}H(S(T))|\mathcal{F}_t] \tag{28}$$

From the discussion in *section 2*, we have

$$\begin{cases} F_t(t,x) + rF_x(t,x) + \frac{1}{2}\sigma^2 F_{xx}(t,x) = 0, \text{ for all } 0 \le t \le T \text{ and } x \in \mathbb{R}, \\ F(T,x) = e^{-rT}H(x) \text{ for all } x \in \mathbb{R} \end{cases} \tag{29}$$

If we do substitution: $\widetilde{F}(t,x) = e^{rt}F(t,x)$ ($F = e^{-rt}\widetilde{F}$), observe

$$F_t = -re^{-rt}\widetilde{F} + e^{-rt}\widetilde{F}_t \tag{30}$$

$$F_x = e^{-rt}\widetilde{F}_x, \;\; F_{xx} = e^{-rt}\widetilde{F}_{xx} \tag{31}$$

Then,

$$\begin{cases} -r\widetilde{F}(t,x) + \widetilde{F}_t(t,x) + rx\widetilde{F}_x(t,x) + \frac{1}{2}\sigma^2 x^2 \widetilde{F}_{xx}(t,x) = 0, \text{ for all } 0 \le t \le T \text{ and } x \in \mathbb{R}, \\ \widetilde{F}(T,x) = H(x) \text{ for all } x \in \mathbb{R} \end{cases} \tag{32}$$

This is the famous *Black-Scholes equation.*

4 Verification and Solution of Risk-neutral Pricing under Black-Scholes Model

Let's carry out so called *verification theorem*, which says: if we have a function $u(t,x)$ being a solution of second order parabolic PDE (32), then $u(t,S(t))$ is the price of option of maturity T with payoff function $H(\cdot)$ under risk-neutrality.

Under risk-neutral measure $\mathbb{Q}$, the SDE is of the following format:

$$d\widetilde{S}(t) = r\widetilde{S}(t)dt + \sigma\widetilde{S}(t)d\widetilde{W}(t), \quad 0 \le t \le T$$

It can be solved easily (we have solved more general SDE with this structure) that

$$\widetilde{S}(t) = \widetilde{S}(0)\exp\left\{\sigma\widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\right\} \tag{33}$$

Let's apply Itō's lemma to $d(e^{-rt}u(t,\widetilde{S}(t)))$,

$$\begin{aligned} d[e^{-rt}u(t,\widetilde{S}(t))] =& \{-ru(t,\widetilde{S}(t)) + u_t(t,\widetilde{S}(t)) + rxu_x(t,\widetilde{S}(t)) + \frac{1}{2}\sigma^2x^2u_{xx}(t,\widetilde{S}(t))\} \\ &+ e^{-rt}u_x(t,\widetilde{S}(t))\sigma\widetilde{S}(t)d\widetilde{W}(t) \end{aligned}$$

The 'dt' term is zero by Feynman-Kac PDE, thus

$$e^{-rt}u(t,\widetilde{S}(t)) = u(0,\widetilde{S}(0)) + \int_0^t e^{-rs}u_x(s,\widetilde{S}(s))\sigma\widetilde{S}(s)d\widetilde{W}(s), \quad 0 \le t \le T$$

Assuming that

$$\mathbb{E}\big[\int_0^T e^{-rs}u_x(s,\widetilde{S}(s))\sigma^2\widetilde{S}^2(s)ds\big] < +\infty,$$

$e^{-rt}u(t,S(t))$ is a $\mathbb{Q}$-martingale. Thus, if $0 \le t \le T$,

$$e^{-rt}u(t,\widetilde{S}(t)) = \widetilde{\mathbb{E}}[e^{-rT}u(T,\widetilde{S}(T)) \,|\, \mathcal{F}_t] = \widetilde{\mathbb{E}}[e^{-rt}H(\widetilde{S}(T)) \,|\, \mathcal{F}_t] \tag{34}$$

or,

$$u(t,\widetilde{S}(t)) = e^{-r(T-t)}\widetilde{\mathbb{E}}[H(\widetilde{S}(T)) \,|\, \mathcal{F}_t] \tag{35}$$

which is exactly the option pricing formula under risk-neutral measure $\mathbb{Q}$. Since we know, for any $t \in [0,T]$,

$$\widetilde{S}(T) = \widetilde{S}(t)\exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + (r - \frac{1}{2}\sigma^2)(T-t)\right\} \tag{36}$$

and that the two terms in this product are independent. It follows from *Lemma* 1.1,

$$\widetilde{\mathbb{E}}[H(\widetilde{S}(T)) \,|\, \mathcal{F}_t] = \widetilde{\mathbb{E}}\Big[H(\widetilde{S}(t)\exp\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + (r - \frac{1}{2}\sigma^2)(T-t)\}) \,|\, \mathcal{F}_t\Big] = g(t,\widetilde{S}(t))$$

where,

$$
\begin{aligned}
g(t,x) &= \widetilde{\mathbb{E}}[H(x\exp\{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+(r-\frac{1}{2}\sigma^2)(T-t)\})] \\
&= \int_{-\infty}^{\infty} H(xe^{\sigma z}e^{(r-\frac{1}{2}\sigma^2)(T-t)})e^{\frac{-z^2}{2(T-t)}}\frac{dz}{\sqrt{2\pi(T-t)}}
\end{aligned}
$$

because

$$
\exp\{-\frac{z^2}{2(T-t)}\}/\sqrt{2\pi(T-t)}
$$

is the probability density of $\widetilde{W}(T)-\widetilde{W}(t)$.

As an example, let's price an European call (underlying asset is $S(t)$) with strike price K and maturity T. Then, the payoff function is $H(x)=(x-K)^+$. If we denote the option price at t by $c(t,x)$, then from previous section, c satisfies

$$
\begin{cases}
-rc(t,x)+c_t(t,x)+rxc_x(t,x)+\frac{1}{2}\sigma^2x^2c_{xx}(t,x)=0, \text{ for all } 0\le t\le T \text{ and } x\in\mathbb{R}, \\
c(T,x)=(x-K)^+ \text{ for all } x\in\mathbb{R}
\end{cases}
\tag{37}
$$

and

$$
\begin{aligned}
c(t,x) &= e^{-r(T-t)}\int_{-\infty}^{\infty}\left(xe^{\sigma z}e^{(r-\frac{1}{2}\sigma^2)(T-t)}-K\right)^+\frac{e^{-z^2/2(T-t)}dz}{\sqrt{2\pi(T-t)}} \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}\left(xe^{\sigma\sqrt{T-t}y}e^{(r-\frac{1}{2}\sigma^2)(T-t)}-K\right)^+\frac{e^{-y^2/2}dy}{\sqrt{2\pi}} \\
&= e^{-r(T-t)}\int_{\frac{\ln\frac{K}{x}-(r-\frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}}}\left(xe^{\sigma\sqrt{T-t}y}e^{(r-\frac{1}{2}\sigma^2)(T-t)}-K\right)\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}
\end{aligned}
$$

After some calculation it can be shown that

$$
c(t,x)=xN(d_+(T-t,x))-Ke^{-r(T-t)}N(d_-(T-t,x)))
$$

where

$$
\begin{aligned}
d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left[\frac{x}{K}\right]+(r+\frac{1}{2}\sigma^2\tau)\right] \\
d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{x}{K}+(r+\frac{1}{2}\sigma^2\tau)\right] \\
N(x) &= \int_{-\infty}^{x}e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}, \quad x\in\mathbb{R}
\end{aligned}
$$

5 Martingale Representation and Δ-hedge

Theorem 5.1 *(Martingale Representation)* Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if $\{M(t)\}_{0\le t\le T}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\ge 0}$, the filtration generated by Brownian motion W, then there exists a process $\{\Gamma(t) = (\Gamma_1(t), ..., \Gamma_d(t))\}_{0\le t\le T}$ adapted to $\{\mathcal{F}_t\}_{t\ge 0}$ such that

$$M(t) = M(0) + \int_0^t \Gamma(u)\cdot dW(u) \tag{38}$$

When the market price of risk equation has a unique solution satisfying the conditions of *Girsanov's theorem*, the martingale representation theorem implies that every contingent claim can be hedged if the filtration is generated by Brownian motion. Then the market is complete.

In single risky asset model concerned in previous sections, we have a unique solution $\theta = \frac{\mu - r}{\sigma}$, thus the *Girsanov's theorem* applies. For any H that is an $\mathcal{F}_T$-measurable payoff. We want $V(T) = H$, i.e.,

$$d[e^{-rt}V(t)] = \Delta_1(t)d\left(e^{-rt}S(t)\right) = e^{-rt}\Delta_1(t)\sigma S(t)d\widetilde{W}(t) \tag{39}$$

Integrating both sides,

$$e^{-rT}V(T) = V(0) + \int_0^T e^{-ru}\Delta_1(u)\sigma S(u)d\widetilde{W}(u) \tag{40}$$

Take expectation with respect to $\mathbb{Q}$, $V(0) = \widetilde{\mathbb{E}}[e^{-rT}V(T)] = \widetilde{\mathbb{E}}[e^{-rT}H]$. Martingale representation says, there is a $\gamma(u)$ so that

$$e^{-rT}V(T) = \widetilde{\mathbb{E}}[e^{-rT}H] + \int_0^T \gamma(u)d\widetilde{W}(u) \tag{41}$$

Hence,

$$\Delta_1(u) = \frac{\gamma(u)}{e^{-ru}\sigma S(u)} \tag{42}$$

Suppose the option price $C(t)$ is known to be a function $F(t, S(t))$ of the price. Then $e^{-rt}V(t) = e^{-rt}F(t, S(t))$. We know

$$d(e^{-rt}V(t)) = e^{-rt}\Delta_1(t)\sigma S(t)d\widetilde{W}(t) \tag{43}$$

Recall $dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$, then

$$dF(t, S(t)) = g(t, S(t))dt + F_X(t, S(t))\sigma S(t)d\widetilde{W}(t) \tag{44}$$

where $g(t.x) = xrF_x(t,x) + F_t(t,x) + \frac{1}{2}\sigma^2x^2F_{xx}(t,x)$. By Itō (using $de^{-rt} = -re^{-rt}dt$ and $de^{-rt}dF(t, S(t)) = 0$)

$$\begin{aligned} d[e^{-rt}F(t,S(t))] &= F(t,S(t))d[e^{-rt}] + e^{-rt}dF(t,S(t)) \\ &= e^{-rt}[g(t,S(t)) - rF(t,S(t))]dt + F_x(t,S(t))e^{-rt}\sigma S(t)d\widetilde{W}(t) \end{aligned}$$

Equate $d[e^{-rt}V(t)] = d(e^{-rt}F(t, S(t)))$, the $'dt'$ coefficient is zero by *Feynman-Kac* and

$$\Delta_1(t) = F_x(t, S(t)) \tag{45}$$

6 Reference

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