Stochastic Integration – I

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In the last lecture, we have shown that Brownian motion has no finite bounded variation, which causes difficulty to define *Riemann-Stieltjes* integral. But it does have finite quadratic variation (in L^2-norm), i.e.,

$$\lim_{||\Pi||\to 0} \mathbb{E}\big[([W,W]_T^{\Pi} - T)^2\big] = 0 \tag{1}$$

where $\Pi := \Pi(t_0, t_1, ..., t_n)$ of $[0,T]$ defines a partition, and $||\Pi|| = \max_{0\le i\le n-1}(t_{i+1} - t_i)$. Based on this, we calculated a special stochastic integral $I_T = \int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T$. Now, we want to formally define the stochastic integral, where the integral can be more general process then just Brownian motion.

1 Itō's integral with respect to Brownian motion

Let $\{W(t)\}_{t\ge 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\alpha(t)\}_{t\ge 0}$ be another stochastic process defined on the same probability space. Our goal is to define an integral of the form:

$$\int_0^t \alpha(s)dW(s) \tag{2}$$

1.1 The Itō's Integral for Simple Process

To define such integral, we may recall how we define *Lebesgue Integral* in the first class: we start by working on simple functions and then pass to the limit to get the desired result. This is a very useful method in measure theory – *"simple function technique"*. Let's mimic what we did before and hope it can work out.

Definition 1.1 A stochastic process is *simple* if it can be written in the form:

$$\alpha(t)(\omega) := \sum_{i=0}^{N-1} y_i(\omega)\mathbf{1}_{[t_i,t_{i+1})}(t) + y_N(\omega)\mathbf{1}_{[t_M,\infty)}(t) \tag{3}$$

for some sequence $0 = t_0 < t_1 < t_2 \cdots < t_N$ and random variable $y_0, y_1, ..., y_N$.

Thus, $\alpha(t)$ takes the random value y_0 on the interval $t_0 \leq t < t_1$, the value y_1 on $t_1 \leq t < t_2$, e.t.c.. Hence, for every ω, the path $\alpha(t)(\omega)$ is a piecewise constant function of t with change points at $t_1, t_2, ..., t_N$. In this case, we define the Itō's integral as we did in the *Stieltjes integral*, but now of course, everything is a random variable. Note that we can write the simple process above as:

$$\alpha(t)(\omega) = \sum_{i=1}^{N-1} \alpha(t_i)(\omega)\mathbf{1}_{[t_i, t_{i+1})}(t) + \alpha(t_N)\mathbf{1}_{[t_N, \infty)}(t) \tag{4}$$

Definition 1.2 If α is a simple process as in (4), the Itō's integral of α w.r.t Brownian motion W is defined as:

$$\int_0^t \alpha(s)dW(s) := \sum_{i=0}^{N} \alpha(t_i)(\omega)\big[W(t_{i+1} \wedge t)(\omega) - W(t_i \wedge t)(\omega)\big] \tag{5}$$

We have written all terms explicitly as function of ω, to emphasize that the integral is a random variable. But, in general, showing the dependence on ω is clumsy and we write, simply:

$$\int_0^t \alpha(s)dW(s) = \sum_{i=1}^{N} \alpha(t_i)\big[W(t_{i+1} \wedge t) - W(t_i \wedge t)\big] \tag{6}$$

Note also that $\int_0^t \alpha(s)dW(s)$ is being defined here for all $t \geq 0$, not just for a single fixed t.

As we mentioned, the stochastic integral can be viewed as betting on martingale. Let's illustrate this idea in more details when the martingale is Brownian motion. Suppose we are allowed to bet on the increments of a Brownian motion at fixed times $0 = t_0 < t_1 < \cdots < t_N$. The bet at time t_i is on the future increments $W(s) - W(t_i)$, $t_i < s \leq t_{i+1}$, the amount of the best is $\alpha(t_i)$ and the amount we win or lose by times s, $t_i < s \leq t_{i+1}$ on this bet is $\alpha(t_i)[W(s) - W(t_i)]$. Then, if $t_k < t \leq t_{k+1}$,

$$\int_0^t \alpha(s)dW(s) = \sum_{i=0}^{k-1}[W(t_{i+1}) - W(t_i)] + \alpha(t_k)[W(t) - W(t_k)] \tag{7}$$

is the total of all that we have won or lost on best taken up to time t. In short, one may think of a stochastic integral as totaling the gains obtained by betting on Brownian motion increments.

1.2 Adapted Process

Let W be a Brownian motion and let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration for W; recall that this means (i) $W(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$; (ii) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $s < t$.

Think of $\mathcal{F}_t$ as the information available at time t to a gambler who wants to bet on the increments of W. If $\{\alpha(t)\}_{t \geq 0}$ is a simple process representing his betting scheme it follows that for each t_i, $\alpha(t_i)$, he amount bet at t_i, is $\mathcal{F}_{t_i}$-measurable. For any t,

$$\alpha(t) = \alpha(t_k) \tag{8}$$

where t_k is the largest of numbers $t_0 < t_1 < \cdots t_N$ such that $t_k \leq t$. These $\alpha(t)$ is $\mathcal{F}_{t_k}$–measurable, and since $t_k \leq t$ and $\mathcal{F}_{t_k} \subseteq \mathcal{F}_t$, it is also $\mathcal{F}_t$-measurable.

Definition 1.3 A stochastic process $\{\beta(t)\}_{t\geq 0}$ defined on the same probability space as W is said to be *adapted to* $\mathcal{F}_t$ if $\beta(t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$.

Thus, according to the previous discussion, if we are interested in stochastic integrals as models of betting on the increments of a Brownian motion, it is natural to consider only adapted integrand $\alpha(\cdot)$.

Restriction to adapted integrands is the key idea that allows the Itō's integral to be extended to more general integrands than simple processes. The irregularity of Brownian paths does not allow one to define stochastic integration by path-wise *Stieltjes integration*. But when adaptedness is imposed, it is possible to develop a useful theory.

1.3 A study of $\int_0^t \alpha(s)dW(s)$ for adapted, simple integrands $\alpha(\cdot)$

Theorem 1.1 Let $\{\alpha(t)\}_{0\leq t\leq T}$ be a simple process which is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for a Brownian motion W. Assume

$$\mathbb{E}\Big[\int_0^T \alpha^2(s)ds\Big] < +\infty \tag{9}$$

Then,

1. $\int_0^t \alpha(s)dW(s)$, $0 \leq t \leq T$, is a martingale w.r.t $\mathcal{F}_t$;
2. *(Itō Isometry)*

$$\mathrm{Var}\left(\int_0^T \alpha(s)dW(s)\right) = \mathbb{E}\Big[\left(\int_0^T \alpha(s)dW(s)\right)^2\Big] = \mathbb{E}\Big[\int_0^T \alpha^2(s)ds\Big] \tag{10}$$

3. Let $X(t) = \int_0^t \alpha(s)dW(s)$, the quadratic variation process of X is $[X,X](t) = \int_0^t \alpha^2(s)ds$;
4. For each ω, $\big[\int_0^t \alpha(s)dW(s)\big](\omega)$ is a continuous function of t.

Remark 1.2 Assumption (9) is important for property 1 of the theorem and also, obviously, for property 2. If

$$\alpha(t) = \sum_{i=0}^{N} \alpha(t_i)\mathbf{1}_{[t_i,t_{i+1})}(t) \tag{11}$$

where $0 = t_0 < \cdots < t_N$, then

$$\alpha^2(t) = \sum_{i=0}^{N} \alpha^2(t_i)\mathbf{1}_{[t_i,t_{i+1})}(t) \tag{12}$$

and so,

$$\begin{aligned}
\mathbb{E}[\int_0^T \alpha^2(t)dt] &= \mathbb{E}[\int_0^T \sum_{i=0}^{N} \alpha^2(t_i)\mathbf{1}_{[t_i,t_{i+1})}(t)] \\
&= \mathbb{E}\big[\sum_{i=0}^{N} \alpha^2(t_i)(T\wedge t_{i+1} - T\wedge t_i)\big] \qquad (13)\\
&= \sum_{i=0}^{N} \mathbb{E}[\alpha^2(t_i)](T\wedge t_{i+1} - T\wedge t_i)
\end{aligned}$$

Thus, when condition (9) is true, $\mathbb{E}[\alpha^2(t_i)] < +\infty$ for any $t_i < T$.

One other fact plays an important rule in the theorem and is a consequence of the assumption that α is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, this fact is: for each i, $\alpha(t_i)$ and $W(t_{i+1}\wedge t) - W(t_i \wedge t)$ are independent. Indeed, when $t \leq t_i$, $W(t_{i+1}\wedge t) - W(t_i\wedge t) = 0$ and so $\alpha(t_i)$ and $W(t_{i+1}\wedge t) - W(t_i\wedge t)$ are independent. When $t > t_i$, $W(t_{i+1}\wedge t) - W(t_i\wedge t) = W(t_{i+1}) - W(t_i)$, and this is independent of $\mathcal{F}_{t_i}$ by the assumption $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration for W. Since $\alpha(t_i)$ is $\mathcal{F}_{t_i}$-measurable, by the assumption that the process $\{\alpha(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, so $W(t_{i+1}\wedge t) - W(t_i\wedge t)$ must be independent of $\alpha(t_i)$.

An immediate consequence of this observation is

$$\mathbb{E}[\alpha(t_i)(W(t_{i+1}\wedge t) - W(t_i\wedge t))] = \mathbb{E}[\alpha(t_i)]\mathbb{E}[W(t_{i+1}\wedge t) - W(t_i\wedge t)] = 0 \qquad (14)$$

Another important consequence is that if $i \neq j$

$$\alpha(t_i)[W(t_{i+1}\wedge t) - W(t_i\wedge t)] \text{ and } W(t_{j+1}\wedge t) - W(t_j\wedge t) \qquad (15)$$

are uncorrelated. Suppose, for example, that $i < j$, so that $i+1 \leq j$. Then since $\alpha(t_i)$ is $\mathcal{F}_{t_i}$ and hence $\mathcal{F}_{t_j}$-measurable, $W(t_{i+1}\wedge t) - W(t_i\wedge t)$ is $\mathcal{F}_{t_{i+1}}$-measurable and hence $\mathcal{F}_{t_j}$-measurable, and since $\alpha(t_j)$ is $\mathcal{F}_{t_j}$-measurable, $\alpha(t_i)[W(t_{i+1}\wedge t) - W(t_i\wedge t)]\alpha(t_j)$ is $\mathcal{F}_{t_j}$-measurable and hence independent of $W(t_{j+1}\wedge t) - W(t_j\wedge t)$, thus

$$\begin{aligned}
&\mathbb{E}[\alpha(t_i)[W(t_{i+1}\wedge t) - W(t_i\wedge t)]\cdot\alpha(t_j)[W(t_{j+1}\wedge t) - W(t_j\wedge t)]] \\
=&\mathbb{E}[\alpha(t_i)[W(t_{i+1}\wedge t) - W(t_i\wedge t)]\alpha(t_j)]\mathbb{E}[W(t_{j+1}\wedge t) - W(t_j\wedge t)] \qquad (16)\\
=&0
\end{aligned}$$

Proof. Let $\alpha(t) = \sum_{i=0}^{N} \alpha(t_i)\mathbf{1}_{[t_i,t_{i+1})}(t)$ define a *simple process* that is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $X(t) = \int_0^t \alpha(s)dW(s)$.

For the second claim:

$$
\begin{aligned}
&\mathbb{E}\big[\Big(\int_0^T \alpha(s)dW(s)\Big)^2\big]\\
=&\mathbb{E}\Big[\Big(\sum_{i=0}^N \alpha(t_i)(W(t_{i+1}\wedge T)-W(t_i\wedge T))\Big)^2\Big]\\
=&\mathbb{E}\big[\sum_{i=1}^N \alpha^2(t_i)(W(t_{i+1}\wedge T)-W(t_i\wedge T))^2\big]\\
+&\mathbb{E}\big[\sum_{i,j=0}^N \alpha(t_i)[W(t_{i+1}\wedge t)-W(t_i\wedge t)]\cdot\alpha(t_j)[W(t_{j+1}\wedge t)-W(t_j\wedge t)]\big]\\
&=\mathbb{E}\big[\sum_{i=1}^N \alpha^2(t_i)(W(t_{i+1}\wedge T)-W(t_i\wedge T))^2\big]\\
&=\mathbb{E}\big[\int_0^T \alpha^2(s)ds\big]
\end{aligned}
$$

Notice here we used (13) and (16).

For the first claim $X(t)=\int_0^t \alpha(s)dW(s)=\sum_{i=0}^N \alpha(t_i)[W(t_{i+1}\wedge t)-W(t_i\wedge t)]$. Since a sum of martingales is a martingale, to show $\{X(t)\}_{0\le t\le T}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\ge 0}$, it suffice to show that $y_i(t):=\alpha(t_i)(W(t_{i+1}\wedge t)-W(t_i\wedge t))$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\ge 0}$, for each i. For this, we need to check that $\mathbb{E}|y_i(t)|<\infty$ for all t, and $\mathbb{E}[y(t)|\mathcal{F}_s]=y(s)$. When $0\le s\le t$,

$$
\begin{aligned}
\mathbb{E}[y_i^2(t)]&=\mathbb{E}[\alpha^2(t_i)(W(t_{i+1}\wedge t)-W(t_i\wedge t))^2]\\
&=\mathbb{E}[\alpha^2(t_i)](t_{i+1}\wedge t-t_i\wedge t)
\end{aligned}
$$

and this is finite because of (13) above and the assumption that $\mathbb{E}[\int_0^T \alpha^2(s)ds]<+\infty$. Since $\mathbb{E}[y_i^2(t)]<+\infty$, $\mathbb{E}|y_i(t)|<+\infty$.

Let $0\le s\le t$. If $t\le t_i$, $y(t)=0$ and $y(s)=0$ and, trivially, $\mathbb{E}[y(t)|\mathcal{F}_s]=y(s)$. If $0\le s<t_i\le t$,

$$
\begin{aligned}
\mathbb{E}[y_i(t)|\mathcal{F}_s]&=\mathbb{E}\big[\mathbb{E}[\alpha(t_i)[W(t_{i+1}\wedge t)-W(t_i\wedge t)]|\mathcal{F}_{t_i}]|\mathcal{F}_s\big]\\
&=\mathbb{E}[\alpha(t_i)\mathbb{E}[W(t_{i+1}\wedge t)-W(t_i\wedge t)|\mathcal{F}_{t_i}]|\mathcal{F}_s]\\
&=0=y(s)
\end{aligned}
$$

If $t_i\le s<t$, then $\alpha(t_i)$ is $\mathcal{F}_s$-measurable and $t_i\wedge t=t_i$, so

$$
\begin{aligned}
\mathbb{E}[y_i(t)|\mathcal{F}_s]&=\mathbb{E}[\alpha(t_i)[W(t_{i+1}\wedge t)-W(t_i)]|\mathcal{F}_s]\\
&=\alpha(t_i)\big[W(t_{i+1}\wedge s)-W(t_i)\big]=y_i(s)
\end{aligned}
$$

because $W(t_i)$ is $\mathcal{F}_s$-measurable and because W is a martingale with respect to $\{\mathcal{F}_t\}_{t\ge 0}$. Thus, we have verified $\mathbb{E}[y_i(t)|\mathcal{F}_s]=y_i(s)$ in all cases.

For the proof of the third claim, we will not be fully rigorous. Notice what we want to prove is equivalent to:

$$[X, X]^{\Pi}(t) = \sum_{i=0}^{n-1} [X(s_{i+1}) - X(s_i)]^2 \to \int_0^t \alpha^2(s)ds \text{ (in } L_2\text{-norm) as } ||\Pi|| \to 0 \tag{17}$$

where Π denotes the partition $0 = s_0 < s_1 < \cdots < s_n = t$ of $[0, t]$. It suffices to do the case $t = T$, as the argument is the same for any t.

By adding points to Π if necessary assume it contains all the points t_j at which α changes value. For every i, there is some j so that $t_j \le s_i < s_{i+1} \le t_{j+1}$ and hence

$$X(s_{i+1}) - X(s_i) = \alpha(t_j)[W(s_{i+1}) - W(s_i)]$$

Therefore,

$$\begin{aligned} [X, X]^{\Pi}(T) &= \sum_{j=0}^{N} \sum_{i; t_j \le s_i < t_{j+1}} \alpha^2(t_j)[W(s_{i+1}) - W(s_i)]^2 \\ &= \sum_{j=0}^{N} \alpha^2(t_j) \sum_{i; t_j \le s_i < t_{j+1}} [W(s_{i+1}) - W(s_i)]^2 \end{aligned}$$

As $||\Pi|| \to 0$,

$$\sum_{i; t_j \le s_i < t_{j+1}} [W(s_{i+1}) - W(s_i)]^2 \to t_{j+1} - t_j$$

Thus,

$$[X, X]^{\Pi}(T) \to \sum_{j=0}^{N} \alpha^2(t_j)(t_{j+1} - t_j) = \int_0^T \alpha^2(s)ds$$

as $||\Pi|| \to 0$ (the convergence is again in L^2-norm).

The last claim is trivial since $W(t_i \wedge t)$ is continuous in t, the same is true for:

$$\int_0^t \alpha(s)dW(s) = \sum_{i=0}^{N} \alpha(t_i)[W(t_{i+1} \wedge t) - W(t_i \wedge t)]$$

$\square$

2 Extending Itō's Integral

In *Theorem 1.1*, we shall pay special attention the second claim – *Itō isometry*. If X is a random variable, define

$$||X||_{L^2} = \sqrt{\mathbb{E}[|X|^2]}$$

$||X||_{L^2}$ is an example of what is called a norm and maybe thought of as a measure of the size of X. In fact, it is related mathematically to the Euclidean norm $||x|| = \sqrt{\sum_{i=1}^{d} x_i^2}$ of a vector $x = (x_1, ..., x_d)$. The reason for the square root in the definition is that if α is a scalar, it is natural to require the size of αX to be $|\alpha|$ times the size of X. Indeed,

$$||\alpha X||_{L^2} = \sqrt{\mathbb{E}[(\alpha X)^2]} = \sqrt{\alpha^2 \mathbb{E}[X^2]} = |\alpha| ||X||_{L^2}$$

If $\{\alpha(t)\}_{0 \leq t \leq T}$ is a stochastic process on $[0, T]$, define

$$||\alpha(\cdot)||_{\mathscr{L}^2(T)} = \sqrt{\mathbb{E}\big[\int_0^T \alpha^2(s)ds\big]}$$

This is a 'legitimate' norm on $\alpha(\cdot)$, measuring its size. Having theses terminology above, the second claim of *Theorem 1.1* says:

$$|| \int_0^T \alpha(s)dW(s)||_{L^2} = ||\alpha(\cdot)||_{\mathscr{L}^2(T)} \tag{18}$$

A map from one set to another that preserves a notion of size is called *isometry*. In this sens, the stochastic integral, thought of as a map that takes a process $\alpha(\cdot)$ to a random variable $\int_0^t \alpha(s)dW(s)$, is an isometry when sizes are measure, respectively, by $|| \cdot ||_{\mathscr{L}^2(T)}$ and $|| \cdot ||_{L^2}$.

This talk of isometries is not just idle chatter we make up to impress ourselves. It is the key to extending the Itō integral from adapted simple integrands to more general integrands. The set of r.v.s X on a probability space such that $||X||_{L^2} < +\infty$ has a property called completeness (*Hilbert space* is a *Banach space*), which says that if $\{X_n\}$ is a *Cauchy sequence*, i.e.,

$$\lim_{n,m \to \infty} ||X_n - X_m|| = 0 \tag{19}$$

then there exists some r.v. X, also satisfying $||X||_{L^2} < \infty$ such that $\lim_{n \to \infty} ||X_n - X||_{L^2} = 0$. We won't go into this, but we only want to point out an important consequence. Let $\{\alpha(t)\}_{0 \leq t \leq T}$ be an adapted process satisfying:

$$||\alpha(\cdot)||_{\mathscr{L}^2(T)} = \sqrt{\mathbb{E}\big[\int_0^T \alpha^2(s)ds\big]} < +\infty \tag{20}$$

Notice, it is not assume now that α is a simple process. However, suppose there is a sequence $\{\alpha_n(\cdot)\}$ of simple adapted process such that

$$\lim_{n \to \infty} ||\alpha_n(\cdot) - \alpha(\cdot)||_{\mathscr{L}^2(T)} = 0 \tag{21}$$

Then because of Itō isometry, there is a process $X(t)$ such that

$$|| \int_0^t \alpha_n(s)dW(s) - X(t)||_{L^2} \to 0 \tag{22}$$

for all $t \leq T$. Now, we can give the general theorem:

Theorem 2.1 If $\{\alpha(t)\}_{0\leq t\leq T}$ is adapted to $\mathcal{F}_t$ and

$$\mathbb{E}[\int_0^T \alpha^2(s)ds] < +\infty$$

there is a sequence of simple, adapted process $\{\alpha_n(\cdot)\}$ such that $\lim_{n\to\infty} ||\alpha_n(\cdot)-\alpha(\cdot)||_{\mathscr{L}^2(T)} = 0$. Let $X(t)$, $0 \leq t \leq T$ be define as in (22). Then, we denote $X(t)$ by

$$X(t) = \int_0^t \alpha(s)dW(s)$$

Moreover, $X(t)$ can be defined so that it is continuous in t and

1. $\{X(t)\}_{0\leq t\leq T}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{0\leq t\leq T}$;
2. *Itō isometry*

$$\mathbb{E}\Big[\Big(\int_0^t \alpha(s)dW(s)\Big)^2\Big] = \mathbb{E}[\int_0^t \alpha^2(s)ds],\ 0 \leq t \leq T$$

3. The quadratic variation of X is

$$[X,X](t) = \int_0^t \alpha^2(s)ds,\ 0 \leq t \leq T.$$

Remark 2.2 There are two essential points to retain from this discussion: (i) $\int_0^t \alpha(s)dW(s)$ can be defined as limit for processes $\alpha(\cdot)$ which are not simple, as long as $\alpha(\cdot)$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies (21); (ii) the integral $\int_0^t \alpha(s)dW(s)$ has the properties which we already saw were true when α is simple adapted process.

3 Worked out Example Again

Our goal is to compute

$$I_T = \int_0^T W(t)dW(t) \tag{23}$$

which we have computed last time informally. Let's now give a rigorous derivation and make some comments.

Let W be a Brownian motion and let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration for W. W is certainly adapted and for any $T > 0$,

$$\mathbb{E}[\int_0^T W^2(s)ds] = \int_0^T \mathbb{E}[W^2(s)]ds = \int_0^T sds = \frac{1}{2}T^2 < \infty \tag{24}$$

Note the interchange of integral and expectation is valid because of *Fubini's theorem*. Therefore, $I_T = \int_0^T W(s)dW(s)$ is well-defined.

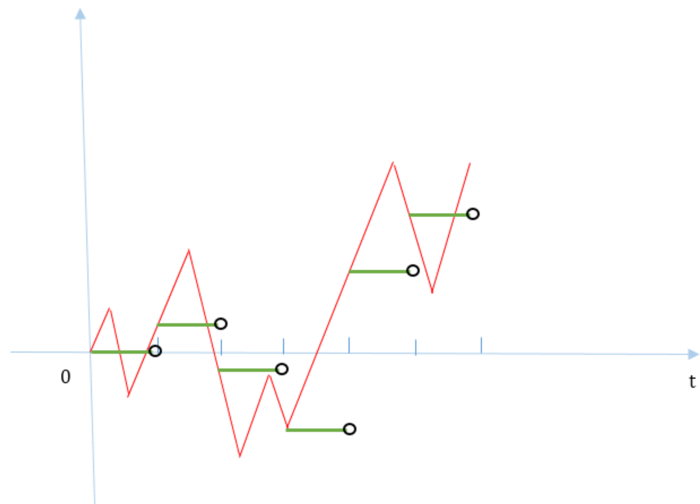

Figure 1: $\alpha^n(\cdot)$

For any n, let

$$\alpha^n(t) = \sum_{i=0}^{n-1} W(\frac{iT}{n})\mathbf{1}_{[\frac{iT}{n},\frac{(i+1)T}{n})}(t) \tag{25}$$

The figure shows how $\alpha^n(\cdot)$ is related to W (see *Figure* 1) Over each sub-interval, the value of α is the value of W at the left endpoint of the sub-interval.

It is very important that $\alpha^n(t) = W(\frac{iT}{n})$, when $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$, because then $\alpha^n(\cdot)$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. Indeed, if $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$, then $\alpha^n(t)$ is $\mathcal{F}_{\frac{iT}{n}}$-measurable, and hence, since $\mathcal{F}_{\frac{iT}{n}} \subseteq \mathcal{F}_t$, it is $\mathcal{F}_t$-measurable. If we choose to let $\alpha^n(t)$ be the value at another point in $[\frac{iT}{n}, \frac{(i+1)T}{n})$ when $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$, $\alpha^n(\cdot)$ would no longer be adapted to $\mathcal{F}_t$. For example, suppose we set $\tilde{\alpha}^n(s) = W(\frac{iT}{n} + \frac{iT}{2n})$, the value of W at the middle point of $[\frac{iT}{n}, \frac{(i+1)T}{n})$, when $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$. Then $\tilde{\alpha}^n(\frac{iT}{n}) = W(\frac{iT}{n} + \frac{iT}{2n})$ is $\mathcal{F}_{\frac{iT}{n}+\frac{iT}{2n}}$-measurable, but not $\mathcal{F}_{\frac{iT}{n}}$-measurable. Hence, it will not be adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

Because the paths of W is continuous,

$$\lim_{n\to\infty} \alpha^n(t)(\omega) = W(t)(\omega) \text{ for every } \omega \in \Omega \text{ and } t < T \tag{26}$$

It is not too hard to show in fact that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T [\alpha^n(s) - W(s)]^2 ds\Big] = 0 \tag{27}$$

for any $T > 0$. Or, equivalently,

$$\lim_{n\to\infty} ||\alpha_n(\cdot) - W(\cdot)||_{\mathscr{L}^2(T)} = 0 \tag{28}$$

We won't prove it here. But it says that $\int_0^T W(s)dW(s)$ is the limit (in the sense of L^2-norm) of

$$\int_0^T \alpha^n(s)dW(s) = \sum_{i=0}^{n-1} W(\frac{iT}{n})[W(\frac{(i+1)T}{n}) - W(\frac{iT}{n})] \tag{29}$$

as $n \to \infty$, by Itō Isometry. The limit (29) is computed carefully last time, we won't repeat it here. The result is:

$$\int_0^T W(s)dW(s) = \frac{1}{2}W^2(T) - \frac{1}{2}T \tag{30}$$

Remark 3.1

Theorem 2.1 says that I_T should be a martingale in the parameter T. This is true because we know that $W^2(T) - T$ is a martingale;

let G be a continuously differentiable function and recall the definition of *Stieltjes integral*. Since

$$\frac{d}{ds}G^2(s) = 2G(s)G'(s) \tag{31}$$

then

$$\int_0^T G(s)dG(s) = \int_0^T G(s)G'(s)ds = \int_0^T \frac{1}{2}\frac{d}{ds}G^2(s)ds = \frac{1}{2}[G^2(T) - G^2(0)] \tag{32}$$

when $G(0) = 0$, we get

$$\int_0^T G(s)dG(s) = \frac{1}{2}G^2(T) \tag{33}$$

Notice how this differs from (30). Itō integral does not behave like *Stieltjes integral*. The term $-\frac{1}{2}T$ appears because of the quadratic variation of Brownian motion and to make the result as a martingale.

4 Stratonovich Integral

As we mentioned in the recitation, the adaptedness of the integrand process is very essential because gambler can only adopt the information in the past for betting. Let's in this section explore what is going to happen if we violate this assumption. We have the following setting $\{W(t)\}_{t\geq 0}$ is a Brownian motion, let T be a fixed positive number let $\Pi = \{t_0, t_1, t_2, ..., t_n\}$ be a partition of $[0, T]$. For each j, let's define

$$t_j^* = \frac{t_{j+1} + t_j}{2} \tag{34}$$

be the middle point of interval $[t_j, t_{j+1}]$. We define the *half-sample quadratic variation* corresponding to Π to be:

$$Q_{\Pi/2} := \sum_{j=1}^{n-1} (W(t_j^*) - W(t_j))^2 \tag{35}$$

we want to show that $Q_{\Pi/2}$ has limit $\frac{T}{2}$ as $||\Pi|| \to 0$ (in L^2 sense). This is not so hard as we already did the quadratic variation calculation, let's mimic what we did there:

$$\begin{aligned}
&\mathbb{E}\Big[\left(\sum_j (W(t_j^*) - W(t_j))^2 - \frac{t}{2}\right)^2\Big] \\
=&\mathbb{E}\Big[\left(\sum_j (W(t_j^*) - W(t_j))^2 - \sum_j \frac{t_{j+1}-t_j}{2}\right)^2\Big] \\
=&\sum_{j,k} \mathbb{E}\Big[\left(\sum_j (W(t_j^*) - W(t_j))^2 - \sum_j \frac{t_{j+1}-t_j}{2}\right)\left(\sum_j (W(t_k^*) - W(t_k))^2 - \sum_k \frac{t_{k+1}-t_k}{2}\right)\Big] \\
=&\sum_j \mathbb{E}\big[W^2(\frac{t_{j+1}-t_j}{2} - \frac{t_{j+1}-t_j}{2})^2\big] \\
=&\sum_j 2\left(\frac{t_{j+1}-t_j}{2}\right)^2 \le \frac{T}{2} \max_{1\le j\le n} |t_{j+1}-t_j|
\end{aligned}$$

As $||\Pi|| \to 0$, above expression goes to 0. Thus,

$$Q_{\Pi/2} \to \frac{T}{2} \tag{36}$$

Remark 4.1 I skip several steps in between, because we did it in quadratic variation case. Basically, I use the independence if $j \neq k$ and moment generation function, i.e., if $X \sim N(\mu, \sigma^2)$, $\mathbb{E}[(X-\mu)^2] = 3\sigma^2$. Please fill the gap by yourself.

Now, let's define the *Stratonovich integral* of $W(t)$ w.r.t to $W(t)$

$$\int_0^T W(t) \circ dW(t) := \lim_{||\Pi||\to 0} \sum_{j=1}^{n-1} W(t_j^*)(W(t_{j+1}) - W(t_j)) \tag{37}$$

Notice the difference between It$\bar{o}$'s integral and this integral is just now we choose a non-adapted (not $\mathcal{F}_{t_j}$-measurable) betting scheme $W(t_j^*)$. And the $\circ$ is used just to distinguish these two integral, NOT composition, or other meanings. Let's carry out the calculation:

$$\begin{aligned}
&\sum_j W(t_j^*)(W(t_{j+1}) - W(t_j)) \\
=&\sum_j \left[W(t_j^*)(W(t_{j+1}) - W(t_j)) + W(t_j)(W(t_j^*) - W(t_j))\right] + \sum_j \left(W(t_j^*) - W(t_j)\right)^2
\end{aligned}$$

But what is the first term, this converges to $I_T = \int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$ (in L^2-sense). Why, because this is just the it$\bar{o}$'s integral, which we did a moment ago, right? (We make one more partition in each interval, but still adapted betting scheme). What is the second term? Well, it is the half quadratic variation that we also did above, it converges to $\frac{1}{2}T$ in L^2-norm. All in all,

$$\int_0^T W(t) \circ dW(t) := \lim_{||\Pi|| \to 0} \sum_{j=1}^{n-1} W(t_j^*)(W(t_{j+1}) - W(t_j)) = \frac{1}{2}W^2(T) \tag{38}$$

So what? Why we shouldn't do so! We still have a result! But notice, this is never a martingale! However, you really want to have It$\bar{0}$'s integral as a martingale to connect to the discrete phenomena that betting on martingale is a martingale. That's why we stick on It$\bar{o}$'s integral.

5 Reference

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