Discrete-time Martingales

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1 Introduction to Martingales

Martingales are models of gambling games that are fair to risk-neutral players. As we discussed in the consecutive coin toss case[1], *filtratoin*, the increasing sequence of sub-σ-algebras, represents the knowledge that one perceive upto certain point. Similarly, as far as gambling is concerned, we let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be a filtration, where $\mathcal{F}_k$ represents the information available to the player after k-th play. In the meanwhile, let $X_1, X_2, ...$ stands for the player's fortune, i.e., X_k is what player has after the k-th play. The conditional expectation $\mathbb{E}[X_{k+1}|\mathcal{F}_k]$ is what the player expects to have after the next play given his knowledge at the end of play k. He consider the game is fair if

$$\mathbb{E}[X_{k+1}|\mathcal{F}_k] = X_k$$

Why? because the expectation of the wealth at next play will be identical the the present value given all prior information. This is the *martingale property*. Let's make a formal definition:

Definition 1.1 (*Martingale*) A sequence of random variables $\{X_n\}_{n\geq 0}$ is a *discrete-time martingale* with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ if

1. X_n is $\mathcal{F}_n$-measurable for all n[2];
2. $\mathbb{E}[|X_n|] < \infty$ for all n;
3. $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$, for all n

Remark 1.1 (ii) is an integrability requirement to have expectation converge. And, (iii) implies X_n is $\mathcal{F}_n$-measurable, we make it (i) just to emphasis.

[1] I wrote a comment on page. 5 of the first recitation notes. The sequence representing stock price going up and down is equivalent as row of coin tosses with head and tail, those of you who attended the second recitation will know

[2] Intuitively speaking, the measurability can be interpreted as follows: X_n is $\mathcal{F}_n$-measurable meaning knowing the sub-σ-algebra, $\mathcal{F}_n$, or the partial information upto n, is identical to knowing the value of X_n.

The contrast of martingale has two possibilities, if the player has a wining strategy, then the expectation of the next play given all current and past knowledge maybe higher than the current value, this is called *sub-martingale*; and from the perspective of the casino, it is a losing strategy, a *sup-martingale*. Here comes the formal definition again:

Definition 1.2 (*sup-, sub-martingale*) A sequence of random variables $\{X_n\}_{n\geq 1}$ is a *discrete (sup-)sub-martingale* if

1. X_n is $\mathcal{F}_n$-measurable for all n;
2. $\mathbb{E}[|X_n|] < \infty$ for all n;
3. $\mathbb{E}[X_{n+1}|\mathcal{F}_n](\leq) \geq X_n$, for all n, resp..

2 Examples of Martingales

The simplest example of martingale maybe the conditional expectation:

Example 2.1 Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, X is a random variable that is integrable, i.e., $\mathbb{E}[|X|] < +\infty$. Let also $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$ be a filtration, and define

$$X_n = \mathbb{E}[X|\mathcal{F}_n],\ n = 0, 1, 2, ...$$

Notice, this is a well-defined $\mathcal{F}_n$-measurable random variable, thus (i) is satisfied, also if you recall the way we define conditional expectation on a filtration, (ii) is satisfied automatically. We are only left to check (iii): let's take expectation of X_{n+1} conditioning on $\mathcal{F}_n$, and using *tower property*:

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_{n+1}]|\mathcal{F}_n\big] = \mathbb{E}[X|\mathcal{F}_n] = X_n$$

Thus, X_{n+1} satisfies *martingale property*. Indeed, $\mathbb{E}[X|\mathcal{F}_n]$, $n = 0, 1, 2, ...$, are all m*martingales*.

Let's go to a casino: think of a game in which, on every toss you lose a dollar if it is a tail (with probability $\frac{1}{2}$) and win a dollar if it is a head, and you are monitoring your total fortune, assuming initial capital is 0. This is similar to the example we have in the first recitation just with the modification that we have to define an additional random variable to record the accumulation of fortune:

Example 2.2 Let ξ_1, ξ_2, ..., be independent and identically distributed with $\mathbb{P}(\xi_i = 1) = \frac{1}{2}$, $\mathbb{P}(\xi_i = -1) = \frac{1}{2}$. Let $W_n = \sum_{i=1}^{n} \xi_i$ and $\mathcal{F}_n = \sigma(\xi_1, \xi_2, ..., \xi_n)$. Let $W_0 = 0$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Observe that $\mathbb{E}[\xi_i] = 0$ for all i. Then $\{W_n\}_{n\geq 0}$ is $\mathcal{F}_n$-measurable for every n, one can check this very technically **or just intuitively knowing that observing the history toss completely determine your fortune**. Also in the finite outcome space, we usually don't worry about integrability. So, only (*iii*) needs to be checked,

$$\begin{aligned}
\mathbb{E}[W_{n+1}|\mathcal{F}_n] &= \mathbb{E}[W_n + \epsilon_{n+1}|\mathcal{F}_n] \\
&= \mathbb{E}[W_n|\mathcal{F}_n] + \mathbb{E}[\xi_{n+1}|\mathcal{F}_n] \text{ (by linearity of conditional expectation)} \\
&= W_n + \mathbb{E}[\xi_{n+1}],\ \text{(take out the thing that you know and independent property)} \\
&= W_n (\text{since } \mathbb{E}[\xi_{n+1}] = 0)
\end{aligned} \tag{1}$$

Most of those in the brackets are listed in the first recitation –*section 2.2.4: conditioning on a general σ-algebra*, they all can be proved rigorously just using the definition of conditional expectation.

Remark 2.3 Observe from (1), if the coin is not fair, we can have a (sup-)sub-martingale, depending on the expectation of single toss. This gambling convince you what we said in the very beginning, if it is a fair game, it should be fair coin, otherwise, either you are favoured or the casino.

A more interesting but also involved example will be betting on martingale:

Example 2.4 Let $\{W_n\}$ be the martingale of *example* 2.2. Suppose now that you can bet any amount Δ_{n-1} on play n with the restriction that for every $n \geq 1$ Δ_{n-1} must be $\mathcal{F}_{n-1}$-measurable (so that you can only use the information known up to time $n-1$ in deciding the amount Δ_{n-1}) and let $\mathbb{E}[|\Delta_k|] < +\infty$ for all $k \geq 0$. The amount you win on play n is therefore

$$\Delta_{n-1}\xi_n = \Delta_{n-1}(W_n - W_{n-1})$$

Your total winnings after n plays is:

$$X_n = \sum_{k=1}^{n} \Delta_{k-1}\xi_k = \sum_{k=1}^{n} \Delta_{k-1}(W_k - W_{k-1})$$

Set your initial fortune to be 0, i.e., $X(\omega) = 0, \forall\omega$. Then, the claim is $\{X_n\}_{n\geq 0}$ is a *martingale* with respect to $\{\mathcal{F}_n\}_{n\geq 0}$, since (i)$X_n$ contains only terms that depend on $\xi_1, ..., \xi_n$, it is $\mathcal{F}_n$-measurable; (ii) integrable because of both Δ and ξ have finite expectation; (iii)

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_n + \Delta_n\xi_{n+1}|\mathcal{F}_n] \\
&= \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[\Delta_n\xi_{n+1}|\mathcal{F}_n] \\
&= X_n + \Delta_n\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] \\
&= X_n + \Delta_n\mathbb{E}[\xi_{n+1}] \\
&= X_n
\end{aligned}$$

Again, all we used is just the properties of conditional expectation.

Remark 2.5 In the class, you may encounter the notion, *predictable process*, it says $\{\phi_n\}$ is *predictable* if ϕ_{n+1} is $\mathcal{F}_n$-measurable. Here, if we change the way Δ_n is defined, i.e., $\phi_{n+1} = \Delta_n$, then ϕ_{n+1} is a predictable process, the derivation is still right.

3 Simple Properties of Martingales*

Here is just a list of simple properties of martingale, some of them you may learn in the class, some are not. But they are quite straightforward, it never hurts to know them. First of all, let's recall *Jensen's inequality* in the context of probability:

Theorem 3.1 If X is an integrable random variable and $f(\cdot)$ is a convex function, then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

By convex function, we mean the second derivative is non-negative. For example, $f(x) = x^2$ is a convex function.

Now, let $\{X_n\}_{n\geq 0}$ be a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$,

Proposition 3.2 If $\{X_n\}_{n\geq 0}$ is a discrete-time martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$, then γ_n is a sub-martingale, where $\gamma_n = f(X_n)$ with convex function $f(\cdot)$.

Proof.

$$\mathbb{E}[\gamma_{n+1}|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|\mathcal{F}_n] \geq f(\mathbb{E}[X_{n+1}|\mathcal{F}_n]) = f(X_n) = \gamma_n$$

□

Proposition 3.3 If $\{X_n\}_{n\geq 0}$ is a discrete-time martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$, then

$$\mathbb{E}[X_n] = \mathbb{E}[X_0] \tag{2}$$

for all n, and

$$\mathbb{E}[X_m|X_n] = X_n \tag{3}$$

whenever $m > n$.

Proof. For (2),

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n|\mathcal{F}_{n-1}]] = \mathbb{E}[X_{n-1}]$$

for all $n \geq 1$. Thus, the assertion follows. For (3), if $m > n+1$,

$$\begin{aligned}
\mathbb{E}[X_m|\mathcal{F}_n] &= \mathbb{E}[\mathbb{E}[X_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \mathbb{E}[X_{m-1}|\mathcal{F}_n], \\
&= \mathbb{E}[\mathbb{E}[X_{m-1}|\mathcal{F}_{m-2}]|\mathcal{F}_n] = \mathbb{E}[X_{m-2}|\mathcal{F}_n], \text{ (if } m-1 > n+1) \\
&= \cdots\cdots = \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n
\end{aligned}$$

□

The converse to (3) is also true. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration. Fix $N > 1$ and suppose Z is $\mathcal{F}_N$-measurable, define

$$X_n = \mathbb{E}[Z|\mathcal{F}_n], \ n = 1, ..., N$$

Obviously, by *tower property*, X_n satisfies (3), i.e.,

$$\mathbb{E}[X_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_m]|\mathcal{F}_n] = \mathbb{E}[Z|\mathcal{F}_n] = X_n$$

We can claim: $\{X_n\}_{n\geq 0}$ is a discrete-time martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$. And indeed this is true as we examined in in the *example* 2.1.

4 Stopping Time

Instead of giving the dry definition of stopping time, let's motivate ourselves a little bit. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a discrete-time filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Imagine that the outcomes of Ω represents the possible ways a market can evolve on discrete time spot from 0 up to T ($t_N = T$). And $\mathcal{F}_n$ (instead of writing $\mathcal{F}_{t_n}$), $n = 0, 1, 2, ..., N$, is the history of the market up to t_n. If you took an advanced finance class in options, you know an option is of *American type* if the holder can exercise it any time at his/her discretion before the expiration T. Of course, the holder, in trying to maximize gain, will choose an exercise time $\tau(\omega)$ depending on the path ω the market actually take. So τ will be a random variable, since it is a function on a probability space.

In modelling markets, we always make the reasonable assumption that the investor cannot see into the future - they are not *clairvoyant*. We want to translate this qualitative assumption into a precise mathematical condition on τ. Consider the set $\{\omega; \tau(\omega) = t_n\}$ of market paths for which decision is to exercise at t_n. If clairvoyance is prohibited, this decision must be based only on observations of the market up to time t_n. Naively, this would seem to be the right mathematical restriction on τ. **But, if the setting is all continuous and τ is a continuous random variable, $\{\tau(\omega) = t\}$ has probability 0 for each t, and putting a condition only on sets of probability zero is not useful[3].** To fix this, consider instead $\{\omega : \tau(\omega) \leq t_n\}$; again, if looking into the future is not allowed, this must belong to $\mathcal{F}_n$ for all n. This leads to an important definition:

Definition 4.1 (*Stopping-time*) Given probability space$(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\{\mathcal{F}_n\}_{n\geq 0}$, a $\{\mathcal{F}_n\}_{n\geq 0}$-*stopping-time* is a random variable $\tau : \Omega \mapsto \mathbb{R}_+$ with the property that

$$\{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n, \ \forall n \geq 0$$

Remark 4.1 When the underling filtration is clear from the context, we shall say *stopping-time* instead of $\{\mathcal{F}_n\}_{n\geq 0}$*stopping-time*. Also, we write the event $\{\tau \leq n\}$ instead of $\{\omega : \tau(\omega) \leq n\}$ as there is no ambiguity.

Let's assume $\{X_n\}_{n\geq 0}$ is a sequence of random variables and let $\mathcal{F}_n^X = \sigma(X_i, i \in \{0, ..., n\})$ (instead of writing X_{t_i} we write X_i) define the filtration it generates (see last recitation *section: partial information and filtration*). At the intuitive level, a random time τ will be a stopping time w.r.t. $\{\mathcal{F}_n^X\}_{n\geq 0}$ if for every t_n, one can determine whether or not $\tau(\omega) \leq t_n$ from knowing the values X_i for $t_i \leq t_n$; if deciding whether $\tau(\omega) \leq t_n$ depends on knowing X_j, $t_j \geq t_n$, then τ will not be a $\{\mathcal{F}_n^X\}_{n\geq 0}$-stopping-time. This principle is a good guide, although technical issues arises in applying it, especially in the continuous-time case.

Let's recall *example* 2.2. Very important scheme of gambling is to quit once some goal is achieved. Consider the fortune process, $W_0, W_1, W_2, ...$, which is a martingale, let τ be a $\{\mathcal{F}_n^\xi\}_{n\geq 0}$-stopping-time[4] Also, we specify the gambling scheme Δ_n to be the following:

$$\Delta_n = \begin{cases} 1, & \text{if } n < \tau; \\ 0, & \text{if } n \geq \tau. \end{cases}$$

[3]It will be explained in the future, when continuous-time stochastic process is concerned

[4]The filtration is generated by $\{\xi_i\}_{i\geq 0}$, the outcome of coin toss.

In other words, you will always bet 1 dollar until quit. Then the stopped martingale is given by, the same as in *example* 2.2, taking $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} \Delta_{i-1}(W_i - W_{i-1}), \ \forall n$$

which implies

$$X_{n+1} - X_n = \Delta_n(W_{n+1} - W_n), \ \forall n$$

Thus, if $n \geq \tau$, $X_{n+1} - X_n = 0$, while if $n < \tau$, then $X_{n+1} - X_n = W_{n+1} - W_n$. As a consequence, if $\tau \leq n$, $X_n = W_\tau$, while if $n < \tau$, then $X_n = W_n$. Since $\{W_n\}_{n\geq 1}$ is a martingale, $\{X_n\}_{n\geq 0}$ is a martingale.

In general, we have the following theorem

Theorem 4.2 Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let τ be a stopping time and $\{X_n\}_{n\geq 0}$ a discrete-time (sup-, sub-)martingale, then the stopped process $\{\bar{X}_n\}_{n\geq 0}$ is a (sup-, sub-)martingale, where

$$\bar{X}_n := \begin{cases} X_n, & \text{if } n < \tau; \\ X_\tau, & \text{if } n \geq \tau. \end{cases}$$

Or sometimes, we denote $\bar{X}_n = X_{n\wedge\tau}$

Proof. (*we only prove the case when* $\{X_n\}_{n\geq 0}$ *is a martingale*). Let's first check integrability condition,

$$|\bar{X}_n| \leq \max_{0\leq k\leq n} |X_k| \leq |X_0| + |X_1| + \cdots + |X_n|$$

We conclude that $\mathbb{E}[|\bar{X}_n|] \leq \sum_{i=0}^{n} \mathbb{E}[|X_i|] < +\infty$, since X_i is integrable for all i.

Next, we check the martingale property. Notice $\bar{X}_n = X_n$ and $\bar{X}_{n+1} = X_{n+1}$ if $\tau > n$, and $\bar{X}_{n+1} = \bar{X}_n$ if $\tau \leq n$, we can express above relationship by one equation:

$$\bar{X}_{n+1} = \bar{X}_n + (X_{n+1} - X_n)\mathbf{1}\{\tau > n\}$$

Thus,

$$\begin{aligned} \mathbb{E}[\bar{X}_{n+1}|\mathcal{F}_n] &= \bar{X}_n + \mathbb{E}[(X_{n+1} - X_n)\mathbf{1}\{\tau > n\}|\mathcal{F}_n] \\ &= \bar{X}_n + \mathbf{1}\{\tau > n\}\mathbb{E}[(X_{n+1} - X_n)|\mathcal{F}_n] \\ &= \bar{X}_n + \mathbf{1}\{\tau > n\} \times 0 \\ &= \bar{X}_n \end{aligned}$$

Indeed, $\{\bar{X}_n\}_{n\geq 0}$ is a martingale. $\square$

Remark 4.3 Since $\bar{X}_0 = X_0$, we conclude that $\mathbb{E}[\bar{X}_n] = \mathbb{E}[X_0]$, $n \geq 0$.

5 Theorems of Joseph L. Doob

This section is devoted to illustrate several classical and significant theorems due to *Joseph L. Doob*, famous American mathematician working on probability theory.

Theorem 5.1 (*Doob's Maximal Inequality*) Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{X_i\}_{n\geq 0}$ be a sub-martingale. Define $S_n = \max_{1\leq i\leq n} X_i$ be the running maximum of X_i. Then for any $\lambda > 0$,

$$\mathbb{P}(S_n \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}[X_n^+]$$

where $X_n^+ = \max\{X_n, 0\}$.

Proof. Define $\tau = \inf\{i \geq 1 : X_i \geq \lambda\}$, obviously it is a stopping time[5]. Then

$$\mathbb{P}(S_n \geq \lambda) = \sum_{i=1}^{n} \mathbb{P}(\tau = i)$$

For each $1 \leq i \leq n$, the event $\{\tau = i\}$ is the same as the event that the process hit λ and it is at time i, thus

$$\mathbb{P}(\tau = i) = \mathbb{E}[\mathbf{1}_{\{X_i\geq\lambda\}}\mathbf{1}_{\{\tau=i\}}] \leq \frac{1}{\lambda}\mathbb{E}[X_i^+\mathbf{1}_{\{\tau=i\}}] \tag{4}$$

Since $\{X_i\}_{n\geq 0}$ is a sub-martingale and $\max\{0, x\}$ is an increasing convex function, $\{X_n^+\}_{n\geq 0}$ is a sub-martingale as well. Therefore,

$$\mathbb{E}[X_n^+\mathbf{1}_{\{\tau=i\}}|\mathcal{F}_i] = \mathbf{1}_{\{\tau=i\}}\mathbb{E}[X_n^+|\mathcal{F}_i] \geq \mathbf{1}_{\{\tau=i\}}\left(\mathbb{E}[X_n|\mathcal{F}_i]\right)^+ \geq \mathbf{1}_{\{\tau=i\}}X_i^+$$

and hence

$$\mathbb{E}[X_i^+\mathbf{1}_{\{\tau=i\}}] \leq \mathbb{E}[X_n^+\mathbf{1}_{\{\tau=i\}}]$$

Substituting this inequality into (4) and then summing over $1 \leq i \leq n$, we get the desired result. □

Now, we prove the *Doob's upcrossing inequality* as a preparation for *Martingale convergence theorem*. *Doob's upcrossing inequality* gives a uniform bound for number of upcrossings (will be defined shortly) for a (sub-, sup-) martingale sequence. Let's focus on sub-martingale case, similar arguments work for sup-martingales. Let $\{X_n\}_{n\geq 0}$ be a sub-martingale sequence, and $a < b$ be two real numbers. An *upcrossing* is a pair (X_k, X_l) such that

$$X_k \leq a < b \leq X_l$$

That is to say, the sub-martingale sequence finishes an upcrossing if it starts at some time when the value is below a then after a few steps it jumps above b. Let $U_N(a, b)$ denote the number of upcrossings up to time N. We have the following theorem:

[5]This is called *first hitting time*, usually of the form that a stochastic process hits some set. It can be rigorously proved as a stopping time. Intuitively, it makes sense that τ only depends on the trajectory of the process.

Theorem 5.2 (*Doob's Upcrossing Inequality*) For any $N \in \mathbb{N}$,

$$\mathbb{E}[U_N(a,b)] = \frac{1}{b-a}\mathbb{E}[(X_N - a)^+] \leq \frac{1}{b-a}\left(|a| + \mathbb{E}[X_N^+]\right)$$

Proof. Without loss of generality, let's set $X_0 \leq a$ (this will not destroy sub-martingale property of the process $\{X_n\}_{n\geq 0}$). We want to estimate $U_N(a,b)$, define the sequence $\bar{X}_n = (X_n - a)^+$. Obviously, $\{\bar{X}_n\}_{n\geq 0}$ as we argued just a moment ago. Then, $U_N(a,b)$ is equal to $\bar{U}(0,\beta)$, number of upcrossings of the interval $(0,\beta)$ by the sequence $\{\bar{X}_n\}_{n\geq 0}$, where $\beta = b - a$. Because of the way we set X_0, we have now $\bar{X}_0 = 0$. Consider the sequence $\{Z_n\}_{n\geq 0}$ defined as follows:

$$Z_0 = 0, \; Z_{n+1} = Z_n + \phi_n(\bar{X}_{n+1} - \bar{X}_n), \; n = 0, 1, 2, ...$$

where $\{\phi_n\}_{n\geq 0}$ is an $\mathcal{F}_n$-adapted process that is binary valued, it can only take 0 or 1. Then

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = Z_n + \phi_n \mathbb{E}[\bar{X}_{n+1} - \bar{X}_n|\mathcal{F}_n] \leq Z_n + \mathbb{E}[\bar{X}_{n+1} - \bar{X}_n|\mathcal{F}_n]$$

Because $\mathbb{E}[\bar{X}_{n+1}|\mathcal{F}_n] \geq \bar{X}_n$ ($\{\bar{X}_n\}_{n\geq 0}$ is a sub-martingale). It implies $\mathbb{E}[Z_n] \leq \mathbb{E}[\bar{X}_n]$ for all n[6].

We define a sequence of stopping times as follows:

$$\begin{aligned}
&L_1 = 0; \\
&T_1 = \min\{n > L_1 : \bar{X}_n \geq \beta\}; \\
&L_2 = \min\{n > T_1 : X_n = 0\}; \\
&T_2 = \min\{n > L_2 : X_n \geq \beta\} \\
&\cdots\cdots
\end{aligned}$$

And $\{\phi_n\}$ switching 0 and 1 in the following way:

$$\phi_n := \begin{cases} 1, & \text{if } L_k \leq n < T_k; \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
Z_N &= \sum_{n=0}^{N-1} \phi_n(\bar{X}_{n+1} - \bar{X}_n) \\
&= \sum_{n=L_1}^{T_1-1} (\bar{X}_{n+1} - \bar{X}_n) + \sum_{n=L_2}^{T_2-1} (\bar{X}_{n+1} - \bar{X}_n) + \cdots + \sum_{n=L_k}^{T_k-1} (\bar{X}_{n+1} - \bar{X}_n)
\end{aligned}$$

where $k = \bar{U}_N(0,\beta)$ is the number of upcrossings of $(0,\beta)$ until N. By construction, each sum is at least β, thus,

$$Z_N \geq \beta \bar{U}_N(0,\beta)$$

[6]when $n = 0$, $\mathbb{E}[Z_1 - Z_0|\mathcal{F}_0] = \mathbb{E}[Z_1] \leq \mathbb{E}[\bar{X}_1 - \bar{X}_0|\mathcal{F}_0] = \mathbb{E}[\bar{X}_1]$, do this inductively.

Taking expectation on both sides,

$$\mathbb{E}[\bar{U}_N(0,\beta)] \leq \frac{\mathbb{E}[Z_n]}{\beta}$$

Now, lets recover $U_N(a,b)$ from $\bar{U}_N(a,b)$,

$$\mathbb{E}[U_N(a,b)] \leq \frac{\mathbb{E}[Z_N]}{b-a}$$

Since $\mathbb{E}[Z_N] \leq \mathbb{E}[\bar{X}_N]$,

$$\frac{\mathbb{E}[Z_N]}{b-a} \leq \frac{\mathbb{E}[\bar{X}_N]}{b-a} = \frac{\mathbb{E}[(X_n-a)^+]}{b-a} \leq \frac{|a|+\mathbb{E}[X_N^+]}{b-a}$$

The assertion follows. $\square$

Theorem 5.3 (*Martingale Convergence Theorem*) If $\{X_n\}_{n\geq 0}$ is a sub-martingale with $\sup_n \mathbb{E}[X_n^+] < +\infty$, or a super-martingale with $\sup_n \mathbb{E}[X_n^-] < +\infty$, then the sequence $\{X_n\}$ is convergent with probability 1 to some random limit X_∞.

Remark 5.4 Again, we only need to prove the case that $\{X_n\}_{n\geq 0}$ is a sub-martingale with $\sup_n \mathbb{E}[X_n^+] < +\infty$, the other case will follow, since if $\{X_n\}_{n\geq 0}$ isa sup-martingale, then $\{-X_n\}_{n\geq 0}$ is a sub-martingale, and the condition $\sup_n \mathbb{E}[X_n^-] < +\infty$ is the same as $\sup_n \mathbb{E}[(-X_n)^+] < +\infty$.

Proof. Define random vriables:

$$X_* = \liminf_{n\to\infty} X_n,$$

with the values ∞ allowed. If the sequence $\{X_n\}_{n\geq 0}$ is not convergent almost surely, then the event

$$A = \{\omega \in \Omega : X_*(\omega) < X^*(\omega)\}$$

has positive probability. Observe that A can be represented as union of countably many events:

$$A = \bigcup_{a,b\in\mathbb{Q}} \{\omega \in \Omega : X_*(\omega) < a < b < X^*(\omega)\}$$

If $\mathbb{P}(A) > 0$ then some of these events must have positive probability. We can then choose $\bar{a} < \bar{b}$ such that

$$\mathbb{P}\{\omega \in \Omega : X_*(\omega) < \bar{a} < \bar{b} < X^*(\omega)\} > 0$$

In this event, the number of upcrossing $U_N(\bar{a},\bar{b})$ tends to infinity, as $N \to \infty$. Therefore,

$$\lim_{N\to\infty} \mathbb{E}[U_N(\bar{a},\bar{b})] = +\infty$$

This contradicts *Doob's maximal inequality.* Consequently, $X_* = X^*$ with probability, and the sequence $\{X_n\}$ is convergent almost surely. $\square$

Theorem 5.5 (*Doob's Decomposition Theorem*) If $\{X\}_{n\geq 0}$ is a sub-martingale on $(\Omega, \mathcal{F}_n, \mathbb{P})$, then X_n can be written as $X_n = Y_n + A_n$ with following properties:

1. $\{Y_n\}_{n\geq 0}$ is a martingale;
2. $A_{n+1} \geq A_n$ for almost all ω for every $n \geq 0$ with $A_0 \equiv 0$;
3. For every $n \geq 1$, A_n is $\mathcal{F}_{n-1}$ measurable.

Furthermore, X_n determines Y_n and A_n uniquely.

Proof. If $X_n = Y_n + A_n$ with Y_n and A_n satisfying 1, 3 and 4. Then

$$A_n - A_{n-1} = \mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] \tag{5}$$

are uniquely determined. Since $A_1 = 0$, all the A_n's are uniquely determined as well. Property (2) is equivalent to the sub-martingale property of $\{X_n\}_{n\geq 0}$. To establish the representation, we define A_n inductively by (5). It is easy to verify that $Y_n = X_n - A_n$ is a martingale and the monotonicity of A_n is a consequence of sub-martingale property. $\square$

6 Reference

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