Asset Pricing Theory in Continuous-time

Jianing Yao
Department of MSIS-RUTCOR
Rutgers University, the State University of New Jersey
Piscataway, NJ 08854 USA

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1 Change of Measure

1.1 Introduction

So far, in our discussion of probability and stochastic process, we have worked on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is fixed. Now, we want to consider a single outcome space Ω with a given σ-algebra $\mathcal{F}$, but allow for different probability measures, say $\mathbb{P}$ and $\mathbb{Q}$, and study the relation between them. This is particularly important for mathematical finance. In derivative pricing theory, Ω is a set of possible market histories and $\mathbb{P}$ is a probability measure modeling our beliefs about the relative likelihoods with which different histories occur. In discrete-time/discrete state space models (e.g., *Markov Chain*), Ω is a finite list of possible histories and we never bothered to put a measure $\mathbb{P}$ on Ω to represent the actual probabilities of the histories. This was because the only measure relevant to derivative pricing is the risk-neutral measure, which does not necessarily reflect actual probabilities. However, there was one crucial probabilistic assumption underlying our choice of Ω, even if we did not specify $\mathbb{P}$ – namely, each history in Ω has a positive probability of occurring and the probability of Ω is 1, that is Ω includes all possibilities that can occur.

On the other hand, for continuous time modeling we can no longer avoid starting with an probability model. To see why, consider the case in which we think prices are continuous function of time. Then the set of plausible market histories is contained in the set of all continuous functions. In these models, we want to allow more than just a discrete set of continuous functions as possible future histories and it makes no sense to ascribe strictly positive probabilities to individual continuous function. We need a way to single out a subset of continuous functions that occur with probability one and this is done by building a probability model, that is by specifying a $\mathbb{P}$. For example, the *Black-Scholes* model

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \tag{1}$$

or, equivalently,

$$S(t) = S(0)\exp\{\sigma W(t) + [\alpha - \frac{1}{2}\sigma^2]t\} \tag{2}$$

implies that with probability one $\ln S(t) = \sigma W(t) + [\alpha - \frac{1}{2}\sigma^2]t$ is a function whose quadratic variation process $[\ln S(t), \ln S(t)] = \sigma^2 t$. Continuous function with this quadratic variation process are a very special subset of all continuous functions. If your competitor uses a *Black Scholes* model with $\sigma'^2 = \sigma^2$ he or she believes that the set of paths for which $[\ln S(t), \ln S(t)] = \sigma^2 t$ has probability zero, which you believe they have probability one. It will be no surprise that you will come up with different prices for derivatives.

Now once you have chosen a $\mathbb{P}$ and so determined the set of histories you think will occur with probability one, you will not price use $\mathbb{P}$ itself but a related risk-neutral $\mathbb{Q}$ (yet to be defined). This brings us back to the issue we began with, how to compare and move between two different measures on a measure space.

1.2 Equivalent Probability Measures

Definition 1.1 Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$. $\mathbb{Q}$ is *absolutely continuous with respect to* $\mathbb{P}$, written $\mathbb{Q} \ll \mathbb{P}$, if

$$\mathbb{P}(A) = 0 \text{ implies } \mathbb{Q}(A) = 0, \text{ for } A \in \mathcal{F} \tag{3}$$

$\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* if $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} >> \mathbb{Q}$, or, in other words,

$$\mathbb{P}(A) = 0 \text{ if and only if } \mathbb{Q}(A) = 0, \text{ for } A \in \mathcal{F} \tag{4}$$

Example 1.1 Let $\Omega = [0, 1]$ and define, for *Borel subsets* $A \subseteq [0, 1]$,

$$\begin{aligned}
\mathbb{P}_1(A) &= \int_0^1 \mathbf{1}_A(x)dx \\
\mathbb{P}_2(A) &= \int_0^1 \mathbf{1}_A(x)2xdx \\
\mathbb{P}_3(A) &= 2\int_{\frac{1}{2}}^1 \mathbf{1}_A(x)dx = 2\mathbb{P}_1(A \cap [\frac{1}{2}, 1])
\end{aligned}$$

For any $A \subset [0, 1]$, we can visualize them:

$$\mathbb{P}_1(A) = \text{ area of region in } (x, y)\text{-plane above } A \text{ and between } y = 0 \text{ and } y = 1$$

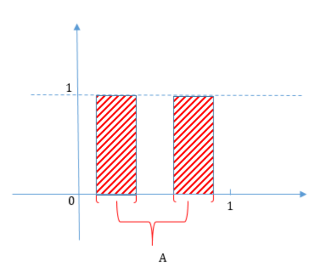

Figure 1: $\mathbb{P}_1(A)$

$$\mathbb{P}_2(A) = \text{ area of region in } (x,y)\text{-plane above } A \text{ and between } y=0 \text{ and } y=2x$$

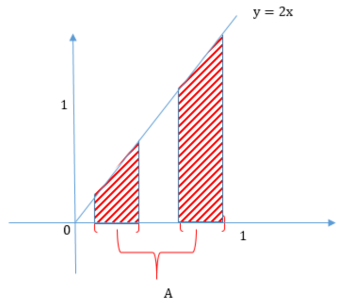

Figure 2: $\mathbb{P}_2(A)$

$$\mathbb{P}_3(A) = \text{ area of region in } (x,y)\text{-plane above } A \cap [\tfrac{1}{2},1] \text{ and between } y=0 \text{ and } y=2$$

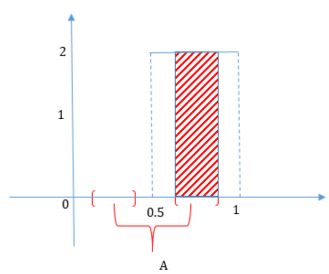

Figure 3: $\mathbb{P}_3(A)$

Note that $\mathbb{P}_3 \ll \mathbb{P}_1$ because if $\mathbb{P}_1(A) = 0$, $\mathbb{P}_3(A) = 2\mathbb{P}_1(A \cap [\frac{1}{2},1]) = 0$. However, $\mathbb{P}([0,\frac{1}{2}]) = \frac{1}{2}$ but $\mathbb{P}_3([0,\frac{1}{2}]) = 0$, so $\mathbb{P}_1$ is **not** absolutely continuous w.r.t. $\mathbb{P}_3$. As a result, $\mathbb{P}_3 = \mathbb{P}_1$ are not equivalent. However, from the pictures, it si clear $\mathbb{P}_2 \ll \mathbb{P}_1$ and $\mathbb{P}_1 \ll \mathbb{P}_2$, thus, $\mathbb{P}_1 \sim \mathbb{P}_2$.

1.3 Building New Measures from Old and Equivalent probability measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. Let X be a random variable defined on this space. This cumulative distribution function of X is then

$$F_X^{\mathbb{P}}(x) = \mathbb{P}\left(\{\omega : X(\omega) \leq x\}\right) \tag{5}$$

Previously, we simply wrote F_X for this function, but since we are now interested in changing probability measures, and since the cumulative distribution function clearly depends on $\mathbb{P}$, we write $F_X^{\mathbb{P}}$ to explicitly show this dependence.

Recall that in the general theory, the expectation of X is defined as an operation of integration on Ω with respect to $\mathbb{P}$,

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) \tag{6}$$

This clearly also depends on the measure $\mathbb{P}$ we are using and so, when it matters, we will write this as $\mathbb{E}_{\mathbb{P}}[X]$.

Remark 1.2 In practice, we do not usually compute $\mathbb{E}_{\mathbb{P}}[X]$ by using (6). For example, when there is a density $f_X^{\mathbb{P}} = \frac{d}{dx} F_X^{\mathbb{P}}$, we usually use the formula:

$$\mathbb{E}_{\mathbb{P}}[X] = \int_{-\infty}^{\infty} x f_x^{\mathbb{P}}(x) dx \tag{7}$$

Assume Z is a random variable defined on Ω, such that

$$\mathbb{P}(Z \geq 0) = 1 \text{ and } \mathbb{E}_{\mathbb{P}}[Z] = 1 \tag{8}$$

For $A \in \mathcal{F}$, define

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z] \tag{9}$$

Theorem 1.3 All followings hold:

1. $\mathbb{Q}$ is a probability measure and $\mathbb{Q} \ll \mathbb{P}$;

2. If X is a random variable on $(\Omega, \mathcal{F})$

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[XZ] \tag{10}$$

   when either side makes sense;

3. $\mathbb{Q} \sim \mathbb{P}$ if and only if $\mathbb{P}(Z > 0) = 1$ and when $\mathbb{P}(Z > 0) = 1$,

$$\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A \cdot Z^{-1}] \tag{11}$$
$$\mathbb{E}_{\mathbb{P}}[X] = \mathbb{E}_{\mathbb{Q}}[X \cdot Z^{-1}] \tag{12}$$

Remark 1.4 Instead of giving the proof (which you can find from many monograph), we only point out:

- the condition $\mathbb{P}(Z \geq 0) = 1$ in (8) guarantees $\mathbb{Q}(A) \geq 0$ in (9) for all $A \in \mathcal{F}$;

- the condition $\mathbb{E}_{\mathbb{P}}[Z] = 1$ in (8) implies $\mathbb{Q}(\Omega) = \mathbb{E}[\mathbf{1}_{\Omega} \cdot Z] = \mathbb{E}_{\mathbb{P}} = 1$, which we need for $\mathbb{Q}$ to be a probability measure;

- finite additivity of $\mathbb{Q}$ is an easy consequence of the fact that if A_1 and A_2 are disjoint $\mathbf{1}_{A_1 \cup A_2} = \mathbf{1}_{A_1} + \mathbf{1}_{A_2}$. Hence,

$$\mathbb{Q}(A_1 \cup A_2) = \mathbb{E}_{\mathbb{P}}[(\mathbf{1}_{A_1} + \mathbf{1}_{A_2})Z] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_1} Z] + \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{A_2} Z] = \mathbb{Q}(A_1) + \mathbb{Q}(A_2)$$

- if $\mathbb{P}(A) = 0$, then $\mathbb{P}(\mathbf{1}_A Z = 0) = 1$ and $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z] = 0$, which shows $\mathbb{Q} \ll \mathbb{P}$;
- (11) follows from

$$\mathbb{E}_{\mathbb{Q}}[XZ^{-1}] = \mathbb{E}_{\mathbb{P}}[XZ^{-1}Z] = \mathbb{E}_{\mathbb{P}}[X]$$

Example 1.5 Let $\Omega = [0,1]$, and let $Z(x) = 2x$. Define $\mathbb{P}_1$ and $\mathbb{P}_2$ as in *Exercise* 1.1. Note that

$$\mathbb{P}_1(A) = \int_0^1 \mathbf{1}_A(x) 2x dx = \int_0^1 \mathbf{1}_A(x) Z(x) dx = \mathbb{E}_{\mathbb{P}_1}[\mathbf{1}_A Z]$$

Since $\mathbb{P}_1(Z = 0) - \mathbb{P}_1(\{x = 0\}) = 0$, we see from above theorem that $\mathbb{P}_1 \sim \mathbb{P}_2$. For this example,

$$\mathbb{E}_{\mathbb{P}_2}[X] = \int_0^1 X(x) 2x dx = \int_0^1 X(x) Z(x) dx = \mathbb{E}_{\mathbb{P}_1}[XZ]$$

Thus,

$$\mathbb{E}_{\mathbb{P}_2}[XZ^{-1}] = \int_0^1 X(x) Z^{-1}(x) Z(x) dx = \int_0^1 X(x) dx = \mathbb{E}_{\mathbb{P}_1}[X]$$

Example 1.6 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Y be a random variable on it that is normal distributed with mean 0 and variance σ^2 under $\mathbb{P}$. This means

$$F_Y^{\mathbb{P}}(x) = \int_{-\infty}^{x} e^{-Y^2/2\sigma^2} \frac{dY}{\sqrt{2\pi}\sigma} \tag{13}$$

Let $Z = e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}$. Because $Y \sim N(0, \sigma^2)$ under $\mathbb{P}$,

$$\mathbb{E}_{\mathbb{P}}[e^{\lambda Y}] = e^{\frac{1}{2}\lambda^2\sigma^2}$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}[Z] = \mathbb{E}_{\mathbb{P}}[e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}] = 1 \tag{14}$$

and hence

$$\mathbb{Q}(A) = \mathbb{E}[\mathbf{1}_A e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}] \tag{15}$$

defines a probability measure $\mathbb{Q}$. Since $\mathbb{P}(Z > 0) = 1$, $\mathbb{Q} \sim \mathbb{P}$. Under $\mathbb{Q}$, Y has cumulative distribution function

$$F_Y^{\mathbb{Q}}(x) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{Y \le x\}} e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}]$$

what is $F_Y^{\mathbb{Q}}$ more explicitly? To find out, we compute the moment generating function of Y under $\mathbb{Q}$. This is, for $t \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{tY}] &= \mathbb{E}_{\mathbb{P}}[e^{tY} e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}] \\
&= e^{-\frac{1}{2}\lambda^2\sigma^2}\mathbb{E}_{\mathbb{P}}[e^{(\lambda+t)Y}] \\
&= e^{-\frac{1}{2}\lambda^2\sigma^2} e^{(t+\lambda)^2\sigma^2/2} \\
&= e^{t\lambda\sigma^2 + \frac{1}{2}t^2\sigma^2}
\end{aligned}$$

This is the moment generating function of a normal random variable with mean $\sigma^2\lambda$ and variance σ^2. Hence, under $\mathbb{Q}$, $Y \sim N(\lambda\sigma^2, \sigma^2)$ and

$$F_Y^{\mathbb{Q}}(x) = \int_{-\infty}^{x} e^{-(Y-\lambda\sigma^2)/2\sigma^2} \frac{dY}{\sqrt{2\pi}\sigma}$$

Example 1.6 is summarized in the following statement:

Theorem 1.7 If $Y \sim N(0, \sigma^2)$ under $\mathbb{P}$, and if $\mathbb{Q}$ is defined by $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A e^{\lambda Y - \frac{1}{2}\lambda^2\sigma^2}]$, then $\mathbb{Q} \sim \mathbb{P}$ and $Y \sim N(\lambda\sigma^2, \sigma^2)$ under $\mathbb{Q}$.

Let's apply this theorem:

Example 1.8 Let $Y \sim N(0, \sigma^2)$ under $\mathbb{P}$, we want to calculate: $\mathbb{E}_{\mathbb{P}}[e^{\lambda Y}\mathbf{1}_{\{Y \leq b\}}]$. Proceed in the following way: define $\mathbb{Q}$ as in *Theorem* 1.7. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[e^{\lambda Y}\mathbf{1}_{\{Y \leq b\}}] &= e^{\lambda^2\sigma^2/2}\mathbb{E}_{\mathbb{P}}[e^{-\lambda Y - \lambda^2\sigma^2/2}\mathbf{1}_{Y \leq b}] \\
&= e^{\lambda^2\sigma^2/2}\mathbb{Q}(Y \leq b) \\
&= e^{\lambda^2\sigma^2/2}\mathbb{Q}\left(\frac{Y - \lambda\sigma^2}{\sigma} \leq \frac{b - \lambda\sigma^2}{\sigma}\right)
\end{aligned}$$

Since $Y \sim N(\lambda\sigma^2, \sigma^2)$ under $\mathbb{Q}$, $\frac{Y-\lambda\sigma^2}{\sigma} \sim N(0,1)$ under $\mathbb{Q}$. Thus,

$$\mathbb{E}_{\mathbb{P}}[e^{\lambda Y}\mathbf{1}_{Y \leq b}] = e^{\lambda^2\sigma^2/2}\int_{-\infty}^{\frac{b-\lambda\sigma^2}{\sigma}} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} = e^{\lambda^2\sigma^2/2}N(\frac{b-\lambda\sigma^2}{\sigma}) \tag{16}$$

where $N(x)$ denotes the cumulative distribution function of the standard normal.

1.4 Girsanov's Theorem

Before giving the theorem, let's introduce some useful notation. If $\mathbb{P}$ and $\mathbb{Q}$ are probability measures and $\shortparallel = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z]$ for $A \in \mathcal{F}$, we denote Z by $\frac{d\mathbb{Q}}{d\mathbb{P}}$. Thus, we write

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A \frac{d\mathbb{Q}}{d\mathbb{P}}] \tag{17}$$

This notation makes the formula $\mathbb{E}_{\mathbb{Q}}[Z] = \mathbb{E}_{\mathbb{P}}[XZ]$ easy to remember if we write

$$\mathbb{E}_{\mathbb{Q}}[X] = \int_{\Omega} X(\omega)d\mathbb{Q}(\omega) = \int_{\Omega} X(\omega)\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)d\mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}[X\frac{d\mathbb{Q}}{d\mathbb{P}}]$$

Theorem 1.9 (*Radon-Nikodym Theorem*) If $\mathbb{Q} \ll \mathbb{P}$, there exists a Z satisfying $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}_{\mathbb{P}}[Z] = 1$ such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z]$

Remark 1.10 This is a kind of converse to *Theorem* 1.3. When $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exists, it is called the *Radon-Nikodym derivative of* $\mathbb{Q}$ *w.r.t.* $\mathbb{P}$.

Girsanov's theorem is a generalization of *Theorem* 1.7 to Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and on which are given a Brownian motion W and a filtration of $\{\mathcal{F}_t\}_{t\geq 0}$ for W. Let $\{\theta(t)\}_{0\leq t\leq T}$ be an $\mathcal{F}_t$-adapted process with

$$\mathbb{P}\left(\int_0^T \theta^2(s)ds < +\infty\right) = 1 \tag{18}$$

so that $\int_0^t \theta(s)dW(s)$ is defined for $0 \leq t \leq T$. Define

$$Z(t) = \exp\{-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\},\ 0 \leq t \leq T \tag{19}$$

By Itō's rule

$$dZ(t) = -\theta(t)Z(t)dW(t)$$

and hence

$$Z(t) = 1 - \int_0^t \theta(s)Z(s)dW(s),\ 0 \leq t \leq T$$

If

$$\mathbb{E}[\int_0^t \theta^2(s)Z^2(s)ds] < +\infty,\ 0 \leq t \leq T$$

then $\{Z(t)\}_{0\leq t\leq T}$ is a martingale and $\mathbb{E}_{\mathbb{P}}[Z(t)] = 1$, $\forall t \in [0, T]$. This implies that

$$\begin{aligned}\mathbb{Q}(A) &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z(t)] &(20)\\ &= \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A \exp\{-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\}],\ 0 \leq t \leq T &(21)\end{aligned}$$

defines a probability measure on Ω. Clearly, $\mathbb{P}(Z(t) > 0) = 1$, $\forall t \in [0, T]$, so in fact $\mathbb{P} \sim \mathbb{Q}$.

Example 1.11 Suppose $-\theta(t) = \lambda$ for all t. Then $Z(t)$ is

$$Z(t) = e^{\lambda W(t) - \frac{1}{2}\lambda^2 t}$$

This is exactly the *Radon-Nikodym derivative* that appearing in *Theorem* 1.7, with $W(t)$ in place of Y and t in place of σ^2. We know then that if $\mathbb{Q}(A) = \mathbb{E}[\mathbf{1}_A Z(t)]$, $W(t) \sim N(\lambda t, t)$ under $\mathbb{Q}$ and hence $\{W(t)\}_{0\leq t\leq T}$ cannot be a Brownian motion under $\mathbb{Q}$.

Return to the general case of $\mathbb{Q}$ defined as in (20). From *Example* 1.11, it cannot be expected that W is also a Brownian motion under $\mathbb{Q}$. *Girsanov's Theorem* addresses how W behaves under $\mathbb{Q}$.

Theorem 1.12 (*Girsanov's Theorem*) Assume $\mathbb{E}_{\mathbb{P}}[Z(t)] = 1$ for $t \in [0, T]$, where Z is defined as in (19) and let $\mathbb{Q}$ be defined as in (20. Then

$$\bar{W}(t) := W(t) + \int_0^t \theta(s)ds, \; 0 \le t \le T \tag{22}$$

is a Brownian motion under $\mathbb{Q}$.

Example 1.13 Let's continue *Example* 1.11, if $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A e^{\lambda W(t) - \frac{1}{2}\lambda^2 t}]$, then *Girsanov* says that

$$\bar{W}(t) = W(t) - \lambda t, \; 0 \le t \le T \tag{23}$$

is a Brownian motion under $\mathbb{Q}$. Thus, for example, if $t < T$, but $t > 0$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{W(t)<b\}} e^{\lambda W(T)}] &= e^{\frac{1}{2}\lambda^2 T} \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{W(t)<b\}} e^{\lambda W(T) - \frac{1}{2}\lambda^2 T}] \\
&= e^{\frac{1}{2}\lambda^2 T} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{W(t)<b\}}] \\
&= e^{\frac{1}{2}\lambda^2 T} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\bar{W}(t)+\lambda t<b\}}] \\
&= e^{\frac{1}{2}\lambda^2 T} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\bar{W}(t)<b-\lambda t\}}]
\end{aligned}$$

Under $\mathbb{Q}$, $\frac{\bar{W}(t)}{\sqrt{t}} \sim N(0,1)$, so

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{W(t)<b\}} e^{\lambda W(T)}] = e^{\frac{1}{2}\lambda^2 T} N(\frac{b - \lambda t}{\sqrt{t}})$$

2 A class of It$\bar{o}$ process Market Models

2.1 Preliminary, useful result

Let $(W_1, ..., W_m)$ be multi-dimensional Brownian motion and $\{\mathcal{F}_t\}_{t \ge 0}$ be a filtration for this Brownian motion. It$\bar{o}$'s rule shows that if

$$dX(t) = R(t)X(t)dt + \sum_{i=1}^{m} \gamma_i(t) dW_i(t) \tag{24}$$

then,

$$d\left(e^{-\int_0^t R(s)ds} X(t)\right) = \sum_{i=1}^{m} e^{-\int_0^t R(s)ds} \gamma_i(t) dW_i(t) \tag{25}$$

and, hence, if $\mathbb{E}[\int_0^t e^{-2\int_0^t R(s)ds} \gamma^2(s)ds] < +\infty$,

$$e^{-\int_0^t R(s)} X(t) \text{ is a martingale.} \tag{26}$$

2.2 The general model

The general model consists of

- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a multi-dimensional Brownian motion $W = (W_1, ..., W_m)$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for W;
- $\mathcal{F}_t$-adapted process: $\{R(t)\}_{t\geq 0}$ is risk free rate process, $\{\alpha_i(t)\}_{t\geq 0}$. $i = 1, ..., n$, the mean rate of growth of asset i, $\{\sigma_{ij}(t)\}$, $1 \leq i \leq n$, $1 \leq j \leq m$, $\{S_1(t)\}_{t\geq 0}$, ..., $\{S_n(t)\}$, price processes of n risky assets satisfying:

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t),\ 1 \leq i \leq m \tag{27}$$

- time horizon T.

Remark 2.1 1 dollar, invested in the money market at $t = 0$, is worth $e^{\int_0^t R(s)ds}$ at time t. Thus,

$$D(t) = e^{-\int_0^t R(s)ds}$$

defines the discount factor for discounting by the risk-free rate. Also, from last recitation:

$$dS_i(t)dS_j(t) = \sum_{k=1}^{m} \sigma_{ik}(t)\sigma_{jk}(t)dt$$

when this differ from zero, asset processes S_i and S_j are correlated.

2.3 The Wealth equation for a self-financing portfolio

A portfolio process is an adapted and vector valued process $\Delta(t) = [\Delta_1(t), ..., \Delta_n(t)]$, $0 \leq t \leq T$, where $\Delta_i(t)$ represents the number of shares to be held in asset i over the infinitesimal time interval $[t, t + dt]$. Let $X_\Delta(t)$ be the dollar amount of wealth resulting from using portfolio process Δ when the initial endowment is $X_\Delta(0)$ and the self-financing condition is imposed. The self-financing condition means that at each time t:

$$X_\Delta(t) - \sum_{i=1}^{n} \Delta_i(t)S_i(t)$$

which is the amount of wealth not invested in risky assets, is invested in the risk-free money market, and there are no additional income sources. It follows that

$$dX_\Delta(t) = R(t)[X_\Delta(t) - \sum_{i=1}^{n} \Delta_i(t)S_i(t)]dt + \sum_{i=1}^{n} \Delta_i(t)dS_i(t) \tag{28}$$

This is the first version of the wealth equation. It will be useful to rewrite the wealth equation in other forms. First note, by the same type of calculation as in the first subsection, using $dD(t) = -R(t)D(t)dt$,

$$\begin{aligned} d(D(t)S_i(t)) &= -R(t)D(t)S(t)dt + D(t)dS_i(t) \\ &= D(t)[-R(t)S_i(t)dt + dS_i(t)] \end{aligned}$$

and

$$\begin{aligned} & d[D(t)X\Delta(t)] \\ =& -R(t)D(t)X_\Delta(t)dt + D(t)dX_\Delta(t) \\ =& -R(t)D(t)X_\Delta(t)dt + D(t)R(t)[X_\Delta(t) - \sum_{i=1}^{m} \Delta_i(t)S_i(t)]dt + D(t)\sum_{i=1}^{m} \Delta_i(t)dS_i(t) \\ =& \sum_{i=1}^{m} \Delta_i(t)[D(t)(-R(t)S_i(t)dt + dS_i(t))] \end{aligned}$$

Using the result above,

$$d[D(t)X_\Delta(t)] = \sum_{i=1}^{m} \Delta_i(t)d[D(t)S_i(t)] \tag{29}$$

2.4 Risk-neutral measures and risk-neutral pricing

Begin with the general model we defined a moment ago:

Definition 2.1 An equivalent risk-neutral measure $\mathbb{Q}$ for the model stated in *section* 2.2 is a probability measure on $(\Omega, \mathcal{F})$ satisfying:

- $\mathbb{Q} \sim \mathbb{P}$ ($\mathbb{P}$ is the original measure for the model);
- $\{D(t)S_i(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale under $\mathbb{Q}$ for each $1 \leq i \leq n$, i.e.,

$$\mathbb{E}_{\mathbb{Q}}[D(T)S_i(T)|\mathcal{F}_t] = D(t)S_i(t), \text{ for } t \leq T, \; i = 1, ..., n. \tag{30}$$

Remark 2.2 From now on, we use $\tilde{\mathbb{E}}[Y] = \mathbb{E}_{\mathbb{Q}}[Y]$.

Theorem 2.3 (*First Fundamental Theorem of Asset Pricing*) This market model is arbitrage-free if there is a risk-neutral measure $\mathbb{Q}$.

We do not give a rigorous proof, because it requires discussing technical issues and giving refined definitions that are beyond our scope. However, the essential insight is this:

under $\mathbb{Q}$, $\{D(t)X_\Delta(t)\}_{t\geq 0}$ *is a martingale for any portfolio process* Δ. (**)

The reason that this is true is due to (29). Since $D(t)S_i(t)$ is a martingale, for each i, we have (very loosely expressed):

$$\tilde{\mathbb{E}}[d[D(t)S_i(t)]|\mathcal{F}_t] = \tilde{\mathbb{E}}[D(t+dt)S_i(t+dt) - D(t)S_i(t)|\mathcal{F}_t] = 0$$

Thus, from (29),

$$\begin{aligned}
\tilde{\mathbb{E}}[d[D(t)X_\Delta(t)]] &= \tilde{\mathbb{E}}[\sum_{i=1}^{m} \Delta_i(t) d[D(t)S_i(t)]|\mathcal{F}_t] \\
&= \sum_{i=1}^{m} \Delta_i(t) \tilde{\mathbb{E}}[d[D(t)S_i(t)]|\mathcal{F}_t] \\
&= 0
\end{aligned}$$

Thus, at least at the formal, infinitesimal level,

$$\tilde{\mathbb{E}}[D(t+dt)X_\Delta(t+dt)|\mathcal{F}_t] = D(t)X_\Delta(t)$$

Let's recall the definition of arbitrage:

1. $X(0) < 0$ and $X(T)(\omega) \geq 0$ for all $\omega \in \Omega$; or
2. $X(0) = 0$, and $X(T)(\omega) \geq 0$ for all $\omega \in \Omega$ and there is at least one $\omega' \in \Omega$, such that $X(T)(\omega') > 0$.

Now, if (**) is true, there can be no arbitrage. For example, if $\mathbb{P}(X_\Delta(T) \geq 0) = 1$, then $\mathbb{Q}(X_\Delta(T) \geq 0) = 1$. Also, since $\mathbb{Q} \sim \mathbb{P}$ and

$$X_\Delta(0) = D(0)X_\Delta(0) = \tilde{\mathbb{E}}[D(T)X_\Delta(T)] \geq 0$$

Since $\mathbb{Q}(D(T) > 0) = 1$. Or, if $\mathbb{P}(X_\Delta(T) \geq 0) = 1$ and $\mathbb{P}(X_\Delta(T) > 0)$, then $\mathbb{Q}(X_\Delta(T) \geq 0) = 1$ and $\mathbb{Q}(X_\Delta(T) \geq 0) > 0$ and so

$$X_\Delta(0) = \tilde{\mathbb{E}}[D(T)X_\Delta(T)] > 0$$

In either case, arbitrage is not possible.

Assuming $\mathbb{Q}$ exists, we can use it for pricing replicable claims. A contingent claim H, considered as a pay-off at time T is just some $\mathcal{F}_T$-measurable random variable. We say H can be *replicated* if:

there exists a portfolio process Δ such that $X_\Delta = H$ with probability 1.

In this case, Δ is called a replicating portfolio. Since an investment of $X_\Delta(t)$ at time t can lead to a terminal value of H using a self-financing trading strategy there will be arbitrage unless the price, call it $V(t)$, of the contingent claim at t equals $X_\Delta(t)$; to repeat:

$$V(t) = X_\Delta(t) \tag{31}$$

is the price of the contingent claim at t. But $\{D(t)X_\Delta(t)\}_{0\leq t\leq T}$ is a martingale under $\mathbb{Q}$ so

$$D(t)X_\Delta(t) = \tilde{\mathbb{E}}[D(T)X_\Delta(T)|\mathcal{F}_t] = \tilde{\mathbb{E}}[D(T)H|\mathcal{F}_t]$$

Hence, if $\mathbb{Q}$ is a risk-neutral measure and if H is a replicable contingent claim, its no-arbitrage price process is:

$$V(t) = D^{-1}(t)\tilde{\mathbb{E}}[D(T)H|\mathcal{F}_t] \tag{32}$$

This is the general pricing formula in a risk-neutral model.

2.5 Two questions

Question 1: when does a risk-neutral measure exist for the model mentioned above?

Definition 2.2 Our model is called *complete* if every contingent claim can be hedged.

Question 2: when is our model complete?

These two questions are the issues we address next.

2.6 Existence of $\mathbb{Q}$

Finding a $\mathbb{Q}$ depends on *Girsanov's theorem*, which is stated here in generality:

Theorem 2.4 Assume the general model. Let $\Theta(t) = [\theta_1(t), ..., \theta_m(t)]$ be an adapted process and define:

$$Z(t) = \exp\{-\int_0^t \sum_{i=1}^m \theta_i(u) dW_i(u) - \frac{1}{2}\int_0^t \theta_i^2(u) du\} \tag{33}$$

Assume $\mathbb{E}_{\mathbb{P}}[Z(T)] = 1$ and define:

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A Z(T)] \tag{34}$$

Then, under $\mathbb{Q}$, $\bar{W}(t) = [\bar{W}_1(t), ..., \bar{W}_d(t)]$, where

$$\bar{W}_i(t) = W_i(t) + \int_0^t \theta_i(u) du, \ 1 \leq i \leq n \tag{35}$$

is a Brownian motion.

To apply this to find a risk-neutral $\mathbb{Q}$, begin with

$$dS_i(t) = \alpha_i(t) S_i(t) dt + S_i(t) \sum_{i=1}^m \sigma_{ij}(t) dW_j(t) \tag{36}$$

Define

$$\tilde{W}_i(t) = W_i(t) + \int_0^t \theta_i(u) du \tag{37}$$

for some processes θ_i, $1 \leq i \leq m$, yet to be specified. Then

$$d\tilde{W}_i(t) = dW_i(t) + \theta_i(t) dt \tag{38}$$

or

$$dW_i(t) = d\tilde{W}_i(t) - \theta_i(t) dt \tag{39}$$

Hence,

$$\begin{aligned} dS_i(t) &= \alpha_i(t)S_i(t)dt + S_i(t)\sum_{i=1}^{m}\sigma_{ij}(t)[d\tilde{W}_j(t) - \theta_j(t)dt] \\ &= [\alpha_i(t) - \sum_{j=1}^{m}\sigma_{ij}\theta_j(t)]S_i(t)dt + S_i(t)\sum_{i=1}^{m}\sigma_{ij}d\tilde{W}_j(t) \end{aligned}$$

Suppose we can find $\theta_1, ..., \theta_m$ so that

$$\alpha_i(t) - \sum_{i=1}^{m}\sigma_{ij}(t)\theta_j(t) = R(t),\ 1 \le i \le n \tag{40}$$

That is, we can solve

$$\sum_{i=1}^{m}\sigma_{ij}(t)\theta_j(t) = \alpha_i(t) - R(t),\ 1 \le i \le m \tag{41}$$

for Θ. Then

$$dS_i(t) = R(t)S_i(t)dt + S_i(t)\sum_{i=1}^{m}\sigma_{ij}(t)d\tilde{W}_j(t),\ 1 \le i \le m \tag{42}$$

It follows that

$$d[D(t)S_i(t)] = D(t)S_i(t)\sum_{i=1}^{m}\sigma_{ij}(t)d\tilde{W}_j(t) \tag{43}$$

Therefore, if we can find a measure $\mathbb{Q}$ under which $[\tilde{W}_1(t), ..., \tilde{W}_m(t)]$ is a m-dimensional Brownian motion, $\{D(t)S_i(t)\}_{t\ge 0}$ will be a martingale for each i, $1 \le i \le m$, and hence $\mathbb{Q}$ will be risk-neutral.

Theorem 2.5 Suppose (41) has a solution $\Theta(t)$, $0 \le t \le T$. Define

$$Z(t) = \exp\{-\int_0^t\sum_{i=1}^{n}\theta_j(u)dW_j(u) - \frac{1}{2}\int_0^t\sum_{i=1}^{n}\theta_j^2(u)du\} \tag{44}$$

If $\mathbb{E}_{\mathbb{P}}[Z(T)] = 1$ and we define $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T) \tag{45}$$

then $\mathbb{Q}$ is an equivalent, risk-neutral measure.

Remark 2.6 The equation of (41) is called the *market price of risk equation*.

Let's give an application: assume we have one risky asset and one dimensional Brownian motion, then

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

The market price of risk equation is:

$$\sigma(t)\theta(t) = \alpha(t) - R(t)$$

which has the solution:

$$\theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)} \tag{46}$$

if it is assumed $\sigma(t) > 0$ with probability 1 for all t.

Theorem 2.7 Assume there is a constant $K < \infty$ such that if $\theta(t)$ is defined as in (46) and

$$\mathbb{P}\left(|\theta(t)| \leq K \text{ for } 0 \leq t \leq T\right) = 1 \tag{47}$$

Then

$$\mathbb{E}[\exp\{-\int_0^T \theta(u)dW(u) - \frac{1}{2}\int_0^T \theta^2(u)du\}] = 1 \tag{48}$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\{-\int_0^T \theta(u)dW(u) - \frac{1}{2}\int_0^T \theta^2(u)du\} \tag{49}$$

defines a risk-neutral measure.

Corollary 2.8 If $R(t) = r$, $\alpha(t) = \alpha$ and $\sigma(t) = \sigma > 0$ are constant, then there exists a risk-neutral measure.

To finish up, let's do another application: assume we have 2 risky assets, 2-dimensional Brownian motion, $\alpha_i(t) = \alpha_i$, $R(t) = r$, $\sigma_{ij}(t) = \sigma_{ij}$ are all constant. The model is then

$$dS_1(t) = \alpha_1 S_1(t)dt + S_1(t)[\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)]$$
$$dS_1(t) = \alpha_2 S_2(t)dt + S_2(t)[\sigma_{21}dW_1(t) + \sigma_{22}dW_2(t)]$$

The market price of risk equation are then:

$$\sigma_{11}\theta_1(t) + \sigma_{12}\theta_2(t) = \alpha_1 - r$$
$$\sigma_{21}\theta_1(t) + \sigma_{22}\theta_2(t) = \alpha_2 - r$$

This will have a solution, a unique constant solution, $\theta_i(t) = \theta_i$, $i = 1, 2$, if $\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} \neq 0$. Since the θ_i's are constant

$$Z(T) = \exp\{-\theta_1(T)W_1(T) - \theta_2 W_2(T) - \frac{1}{2}[\theta_1^2 + \theta_2^2]T\}$$

and one can check directly that $\mathbb{E}_{\mathbb{P}}[Z(T)] = 1$. Hence, a risk-neutral measure will exist.

3 Reference

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