Stochastic Integration – II

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1 Second Order Taylor Polynomials and Remainder

1.1 Functions of a scalar variable

Let f be a function defined on $\mathbb{R}$ and let $a \in \mathbb{R}$, the second order Taylor polynomial of f at a is

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

It is the unique quadratic function satisfying $P_{2,a}(a) = f(a)$, $P'_{2,a}(a) = f'(a)$ and $P''_{2,a}(a) = f''(a)$. The remainder $R_a(x)$ is defined by

$$\begin{aligned} f(x) &= P_{2,a}(x) + R_a(x) \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + R_a(x) \end{aligned}$$

$P_{2,a}$ should be thought of as a quadratic approximation to f near a; $R_a(x)$ is the difference between f and $P_{2,a}$. How good is it?

Theorem 1.1 If f, f' and f'' exist and are continuous everywhere,

$$\lim_{x \to a} \frac{|R_a(x)|}{(x-a)^2} = 0 \tag{1}$$

Thus $|R_a(x)|$ is an order of magnitude smaller than the quadratic terms in $P_{2,a}(x)$ as $x \to a$.

1.2 Multi-variable Function

Now, we consider function of multiple arguments, $f : \mathbb{R}^{\ltimes} \mapsto \mathbb{R}$, i.e., $f(x_1, x_2, ..., x_n)$ for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Definition 1.1 $f \in C^2$ if f_{x_i}, f_{x_i,x_j} exist and are continuous for all i and j, $1 \le i,j \le n$. Here,

$$f_{x_i}(x) = \frac{\partial f}{\partial x_i}(x),\ f_{x_i,x_j}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

The second order Taylor polynomial of f at $a = (a_1, ..., a_n)$ is

$$P_{2,a}(x) = f(a) + \sum_{i=1}^{n} f_{x_i}(a)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i,x_j}(a)(x_i - a_i)(x_j - a_j)$$

In the case $n = 2$, $P_{2,a}(x)$ written out is

$$\begin{aligned} P_{2,a}(x) =& f(a) + f_{x_1}(a)(x_1 - a) + f_{x_2}(a)(x_2 - a_2) + \frac{1}{2}f_{x_1,x_2}(a)(x_1 - a)^2 \\ &+ f_{x_1,x_2}(a)(x_1 - a)(x_2 - a_2) + \frac{1}{2}f_{x_2,x_2}(a)(x_2 - a_2)^2 \end{aligned} \tag{2}$$

Similarly, $R_{2,a}(x)$ is defined as

$$f(x) = P_{2,a}(x) + R_a(x) \tag{3}$$

Theorem 1.2 If $f \in C^2$,

$$\lim_{x \to a} \frac{|R_a(x)|}{\sum_{i=1}^{n}(x_i - a_i)^2} = 0 \tag{4}$$

Again, $R_a(x)$ is of an order smaller than the term in $P_{2,a}(x)$ as $x \to a$.

1.3 Convenient Form

For function of a scalar variable, we can express Taylor formula as follows:

$$f(y + \Delta y) - f(y) = f'(y)\delta y + \frac{1}{2}f"(y)\Delta y^2 + o((\Delta y)^2) \tag{5}$$

Here, $\lim_{\Delta y \to 0} \frac{o(\Delta y)^2}{(\Delta y)^2} = 0$. For function of n-variables, replacing a by $y = (y_1, ..., y_n)$ and x by $x = y + \Delta y$, where $\Delta y = (\Delta y_1, ..., \Delta y_n)$,

$$f(y + \Delta y) - f(y) = \sum_{i=1}^{n} f_{x_i}(y)\Delta y_i + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i,x_j}(y)\Delta y_i \Delta y_j + o(\sum_{i=1}^{n}(\Delta y_i)^2) \tag{6}$$

2 Stochastic Calculus

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space, W is a Brownian motion defined on it and $\{\mathcal{F}_t\}_{t \ge 0}$ is a filtration for W.

2.1 Itō Process

Definition 2.1 A stochastic process of the form:

$$X(t) = X(0) + \int_0^t \beta(s)ds + \int_0^t \alpha(s)dW(s), \ t \leq T. \tag{7}$$

where (i) $\{\beta(s)\}_{0\leq s\leq T}$, $\{\alpha(s)\}_{0\leq s\leq T}$ are stochastic process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$; (ii) the integrals in (7) are well-defined; and (iii) $X(0)$ is $\mathcal{F}_0$-measurable, is called an Itō process.

Another notation we used to express (7) is the differential form:

$$dX(t) = \beta(t)dt + \alpha(t)dW(t) \tag{8}$$

But why bother with this formal, differential notation? Because differential can be given intuitive meanings that help one understand Itō calculus conceptually and that help guide modeling and computation using Itō process. As in ordinary calculus, $'dt'$ should be thought of as an infinitesimally small increment in the t variable. Differentials of a stochastic process should be thought of as forward increments of the process over a time interval of duration $'dt'$; thus

$$dX(t) = X(t+dt) - X(t), \ dW(t) = W(t+dt) - W(t) \tag{9}$$

Here, forward means that the differential at t is the increment over the time interval $[t, t+dt]$ going forward into the future. Then (8) can be interpreted as saying:

$$X(t+dt) - X(t) = \beta(t)dt + \alpha(t)[W(t+dt) - W(t)] \tag{10}$$

This expression makes clear that the change in X due to W over an infinitesimally small interval is the product of an $\mathcal{F}_t$-measurable random variable $\alpha(t)$ – remember, we assume $\alpha(\cdot)$ is adapted to the filtration – times the forward increment $W(t+dt) - W(t)$, which is independent of $\alpha(t)$. That Itō integral are built by adding up such products is a central concept of stochastic calculus. An integral is a limit of sum of increments, so formally, $X(t) - X(0) = \int_0^t dX(s)$. By replacing $dX(s)$ by $\beta(s)ds + \alpha(s)dW(s)$, one is led from the formal expression (8) back to (7). If $\alpha = 0$, the differential notation $dX(t) = \beta(t)dt$ gives a different way to express the ordinary integral $X(t) - X(0) = \int_0^t \beta(s)ds$.

2.2 Differentials and Modeling

Usually one constructs Itō process models starting from a differential point of view. To illustrate, we write down a generalized *Black-Scholes-Merton* model for the price of a risky asset. Let $\{S(t), 0 \leq t \leq T\}$ denote the price process. The return on owning one share over the time interval $[t, t+dt]$ is

$$\frac{S(t+dt) - S(t)}{S(t)} \tag{11}$$

Think of this as a random value that consists of a known expected rate of return

$$\alpha(t) = \frac{\mathbb{E}[\frac{S(t+dt)-S(t)}{S(t)}|\mathcal{F}_t]}{dt} \tag{12}$$

(We condition on $\mathcal{F}_t$ because the rate is known given the information in $\mathcal{F}_t$) plus an random fluctuation of zero mean around this rate. One way to model this is

$$\frac{dS(t)}{S(t)} = \frac{S(t+dt)-S(t)}{S(t)} = \alpha(t)dt + \sigma(t)dW(t) \tag{13}$$

where $\sigma(\cdot)$ is an $\mathcal{F}_t$-adapted volatility process. The fluctuation $\sigma(t)dW(t) = \sigma(t)(W(t+dt) - W(t))$ has variance $\sigma^2(t)dt$ and, since $W(t+dt) - W(t)$ is independent of the past $\mathcal{F}_t$, it has zero mean. The random input $W(t+dt) - W(t)$ driving the fluctuation in $dS(t)$ is independent of $\mathcal{F}_t$ and hence is not predictable in any way from past information. The model one gets from (13) is thus

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \tag{14}$$

where $\alpha(\cdot)$ and $\sigma(\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. This equation is an example of a stochastic differential equation, written in integral form, it is equivalent to:

$$S(t) = S(0) + \int_0^t \alpha(s)S(s)ds + \int_0^t \sigma(s)S(s)dW(s) \tag{15}$$

These equations do not give explicit formula for $S(t)$, $t \geq 0$. Rather they specify a condition that $S(t)$ should satisfy. Whether, a solution $S(t)$, $t \geq 0$, to these equation exists is a different matter that will be treated later. When α and σ are non-random and constant, we get the standard *Black-Scholes* price model:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \tag{16}$$

2.3 Products of Differentials

Since integrals are limits of sums over finer and finer partitions, it makes intuitive sense to think of

$$\int_0^t (dW(s))^2 = \lim_{||\Pi||\to 0} \sum_{i=}^{n} [W(t_{i+1}) - W(t_i)]^2 \tag{17}$$

where Π is a partition of $[0,t]$. But we know this limit is quadratic variation of Brownian motion leading to the formal identity:

$$\int_0^t (dW(s))^2 = [W,W](t) = t,\ t > 0 \tag{18}$$

Again, one should think of this identity in a purely formal way. By considering its differential form, it suggests the formal identity:

$$(dW(t))^2 = dt \tag{19}$$

To generalize this idea, we will express rigorous identity for quadratic variation and quadratic cross variation using products of differentials; that is

$$dX(t)dY(t) = d[X,Y](t) \tag{20}$$

where

$$[X,Y] = \lim_{||\Pi||\to 0} \sum_{i=1}^{n-1} [X(t_{i_1}) - X(t_i)][Y(t_{i+1}) - Y(t_i)] \tag{21}$$

It is easy to shown

$$\lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} [t_{i+1} - t_i]^2 = 0 \tag{22}$$

$$\lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} [t_{i+1} - t_i][W(t_{i+1}) - W(t_i)] = 0 \tag{23}$$

We will write

$$(dt)^2 = d[t,t] = 0 \tag{24}$$

$$dtdW(t) = d[t,W] = 0 \tag{25}$$

At the formal level, it is valid to apply the usual algebraic rule to products of differentials. Thus, if $dX(t) = \beta(t)dt + \alpha(t)dW(t)$,

$$dX(t)dt = \beta(t)(dt)^2 + \alpha(t)dW(t)dt = 0 \tag{26}$$

$$dX(t)dW(t) = \beta(t)dtdW(t) + \alpha(t)(dW)^2(t) = \alpha(t)dt \tag{27}$$

and

$$(dX(t))^2 = \beta^2(t)(dt)^2 + 2\beta(t)\alpha(t)dtdW(t) = \alpha^2(t)(dW(t))^2 = \alpha^2(t)dt \tag{28}$$

3 It$\bar{o}$'s Rule

Let $X(t)$, $0 \le t \le T$, be an It$\bar{o}$ process, and let f be a function of a real variable. We want to ask: is $f(X(t))$ an It$\bar{o}$ process; if so, what is $d[f(X(t))]$?

Let's first investigate the case when $dX(t) = \beta(t)dt$, that is, when

$$X(t) = X(0) + \int_0^t \beta(s)ds \tag{29}$$

our question above is answered by the chain rule from calculus.

Proposition 3.1 If f' exists everywhere and is continuous, then, for (29),

$$d(f(X(t))) = f'(X(t))dX(t) = f'(X(t))\beta(t)dt \tag{30}$$

Proof. By the fundamental theorem of calculus and the chain rule,

$$f(X(t)) - f(X(0)) = \int_0^t \frac{d}{ds} f(X(s))ds = \int_0^t f'(X(s))X'(s)ds \tag{31}$$

So $f(X(t)) - f(X(0)) = \int_0^t f'(X(s))\beta(s)ds$, which, in differential notation, is $d[f(X(t))] = f'(X(t))\beta(t)dt$. $\square$

Now, let's address the stochastic case: assume now that

$$dX(t) = \beta(t)dt + \alpha(t)dW(t) \tag{32}$$

and that $f \in C^2$. We will assume that $f(X(t))$ is Itō process. To formally compute $d[f(X(t))]$, we will formally approximate

$$f(X(t+dt)) - f(X(t)) \tag{33}$$

and keep only term of order dt; that is, any term which go to zero faster then dt will be discarded. Write

$$X(t+dt) = X(t) + dX(t) \tag{34}$$

By the Taylor polynomial approximation with $X(t)$ in place of y and $dX(t)$ in place of Δy,

$$\begin{aligned} d[f(X(t))] &= f(X(t+dt)) - f(X(t)) && (35)\\ &= f'(X(t))dX(t) + \frac{1}{2} f"(X(t))(dX(t))^2 + o((dX(t))^2) && (36)\end{aligned}$$

But from last section, we know that $(dX(t))^2 = \alpha^2(t)dt$ and so the $o((dX(t))^2)$ term goes to zero faster than dt and may be neglected. Thus

$$d[f(X(t))] = f'(X(t))\beta(t)dt + \frac{1}{2} f"(X(t))\alpha^2(t)dt + f'(X(t))\alpha(t)dW(t) \tag{37}$$

Let's summarize it as a theorem

Theorem 3.2 *(Itō's rule, case a)* Let $dX(t) = \beta(t)dt + \alpha(t)dW(t)$ be an Itō process. Let $f \in C^2$. Then $f(X(t))$ is an Itō process and

$$d[f(X(t))] = [f'(X(t))\beta(t) + \frac{1}{2} f"(X(t))\alpha^2(t)]dt + f'(X(t))\alpha(t)dW(t) \tag{38}$$

In integral form:

$$f(X(t)) = f(X(0)) + \int_0^t [f'(X(s))\beta(s) + \frac{1}{2} f"(X(s))\alpha^2(s)]ds + \int_0^t f'(X(s))\alpha(s)dW(s) \tag{39}$$

Remark 3.3 We almost proved the above theorem (except some technical stuff), let's switch to some comments instead of going into the details of the proof:

1. The novel term appearing in (38) and (39) that does not appear in the chain rule for function of ordinary integrals is the term

$$\frac{1}{2}f"(X(t))\alpha^2(t)dt \tag{40}$$

The fact this term appear is due to entirely to the fact that the quadratic variation of Brownian motion up to time t equals t for all t. This term called *Itō correction term*;

2. It can happen that $f \in C^2$ and

$$\mathbb{E}[\int_0^T [f'(X(t))\alpha(t)]^2 dt] = +\infty \tag{41}$$

However, we require this expectation to be finite to define the stochastic integral term:

$$\int_0^t f'(X(s))\alpha(s)dW(s),\ t \leq T \tag{42}$$

But, there is really no problem. One can extend the stochastic integral and define $\int_0^t \gamma(s)dW(s)$, $t \leq T$, assuming only that γ ia adapted and

$$\mathbb{P}\left(\int_0^T \gamma^2 ds < +\infty\right) = 1 \tag{43}$$

This condition will be satisfied for $\gamma(s) = f'(X(s))\alpha(s)$ if $\mathbb{P}\left(\int_0^T \alpha^2(s)ds < +\infty\right) = 1$. However, if it is really infinity, then it may not be a martingale.

Let's give an example:

Proposition 3.4

$$S(t) = S(0)\exp\{\int_0^t \sigma(s)dW(s) + \int_0^t [\alpha(s) - \frac{1}{2}\sigma^2(s)]ds\} \tag{44}$$

solves the price model:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \tag{45}$$

Proof. To show this let $f(x) = S(0)\exp(x)$ and set

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t [\alpha(s) - \frac{1}{2}\sigma^2(s)]ds \tag{46}$$

Then, $S(t) = f(X(t))$ and $f'(x) = f"(x) = f(x)$. Hence, by Itō's rule

$$\begin{aligned}
dS(t) &= [f'(X(t))[\alpha(t) - \frac{1}{2}\sigma^2(t)] + \frac{1}{2}f"(X(t))\sigma^2(t)]dt + f'(X(t))\sigma(t)dW(t) \\
&= S(t)[\alpha(t) - \frac{1}{2}\sigma^2(t) + \frac{1}{2}\sigma^2(t)]dt + S(t)\sigma(t)dW(t) \\
&= S(t)\alpha(t)dt + S(t)\sigma(t)dW(t)
\end{aligned}$$

□

But actually, it is the unique solution, that is, if we directly solve this stochastic differential equation, we will get (44). How? again, by Itō's formula. Let's apply it to $\ln S(t)$,

$$\begin{aligned} d(\ln S(t)) &= \frac{1}{S(t)} d(S(t)) - \frac{1}{2S^2(t)} (dS(t))^2 \\ &= \frac{1}{S(t)} \left[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right] - \frac{1}{2S^2(t)} \sigma^2(t)S^2(t)dt \\ &= (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t) \end{aligned}$$

Let's take integral from 0 to T,

$$\ln \frac{S(T)}{S(0)} = \int_0^T (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \int_0^T \sigma(t)dW(t) \tag{47}$$

that is equivalent to:

$$S(T) = S(0) \exp\{\int_0^T (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \int_0^T \sigma(t)dW(t)\} \tag{48}$$

Problem solved!!

To close our discussion, let's give another variants of Itō's rule:

Theorem 3.5 *(Itō's rule, case a)*Let $dX(t) = \beta(t)dt + \alpha(t)dW(t)$ be an Itō process. Let $f(t,x) \in C^{1,2}$, i.e., $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous, then for every $T \geq 0$,

$$\begin{aligned} f(T, X(T)) &= f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) \\ &+ \frac{1}{2} \int_0^T f_{xx}(x, X(t))d[X, X](t) \\ &= f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\beta(t)dW(t) \\ &+ \int_0^T f_x(t, X(t))\alpha(t)dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t))\alpha^2(t)dt \end{aligned}$$

The proof is similar to the previous case with algebraic complications.

4 Reference

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