Sequence

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1 Convergence of Sequence

Let's first give the definition of a sequence:

Definition 1.1 A *sequence* $\{p_n\}$ in X is a function $f : \mathbb{N} \mapsto X$, i.e., maps some point n to p_n, a point of X.

Now, we can talk about the concept of *convergence of a sequence*:

Definition 1.2 $\{p_n\}$ *converges* if $\exists p \in X$ such that $\forall \epsilon > 0$, $\exists N$, for $n \geq N$,

$$d(p_n, p) < \epsilon$$

We usually write $p_m \to p$ or $\lim_{n\to\infty} p_n = p$. Instead of saying p_n converges to p, we also use the term *limit*, i.e., p is a *limit* of p_n. Let's give a straight forward example:

Example 1.1 Consider the following sequence $\{p_n\}$,

$$p_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

then we can claim: $p_n \to 1$. To prove that, given ϵ, we need to find N that makes that work.

Proof. Given $\epsilon > 0$, choose $N = [\frac{1}{\epsilon}] + 1$. For $n \geq N$,

$$|p_n - p| = |\frac{n+1}{n} - 1| = |\frac{1}{n}| < \epsilon$$

$\square$

To study the convergence of sequence, let's work on the following true or false questions and justify if the statement is true.

- *A.* $p_n \to p$ and $p_n \to p'$ implies $p = p'$;

- *B.* $\{p_n\}$ is bounded implies p_n converges;
- *C.* $\{p_n\}$ converges implies $\{p_n\}$ is bounded;
- *D.* $p_n \to p$ implies p is a limit point of the range of $\{p_n\}$;
- *E.* p is a limit point of $E \subset X$ implies $\exists$ a sequence $\{p_n\}$ in E such that $p_n \to p$;
- *F.* $p_n \to p$ if and only if every $n.b.h.d$ of D contains all but finitely many p_n.

Among them, A, C, E and F are true. We now prove them one by one:

Proof. (A) Assume that $p_n \to p$, $p_n \to q$, let $\epsilon = d(p, q)$. Then $\exists N_p$, for $n \geq N_p$,

$$d(p_n, p) < \frac{\epsilon}{2}$$

Also, $\exists N_q$, for $n \geq N_q$,

$$d(p_n, q) < \frac{\epsilon}{2}$$

Let $N = \max\{N_p, N_q\}$, then for $n \geq N$,

$$\epsilon = d(p, q) \leq d(p, p_n) + d(q, p_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Contradiction! □

Proof. (C) Use $\epsilon = 1$, then $\exists N$, for $n \geq N$, $d(p_n, p) < 1$. Let $R = \max_{1, d(p_1,p), d(p_N,p)}$. So all $\{p_n\} \in B_R(p)$. □

Proof. (E) Choose p_n in $B_{\frac{1}{n}}(p)$, then $p_n \to p$. □

Proof. (F) Trivial. □

2 Subsequence and Cauchy Sequence

As a warm up, we first give simple but important theorem without proof (readers can exercise on it).

Theorem 2.1 Consider sequences $\{s_n\}$, $\{t_n\}$, which have the limits s and t, respectively, the followings are true:

- $\lim_{n\to\infty}(s_n + t_n) = s + t$;
- $\lim_{n\to\infty} cs_n = cs$;
- $\lim_{n\to\infty}(c + s_n) = c + s$;
- $\lim_{n\to\infty} s_n \cdot t_n = s \cdot t$.

We have the definition of a sequence, but sometimes we are interested a sub-collection of the elements in a sequence, this motivates the definition of subsequence, just as a set and its subset.

Definition 2.1 Given $\{p_n\}$ a sequence, let $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$, then $\{p_{n_i}\}$ is a subsequence.

Example 2.2 $\{p_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\}$, a subsequence of $\{p_n\}$ can be $\{\frac{2}{3}, \frac{4}{5}, \frac{5}{6}, \cdots\}$.

One may ask if $p_n \to p$, must any subsequence converges to p ? The answer is yes, because every $n.b.h.d$ of p contains all but finitely many points of p_n. A more interesting question will be: if the sequence does not converge, can it have a subsequence that converges? Let's give an example:

Example 2.3 $\{p_n\} = \{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \cdots\}$, it actually has two subsequences that converge, $\{\pi, \cdots\}$, $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$.

Subsequence can converge when sequence itself does not have a limit. Let's continue on questioning the subsequences. Must every sequence contains a convergent subsequence? No, natural number will be an anti-example. Lastly, if a sequence is bounded, must it have a convergent subsequence? The answer is negative again, consider the following example: $\{3, 3.1, 3.14, 3.141, \cdots\}$. Among them, the last question can be corrected to be true if we have a sequentially compacted space.

Definition 2.2 A *metric space* is *sequentially compacted* if every sequence has a convergent subsequence.

The following theorem says: in compact space, every sequence has a subsequence convergent to **a point of** X.

Theorem 2.4 If X is compact, X is sequentially compacted.

Proof. Let $R = \text{range}\{p_n\}$.

- If R is finite, then some p in $\{p_n\}$ is achieved infinitely many times. We can choose this subsequence;
- If R is infinite, then, since X is compact, any infinite subsequence has a limit point. Call this limit point p, we can construct a subsequence by choosing point from $B_{\frac{1}{n}}(p)$ while letting $n \to \infty$.

$\square$

Remark 2.5 In fact, sequentially compactness also implies compactness. But the proof is a little bit involved, we will skip. Readers, if interested in, can consult *wikiproof*.

There is a nice *corollary* follows immediately:

Corollary 2.6 (*Bolzano-Weierstrass Theorem*) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

3 Complete Space

As one of the most important concepts in real analysis, *Cauchy sequence* leads to many cheerful results. We will here touch some basics, in advanced class, you will see more applications of it. Up to now, for saying a sequence is convergent, we need to know what is the limit. But, what if we don't know the limit? The idea is the following: if the sequence does converge, then $\{p_n\}$ must becomes closer and closer. Let's give the definition:

Definition 3.1 The $\{p_n\}$ is *Cauchy* if $\forall \epsilon > 0$, $\exists N$ such that for $m, n \geq N$, we have

$$d(p_n, p_m) < \epsilon$$

Theorem 3.1 Cauchy sequence is a bounded sequence.

Proof. By definition, given any $\epsilon > 0$, there is an integer N such $|x_n - x_m| < \epsilon$ for all $n, m \geq N$. So, $\forall n, m \geq N$,

$$|x_n| - |x_m| \geq |x_n - x_m| < \epsilon$$

Taking $m = N$ and transposing, we have

$$|x_n| < |x_m| + \epsilon, \text{ for all } n \geq N$$

Thus, for all n,

$$|x_n| \leq \max\{|x_1|, ..., |x_{N-1}|, |x_N| + \epsilon\}.$$

A bound for $\{x_n\}$ is then $M = \max\{|x_1|, \cdots, |x_{N-1}|, |x_N| + \epsilon\}$. □

We can validate our reasoning above:

Theorem 3.2 If $\{p_n\}$ is convergent, it implies $\{p_n\}$ is *Cauchy*.

Proof. Given $\epsilon > 0$, $\exists N$ such that $n > N$ implies $d(p_n, p) < \frac{\epsilon}{2}$, $d(p_m, p) < \frac{\epsilon}{2}$. Thus, for this particular N, if $m, n > N$,

$$d(p_n, p_m) < d(p_n, p) + d(p_m, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Remark 3.3 The converse is not true in general.

To have the reverse implication true we need to have a *complete space.*

Definition 3.2 A metric space is complete, if every Cauchy sequence converges to some $x \in X$.

Obviously, $\mathbb{Q}$ is not complete, but $\mathbb{R}$ is complete. Are there any characterizations of complete space? Here is one:

Theorem 3.4 Compact metric space is in fact complete.

Proof. Since X is compact, its sequentially compact. There exists $\{x_{n_k}\}$ converging to a point of $x \in X$. On the other hand, we have for $\epsilon > 0$, $\{x_n\}$ is Cauchy implying there exists N_1 such that if $i, j \geq N_1$, $d(x_i, x_j) < \frac{\epsilon}{2}$, while $\{x_{n_k}\} \to x$ implies $\exists N_2$ such that for $n_k \geq N_2$, $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$, pick $n_k > N$, if $n \geq n > N$, then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

Corollary 3.5 $[0, 1]$ is complete and k-cell are complete.

Corollary 3.6 $\mathbb{R}^n$ is also complete.

Proof. If $\{x_n\}$ is Cauchy and bounded, it is contained in a ball. If it is in a ball, it must be in some k-cell, the assertion follows from above *corollary.* $\square$

Example 3.7 Does $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ converge ? Consider $n > m$,

$$|x_n - x_m| = \left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right) \geq \frac{n-m}{n} = 1 - \frac{m}{n}$$

Let $n = 2m$, $|x_{2n} - x_n| > \frac{1}{2}$, sequence is not Cauchy thus does not converge.

In general, metric space is not complete, but we really want to have Cauchy property of sequence, what can we do. Thank about $\mathbb{Q}$ and $\mathbb{R}$. Here is a question you may ask yourself, if we have an arbitrary metric space that is not complete, can it be embedded in a larger metric space that is complete. For example, $\mathbb{Q}$ can be embedded in $\mathbb{R}$.

Theorem 3.8 Every metric space (X, d) has a completion (X^*, d).

Proof. (Sketch of the idea) Given X(think about $\mathbb{Q}$), let (X^*) be the set of all Cauchy sequence under an equivalence relation $\sim$, where $\{p_n\}$ and $\{q_n\}$ are equivalent if $\lim_{n\to\infty} d(p_n, q_n) = 0$. For $P, Q \in X^*$, Let

$$\Delta(P, Q) = \lim_{m\to\infty} d(p_n, q_n)$$

Here $\{p_n\}$ and $\{q_n\}$ are representative of P and Q, because of the equivalence class. Then X^* is complete, with X isometrically embedded in X^*[1] $\square$

Remark 3.9 This can be thought as another way to construct $\mathbb{R}$ from $\mathbb{Q}$.

4 Bounded Sequence

We will talk about other concepts of sequences. Firstly, let's define the monotonicity of the sequence.

[1] *isometrically embedded:* there is bijection from subset of X^* that preserve the distance

Definition 4.1 A sequence $\{s_n\}$ is said to be *monotonically increasing* if $s_n \leq s_{n+1}$, $\forall n$; A sequence $\{s_n\}$ is said to be *monotonically decreasing* if $s_n \geq s_{n+1}$, $\forall n$.

Theorem 4.1 Bounded monotonic sequences converge to their *supremum* or *infinimum*(depending on increasing or decreasing).

Proof. Given $\{s_n\}$, let $s = \sup\{\text{range}\{s_n\}\}$. $\forall \epsilon > 0$, $\exists N$ such that $n > N$, $s - \delta \leq s_n \leq s$. But then for all $n > N$, $s_N \leq s_n < s$. So, this N works for ϵ. The argument for decreasing sequence is the same. □

Some sequences actually diverges just because they go on and on. That is, $s_n \to +\infty$, if for all $M \in \mathbb{R}$, $\exists N$ such that $n > N$, $|s_n| > M$. Similarly, $s_n \to -\infty$, if for all $M \in \mathbb{R}$, $\exists N$ such that $n > N$, $|s_n| < M$.

Given $\{s_n\}$, let $E =$ subsequential limits(here, we allow $+\infty$, $-\infty$). Let $s^* = \sup E$ (sometimes, called $\limsup s_n$ or upper limit) and $s_* = \inf E$ (called, $\liminf s_n$ or lower limit). Alternative way to think of them are:

$$\limsup s_n = \lim_{n\to\infty} (\sup_{k>n} s_n);$$
$$\liminf s_n = \lim_{n\to\infty} (\inf_{k>n} s_n).$$

What you do is actually to chop off the initial behavior of the sequence and look at those tail elements of the sequence. Obviously, if $s_n \to s$, then

$$\limsup s_n = \liminf s_n = \lim s_n = s$$