Discrete time model for Financial Market

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1 Financial Derivatives

We have spent about two classes on purely probability theory, where we finished up with discrete-time stochastic process. But how can this technical stuff related to financial market, in this section, we will give some background information of financial derivatives and point out its linkage between stochastic process.

1.1 Overview

A financial derivative is a contract between two parties for the exchange of money and/or assets at a future date. The date of exchange is called the *exercise time*. The speciality of the financial derivative is that the dollar value of the exchange is derived from the values, at exercise time, of *underlying variables*, according to a formula specified in the contract. They underlying variables are usually prices of assets traded in a market. We shall notice that the asset prices fluctuate as time progresses is unpredictable. Since the value at exercise time of a derivative contract is contingent on underlying variables, financial derivatives are also called *contingent claims*.

In principle, the underlyings of a derivative could be anything, including other financial derivatives, or even the weather. Typically though, financial derivatives ultimately depend on the values of basic assets traded in a financial market. By *'basic'*, we mean assets which involve ownership of real wealth (stocks, foreign currency, commodities)m or the contractual promise of future earnings, and which are actively traded on open markets. In this course, it is always assume that the underlyings are assets in financial market.

The defining feature of financial market is risk; we do not know and cannot predict exactly what future prices will be. Thus, returns on investments in financial assets are uncertain. We refer to this uncertain as 'risk', a term employed here only in general sense. There are serious theories attempting to define quantitative measure of risk and to analyze their behaviour, but it is beyond the scope of this class (maybe you have already learn it in "optimization models in finance", by Prof. Andrzej Ruszczyński).

We already learnt the discrete-time stochastic process, it can be used to model stock price movement in a discrete-time setting, i.e., $S(0), S(h), S(2h), ...$[1]. A discrete-time model is appropriate when trading or exercising the derivative is allowed only at a discrete set of times. However, to treat common financial derivatives, such as options or futures, which can be traded at any time up to the exercise time, continuous-time models capture market behaviour more accurately. We will learn it soon. But for now, let's stick with discrete-time model, our objective is to use them for derivative pricing.

1.2 Principle of derivative pricing

Much derivative pricing theory is based on the principle that markets should not admit arbitrage. Loosely speaking, an arbitrary is a strategy for allocating investments that produces money from nothing. Thus, if you take advantage of an arbitrage, money flows to you from other investor just because of the way prices are structured, not because your investments are producing wealth. For example, suppose you can find a bank wiling to loan you money at 5% interest and another bank offering 6% interest on your savings account. By borrowing from the first bank and depositing in the second you will make money with no initial investment of your own. Obviously such a situation is not stable and cannot last. Investors, called arbitrageurs, are constantly on the lookout for such arbitrage opportunities, and as they begin to exploit these opportunities, their actions will raise and lower demand in such a way to push markets back towards no-arbitrage conditions. At least that is the belief ! So while it is not true that there are no arbitrage opportunities in real markets, one expects them to be small and not last too long, leading to markets in which the no-arbitrage assumption is approximately correct. Of course, the reasoning here involves a bit of paradox. We postulate there is no arbitrage, because, in fact, it exists, but arbitrageurs rapidly take advantage of it !

1.3 Backgrounds on financial derivatives

In each derivative contract there is a buyer and seller, the buyer is said to have the long side of the derivative, or to be long in the derivative, or to be the holder the seller is said to be short the derivative.

As the simplest example, a *forward contract* is an agreement to transact a purchase in the future. The long party to a forward contract agrees to pay a stipulated sum of money for a stipulated quantity of an asset, T units of time into the future. T is called the time to maturity, the day on which the exchange takes place is called the *delivery date*. No money exchanges hands at the time the future contract is entered into, and both parties are obligated to fulfil the terms of the contact. The *payoff function* of the forward contract is the monetary value of the exchange to the long party at the delivery date. If $S(T)$ denotes the price per unit of asset, and if K is the price per unit agreed to in the forward contract, the payoff (per unit) is $S(T) - K$, since the buyer is paying K for a unit worth $S(T)$. Thus the payoff function is $V(s) = s - K$ as a function of price s.

[1]We use S for stock price

An *option* is a contract which gives it holder the right but not the obligation, to exercise the contract. The holder wil of course not exercise if doing so would incur a loss, so an option contract is equivalent to the long and short parties agreeing to exchange zero goods and money if exercising would mean a loss for the long party. In an option contract, the last date on which the option can be exercised is called the expiration date or expiry date. An option is of *European type* if it can only be exercised at expiry, while it is of *American type* if the buyer is free to choose any exercise time before expiry.

The basic (also called 'vanilla') options are European and American calls and put written on a single underlying. Typical underlyings of vanilla options are stocks and market indices. A call option gives its holder the right, but not obligation, to buy a unit of the underlying asset at specified price K, called the strike price. If T is the expiry date and $S(T)$ is the price of the underlying at T, the payoff per unit of asset, of a European call is

$$(S(T) - K)^+ := \max\{S(T) - K, 0\}$$

Indeed, if the price $S(T) < K$, the holder can buy the asset more cheaply in the market and will not exercise; if $S(T) > K$, the holder can exercise the option, buy a unit for price K, and immediately sell it in the market at prices $S(T)$ for a profit of $S(T) - K$. For the case of American option, define τ to be the exercise time if the holder chooses to exercise and to be T if he does not. Then the payoff is $(S(\tau) - K)^+$.

A put option gives the holder the right, but not the obligation to sell a unit of asset for the strike price K. For an European put, the payoff is $(K - S(T))^+$ and a similar formula can be obtained for the American put.

Remark 1.1 There are many other financial derivatives, such as *Asian option*, *future*, e.t.c., but in this course, we may not encounter them.

2 Discrete Model

2.1 Set-up and Basic Models

A discrete-time model for a market of N risky asset is composed of the following elements:

1. Trading times: $\mathcal{T} = \{t_0 = 0, t_1 = h, ..., t_k = kh, ..., t_n = T\}$;

2. Outcome space: $\Omega = \{\omega_1, \omega_2, ..., \omega_m\}$;

3. Risk-free asset: $B(0) = 1$ and $B(t_i) = e^{rih}$, $1 \leq i \leq n$ (*continuously compounded*);

4. Risky assets: For each risky asset i, $1 \leq i \leq N$, and each $\omega \in \Omega$, a function $S_i(t)(\omega)$, $t \in \mathcal{T}$, called the *price path*, that gives the price of asset i at time t if the market is in state ω;

Remark 2.1 (1) The trading time can be arbitrary, not necessary to be equally separated. We do this just for the convenience of notation; (2) The elements ω of this set label all the possible future states of the market admitted by the model. It is at this point that we incorporate risk. We don't know what the future state of the market will be, but it is the job of the model to list the possibilities. We will begin with the study of models for which Ω is a finite set. We don't yet put probabilities on these outcomes as part of our model, but there is a probabilistic idea lurking in the background.

Let's give several classical discrete models, which we will analyse in details later on:

(i) One period binomial model with one risky asset: One period means that the only trading times are $t_0 = 0$, and $t_n = T$, binomial means that the future economy has two states, which are labelled 1 and -1, hence $\Omega = \{d, u\}$. There is one risky asset, whose price at t is denoted by $S(t)(\omega)$, $t \in \mathcal{T}$. The model is specified by four parameters: $S(0)$ initial asset price, r risk free rate and return ration $0 < d < u$. If the future state of the economy is u ('up' state), $S(T)(u) = uS(0)$; if it is d ('down' state), $S(T)(d) = dS(0)$.

(ii) One period, multiple risky assets, general outcome space: The only trading times are 0 and T, and there is a money market with risk free rate r. But now we let Ω be arbitrary, and suppose that there are N assets. These are defined by N constants, $S_1(0), ..., S_N(0)$, the asset price at time 0, and N functions on Ω, $S_1(T)(\omega), ..., S_N(T)(\omega)$, representing asset prices at time T. Here is a particular example with $N = 1$. Let $\Omega = [a, b]$, where $0 < a < b$, and suppose $S_1(T)(\omega) = \omega S_1(0)$. This model allows for a continuous range of possible assets returns. Suppose $N = 2$, we have two risky assets S_1 and S_2, we want to model in such a way that two assets can fluctuate independently of one another. The binomial model for Ω is not rich enough to support a model of this situation.

(iii) Multi–period, binomial tree for one risky asset: This model extends the binomial model through multiple periods. In this model a constant rate of interest rate r is assumed, and is compounded continuously, thus, $B(t_k) = e^{rkh}$. Let

$$\Omega := \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \{u, d\} \text{ for each } i\}$$

When the economy moves up in period i, the return on the asset is u, when it moves down the return is d, where $0 < d < u$. Thus, the price paths for the single risky asset maybe generated through the recursion:

$$S(t_{i+1})(\omega) = \begin{cases} uS(t_i)(\omega), & \text{if } \omega_{i+1} = u; \\ dS(t_i)(\omega), & \text{if } \omega_{i+1} = d. \end{cases}$$

Notice that, as it should, $S(t_k)(\omega_1, ..., \omega_n)$ depends only on the market movements $\omega_1, ..., \omega_k$ in the first k periods. *Figure 1* shows a typical representation of a binomial tree model of three period:

2.2 Portfolio and Portfolio Processes

Consider an economy at time $t \in \mathcal{T}$ with N risky asset, whose prices are $S_1(t)(\omega), ..., S_N(t)(\omega)$, and with a risk free money market account of value $B(t)$. Let

$$A(t)(\omega) := (B(t), S_1(t)(\omega), ..., S_N(t)(\omega))$$

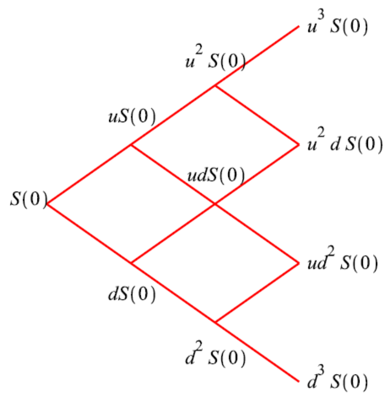

Figure 1: Binomial tree of three period

This is called the *asset price vector* at (t, ω). A portfolio is a list of the amounts of each asset held by an investor. It maybe represented as a vector in $\mathbb{R}^{N+1}$,

$$\Delta = [\Delta_0 \ \Delta_1 \ ... \ \Delta_N]^\top$$

where Δ_0 is the number of the money market account, and, for $i \geq 1$, Δ_i being the number of units of asset i, held in the portfolio. Thus, the monetary value of a portfolio Δ at time t, given the history of economy is ω, is

$$\Delta_0 B(t) + \sum_{i=1}^{N} \Delta_i S_i(t)(\omega) = A(t)(\omega) \cdot \Delta$$

The '$\cdot$' means inner product.

In a multi-period models, one can update the portfolio at each trading time and adjust the portfolio according to what happens in the market. In this case, we call it a *portfolio process*, which is a function which assigns to $\omega \in \Omega$ and trading times t_i for $i < n$, a vector

$$\Delta(t_i)(\omega) = [\Delta_0(t_i)(\omega) \ \cdots \ \Delta_N(t_i)(\omega)]$$

This vector represents the portfolio the investor chooses to hold over the time period from t_i to t_{i+1}. Let the *monetary value* of the portfolio $\Delta(t_i)(\omega)$ at time t_i be denoted by $W(t_i)(\omega)$, W is called the *wealth process*. Then

$$W(t_i)(\omega) = \Delta(t_i)(\omega) \cdot A(t_i)(\omega)$$

At the end of the period over which the portfolio is held, it is worth

$$\Delta(t_i) \cdot A(t_{i+1})(\omega)$$

The portfolio process $(\Delta(t_0), ..., \Delta(t_n))$ is called *self-financing* if for each i, $0 \leq i \leq n-1$,

$$W(t_{i+1}) = \Delta(t_i) \cdot A(t_{i+1})(\omega)$$

It means there is no infusion of new cash to invest at each time spot. All the wealth in a self-financing portfolio results from investment earning on the original amount of money $W(t_0) = \Delta(t_0) \cdot A(t_0)$ in the portfolio. $W(t_0)$ is called the *initial endowment*.

2.3 Arbitrage

As we mentioned before, we shall always assume there exists no arbitrage. Let's first give the definition of *no arbitrage*,

Definition 2.1 An arbitrage, is an admissible (not allowing to look into the future when adjust the weight of each assets in the portfolio), self-financing portfolio process, whose wealth process $W(t)$ satisfies one of the following properties:

1. $W(0) < 0$ and $W(T)(\omega) \geq 0$ for all $\omega \in \Omega$; or
2. $W(0) = 0$, and $W(T)(\omega) \geq 0$ for all $\omega \in \Omega$ and there is at least one $\omega' \in \Omega$, such that $W(T)(\omega') > 0$.

In the first case, we start by borrowing and end up with at least 0 no matter what happens in the economy; in the second case, we start with nothing and, at the end, we are sure not to have lost anything and for at least one possible outcome of the future, we end up with a profit.

(i) Condition for arbitrage in the one-period binomial model: In this case, we claim: **there is no arbitrage if and only if** $d < e^{rT} < u$.

To justify it, we first express very explicitly what must occur mathematically for an arbitrage opportunity to exist. In the one period model, we get to choose a portfolio at $t_0 = 0$ only. Call this portfolio $\Delta = [\Delta_0\ \Delta_1]^\top$. At the beginning of the period, $W(0) = \Delta_0 + S(0)\Delta_1$, while by the end, it becomes $W(T)(\omega) = e^{rT}\Delta_0 + S(T)(\omega)\Delta_1$. There are just two possibilities: $W(T)(u) = e^{rT}\Delta_0 + uS(0)\Delta_1$ and $W(T)(d) = e^{rT}\Delta_0 + dS(0)\Delta_1$. Hence, Δ is an arbitrage if either

$$\Delta_0 + S(0)\Delta_1 < 0 \text{ and } e^{rT}\Delta_0 + yS(0)\Delta_1 \geq 0, \text{ for } y = u \text{ or } d,$$

or

$$\Delta_0 + S(0)\Delta_1 = 0 \text{ and } e^{rT}\Delta_0 + yS(0)\Delta_1 \geq 0 \text{ for } y = u \text{ or } d, \\ \text{and at least one is } > 0,$$

Graphically, in the (Δ_0, Δ_1)-plane, we can draw the $\Delta_0 + S(0)\Delta_1 \leq 0$, it is the region below the line $\Delta_1 = -\Delta_0/S(0)$ (see *figure* 2). The next figure (*figure* 3) shows the region $e^{rT}\Delta_0 + dS(0)\Delta_1 \geq 0$ and $e^{rT}\Delta_0 + uS(0)\Delta_1 \geq 0$. In this figure, the line marked L_d is the graph of $e^{rT}\Delta_0 + dS(0)\Delta_1 = 0$, and the line marked L_u is the graph of $e^{rT}\Delta_0 + uS(0)\Delta_1 \geq 0$.

At every point of the shaded region in *figure* 3, including the boundaries, except for the origin $(0,0)$, at least one of $e^{rT}\Delta_0 + dS(0)\Delta_1$ and $e^{rT}\Delta_0 + uS(0)\Delta_1$ is strictly positive. The origin $(0,0)$ does not correspond to an arbitrage. Thus, the one period, binomial model admits arbitrage if and only if the shaded regions of *figure* 2 and *figure* 3 intersect in a point other than the origin. It is clear then that there is no arbitrage if and only if

$$\text{slop}(L_u) > \text{slope}(L_0) > \text{slope}(L_d)$$

as in *figure* 4. This is equivalent to

$$-\frac{e^{rT}}{uS(0)} > -\frac{1}{S(0)} > -\frac{e^{rT}}{dS(0)}$$

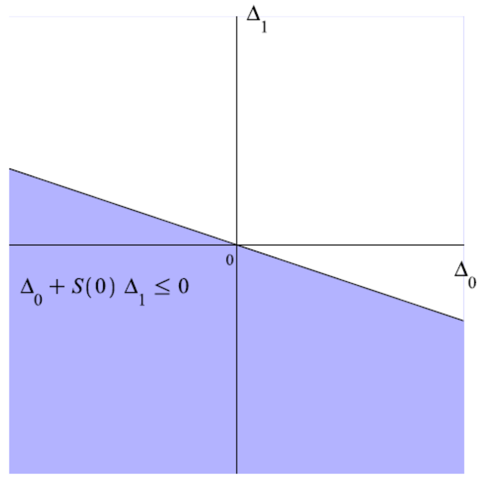

Figure 2: Region of portfolios corresponding to non-positive initial endowment

which proves the claim.

(ii) Multi-period, binomial model: In multi-period binomial model, exactly the same condition needs to be hold, $d < e^{rh} < u$, with the adjustment of duration. Let's give an informal argument. Let $W(t_i)(\omega)$ be a self-financing wealth process. One possibility is that $W(0) < 0$ and $W(t_n) \geq 0$ for all ω. Pick any $\omega = (\omega_1, ..., \omega_k)$, and consider the sequence $W(0), W(t_1)(\omega), W(t_2)(\omega)$,.... Since one can not look into the future, it must be true that each $W(t_i)$ really only depends on $(\omega_1, ..., \omega_i)$, so it can be written as $W(t_i)(\omega_1, ..., \omega_i)$. Since this is the wealth process of an arbitrage portfolio, by following the wealth process along the nodes of the binomial tree there must be some k, $0 \leq k < n$ such that $W(t_k)(\omega_1, ..., \omega_k) < 0$ but $W(t_{k+1})(\omega_1, ..., \omega_k, \eta) \geq 0$, whether $\eta = u$ or $\eta = d$. This means we can achieve an arbitrage in this single period, binomial model starting at the node corresponding to $(\omega_1, ..., \omega_k)$ in the binomial tree. But, we know this cannot happen if $d < e^{rh} < u$. Hence an arbitrage portfolio starting from $W(0) < 0$ is not possible. A similar argument rules out arbitrage starting from $W(0) = 0$.

3 Pricing Contingent claims

3.1 Elementary theory for one-period model

Suppose a financial market model is given that admits no arbitrage. As usual, T will denote the end of the final period. A function V on Ω is called a contingent claim. For example, the payoff $(S(T)(\omega) - K)^+$ of a *European call* at *strike* K and *expiry* T, as a function on Ω, is a contingent claim. Although any such function V can be a contingent claim, at least mathematically, in practice this term is only applied to recognized financial derivatives. The interpretation is that V represents a derivative that pays V to the holder at time T. The question is what is a fair price to charge to someone who wishes to buy this derivative at

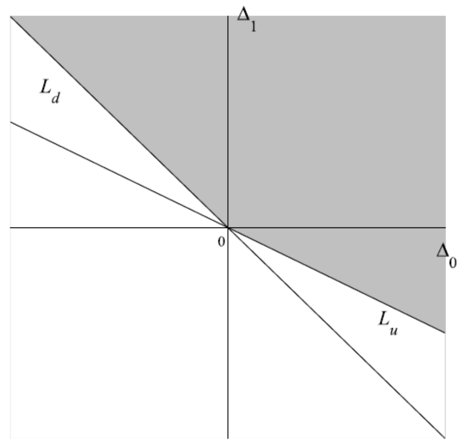

Figure 3: Portfolios always resulting in non-negative wealth at T

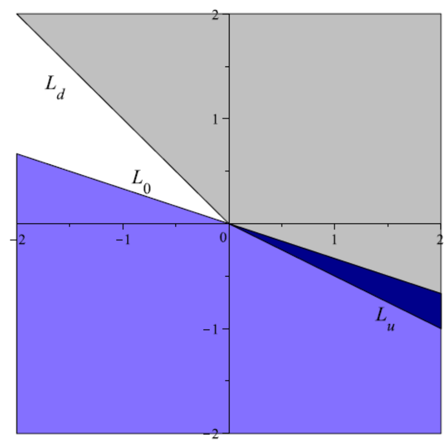

Figure 4: Situation allowing no arbitrage, binomial model

time $t = 0$?

When $\Omega = \{\omega_1, ..., \omega_m\}$, it will be convenient to represent a contingent claim V by the m-dimensional vector,

$$[V(\omega_1) \;\cdots\; V(\omega_m)]^\top$$

Let $V(T)$ be a contingent claim (the parameter T is inserted here to indicate the payoff is at T.) An admissible , self-financing portfolio Δ, with associated wealth process $W_\Delta(t)(\omega)$ is said to replicate $V(T)$ if

$$W_\Delta(T)(\omega) = V(T)(\omega) \text{ for all } \omega \in \Omega$$

A contingent claim for which there exists a replicating portfolio is said to be *attainable.* A market model is said to be *complete* if every contingent claim is *attainable.* In the one-period model, how attainability looks like? Well, it is just simple linear algebra. Let there be N risky assets and a money market account. Recall the asset price vector $A(T)(\omega) = (B(T), S_1(T)(\omega), ..., S_N(T)(\omega)$. List these for all ω in the asset price matrix

$$A(T) = \begin{pmatrix} B(T) & S_1(T)(\omega_1) & \cdots & S_N(T)(\omega_1) \\ \cdots & \cdots & \cdots & \cdots \\ B(T) & S_1(T)(\omega_m) & \cdots & S_N(T)(\omega_m) \end{pmatrix}$$

A portfolio Δ held over the period from 0 to T will, for each ω, have the value $A(T)(\omega) \cdot \Delta$. Thus $V(T)$ will be attainable if there exists a vector Δ such that

$$A(T) \cdot \Delta = [V(\omega_1) \ \cdots \ V(\omega_m)]^\top$$

Thus the set of attainable contingent claim is equal to the column space of the matrix $A(T)$, and the market is complete if the rank of $A(T)$ is m (we implicitly assume that the number of scenarios is larger than the number of stocks in the portfolio).

Example 3.1 Consider a market with a money market account at rate r and an asset with price $S(t)(\omega)$. The payoff function is therefore $V = S(T) - K$ (maturing at T with delivery price K). The portfolio $\Delta_0 = -e^{-rT}K$, $\Delta_1 = 1$ replicates this forward payoff. It requires borrowing $e^{-rT}K$ at the risk-free rate and purchasing one share of stock. At time T, the owner of this portfolio owe $e^{rT}e^{-rT}K = K$ to the lender and own a share of asset worth $S(T)(\omega)$. Hence, $W(T)(\omega) = S(T)(\omega) - K$. This is true whatever be the model for $S(T)(\omega)$. The value of this replicating portfolio at time $t = 0$ is

$$W(0) = S(0) - e^{-rT}K$$

Let's again visit our one-period binomial model:

Example 3.2 We assume no arbitrage is allowed, that is $d < e^{rT} < u$. We claim: **there is an unique replicating portfolio for each contingent claim in this market.**

Indeed, there are two unknowns and two equations,

$$\begin{aligned} e^{rT}\Delta_0 + dS(0)\Delta_1 &= V(d) \\ e^{rT}\Delta_0 + uS(0)\Delta_1 &= V(u) \end{aligned}$$

it has the unique solution

$$\begin{aligned} \Delta_0 &= \frac{1}{e^{rT}(u-d)} \left(uV(1) - dV(1)\right) \\ \Delta_1 &= \frac{1}{S(0)u - S(0)d}(V(u) - V(d)) \end{aligned}$$

This means we need to buy Δ_1 shares of stocks and the rest of them in bonds to replicate the contingent claim. The value of this portfolio at time $t = 0$ is thus

$$W(0) = \Delta_0 + S(0)\Delta_1 = \left(\frac{1 - de^{-rT}}{u-d}\right) V(d) + \left(\frac{ue^{-rT} - 1}{u-d}\right) V(u)$$

Consider a financial market that does not admit arbitrage. Now introduce a contingent claim $V(T)$ for which there is a replicating portfolio with wealth process W. Let $V(0)$ denote the price at time 0 for the contingent claim. The unique value of $V(0)$ for which there is no arbitrage is

$$V(0) = W(0)$$

This is easy to see: if $W(0) < V(0)$, you can sell a contingent claim for $V(0)$, invest $W(0)$ of this according to the replicating portfolio, and invest $X(0) - V(0)$ at risk-free rate. At time T, the replicating portfolio will yield V and allow you to pay the contingent claim to the holder, and you are left with a guaranteed profit of $e^{rT}(W(0) - V(0))$. If at $t = 0$, $V(0) < W(0)$, you can buy a contingent claim and borrow a replicating portfolio – that is to borrow Δ_0 dollars and borrow Δ_1 units of the risky asset. The initial value of this strategy to you is $V(0) - W(0) < 0$, since $W(0)$ is the value of the replicating portfolio. At time T, you will owe the value $W(T)$ to the lender of the replicating portfolio. But the contingent claim you own provides you with exactly $V(T) = W(T)$ to pay off this debt. Thus, you start from a position of negative wealth and end up with zero no matter what happens, which is again an arbitrage!

Example 3.3 We saw in *example 3.1*, the forward can be replicated by a portfolio with initial value $e^{-rT}K - S(0)$. Hence, by no-arbitrage principle, this is what the forward contract is worth at time $t = 0$. Forward contracts are set so that there is no initial exchange of money. This means that K should be set so that $K = e^{rT}S(0)$. This is called the forward price, and sometimes denoted $F_0 = e^{rT}S(0)$.

Example 3.4 In the one period binomial model. By *example 3.2* and the no-arbitrage principle, the price at time $t = 0$ of contingent claim V is

$$V(0) = e^{-rT}\left(\frac{u - e^{rT}}{u - d} \ \frac{e^{rT} - d}{u - d}\right) \cdot (V(d) \ V(u))^{\top}$$

The vector $e^{-rT}\left(\frac{u-e^{rT}}{u-d} \ \frac{e^{rT}-d}{u-d}\right)$ is called the *state-price vector*. Note that its components are positive and sum to e^{-rT}, which is the discount factor for computing the present value, at $t = 0$, of future cash payments at $t = T$.

3.2 The Fundamental Theorem of Asset Pricing for One-period, Finite Scenarios Models

Consider a one-period model in which Ω is finite and there are N risky assets with price processes $S_1, ..., S_N$. First we explain the general concept of a *state-price vector*. We saw in *example 3.4* that for the one period binomial model there is a vector

$$\mathbf{p} = (p_1 \ p_2) = e^{-rT}\left(\frac{u - e^{rT}}{u - d} \ \frac{e^{rT} - d}{u - d}\right)$$

so that the no-arbitrage price of any contingent claim defined by $V(T) = (V(T)(d)\ V(T)(u))^\top$ is $V(0) = \mathbf{p} \cdot V(T)$. In particular, a money market investment of $B(0) = 1$ at $t = 0$ pays e^{rT} at time T, so in this case $V(T)(d) = V(T)(u) = e^{rT}$. Indeed,

$$(p_1\ p_2) \cdot (e^{rT}\ e^{rT}) = e^{rT}(p_1 + p_2) = 1 = B(0)$$

Likewise, an investment in one unit of stock pays out $V(T)(d) = dS(0)$, $S(T)(u) = uS(0)$ and we have

$$(dS(0),\ uS(0)) \cdot (p_1,\ p_2) = S(0)$$

These are two linear equations in two unknowns p_1, p_2 and hence they determine the price vector. A state-price vector for a more general, one-period model is defined by generalizing these conditions on the assets of a model.

Consider a one period model with $\Omega = \{\omega_1, ...\omega_m\}$, one risk-free asset with rate r, and N risky assets. In this model, the value of the money market is given by $B(0) = 1$ and $B(T)(\omega) = e^{rT}$ for all ω. The ratio e^{-rT} is called the *discount factor*. A *state price vector* is vector $\mathbf{p} = (p_1, ..., p_m)$ such that

$$\sum_{i=1}^{m} p_i = e^{-rT}$$

and

$$S_i(0) = (S_i(T)(\omega_i), ..., S_i(T)(\omega_m)) \cdot \mathbf{p} \text{ for } 1 \leq i \leq N$$

In vector-matrix notation, a state-price vector $\mathbf{p}$ is any solution, whose components are all positive,

$$\mathbf{p} \cdot A(T) = (p_1\ p_2) \cdot \begin{pmatrix} B(T) & S_1(T)(\omega_1) & \cdots & S_N(T)(\omega_1) \\ \cdots & \cdots & \cdots & \cdots \\ B(T) & S_1(T)(\omega_m) & \cdots & S_N(T)(\omega_m) \end{pmatrix} = (1\ S_1(0)\ ...\ S_N(0))$$

If $\mathbf{p}$ is a state-price vector,

$$\mathbf{q} = (q_1\ ...\ q_m) := e^{rT}(p_1\ ...\ p_m)$$

is a vector with positive components that solves

$$\sum_{i=1}^{m} q_i = 1$$

and

$$S_i(0) = e^{-rT}\ (S_i(T)(\omega_1)\ ...\ S_i(T)(\omega_m)) \cdot \mathbf{q} \text{ for } 1 \leq i \leq N$$

Conversely, if $\mathbf{q}$ solves these equations, $e^{-rT}\mathbf{q}$ is a state-price vector. It will be convenient to always represent state price vectors in this form. Now, we come to a major result:

Theorem 3.5 *(Fundamental Theorem of Asset Pricing for the One-period Model)* A one-period model with $\Omega = \{\omega_1, ..., \omega_m\}$, N risky assets, and a risk-free asset with interest rate r is arbitrage-free if and only if there exists a state price vector:

$$e^{-rT}(q_1 \; ... \; q_m) \tag{1}$$

In this case the no-arbitrage price at time $t = 0$ of an attainable contingent claim with payoff $V(T)(\omega)$ at time T is

$$V(0) = e^{-rT}\mathbf{q} \cdot V(T) = e^{-rT} \sum_{i=1}^{m} q_i V(T)(\omega)$$

Proof. (*partial proof*) Suppose a state-price vector exists, we shall show there cannot be an arbitrage. For arbitrary portfolio $\Delta = (\Delta_0, ..., \Delta_p)^\top$, let $W(T)(\omega_i) = A(T)(\omega_i) \cdot \Delta$ be the value of this portfolio at (T, ω_i). Then

$$(W(T)(\omega_1) \; ... \; W(T)(\omega_m))^\top = A(T) \cdot (\Delta_0 \; ... \; \Delta_N)^\top$$

Thus,

$$\begin{aligned}
\mathbf{p} \cdot (W(T)(\omega_1) \; ... \; W(T)(\omega_m))^\top &= \mathbf{p} \cdot A(T) \cdot (\Delta_0 \; ... \; \Delta_N)^\top \\
&= (1 \; S_1(0) \; ... \; S_N(0)) \cdot (\Delta_0 \; ... \; \Delta_N)^\top \\
&= \Delta_0 + \sum_{i=1}^{N} \Delta_i S_i(0) = W(0)
\end{aligned}$$

The last expression is the value of the portfolio at time 0. Now suppose that $W(T)(\omega_i) \geq 0$ for all ω_i, then the first term,

$$\mathbf{p} \cdot (W(T)(\omega_1) \; ... \; W(T)(\omega_m))^\top$$

since all the components of $\mathbf{p}$ are positive, and so it follows that $W(0) \geq 0$. Therefore, there is no portfolio that gives an arbitrage of the first type. A similar argument shows there can be no arbitrage of the second type.

It is more subtle to show that if there is no arbitrage, a state-price vector must exist. This requires some theory of convex analysis and will be omitted.

Once we have a state-price vector, it is also relatively straightforward to derive the pricing formula. Let Δ replicate the contingent claim $V(T)$, and let $W(0)$ and $W(T)(\omega)$ be the values of this replicating portfolio at times 0 and T. Of course, $W(T)(\omega) = V(T)(\omega)$ for all ω. By the no-arbitrage principle, $V(0) = W(0)$ is the no-arbitrage price of the claim. Working the equations, taken in the reverse direction,

$$W(0) = \mathbf{p} \cdot W(T) = e^{-rT}\mathbf{q} \cdot V(T)$$

This completes the proof. □

3.3 Written in the form of $\mathbb{Q}$-Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $c_1, c_2, ...$ are real numbers and if $A_1, A_2, ...$ are disjoint events in $\mathcal{F}$ then

$$X = \sum_{i=1}^{\infty} c_i \mathbf{1}_{A_i}$$

is a *discrete random variable.* The sum maybe finite:

$$X = \sum_{i=1}^{K} c_i \mathbf{1}_{A_i}$$

X takes values only in the set $\{c_1, c_2, ...\}$ and $\mathbb{P}(X = c_i) = \mathbb{P}(A_i)$. Conversely, assume X is a random variable whose possible values lie in the set of distinct numbers, $\{c_1, c_2, ...\}$. Let $A_i = \{\omega : X(\omega) = c_i\}$, for each i. Then

$$X = \sum_{i=1}^{\infty} c_i \mathbf{1}_{A_i}$$

All random variables on discrete Ω are discrete.

Example 3.6 *one period model:* $\Omega = \{u, d\}$. Assume $\mathbb{P}(\{1\}) = p$, $\mathbb{P}(\{-1\}) = q = 1 - p$. Let

$$S(T)(\omega) = uS(0)\mathbf{1}_u(\omega) + dS(0)\mathbf{1}_d(\omega)$$

Then, a contingent claim:

$$V(T)(\omega) = V(T)(u)\mathbf{1}_u(\omega) + V(T)(d)\mathbf{1}_d(\omega)$$

Let us recall the definition of expectation, if $X = \sum_{i=1}^{\infty} c_i \mathbf{1}_{A_i}$, the integral of X with respect to $\mathbb{P}$ is defined to be

$$\int_{\Omega} X(\omega)\mathbb{P}(d\omega) = \sum_{i=1}^{\infty} c_i \mathbb{P}(A_i)$$

if the sum converges. This integral is called the expected value of X w.r.t $\mathbb{P}$ and written $\mathbb{E}_P$.

Example 3.7 In the example above,

$$\mathbb{E}_P[S(T)] = qdS(0) + puS(0)$$

For a contingent claim

$$\mathbb{E}_P[V(T)] = qV(T)(d) + pV(T)(d)$$

Let $d < e^{rT} < u$. Then there exists unique probability measure $\mathbb{Q} = (q_1\ q_2)$ such that

$$\mathbb{Q}(u) = q_1 = \frac{ue^{-rT} - 1}{u - d},\ \mathbb{Q}(d) = q_2 = \frac{1 - de^{-rT}}{u - d},$$

And we can write

$$S(0) = e^{-rT}[q_1 S(T)(u) + q_2 S(T)(d)] = e^{-rT}\mathbb{E}_Q[S(T)]$$

and, for any contingent claim, the no arbitrage price is:

$$V(0) = e^{-rT}[q_1 V(T)(u) + q_2 V(T)(d)] = e^{-rT}\mathbb{E}_Q[V(T)]$$

Let's restate the *Theorem 3.5.*

Definition 3.1 In one-period model with $\Omega = \{\omega_1, \omega_2, ..., \omega_m\}$, risky assets $S_1(t), ..., S_N(t)$, and a money market account at risk-free rate r. A measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is said to be *risk neutral* if $\mathbb{Q}(\{\omega_j\}) > 0$ for each j, and if for each risky asset i, $1 \leq i \leq N$,

$$S_i(0) = e^{-rT}\mathbb{E}_Q[S_i(T)]$$

Theorem 3.8 If $\mathbb{Q}$ is a risk-neutral measure,

$$e^{-rT}\left(\mathbb{Q}(\{\omega_1\})\ ...\ \mathbb{Q}(\{\omega_m\})\right)$$

is a *state-price vector*. Conversely, a *state-price vector* defines a *risk-neutral measure.*

Theorem 3.9 The one-period model is arbitrage free if and only if there exists a risk-neutral measure $\mathbb{Q}$. In this case, if $V(T)$ is an attainable contingent claim, its unique , no arbitrage price is

$$V(0) = e^{-rT}\mathbb{E}_Q[V(T)]$$

3.4 Multi-period binomial model

We have the outcome space Ω, whose elements describe the possible future market histories is the set $\{u, d\}^n$ of all sequences $\omega = (\omega_1, ..., \omega_n)$, where each ω_i is either d or u and indicates which one of two possible movements the market makes; if $S(t_j)(\omega)$ denotes the price of risky asset, $\omega_i = d$ implies $S(t_i)(\omega) = dS(t_{i-1})(\omega)$ and $\omega_i = u$ implies $S(t_i)(\omega) = uS(t_{i-1})(\omega)$. Here $0 < d < u$ are fixed parameters, the same for all time periods. The risk-free money market account is specified by $B(t_i) = e^{rhi}$. It shall be assumed henceforth that $d < e^{rT} < u$, which implies the model is arbitrage free.

A contingent claim with payoff at time T is specified by a function $H(\omega)$ on the set Ω: for each $\omega = (\omega_1, ..., \omega_n) \in \Omega$, $H(\omega)$ is what the contingent claim pays at time T if the market has followed the path ω. It will be assumed that a contingent claim can be traded at all times. The price of the contingent claim at time t_i as a function of ω is denoted $V(t_i)(\omega)$. Of course, we have,

$$V(T)(\omega) = H(\omega),\ \text{for all}\ \omega \tag{2}$$

The price $V(t_i)(\omega)$ can only depend on the market history $(\omega_1, ..., \omega_i)$ up to time t_i, in other words, it depends on the σ-algebra $\mathcal{F}_i$, so really we can and will write $V(t_i)(\omega) = V(t_i)(\omega_1, ..., \omega_i)$. The question we want to answer is: what should the price process $V(t_i)(\omega)$, $0 \le i \le n$, $\omega \in \Omega$, be in order that the market remains arbitrage-free. It turns out that the no-arbitrage condition implies a unique price for all t_i and ω. It will be shown how to calculate this price.

Fix a k and suppose, hypothetically, that the price $V(t_{k+1})(\omega_1, ..., \omega_k, \omega_{k+1})$, for all $(\omega_1, ..., \omega_k, \omega_{k+1})$, are known for the contingent claim at the end of the next period. Now suppose that we have arrived at time t_k and have observed the market history $(\omega_1, ..., \omega_k)$ upto that point. There are then two possibilities for the market history up to t_{k+1}; either $(\omega_1, ..., \omega_k, -1)$ or $(\omega_1, ..., \omega_k, 1)$.

Owning the contingent claim is equivalent to owning a claim that pays $V(t_{k+1})(\omega_1, ..., \omega_k, -1)$ at time t_{k+1} in the first case, or $V(t_{k+1})(\omega_1, ..., \omega_k, 1)$ in the second. We know from the analysis of the one-period problem, that one can achieve an arbitrage over the $(k+1)^{\mathrm{st}}$ unless the price of the claim at t_k and $(\omega_1, ..., \omega_k)$ is

$$V(t_k)(\omega_1, ..., \omega_k) = e^{-rh}[\tilde{q}V(t_{k+1})(\omega_1, ..., \omega_k, -1) + \tilde{p}V(t_{k+1})(\omega_1, ..., \omega_k, 1)] \tag{3}$$

where

$$\tilde{q} = \frac{u - e^{rh}}{u - d}, \ \tilde{p} = \frac{e^{rh} - d}{u - d}$$

Equation (3) is a backward-in-time recursive formula, called a *dynamic programming equation*, that the price of the contingent claim must satisfy at all times t_k, $0 \le k \le n-1$ in order that there be no arbitrage.

Equation (3) and (2) uniquely determine the price, at all times and all ω, of a contingent claim paying $H(\omega)$ at time $T = nh$. By applying (3) for $k = n-1$,

$$V(t_{n-1})(\omega_1, ..., \omega_{n-1}) = e^{-rh}[\tilde{q}V(T)(\omega_1, ..., \omega_{n-1}, -1) + \tilde{p}V(T)(\omega_1, ..., \omega_{n-1}, 1)] \tag{4}$$

$$= e^{-rh}[\tilde{q}H(\omega_1, ..., \omega_{n-1}, -1) + \tilde{p}H(\omega_1, ..., \omega_{n-1}, 1)] \tag{5}$$

This gives the price at time t_{n-1} for all states of the market. Now the price at time t_{n-2} can be found from $V(t_{n-1})$ by using (3) with $k = n-2$. Working backwards this way step by step, one finally is able to compute the price of the contingent claim for all times and all states.

Notice that since $\tilde{q}$ and $\tilde{p}$ remain the same for all period, we can have risk-measure $\mathbb{Q}$, and all price path can be assigned a probability in terms of $\tilde{p}$ and $\tilde{q}$, thus (3) can be written in terms of $\mathbb{Q}$-conditional-expectation:

$$V(t_k) = e^{-rh}\mathbb{E}_Q[V(t_{k+1})|\mathcal{F}_i] \tag{6}$$

and also (4),

$$V(t_{n-1}) = e^{-rh}\mathbb{E}_Q[H(T)|\mathcal{F}_{t_{n-1}}] \tag{7}$$

Also, this model can be easily extended to finite outcome spaces. We just need to generalize the one-period model, which we already did in *theorem 3.5*. Then, *equation (6) and (7)* remain the same.

4 Reference

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