Swap Pricing With Collateral

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1 Initial Set-up

1.1 Concepts

In this section, let's discuss some generic pricing of collateralized trades. Under collateral agreement, the firm receives the collateral from the counter-party when the present value of the contracat is positive, and needs to pay the margin called "collateral rate" on the outstanding collateral to the payer. One can think of the counter-party deposits his capital in your collateral account. The most commonly used collateral is a currency of developed countries(relatively stable), such as USD, EUR and JPY. Similar to trading futures, the mark-to-market of the contracts is to be made quite frequently.

In order to make the pricing problem tractable, we will impose the following assumption:

Assumption 1.1 *The collateralization is perfect and continuous with zero threshold by cash. That is, mark-to-market and collateral posting is to be made continuously, in the meanwhile, the posted ammount of cash is* 100% *of the contract's value.*

For a justification of the assumption above, indeed, the daily adjustment of the collateral becomes very popular nowadays in the market, hence the approximation should not be far from th reality. Mathematically speaking, such simplification keeps us away from involving credit risk,otherwise nonlinearity can enter to the pricing.[1] As a result, we can still decompose the cashflows of a collateralized swap and treat them as a portfolio of independently collateralized strips of payments.

1.2 Collateral Model – Domestic Currency

Let's now model the collateral account $V(s)$, in domestic currency, as a stochastic process, that is affected by re-investing on risk-free asset, paying collateral rate and adjusting positions in derivative. To be mathematically sounding, we introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})^2$, $V(s)$ satisfies the following ODE almost surely, i.e.,

$$dV(s) = y(s)V(s)ds + a(s)dh(s),\ s \in [t, T],\ a.s., \tag{1}$$

where $y(s) = r(s) - c(s)$ represents the difference of risk-free rate $r(s)$ and the collateral rate $c(s)$ at time s, $h(s)$ denotes the time s value of the derivative that matures at T with cashflow $h(T)$, and $a(s)$ is the number of positions of the derivatives. One can verify the following *proposition*

Proposition 1.2 *The solution to* (1) *is:*

$$V(T) = e^{\int_t^T y(u)du} V(t) + \int_t^T e^{\int_s^T y(u)du} a(s)dh(s),\ a.s., \tag{2}$$

for $T \geq t \geq 0$.

If we fix the initial value and trading strategy as,

$$V(s) = h(t),\ a(s) = \exp\Big(\int_t^s y(u)du\Big)$$

then (2) gives,

$$V(T) = e^{\int_t^T y(s)ds} h(T),\ a.s..$$

Risk-neutral pricing theory implies $e^{-\int_0^t r(s)ds V(s)}$ is a martingale, thus,

$$\begin{aligned} h(t) = V(t) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r(s)ds} V(T) \,\big|\, \mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T (r(s)-y(s))ds} h(T) \,\big|\, \mathcal{F}_t\big] \\ &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T c(s)ds} h(T) \,\big|\, \mathcal{F}_t\big] \end{aligned} \tag{3}$$

[1] If default risk is concerned here, then one has to involve the survival function that is non-linear.
[2] Here, we suppose $\mathbb{Q}$ is the risk-neutral measure for the convenience of pricing.

1.3 Collateral Model – Foreign Currency

Next, we move to the case where the collateral is posted by a foregin currency. In this case, the collateral account process $V^f(s)$ satisfies:

$$dV^f(s) = y^f(s)V^f(s)ds + a(s)d[h(s)/f_x(s)],\ s \in [t, T],\ a.s., \tag{4}$$

where $f_x(s)$ is the foreign exchange rate at time s and $y^f(s) - c^f(s)$ denotes the difference of the risk-free and collateral rate of the foreign currency. We have, again,

Proposition 1.3 *The solution to* (4) *is:*

$$V^f(T) = e^{\int_t^T y^f(u)du} V^f(t) + \int_t^T e^{\int_s^T y^f(u)du} a(s)d[h(s)/f_x(s)],\ a.s., \tag{5}$$

for $T \geq t \geq 0$.

If we choose trading strategy:

$$V^f(t) = h(t)/f_x(t),\ a(s) = e^{\int_t^s y^f(u)du},$$

we have

$$V^f(T) = e^{\int_t^T y^f(s)ds} h(T)/f_x(T)$$

As a result,

$$\begin{aligned} h(t) = V^f(t) f_x(t) &= \mathbb{E}^{\mathcal{Q}}\left[e^{-\int_t^T r(s)ds} V(T) f_x(T) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathcal{Q}}\left[e^{-\int_t^T r(s)ds} e^{-\int_t^T (r^f(s) - c^f(s))ds} h(T) \,\middle|\, \mathcal{F}_t\right] \end{aligned} \tag{6}$$

1.4 Interpretation of the analysis

The conclusion of this section is clear that *LIBOR discounting* is no longer proper for pricing of collateralized trades. In the domestic currency case, we observe that we have to discount future cashflow by the collateral rate that is usually lower than the LIBOR rate. In terms of funding cost, we can have the following interpretation: when there is a receipt of a future cahsflow, we will receive a collateral from counter-party, in return, we have to pay the collateral rate and pay back the collateral at the end, it is equivalent to a loan for which we fund the position at the expense of the collateral rate; when there is a payment of a future cashflow, we have to post a collateral, which can be thought as a loan provided by counterparty with the same rate.

As a result, compared to the trade without collateral, we get more in the case of positive present value since we can fund the loan cheaply, but lose morein the case of negative value due to the lower return from the loan lent to the client.

2 Pricing Swaps

2.1 Discounting Curve – Overnight Index Swap(OIS)

Before pricing swaps, we need to first derive the discounting curve. For pricing collateral trades, it is very important to find out the forward curve of overnight rate. The *overnight index swap(OIS)*, which exchanges the fixed coupon agains the daily compounded overnight rate, can provide the important information. To proceed let's make the following assumption for simplication:

Assumption 2.1 *OIS is continuously and perfectly collateralized with zero threshold, the daily compounding can be approximated well by continuous compounding.*

Under above assumption and pricing formula (3), we can have[3]

$$S_N \sum_{n=1}^{N} \Delta_n \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_n} c(s)ds}\right] = \sum_{n=1}^{N} \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T_n} c(s)ds}\left(e^{\int_{T_{n-1}}^{T_n} c(s)ds} - 1\right)\right] \tag{7}$$

where S_N is the time t N-tenor[4] swap rate with fixed leg tenor Δ_n and $c(t)$ is the overnight rate(thus, collateral rate). A simple algebraic manipulation gives:

$$S_N \sum_{n=1}^{N} \Delta_n P(t, T_n) = P(t, T_0) - P(t, T_N) \tag{8}$$

where

$$P(t, T) = \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^{T} c(s)ds}\right].$$

Now, from (8), we can bootstrap the set of discounting factor $P(t, T)$ as well as the overnight forward curve by appropriate splining method.

[3]For notation simplicity, we sometimes write $\mathbb{E}^{\mathcal{Q}}[\cdot \mid \mathcal{F}_t]$ as $\mathbb{E}_t^{\mathcal{Q}}[\cdot]$.

[4]N-tenor means the following schedule $T_0, ..., T_N$ for one leg.

2.2 Single Currency Case

Let's try to come up with pricing mechanism when the swap traded is collateralized in a single currency. Our intention is to get the forward LIBOR curve. For interest rate swap(IRS), we have:

$$C_M \sum_{m=1}^{M} \Delta_m P(t, T_m) = \sum_{m=1}^{M} \delta_m P(t, T_m) \mathbb{E}_t^c[L(T_{m-1}, T_m)]. \tag{9}$$

Here, the day count factor for floating leg and fixed leg are different, but same number of tenors, although, in general, they are not necessary to be the same. In addition, $\mathbb{E}_t^c[\cdot]$ is the conditional expectation as $P(t, T)$ is the numéraire. As for tenor swap(TS), the follwoing equality must hold:

$$\sum_{n=1}^{N} \delta_n (\mathbb{E}_t^c[L(T_{n-1}, T_n)] + \tau_N) P(t, T_n) = \sum_{m=1}^{M} \delta_m P(t, T_m) \mathbb{E}_t^c[L(T_{m-1}, T_m)], \tag{10}$$

notice τ_N is the spread of tenor swap. Since all $P(t, T)$ appearing in (9)-(10) are known from OIS market, we can bootstrap the entire forward with help of interpolation, given combination of various maturity swaps.

2.3 Multiple currnecy case – constant Cross Currency Cwap(CCS)

We will use USD and JPY swaps to demonstrate the idea. The required conditions from JPY-collateralized JPY swaps are given by:

$$S_N \sum_{n=1}^{N} \Delta_n P(t, T_n) = P(t, T_0) - P(t, T_N), \tag{11}$$

$$C_M \sum_{m=1}^{M} \Delta_m P(t, T_m) = \sum_{m=1}^{M} \delta_m P(t, T_m)] \mathbb{E}_t^c[L(T_{m-1}, T_m)], \tag{12}$$

$$\sum_{n=1}^{N} \delta_n (\mathbb{E}_t^c[L(T_{n-1}, T_n)] + \tau_N) P(t, T_n) = \sum_{m=1}^{M} \delta_m P(t, T_m) \mathbb{E}_t^c[L(T_{m-1}, T_m)], \tag{13}$$

and those of USD-collateralized USD swaps are:

$$S_N^{\$}\sum_{n=1}^{N}\Delta_n^{\$}P^{\$}(t,T_n) = P^{\$}(t,T_0) - P^{\$}(t,T_N), \tag{14}$$

$$C_K^{\$}\sum_{k=1}^{M}\Delta_k^{\$}P^{\$}(t,T_k) = \sum_{n=1}^{N}\delta_n^{\$}P^{\$}(t,T_n)]\mathbb{E}_t^{\$}[L^{\$}(T_{n-1},T_n)], \tag{15}$$

$$\sum_{n=1}^{N}\delta_n^{\$}(\mathbb{E}_t^{\$}[L^{\$}(T_{n-1},T_n)] + \tau_N^{\$})P^{\$}(t,T_n) = \sum_{m=1}^{M}\delta_m^{\$}P^{\$}(t,T_m)\mathbb{E}_t^{\$}[L^{\$}(T_{m-1},T_m)], \tag{16}$$

From above conditions, we can derive discounting curves and forward curves both in JPY and USD.

Now, let us consider USD-collateralized JPY interest rate. The common practice in the market for USDJPY CCS is collateralized by USD cash. It is known, for CCS, the two legs in JPY currency are

$$V^{JPY}(t) = \sum_{n=1}^{N}\delta_n(\mathbb{E}_t[L(T_{n-1},T_n)] + b_N)P(t,T_n) - P(t,T_0) + P(t,T_N) \tag{17}$$

$$V^{USD}(t) = (\sum_{n=1}^{N}\delta_n^{\$}\mathbb{E}_t^{\$}[L^{\$}(T_{n-1},T_n)]P^{\$}(t,T_n) - P^{\$}(t,T_0) + P^{\$}(t,T_N))/N^{\$} \tag{18}$$

where $N^{\$} = N_{JPY}/f_x(t)$, N_{JPY} is the JPY notional per USD and $f_x(t)$ is the USDJPY exchange rate time t. V^{USD} is already known, however, it is impossible to uniquely determine the discount factor $P(t,T_n)$ and forward curve building blocks $\mathbb{E}_t[L(T_{n-1},T_n)]$. However, if there exist USD-collateralized JPY IRS and

TS markets[5], we get the additional information as:

$$\bar{C}_M \sum_{m=1}^{M} \Delta_m P(t, T_m) = \sum_{m=1}^{M} \delta_m P(t, T_m) \mathbb{E}_t[L(T_{m-1}, T_m)], \tag{19}$$

$$\sum_{n=1}^{N} \delta_n (\mathbb{E}_t[L(T_{n-1}, T_n)] + \bar{\tau}_N) P(t, T_n) = \sum_{m=1}^{M} \delta_m P(t, T_m) \mathbb{E}_t[L(T_{m-1}, T_m)]. \tag{20}$$

Here, $\bar{C}_N$ and $\bar{\tau}_N$ denote the par rates of the USD-collateralized JPY swaps, which differ from C_M and τ_N, the par rates of JPY collateralized swaps in general. We can now establish the following relationship:

$$\sum_{n=1}^{N} \delta_n (b_N - \bar{\tau}_N) P(t, T_n) + \bar{C}_M \sum_{m=1}^{M} \Delta_m P(t, T_m) - V^{USD}(t) = P(t, T_0) - P(t, T_N). \tag{21}$$

As a result, we can derive $P(t, T)$ and the forward LIBOR with both tenors by applying appropriate spline method.

Finally, let briefly go through the case when we have JPY-collateralized USD swap markets. Since we didn't assume that JPY overnight rate is risk-free, the difference between the risk-free and collateral rates appears in the expression of the present value as given in (6):

$$\bar{C}^{\$}_K \sum_{k=1}^{K} \Delta^{\$}_k P^{\$}(t, T_k) \mathbb{E}^{\$}_t[e^{\int_t^{T_k} y(s)ds}] = \sum_{n=1}^{N} \delta^{\$}_n P^{\$} \mathbb{E}^{\$}_t[e^{\int_t^{T_n} y(s)ds} L^{\$}(T_{n-1}, T_n)]. \tag{22}$$

In the same way, if there exists JPY-collateralized USDJPY CCS, we also have the following condition:

$$\begin{aligned} &\sum_{n=1}^{N} \delta^{\$}_n P^{\$}(t, T_n) \mathbb{E}^{\$}_t[e^{\int_t^{T_n} y(s)ds} L^{\$}(T_{n-1}, T_n)] \\ =&N^{\$}\Big(\sum_{n=1}^{N} \delta_n (\mathbb{E}^c_t[L(T_{n-1}, T_n)] + \bar{b}_N) P(t, T_n) - P(t, T_0) + P(t, T_N)\Big), \end{aligned}$$

[5]Indeed, it seems that the US banks tend to ask their counter-parties to post USD collateral even for the JPY IRS and TS

where $\bar{b}_N$ is the par spread of the length-N CCS. Since the right hand side and USD discount factors are already known, we can determine the set of:

$$\mathbb{E}_t^{\$}[e^{\int_t^{T_n} y(s)ds}],\ \ \mathbb{E}_t^{\$}[e^{\int_t^{T_n} y(s)ds} L^{\$}(T_{n-1}, T_n)]$$

3 Appendix

Proof of proposition 1.2:
The integration by parts formula implies:

$$\int_t^T V(s) d(e^{\int_s^T y(u)du}) = V(s) e^{\int_s^T y(u)du}\Big|_t^T - \int_t^T e^{\int_s^T y(u)du} dV(s)$$

Equivalently,

$$\begin{aligned}-\int_t^T V(s)y(s)e^{\int_s^T y(u)du} ds =& V(T) - e^{\int_t^T y(s)ds} V(t) - \int_t^T e^{\int_s^T y(u)du} dV(s) \\ =& V(T) - e^{\int_t^T y(s)ds} V(t) - \int_t^T e^{\int_s^T y(u)du} y(s)V(s)ds \\ & - \int_t^T e^{\int_s^T y(u)du} y(s)a(s)dh(s)\end{aligned}$$

The result follows immediately.