Metric Space, Limit Point and Open/Closed set

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1 Metric Space

Euclidean space is the first one that we encounter in elementary mathematics, actually there are various abstract topological spaces. Among them, we will discuss the most commonly used one – *metric space*, which can be thought as a generalization of Euclidean space. Remember on $\mathbb{R}^n$, *Euclidean distance* function $d(\cdot,\cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ is adopted to measure the distances among all its element, the notion of *metric space* says: it is a set for which distances between all members of the set are defined. In other words, metric space is the space where we have metric on it. This metric should preserve all properties in *Euclidean space*, let's give the definition:

Definition 1.1 (Metric Space) A set X is a *metric space* if $\exists$ *metric* $d : X \times X \mapsto \mathbb{R}$ such that $\forall p, q \in X$,

- *Non-negative:* $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if, $p = q$;
- *Symmetric:* $d(p, q) = d(q, p)$;
- *Triangle Inequality Holds:* $d(p, q) \leq d(p, r) + d(r, q)$.

Example 1.1 On $\mathbb{R}^n$, besides *Euclidean metric*, one can design different metric as long as they are legitimate, for example $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, for $x, y \in X$ is a valid metric.

Example 1.2 On the space of *Lebesgue integral functions* on interval $[a, b]$ (denoted as $\mathcal{L}[a, b]$), $d : \mathcal{L}[a, b] \times \mathcal{L}[a, b] \mapsto \mathbb{R}$ defined as following

$$d(f, g) = \int_a^b |f(x) - g(x)| dx, \ for \ f, g \in \mathcal{L}[a, b]$$

is a well-defined metric. Or

Example 1.3 On abstract space X, the discrete metric:

$$\delta(p,q) = \begin{cases} 0 & p = q, \\ 1 & p \neq q. \end{cases}$$

also makes them a metric space.

After we establish the notion of metric, we can now introduce the most fundamental element of any topological space – the *open ball*. Given a metric space (X, d)[1], for any $x \in X$, an open ball (or sometimes, called, *neighborhood*) around x of radius r is the set,

$$N_r(x) := \{y \mid d(x,y) < r\}.$$

Similarly, the *closed ball* is define as:

$$\bar{N}_r(x) := \{y \mid d(x,y) \leq r\}.$$

Open ball tells us which points are *"close"*, we can visualize in *Figure 1.* (Try to visualize the open ball in discrete metric)

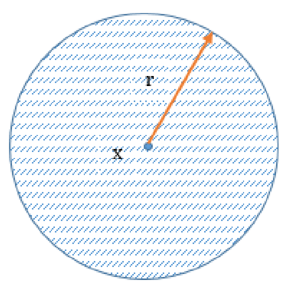

Figure 1: Neighborhood of x with radius r

2 Limit point, interior point and isolated point

Closed ball and open allow us to define all kinds of points in topological space, which leads to the concept of *open set* and *closed set*. We shall investigate them one by one while providing examples. Note, we will work on the metric space (X, d).

Definition 2.1 A point $p \in X$ is a *limit point* of a set $E \in X$, if every $n.b.h.d$ (shorthand for neighborhood) of p contains a point $q \in E$ such that $p \neq q$.

Remark 2.1 In this definition, p does not need to be in E.

Example 2.2 In $\mathbb{R}$, consider the set $G = \{\frac{1}{n} : n \in \mathbb{N}\}$, the limit point of set G is 0.

Example 2.3 In $\mathbb{R}^2$, consider the set B (shaded are plus point a) in *Figure 2*, among a, b, c, d, e, z, which of them are limit points?

[1]Metric space is usually defined by a pair, where X stands for the set and d represents the metric function

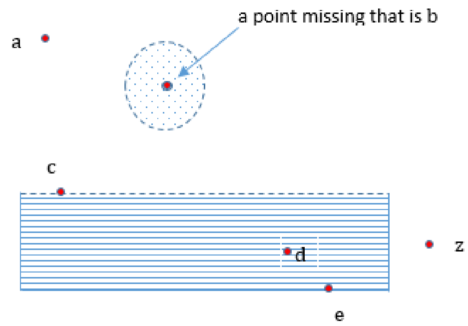

Figure 2: Set B with various point

A point p is not a limit point of E, if $\exists$ $n.b.h.d$ N of p such that it doesn't contain any point of E. It leads to the definition of *isolated point*:

Definition 2.2 p is an *isolated point* of E if $p \in E$ and p is not a limit point of E.

Finally, we introduce the *interior point*,

Definition 2.3 p is an *interior point* of E if $\exists$ a $n.b.h.d$ N of p such that $N \subset E$.

Remark 2.4 Unlike limit point, interior point is in the set. As an exercise, can you identify, in above examples, which points are isolated points and interior points, respectively.

In $\mathbb{R}$, consider $\emptyset$, $\mathbb{R}$, $\mathbb{Q}$, with Euclidean metric and discrete metric, what are limit points and interior points? Under Euclidean metric, $\emptyset$ has no limit point/interior point, every points of $\mathbb{R}$ are limit points and interior points, every points of $\mathbb{Q}$, however, can only be limit points. In terms of discrete metric, there is no limit point/interior point, for set $\mathbb{R}$, there exists no limit point but every point is an interior point, the same for $\mathbb{Q}$.

There is a equivalent characterization of limit point,

Theorem 2.5 If p is a limit point of E, then every $n.b.h.d$ of p contains infinitely many points of E.

Proof. (*By contradiction*) $\exists$ $n.b.h.d$ of p with only finitely many points of E, call them $e_1, ..., e_n$. Let $r = \min_{i=1,...,n}\{d(p, e_i)\}$. Then $N_r(p)$ has no poinst of E. Contradiction! $\square$

Now, we are in a position to introduce the *open set* and *closed set*:

Definition 2.4 A set E is open if every point of E is an interior point of E.

We shall firstly examine what we used to think about the *real line* is true.

Example 2.6 In $\mathbb{R}$, $(a, b) = \{x \ : \ a < x < b\}$ is open, $\emptyset$ is open and the entire $\mathbb{R}$ is open.

Definition 2.5 A set E is closed if E contains all its limit points.

Example 2.7 In $\mathbb{R}$, a single point p is closed (there is no limit point of a set consists only one point, so it contains all its limit point). $[a, b]$ is closed, neither $[a, b)$ nor $(a, b]$ are closed/open, they are half-open intervals. We know from previous example $\mathbb{R}$ is open, but it is also closed. Actually, it is the only set that both open and closed (the slang for this is *"clopen"*).

As pointed out, *open set* is fundamental element of any topological space. We usually define the base (basis) for *"topology"* that is a collection $\{V_\alpha\}$ such that for all open U and $\forall x \in U$, $\exists V_\alpha$, $x \in V_\alpha \subset U$. So every open set is the union of bases elements. From more practical point of view, the reason we care so much about open set is that one can perturb point in the set $E \in X$ and it still remain in the set E.

3 Relation between Open set and Closed set

Let's first introduce the definition of *"closure"*:

Definition 3.1 Let A' be a set of limit points of $A \in X$, the closure of A is $\bar{A} = A \cup A'$.

We can claim:

Theorem 3.1 $\bar{A}$ is closed set.

Proof. If p is a limit point of $\bar{A}$, it is sufficient to show that p is in $\bar{A}$, which mean we can either show p is in A or p is a limit point of A. Consider a $n.b.h.d$ N of p, assume p is not in A, we will show N contains a point of A. Since N is a $n.b.h.d$ of p and p is a limit point of $\bar{A}$, therefore, N contains a point q of $\bar{A}$. If q is already in A, we found the desired point $q' = q$. Otherwise, if $q \notin A$, then q is a limit point of A, consider N' the $n.b.h.d$ of q such that $N' \subset N$ (such N' exists because of the openness of N), thus $q' = q \in N$, the desired point. $\square$

As we already observe that we used the fact that neighborhood is open, we need to justify it.

Lemma 3.2 $n.b.h.d$ is open

Proof. Suppose the neighborhood is centered at p with radius r, we define $a := d(p, q) < r$, for some $q \in N_r(p)$. Let $r' = r - a$, we claim $N_{r'}(q) \subset N_r(p)$. Because, for $x \in N_{r'}(q)$, if $d(x, q) < r'$, done, otherwise, $d(x, p) \leq d(x, q) + d(q, p) < r' + a = r$. $\square$

Two more results regarding the closure:

Theorem 3.3 E is closed if and only if $E = \bar{E}$.

Proof. ("$\Rightarrow$") If E is closed, then $E' \subset E$, thus $E \cup E' \subset E$, or, equivalently, $\bar{E} \subset E$, but $E \subset \bar{E}$, so $E = \bar{E}$. ("$\Leftarrow$") $E = \bar{E}$ implies E contains all its limit points. By definition, E is closed. $\square$

Theorem 3.4 If $E \subset F$, where F is closed, then $\bar{E} \subset F$. (equivalent statement is: $\bar{E}$ is the smallest closed set containing E).

Proof. Since F is a closed set thus containing all its limit points, in particular, it contains limit point of E, thus $\bar{E} \subset F$. $\square$

It seems like the relation between open set and closed set are very "intimate", let's explore further.

Theorem 3.5 E is open if and only if E^c is closed.

Proof. (*sketch of the proof*) Openness of E is identical to, for any $x \in E$, it is an interior point, by definition, that means $\forall x \in E$, $\exists$ *n.b.h.d* N of x, where N is disjoint from E^c. That is, $\forall x \in E$, x is not a limit point of E^c, thus E^c contains all its limit point. $\square$

Next, we shall examine the unions & intersection of open/closed sets. We firstly take a look at the following question: define $K_i = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$, is $\cup_{n=1}^{\infty} K_n$ close or open? The answer is: it is open, $(-1, 1)$. The moral is that although finite union of closed sets is closed, if there are infinitely many, it may not be closed. The same happens for the infinitely many intersection of open sets, for example, $\cap_{n=1}^{n}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is closed. As a conclusion, we have:

Theorem 3.6 The followings are true:

- Arbitrary union of open sets are open;
- Arbitrary intersection of closed set is closed;
- Finite intersection of open sets are open;
- Finite union of closed sets are closed.

Lastly, let's give one more definition related to limit point.

Definition 3.2 E is *dense* in metric X, if every point of X is a limit point of E or in E, that is, $\bar{E} = X$, or every open sets of X contains a point of E.

An classical and obvious example is $\mathbb{Q}$ is *dense* in $\mathbb{R}$.