Production & Operations Management — Recitation 4

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Overview

Solving LP problem

Two major steps need to be made:

- Step 1: Extract information from the problem statement and formulate the model
- Step 2: Analyze and solve the LP formulated by using:
  (i) **Graphical Solution: Only applicable in 2-dimension**;
  (ii) Computer Solution: make use of Excel Solver (or others)($\surd$).

Applicability of Graphical Solution

- The graphical method is only applicable when we have 2 variable, i.e., x_1, x_2 . This is highly unfortunate that in most of the real life problem, we can not rely on the graphical method;
- However, it sheds light on understanding the LP models, their solution, and solution interpretation.

Graphical Solution Procedure

After formulation of the model, we need to

- Plot the constraints;
- Identify the feasible region (or, solution space);
- Determine the optimal solution either by (a) using Isoprofit (or isocost) objective function; or (b) enumeration (comparing the extreme point of the feasible region, that is, evaluating and comparing the O.F. at corner points).

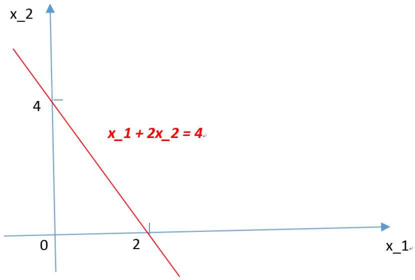
Plot the Constraints - 1

Recall that the constraints (including non-negative constraints) are system of equalities or inequalities. What we need to do is to realize those equalities or inequalities on the graph.
Take one constraint:

$$ax_1 + bx_2 \leq (\geq, \quad =)c$$

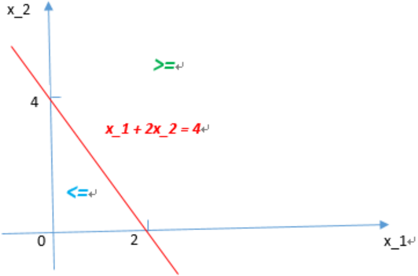
No matter they are equalities of inequalities, treat them as equality and graph it on the plane. For example, $x_1 + 2x_2 = 4$ can be graphed as follows:

Plot the Constraints - 2



Since a line is determined by two points, we just need to know two particular points on the line. How? Just set $x_1 = 0$, solve for x_2, you get one point (x_1, x_2), then you set $x_2' = 0$, solve for x_1', you get the other point (x_1', x_2'). We finish by drawing a line to connect them.

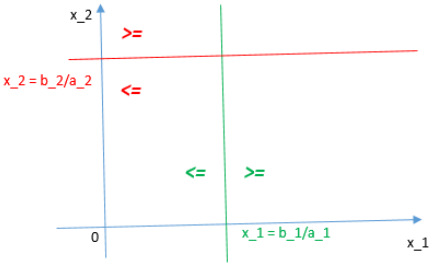
Plot the Constraints - 3



While equality corresponds to a line, inequality corresponds to a region. The plane is divided into two parts by the line, we can have the lower part represented by "less than or equal" ($\leq$) and the upper part represented by "great than or equal" ($\geq$).

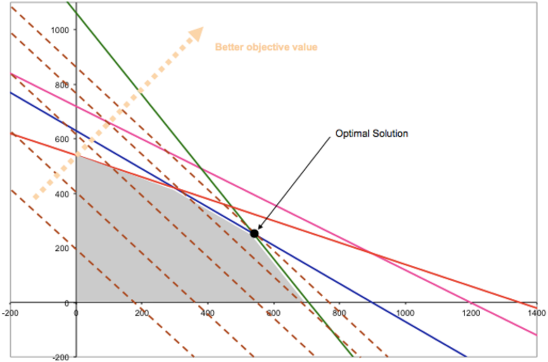
Plot the Constraints - 4

A particular linear constraint is: $a_i x_i = (\leq, \ \geq) b_i$, $i = 1, 2$. We just need to draw the a vertical line (if it is for x_1) or horizontal line (if it is for x_2), and identify the region corresponding to different operators $(\leq, \ \geq, \ =)$



Plot the Constraints - 5

Follow this procedure, we can realize all the constraint. For example, in the bag problem,



Feasible Region

Definition

The set of points in the graph that satisfy all constraints

Intuitively, they are all candidate optimal solutions since they satisfy the feasibility criterion. At least one of them will be the optimal solution (as long as we have a non-empty feasible region). As we are in the $x - y$ coordinate system, each point on the plane is a pair (x_1, x_2). The goal is, of course, to find the (x_1, x_2)'s that can maximize (or minimize) the objective function.

Optimal Solution - Extreme Point

The LP theory says at least one of the "extreme point" is the optimal solution. Thus, what we can do is to solve for each extreme point, the intersection of two linear equations. Then, we evaluate the objective function by plugging (x_1, x_2)'s. Comparison will give us the largest one (or smallest one) according to the definition of optimality (maximization or minimization).

Optimal Solution - Taught in the class 1

Steps:

- Draw the objective function as isoprofit or isocost line (*assume a value for the OF and solve for* x_1 *and* x_2);
- Move the line parallel to itself until it touches the 'farthest' from the origin (if maximizing) or it touches the closest point to the origin (if minimizing);
- Solve for the two constraints meeting at the point as simultaneous equation.

Steps continued:

- We conduct two steps for each corner (1)solve for the intersecting constraints;(2) calculate the O.F.
- Select the corner that produces the highest O.F (if maximizing), or the corner that produces the lowest O.F. (if minimizing).

Optimal Solution - Isoprofit/ Iscost - 1

Definition

Isoprofit/isocost is a line on which objective function has the same value.

Think about the objective function,

$$\max.\ \mathrm{OF} = c_1 x_1 + c_2 x_2$$

We rewrite it as:

$$x_2 = -c_1/c_2 x_1 + \mathrm{OF}/c_2$$

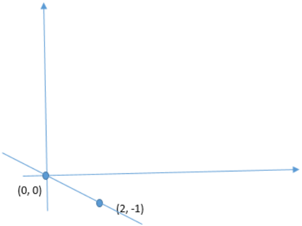
Actually, we want to maximize the "intercept".

Optimal Solution - Isoprofit/ Isocost - 2

How can we maximize the intercept ? Dropping of the OF$/c_2$ for a while, we can get a line $x_2 = -c_1/c_2 x_1$ that getting through the origin. For example, imagine we have $OF = x_1 + 2x_2$, then we rewrite it as:

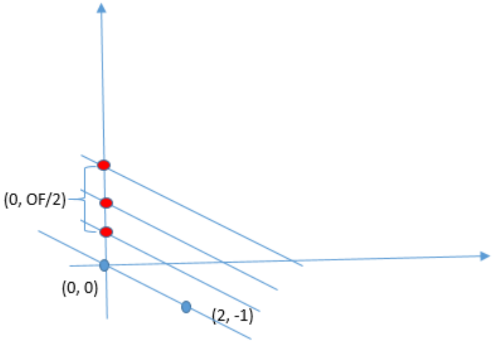
$$x_2 = -\frac{1}{2} + \frac{OF}{2}$$

$x_2 = -\frac{1}{2}x_1$ is a line through the origin, so to fix the line, we only need one more point, plug whatever x_1, try $x_1 = 2$, we get $x_2 = -1$. Visualize it on the graph:



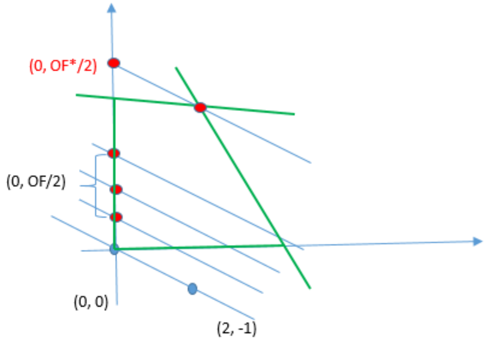
Optimal Solution - Isoprofit/ Isocost - 3

We add back the term $\frac{OF}{2}$, which is actually telling us that we can shift this line to as shown in the graph



Optimal Solution - Isoprofit/ Isocost - 3

Imagine that we have the green area as feasible region, to obtain optimal, we need to maximize our intercept. We are only allowed to move the line in the green region.



By doing this, we find the optimal solution for objective function and also

Example: Problem Statement

Problem statement: Our firm makes two product, type 1 and type 2. The selling price for them are \$60 and \$ 50 respectively. There are two steps to make them, assembly and inspection, for one unit type 1, it needs 4 hours for assembly and 2 hours for inspection, in the meanwhile it needs a storage space 3 cubic feet. For one unit type 2, it needs 10 hours for assembly and 1 hour for inspection, a storage space 3 cubic feet/unit is required. we have the in total 100 hours for assembly and 22 hours for inspection, a storage space 39 cubic feet is available. How can we maximize the profit?

Example: the complete LP program

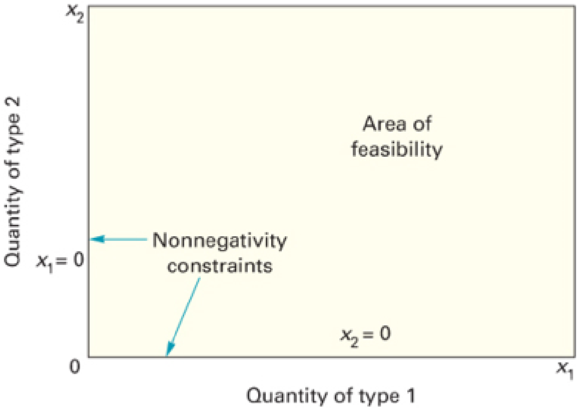
The LP program looks like:

$$
\begin{aligned}
x_1 &= \text{ quantity of type 1 made in next production cycle} \\
x_2 &= \text{ quantity of type 1 made in next production cycle}
\end{aligned}
$$

$$\text{Max } 60x_1 + 50x_2$$

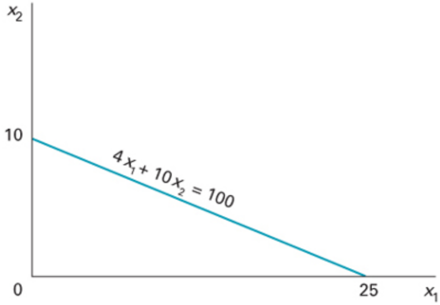
$$
\begin{aligned}
\text{Subject to: } & 4x_1 + 10x_2 \leq 100 \\
& 2x_1 + x_2 \leq 22 \\
& 3x_1 + 3x_2 \leq 39 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

1. **Begin by placing non-negative constraints on a graph**



Example : Plotting the constraints c.t.d

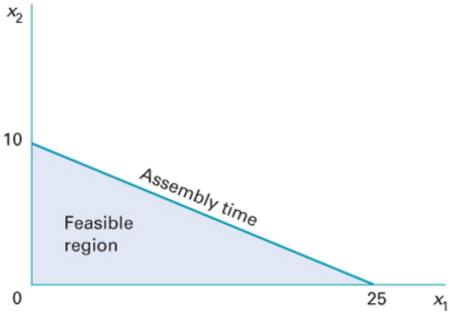
2. Setting the inequality as equality an draw the line



To get the intersection point of a variable, set the other variable to zero and solve the equation.

Example : Plotting the constraints c.t.d

Shade the area

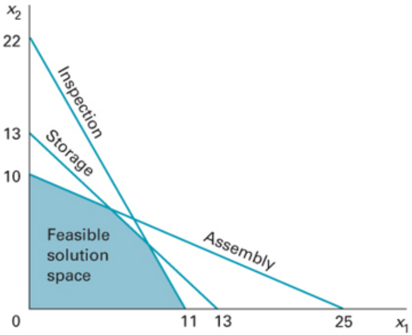


Remark

This is a $\leq$ relationship, so the lower part is feasible region for this particular inequality.

Example : Feasible Region

4. Complete the rest of the constraints to form the feasible region

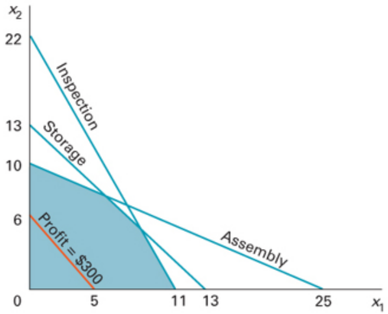


Remark

They are the set of points that satisfy all constraints simultaneously.

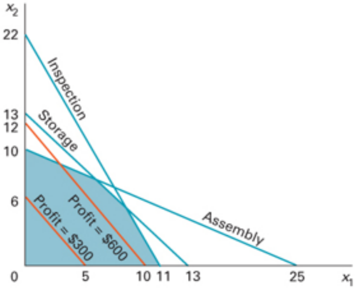
5. Following the procedure below:

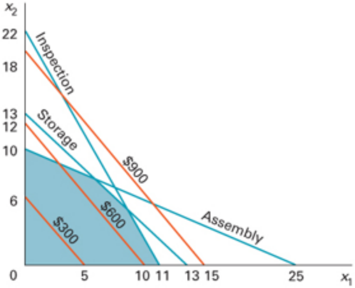
- Set the objective function to some quantity (profit or cost), here we try 3000, because it is divisible by 60 and 50;
- Solve for x_1, x_2 by fixing one of them as 0 each time. Here, we get $(50, 0)$, $(0, 60)$;
- Connect these two points to get the isoprofit line.



Example : Identify the optimal solution

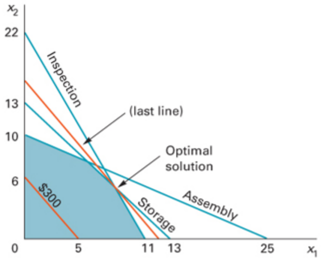
6. Shift the isoprofit line





7. Find the optimal solution

- Move the isoprofit line to the "farthest" point within the feasible region;
- This optimum point will occur at the intersection of two constraints, solve for the values of x_1 and x_2 where this occurs;
- The optimal value of objective function is an evaluation at (x_1, x_2) that is obtained in last step.



Example : Solution

8. Computation of Optimal Solution. Convert the inequality system to equality system, since we only need to solve for the points on the them.

$$2x_1 + 1x_2 = 22$$
$$3x_1 + 3x_2 = 39$$

We get $x_1 = 9$, $x_2 = 4$. So the optimal value (profit)

$$\text{Profit} = 9 \times 60 + 4 \times 50 = 740$$

Example : Check by enumeration

9. We can evaluate the objective function at each extreme point and choose the biggest one to back test

Remark

It must be the largest value coincide to the solution by shifting isoprofit line, otherwise, you make a mistake either in enumeration or isoprofit line approach.

References

Ahmed, Mahmoud (2014)
Lecture Slides of Operations Management

Thank You !!!