Exotic Derivatives

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In this notes, we discuss pricing of exotic derivatives. It is the best if some analytic, or even semi-analytic formula can be found, otherwise, we have to resort to Monte Carlo simulation for pricing. Particular attentions are paid on Constant Variance Swap(CMS) as it is actively traded on the market.

1 In-Advanced Swaps & In-Advanced Caps

Remember in the notes "Interest Rate Basics - under Singale Curve Framework", we have discussed LIBOR in-arrears and its application to swaps already. The key point there is to make a convexity adjustment. Let's re-cap a little bit here and also discuss the case when it is applied to cap.

For a payer IRS with resets dates $\mathcal{T} = \{T_0, T_1, ..., T_M\}$ and fixed leg rate K[1], the present value at time 0, of an in-advance swap with notional 1 is expressed as:

$$PFS = \mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=0}^{M-1} D(0,T_i)\tau_{i+1}\big(L(T_i,T_{i+1}) - K\big)\Bigg] := 0 \tag{1}$$

That has to be 0. Thus, the fair fixed leg rate is equal to:

$$K = \frac{\sum_{i=1}^{M-1} \tau_{i+1} P(0,T_i)\mathbb{E}^{T_i}\big[L(T_i,T_{i+1})\big]}{\sum_{i=0}^{M-1} P(0,T_i)\tau_{i+1}} \tag{2}$$

[1]For simplicity, we assume here the schedule for floating leg is the same as the fixed leg.

The expectation has to be evaluated with a model specified, because $F_{i+1}(T_i) := F(T_i; T_i, T_{i+1})$ is only a martingale under T_{i+1}-forward measure. Recall the following result of change of measure, from T_i-forward measure to T_{i+1}-forward measure,

$$\mathbb{E}^{T_i}\Big[L(T_i, T_{i+1})\Big] = F_{i+1}(0) + \frac{\tau_{i+1} Var\big[F_{i+1}(T_i)\big]}{1 + \tau_{i+1} F_{i+1}(0)} \tag{3}$$

The most convenient model here is the log-normal forward rate model,

$$dF_{i+1}(t) = \sigma_{i+1}(t) F_{i+1}(t) dW_{i+1}(t) \tag{4}$$

where $W_{i+1}(\cdot)$ is a $\mathbb{Q}^{T_{i+1}}$-Brownian motion. The volatility function here can be deduced from cap. Thus, (3) can be evaluated as well as the fair fixed rate K.

Let's now consider the in-advance caps with strike K, the present value is thus,

$$IAC(0, \mathcal{T}, N = 1, K) = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=0}^{M-1} D(0, T_i)\tau_{i+1}\big(L(T_i, T_{i+1}) - K\big)^+\Big] \tag{5}$$

The following derivation is standard,

$$\begin{aligned}
IAC(0, \mathcal{T}, N = 1, K) &= \mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=0}^{M-1} D(0, T_i)\Big(\frac{1}{P(T_i, T_{i+1})} - (1 + \tau_{i+1} K)\Big)^+\Big] \\
&= \sum_{i=0}^{M-1} \tau_{i+1} P(0, T_{i+1}) \mathbb{E}^{T_{i+1}}\Big[\frac{1}{P(T_i, T_{i+1})}\Big(F_{i+1}(T_i) - K\Big)^+\Big] \\
&= \sum_{i=0}^{M-1} \tau_{i+1} P(0, T_{i+1}) \mathbb{E}^{T_{i+1}}\Big[(1 + \tau_{i+1} F_{i+1}(T_i))\Big(F_{i+1}(T_i) - K\Big)^+\Big] \\
&= \sum_{i=0}^{M-1} \tau_{i+1} P(0, T_{i+1}) \mathbb{E}^{T_{i+1}}\Big[\Big(F_{i+1}(T_i) - K\Big)^+\Big] \\
&+ \sum_{i=0}^{M-1} \tau_{i+1}^2 P(0, T_{i+1}) \mathbb{E}^{T_{i+1}}\Big[F_{i+1}(T_i)\Big(F_{i+1}(T_i) - K\Big)^+\Big]
\end{aligned}$$

Under log-normal forward rate assumption, i.e., dynamics (4), the first term is a "Black" term, the second term requires a direct integration, which can also be computed explicitly.

Remark 1.1. *We assume LMM because of its simplicity, other short rate models can also be used here. In practice, since the model is assumed for a correction, it is desirable to have some closed form solution.*

2 Autocaps, Caps with Deferred Caplets and Ratchets

An *autocap* is similar to a cap, but at most $N \leq M$ caplets can be exercised, and they have to be automatically exercised when in the money. The discounted payoff at time 0 is thus,

$$\sum_{i=1}^{M} \tau_i D(0, T_i)\big(F_i(T_{i-1}) - K\big)^+ \mathbf{1}_{A_i}, \\ A_i = \{ \textit{ at most } N \textit{ among } F_1(T_0), ..., \ F_i(T_{i-1}) \textit{ are larger than } K.\} \tag{6}$$

Notice $A_i \in \mathcal{F}_{T_{i-1}}$, that is, only at time T_{i-1}, we know whether the payoff is still alive or not. In addition, by definition, A_i not only depends on $F_i(T_{i-1})$ but also all $F_j(T_{j-1})$, for $j \leq i$. Therefore, the decomposed expectation below is a functional of a span of forward rates, i.e.,

$$g^i\big(F_0(T_1), ..., F_i(T_{i-1})\big) = \mathbb{E}^{\mathbb{Q}}\big[\tau_i D(0, T_i)\big(F_i(T_{i-1}) - K\big)^+ \mathbf{1}_{A_i}\big]$$

where g^i is a Borel measurable function taking i arguments. The analytical formula is very unlikely to exist, thus we have to use Monte Carlo simulation. Before that, let's formally write down the risk-neutral expectation to be evaluated,

$$\mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=1}^{M} \tau_i D(0, T_i)\big(F_i(T_{i-1}) - K\big)^+ \mathbf{1}_{A_i}\Big] \tag{7}$$

$$= P(0, T_M) \sum_{i=1}^{M} \tau_i \mathbb{E}^{T_M}\left[\frac{\big(F_i(T_{i-1}) - K\big)^+ \mathbf{1}_{A_i}}{P(T_i, T_M)}\right] \tag{8}$$

We models forward rate directly, assuming it has log-normal dynamic, for $k = \beta(t), \beta(t) + 1, ..., M$,

$$dF_k(t) = -\sigma_k(t) F_k(t) \sum_{j=k+1}^{M} \frac{\rho_{k,j} \tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t) F_k(t) dW_k^M(t). \tag{9}$$

where $\beta(t) = i + 1$ for $t \in [T_{i-1}, T_i]$. At time t, the following paths need to be generated under T_M-forward measure,

$$F_{\beta(t)}(t), ..., F_M(t), \tag{10}$$

For example, if $t \in [0, T_0]$, the paths generated are:

$$F_1(t), F_2(t), ..., F_M(t). \tag{11}$$

When T_0 is arrived, we compare $F_1(T_0)$ with K and check whether indicator function $\mathbf{1}_{A_1}$ is still valid. From T_0, the paths to be simulated are:

$$F_2(t), ..., F_M(t), \tag{12}$$

Then, at T_1, one compute $(F_2(T_1) - K)^+$ and $\mathbf{1}_{A_2}$. Repeat this procedure until the end, then one simulation is done. Notice, one can stop early if, for some j, $\mathbf{1}_{A_j}$ is 0 already, because, in this case, $\mathbf{1}_{A_k}$, $k \geq j$, will all equal to zero. After a large number of simulation, we average out to get the estimator of the expectation.

For caps with deferred caplets, the cashflow is generated at terminal time T_M. We have the following discounted payoff at time 0,

$$\sum_{i=1}^{M} \tau_i D(0, T_M)\big(F_i(T_{i-1}) - K\big)^+ \tag{13}$$

The risk-neutral pricing leads to:

$$\mathbb{E}\Bigg[\sum_{i=1}^{M} \tau_i D(0, T_M)\big(F_i(T_{i-1}) - K\big)^+\Bigg] = P(0, T_M) \sum_{i=1}^{M} \tau_i \mathbb{E}^{T_M}\Bigg[\bigg(F_i(T_{i-1}) - K\bigg)^+\Bigg] \tag{14}$$

Having last expression, we are at home, applying BS formula while assuming the underlying follows LIBOR market model of log-normal type, a closed-form solution can be obtained.

Lastly, we briefly discuss *ratchets*, in particular, *one-way floaters*. It works in the following way:

- institution A pays to B (a percentage γ of) a reference floating rates (plus a constant spread S) at dates $\mathcal{T} = \{T_1, T_2, ..., T_M\}$. Mathematically, for $T_i \in \mathcal{T}$, A pays B,

$$\bigg(\gamma F_i(T_{i-1}) + S\bigg)\tau_i \tag{15}$$

- Institution B pays to A a coupon that is given by the reference rate plus a spread X at $\mathcal{T}$, floored and capped respectively by previous coupon and be by previous coupon plus an increment Y. That is, for $T_i \in \mathcal{T}$,

$$c_i = \begin{cases} \big(F_i(T_{i-1}) + X\big)\tau_i, & \textit{if } c_{i-1} \leq (F_i(T_{i-1} + X)\tau_i \leq c_{i-1} + Y, \\ c_{i-1}, & \textit{if } \big(F_i(T_{i-1}) + X\big)\tau_i < c_{i-1}, \\ c_{i-1} + Y, & \textit{otherwise.} \end{cases} \tag{16}$$

Notice, at the first payment T_1, institution V pays to A the coupon:

$$\big(F_1(T_0) + X\big)\tau_1. \tag{17}$$

The pricing formula for such derivative is:

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^{M} D(0, T_i)\big[c_i - (\gamma F_i(T_{i-1}) + S)\tau_i\big]\Big] \\
&= P(0, T_M) \sum_{i=1}^{M} \mathbb{E}^{T_M}\left[\frac{c_i - \big(\gamma F_i(T_{i-1}) + S\big)\tau_i}{P(T_i, T_M)}\right]
\end{aligned}$$

Again, it is a path-dependent contingent claim, the most efficient method is to use Monte Carlo simulation with LIBOR market model.

Remark 2.1. *As in the case of in-advance swap and in-advance cap, the LIBOR market model with log-normal dynamics has to be calibrated first. For caps with deferred caplets, only cap is sufficient for deduce the volatility function. For auto-cap and ratchets, since they depends on the joint distribution of a span of forward rates, the correlation matrix needs to be determined as well, thus one has to calibrate by both caps and Swaption.*

Remark 2.2. *The choice of underlying model is really at one's discretion. The LIBOR market model is chosen for its convenience for illustration.*

3 Constant Maturity Swaps Related

This section is devoted to CMS and its related derivatives. Let's firstly define the *CMS Swaplet*, that is a swap leg paying the N years swap rate plus a margin m on the payment dates $\{T_1, ..., T_M\}$. Specifically, for any $T_j \in \{T_1, ..., T_M\}$, the leg pays,

$$\tau_j(S_j^{1,N} + m) \tag{18}$$

where τ_j counts the days between T_{j-1} and T_j. The CMS leg is said to be *set-in-advance*, which is standard, if $S_j^{1,N}$ is the swap rate begins at T_{j-1} and ends N years later. To be explicit, we denote such swap rate by $S^{1,N}(T_{j-1})$. This rate is fixed on the date T_{j-1} if we don't assume lags, and pertains throughout the interval, with accrued interest (18) paid on T_j. Otherwise, CMS legs is said to be *set-in-arrears*. The swap rate begins on the end date T_j instead of T_{j-1}, in which case we write $S^{1,N}(T_j)$.

We also introduce two derivatives whose underlying asset is CMS Swaplet, they are CMS caps and CMS floors. The corresponding caplet and floorlet, paid at T_j, for any $j = 1, ..., M$, is

$$\textit{caplet:}\;\; \tau_j\big(S_j^{1,N} - K\big)^+, \quad \textit{floorlet:}\;\; \tau_j\big(K - S_j^{1,N}\big)^+. \tag{19}$$

Our analysis will focus on (18)-(19) because they are building blocks of CMS Swap and CMS cap/floor. Also, we will focus on the set-in-advance swap rate, i.e., $S_j^{1,N} = S^{1,N}(T_{j-1})$.

3.1 Reference Swap and Swaption Recap

Since the Swaplet is directly related to the reference swap, we will recall some essentials of IRS and Swaption. The definition of swap rate $S_j^{1,N}$ is the par rate for a standard swap that starts at date T_0^S and ends N years later at T_N^S. To be explicit, let $\{T_1^S, ..., T_N^S\}$ be the fixed leg pay dates of the reference swap, then it pays $\delta_j R$ at time T_j^S, for $j = 1, ..., N$, where δ_j counts the days between T_{j-1}^S and T_j^S. In return, the payer receives the floating leg payments at T_j^S, for $j = 1, ..., N$. At time $t \leq T_0^S$, the value of the payer swap is:

$$PFS(t) = P(t, T_0^S) - P(t, T_N^S) - R \sum_{j=1}^{N} \delta_j P(t, T_j^S), \tag{20}$$

The fair forward swap rate at time t is thus:

$$S^{1,N}(t) := R = \frac{P(t, T_0^S) - P(t, T_N^S)}{\sum_{j=1}^{N} \delta_j P(t, T_j^S)}. \tag{21}$$

which makes the swap at par at time t, i.e., $PFS(t) = 0$. Notice, $L(t) := \sum_{j=1}^{N} \delta_j P(t, T_j^S)$ is an interesting quantity here. It represents the time t value of receiving \$1 per year for N years(paid annually or semi-annually, according to the reference swap's frequency).

Now, consider a standard European payer option on the reference swap, a payer Swaption giving one right to enter a swap at maturity with strike rate K. The exercise date of the Swaption is T_0^S, thus, the payoff at time T_0^S is,

$$\begin{aligned}
V(T_0^S, S^{1,N}(T_0^S)) :=& \left(\sum_{j=1}^{N} \tau_j \mathbb{E}_{T_0^S}^{\mathbb{Q}} \Big[D(T_0^S, T_i^S) \big(L(T_{j-1}^S, T_j^S) - K \big) \Big] \right)^+ \\
=& \left(\sum_{j=1}^{N} \tau_j P(T_0^S, T_j^S) \mathbb{E}_{T_0^S}^{T_j^S} \Big[L(T_{j-1}^S, T_j^S) - K \Big] \right)^+ \\
=& \left(\sum_{j=1}^{N} \tau_j P(T_0^S, T_j^S) \Big(F(T_0^S; T_{j-1}^S, T_j^S) - K \Big) \right)^+ \\
=& \big(S^{1,N}(T_0) - K \big)^+ L(T_0^S)
\end{aligned}$$

Then, the time $t = 0$ price of the contract is:

$$\begin{aligned} V(0, S^{1,N}(0)) =&\mathbb{E}^{\mathbb{Q}}\Big[D(t, T_0^S)\big(S^{1,N}(T_0^S) - K\big)^+ L(T_0^S)\Big] \\ =&L(0)\mathbb{E}^{swap}\big[\big(S^{1,N}(T_0^S) - K\big)^+\big] \end{aligned} \tag{22}$$

As proved previously, the forward swap rate $S^{1,N}(t)$ is a martingale under swap measure,

$$\mathbb{E}_r^{swap}\big[S^{1,N}(t)\big] = S^{1,N}(r), \quad 0 \le r \le t \le T_0^S. \tag{23}$$

To complete the pricing with a analytical formula, one can assume a market model for the forward swap rate, which has to be driftless, for instance

$$dS^{1,N}(t) = \sigma^{1,N}(t)S^{1,N}(t)dW^{swap}(t), \quad 0 \le t \le T_0^S. \tag{24}$$

Alternatively, we can use SABR model,

$$dS^{1,N}(t) = \alpha^{1,N}(t)\big(S^{1,N}(t)\big)^\beta dW_1^{swap}(t), \tag{25}$$

$$d\alpha^{1,N}(t) = \nu\alpha^{1,N}(t)dW_2^{swap}(t) \tag{26}$$

$$dW_1^{swap}(t)dW_2^{swap}(t) = \rho dt. \tag{27}$$

which has an approximation formula for volatility to be plugged into BS formula.

3.2 CMS Caplets

We now come back to CMS pricing, starting from CMS caplets for a reason. Recall, the payoff of a CMS caplet is,

$$\big(S^{1,N}(T_{p-1}) - K\big)^+, \tag{28}$$

that is paid at T_p, for $p \in \{1, 2, ..., M\}$. The present value, at time t, is,

$$V_{caplet}^{CMS}(t) = L(t)\mathbb{E}^{swap}\Bigg[\frac{(S^{1,N}(T_{p-1}) - K)^+ P(T_{p-1}, T_p)}{L(T_{p-1})}\Bigg] \tag{29}$$

Observe the ratio $P(T_{p-1}, T_p)/L(T_{p-1})$ is another martingale under swap measure,

$$\mathbb{E}^{swap}\Bigg[\frac{P(T_{p-1}, T_p)}{L(T_{p-1})}\Bigg] = \frac{P(0, T_p)}{L(0)}. \tag{30}$$

By dividing $P(T_{p-1}, T_p)/L(T_{p-1})$ by (30), we have

$$V_{caplet}^{CMS}(0) = P(0, T_p)\mathbb{E}^{swap}\Bigg[\big(S^{1,N}(T_{p-1}) - K\big)^+ \frac{P(T_{p-1}, T_p)/L(T_{p-1})}{P(0, T_{p-1})/L(0)}\Bigg] \tag{31}$$

which can be re-written as:

$$V_{caplet}^{CMS}(0) = \frac{P(0,T_p)}{L(0)}\Bigg(L(0)\mathbb{E}^{swap}\Big[\Big(S^{1,N}(T_{p-1}) - K\Big)^+\Big]\Bigg)$$
$$+ P(0,T_p)\mathbb{E}^{swap}\Bigg[\Big(S^{1,N}(T_{p-1}) - K\Big)^+\Bigg(\frac{P(T_{p-1},T_p)/L(T_{p-1})}{P(0,T_{p-1})/L(0)} - 1\Bigg)\Bigg] \tag{32}$$

The first term is the price of a European Swaption with notional $P(0,T_p)/L(0)$, which is quoted on the market. The second term is a *convexity correction* term.

The martingale property of forward swap rate under swap measure suggests one to express the correction term as a function of forward swap rate, at least, approximately. To be concrete, we want to write

$$\frac{P(T_{p-1},T_p)}{L(T_{p-1})} = G\big(S^{1,N}(T_{p-1})\big), \quad \frac{P(0,T_{p-1})}{L(0)} = G\big(S^{1,N}(0)\big), \tag{33}$$

for a Borel measurable function $G : \mathbb{R} \mapsto \mathbb{R}$. If this can be done, then the convexity correction terms amounts to evaluate the following expectation at time 0:

$$CC = P(0,T_{p-1})\mathbb{E}^{swap}\Bigg[\big(S^{1,N}(T_{p-1}) - K\big)^+\Bigg(\frac{G(S^{1,N}(T_{p-1}))}{G(S^{1,N}(0))} - 1\Bigg)\Bigg] \tag{34}$$

To justify the existence of $G(\cdot)$. Let's suppose the reference forward starting swap has each period of equal length $\delta = 1/q$, where q is the number of periods per year. As mentioned previously, $L(t)$ can be thought as a contract that enables receiving \$1 per year for N years following schedule $\{T_1^S, ..., T_N^S\}$. Such contract can be priced firstly discounted back to T_0^S and then t, i.e.,

$$L(t) = P(t,T_0^S)\sum_{i=1}^{N} 1/q \times \frac{P(t,T_i^S)}{P(t,T_0^S)}. \tag{35}$$

Remember $S^{1,N}(t)$ is a average rate that is compatible with the yield rate from T_1^S to T_N^S(the swap rate is essentially the fixed rate exchanged against to LIBOR). Thus, we can write the following approximation:

$$L(t) \approx P(t,T_0^S)\sum_{i=1}^{N}\frac{1/q}{\big(1 + S^{1,N}(t)/q\big)^i} = \frac{P(t,T_0^S)}{S^{1,N}(t)}\Bigg(1 - \frac{1}{\big(1 + S^{1,N}(t)/q\big)^N}\Bigg), \tag{36}$$

By the same token, the zero coupon bond for the pay date T_p^S, for $p = 1, ..., N$, can be approximated by:

$$P(t,T_p) \approx \frac{P(t,T_0^S)}{\big(1 + S^{1,N}(t)/q\big)^{\Delta}}, \ where \ \Delta = \frac{T_p^S - T_0^S}{T_1^S - T_0^S}. \tag{37}$$

Consequently, (36)-(37) lead to:

$$G\big(S^{1,N}(t)\big) = \frac{P(t, T_{p-1}^S)}{L(t)} \approx \frac{S^{1,N}(t)}{\big(1 + S^{1,N}(t)/q\big)^{\Delta}} \frac{1}{1 - \frac{1}{\big(1+S^{1,N}(t)/q\big)^{N}}}. \tag{38}$$

As desired, we obtain the apporimxating function $G(\cdot)$.

Remark 3.1. *The bond approximation scheme we used is a flat approximation which assumes the initial and final yield curves are flat, at least over the tenor of the reference Swaption. There are other approaches,* exact yield model*,* parallel shifts model*,* non-parallel shifts model *e.t.c.. Since the correction term is of small order, the choice of the method generally won't affect too much.*

As a next step, we want to further simplify (34) to be represented only as a function of payer Swaption, for which analytical formula exists (22). The key here is the integral version of Taylor expansion.

For a sufficient regular function f (second order continuously differentiable, with bounded derivatives),

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(t)dt \\ &= f(x_0) + [(t-x)f'(t)]\big|_{t=x_0}^{t=x} - \int_{x_0}^{x} (t-x)f''(t)dt \\ &= f(x_0) + (x-x_0)f'(x_0) - \int_{x_0}^{x} (t-x)f''(t)dt \end{aligned} \tag{39}$$

Consider f below,

$$f(x) := \begin{cases} (x-K)\left(\frac{G(x)}{G\big(S^{1,N}(0)\big)} - 1\right), & x > K, \\ 0, & x \le K. \end{cases} \tag{40}$$

Observe $f(K) = 0$ and $x > K$, $(x-K)^+ = (x-K)$, apply (39) to (40),

$$f(x) = f'(K)(x-K)^+ + \int_K^y (x-K)^+ f''(t)dt. \tag{41}$$

Then, the convexity correction term can be re-written as,

$$\begin{aligned} CC = P(0, T_p)\Big\{ & f'(K)\mathbb{E}\big[\big(S^{1,N}(T_{p-1}) - K\big)^+\big] \\ & + \int_K^{\infty} f''(x)\mathbb{E}\big[\big(S^{1,N}(T_{p-1}) - x\big)^+\big]dx \Big\} \end{aligned} \tag{42}$$

Substituting the correction term back into the pricing formula yields,

$$V^{CMS}_{caplet}(0) = \frac{P(0,T_p)}{L(0)}\left\{(1+f'(K))C(K) + \int_K^\infty C(x)f''(x)dx\right\}, \tag{43}$$

where $C(x)$ is the time 0 European payer Swaption price as defined in (22) with strike x.

Formula (43) is significant, it allows pricing CMS caplet in terms of Swaption at different levels of strikes. If we use classical SABR model for forward swap rate process, on the wings the pricing maybe not satisfactory. The AKS improved version of SABR gives much better performance on the wing, which makes the above formula very friendly to work with. However, in general, the pricing formula (43) is a bit computationally heavy because of integration, summation, along different strike, each of which needs a Swaption evaluation. Therefore, we will later on introduce an integration free approach.

3.3 CMS Floorlets and Swaplets

By repeating the above arguments, we can derive the CMS floorlet,

$$V^{CMS}_{floor}(0) = \frac{P(0,T_p)}{L(0)}\left\{(1+f'(K))P(K) - \int_{-\infty}^K P(x)f''(x)dx\right\}, \tag{44}$$

where $P(x)$ is the price of a receiver Swaption with strike x,

$$P(x) = L(0)\mathbb{E}^{swap}\left[\left(x - S^{1,N}(T_{p-1})\right)^+\right]. \tag{45}$$

Notice, the call-put parity between CMS caplets and floorlets remains valid, i.e.,

$$\left(S^{1,N}(T_{p-1}) - K\right)^+ - \left(K - S^{1,N}(T_{p-1})\right)^+ = S^{1,N}(T_{p-1}) - K. \tag{46}$$

Generally, it says the payoff of a CMS caplet(with strike x) minus a CMS floorlet(with strike x), is equal to the payoff a CMS swaplet minus x. In particular,

$$V^{CMS}_{cap}(0) - V^{CMS}_{floor}(0) = V^{CMS}_{swaplet}(0) - xP(0,T_p). \tag{47}$$

Therefore, we have CMS Swaplets,

$$V^{CMS}_{swaplet}(0) = P(0,T_p)S^{1,N}(0) + \frac{P(0,T_p)}{L(0)}\left\{\int_R^\infty C(x)f''_{atm}(x)dx + \int_{-\infty}^R P(x)f''_{atm}(x)dx\right\} \tag{48}$$

where $R := S^{1,N}(0)$ the initial forward swap rate.

3.4 Further Approcimation – Analytical Formula

As mentioned earlier, the pricing now still requires a integral of Swaption along strike direction. We now provide a further approximation of the pricing so that integration, or summation, can be avoided. The function $G(\cdot)$ is a sufficiently regular function, without jumps and spikes, in addition, $S^{1,N}(T_{p-1})$ can not be too far from the initial swap rate $S^{1,N}(0)$ in general. Therefore, it is reasonable to expand $G(\cdot)$ as

$$G(x) \approx G\big(S^{1,N}(0)\big) + G'\big(S^{1,N}(0)\big)\big(x - S^{1,N}(0)\big) + \cdots . \tag{49}$$

for x in the neighborhood of $S^{1,N}(0)$. Substitute in f,

$$f(x) \approx \frac{G'\big(S^{1,N}(0)\big)}{G\big(S^{1,N}(0)\big)}\big(x - S^{1,N}(0)\big)\big(x - K\big). \tag{50}$$

The advantage of such approximation is the vanish of the second order $f''(x) = 0$. Now, we can rewrite our CMS caplet pricing formula:

$$V_{caplet}^{CMS}(0) = \frac{P(0,T_P)}{L(0)}C(K) + G'\big(S^{1,N}(0)\big)\Big\{(K - S^{1,N}(0)C(K) + 2\int_K^\infty C(x)dx\Big\} \tag{51}$$

where we have used $G(S^{1,N}(0)) = P(0,T_p)/L(0)$. Let's make the following observation:

$$\begin{aligned}\int_K^\infty V(0,x)dx &= \int_K^\infty L(0)\mathbb{E}^{swap}\Big[\big(S^{1,N}(T_{p-1}) - K\big)^+\Big]dx\\ &= L(0)\mathbb{E}\Big[\int_K^\infty \Big(S^{1,N}(T_{p-1}) - x\Big)^+ dx\Big]\\ &= \frac{1}{2}L(0)\mathbb{E}^{swap}\Big[\Big(\big(S^{1,N}(T_{p-1}) - K\big)^+\Big)^2\Big].\end{aligned}$$

Putting these all together yields

$$V_{caplet}^{CMS}(0) = \frac{P(0,T_P)}{L(0)}C(K) + G'\big(S^{1,N}(0)\big)L(0)\mathbb{E}^{swap}\Big[\big(S^{1,N}(T_{p-1}) - S^{1,N}(0)\big)\big(S^{1,N}(T_{p-1}) - K\big)^+\Big] \tag{52}$$

for CMS caplet price. An identical argument yields the formula for floorlet,

$$V_{floorlet}^{CMS}(0) = \frac{P(0,T_P)}{L(0)}P(K) - G'\big(S^{1,N}(0)\big)L(0)\mathbb{E}^{swap}\Big[\big(S^{1,N}(0) - S^{1,N}(T_{p-1})\big)\big(K - S^{1,N}(T_{p-1})\big)^+\Big]. \tag{53}$$

By put-call parity, the CMS swap payment is then:

$$V_{swaplet}^{CMS}(0) = P(0, T_p) S^{1,N}(0) + G'\big(S^{1,N}(0)\big) L_0 \mathbb{E}^{swap}\Big[\big(S^{1,N}(T_{p-1}) - S^{1,N}(0)\big)^2\Big]. \tag{54}$$

To obtain an explicit formula, we can work under a specific market model – forward swap rate model, which will give us an explicit formula that is efficient to evalute.