Compact Set

Jianing Yao
Department of MSIS-RUTCOR
Rutgers University, the State University of New Jersey
Piscataway, NJ 08854 USA

January 28, 2015

1 Compact Set

Finite set is usually easy to handle, one can dis-tangle almost all issues by fingers. Unfortunately, we cannot always have such nice set. The notion of *"compactness"* saves the world, indeed, it is the next bets thing to be finite.

Given metric space (X, d), let us first define the open cover of a set:

Definition 1.1 An *open cover* of $E \in X$ is a collection of open sets $\{G_\alpha\}$, whose union contains E.

Also we can have the notion of *sub-cover*:

Definition 1.2 A *Sub-cover* of $\{G_\alpha\}$ is a sub-collection $\{G_{\alpha_r}\}$ that still covers E.

Let's give an example:

Example 1.1 In $\mathbb{R}$, $[\frac{1}{2}, 1)$ has an open cover $\{V_n\}_{n\geq 3}$, where $V_n = (\frac{1}{n}, 1 - \frac{1}{n})$. Of course, we can think of a lot of open covers for this set, for example, the trivial singleton $\{(0, 2)\}$ is an open cover. And for a more complicated one, $\{W_x\}_{x\in[\frac{1}{2},1)}$ is an open cover, where $W_x = N_{\frac{1}{10}}(x)$.

After having this example above, we have a natural question to ask: *given an open cover, do we need all these open sets to still cover the set?*

For *Example 1.1*, we can have a sub-cover $\{V\}_{n\geq 22}$ that still covers $[\frac{1}{2}, 1)$. Or

$$A = \{W_{\frac{5}{10}}, W_{\frac{6}{10}}, W_{\frac{7}{10}}, W_{\frac{8}{10}}, W_{\frac{9}{10}}\}$$

forms a sub-cover as well, and actually, we have finitely many open sets in this sub-collections.

Example 1.2 For $[0, 1]$ in $\mathbb{R}$, it has a finite sub-cover: $\{W_0, W_1, V_{11}\}$.

Now, we can give the definition of *compact set*:

Definition 1.3 A set K is *compact* in X if every open cover of K contains a finite sub-cover.

Example 1.3 $\mathbb{Z}$ in $\mathbb{R}$ is not compact, because it doesn't have finite sub-cover (Easy to verify).

From the definition, we notice that to prove the compactness of a set **every** open cover needs to be checked. This is hopeless and not practical, we shall give some equivalent characterization of compact set.

Theorem 1.4 Finite sets are in fact compact.

Proof. Consider some open cover $\{G_\alpha\}$ covering $\{x_1, ..., x_N\}$ for all x_i. Choose one G_{α_i} that contains x_i, then $\{G_{\alpha_i}\}_{i=1}^N$ covers the set. □

Theorem 1.5 Compact sets are bounded.

Remark 1.6 A set K is *bounded* if $K \subset N_r(x)$ for some $x \in X$.

Proof. Suppose K is compact, let $B(x) = N_1(x)$ and take $\{B(x)\}_{x \in K}$, which is an open cover of K. By compactness of K, there exists a finite sub-cover $\{B(x_i)\}_{i=1}^n$. Set $R = \max_{1 \leq (i,j) \leq N}\{d(x_i, x_j)\}$, then $N_{R+2}(x_i)$ contains all K. □

2 Compactness is an Intrinsic Property

In this section, we want to illustrate the notion of compact is intrinsic of the set, has nothing to do with which metric space the set is in. But let's firstly examine the open set. In *Figure 1*, (a, b) is certainly open in $\mathbb{R}$, but it is not open in $\mathbb{R}^2$. Why? Because in $\mathbb{R}^2$, if we want to

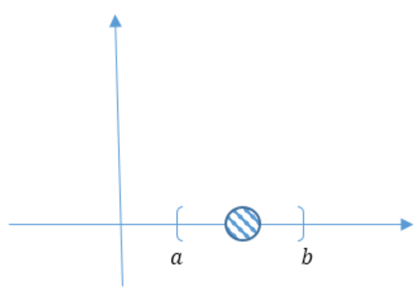

Figure 1: Relative Openness

prove (a, b) is open, then every point is an interior point, which is certainly not, as shown in the figure. Thus, we have the conclusion openness depends on the metric space the set is in.

Before proceeding to the compact set, let's discuss the concept of *"relative open set"*. If $Y \subset X$, where (X, d) is a metric space, then Y is certainly a metric space (Y inherits the metric from X). In *Figure 2*, The neighborhood of x in X looks like the blue dashed line circle, but the neighborhood of x in Y is the red dashed line incomplete circle, because by definition the open ball is defined as $N_r(x) = \{y : d(x, y) < r\}$, but $y \in Y$. Indeed, the open ball of x with radius r is different with respect to different metric space.

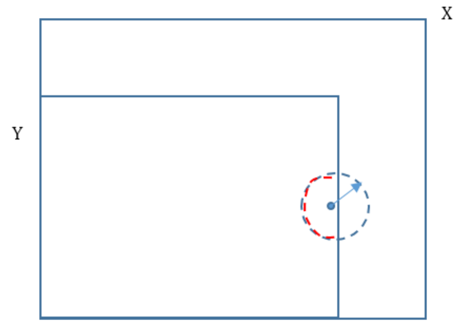

Figure 2: Relative Openness C.t.d

Then, how about the general sets open in X versus open in X. Let's again draw a picture (*Figure* 3), we observe that the blue solid curve including part of the boundary of E is open in Y, but it is certainly not open in X. This motivates the definition of openness w.r.t metric space:

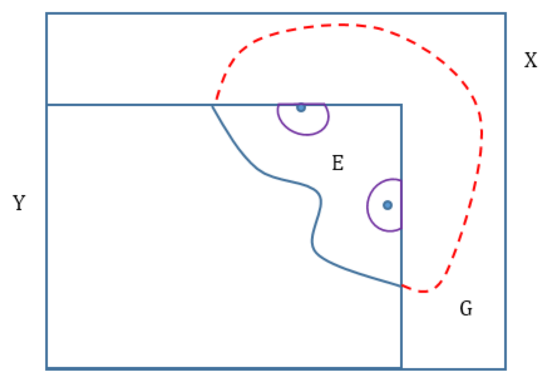

Figure 3: Relative Openness C.t.d

Definition 2.1 A set U is *open* in Y (*or, relative open in* Y), if every point of U is an interior point of U.

Remark 2.1 The only difference here is that we need to specify the interior point is defined using *n.b.h.d* in Y.

The next theorem allows us to pass from open set in one space to open set in another space.

Theorem 2.2 If $E \subset Y \subset X$, then E is open in Y if and only if $E = Y \cap G$ for some G open in X.

Proof. ("$\Leftarrow$") Take a point of x of E, if $N_r(x)$ is in G, then $N_r(x) \cap Y$ is a *"n.b.h.d"* of x in E. ("$\Rightarrow$") Every point x has $N_{r_X}(x) \subset Y \cap E$, then $\cup_{x \in E} N_{r_X}(x)$ in X is open, call it G. $\square$

Finally, we have the following theorem, which basically says that compactness is an intrinsic property of a set.

Theorem 2.3 If $K \subset Y \subset X$, K is compact in Y if and only if K is compact in X.

Proof. ("$\Rightarrow$") Assume K is compact in Y, consider an open cover $\{U_\alpha\}$ of K in X. Let $V_\alpha = U_\alpha \cap Y$, then $\{V_\alpha\}$ certainly cover K in Y. By compactness, there exists finite sub-cover $\{V_{\alpha_i}\}_{i=1}^n$ cover K in Y. Thus, $\{U_{\alpha_i}\}$ covers K in X, as desired.

("$\Leftarrow$") Consider an open cover $\{V_\alpha\}$ of K in Y, by earlier theorem, there exists U_α such that $U_\alpha \cap Y = V_\alpha$, these $\{U_\alpha\}$ covers K in X, so there exists finite sub-cover $\{U_{\alpha_i}\}_{i=1}^n$, then $\{V_{\alpha_i}\}_{i=1}^n$ is a finite sub-cover of $\{V_\alpha\}$ for K in Y. $\square$

Let's conclude this section by an interesting observation: in previous lecture, we talk about the bases of a topological space. The insight is, if one has a metric space that has a countable number of bases, then the space is small in some sense. It turns out that if one has a metric space that is compact, then the space has a countable bases (countable dense subsets).

3 Relation of Compact sets to Closed Sets

In this section, we are going to investigate the relationship between compact set and closed set. We already know that compact set is bounded, actually it is also closed.

Theorem 3.1 Given metric space (X, d), any compact set $K \subset X$ is closed.

Proof. It is equivalent to show that K^c is open. Consider a point p not in K, we will show, p has no *n.b.h.d* that does not intersect K(p is interior to K^c). For any $q \in K$, define $V_q = N_{\frac{r}{2}}(q)$, $U_q = N_{\frac{r}{2}}(p)$, where $r = d(p, q)$. Notice $\{V_q\}_{q \in K}$ covers K, by the compactness of K, there exists a finite sub-cover $\{V_{q_i}\}_{i=1}^n$. Thus,

$$W = U_{q_1} \cap U_{q_2} \cap \cdots \cap U_{q_n}$$

is an open set with radius $\min(p, q_i)$. Observe $W \cap V_{q_i} = \emptyset$ for each i, because $W \subset U_{q_i}$ and $U_{q_i} \cap V_{q_i} = \emptyset$. W is the desired *n.b.h.d.* We are done because the choice of $p \in K^c$ is arbitrary. $\square$

By the virtue of above theorem, we find $(0, 1)$ is not compact. The entire $\mathbb{R}$ is not compact, because it its not bounded though it is closed. As we observe in the last section, to prove compactness of a set via definition is not that easy, because you need to prove for **every open cover**, there exists a finite sub-covers. Now, we can at least give one more characterization of compact set.

Theorem 3.2 A closed subset B of a compact set K is also compact.

Proof. Let $\{U_\alpha\}$ be an open cover of B, notice B^c is open, thus $\{U_\alpha\} \cup \{B^c\}$ is an open cover of K. Therefore, by compactness, there exists a finite sub-cover $\{U_{\alpha_i}\}_{i=1}^n \cup \{B^c\}$ of $\{U_\alpha\} \cup B^c$. Since B^c does not cover B, thus the finite sub-cover of B is actually $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$. □

As an immediate *corollary*, readers should prove by themselves:

Corollary 3.3 Given closed set F and compact set K in X, $F \cap K$ is compact.

4 Heine-Borel Theorem

Up to now, we already shown the necessary conditions for compactness and one equivalent characterization. What can be a sufficient condition for a set to be compact, at least in some space. This is resolved by the great *Heine-Borel Theorem*, but before that, we shall have some preliminaries. Firstly, we introduce the concept of *nested closed intervals (k-cells)*, i.e., $I_n = [a_n, b_n]$, for $n \geq 1$, where if $m \geq n$, then $a_n \leq a_m \leq b_m \leq b_n$.

Theorem 4.1 (*Nested Interval Theorem(NIC)*) The *nested closed intervals (k-cells)* in $\mathbb{R}$ ($\mathbb{R}^k$) are not empty.

Proof. Let $x = \sup_i\{a_i\}$, it exists because they are bounded above by b_1. Clearly, $x \geq a_i$ for all i and $x \leq b_n$, because b_n is an upper-bound for all a_n. □

As a little digression, we can use above fact to prove $\mathbb{R}$ is uncountable. The reasoning goes as follows, assume $\mathbb{R}$ is countable, i.e., $\{x_1, x_2, \cdots\}$, in *Figure 4*, Choose I_1 misses x_1,

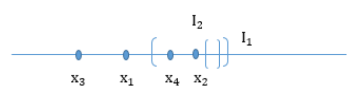

Figure 4: Nested Interval

$I_2 \subset I_1$ misses x_1, x_2, $I_3 \subset I_2$ misses x_1, x_2 and x_3, ... We can continue this process of nested sequence, by above theorem, we can always find $x \in I_n$, that x is not in the list.

Since we have the above nice theorem, let's focus our attention on $\mathbb{R}$ (or, more generally, $\mathbb{R}^n$), the first sufficient condition for compactness is as following:

Theorem 4.2 Any closed interval $[a, b]$ is compact in $\mathbb{R}$.

Proof. (By contradiction) Suppose $[a, b]$ is not compact, then there is an open cover $\{G_\alpha\}$ that has no finite sub-cover. Pick half point $c_1 \in (a, b)$, as in *Figure 5*: Then $\{G_\alpha\}$ covers both $[a, c_1]$, $[c_1, b]$, at least one of them has no finite sub-cover. Without loss of generality, let's assume $I_1 = [a, c_1]$ has no finite sub-cover, then we split it again at half point c_2, it becomes $[a, c_2]$, $[c_2, c_1]$, where we assume $[c_2, c_1]$ has no finite sub-cover, continue this process and we will obtain: $I_1 \supset I_2 \supset I_3 \supset \cdots$ nested closed intervals. Observe (1) each halved with each step and (2) none of them has finite sub-cover of $\{G_\alpha\}$. By the NIC, there exists

Figure 5: Construction for Proof

$x \in I_n$, for all n. Thus, x is in some open cover $G_{\hat{\alpha}}$, x is an interior point of this set, i.e., $\exists r > 0$, $N_r(x) \subset G_{\hat{\alpha}}$. Notice, by (1), some $I_n \subset N_r(x)$ for some n meaning $G_{\hat{\alpha}}$ covers I_n, which contradicts (2). □

Eventually, we can give the *Heine-Borel Theorem*,

Theorem 4.3 (*Heine-Borel*) In $\mathbb{R}$ (or $\mathbb{R}^n$), K is compact if and only if K is closed.

Proof. The forward direction is already shown, we only need to prove the other way around. Since K is bounded, then $K \in (-r, r) \subset [-r, r]$ for some $r > 0$. Since $[-r, r]$ is closed, it is compact by *Theorem 4.2*. Also K itself is closed and a subset of $[-r, r]$ (that is compact), by *Corollary 3.3*, K is also compact. □

Remark 4.4 This is true for $\mathbb{R}^n$ but not necessary for arbitrary metric space. In fact, if a metric space can have this theorem hold, we say the metric space enjoys *Heine Borel Property*.

Example 4.5 Let's denote the space of all continuous bounded functions on real line as $C_b(\mathbb{R})$, and the metric for this space is, for any $f, g \in C_b(\mathbb{R})$,

$$d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

Let's define the function f_n as follows(*Figure 6*). We claim the set $A = \{f_n \mid n \in \mathbb{Z}\}$ is

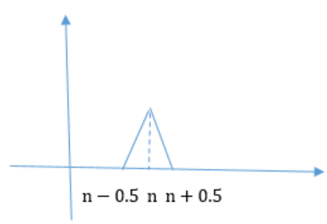

Figure 6: Counter Example

bounded and closed but not compact.

Here comes another necessary and sufficient condition for $K \subset X$ to be compact regardless what metric space it is.

Theorem 4.6 K is compact if and only if every infinite subset E of K has limit point in K.

Proof. ("$\Rightarrow$") Proof by contradiction. Assume no point of K is a limit point of E, then each point $q \in E$ has a *n.b.h.d* V_q containing exactly one point q of E. Thus, $\{V_q\}$ covers E with no finite sub-cover, contradiction!.

(For "$\Leftarrow$", we will only prove it for $\mathbb{R}^n$, but it's true for all metric space) We only need to show K is closed and bounded (by *Heine-Borel*). Suppose K is not bounded, then we can choose a sequence x_n such that $|x_n| > n$, thus, it has no limit point. On the other hand, suppose K is not closed, then there exists $p \notin K$ that is a limit point of E. Choose x_n such that $d(x_n, p) \leq \frac{1}{n}$, x_n has a limit point of p and no others. □

Corollary 4.7 (*Bolzano-Weierstrass Theorem*) Every bounded infinite subsets of $\mathbb{R}^n$ has a limit point in k-cell.

Proof. If the subset E is bounded, E is in some compact k-cell. □

As a generalization of *nested closed interval theorem*, we have one due to *Cantor*,

Theorem 4.8 (*Cantor, Finite Intersection Property*) Suppose $\{K_\alpha\}$ are compact subsets in some metric space (X, d). If any *finite sub-collection* has non-empty intersection, then the intersection of all of them is not empty.

Proof. Let $U_\alpha = K_\alpha^c$, and it is open since compact set is closed. Fix some $K \in K_\alpha$, if $\cap_\alpha K_\alpha = \emptyset$, then $\{U_\alpha\}$ cover K. Then, there exists a finite sub-cover $\{U_{\alpha_i}\}_{i=1}^n$ covering K. Then

$$K \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$$

Contradiction against the hypothesis. □