# Series

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## 1 Series

Let's start this section with some examples:

**Example 1.1** One can prove by *Fourier Series* or *Euler's formula*:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

**Example 1.2** How about the following sum of infinite series:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Someone argues it is equal to 0, someone would say it is 1, it depends on how they combine the terms.

**Example 1.3** The following one is what we learned before:

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots = \frac{3}{2}$$

which is a special case of :

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

Why? Check this out:

$$(1 + x + x^2 + \cdots)(1 - x) = 1$$

It is not necessary that sum of arbitrary infinite series will lead to a finite number, at least, sometimes it is controversial. We will give a justification of such phenomenon by abstraction. Given $\{a_n\}$, define

$$\frac{n=p}{q} a_n = a_p + a_{p+1} + \cdots + a_q$$

or usually, we define the *n-th partial sum* as:

$$s_n = \sum_{k=1}^{n} a_k$$

This partial sum is a sequence, written as:

$$\sum_{n=1}^{\infty} a_n$$

This may or may not converge, if it does and converges to $s$,

$$\sum_{n=1}^{\infty} a_n = s = \lim_{k\to\infty} = s_k = \lim_{n\to\infty} \left( \sum_{k=1}^{n} a_k \right)$$

Now, the natural question to ask is: when does a series converge, or, equivalently, when the sequence of partial sum converge?

**Example 1.4** Given $a_n = \frac{1}{n}$, does

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

converge? No, it is so called *harmonic series* that never converges. How can we justify this? We can ask ourselves if $\{s_n\}$ is Cauchy (actually, it is sufficient, since the series is real-valued). For $n < m$,

$$s_m - s_n = a_{n+1} + \cdots + a_m$$

In particular, if we choose $m = 2n$, then

$$s_{2n} - s_n > \frac{1}{2}$$

Therefore, it is **NOT** Cauchy, thus not convergent.

The next theorem will render us a way to determine the convergence of the series:

**Theorem 1.5** (*Cauchy criterion for series*) $\sum_{n=1}^{\infty} a_n$ converges if and only if $\forall \epsilon > 0$, $\exists N$, such that, for $n > N$,

$$|\sum_{k=n}^{m} a_k| < \epsilon$$

What if we let $m = n$,

**Corollary 1.6** (*term test for the convergence*) $\sum_{n=1}^{\infty} a_n$ converges implies $\lim_{n\to\infty} a_n = 0$.

The converse is not true in general, recall the harmonic series. As a special case, if the series is non-negative, we have

**Theorem 1.7** If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sum is bounded.

*Proof.* If $a_n \geq 0$, partial sum is monotonically increasing. Since it is also bounded, we conclude that it converges. $\square$

If you know one series, but don't anything about another series. We can compare them and hopefully tell something about the new sequence.

**Theorem 1.8** (*comparison test*)

- If $|a_n| \leq c_n$ for $n$ large enough ($n \geq N_2$) and $\sum_{n=1}^{\infty} c_n$ converge, then $\sum_{n=1}^{\infty} a_n$ converges;
- If $a_n \geq d_n \geq 0$ for $n$ large enough, if $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

*Proof.* For the first claim. Fix a $\epsilon > 0$. Since $\sum_{n=1}^{\infty} c_n$ converges, $\exists N_1$ such that $m, n \geq N_1$ implies $|\sum_{n=1}^{m} c_n| < \epsilon$. Let $N = \max\{N_1, N_2\}$, for $m, n \geq N$, $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \epsilon$ as desired.

For the second claim. Use the above result to show the contrapositive is: if $\sum_{n=1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} d_n$ converges. $\square$

Now, let's investigate what is the series to be compared. Since we know the most about the *geometric series*:

**Theorem 1.9** If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$; if $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges.

*Proof.* If $|X| < 1$, let $s_n = 1 + x + x^2 + \cdots + x^n$, then

$$s_n = \frac{1 - x^{n_1}}{1 - x}$$

Take the limit of it:

$$\lim_{n \to \infty} s_n = \frac{1}{1-x} \lim_{n \to \infty} (1 - x^{n+1}) = \frac{1}{1-x}.$$

If $|x| \geq 1$, terms do not go to 0, so $x_n$ diverges. $\square$

Let's check this one, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, does it converge or diverge, for what $p$? Can we use comparison?

**Theorem 1.10** (*Cauchy*) Suppose $a_1 \geq a_2 \geq a_3 \cdots \geq 0$, monotonically decreasing, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

*Proof.* (*proof idea*) Let

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n = (a_1) + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots \\ t_k &= a_1 + 2a_2 + \cdots + 2^k a_{2^k} = (a_1) + (a_2 + a_2) + (a_3 + a_3 + a_3 + a_3) + \cdots \end{aligned}$$

If $n < 2^k$, then $s_n < t_k$. By comparison, the convergence of $\{t_k\}$ implies the convergence of $\{s_n\}$. For the other direction,

$$2s_n = 2a_1 + 2a_2 + 2(a_3 + a_4) + 2(a_5 + \cdot + a_8) + \cdots$$
$$t_k = a_1 + (a_2 + a_2) + 4a_4 + 8a_5 + \cdots$$

If $n > 2^k$, we have another way around comparison, the assertion follows. □

Application of this theorem can address the question about $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

**Theorem 1.11** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, but diverges if $p \leq 1$.

*Proof.* If $p \leq 0$, then the term does not go to 0, diverge. If $p > 0$, use above theorem to pick out some terms:

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=1}^{\infty} 2^{(1-p)k}$$

It is geometric, it converges if and only if $2^{1-p} < 1$, that is $p > 1$. □

Another series always arises is $\sum_{n=0}^{\infty} \frac{1}{n!}$.

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = e$$

Why it converges? It's partial sums are bounded by $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, about 3. How fast is the convergence?

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots = \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$$

The terms inside the brackets is the geometric series, which converges to $\frac{n+1}{n}$. That is to say, $e - s_n < \frac{1}{n!n}$, as $n$ becomes larger, the difference become small rapidly. What does this imply? We can use this to show that $e$ is irrational. Why? If $e = \frac{m}{n}$, then

$$n!(e - s_n) < \frac{1}{n}$$

If $e$ were rational, then the left hand side will be integer, but it can not be happen, there is no integer in the interval $(0, 1)$.

# 2 Series Convergent Test

We will continue to investigate the criterion to have a series to be convergent.

**Theorem 2.1** (*Root Test*) Given $\sum_{n=1}^{\infty} a_n$, let $\alpha = \limsup \sqrt[n]{|a_n|}$, then $\alpha > 1$ implies convergent, $\alpha < 1$ implies diverge and $\alpha = 1$ means test is inconclusive.

*Proof.* If $\alpha < 1$, choose $\beta$ such that $\alpha < \beta < 1$, then $\exists N$ such that for $n \geq N$,

$$\sqrt[n]{|a_n|} < \beta$$

So $|a_n| < \beta^n$ for $n \geq N$. But $\sum_{n=1}^{\infty} \beta^n$ converges, so $\sum_{n=1}^{n} |a_n|$ converges as well.

If $\alpha > 1$, there exists a subsequence, $\sqrt[n]{|a_k|} \to \alpha > 1$. So $|a_{n_k}| > 1$ for infinitely many terms, so the terms do not go to 0, the series diverges.

If $\alpha = 1$, notice $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges while $\frac{1}{n^2}$ converges, for both of them $\alpha = 1$. Thus, the conclusion is not for sure. □

**Theorem 2.2** (*Ratio Test*) Given $\sum_{n=1}^{\infty} a_n$, if $\limsup |\frac{a_{n+1}}{a_n}| < 1$, $\sum_{n=1}^{\infty} a_n$ diverges, if $\limsup |\frac{a_{n+1}}{a_n}| > 1$, $\sum_{n=1}^{\infty} a_n$ converges.

*Proof.* (*by comparison*) Fir the first case, we have $|\frac{a_{n+1}}{a_n}| < \beta < 1$ for $m \geq N$, for some $N$. From that,

$$|a_{n+1}| < \beta |a_n| < \beta^2 |a_{n-1}| < \beta^3 |a_{n-2}| < \cdots$$

Thus,

$$|a_{N+k}| < \cdots < \beta^k a_N$$

As a result: $\sum_{k=1}^{\infty} a_{N+k} \leq a_N \sum_{k=1}^{\infty} \beta$, the right hand side is bounded, thus the assertion follows. For divergence case, notice that the term does not go to 0. □

We have developed a lot of tests for series, one of the most importance series one often encounter is the *power series*, i.e., $c_n \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

where $z \in \mathbb{C}$. What kind of $z$ can make it converge?

**Theorem 2.3** If $\alpha = \limsup \sqrt[n]{|c_n|}$, let $R = \frac{1}{\alpha}$ (radius of convergence), then $\sum_{n=1}^{\infty} c_n z^n$ converges if $|z| < R$, it diverges if $|Z| > R$.

*Proof.* (*proof idea*) $\limsup \sqrt[n]{|a_n|} = \limsup |z| \cdot \sqrt[n]{|c_n|} < 1$, but $\alpha = \limsup \sqrt[n]{|c_n|}$. □

What if we have two sequences $\{a_n\}$, $\{b_n\}$, can we say something about $\sum_{n=1}^{\infty} a_n b_n$? Let's carry out so called *summation by parts*, the analogy of *integration by parts*. Define $A_n = \sum_{k=1}^{n} a_k$ for $n > 0$, set $A_{-1} = 0$. Then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

If we set $v = A_n$, $u = b$, then $dv = a$, $du = b_{n+1} - b_n$. As a result:

$$\int u dv = -\int_p^{q-1} v du + uv|_p^q$$

**Theorem 2.4** If $A_n$ is bounded and $b_n$ is positive and decreasing, approaching to 0, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

*Proof.* (*idea*) Suppose $|A_n| \leq M$. Given $\epsilon > 0$, $\exists N$ such that $b_N \leq \frac{\epsilon}{2M}$. For $q \geq p \geq N$,

$$\begin{aligned} |\sum_{n=p}^{q} a_n b_n| &\leq M|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q - b_p| \\ &\leq 2Mb_p \leq 2Mb_n = \epsilon \end{aligned}$$

as desired. □

Isn't that cool! It can be cooler if you apply it to an example:

**Corollary 2.5** Given $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$, suppose $c_i$ alternates in signs and approaches to 0, $\sum_{n=1}^{\infty} c_n$ converges.

*Proof.* Let $a_n = (-1)^{n+1}$, $b_n = |c_n|$, apply above theorem, the assertion follows. □

If we consider the *sum of the series*, it means:

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$

What does *product of the series* mean? The motivation is from the power seires:

$$\begin{aligned} &(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ =&(a_0 b_0) + (a_1 b_0 + a_0 b_1)z + (a_2 b_0 + a_1 b_1 + a_0 b_2)z^2 + \cdots \end{aligned}$$

If we define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Then, the *product of the series* is $\sum_{n=1}^{\infty} c_n$. Why we consider this as the *product of the series* not $\sum_{n=1}^{\infty} a_n b_n$? It is because if $\{a_n\}$ converges to $a$ and $\{b_n\}$ converges to $b$, $\sum_{n=1}^{\infty} a_n b_n$ will not converge to $ab$. Thus, it is not a good notion of product. But the one we proposed, under some circumstance (mild condition), does converge.

**Theorem 2.6** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges *absolutely* to $A$, $B$, respectively, then $\sum_{n=1}^{\infty} c_n$ converges to $AB$.

*Proof.* Can be founded in Rudin's book. □

This motivates the concept of *absolutely convergence*:

**Definition 2.1** $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

**Example 2.7**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Above series converges but it is not absolutely convergent, because then it is a harmonic series.

An important theorem associated with it is:

**Theorem 2.8** If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

*Proof.* (*sketch*) We know

$$|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k|$$

But the right hand side is small by *Cauchy criterion* for $\sum_{n=1}^{\infty} |a_n|$. □

The last topic will be the *re-arrangement*.Suppose $\sum_{n=1}^{\infty} a_n = A$, if we re-arrange the term, must it converge? No! Even worse, if it does converge, it also not necessarily converges to $A$. Here is an example:

**Example 2.9** Recall

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

It converges to $\ln 2$, because the *Taylor expansion* of $\ln x$,

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

However, if we can re-arrange the term, we can have it converge to any real number that we want.

**Theorem 2.10** If $\sum_{n=1}^{\infty} a_n$ converges but not absolutely, then a re-arrangement can have any $\limsup$ and $\liminf$. Alternatively, if the convergence is absolute, then all re-arrangement will lead to the same limit.

For example, if we want to have above series converge to $\pi$, what will be the re-arrangement?

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots$$

The summation of odd term will arrive at $\pi$ sometimes, and we stop at that term. Let's now start to subtract term

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots - \frac{1}{2} - \frac{1}{4} - \cdots$$

until it less than $\pi$. We continue this process, add and subtract, we can have a series converge to $\pi$.