Measure and Integration

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Chapter 1

Sets Theory

1.1 Extended Real Numbers

In this section, we are going to extend the real number system encountered in calculus. In this case, the limit of a set and also the limit of a convergent sequence can be $\pm\infty$ which is not well-defined before.

1.1.1 Algebraic Property of Extended Real Number

In the field of measure and integration, we usually work on the extension of real numbers, namely, $-\infty$ and $+\infty$ are included:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

In the meanwhile, we define the order relation on $\mathbb{R}^*$:

$$\forall x \in \mathbb{R}, \ -\infty < x < +\infty$$

also the algebraic operation on $\mathbb{R}^*$:

- *Addition:* $\forall x \in \mathbb{R}$,

$$\begin{aligned}
(-\infty) + x &= -\infty \\
(+\infty) + x &= +\infty \\
(+\infty) + (+\infty) &= \infty \\
(-\infty) + (-\infty) &= -\infty
\end{aligned}$$

- *Multiplication:* $\forall x > 0, x \in \mathbb{R}$,

$$x(+\infty) = (+\infty)x = +\infty$$
$$x(-\infty) = (-\infty)x = -\infty$$

For $x < 0$,

$$x(+\infty) = (+\infty)x = -\infty$$
$$x(-\infty) = (-\infty)x = +\infty$$

Moreover,

$$(+\infty)0 = (-\infty)0 = 0$$
$$(\pm\infty)(+\infty) = (\pm\infty)$$
$$(\pm\infty)(-\infty) = (\mp\infty)$$

- $(-\infty) + (+\infty)$ and $(+\infty) + (-\infty)$ are undefined.

1.1.2 Limit of set and Convergent Sequence

Consider the subset of $\mathbb{R}^*$, let $A \in \mathbb{R}^*$ be non-empty:

- If $A \cap \mathbb{R}$ is not bounded above, $\sup(A) := +\infty$;
- If $A \cap \mathbb{R}$ is not bounded below, $\inf(A) := -\infty$;

Therefore, $\sup(A)$ and $\inf(A)$ always exists for every non-empty subset A of $\mathbb{R}^*$. In term of series, let $\{x_n\}_{n\geq 1}$ be any monotonically increasing sequence in $\mathbb{R}^*$ which is not bounded above, we say $\{x_n\}_{n\geq 1}$ is convergent to $+\infty$ and write

$$\lim_{n\Rightarrow\infty} x_n = +\infty$$

It immediately follows that every monotone sequence in $\mathbb{R}^*$ is convergent. Besides, for any sequence $\{x_n\}_{n\geq 1}$ in $\mathbb{R}^*$, sequence $\{\sup_{k\geq j} x_k\}_{j\geq 1}$ or $\{\inf_{k\geq j} x_k\}_{j\geq 1}$ always converge.

Definition 1.1.1 (Limit Superior and Limit Inferior) The limit superior of $\{x_n\}_{n\geq 1}$ is defined as:

$$\limsup_{n\Rightarrow\infty} x_n = \lim_{j\Rightarrow\infty} (\sup_{k\geq j} x_k)$$

while the limit inferior of the sequence is:

$$\liminf_{n\Rightarrow\infty} x_n = \lim_{j\Rightarrow\infty} (\inf_{k\geq j} x_k)$$

We can easily observe that:

$$\limsup_{n \Rightarrow \infty} x_n \geq \liminf_{n \Rightarrow \infty} x_n$$

Definition 1.1.2 $\{x_n\}_{n \geq 1}$ is convergent to $x \in \mathbb{R}^*$ if

$$\lim_{n \Rightarrow \infty} x_n = \limsup_{n \Rightarrow \infty} x_n = \liminf_{n \Rightarrow \infty} x_n := x$$

Definition 1.1.3 Let $\{x_k\}_{k \geq 1}$ be a sequence in $\mathbb{R}^*$ such that for every $n \in \mathbb{N}$, the *partial sum* $S_n = \Sigma_{k=1}^{n} x_k$ is well defined, if $\{S_n\}_{n \geq 1}$ is convergent to $x \in \mathbb{R}^*$, then we say $\Sigma_{k=1}^{\infty} x_k$ is convergent to x, it is also called the sum of the infinite series $\{x_k\}_{k \geq 1}$.

1.2 Basics of Measure Theory

Before trying to define a measure of a set one must first study the structure of sets that are *measurable*, i.e., those sets for which it will prove to be possible to associate a numerical value in an unambiguous way. (**Not necessarily all sets are measurable!**)

1.2.1 Semi-Algebra & Algebra

Let X be a noon-empty set and C be a collection of subset of X, i.e., $C \subseteq \mathscr{P}(X)$:

Definition 1.2.1 C is called a *semi-algebra* if:

- $\emptyset, X \in C$;
- If A and B belongs to C, then $A \cap B \in C$;
- If $A \in C$, then $A^C = \cup_{i=1}^{n} C_i$, where $C_i \in C$ and $C_i \cap C_j = \emptyset$(they are pairwise disjoint)

It seems to be a little bit abstract, so we show some examples of semi-algebra:

Example 1.2.1 Let $C = \mathscr{P}(X)$, then obviously (i) $\emptyset, X \in C$; (ii) if A and B belongs to C, $A \cap B \in C$; (iii) if $A \in C$, then $A^C \subseteq X$, thus A^C can be easily partitioned into two disjoint subsets that element of power sets of X.

Example 1.2.2 Let $X = \mathbb{R}$, C is the collection of all intervals in $\mathbb{R}$. (i) $\emptyset \in C$, that is $\emptyset = (a, a)$ for $a \in \mathbb{R}$; (ii) $I, J \in C$, then $I \cap J \in C$, one of the case is that I and J are disjoint then, the intersection yields $\emptyset$ which belongs to C, or we have an actual intersection that can be of all kinds (e.g., half open half closed) that belongs to C; (iii) if $I \in C$, then $I^C = \cup_{i=1}^{n} C_i$, where $C_i \in C$, $C_i \cap C_j = \emptyset$, for example, if $I = (a, b)$, then $\mathbb{R} \setminus I = (-\infty, a] \cup [b, +\infty)$, or $I = [a, b)$, then $\mathbb{R} \setminus I = (-\infty, a) \cup [b, +\infty)$ .etc,.

Remark 1.2.3 By similar argument as the above example, if $X = \mathbb{R}^2$, C is the collection of all rectangles in $\mathbb{R}^2$, C is the semi-algebra of $\mathbb{R}^2$.

Now let us introduce the notion of *algebra*. Again, X is a non-empty set but $\mathscr{F}$ is a collection of subsets of X:

Definition 1.2.2 $\mathscr{F}$ is an *algebra* of X if the following axioms are satisfied:

- $\emptyset, X \in \mathscr{F}$;
- If A and B belongs to $\mathscr{F}$, then $A \cap B \in \mathscr{F}$;
- If $A \in \mathscr{F}$, then $A^C \in \mathscr{F}$.

Remark 1.2.4 Notice every algebra is also a semi-algebra since the complement of any subset is made up by just one subset of the algebra, but not every semi-algebra is an algebra. For example, if $X = \mathbb{R}$, C is the collection of all intervals. We already knew that C is a semi-algebra, let's check whether it is an algebra. $I = (a, b) \in C$ but I^C is not an interval, actually, $I^C = (-\infty, a] \cup [b, +\infty)$, thus C is not an algebra.

We give an example when $\mathscr{F}$ is an algebra:

Example 1.2.5 Let $X = \mathbb{R}$, C is the collection of all intervals,

$$\mathscr{F} = \{E \in \mathbb{R} \mid E^C = \sqcup_{j=1}^{n} I_j, I_j \in C\}$$

(i) Since $C \in \mathscr{F}$ (the union of single set that belongs to C), $\emptyset$ and $\mathbb{R}$ is included in $\mathscr{F}$; (ii) assume that E_1, $E_2 \in \mathscr{F}$,

$$\begin{aligned}(E_1 \cup E_2)^C &= E_1^C \cup E_2^C \\ &= (\sqcup_{j=1}^{n}) \cup (\sqcup_{k=1}^{m} J_k) \\ &= \sqcup_{j=1}^{n} \sqcup_{k=1}^{n} (I_j \cap J_k)\end{aligned}$$

Note: this is for the situation that $(\sqcup_{j=1}^{n})$ *and* $(\sqcup_{k=1}^{m} J_k)$ *maybe intersects with each other, otherwise it is just the finite union of disjoint intervals that can be easily treated.* (iii) $E \in \mathscr{F}$, then $E^C = \sqcup_{j=1}^{m} I_j \in \mathscr{F}$. This validates $\mathscr{F}$ is an algebra of $X = \mathbb{R}$.

Remark 1.2.6 Another important observation is that, if E, $F \in \mathscr{F}$, then

$$(E \cup F) = E^C \cap F^C \in \mathscr{F} \text{ since } E^C, F^C \in \mathscr{F} \text{ and } \mathscr{F} \text{ is closed under union}$$

Thus, if $\mathscr{F}$ is an algebra, not only the intersection of E, $F \in \mathscr{F}$ belongs to the algebra, the union also belongs to the algebra.

Since we knew that every algebra is a semi-algebra, we define the following sets:

$$\mathscr{I} = \{\mathscr{F} \mid \mathscr{F} \subseteq \mathscr{P}(X),\ \mathscr{F} \text{ is an algebra, } C \in \mathscr{F}\}$$
$$\mathscr{A} = \cap_{\mathscr{F} \in \mathscr{I}}$$

We claim that: (i) $C \subseteq \mathscr{A}$; (ii) $\mathscr{A}$ is an algebra.

Proof. The first claim is trivial. For the second part, we will show that $\mathscr{A}$ satisfies the axioms of an algebra:

- Since $\emptyset \in \mathscr{F}$ and $X \in \mathscr{F}$, $\emptyset$, $X \in \mathscr{A}$;
- If $E \in \mathscr{A}$, then $E \in \mathscr{F}$, $\forall \mathscr{F} \in \mathscr{I}$. Since $\mathscr{F}$ is an algebra, $E^C \in \mathscr{F}$, $\forall \mathscr{F} \in \mathscr{I}$. As a result,
$$E^C \in \cap_{\mathscr{F} \in \mathscr{I}} \mathscr{F} = \mathscr{A}$$
- If $E, F \in \mathscr{A}$, then $E, F \in \mathscr{F}$, $\forall \mathscr{F} \in \mathscr{I}$. Since $\mathscr{F}$ is an algebra, $E \cap F \in \mathscr{F}$, $\forall \mathscr{F} \in \mathscr{I}$. As a result,
$$E \cap F \in \mathscr{A}$$

This proves $\mathscr{A}$ is an algebra containing C. □

Remark 1.2.7 Obviously, by this construction, $\mathscr{A}$ is the smallest algebra of subset of X such that $C \in \mathscr{A}$, i.e., if $\mathscr{F}$ is any algebra $C \subseteq \mathscr{F}$, then $\mathscr{A} \subseteq \mathscr{F}$.

Let's formulate above discussions as a theorem:

Theorem 1.2.8 Let X be any set and let C be any class of subsets of X, define:

$$\mathscr{F}(C) := \cap \mathscr{A}$$

where the intersection is taken over all algebras $\mathscr{A}$ of subsets of X such that $C \in \mathscr{A}$. Then the followings hold:
(i)$C \subseteq \mathscr{F}(C)$ and $\mathscr{F}(C)$ is also an algebra of subsets of X;
(ii) If $\mathscr{A}$ is any algebra of subsets of X such that $C \subseteq \mathscr{A}$, then $\mathscr{F}(C) \subseteq \mathscr{A}$.

Remark 1.2.9 $\mathscr{F}(C)$ is the smallest algebra of subsets of X containing C and is called *the algebra generated by* C.

Example 1.2.10 Let X be any kinds of set and $C \subseteq X$ to be the singleton of X, i.e., $C := \{\{x\}|x \in X\}$. Claim (i): $\mathscr{F}(C) = \{A \subseteq X \mid A \text{ or } A^C \text{ is finite}\}$ is an algebra. Claim (ii): $\mathscr{F}(C)$ is actually the smallest algebra that contains C.

Proof. For the first claim, we check the axioms of being an algebra again: (i) $\emptyset \in \mathscr{F}(C)$, $X \in \mathscr{F}(C)$, since $\emptyset$ is always finite; (ii) if $E \in \mathscr{F}(C)$, E^C is finite, thus $E^C \in \mathscr{F}(C)$; (iii) if $E, F \in \mathscr{F}(C)$, how can we prove $E \cap F \in \mathscr{F}(C)$ (or $E \cup F \in \mathscr{F}(C)$)?

- Case 1: both E, F are finite, then $E \cup F$ is finite. As a result, $E \cup F \in \mathscr{F}(C)$;
- Case 2: either E or F is not finite. Suppose E is not finite, if $E \in \mathscr{F}$, then E^C is finite. Also, $E \subseteq E \cup F$ implies $(E \cup F)^C \subseteq E^C$. Thus, $(E \cup F)^C$ is finite so that $E \cup F \subseteq \mathscr{F}(C)$.

this completes the proof that $\mathscr{F}(C)$ is indeed an algebra. We can also observe that $C \in \mathscr{F}(C)$, now we prove the second claim, that is, let $\mathscr{A}$ be any algebra such that $C \subseteq \mathscr{A}$, we shown $\mathscr{F}(C) \subseteq \mathscr{A}$. Pick $A \in \mathscr{F}(C)$, suppose that A is finite, then $A = \cup_{i=1}^{n} x_i$. Since $\{x_i\} \in C$ and $C \in \mathscr{A}$, $\{x_i\} \in \mathscr{A}$. This proves $A \in \mathscr{A}$. $\square$

From above we see, we can describe the algebra generated by certain sets by explicit calculation. Now let's look at a special structure of subsets that can be used to generate an algebra:

Theorem 1.2.11 Let C a semi-algebra of subsets of a set X, then $\mathscr{F}(C)$, the algebra generated by C, is given by:

$$\mathscr{F}(C) = \{E \in X \mid E = \sqcup_{i=1}^{n} C_i,\, C_i \in C\}$$

Proof. We check the axioms one by one. But before that we should notice that $C \in \mathscr{F}$, because C is a union of itself.
(i) since $C\mathscr{F}(C)$, thus $\emptyset,\, X \in \mathscr{F}(C)$;
(ii) if $E \in \mathscr{F}(C)$, it implies $E = \sqcup_{i=1}^{n} C_i,\, C_i \in \mathscr{F}(C)$. Recall that $C_i \in C$ and C is a semi-algebra, it indicates $C_i^C = \sqcup_{j=1}^{k_i} A_j^i$, for some $A_j^i \in C$. As a result,

$$E^C = \cap_{i=1}^{n} C_i^C = \cap_{i=1}^{n} [\sqcup_{j=1}^{k_i} A_j^i] = \sqcup (A_j^i \cap A_l^k)$$

where $(A_j^i \cap A_l^k) \in C$;
(iii) if $E, F \in \mathscr{F}(C)$, then $E \cap F \in \mathscr{F}(C)$. Since

$$E = \sqcup_{i=1}^{n} A_i, \quad A_i \in C, \quad F = \sqcup_{j=1}^{n} B_j, \quad B_j \in C$$

then,

$$E \cap F = (\sqcup_{i=1}^{n} A_i) \cap (\sqcup_{j=1}^{n} B_j) = \sqcap_{i,j}^{m,n} (A_i \cap B_j)$$

where $(\sqcup_{j=1}^{n} B_j) \in C$. Thus $E \cap F \in \mathscr{F}(C)$. This proves that $\mathscr{F}(C)$ is indeed an algebra containing C. As the next step, we prove it is the smallest one, that is, again, for $C \subseteq \mathscr{A}$, $\mathscr{F}(C) \subseteq \mathscr{A}$. Let $E \in \mathscr{F}(C)$, it results that $E = \sqcup_{i=1}^{n} A_i,\, A_i \in C \subseteq \mathscr{A}$, which implies $E \in \mathscr{A}$. Thus, $\mathscr{F}(C) \subseteq \mathscr{A}$. $\square$

Observing the algebra generated by semi-algebra, let's see other alternatives.

Theorem 1.2.12 Let $\mathscr{C}$ be any collection of subsets of a set X and $E \subseteq X$. Let

$$\mathscr{C} \cap E := \{C \cap E \mid C \in \mathscr{C}\}$$

Then, $\mathscr{F}(\mathscr{C}) \cap E = \mathscr{F}(\mathscr{C} \cap E)$

Proof. Since $\mathscr{C} \in \mathscr{F}(\mathscr{C})$, then $\mathscr{C} \cap E \subseteq \mathscr{F}(\mathscr{C}) \cap E$. Also, we shall observe that $(\mathscr{F}(\mathscr{C} \cap E)$ is an algebra of subset of E. Why? (i)$\emptyset = \emptyset \cap E \in \mathscr{F}(\mathscr{C} \cap E)$, on the other hand, $E = X \cap E \in \mathscr{F}(\mathscr{C} \cap E)$; (ii) if $A, B \in \mathscr{F}(\mathscr{C}) \cap E$, it implies that $A = G \cap E$, where $G \in \mathscr{F}(\mathscr{C})$ and $B = H \cap E$, where $H \in \mathscr{F}(\mathscr{C})$. Then, $A \cap B = (G \cap H) \cap E$, where $G \cap H \in \mathscr{F}(\mathscr{C})$. Thus, $A \cap B \in \mathscr{F}(\mathscr{C}) \cap E$; (iii) if $A \in \mathscr{F}(\mathscr{C}) \cap E$, $A = G \cap E$, where $G \in \mathscr{F}(\mathscr{C})$. Since $A^C \in E$, $A^C = G^C \cap E \in \mathscr{F}(\mathscr{E}) \cap E$. This shows us that $\mathscr{F}(\mathscr{C} \cap E) \subseteq \mathscr{F}(\mathscr{C}) \cap E$. For the other way around, we leave as an exercise. $\square$

Remark 1.2.13 What the theorem tells is that: if we restrict the class $\mathscr{C}$ to subsets of E and generate the algebra of subsets of E by $C \cap E$, then it is the same as generating the algebra first and then restricting to subsets of E.

The structure of algebra has a special property, namely, any countable union of algebra can be represented as a countable union of disjoint algebra. We formulate it as the following theorem:

Theorem 1.2.14 Let $\mathscr{A}$ be an algebra of subsets of a set X, let

$$E = \cup_{n=1}^{\infty} A_n$$

where each $A_n \in \mathscr{A}$. Then there exists sets $B_n \in \mathscr{A}$, $n \geq 1$, such that $B_n \cap B_m = \emptyset$ for $m \neq n$ and

$$E = \sqcup_{n=1}^{\infty} B_n$$

Proof. This is basically a proof by construction, suppose $\mathscr{A}$ is an algebra, and $A_1, A_2, ... \in \mathscr{A}$, also $E = \cup_{n=1}^{\infty} A_n$. Now we define $\{B\}_{n=1,2,...}$ in the following way:

$$\begin{aligned}
&B_1 := A_1 \\
&B_2 := A_2 \setminus A_1 \\
&B_3 := A_3 \setminus (A_1 \cup A_2) \\
&\cdots \\
&B_n := A_n \setminus (\cup_{i=1}^{n-1} A_i)
\end{aligned}$$

This implies: $B_n = A_n \cap (\cup_{i=1}^{n-1} A_i)^C$, $\forall n$. By the virtue of being an algebra, $B_n \mathscr{A}$ for all n. Also, $B_n \cup B_m = \emptyset$ for $m \neq n$. Furthermore,

$$\sqcup_{i=1}^{n} B_i = \cup_{i=1}^{n} A_i = E$$

$\square$

1.2.2 Sigma Algebra

Based on above discussion, we put forward the most important and useful structure of sets in the measure theory - *Sigma Algebra* (σ-algebra). Let X be a noon-empty set and S be a collection of subset of X, i.e., $S \subseteq \mathscr{P}(X)$:

Definition 1.2.3 S is called a *sigma-algebra* if:

- $\emptyset, X \in S$;
- If $A \in S$, then $A^C \in S$;
- If $A_i \in S$, for $i = 1, 2, ...,$ then $\cup_{i=1}^{\infty} A_i \in S$.

Remark 1.2.15 Every sigma algebra is an algebra and every algebra is a semi-algebra

Let's show some examples of sigma algebra:

Example 1.2.16 Let X be any uncountable set and let

$$\mathscr{F} := \{E \subseteq X \mid \text{either } E \text{ or } E^C \text{ is finite}\}$$

then through previous example, we know $\mathscr{F}$ is an algebra, but is this a sigma algebra. We only need to check whether the third requirement is met: for $E_1, E_2, ..., \in \mathscr{F}$,

$$\cup_{n=1}^{\infty} E_n \in \mathscr{F}$$

But this will fail if we consider $E_i = \{x_i\}$, $\forall i$. Obviously, $E_i \in \mathscr{F}$, but $\cup_{i=1}^{\infty} E_i$ is not necessarily in $\mathscr{F}$, because in this case E^C can only be infinite to have X be uncountably many.

Example 1.2.17 Let X be any set, then $\{X, \emptyset\}$ and $\mathscr{P}(X)$ are obvious examples of sigma algebra of subsets of X. Now let's consider the following set:

$$\mathscr{S} := \{A \subseteq X \mid \text{either } A \text{ or } A^C \text{ is countable}\}$$

we claim that S is a sigma-algebra of subsets of X.

Proof. We can see immediately $\emptyset$ and X is in the collection S; also if $A \in S$ then $A^C \in S$. The criterion needs to be checked is that if $A_n \in S$ for $n = 1, 2, ...,$ $\cup_{i=1}^{\infty} A_n \in S$. There are two cases:

- *case 1:* all A_n's are countable, then $\cup_{i=1}^{\infty} A_n$ is countable so that it belongs to S[1];
- *case 2:* there exists n_0 such that $A_{n_0} \in S$ and not countable, while $A_{n_0}^C$ is countable. Observe that $A_{n_0} \subseteq \cap_{n=1}^{\infty} A_n$, since A_{n_0} is one of the member of $\{A_n\}_{n=1,2,...}$. It implies $(\cup_{n=1}^{\infty} A_n)^C \subseteq A_{n_0}^C$ where the later set is countable, thus $(\cup_{n=1}^{\infty} A_n)$ is countable thus in S.

□

[1] the countable union of countable set is countable

Let's reconsider the generation of collection of subsets, namely, given a collection C of subsets of a set X, does there exist a sigma algebra of subsets X that includes C? Can we find the smallest one? The answer is positive for the theorem below:

Theorem 1.2.18 Let X be any set and let C be any class of subsets of X, define:

$$S(C) := \cap S,$$

where the intersection is taken over all sigma algebras $\mathscr{S}$ of subsets of X such that $C \in S$. Then the followings hold:
(i)$C \subseteq S(C)$ and $S(C)$ is also an algebra of subsets of X;
(ii) If $\mathscr{S}$ is any algebra of subsets of X such that $C \subseteq \mathscr{S}$, then $S(C) \subseteq \mathscr{S}$.

Proof. First of all, $\emptyset$, $X \in S(C)$, since every S is a sigma algebra such that $\emptyset, X \in S\ \forall S$, thus, they are also included in the intersection of S. Secondly, if $A \in S(C)$, then $A \in S$ for all S. Also, $A^C \in S$, $\forall S$. Thus, $A^C \in \cap S = S(C)$. On the top of that, assume $A_n \in S(C)$, $\forall n$, then $A_n \in S$, $\forall S$. As a result, $\cup_{n=1}^{\infty} A_n \in S$ for all S. Therefore, $\cup_{n=1}^{\infty} A_n \in S(C)$. These validate that $S(C)$ is a sigma algebra. $C \in S(C)$ is obvious. $S(C)$ is also the smallest for it is the intersection of all sets S. □

Remark 1.2.19 By the above theorem, $S(C)$ is the smallest sigma algebra of subsets of X containing C and is called *σ-algebra generated by C*.

Example 1.2.20 Let X be any kinds of set and $C \subseteq X$ to be the singleton of X, i.e., $C := \{\{x\}|x \in X\}$. The sigma algebra generated by C is

$$\mathscr{S}(C) = \{E \subseteq X \mid E \text{ or } E^C \text{ is countable}\}$$

Proof. We have already proved that S is a sigma algebra, and also it is trivial that $C \subseteq S$. There is only one thing left to prove that is: S is the smallest, i.e., let $\mathscr{S}$ be any sigma algebra such that $C \in \mathscr{S}$. We need to show that $S(C) \subseteq \mathscr{S}$. Let $A \in S$, either A is countable, i.e., $A = \{x_1, x_2, ...\} = \cup_{i=1}^{\infty}\{x_i\} \in \mathscr{S}$, or A^C is countable so that $A^C \in \mathscr{S}$ which implies $A \in \mathscr{S}$. □

At last, let's discuss the *σ-algebra of Borel subsets*. Let X be any topological space, μ denote the class of all open subsets of X and C denotes the class of the all closed subsets of X. The pair (X, F) indicates the topological space X, where F is the topology. It satisfies the following conditions:

- both the empty set and X are elements of F;
- any union of elements of F is an element of T;
- any intersection of finitely many elements of T is an element of T.

Remark 1.2.21 To be a topology is not the same as an algebra or σ-algebra.

Theorem 1.2.22 The sigma algebra generated by μ and C are the same, i.e., $S(\mu) = S(C)$.

Proof. Let $E \in \mu$, thus E^C is closed, and $E^C \in C \in S(C)$. Thus, $E \in S(C)$. Since E is arbitrary, $\mu \subseteq S(C)$. For $S(\mu)$ is the smallest sigma algebra containing μ, $S(\mu) \subseteq S(C)$. Converse argument is exactly the same (sketch: $A \in C \Rightarrow A^C \in \mu \subseteq S(\mu) \Rightarrow A \in S(\mu) \Rightarrow S(C) \subseteq S(\mu)$). Thus, $S(C) = S(\mu)$. □

There are several interesting observations at this point:

Theorem 1.2.23 Assume $C \subseteq (X)$, then $S(A(C)) = S(C)$.

Proof. By definition, $C \subseteq A(C) \subseteq S(A(C))$. This implies that $S(C) \subseteq S(A(C))$. Also $C \subseteq S(C)$, $S(C)$ is also the algebra, then $A(C) \subseteq S(C)$. Thus, $S(A(C)) \subseteq S(C)$. □

Theorem 1.2.24 If $Y \subseteq X$, then $S(C \cap Y) = S(C) \cap Y$.

Proof. Note that $C \subseteq S(C)$, thus $C \cap Y \subseteq S(C) \cap Y$. If we can show the latter is a sigma-algebra, then we can prove that $S(C \cap Y) \subseteq S(C) \cap Y$. Let's check: (i) because $\emptyset = \emptyset \cap Y$, thus $\emptyset \in S(C) \cap Y$. Similarly, $Y = X \cap Y$. thus $Y \in S(C) \cup Y$; (ii) if $E \in S(C) \cap Y$, then $E = A \cap Y$, where $A \in S(C)$. $E^C \in Y$ means $E^C \cap Y = A^C \cap Y \subseteq S(C) \cap Y$; (iii) $E_n \in S(C) \cap Y$, thus $E = A_n \cap Y$, $A_n \in S(C)$. This implies, $\cup E_n = (\cup A_n) \cap Y$. We also need to prove another way around of the inclusion: (sketch: $S(C) \cap Y \subseteq S(C \cap Y)$. Let $A := \{E \subseteq X \mid E \cap Y \in S(C \cap Y)\}$, we only need to show that A is a sigma algebra and $C \subseteq A$(this will imply $S(C) \subseteq A$). (i) $\emptyset, X \in A$ trivially; (ii) $E \in A \Rightarrow E \cap Y \in S(C \cap Y) \Rightarrow E^C \cap Y \in S(C \cap Y) \Rightarrow E^C \in A$; (iii) $E_n \in A \Rightarrow E_n \cap Y \in S(C \cap Y) \Rightarrow (\cup E_n) \cap Y \in S(C \cap Y)$, hence $\cup E_n \in A$. And clearly, $S(C) \subseteq A$). □

Remark 1.2.25 This is a very useful technique for the future lectures. For example, imagine that $X = \mathbb{R}$, and Y is an interval, and C are collections of open sets. If we want to generate the sigma algebra from the restriction of C to Y, we can firstly generate the sigma algebra of open sets C and then take intersection with Y.

1.2.3 Monotone Class

We have introduced *semi-algebra, algebra, sigma algebra*, in particular, σ-algebra is the foundation to develop measure theory. There is another class of subsets that worth mentioning, that is the *monotone class*.

Definition 1.2.4 Let X be a non-empty set and M be a class of subsets of X. We say M is a *monotone class* if

- $A_n \in M$ and $A_n \subseteq A_{n+1}$ for $n = 1, 2, ...$ implies: $\cup_{n=1}^{\infty} A_n \in M$;
- $A_n \in M$ and $A_{n+1} \subseteq A_n$, for $n = 1, 2, ...$ implies: $\cap_{n=1}^{\infty} A_n \in M$.

Proposition 1.2.26 Every σ-algebra is also a monotone class.

Proof. Let $A_n \in M$, and A_n is increasing collection of sets, i.e., $A_n \subseteq A_{n+1}$, $\forall n \geq 1$. Then, $\cup_{n=1}^{\infty} A_n \in M$ (since M is a σ-algebra). Let $A_n \in M$, where A_n is a decreasing collection of sets, i.e., $A_{n+1} \subseteq A_n$, $\forall n \geq 1$. Then $A_n^C \in M$ so that $\cup_{n=1}^{\infty} A_n^C \in M$. We extact the complement, $(\cap_{n=1}^{\infty} A_n)^C \in M$, which yields $\cap_{n=1}^{\infty} A_n \in M$. $\square$

The converse is not true, for example, let X be any uncountable set and $M := \{A \subseteq X \mid A \text{ is countable}\}$, then M is a monotone class but not σ-algebra. To see why, we first prove that it is a monotone class. Assume $A_n \in M$ and $A_n \subseteq A_{n+1}$, $\forall n$. Note A_n is countable, so $\cup_{n=1}^{\infty} A_n$ is also countable, belonging to M. On the other hand, if $A_n \in M$ and $A_{n+1} \subseteq A_n$ for all n, notice that $\cap_{n=1}^{\infty} A_n$ is countable (because $\cap_{n=1}^{\infty} A_n \subseteq A_n$, $\forall n$. So, $\cap_{n=1}^{\infty}$ is countable for A_n is countable). Thus, $\cap_{n=1}^{\infty} A_n \in M$. Indeed, M is a monotone class. Next, we show that M is not a σ-algebra. This is trivial, since the whole set X is uncountable thus not belonging to M.

Let's consider the generation of subsets again with respect to the monotone class. Let X be any non-empty set and C be any collection of subsets of X. Clearly, $\mathscr{P}(X)$ is a monotone class of subsets of X such that $C \in \mathscr{P}(X)$. Let $M(C) := \cap M$ where the intersection is taken over all those monotone classes M of subsets of X such that $C \in M$. We have the following two claims: *(i) $C \in M(C)$; (ii) $M(C)$ is a monotone class.*

Proof. Assume $A_n \in M(C)$, and $A_{n+1} \subseteq A_n$. Then $A_n \in M$, $\forall n$. Thus, $\cap_{n=1}^{\infty} A_n \in M$ for all M, which implies $\cap_{n=1}^{\infty} A_n \in M(C)$. On the other hand, consider $A_n \in M(C)$ and $A_n \subseteq A_{n+1}$. $A_n \in M$ for all M, then $\cup_{n=1}^{\infty} A_n \in M$ for all M. Thus, $\cup_{n=1}^{\infty} A_n \in M(C)$. This proves that $M(C)$ is a monotone class. $\square$

Remark 1.2.27 Just like $A(C)$, $S(C)$, $M(C)$ is also the smallest monotone class that containing C.

Theorem 1.2.28 Let C be any class of subsets of X, then the following hold: (i) If C is an algebra which is also a monotone class, then C is a σ-algebra; (ii) $C \subseteq M(C) \subseteq S(C)$.

Proof. For the first claim: i) $\emptyset, X \in C$; ii) $A \in C$ implies $A^C \in C$ (i), ii) by the virtue of being an algebra); iii) if $A_n \in C$, then $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} (\cup_{i=1}^{n} A_i)$. Since $(\cup_{u=1}^{n} A_i) \in C$ and is increasing collection of subsets of X, by the definition of monotone class, $\cup_{n=1}^{\infty} A_n \in C$. The second claim is trivial since $C \subseteq M(C), S(C)$ by definition, and every monotone class is not necessarily a sigma algebra. $\square$

We already shown that the monotone class generated by any class of subsets of X is contained in the sigma algebra generated by the same class of subsets of X, i.e., $M(C) \subseteq S(C)$, but when will the inverse inclusion be valid so that we have equivalent relationship. Here is the theorem:

Theorem 1.2.29 (σ-algebra monotone class theorem) Let A be an algebra of subsets of X, then $S(A) = M(A)$.

Proof. we only need to show that $S(A) \subseteq M(A)$. Observe that if we can show that $M(A)$ is an algebra (it is a monotone class already), then by last theorem, $M(A)$ is a σ-algebra. Since $S(A)$ is the smallest sigma algebra containing A, we will have $S(A) \subseteq M(A)$. So we now show $M(A)$ is an algebra. First of all, $\emptyset, X \in M(A)$, since they are belonging to A. Secondly, we check whether $M(A)$ is closed under complement, i.e., $A \in M(A) \Rightarrow A^C \in M(A)$. Consider the following sets:

$$B = \{E \subseteq X \mid E^C \in M(A)\}$$

the above criterion will be met if $M(A) \subseteq B$ and B is a monotone class. (Why? Because B and $M(A)$ are both monotone class containing A, but $M(A)$ is the smallest one)
Let $A_0 \in A$, then $A_0^C \in A \subseteq M(A)$, hence $A_0 \in B$, or equivalently, $A \subseteq B$. On other hand, let $A_n \in B$ such that A_n is an increasing collection of subsets. By definition of the set B, $A_n^C \in M(A)$. Thus, $\cap_{n=1}^{\infty} A_n^C \in M(A)$, which yields $(\cup_{n=1}^{\infty} A_n)^C \in M(A)$(or $\cup_{n=1}^{\infty} A_n \in B$). By the same token, if we let $A_n \in B$ to be a decreasing collection of subsets, then $A_n^C \in M(A)$. So $\cup_{n=1}^{\infty} A_n^C \in M(A)$. As a result, $(\cap_{n=1}^{\infty} A_n)^C \in M(A)$(or $\cap_{n=1}^{\infty} A_n \in B$).

Next, let's validate that $M(A)$ is closed under unions. Fix $F \in M(A)$, and let

$$L(F) := \{A \subseteq X \mid A \cup F \in M(A)\}$$

we only need to show that $M(A) \subseteq L(F)$. We show the argument in two steps: (i)$L(F)$ is a monotone class; (ii) $A \in L(F)$, $\forall F \in M(A)$. For the first claim, assume $E_n \in L(F)$ that is increasing, then $\cup_{n=1}^{\infty} E_n \in L(F)$. Also, $E_n \cap F \in M(A)$, which implies $\cup_{n=1}^{\infty}(E_n \cup F) \in M(A)$. Thus, $(\cup_{n=1}^{\infty} E_n) \in M(A)$. The case of decreasing sequence follows the same argument, we will skip it. Now for the second claim, if $F \in M(A)$, $\forall E \subseteq A$, then $E \cup F \in A \subseteq M(A)$. This implies $E \in L(F)$, thus $E \in L(F)$, that is, $A \subseteq L(F)$ for all $F \in A$. As a result, $M(A) \subseteq L(F)$, $\forall F \in A$. □

1.3 Summary

We begin, generally, with a set X whose elements are called *points*. One may think of X as a subset of $\mathbb{R}^n$, but it can be more general set than that, for example, the set of paths in a path-space on which we are trying to define a 'functional integral'. Then we propose several distinguished collections semi-algebra (C), algebra (A), sigma-algebra (S) and monotone

class (M) of subsets of X:
1. Semi-algebra:

- $\emptyset, X \in C$;
- if $A, B \in C$, then $A \cap B \in C$;
- if $A \in C$, then $A^C = \sqcup_{i=1}^{n} C_i$, where $C_i \in C$.

2. Algebra:

- $\emptyset, X \in A$;
- if $E, F \in A$, then $E \cap F \in A$;
- if $E \in A$, then $E^C \in A$.

Note: From the last two criterion, we can actually have the finite union of sets of algebra is also in the algebra, i.e., $E \cup F \in A$, $\forall E, F \in A$.
3. Sigma-algebra:

- $\emptyset, X \in S$;
- if $E_n \in S$ for $n = 1, 2, ...$, then $\cap_{i=1}^{\infty} E_n \in S$;
- if $E \in S$, then $E^C \in S$.

Note: Similarly, the countable union of sets of sigma-algebra is an element of the sigma-algebra, i.e., $\cup_{i=1}^{\infty} E_n \in S$, $\forall E_n \in S$.
4. Monotone class

- $A_n \in M$ and $A_n \subseteq A_{n+1}$ for $n = 1, 2, ...$ implies: $\cup_{n=1}^{\infty} A_n \in M$;
- $A_n \in M$ and $A_{n+1} \subseteq A_n$, for $n = 1, 2, ...$ implies: $\cap_{n=1}^{\infty} A_n \in M$.

They have the following relationship: sigma algebra $\Rightarrow$ algebra $\Rightarrow$ semi-algebra, and sigma algebra $\Rightarrow$ monotone class. In particular, although monotone class doesn't imply sigma algebra, monotone class + algebra does yield a sigma algebra.

We also consider the extension of general collection of subsets C of X to algebra, sigma algebra and monotone class, $A(C)$, $S(C)$, $M(C)$, respectively. They are defined as:

$$Q(C) = \cap Q$$

where the intersection is taken over all Q of subsets of X such that $C \in Q$. Here, Q can be A, S, M. Then $Q(C)$ is also a(an) algebra (sigma-algebra, monotone class, respectively) and actually is the smallest algebra (sigma algebra, monotone class, respectively) containing C. They are called the algebra(sigma algebra, monotone class, respectively) generated by C. There is also an important theorem, monotone sigma algebra theorem, it says that $M(C) = S(C)$ whenever C is an algebra. In the future development of measure theory, the sigma algebra plays a central role. One may ask why should we consider the extension of a general collection of subsets C, because this enables us to assign measure to those subsets in a reasonable one, which we will see in the later chapters. But, here, we can give some flavors that why sigma-algebra and such extension are important by the following examples:

- In probability theory, we have the *state space* Ω (the set of all possible outcomes of the experiment), the *events* (a property which can be observed either to hold or not to hold after the experiment is done, mathematically, it is defined as a subset of Ω); the family of all events, denoted by $\mathscr{A}$, is the power set of Ω, $\mathscr{A} = 2^{\Omega}$. For simplicity, we now define A a subset of Ω, but actually knowing A, we also know $\emptyset, \Omega, A^C$, this naturally extends to a sigma algebra containing A, then we can assign probability measure on it. But one may ask why not assigning probability measure to its all subsets of Ω, i.e., 2^{Ω}, since obviously σ-algebra may not cover all the subsets. That is because the power set is just to rich to have a measure to each of them, technically speaking, due to the axiom of choice, there exists non-measurable sets.

- Another important example is called the *Borel sigma algebra*. It is generated by the open subsets of $\mathbb{R}^n$, this is called the *Borel sigma algebra*, denoted by $\mathscr{B}$. Alternatively, it is generated by the open balls of $\mathbb{R}^n$, i.e., the family of sets of the form

$$\mathscr{B}(x, R) = \{y \in \mathbb{R}^n; |x - y| < R\}$$

  It is a fact that this Borel sigma algebra also contains the closed sets by the axioms of sigma algebra. But, with the help of axiom of choice, we also can prove that $\mathscr{B}$ does not include all the subsets of $\mathbb{R}^n$. (Axiom of choice is beyond the scope of measure and integration but will believed to be hold through out the notes)