Credit Risk Modeling – Multi-name Derivatives

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In this notes, we discuss the advnaced credit derivatives. We start with multi-name product, CDS portfolio and its related derivatives, then, we pass to the product where the reference entities are correlated. In particular, we will discuss synthetic CDO and tranch calculations.

1 CDS Portfolio Indices and Related Derivatives

CDS portfolio indices is a highly liquid product that enables investors to hedge the risk of a portfolio of credit exposures. In Europe and Asia, the indices are named *iTraxx* while in North America it is under the name of *CDX*. It is an OTC bilateral contract, on contrast to single-name CDS, a *buyer* of indices is the one offering insurance, i.e., receiving spreads, the *seller* side is going short on the credit risks. In the following, we will consider the pricing of CDS portfolio indices, option derivative and CDO.

1.1 Valuation of CDS Portfolio Indices

Let's suppose at time $T_0 = 0$[1], an investor enters into a long credit position of CDS index with maturity T. The CDS index spans M credits, each of which has

[1] We assume the trade date is equal to effecitve date.

a recovery rate R_m for $m = 1, ..., M$. The contract pays a fixed coupon at payment dates $T_1, T_2, ..., T_N = T$, such coupon is called the *contractual index spread*, we denote with $C(T)$. Notice, as usual, the fixed rate is paid on the notional of the CDS index. The protection leg for credit m will be triggered at its default time τ_m, which results in a loss of buyer at value $(1 - R_m)/M$(the reference entities are equally weighted typically). To enter in to this contract, the investor has to make an initial upfront payment of $U_I(0)$. Unlike the single-name CDS with a initial spread making the value as 0, CDS index contractual spread is by convention a multiple of 5, thus, in general, it can not be entered with no cost. That is to say, the fair value, also called the *intrinsic value* denoted by $V_I(0)$, is not always equal to $U_I(0)$.

1.2 Intrinsic Value of Protection Leg and Premium Leg

To calculate the protection leg, we need to find the present value of the summation over all M reference entities with each paying out $(1 - R_m)/M$ at time $\tau_m < T$. Assuming the independence of default times and interest rate, we can work under risk neutral measure $\mathbb{Q}$ to obtain,

$$\begin{aligned} \textit{Protection Leg PV}(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M}\sum_{i=1}^{M} D(0,\tau_m)(1-R_m)\mathbf{1}_{\{t<\tau_m\le T\}}\right] \\ &= \frac{1}{M}\sum_{i=1}^{M}(1-R_m)\int_0^T P(0,u)d(-S_m(0,u)) \end{aligned} \tag{1}$$

where $S_m(0, u) = \mathbb{Q}\big(\tau_m > u\big)$. Using the fact that each individual CDS's protection leg is equivalent to the intial credit spread times *RPV01*, i.e.,

$$(1 - R_m)\int_0^T P(0,u)d(-S_m(0,u)) = S_m(0,T)\textit{RPV01}_m(0,T), \tag{2}$$

(1) can be rewritten as:

$$\textit{Protection Leg PV}(0) = \frac{1}{M}\sum_{i=1}^{M} S_m(0,T)\textit{RPV01}_m(0,T). \tag{3}$$

The above formula suggests we can use individual CDS market quotes to compute the protection leg of CDS index.

As for the premium leg, notice, each occurrence of credit event leads to a reduction of $C(T)/M$. Therefore, the present value of the premium leg is a summation of the reference entities and cashflows, with the payment conditioning on survival, namely,

$$\begin{aligned} \textit{Premium Leg PV}(0) &= C(T)\frac{1}{M}\sum_{m=1}^{M}\Bigg(\sum_{i=1}^{N}\Bigg(\tau(T_{i-1},T_i)P(0,T_i)S_m(0,T_i) \\ &\quad + \int_{T_{i-1}}^{T_i}\tau(T_{i-1},u)P(0,u)\big(-dS_m(0,u)\big)\Bigg)\Bigg) \\ &= \frac{C(T)}{M}\sum_{m=1}^{M} \textit{RPV01}_m(0,T) \end{aligned} \tag{4}$$

For simplicity, we do not take into account of the premium accrued effect.

Consequently, by subtracting the protection leg from premium leg, we obtain the intrinsic value of the CDS index,

$$V_I(0) = \frac{1}{M}\sum_{m=1}^{M}\big(C(T) - S_m(0,T)\big)\textit{RPV01}_m(0,T). \tag{5}$$

The formula above implies CDS index's intrinsic value can be computed in terms of the survival curves of each reference entities and their market qutoes.

1.3 Intrinsic Spread

As shown above, the calculation of intrinsic value of CDS index requires individual CDS spread curves, however, the market quotes CDS index by only one spread curve – *index curve*, denoted by $S_I(0,\cdot)$. By convention, it also assumes a flat term structure of the index curve, which can be found by solving for the value of $S_I(0,T)$ at which a CDS contract with coupon $C(T)$ has the same up-front value as the index, i.e.,

$$U_I(0) = \big(S_I(0,T) - C(T)\big)\textit{RPV01}_I(0,T) \tag{6}$$

where RPV01 with subscript I is adopting a flat credit curve. Remember U_I and $C(T)$ are known quantities. Thus, RPV01$(0,T)$ can be though as a function of

$S_I(0,T)$. Equating $U_I(0)$ to the intrinsic value $V_I(0)$ yields

$$\frac{1}{M}\sum_{m=1}^{M}\big(S_m(0,T)-C(T)\big)RPV01_m(0,T)=\big(S_I(0,T)-C(T)\big)RPV01_I(0,T), \tag{7}$$

where the left hand side can be calculated directly and the right hand side is a function of $S_I(0,T)$. One dimension root finding algorithm can solve for $S_I(0,T)$. On the other hand, if we assumes the flatness and homogeneity of each individual's credit curve, we obtain the follow approximation for RPV01$(0,T)$,

$$RPV01(0,T)\approx\frac{1}{M}\sum_{m=1}^{N}RPV01_m(0,T), \tag{8}$$

from which it follows directly from (7) that,

$$S_I(0,T)\approx\frac{\sum_{m=1}^{M}S_n(0,T)RPV01_m(0,T)}{\sum_{m=1}^{N}RPV01(0,T)}. \tag{9}$$

Observe, in identity (7), the approximation is exact when the up-front payment $U_I(0)$ equals to the intrinsic value $V_I(0)$. In general, we can not have the equivalence between $U_I(0)$ and $V_I(0)$, but the empirical experiment shows such identical relationship is reasonable to assume.

1.4 Options on CDS Indices

The very vanilla derivative that has CDS portfolio index as underlying is the *European style option* on the index, also called *portfolio Swaptions*. Such derivative is actively traded in the market, which provides liquidity of the underlying thus makes pricing and hedging convenient. The reasons to trade such derivatives are the same as other vanilla option derivatives, namely, it renders a cheaper way to take long or short position of credit index and also implies the volatility of the credit market.

1.4.1 Mechanism

An *index option* is a bilateral OTC contract to buy or sell protection on a specified index that matures at time T at an index spread agreed today, say rate K. The option itself expires at $T_E \in (0, T)$. It is a payer option if the option holder has the option to enter into a long protection position on expiry at agreed strike spread, a receiver option if the holder can choose to enter into a short protection position.

Without loss of generality, let's focus on the pricing of payer option and the CDS index has M credits with fixed coupon $C(T)$ that ends at T. On maturity date of option, T_E, the owner can claim the protection for the credits defaulted before time T_E, such a feature is called *front end protection*. If exercised, the portfolio Swaptions is *price-based options*, which gives the cash payment based on the current value of the underlying. In our case, the amount of cash received is:

$$\big(K - C(T)\big) \mathit{RPV01}_I(T_E, T, K). \tag{10}$$

After expiry, the option owner has a long position of the credit, the present value of future cashflow of the CDS index is then,

$$\big(S_I(T_E, T) - C(T)\big) \mathit{RVP01}_I\big(T_E, T, S_I(T_E, T)\big), \tag{11}$$

where the $\mathit{RPV01}_I$ at time T_E is calculated using a flat credit curve at the index spread $S_I(T_E, T)$ on the option expiry date.

Remark 1.1. *Notice the strike in* (10) *is associated with full list of credits as well as original notional before any defaults, while* $S_I(T_E, T)$ *in* (11) *is associated with the the remaining notional and reference entities after removing those defaulted.*

To combine, the payoff of the option is

$$\begin{aligned} V^{Payer}(T_E) := \max \Bigg\{ &\frac{1}{M} \sum_{m=1}^{M} (1 - R_m) \mathbf{1}_{\{\tau_m \le T_E\}} \\ &+ \big(S_I(T_E, T) - C(T)\big) \mathit{RPV01}_I(T_E, T, S_I(T_E, T) \\ &- \big(K - C(T)\big) \mathit{RPV01}_I(T_E, T, S_I(T_E, T), 0 \Bigg\}. \end{aligned} \tag{12}$$

Notice if there is no deviation of K from the time T_E market index spread, i.e., $K = S_I(T_E, T)$, the payoff of ends up with 0 given no default T_E.

1.5 Valuation

Recall (7), the time T_E market index spread can be rewritten as:

$$\begin{aligned}&\big(S_I(T_E,T)-C(T)\big)RPV01_I\big(T_E,T,S_I(T_E,T)\big)\\&=\sum_{m=1}^{M}\mathbf{1}_{\{\tau_m\geq T_E\}}\big(S_m(T_E,T)-C(T)\big)RPV01_m(T_E,T)\end{aligned}\tag{13}$$

The presence of indicator function removes those defaulted credit, because that's what $S_I(T_E,T)$ and $RPV01_I(T_E,T,S_I(T_E,T))$ are associated. Pick an intermediate time $t\in[0,T_E]$, we have the following alternative formula for payoff at maturity:

$$V^{Pay}(T_E)=\max\Big\{H\big(T_E,\{S_m(T_E,T)\}\big)-G(K),0\Big\},\tag{14}$$

where $\{S_m(T_E,T)\}$ stands for the dependence on the series of S_m of H, for $m=1,...,M$. And H is specified below

$$\begin{aligned}H\big(T_E,\{S_m(T_E,T)\}\big)=&\frac{1}{M}\sum_{m=1}^{M}(1-R_m)\mathbf{1}_{\{0<\tau_m\leq t\}}+\frac{1}{M}\sum_{m=1}^{M}(1-R_m)\mathbf{1}_{\{t<\tau\leq T_E\}}\\&+\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}_{\{\tau_m>T_E\}}\big(S_m(T_E,T)-C(T)\big)RPV01_m(T_E,T)\end{aligned}\tag{15}$$

For time t value of the option, it amounts to compute the discounted payofff under risk neutral measure,

$$V^{Pay}(t)=\mathbb{E}_t^{\mathbb{Q}}\Big[D(t,T_E)\max\Big\{H\big(T_E,\{S_m(T_E,T)\}\big)-G(K),0\Big\}\Big]\tag{16}$$

As mentioned earlier, the uncertainty in interest rate is not as much as the uncertainty in survival, we assume the interest rate is deterministic. Thus,

$$V^{payer}(t)=P(t,T_E)\mathbb{E}_t^{\mathbb{Q}}\Big[\max\Big\{H\big(T_E,\{S_m(T_E,T)\}\big)-G(K),0\Big\}\Big],\tag{17}$$

obviously, the time T_E distribution of H function determines the expectation in (17).

Before doing that, let's diagress a bit to explore the expectation of the H functional. There are three terms in $H(T_E, \{S_m(T_E, T)\}$, the latter two of which are random. For the second term, after taking expectation, we obtain

$$\frac{1}{M}\sum_{m=1}^{M}(1-R_m)\mathbb{E}_t^{\mathbb{Q}}\left[\mathbf{1}_{\{t<\tau_m\leq T_E\}}\right] = \frac{1}{M}\sum_{m=1}^{M}(1-R_m)\big(1-S_m(T_E, T)\big). \quad (18)$$

The expectation of the third term is:

$$\frac{1}{M}\sum_{m=1}^{M}\mathbb{E}_t^{\mathbb{Q}}\Bigg[\mathbf{1}_{\{\tau_m>T_E\}}\big(S_m(T_E, T)-C(T)\big)RPV01_m\big(T_E, S_m(T_E, T)\big)\Bigg] \quad (19)$$

Define the *risky annuity numérairé*,

$$A_m(t) - \mathbf{1}_{\{\tau_m>t\}}RPV01_m(t, T_E, T), \quad (20)$$

By change of numérairé technique,

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{\tau_m>T_E\}}\big(S_m(T_E, T)-C(T)\big)RPV01_m\big(T_E, S_m(T_E, T)\big)}{D(T_E, T_E)}\right]\\
=&\frac{\mathbf{1}_{\{\tau_m>t\}}\big(S_m(t, T_E, T)-C(T)\big)RPV01(t, T_E, T)}{D(t, T_E)}\\
=&\frac{\big(S_m(t, T_E, T)-C(T)\big)RPV01(t, T_E, T)}{P(t, T_E)}
\end{aligned}$$

Notice $D(T_E, T_E) = 1$. Therefore, we have

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\, H\big(T_E, \{S_m(T_E, T)\}\big)\,\right] =& \frac{1}{M}\sum_{m=1}^{M}\Big(K(1-R_m)\mathbf{1}_{\{0<\tau_m\leq t\}} + (1-R_m)\big(1-S_m(t, T_E)\big)\Big)\\
&+\frac{1}{MP(t, T_E)}\sum_{m=1}^{M}\big(S_m(t, T_E, T)-C(T)\big)RPV01_m(t, T_E, T)
\end{aligned} \quad (21)$$

Now, we come back to the main task. Assume a homogeneous index portfolio with a single flat spread S and recovery R, here S is a $\mathcal{F}_{T_E}$-measurable random variable, then we can write

$$\begin{aligned}
&H(T_E, \{S_m(T_E, T)\} = \widetilde{H}(T_E, S)\\
:=&\frac{1}{M}\sum_{m=1}^{M}(1-R_m)\mathbf{1}_{\{0<\tau_m\leq t\}} + (S-C(T))RPV01_I(T_E, T, S).
\end{aligned}$$

Since S is flat spread, then by credit triangle, we can compute the risk PV01 explicitly as:

$$RPV01_I(T_E, T, S) = \sum_{i=1}^{N} \Delta(T_{i-1}, T_i) P(T_E, T_i) \exp\left\{ -\frac{S}{1-R}(T_i - T_E) \right\} \tag{22}$$

where N is number of payments to be made in between T_E and T. Now, under risk-neutral measure, we have

$$V^{Payer}(t) = \mathbb{E}_t^{\mathbb{Q}}\left[D(t, T_E) \max\left\{ \widetilde{H}(T_E, S) - G(K), 0 \right\} \right] \tag{23}$$

$$= P(t, T_E) \int_0^{\infty} \max\left\{ \widetilde{H}(T_E, s) - G(K), 0 \right\} f_S(s) ds \tag{24}$$

where $f_S(\cdot)$ is the density function of S that is supported on $[0, +\infty)$. Because of the non-negativity, the very natural and the simplest choice of the dynamic of S is log-normal, which results in:

$$S = S_t \exp\left\{ \sigma(W(T_E) - W(t)) - \frac{1}{2}\sigma^2(T_E - t) \right\}. \tag{25}$$

Then, the pricing can be re-written as:

$$V^{Payer}(t) = P(t, T_E) \int_{-\infty}^{\infty} \max\left\{ \widetilde{H}(T_E, S(S_t, z)) - G(K), 0 \right\} \phi_Z(z) dz \tag{26}$$

where $\phi(\cdot)$ is the *standard normal distribution*. To find the initial S_t, we use (21) to find S_t. Namely, use root find to solve for S_t, such that

$$\int_{-\infty}^{\infty} \widetilde{H}(T_E, S(S_t, z)) \phi(z) dz = \mathbb{E}_t\big[H(T_E, \{S_m(T_E)\}) \big]. \tag{27}$$

It is not surprising we have a put-call parity here for payer and receiver portfolio Swaption, namely,

$$V^{Payer}(t) - V^{Receiver}(t) = P(t, T_E)\mathbb{E}_t\left[H\big(T_E, S_I(T_E, T)\big) \right] - P(t, T_E)G(K) \tag{28}$$

where the expectation can be calculated according to (21). The relationship allows us to save the computation of receiver portfolio Swaption when a payer option is already derived.