American Option

Jianing Yao
Department of MSIS-RUTCOR
Rutgers University, the State University of New Jersey
Piscataway, NJ 08854 USA

April 23, 2015

1 Definitions

American options are options which the owner may exercise at any time between date of purchase ($t = 0$) and expiry ($t = T$). Consider an underlying asset with price $\{S(t)\}_{t\geq 0}$, the *American put option* with strike K and expiry T pays $(S(t) - K)^+$ to the option owner should he or she choose to exercise at time $t \leq T$. Obviously, the owner will only exercise if $S(t) > K$. The option expires as worthless if it is not exercised by time T. The *American put option* with strike K and expiry T pays $(K - S(t))^+$ if exercised at time $t \leq T$ and expires as worthless if never exercised by time T.

The *perpetual American put* with strike K is an American put with $T = \infty$. That is, there is no upper limit to the time at which it can be exercised. The perpetual put is not traded, but it is a nice textbook example because there is an explicit formula for its price.

In general, the payoff of an American option has the form $g(S(t))$ if exercised at time t, where g is the payoff function and $S(t)$ is the price of the underlying asset at time t.

2 Exercise Time

The owner of an American option will decide when to exercise based on the current level of the price, its past history and whatever other information is available about the economy and its past history. It is assumed that investors cannot look into the future. In a model in which the information available to investors at each time is encoded in a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, this translates into the following assumption:

Assumption 1: all exercise times are $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times.

The notation τ will be used to denote a generic exercise time. The holder of an option may choose not to exercise. We assume that if the option has not been exercised by the time T of expiration and if the option is in the money at T, the holder will exercise and receive a positive payoff. Letting an option which is out of the money at T expire is the same as exercising at time T and getting a zero payoff. Therefore, for options with finite expiration time, we shall assume for mathematical convenience that the option is always exercised, with the understanding that exercising at time T and gaining nothing represents not exercising.

Most of the analysis of American options that is presented in this class will be made under the following additional assumptions:

Assumption 2: the price process S is Markov process;
Assumption 3: $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by S.

In this frame work, the decision about whether to exercise at time t must depend only on the observed history of the price up to time t. The analysis we will do shows that, supposing assumption 2 and assumption 3, the best decision about whether to exercise or not at time t will only use the current value $S(t)$ of the price of the underlying. This is a consequence of the Markov property of S.

Consider an American option initiated at time 0 that expires at T, and let t be an intermediate time, $0 < t < T/$ Assuming that the option has not yet been exercised, we will be interested in studying its value for times after t, conditioned on $\mathcal{F}_t$. If we know the value of $S(t)$ and S is Markov, no further information about the past history of S is relevant to the conditional distribution of the price for times after t. This suggests we can treat the problem of valuing the option after time t in the same manner as from time 0. To do this, a further elaboration of assumption 1 will be useful:

Assumption 1': for $u \geq t$, $\mathcal{F}_u^{(t)}$ be the σ-algebra generated by $S(v)$ for times $t \leq v \leq u$.

The information contained in $\mathcal{F}_u^{(t)}$ is equivalent to observing the price $S(v)$ between times t and u. the collection $\{\mathcal{F}_u^{(t)}\}_{u\geq t}$ is the filtration generated by the price process S after time t. Assuming we know $S(t)$, $\mathcal{F}_u^{(t)}$ is the new information about the price process accumulated between times t and u.

With theses definitions, when we condition on $S(t) = x$ and on the option not being exercised by time t, we will consider future exercise times τ that satisfy,

$t \leq \tau \leq T$ *and* τ *is an* $\{\mathcal{F}_u^{(t)}\}_{u\geq t}$*-stopping time.*

3 The value of an American option; the intuition behind pricing

Let $(\Omega, \mathcal{F}, \mathbb{Q})$, $\{S(t)\}_{t\geq 0}$ be a risk-neutral price model. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by S and assume the risk free interest rate is constant r. Consider an American option that pays $g(S(t))$ if exercised at time t and let T be its expiration date. Suppose $0 \leq t \leq T$, $S(t) = x$, and the option has not been exercised by time t, and let $V(t)$ denote its value. For convenience, call the option seller Bob and the option buyer Alice.

What principles can we use to determine $V(t)$? Observe first that whatever $V(t)$ is, it must be true that $V(t) \geq g(S(t))$, because if $V(t) < g(S(t))$, Alice could buy it, immediately exercise and realize an arbitrage with profit $g(S(t)) - V(t)$. Her counter-party, Bob, would suffer an immediate loss of $V(t) - g(S(t))$.

To motivate the first approach to pricing, recall that the price $V(t)$ of an European option with pay off $g(S(T))$ is determined by insisting that $e^{-rt}V(t)$ be a martingale under the risk-neutral measure. This immediately leads to the risk-neutral pricing formula $\widetilde{\mathbb{E}}[e^{-r(T-t)}g(S(T)) \mid \mathcal{F}_t]$, which, in the Markov case under consideration, equals $\widetilde{\mathbb{E}}[e^{-r(T-t)g(S(T))} \mid S(t)]$. This price at t is therefore the expected, discounted option payoff, conditioned on the value of $S(t)$. This pricing formula does not extend immediately to the American case, because the value of the option at time T is not well-defined; the option may have already been exercised and the time at which it is exercised is not known in advance.

Now, fix a time t and an arbitrary stopping time τ satisfying $t \leq \tau \leq T$. Consider an option sold at time t that offers the payoff $g(S(\tau))$ at time τ. For example, the option contract might specify that τ is the first time after t that $S(u)$ hits level b, if this should occur before time T, and that $\tau = T$ otherwise. In this case, when $\tau < T$ the option holder receives $g(b)$ at time τ, and if $\tau = T$ receives $g(S(T))$. Such an option is not an American option, since the exercise time, though random, is fixed in advance. Let $V_\tau(u)$, $u \geq t$, denote its value. The same reasoning behind risk-neutral pricing of an European option requires that $e^{-r(u-t)}V_\tau(u)$, $u \geq t$, is a martingale up to time τ. It follows that

$$V_\tau(t) = \widetilde{\mathbb{E}}\big[e^{-r(\tau-t)}g(S(\tau)) \mid S(t)\big]$$

This is the *expected discounted payoff* of the option conditioned on $S(t)$.

Suppose at time t that Alice is offered a choice between an American option and an option that must be exercised at a given stopping time τ. The price $V(t)$ of the American option, must at least be as large as V_τ. If it were smaller Alice could purchase the American option and exercise at τ, and she would get the same payoff $g(S(\tau))$ at cheaper price than $V_\tau(t)$. This again creates an arbitrage opportunity, since $V_\tau(t)$ is a no-arbitrage price. To put it another way, the American option offers Alice more choices of exercise times than just τ and so she must be willing to pay at least as much for it. But this reasoning applies to any stopping time τ, so $V(t) \geq V_\tau(t)$ for all stopping times τ with $t \leq \tau \leq T$. On the other hand, Alice would not be willing to pay price $V(t)$ for the American option if she could not find an exercise time τ so that $V(t) = V_\tau(t)$, because she would stand to loose an average no matter what exercise strategy she employed. Thus, we conclude that a fair price $V(t)$ for the American option should be the maximum expected discounted payoff $V_\tau(t)$ possible using a stopping time τ, with $t \leq \tau \leq T$.

A second approach looks at the problem from a martingale angle. For this, the concept of a super-martingale is central. Recall that a process $X(t)$ is a super-martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if

1. $\mathbb{E}[|X(t)|] < +\infty$, for all t;
2. X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$;
3. $\mathbb{E}[X(t)|\mathcal{F}_s] \leq X(s)$ almost surely, whenever $0 \leq s \leq t$.

Remark 3.1 Because of the third property,

$$\mathbb{E}[X(t)] = \mathbb{E}\big[\mathbb{E}[X(t) \mid \mathcal{F}_s]\big] \leq \mathbb{E}[X(s)], \text{ if } s \leq t.$$

Thus, super-martingale is decreasing in expectation.

Before going further, let's keep the following facts in mind: (a) martingales are examples of super-martingales in which the expectation does not decrease as a function of time. A strict super-martingale is one in which the expectation is strictly decreasing, at least for sometimes. If $X(t)$ is your fortune in a game, the game is unfavorable to you if X is strict super-martingale; (b) the optional stopping theorem extends to super-martingales, if X is a super-martingale and τ is a stopping time, then $\{X(t \wedge \tau); t \geq 0\}$ is a super-martingale.

Using martingale and super-martingale concepts, we can reason as follows. If Bob sells an American option he will want its discounted price to be a super-martingale – that is, either fair or unfavorable to the buyer, so that it is fair or favorable to him. In particular Bob wants to be able to invest money $V(t)$ he receives for the option so as to cover the payoff whenever Alice might choose to exercise. In a risk-neutral market, Bob's discounted wealth from investing in the market will be a martingale, so, intuitively, if the value of the option is a super-martingale, he should be able to hedge it. On the other hand, Alice will no want to buy the option unless she can arrange a fair game for herself. this does not mean that the discounted price should be a martingale. But it ought to be a martingale, at least up to some stopping time, at which Alice can then exercise, and so not lose the martingale property. How long should the price remain a martingale? Alice will certainly not want to exercise the option so long as $V(t) > g(S(t))$, because if $V(t) > g(S(t))$ she could sell the option to another party for more money that she would get by exercising it. So the price should be a martingale at least up to the first time $V(t) = g(S(t))$. This is a start on a different, somewhat more abstract approach to pricing: insist that the discounted price ba martingale up to the time $V(t) = g(S(t))$, and a super martingale overall.

These two approaches may look quite different at first glance. But there is a deep and elegant connection between them that is at the heart of optimal stopping theory. We define the price of an American option sa the maximum, expected, discounted payoff over all stopping times. We will then show how to find this price and an optimal exercise time, by the martingale approach.

4 Defining the Price

Again, consider an American option with payoff function $g(x)$. taking our cue from the discussion above, define

$$v(t,x) = \max \left\{ \widetilde{\mathbb{E}}[e^{-r(\tau-t)} g(S(\tau)) \,|\, S(t) = x];\ \tau \text{ satisfies } \textit{Assumption 1'} \right\} \tag{1}$$

From the perspective of time t this is the best, discounted, expected payoff of the option can yield. We define the price $V(t)$ of the American option at time t, assuming it has not yet been exercised, as

$$V(t) = v(t, S(t)) \tag{2}$$

The function $v(t,x)$ is called the *value function* of the option pricing problem. A stopping time τ^* for which

$$v(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(\tau^*-t)} g(S(\tau^*)) \,|\, S(t) = x \right]$$

is called an *optimal exercise*, or *optimal stopping time*, for valuing the option starting at t.

Remark 4.1 Not to the *mathematicians*. It is more proper and rigorous to define the value function as:

$$v(t,x) = \sup\left\{\widetilde{\mathbb{E}}[e^{-r(\tau-t)}g(S(\tau))\,|\,S(t)=x];\ \tau \text{ satisfies } \textit{Assumption 1'}\right\}$$

This allows for the possibility that, while one can find exercise times whose return is arbitrarily close to $v(t,x)$, there is no exercise time that actually achieve $v(t,x)$. However, for the optional problems considered here, an optimal exercise time always exists.

5 Optimal Stopping Theory

The problem of choosing a stopping time τ to maximize the expectation $\mathbb{E}[X(\tau)]$ of a stochastic process X stopped at time τ, is called an *optimal stopping problem*. There is a well-developed theory for solving optimal stopping problems, and that is what will be used to solve the American option pricing problem.

We have already noted the lower bound for the value function in previous discussion. Consider an American option with payoff function g. Suppose time t has arrived, the option has not yet been exercised, and $S(t)=x$. The strategy of exercising immediately at t – that is, setting $\tau = t$ – certainly satisfies assumption 1'. Therefore,

$$v(t,x) \geq g(x), \text{ for all } t, 0 \leq t \leq T, \text{ and all } x. \tag{3}$$

Because of this fact, $g(S(t))$ is called the *intrinsic value* of American option at time t.

6 Continuation and Stopping Regions: the optimal exercise time

In this section, we show how to find the optimal exercise time if the value function $v(t,x)$ is known. The idea is simple and follows from the definition of $v(t,x)$. If $v(t,x) = g(x)$, the maximum achievable, expected, discounted payoff is the same as what one can get by exercising the option immediately at time t. If $v(t,x) > g(x)$, then exercising immediately yields a payoff $g(x)$ strictly than the best on can do. Therefore, not exercising the option until $v(t,S(t)) = g(S(t))$ will be optimal.

It is standard to formulate this exercise strategy in terms of *continuation and stopping sets*. The continuation set or region is

$$\mathcal{C} = \{(t,x); v(t,x) > g(x)\}$$

and is so called because one should continue not exercising so long as $(t,S(t)) \in \mathcal{C}$. The stopping set is the complement of $\mathcal{C}$:

$$\mathcal{S} = \{(t,x); v(t,x) = g(x)\}$$

When $(t, S(t)) \in \mathcal{S}$, it is optimal to exercise at t. We also call $\mathcal{S}$ the optimal exercise region. To conclude

$$\tau^* = \min\{u \geq t; (u.S(u)) \in \mathcal{S}\} \wedge T \tag{4}$$

is optimal exercise time when starting at time t.

Since we know how to find the optimal exercise time if the value function is known, the problem of pricing American options reduces to finding the value function. Of course, if we knew the optimal exercise time τ^*, that would give us a handle on $v(t,x)$ because $v(t,x) = \widetilde{\mathbb{E}}[e^{-r(\tau^*-t)}g(S(\tau^*)) \,|\, S(t) = x]$. It seems as if we have a chicken-egg problem here. The resolution to this problem is the martingale approach which I don't think will be covered in the class. But just want you to know that there exists such a class of PDEs called (*Hamilton-Jacobi-Bellman*), which arises in stochastic control problem, the solution of which corresponds to the value function, where our $v(t,x)$ is a special case. For PDE of this kind, the analytical solution exists only for very restricted parameter class, e.g., parameters are all constant, otherwise, we rely on numerical solution to these *obstacle problems*.

7 Reference

1. *Jean Jacod, Philip Protter, "Probability Esentials", Springer, 2004;*
2. *E.Cinlar, "Probability and Stochastics", Springer, 2011*
3. *Daniel, Ocone, "Notes in mathematical finance II 2012"*
4. *Alison Etheridge, "A course in financial calculus", Cambridge, 2002*