EE5630

Euler Equations

&

The Four Variational Problems

Problem-1

The Simplest Variational Problem

Problem 1: Let x be a scalar function in the class of functions with continuous first derivatives. It is desired to find the function x^* for which the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \qquad (4.2\text{-}1)$$

has a relative extremum.

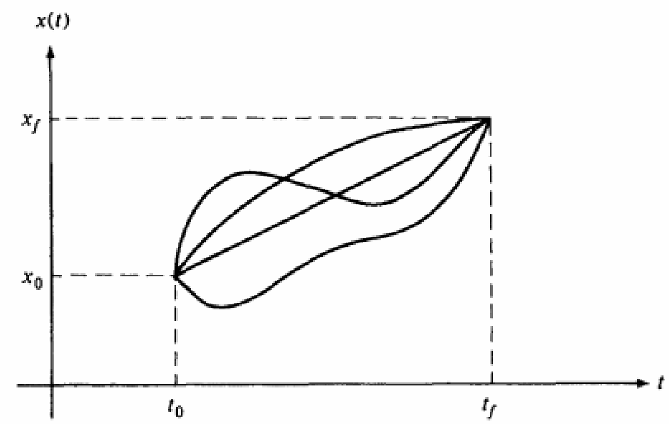

Figure 4-6 Admissible curves for *Problem 1*

Main Steps

1. Define the Increment

$$\begin{aligned}\Delta J(x, \delta x) &= J(x + \delta x) - J(x) \\ &= \int_{t_0}^{t_f} g(x(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), t)\, dt \\ &\quad - \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt.\end{aligned} \tag{4.2-2}$$

2. Taylor Series Expansion

$$\begin{aligned}\delta J(x, \delta x) = \int_{t_0}^{t_f} \Big\{ &\left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right] \delta x(t) \\ &+ \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \delta \dot{x}(t) \Big\}\, dt.\end{aligned} \tag{4.2-5}$$

Main Steps

3. Integration by Parts

$$\delta J(x, \delta x) = \overbrace{\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]}^{B} \delta x(t)\Big|_{t_0}^{t_f}$$

$$+ \int_{t_0}^{t_f} \left\{ \overbrace{\left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]}^{A} \right\} \delta x(t)\, dt. \qquad (4.2\text{-}7)$$

4. Variation at end points ($\delta x(t_0) = 0,\ \delta x(t_f) = 0,$) and Fundamental Theorem

$$\delta J(x^*, \delta x) = 0 = \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] \right\} \delta x(t)\, dt. \qquad (4.2\text{-}8)$$

Main Steps

5. Necessary condition for x* to be extremal

Euler Equation

$$\boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0} \qquad (4.2\text{-}10)$$

6. Split Boundary Conditions

$$x(t_0) = \mathrm{x}_0$$

$$x(t_f) = \mathrm{x}_f$$

Problem-2

Problem 2: Find a necessary condition for a function to be an extremal for the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt; \tag{4.2-24}$$

t_0, $x(t_0)$, and t_f are specified, and $x(t_f)$ is free.

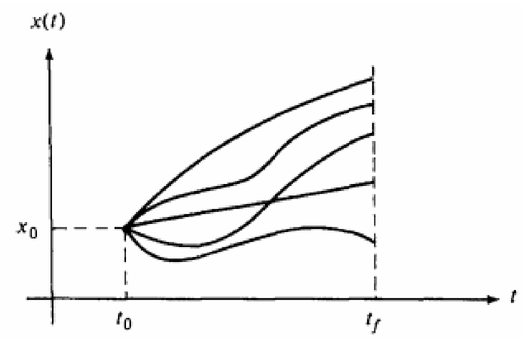

Figure 4-9 Several admissible curves for *Problem 2*

Main Steps

1. Define the Increment

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$
$$= \int_{t_0}^{t_f} g(x(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), t)\, dt$$
$$- \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt.$$

2. Taylor Series Expansion

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \right] \delta x(t) + \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \right] \delta \dot{x}(t) \right\} dt.$$

Main Steps

3. Integration by Parts

B

A

$$\delta J(x, \delta x) = \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]\delta x(t)\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left\{\left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]\right\} \delta x(t)\, dt. \qquad (4.2\text{-}7)$$

4. Variation at end points ($\delta x(t_0) = 0$ for all admissible curves, but $\delta x(t_f)$ is arbitrary.)

=> B term will not vanish.

Main Steps

5. Necessary condition for x* to be extremal

Euler Equation

$$\boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0} \qquad (4.2\text{-}10)$$

6. Split Boundary Conditions

$$x(t_0) = \mathrm{x}_0$$

$$\boxed{\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0.} \qquad (4.2\text{-}27)$$

Problem-3

Problem 3: Find a necessary condition that must be satisfied by an extremal of the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt; \tag{4.2-43}$$

t_0, $x(t_0) = x_0$, and $x(t_f) = x_f$ are specified, and t_f is free.

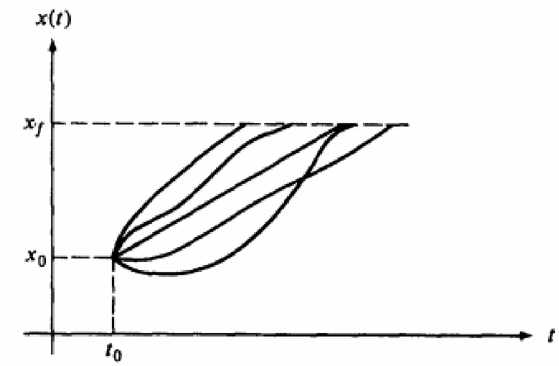

Figure 4-11 Several admissible curves for *Problem 3*

Main Steps

1. Define the Increment

$$\Delta J = \int_{t_0}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\, dt - \int_{t_0}^{t_f} g(x^*(t), \dot{x}^*(t), t)\, dt$$
$$= \int_{t_0}^{t_f} \{g(x(t), \dot{x}(t), t) - g(x^*(t), \dot{x}^*(t), t)\}\, dt \qquad (4.2\text{-}44)$$
$$+ \int_{t_f}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\, dt,$$

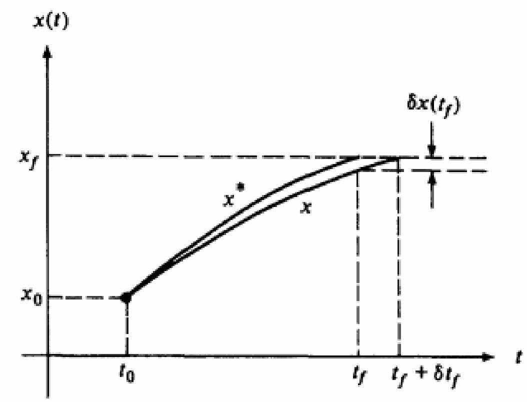

Figure 4-12 An extremal, x^*, and a neighboring comparison curve, x

$$\Delta J = \int_{t_0}^{t_f} \{g(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t)$$
$$- g(x^*(t), \dot{x}^*(t), t)\}\, dt \qquad (4.2\text{-}45)$$
$$+ \int_{t_f}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\, dt.$$

Main Steps

2. Taylor Series Expansion of first integrand:

$$\Delta J = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) \right] \delta x(t) + \left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t) \right] \delta \dot{x}(t) \right\} dt + o(\delta x(t), \delta \dot{x}(t)) + \int_{t_f}^{t_f + \delta t_f} g(x(t), \dot{x}(t), t)\, dt.\dagger \qquad (4.2\text{-}46)$$

Main Steps

3. Integration by Parts

$$
\begin{aligned}
\Delta J = {} & \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\delta x(t_f) + [g(x(t_f), \dot{x}(t_f), t_f)]\,\delta t_f \\
& + \int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) \right. \qquad\qquad (4.2\text{-}48) \\
& \left. - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]\right\} \delta x(t)\, dt + o(\cdot),
\end{aligned}
$$

A

- Solving Further:

C

$$
\begin{aligned}
\delta J(x^*, \delta x) = 0 = &\left\{\left[-\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \dot{x}^*(t_f)\right.\\
&\left.+\, g(x^*(t_f), \dot{x}^*(t_f), t_f)\right\} \delta t_f \\
&+ \int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t)\right.\\
&\left.- \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]\right\} \delta x(t)\, dt.
\end{aligned}
\tag{4.2-53}
$$

A

Main Steps

5. Necessary condition for x* to be extremal

Euler Equation

$$\boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0} \qquad (4.2\text{-}10)$$

6. Split Boundary Conditions

$$x(t_0) = x_0$$

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\dot{x}^*(t_f) = 0. \qquad (4.2\text{-}55)$$

Problem-4

Problem 4: Find a necessary condition that must be satisfied by an extremal for a functional of the form

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt; \tag{4.2-62}$$

t_0 and $x(t_0) = x_0$ are specified, and t_f and $x(t_f)$ are free.

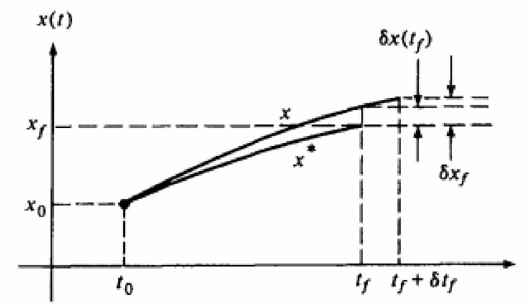

Figure 4-13 An extremal and a neighboring comparison curve for *Problem 4*

Main Steps

1. Similar steps give

$$
\begin{aligned}
\delta J(x^*, \delta x) = 0 = {} & \Big[\underbrace{\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)}_{B}\Big]\,\delta x_f \\
& + \Big[\underbrace{g(x^*(t_f), \dot{x}^*(t_f), t_f) - \Big[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\Big]\dot{x}^*(t_f)}_{C}\Big]\,\delta t_f \\
& + \int_{t_0}^{t_f} \Big\{\underbrace{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\Big[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\Big]}_{A}\Big\}\,\delta x(t)\,dt.
\end{aligned}
\qquad (4.2\text{-}65)
$$

Main Steps

5. Necessary condition for x* to be extremal

Euler Equation

$$\boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0} \qquad (4.2\text{-}10)$$

6. Split Boundary Conditions

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] = 0. \qquad (4.2\text{-}26)$$

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \dot{x}^*(t_f) = 0. \qquad (4.2\text{-}55)$$

Functionals with Several Independent Function

Problems with Fixed End Points

Problem 1a: Consider the functional

$$J(x_1, x_2, \ldots, x_n) = \int_{t_0}^{t_f} g(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t), t)\, dt, \quad (4.3\text{-}1)$$

n Euler Equations

$$\begin{aligned}
&\frac{\partial g}{\partial x_i}(x_1^*(t), \ldots, x_n^*(t), \dot{x}_1^*(t), \ldots, \dot{x}_n^*(t), t) \\
&\quad - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}_i}(x_1^*(t), \ldots, x_n^*(t), \dot{x}_1^*(t), \ldots, \dot{x}_n^*(t), t)\right] \qquad (4.3\text{-}6) \\
&= 0 \quad \text{for all } t \in [t_0, t_f] \quad \text{and} \quad i = 1, \ldots, n.
\end{aligned}$$

Summary

Table 4-1 DETERMINATION OF BOUNDARY-VALUE RELATIONSHIPS

Problem description	*Substitution*	*Boundary conditions*	*Remarks*
1. $\mathbf{x}(t_f)$, t_f both specified (*Problem 1*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (*Problem 2*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (*Problem 3*)	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent (*Problem 4*)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (*Problem 4*)	$\delta\mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f)\,\delta t_f$†	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f

† $\frac{d\boldsymbol{\theta}}{dt}$ denotes the $n \times 1$ column vector $\left[\frac{d\theta_1}{dt} \quad \frac{d\theta_2}{dt} \quad \cdots \quad \frac{d\theta_n}{dt}\right]^T$.