Chapter 5, Section 1 - Summary (EE5630)

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Using the techniques that were developed in Chapter 4, the necessary conditions for optimal control can be determined. As we've done throughout the course, we're looking for an optimal control action, $\mathbf{u}^*$, that causes the system to follow a trajectory, $\mathbf{x}^*$, that minimizes the cost function, J, both given by Equation 1.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t), \\ J(\mathbf{u}) &= h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt, \end{aligned} \tag{1}$$

with initial conditions $\mathbf{x}(t_0) = \mathbf{x_0}$.

In Chapter 3, using dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation (given by Equation 2), we were able to find an optimal control action, $\mathbf{u}^*$, that causes the system to follow a trajectory, $\mathbf{x}^*$, that minimizes the cost function, J. This method of finding $\mathbf{u}^*$ and $\mathbf{x}^*$ is quite useful, but it doesn't directly include any of the constraints, something that any real system has.

$$\begin{aligned} 0 &= J_t + \mathcal{H} \\ &= J_t + g(\mathbf{x}(t), \mathbf{u}(t), t) + J_x^{*T}\left[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)\right] \end{aligned} \tag{2}$$

Another method, Pontryagin's Minimum Principle, allows us to directly embed the constraints of the system into our equations. Thus, making the use of constraints more accessible or straightforward (although maybe not *easier*). The way that we include the constraints is by creating an *augmented* cost function (using g_a instead of g in the Hamiltonian, $\mathcal{H}$) and using ideas from Chapter 4 (variationals and Lagrange multipliers). We also modify the Hamiltonian, $\mathcal{H}$, to include g_a. This is given below

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)^T\left[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)\right]. \tag{3}$$

At the end of some mathematical gymnastics and manipulation of both equations in 1, using the calculus of variations and Lagrange multipliers, we arrive at the necessary conditions of optimality given in Equations 4 and 5 below.

$$\boxed{\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}(t) &= -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial \mathcal{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \end{aligned}\right\} \forall t \in [t_0, t_f]}, \tag{4}$$

and

$$\boxed{\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[\mathcal{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t_f}(\mathbf{x}^*(t), t_f)\right]\delta t_f = 0}. \tag{5}$$