Assignment 3 - EE5630

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Question 1

Find the extremal for the following functional

$$J = \int_0^1 [x^2(t) + \dot{x}^2(t)]dt; \qquad x(0) = 0; x(1) = 1$$

Solution

Let $g = x^2(t) + \dot{x}^2(t)$. To find the extremal, we'll use the Euler equation given below in Equation 1

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 0 \tag{1}$$

Therefore

$$\frac{\partial g}{\partial x} = 2x\left(t\right), \qquad \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 2\ddot{x}\left(t\right)$$

$$\begin{aligned}\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} =& 0 \\ 2x\left(t\right) - 2\ddot{x}\left(t\right) =& 0 \\ x\left(t\right) =& \ddot{x}\left(t\right)\end{aligned}$$

Let's make a guess as to what the solution could be and apply the

$$\begin{aligned}x\left(t\right) =& Ae^t + Be^{-t} \\ x\left(0\right) =& A + B = 0 \\ x\left(1\right) =& Ae + Be^{-1} = 1\end{aligned}$$

Solving the system of equations given by $x\left(0\right)$ and $x\left(1\right)$ we get

$$\begin{aligned}A =& \frac{1}{1+e} \\ B =& \frac{e}{1+e}\end{aligned}$$

Which gives the extremal as

$$\boxed{x\left(t\right) = \frac{1}{1+e}\left[e^t + e^{-t+1}\right]}$$

Question 2

Find the extremal for the following functional:

$$J = \int_0^1 [x(t) + \dot{x}(t) + \frac{1}{2}\dot{x}(t) + x(t)\,\dot{x}(t)]dt; \qquad x(0) = \frac{1}{2};\ x(1) = \text{free}$$

Solution

Let $g = x(t) + \dot{x}(t) + \frac{1}{2}\dot{x}(t) + x(t)\,\dot{x}(t)$, then

$$\frac{\partial g}{\partial x} = 1 + \dot{x}, \qquad \frac{\partial g}{\partial \dot{x}} = 1 + \dot{x} + x \Rightarrow \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = \ddot{x} + \dot{x}$$

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} &= 0 \\ 1 + \dot{x} - (\ddot{x} + \dot{x}) &= 0 \\ \therefore \quad \ddot{x} &= 1 \\ \therefore \quad x(t) &= \frac{1}{2}t^2 + a\,t + b \end{aligned}$$

Applying the boundary conditions we get

$$\begin{aligned} x(0) = b = {}& \frac{1}{2} \\ \& \\ \frac{\partial g}{\partial \dot{x}}(1) &= 1 + (1 + a) + \left(\frac{1}{2} + a + b\right) = 0 \\ &= 1 + (1 + a) + \left(\frac{1}{2} + a + 1\right) \\ \therefore \quad a &= -\frac{3}{2} \end{aligned}$$

Which leaves the solution as

$$\boxed{x(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{1}{2}}$$

Question 3

Determine the necessary conditions that must be satisfied by the extremals for the functional:

$$J(w) = \int_{t_0}^{t_f} [w_3^2(t) + 5]dt$$

and the differential constraints that must be satisfied are:

$$\begin{aligned} \dot{w}_1(t) &= w_2(t) \\ \dot{w}_2(t) &= w_3(t) \end{aligned} \tag{2}$$

Solution

Let $g_a = w_3^2 + 5 + p_1[w_2 - \dot{w}_1] + p_2(t)[w_3 - \dot{w}_2]$, then

$$\begin{aligned}
&\frac{\partial g_a}{\partial w_1} = 0; \qquad \frac{\partial g_a}{\partial \dot{w}_1} = -p_1 \Rightarrow \frac{d}{dt}\frac{\partial g_a}{\partial \dot{w}_1} = -\dot{p}_1 \quad \therefore \dot{p}_1 = 0 \\
&\frac{\partial g_a}{\partial w_2} = p_1; \qquad \frac{\partial g_a}{\partial \dot{w}_1} = -p_2 \Rightarrow \frac{d}{dt}\frac{\partial g_a}{\partial \dot{w}_1} = -\dot{p}_2 \quad \therefore p_1 + \dot{p}_2 = 0 \\
&\frac{\partial g_a}{\partial w_2} = 2w_3 + p_2; \qquad \frac{\partial g_a}{\partial \dot{w}_3} = 0 \Rightarrow \frac{d}{dt}\frac{\partial g_a}{\partial \dot{w}_3} = 0 \quad \therefore 2w_3 + p_2 = 0
\end{aligned}$$

Therefore, the necessary conditions for optimality are

$$\boxed{\begin{aligned} p_1^*(t) &= 0 \\ p_2^*(t) &= -\,p_1^*(t) \\ 2w_3^* + p_2^* &= 0 \\ \dot{w}_1^*(t) &= w_2^*(t) \\ \dot{w}_2^*(t) &= w_1^*(t) \end{aligned}}$$

Question 4

Find the optimal control that transfers the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t)\end{aligned} \tag{3}$$

from any initial state (x_0) to the origin in minimum time. The constraint on the control is as follows:

$$-1 \le u(t) \le +1$$

(Note: This is a special class of **minimum time problem** with linear stationary dynamics and linear regulator (control). The solution of these type of problems is **Bang-Bang Control** and can be obtained analytically! Bang-bang control arises in linear minimum-time problems with constrained input magnitude. 1 The resulting optimal control for these problems needs only take on two values, which are the extreme values of the control.)

Solution

Let $\mathcal{H} = g + p(t)^T [a] = 1 + p_1 x_2 + p_2 u$, then

$$\begin{aligned}\dot{\boldsymbol{x}}^* = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} &\Rightarrow \dot{x}_1^* = \frac{\partial \mathcal{H}}{\partial p_1} = x_2^*; \qquad \dot{x}_2^* = \frac{\partial \mathcal{H}}{\partial p_2} = u^* \\ \dot{\boldsymbol{p}}^* = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} &\Rightarrow \dot{p}_1^* = -\frac{\partial \mathcal{H}}{\partial x_1} = 0; \qquad \dot{p}_2^* = \frac{\partial \mathcal{H}}{\partial x_2} = -\dot{p}_1^*\end{aligned}$$

Because $\mathcal{H}(u^*) \le \mathcal{H}(u)$, we get

$$\begin{aligned}\mathcal{H}(u^*) &\le \mathcal{H}(u) \\ 1 + p_1^* x_2^* + p_2^* u^* &\le 1 + p_1^* x_2^* + p_2^* u \\ p_2^* u^* &\le p_2^* u\end{aligned}$$

Which shows that

$$u(t) = \begin{cases} 1 & \text{if } p_2^* > 1 \\ -1 & \text{if } p_2^* < -1 \\ p_2^* & \text{otherwise} \end{cases}$$

Question 5

Find the optimal control that transfers the system

$$x\left(t\right)=u\left(t\right)$$

from any initial state (x_0) to the origin such that the following cost is minimized:

$$J(u)=\int_{0}^{t_f}\left|u\left(t\right)\right|dt$$

where t_f is free and the constraint on the control is as follows:

$$-1\leq u(t)\leq +1$$

(Note: This is a special class of **minimum fuel/control problem** with linear stationary dynamics and linear regulator (control). The solution of these type of problems is **Bang-Off-Bang Control** and can be obtained analytically!)

Solution

Let $\mathcal{H}=g+p\left(t\right)^{T}\left[a\right]=\left|u\right|+pu$, then

$$\begin{aligned}
p^{*}=&-\frac{\partial\mathcal{H}}{\partial x}=0\Rightarrow p^{*}=c\,\Big|\,c\in\mathbb{R}\\
x\left(t\right)=&x_0+\int_{0}^{t_f}u\left(t\right)\,dt\\
x\left(0\right)=&x_0+\int_{0}^{t_f}u\left(t\right)\,dt=\,0\Rightarrow x_0=-\int_{0}^{t_f}u\left(t\right)\,dt
\end{aligned}$$

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Question 6

The dynamics of a system are given as:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned} \tag{4}$$

and the cost function to be minimized is

$$J = \frac{1}{2}\int_0^2 u^2(t)\,dt$$

Optimal feedback solution is to be found by using Pontryagin's Minimization Principle. Admissible states and controls are not bounded. $X(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $X(2) = \begin{bmatrix} 5 & 2 \end{bmatrix}^T$.

a) Find the necessary conditions that must be satisfied. (Obtain state equations and co-state equations)

b) Try to solve analytically using the necessary conditions and boundary values.

c) Develop a MATLAB code to solve the new ODE system symbolically using syms and dsolve functions.

d) Compare the results obtained in part (b) and (c) by plotting relevant state trajectories.

Solution

a)

Let $\mathcal{H} = \frac{1}{2}u^2 + p_1x_2 + p_2(-x_2 + u)$, then

$$\begin{aligned} \dot{p}_1^* = -\frac{\partial \mathcal{H}}{\partial x_1} = 0; \quad \dot{p}_1^* = -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1^* + p_2^* \\ \frac{\partial \mathcal{H}}{\partial u} = u^* + p_2^* = 0 \Rightarrow u^* = -p_2^* \end{aligned}$$

Which leads to

$$\begin{aligned} x_1^* &= c_1 + c_2\left[1 - e^t\right] + c_3\left[-t - \frac{1}{2}e^{-t} + \frac{1}{2}e^t\right] + c_4\left[1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^t\right] \\ x_2^* &= c_2e^{-t} + c_3\left[-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t\right] + c_4\left[\frac{1}{2}e^{-t} - \frac{1}{2}e^t\right] \\ p_1^* &= c_3 \\ p_2^* &= c_3\left[1 - e^t\right] + c_4e^t \end{aligned} \tag{5}$$

and the system of equations given in (5) we get the necessary conditions

b)

$$\begin{aligned} x_1^* &= -6.103 + 7.289t + 6.696e^{-t} - 0.593e^t \\ x_2^* &= 7.289 - 6.696e^{-t} - 0.593e^t \\ p_1^* &= -7.289 \\ p_2^* &= -7.289\left[1 - e^t\right] - 6.103e^t \end{aligned}$$

d)

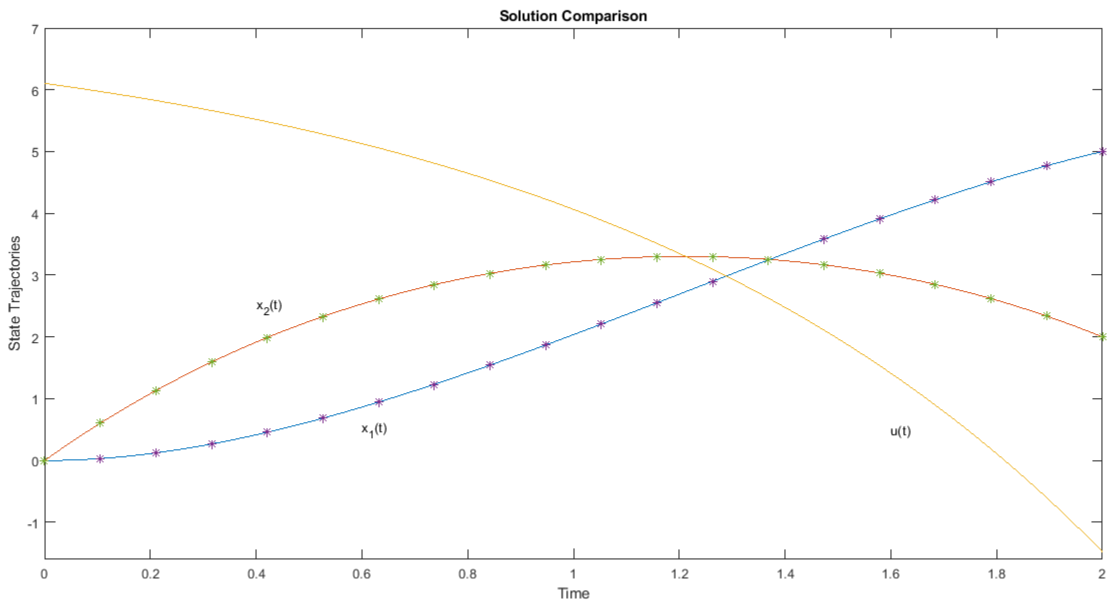

Fig. 1: Plot of state trajectories and control action

Question 7

The dynamics of a system are given as:

$$\begin{aligned}\dot{x}_1(t) &= -2\left[x_1(t)+0.25\right]+\left[x_2(t)+5\right]\exp\left[\frac{25x_1(t)}{x_1(t)+2}\right]-\left[x_1(t)+0.25\right]u(t)\\ \dot{x}_2(t) &= 0.5-x_2(t)-\left[x_2(t)+5\right]\exp\left[\frac{25x_1(t)}{x_1(t)+2}\right]\end{aligned}$$

with initial conditions $\boldsymbol{x}(0)=\begin{bmatrix}0.05 & 0\end{bmatrix}^T$ and the cost function to be minimized is

$$J=\int_0^{0.78}\left[x_1^2(t)+x_2^2(t)+0.1u^2(t)\right]dt$$

Optimal feedback solution is to be found by using Pontryagin's Minimization Principle. Admissible states and controls are not bounded. $X(0)=\begin{bmatrix}0 & 0\end{bmatrix}^T$ and $X(2)=\begin{bmatrix}5 & 2\end{bmatrix}^T$.

Using the state and co-state equatinos obtained in Example 6.2-2 (Kirk):

a) Solve this boundary value problem using the steepest descent algorithm in MATLAB.

b) Solve this boundary value problem using the MATLAB function *bvp4c*.

c) Which method do you like better and why?

Solution

c)

If I were to go purely off of lines of code then I would use the built-in *bvp4c* function that's available in MATLAB. It also happens that *bvp4c* has a lower final cost of approximately 0.009599 compared to gradient descent's cost of 0.028196 which is almost three times larger than *bvp4c*, as shown in Figures 2 and 3.

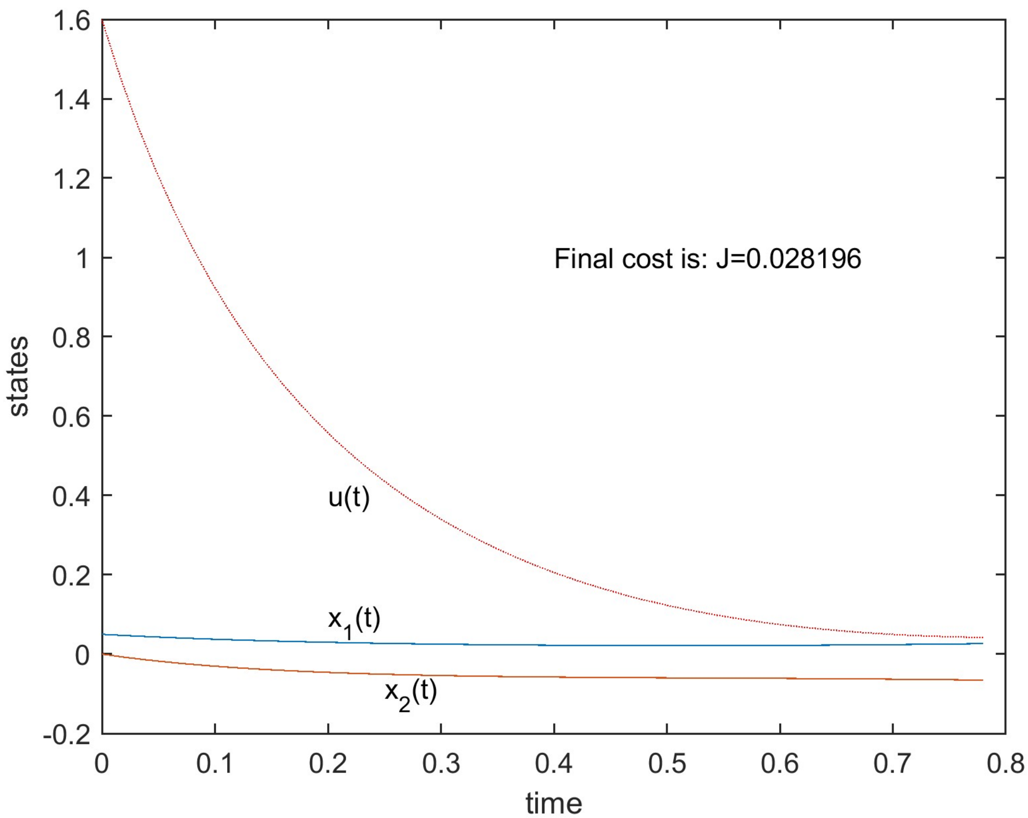

Fig. 2: Cost of gradient descent

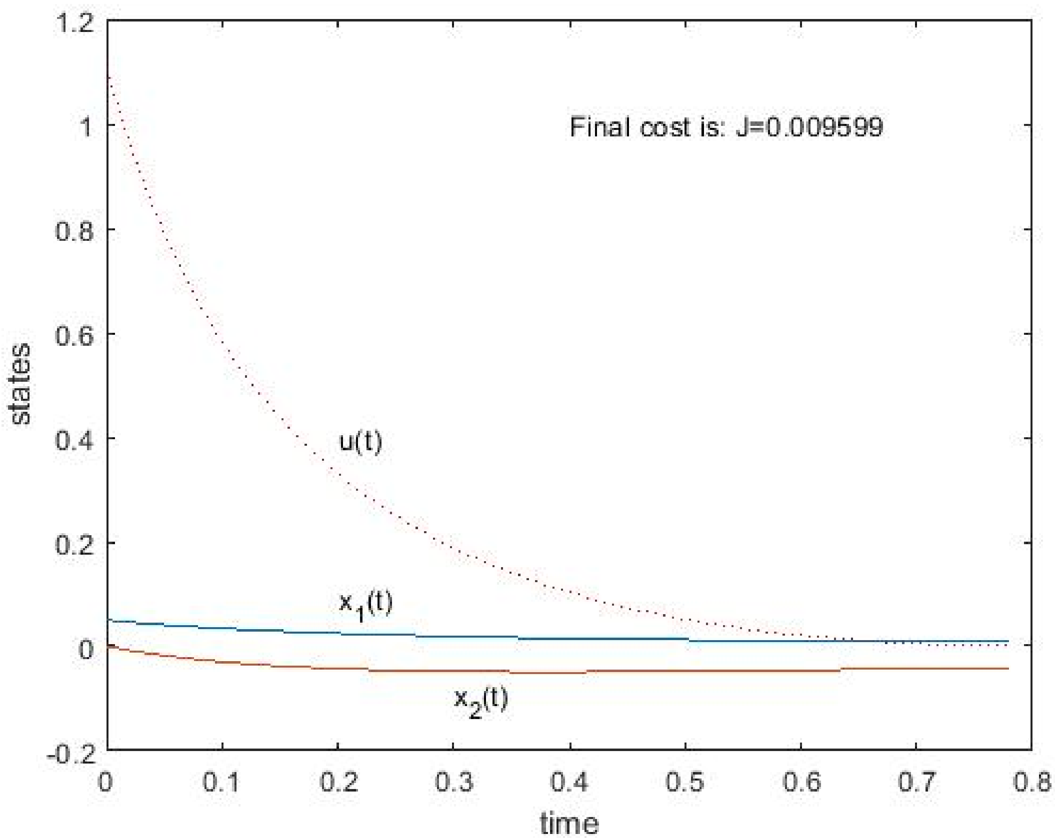

Fig. 3: Cost of MATLAB function *bvp4c*