Assignment : 2 - Solutions

(1)

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + x_2(t) + u(t) \end{aligned}\right\} \Rightarrow a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$J = \int_0^T \underbrace{\frac{1}{2}\left[q_1 x_1^2 + q_2 x_2^2 + u^2 \right]}_{g} dt$$

$\Rightarrow h = 0$

$g = \ldots$

define $\mathcal{H} = g + J_x^{*T} \cdot [a]$

where J^* is the value function (also represented as V in some books)

and J_x^* is the partial derivative w.r.t. state x.

$$J_x^* = \begin{bmatrix} \frac{\partial J^*}{\partial x_1} \\ \frac{\partial J^*}{\partial x_2} \end{bmatrix}$$ in this problem.

Now, if $u^*(t)$ minimizes $\mathcal{H}$ then-

$$\frac{\partial H}{\partial u} = 0$$

$$\Rightarrow \frac{\partial}{\partial u}\left[\frac{1}{2} q_1 x_1^2 + \frac{1}{2} q_2 x_2^2 + \frac{1}{2} u^2 + J_{x_1}^* \cdot a_1 + J_{x_2}^* a_2 \right] = 0$$

$$\Rightarrow u^*(t) + J_{x_2}^* (1) = 0$$

$$\Rightarrow \boxed{u^*(t) = -J_{x_2}^*}$$

#

(2) $\dot{x}(t) = -10\,x(t) + u(t) \quad \} \Rightarrow a$

and $J = \underbrace{\frac{1}{2} x^2(T)}_{h} + \int_0^T \underbrace{\left[\frac{1}{4} x^2(t) + \frac{1}{2} u^2(t)\right]}_{g} dt$

define $\mathcal{H} = g + J_x^{*T} \cdot [a]$

$$\mathcal{H} = \frac{x^2}{4} + \frac{u^2}{2} + J_x^* \left[-10\,x(t) + u(t)\right]$$

now $\frac{\partial H}{\partial u} = 0 \Rightarrow$

$$u^*(t) + J_x^* = 0$$

$$\boxed{u^*(t) = -J_x^*} \quad \text{--- (1)}$$

Now put value of $u^*(t)$ into the HJB to get –

$$0 = J_t^* + \left\{ \frac{x^2}{4} + \frac{J_x^{*2}}{2} - 10 J_x^* x(t) - J_x^{*2} \right\}$$

$$\boxed{0 = J_t^* + \frac{x^2}{4} - \frac{J_x^{*2}}{2} - 10 J_x^* x} \quad \text{--- (2)} \quad \text{HJB PDE.}$$

Now, we observe from boundary condition that -

$$J^*(x(T), T) = \frac{1}{2} x^2(T)$$

hence we assume J^* to be of the quadratic form $\quad J^* \approx \frac{1}{2} K(t)\, x^2(t)$

$$\Rightarrow J_x^* = K(t)\, x(t)$$

$$\Rightarrow \cancel{u^*(t) = -J^*}$$

$$\boxed{u^*(t) = -K(t)\, x(t)} \quad \#$$

(3) $\dot{x}(t) = x(t)\, u(t)$; $\}\to a$

$x(0) = 1$

and $J = \underbrace{x^2(1)}_{h} + \int_0^1 \underbrace{[x(t)\ u(t)]^2}_{g} dt$

a) $\mathcal{H} = g + J_x^* [a]$

$\mathcal{H} = x^2 u^2 + J_x^* \, x u$

$\frac{\partial H}{\partial u} = 0 \Rightarrow 2x^2 u + J_x^* x = 0$

$$\boxed{u^* = -\frac{J_x^*}{2x}} \quad x \neq 0 \quad -(1)$$

(b) putting u^* to get HJB in J -

$0 = J_t^* + \mathcal{H}^*$

$0 = J_t^* + \frac{J_x^{*2}}{4} + \frac{(-)J_x^{*2}}{2}$

$$\boxed{0 = J_t^* + \frac{1}{4} J_x^{*2}} \quad -(2) \quad \text{HJB PDE}$$

(c) Now looking at the boundary condition at $t = 1$ we have

$J^*(x(1), 1) = x^2(1)$

hence we claim the quadratic sol^n to PDE.

$\Rightarrow$ ~~$J^* = x^T(t) K(t) x^T(t)$~~

$\Rightarrow$ $J^* = K(t)\, x^2$ $-(3)$; $K(1) = 1$

putting J^* from (3) into (2) we get -

$0 = \dot{K}(t)\, x^2 + \frac{1}{4} \cdot 4x^2 K^2(t)$

$$\cancel{\dot{K}(t)\,x^2 + 3x^2 K^2(t) = 0}$$

$$\cancel{\frac{dK}{dt} = -3K^2}$$

$$\cancel{\int \frac{dK}{K^2} = -3\int dt}$$

$$\dot{K}(t) = K^2(t)$$

$$\int \frac{dK}{K^2} = \int dt$$

$$-\cancel{(2)}\frac{1}{K(t)} = t + c$$

$$K(t) = -\frac{1}{(t+c)} = \frac{1}{-c-t}$$

boundary condition now $K(1) = 1 \Rightarrow 1 = \frac{1}{-c-1} \Rightarrow c = -2$

$$\Rightarrow \boxed{K(t) = \frac{1}{2-t}}$$

\#

Hence $u^*(t) = -\frac{J_x^*}{2x}$ (from ①)

$$= -\frac{2x\,K(t)}{2x}$$

$$= -K(t)$$

$$\boxed{u^*(t) = \frac{-1}{2-t}}$$

\#

(4) $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

} comparing $\dot{X} = AX + BU$ to the given system dynamics

Now compare the standard cost function to the given one -

standard - $J = \frac{1}{2} X^T(t_f) H X(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (X^T Q X + U^T R U)\, dt$

given - $J = 10 x_1^2(T) + \int_0^T \frac{1}{2}[x_1^2 + 2x_2^2 + u^2]\, dt$

$\Rightarrow$ Now to understand clearly we first look at X -

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $X^T = [x_1 \quad x_2]$

$X^T H X = [x_1 \quad x_2] \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$1\times2 \qquad 2\times2 \qquad 2\times1$

$X^T H X = h_1 x_1^2 + h_2 x_2^2$

$\Rightarrow \frac{1}{2} X^T H X = \frac{1}{2}(h_1 x_1^2 + h_2 x_2^2)$

$\Rightarrow \boxed{h_1 = 20, \quad h_2 = 0} \Rightarrow H = \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}$

similarly -

$\frac{1}{2} X^T Q X = \frac{1}{2}\{x_1^2 + 2x_2^2\}$

$\Rightarrow \boxed{q_1 = 1 \text{ and } q_2 = 2} \Rightarrow Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

and $\frac{1}{2} u^T R u = \frac{1}{2} u^2 \Rightarrow \boxed{R = 1}$

Now use the differential Riccati equation -

$$0 = \dot{K}(t) + Q - K B R^{-1} B^T K + K A + A^T K$$

↳ this is a ODE in $K(t)$ and all other matrices are known.

Now in order to solve this system of ODE in MATLAB you need to use ODE 45 ~~function~~ function and solve -

$$\dot{K}(t) = \cancel{\begin{pmatrix} \dot{k}_1(t) \\ \dot{k}_2(t) \end{pmatrix}} = \left\{ \begin{matrix} \text{- - - -} \\ \text{- - - - -} \end{matrix} \right\} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{matrix} \text{- -} \\ \text{- -} \end{matrix}$$

once you have $K(t)$, then calculate

$$u^*(t) = -R^{-1} B^T K X$$ to get the feedback control.

Plot $X(t)$, $K(t)$ and $U(t)$ for $t \in [0, 10]$.

#

(5) For derivation of HJB check lecture notes / book.