EE5630: Optimal Control - Assignment 2

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Question 1

Given the system dynamics

$$
\begin{aligned}
\dot{x}_1(t) =& x_2(t) \\
\dot{x}_2(t) =& -x_1(t) - x_2(t) + u(t)
\end{aligned}
$$

and the cost function as

$$
J = \int_0^T \frac{1}{2}[q_1 x_1^2(t) + q_2 x_2^2(t) + u^2(t)]dt; \quad q_1, q_2 > 0.
$$

Find $U^*(t)$ expressed as a function of $X(t)$, t, and J_X^* for the given system.

Solution

Let

$$
\begin{aligned}
\mathscr{H} =& g + J_X^{*\top}[a] \\
=& q_1 x_1^2 + q_2 x_2^2 + u^2 + \begin{bmatrix} J_{x_1} & J_{x_2} \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 + x_2 + u \end{bmatrix} \\
=& \frac{1}{2}(q_1 x_1^2 + q_2 x_2^2 + u^2) + J_{x_1} x_2 + J_{x_2}(-x_1 + x_2 + u)
\end{aligned}
$$

Now, find $\frac{\partial \mathscr{H}}{\partial u} = 0$

$$
\begin{aligned}
\frac{\partial \mathscr{H}}{\partial u} =& u^* + J_{x_2} \\
=& 0 \\
\frac{\partial^2 \mathscr{H}}{\partial u^2} =& 1 > 0 \;\therefore \text{minimum}
\end{aligned}
$$

Therefore, the optimal control trajectory is $\boxed{u^*(t) = -J_{x_2}^*(x(t), t)}$.

Question 2

A system has the following first order linear dynamics

$$\dot{x}(t) = -10x(t) + u(t)$$

which needs to be controlled while minimizing the following cost function:

$$J = \frac{1}{2}x^2(T) + \int_0^T [\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)]dt;$$

The admissible state and control trajectories are not constrained by any boundaries. Find the optimal control law using HJB equation for $T = 5$ and assuming the solution of HJB PDE to be of the quadratic form. You can leave your answers in terms for $K(t)$.

Solution

Let

$$\begin{aligned}
\mathscr{H} &= g + J_X^{*\top}[a] \\
&= (\frac{1}{4}x^2 + \frac{1}{2}u^2) + J_x(-10x + u)
\end{aligned}$$

Now, find $\frac{\partial \mathscr{H}}{\partial u} = 0$

$$\begin{aligned}
\frac{\partial \mathscr{H}}{\partial u} &= u^* + J_x \\
&= 0 \\
u^* &= -J_x \\
\frac{\partial^2 \mathscr{H}}{\partial u^2} &= 1 > 0 \;\therefore \text{minimum}
\end{aligned}$$

This system is Linear, Quadratic, and a Regulator (LQR) problem. Therefore, the optimal control trajectory is $\boxed{u^*(t) = -k(t)\,x(t)}$. The minimum cost, J^*, of an LQR problem is given by $J^* = \frac{1}{2}k(t)\,x^2(t)$ and the solution to the Hamilton-Jacobi-Bellman of an LQR problem can be found in the following manner:

1.

$$\begin{aligned}
J_t^* &= \frac{1}{2}\dot{k}(t)\,x(t) \\
J_x^* &= k(t)\,x(t)
\end{aligned}$$

2.

$$
\begin{aligned}
\mathscr{H}^* =& \frac{1}{4}x^2 + \frac{1}{2}(-J_x^*)^2 + J_x^*(-10x - J_x^*) \\
=& \frac{1}{4}x^2 + \frac{1}{2}J_x^{*2} - 10J_x^* x - J_x^{*2} \\
=& \frac{1}{4}x^2 - \frac{1}{2}J_x^{*2} - 10J_x^* x
\end{aligned}
$$

3.

$$
\begin{aligned}
0 &= J_t^* + \mathscr{H}^* \\
&= J_t^* + \frac{1}{4}x^2 - \frac{1}{2}J_x^{*2} - 10J_x^* x \\
&= \frac{1}{2}\dot{k}\, x^2 + \frac{1}{4}x^2 - \frac{1}{2}(kx)^2 - 10(kx)x \\
&= \frac{1}{2}\dot{k}\, x^2 + \frac{1}{4}x^2 - \frac{1}{2}k^2x^2 - 10kx^2 \\
&= \dot{k} + \frac{1}{2} - k^2 - 20k \qquad (1)
\end{aligned}
$$

4. Using Wolfram Alpha and taking the natural log (ln) of Equation 1 yields

$$
\boxed{t + \alpha = \frac{1}{\sqrt{402}} \ln\left[\frac{2k(t) + 20 - \sqrt{402}}{2k(t) + 20 + \sqrt{402}}\right], \qquad \alpha \in \mathbb{R}}
$$

Question 3

The dynamics of a nonlinear scalar system is:

$$
\dot{x}(t) = x(t)u(t); \quad x(0) = 1;
$$

and the cost function to be minimized is

$$
J = x^2(1) + \int_0^1 [x(t)u(t)]^2 dt;
$$

Find the optimal feedback solution by solving the corresponding HJB equation. [Hint: First, find the HJB partial differential equation in terms of J_X, J_t. Then using boundary conditions show that the PDE admits a solution that is quadratic in x. Finally integrate the ODE in $K(t)$ to get the feedback solution.]

Solution

Let

$$
\begin{aligned}
\mathscr{H} &= g + J_X^{*\top}[a] \\
&= x^2u^2 + J_x x\, u
\end{aligned}
$$

Now, find $\frac{\partial \mathscr{H}}{\partial u} = 0$

$$\begin{aligned}
\frac{\partial \mathscr{H}}{\partial u} &= 2x^2 u^* + J_x x \\
&= 0 \\
u^* &= -\frac{J_x}{2x^2} \\
\frac{\partial^2 \mathscr{H}}{\partial u^2} &= 2x^2 > 0 \ \therefore \text{minimum}
\end{aligned}$$

This system is not LQR problem. Therefore, the optimal control trajectory cannot automatically be assumed to be quadratic in x. However, by looking at the initial conditions we can see that the final cost *is* quadratic in x, $J(x(1), 1) = x^2(1)$. This means that it is safe to assume that throughout the process the cost is quadratic in x.

1.

$$\begin{aligned}
J_t^* &= \frac{1}{2}\dot{k}(t)\, x(t) \\
J_x^* &= k(t)\, x(t)
\end{aligned}$$

2.

$$\begin{aligned}
\mathscr{H}^* &= x^2 \left(-\frac{J_x}{2x^2}\right)^2 + J_x^* \left(-\frac{J_x}{2x^2}\right) \\
&= J_x^{*2} - \frac{1}{2} J_x^{*2} \\
&= \frac{1}{2} J_x^{*2}
\end{aligned}$$

3.

$$\begin{aligned}
0 &= J_t^* + \mathscr{H}^* \\
&= J_t^* + \frac{1}{2} J_x^{*2} \\
&= \frac{1}{2}\dot{k}\, x^2 + \frac{1}{2}\left(k\, x\right)^2 \\
&= \frac{1}{2}\dot{k}\, x^2 + \frac{1}{2} k^2\, x^2 \\
&= \dot{k} \, + k^2
\end{aligned} \tag{2}$$

4. Solving Equation 2 yields

$$\boxed{k\left(t\right) = \frac{1}{t + \alpha}, \qquad \alpha \in \mathbb{R}}$$

Question 4

Given the system dynamics of a plant:

$$
\begin{aligned}
x_1(t) =& x_2(t) \\
x_2(t) =& -x1(t) - 2x_2(t) + u(t)
\end{aligned}
$$

and the cost function to minimize is

$$J = 10x_1^2(T) + \int_0^T \frac{1}{2}[x_1^2(t) + 2x_2^2(t) + u^2(t)]dt$$

Final time $T = 10$ and the admissible state and control trajectories are not constrained by any boundaries. Find the optimal control law by solving the Riccati Equation numerically.

a) Identify A, B, Q, R, H

b) Find and plot $K(t)$ using Riccati Equation

c) Find and plot optimal control law

d) Plot the trajectories of states x_1 and x_2 using control in part c) (Hint: Identify A, B, Q, R, H etc and then use $ODE45$ command in matlab to solve the ODE in $K(t)$)

Solution

a)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 1$$

b)

To use MATLAB's `ode()` function, a function for the ODE that we wish to solve must first be written, as shown in Listing 2. As Listing 1 shows, the `ode()` function requires this ODE function as an input.

Listing 1: Script to solve the Riccati equation using the ODE45 solver

```
%% Solve the Riccati Equation numerically
% adapted from:
% http://www.eng.auburn.edu/~tplacek/courses/3600/ode45waterloo.pdf
% https://www.mathworks.com/matlabcentral/answers/94722-how-can-i-solve-
    the-matrix-riccati-differential-equation-within-matlab

%%% Part b) find and plot K(t) using Riccati Equation
A = [ 0,  1;
     -1, -2];
B = [0;
     1];
```

```
Q = [1, 0;
     0, 2];
R = 1;

X0 = [0, 0, 0, 0]';
period = [0, 10];

% ode45 requires that a function be passed as an input
[t1, X] = ode45(@(t, K)riccati(t, K, A, B, Q, R), period, X0);

plot(t1, X)
legend('k1', 'k2', 'k3', 'k4')
```

The API for MATLAB's `ode()` function requires that the ODE function that is being solved accepts a column vector as a parameter and returns a column vector as well. This is why `reshape()` is used at the beginning of the `riccati()` function.

Listing 2: Function to calculate the Riccati equation at one time step

```
function dkdt = riccati(t, K, A, B, Q, R)
% This is the Riccati equation specific to question 4 from assignment 2
% from Linear Systems Analysis (EE5600) at CSULA.
%
% Adapted from:
% https://www.mathworks.com/matlabcentral/answers/94722-how-can-i-solve-
   the-matrix-riccati-differential-equation-within-matlab
    K = reshape(K, size(A));
    dkdt = -Q + K*B*(R^-1)*B' - K*A - A'*K;
    dkdt = dkdt(:);
end
```

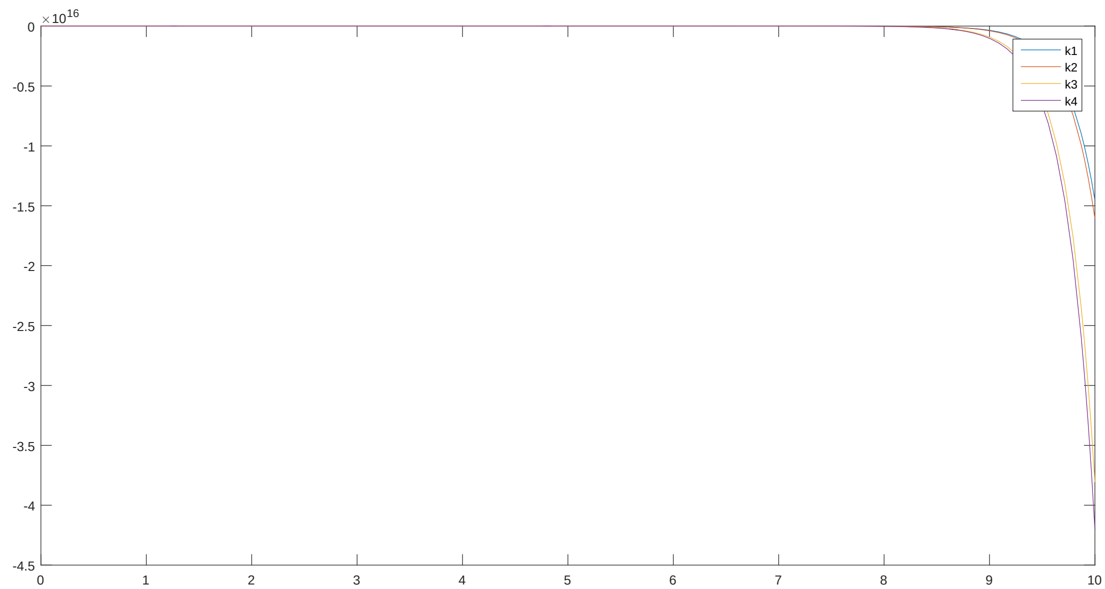

Figure 1: Plot of $K(t)$

c) and d)

The remaining portions of this problem were not completed due to time constraints.

Question 5

Derive the Hamilton-Jacobi-Bellman partial differential equation.

Solution

1. First, the problem must be formulated
   (a) System dynamics: $\dot{X}(t) = a(X(t), U(t), t)$
   (b) Cost function: $J = h(X(t_f), t_f) + \displaystyle\int_{t_0}^{t_f} g(X(t), U(t), t)dt$
   (c) Define the state and control constraints
2. Now, consider a more general cost function (i.e, a *value function*)

(a) $J(X(t), t, \underset{t\leq\tau\leq t_f}{U(\tau)}) = h(X(t_f), t_f) + \int_t^{t_f} g(X(\tau), U(\tau), \tau) d\tau$
for $t \leq \tau \leq t_f$

3. Find the minimum cost function for all admissible $X(t)$ and for all $t \leq \tau \leq t_f$.

   (a) $J^*(X(t), t) = \min_{\substack{U(\tau)\\ t\leq\tau\leq t_f}} \{\int_t^{t_f} g(X(\tau), U(\tau), \tau) d\tau + h(X(t_f),\, t_f)\}$

4. Subdivide the interval

   (a) $J^*(X(t), t) = \min_{\substack{U(\tau)\\ t\leq\tau\leq t_f}} \{\int_t^{t+\Delta t} g\, d\tau + \int_{t+\Delta t}^{t_f} g\, d\tau + h\}$

5. Use the principle of optimality

   (a) $J^*(X(t), t) = \min_{\substack{U(\tau)\\ t\leq\tau\leq t_f}} \{\int_t^{t+\Delta t} g\, d\tau + J^*(X(t+\Delta t),\, t+\Delta t)\}$
   where $J^*(X(t+\Delta t),\, t+\Delta t)$ is the minimum cost of the process for the time interval $t+\Delta t \leq \tau \leq t_f$ with initial state $X(t+\Delta t)$.

6. Expand $J^*(X(t+\Delta t),\, t+\Delta t)$ in a Taylor series about the point $(x(t), t)$ (assuming that the second partial derivatives of J^* exist and are bounded)

   (a)

$$J^*(X(t), t) = \min_{\substack{U(\tau)\\ t\leq\tau\leq t_f}} \{\int_t^{t+\Delta t} g\, d\tau + J^*(X(t),\, t) + [\frac{\partial J^*}{\partial t}(X(t), t)]\Delta t$$
$$+ [\frac{\partial J^*}{\partial X}(X(t), t)]^T [X(t+\Delta t) - X(t)] + \mathcal{O}(\Delta t)\}$$

   where $\mathcal{O}(\Delta t)$ is higher order terms.

7. For small Δt

   (a)

$$\begin{aligned} J^*(X(t), t) = \min_{U(t)} \{ & g(X(t), U(t), t)\Delta t + J^*(X(t), t) \\ & + J_t^*(X(t), t)\Delta t + J_X^{*\top}(X(t), t)[a(X(t), U(t), t)]\Delta t \\ & + \mathcal{O}(\Delta t)\} \end{aligned}$$

8. Remove $J^*(X(t), t)$ and $J_t^*(X(t), t)$ terms (they don't depend on $U(t)$)

(a)

$$\begin{aligned}0 = &J_t^*(X(t),t)\Delta t + \min_{U(t)}\{g(X(t),U(t),t)\Delta t\\&+ J_X^{*\top}(X(t),t)[a(X(t),U(t),t)]\Delta t + \mathcal{O}(\Delta t)\}\end{aligned}$$

9. Divide by Δt and let $\Delta t \to 0$

(a)

$$0 = J_t^*(X(t),t) + \min_{U(t)}\{g(X(t),U(t),t) + J_X^{*\top}(X(t),t)[a(X(t),U(t),t)]\}$$

Which can be shortened to

$$0 = J_t^* + \min_{U(t)}\{g + J_X^{*\top}[a]\}$$

or

$$\boxed{0 = J_t^* + \mathscr{H}^*}$$

Question 6

Write a short note on LQR problem and derive the Riccati equation for LQR problem.

Solution

An LQR problem, such as Questions 1 and 2, are problems such that the plant dynamics are linear and the cost function is quadratic. The plant dynamics can be described as in Equation 3 and the cost function to be minimized given in Equation 4.

$$\dot{X}(t) = A(t)X(t) + B(t)U(t) \tag{3}$$

$$J = \frac{1}{2}X^\top(t_f)HX(t_f) + \int_{t_0}^{t_f}\frac{1}{2}\left[X^\top(t)Q(t)X(t) + U^\top(t)R(t)U(t)\right]dt \tag{4}$$

where H and Q are real, symmetric positive semi-definite matrices, R is a real, symmetric positive definite matrix, and t_0 and t_f are the initial and final times, respectively, and are specified. The control and state trajectories $U(t)$and $X(t)$ are not constrained by any boundaries.

1. Form the Hamiltonian

(a)

$$\begin{aligned}\mathscr{H}(X(t),U(t),J_X^*,t) = &\frac{1}{2}\left[X^\top(t)Q(t)X(t) + U^\top(t)R(t)U(t)\right]\\&+ J_t^*(X(t),t)\cdot[A(t)X(t) + B(t)U(t)]\end{aligned} \tag{5}$$

2. Minimize the Hamiltonian $\left(\frac{\partial \mathscr{H}}{\partial u}=0\right)$

   (a)
$$\frac{\partial \mathscr{H}}{\partial u}\left(X\left(t\right),U\left(t\right),J_X^*,t\right)=R\left(t\right)U\left(t\right)+B^\top\left(t\right)J_X^*\left(X\left(t\right),t\right)\\=0 \tag{6}$$

3. Check that Equation 6 really is a minimum $\left(\frac{\partial^2 \mathscr{H}}{\partial u^2}>0\right)$

   (a)
$$\frac{\partial^2 \mathscr{H}}{\partial u^2}=R\left(t\right)>0$$

   because R is real, positive definite

4. Because $\mathscr{H}$ is a quadratic form in $U\left(t\right)$, the control trajectory that minimizes Equation 6 is a global minimum. By solving Equation 6 for $U^*\left(t\right)$we get

   (a)
$$U^*\left(t\right)=-R^{-1}\left(t\right)B^\top\left(t\right)J_X^*\left(X\left(t\right),t\right) \tag{7}$$

5. Substitute Equation 7 into Equation 5 (the arguments of the matrices will be dropped from here on due to space constraints)

   (a)
$$\begin{aligned}\mathscr{H}\left(X\left(t\right),U\left(t\right),J_X^*,t\right)=&\frac{1}{2}X^\top QX+\frac{1}{2}J_X^{*\top}BR^{-1}B^\top J_X^*\\&+J_t^*AX-J_X^{*\top}BR^{-1}B^\top J_X^*\\=&\frac{1}{2}X^\top QX-\frac{1}{2}J_X^{*\top}BR^{-1}B^\top J_X^*-J_t^{*\top}AX\end{aligned}$$

6. The Hamilton-Jacobi-Bellman equation becomes

   (a)
$$0=J_t^*+\frac{1}{2}X^\top QX-\frac{1}{2}J_X^{*\top}BR^{-1}B^\top J_X^*-J_t^{*\top}AX \tag{8}$$

7. The boundary condition

   (a)
$$J_X^*\left(X\left(t_f\right),t_f\right)=\frac{1}{2}X^\top\left(t_f\right)HX\left(t_f\right)$$

8. Guess a quadratic solution

(a)

$$J_X^* \left(X\left(t\right), t\right) = \frac{1}{2} X^\top K X$$

where K is a real symmetric positive-definite matrix

9. Substitute our guess into the HJB equation, Equation 8

(a)

$$0 = \frac{1}{2} X^\top \dot{K} X + \frac{1}{2} X^\top Q X - \frac{1}{2} X^\top K B R^{-1} B^\top K X + X^\top K A X \tag{9}$$

10. Use the substitution of $KA = \frac{1}{2}\left[KA + (KA)^\top\right] + \frac{1}{2}\left[KA - (KA)^\top\right]$, the property that $(CD)^\top = D^\top C^\top$, and the fact that the transpose of a scalar is the scalar to show that only the symmetric part of KA contributes to Equation 9 to get

(a)

$$0 = \frac{1}{2} X^\top \dot{K} X + \frac{1}{2} X^\top Q X - \frac{1}{2} X^\top K B R^{-1} B^\top K X + \frac{1}{2} X^\top K A X + \frac{1}{2} X^\top A^\top K X$$

which holds for X

11. Removing all X

(a) $\boxed{0 = \dot{K} + Q - KBR^{-1}B^\top K + KA + A^\top K}$

12. With the boundary condition

(a) $\boxed{K\left(t_f\right) = H}$