EE 5630: Optimal Control - Assignment 1

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Question 1.

Example 2.2-1 describes a process in which a performance measure and associated weights can be determined for controlling the attitude, $\theta(t)$, of a manned spacecraft using a gas expulsion system, shown in Figure 1. The objective of the control system is to maintain the attitude of the spacecraft at $\theta(t) = 0$ with small accelerations.

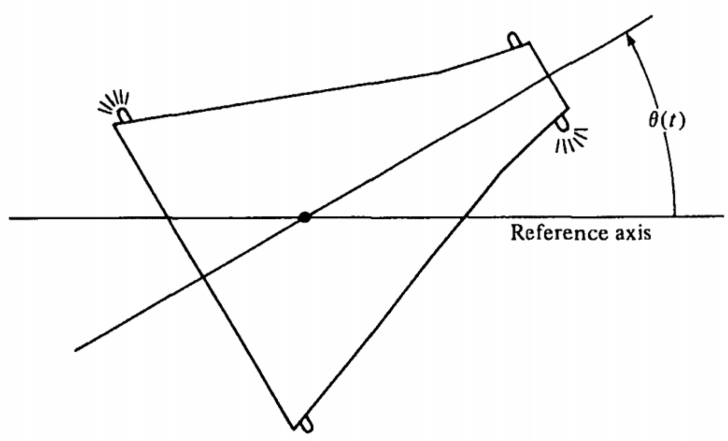

Figure 1: Attitude control of a spacecraft [1]

The dynamics of the system are given by the differential equation given in Equation 1.

$$I\ddot{\theta}(t) = \lambda(t) \tag{1}$$

where I is angular moment of inertia and $\lambda(t)$ is the torque produced by the gas jets. The state space equations are

$$\dot{x_1}(t) = x_2(t) \tag{2}$$
$$\dot{x_2}(t) = u(t) \tag{3}$$

where $u(t) = \frac{1}{I}\lambda(t)$ and the states, x_1 and x_2, are angular position and velocity, respectively. The performance measure is given in Equation 4 below

$$J = \int_0^\infty [q_{11}x_1^2(t) + q_{22}x_2^2(t) + Ru^2(t)]\, dt \tag{4}$$

where q_{11}, q_{22}, and R are the weights.

To find specific values of the weights, we must take into account the desired qualities of the controller. In this problem it desired to maintain a specific attitude and to do so using small accelerations. This means that we are concerned with $x_1(t)$ and with $u(t)$. Therefore, our weighting matrix, Q, can be defined as

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

Notice in Equation 5 that all entries *except* the q_{11} entry are 0. This is because the only state that is to be given any preference is the x_1 state, which corresponds to the q_{11} entry. We're also concerned about the acceleration which is the control action, $u(t)$, in this case. Because $u(t)$ is not a vector, or $R \in \mathbb{R}$, Equation 4 can be rewritten as

$$J = \int_0^\infty [q_{11}x_1^2(t) + Ru^2(t)]\, dt \tag{6}$$

Now, we can see how choosing different values for q_{11} and R affect system performance. Let's focus only on differing values of R. Throughout, we'll use the following conditions for Q and X_0

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \tag{7}$$

Figure 2 shows the response of the system with $R = 0.1$. Because R is a small value, its resonse is given less preference which results in a large acceleration (control action). The large acceleration in turn results in large overshoot along with the spacecraft quickly settling on its desired attitude of $0°$. A fast response is not what we want as it could injure or make the astronauts that are inside the spacecraft ill.

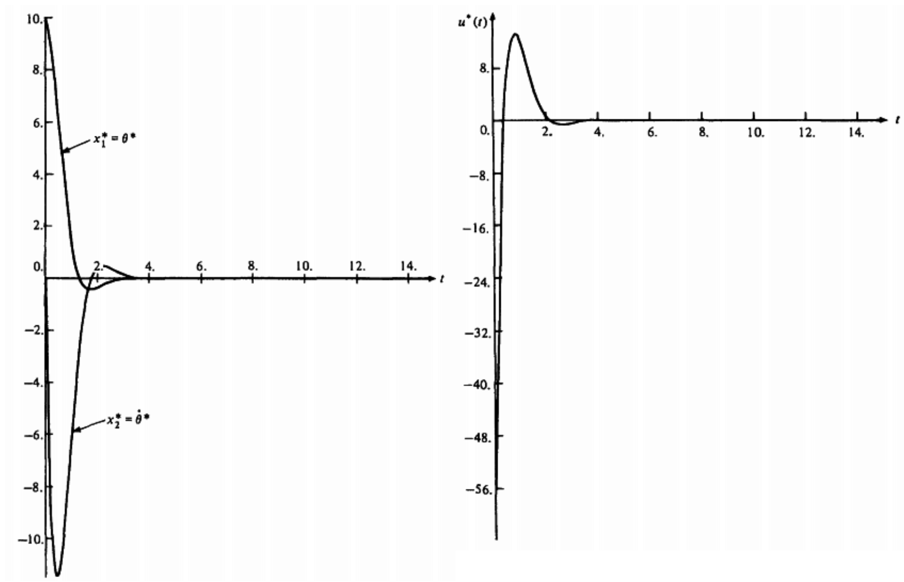

Figure 2: Response of system with $R = 0.1$ [1]

Figure 3 shows the response of the system with $R = 1$. Because R is a larger value than in the previous configuration, its resonse is given more preference which results in a smaller acceleration. This smaller acceleration in turn results in a smaller overshoot and the spacecraft settling on its desired attitude of 0° slower than before (longer settling time).

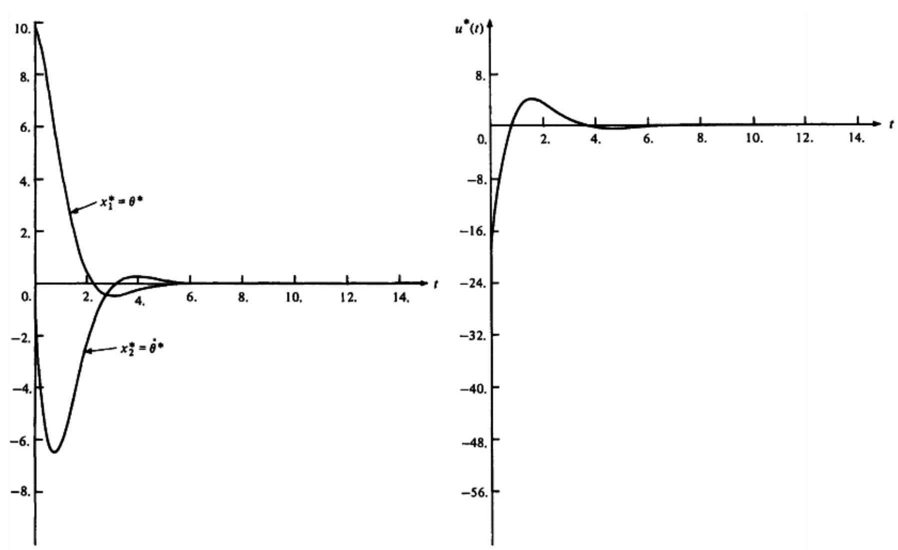

Figure 3: Response of system with $R = 1$ [1]

Figure 4 shows the response of the system with $R = 10$. Because R is a much larger value than the first configuration (100 times as large), its control action was given much more preference and control history and state trajectories that are generated are much closer to the requirements. The overshoot is considerably smaller than in the first configuration ($R = 0.1$) and its settling time is also much longer.

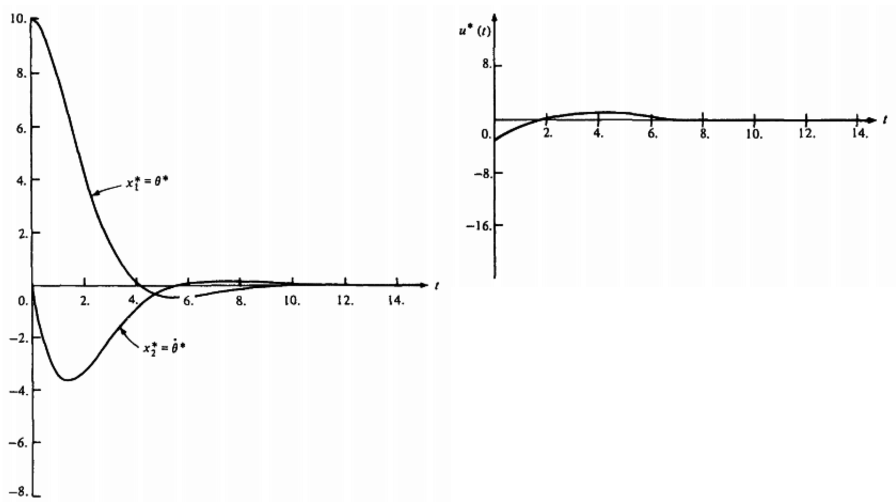

Figure 4: Response of system with $R = 10$ [1]

What **Example 2.2-1** shows is that giving larger weights to a parameter in the performance measure causes that parameter to be given more consideration by the controller. This allows control engineers to tune their controllers to give the desired performance.

Problem 2-1.

The states for the mixing process from **Problem 1-6** are given by

$$\dot{v}_1(t) \begin{cases} m(t) - \dfrac{v_1(t)\,k}{\alpha_1}[h_1(t) - h_2(t)], & \text{if } h_1(t) \le h_2(t) \\ m(t) + \dfrac{v_2(t)\,k}{\alpha_2}[h_2(t) - h_1(t)], & \text{if } h_2(t) > h_1(t) \end{cases} \tag{8}$$

and

$$\dot{v}_2(t) \begin{cases} -\dfrac{v_1(t)\,k}{\alpha_1}[h_1(t) - h_2(t)], & \text{if } h_1(t) \le h_2(t) \\ \dfrac{v_2(t)\,k}{\alpha_2}[h_2(t) - h_1(t)], & \text{if } h_2(t) > h_1(t). \end{cases} \tag{9}$$

a)

The type of problem defined here is a *tracking problem*. Here, the $v_2(t)$ state is to be kept as close to M ft^3 as possible. Therefore, a performace measure that can be used is

$$J = \int_{t_0}^{t_f} [v_2(t) - M]^2 dt \tag{10}$$

where t_0 and t_f are the initial and final times, respectively, and $t_f - t_0 = 1$ day.

b)

A set of physically realistic state and control constraints are

$$\begin{aligned} 0 \le\ & h_1(t) \le H_1, && (11)\\ 0 \le\ & h_2(t) \le H_2, && (12)\\ 0 \le\ & w_1(t) \le W_1, && (13)\\ 0 \le\ & w_2(t) \le W_2, && (14)\\ 0 \le\ & m(t) \le M && (15) \end{aligned}$$

where

- H_1 and H_2 are the maximum heights of tanks 1 and 2, respectively
- W_1 and W_2 are the maximum rates of water entering tanks 1 and 2, respectively

Problem 2-2.

This type of problem is classified as a *terminal control problem* in which a parameter is being *maximized* in which the final total volume of dye in tank 2 is to be as close to N ft^3 as possible.

a)

A performance measure that can be used is

$$J = -v_2(t_f)\,. \tag{16}$$

A minus sign is being used because the quantity $v_2(t_f)$ is being *maximized*.

b)

A set of physically realizable state and control constraints are

$$\begin{aligned} 0 \leq\ & h_1(t)\ \leq H_1, && (17)\\ 0 \leq\ & h_2(t)\ \leq H_2, && (18)\\ 0 \leq\ & w_1(t)\ \leq W_1, && (19)\\ 0 \leq\ & w_2(t)\ \leq W_2, && (20)\\ \int_{t_0}^{t_f} & m(t)\,dt\ \leq N && (21) \end{aligned}$$

where

- H_1 and H_2 are the maximum heights of tanks 1 and 2, respectively
- W_1 and W_2 are the maximum rates of water entering tanks 1 and 2, respectively
- where t_0 and t_f are the initial and final times, respectively, and $t_f - t_0 = 1$ day

Problem 2-4.

This problem is the same as **Example 2.2-1** except that the desired attitude is now $15^\circ \pm 0.1^\circ$ and the spacecraft should reach its desired attitude within 30 seconds.

a)

The *state constraints* are

$$14.9^\circ \leq x_1(t_f) \leq 15.1^\circ \tag{22}$$

The *control constraints* are

$$-U_{max} \leq u(t) \leq U_{max} \tag{23}$$

where U_{max} is the maximum thrust.

b)

An appropriate control measure is

$$J = \int_{t_0}^{t_f} u^2(t)dt \tag{24}$$

Problem 2-5.

This problem is the same as **Example 2.2-1** except that the desired attitude is now $15^\circ \pm 0.1^\circ$ and the spacecraft should reach its desired attitude in minimum time.

b)

The *state constraints* are

$$14.9^\circ \leq x_1(t_f) \leq 15.1^\circ \tag{25}$$

The *control constraints* are

$$-U_{max} \leq u(t) \leq U_{max} \tag{26}$$

b)

An appropriate control measure is

$$J = \int_{t_0}^{t_f} dt \tag{27}$$

Problem 2-6. Figure 5 shows a rocket that is to be approximated by a particle of instantaneous mass $m(t)$. The instantaneous velocity is $v(t)$, $T(t)$ is the thrust, and $\beta(t)$ is the thrust angle. If we assume no aerodynamic drag or gravitational forces, and if we select $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$, $x_3 \triangleq y$, $x_4 \triangleq \dot{y}$, $x_5 \triangleq m$, $u_1 \triangleq T$, $u_2 \triangleq \beta$, the state equations are

$$\dot{x_1}(t) = x_2(t) \tag{28}$$
$$\dot{x_2}(t) = \frac{[u_1(t)\, \cos u_2(t)]}{x_5(t)} \tag{29}$$
$$\dot{x_3}(t) = x_4(t) \tag{30}$$
$$\dot{x_4}(t) = \frac{[u_1(t)\, \sin u_2(t)]}{x_5(t)} \tag{31}$$
$$\dot{x_5}(t) = -\frac{1}{c}u_1(t), \tag{32}$$

where c is a constant of proportionality. The rocket starts from rest at the point $x = 0$, $y = 0$.

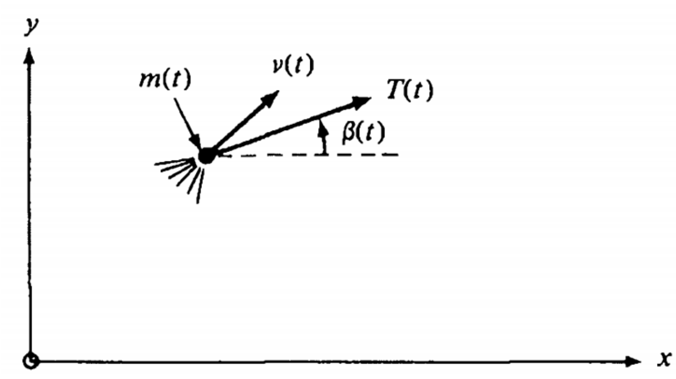

Figure 5: Rocket to be approximated by a particle of instantaneous mass [1]

a)

A set of physically reasonable *state constraints* are

$$0 \leq x_1(t) \tag{33}$$
$$0 < x_5(t) \tag{34}$$
$$\int_{t_0}^{t_f} \dot{x_5}(t)\, dt \leq F_t \tag{35}$$

where F_t is the total fuel. A set of physically reasonable *control constraints* are

$$0 \leq u_1(t) \leq T_{max} \tag{36}$$
$$-\pi \leq u_2(t) \leq \pi \tag{37}$$

where T_{max} is the maximum thrust. The control constraint on $u_2(t)$ stems from the fact that if the rocket were to go beyond these angles ($\pm\pi$) then the rocket would be pointed towards the earth.
b) The objective is to have $y(t_f) = 3$ mi and to maximize $x(t_f)$. A new physical constraint is

$$x_3(t_f) = 3\,\mathrm{mi} \tag{38}$$

A performance measure for this situation is

$$J = -x_1^2(t_f) \tag{39}$$

A minus sign is used here because this is a *maximization* problem. Otherwise, the performance measure would cause $x_1(t_f)$ to be minimized.
c) The objective is to have the rocket reach $x = 500$ mi and $y = 3$ mi within 2.5 min with maximum possible vehicle mass.

A set of constraints for this objective is

$$x_1(t_f) = 500\mathrm{mi} \tag{40}$$
$$x_3(t_f) = 3\mathrm{mi} \tag{41}$$

A performance measure for this objective is

$$J = -qx_5(t_f), \quad q \in \mathbb{R}, \text{ and } q > 0 \tag{42}$$

where q is a weight. In Equation 42 does not need an absolute value operator or to be squared because the state x_5 represents *mass* which can never, as far as we know, be negative.

References

[1] D. E. Kirk, *Optimal control theory: an introduction.* Dover Publications, 1998.