EE5600 - Homework 2

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Question 1

A robot-arm drive system for one joint can be represented by the differential equation

$$\frac{dv(t)}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

where $v(t)$ = velocity, $y(t)$ = position, and $i(t)$ is the control-motor current (Hint: $i(t) = u(t)$). Put the equations in state variable form and set up the matrix form for $k_1 = k_2 = 1$.

Solution

$$\begin{aligned} x_1 &= y \\ x_2 &= v = \dot{y} \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + k_3 u \\ &\text{or} \\ X(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Question 2

The state-space representation of a dynamical system is given as

$$\dot{X}(t) = AX(t) + BU(t) \tag{1}$$

$$Y(t) = CX(t) + DU(t) \tag{2}$$

Prove that:

$$Y(t) = C\left[\Phi(t) X(0) + \int_0^t \Phi(t-\tau) BU(\tau) d\tau\right] + DU(t)$$

where $\Phi(t) = e^{At}$ is the fundamental or state transition matrix.

Solution

First we must find the Laplace transform of Equation 1 and solve for $X(s)$.

$$\begin{aligned} sX(s) - X(0) =& AX(s) + BU(s) \\ sX(s) - AX(s) =& X(0) + BU(s) \\ [s\mathbb{I} - A] X(s) =& X(0) + BU(s) \\ X(s) =& [s\mathbb{I} - A]^{-1} [X(0) + BU(s)] \\ X(s) =& [s\mathbb{I} - A]^{-1} X(0) + [s\mathbb{I} - A] BU(s) \end{aligned} \tag{3}$$

Next, find the inverse Laplace transform of Equation 3.

$$\begin{aligned} X(t) &= e^{At} X(0) + \int_0^t e^{A(t-\tau)} BU(\tau) d\tau \\ &= \Phi(t) X(0) + \int_0^t \Phi(t-\tau) BU(\tau) d\tau \end{aligned} \tag{4}$$

Where $\Phi(t) = e^{At}$. Substituting Equation 4 into Equation 2 yields

$$\boxed{Y(t) = C\left[\Phi(t) X(0) + \int_0^t \Phi(t-\tau) BU(\tau) d\tau\right] + DU(t)}$$

as required.

Question 3

Problem 2.1

Consider the memoryless systems with characteristics shown in Figure 1, in which u denotes the input and y the input. Which of the is a linear system? is it possible to introduce a new output so that the system in Figure 1 (b) is linear?

Solution

a)

The equation for the system is $y = mx$.

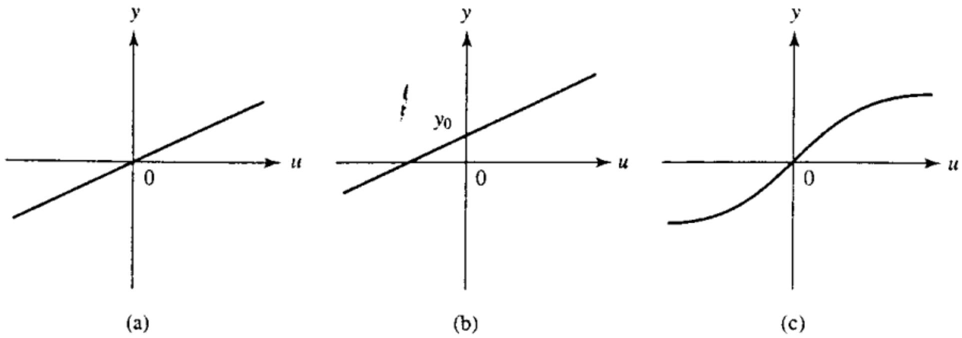

Figure 1: Three different memoryless systems

$$\begin{aligned} f\left(ax_1 + bx_2\right) =& m\left(ax_1 + bx_2\right) \\ =& amx_1 + bmx_2 \\ =& af\left(x_1\right) + bf\left(x_2\right) \\ \therefore& \text{ linear} \end{aligned}$$

b)

The equation for the system is $y = mx + b$.

$$\begin{aligned} f\left(ax_1\right) =& m\left(ax_1\right) + b \\ =& amx_1 + b \\ \neq& af\left(x_1\right) \\ \therefore& \text{ not linear} \end{aligned}$$

This system can be linearized by adding an offset of $-b$ which will change the equation of the system to $y = mx$ which is the same as in (a), which is linear.

c)

The equation for the system is $y = m\left(x\right)x$.

$$\begin{aligned} f\left(x_1 + x_2\right) =& m\left(x_1 + x_2\right)\left(x_1 + x_2\right) \\ =& m\left(x_1 + x_2\right)x_1 + m\left(x_1 + x_2\right)x_2 \\ \therefore& \text{ not linear} \end{aligned}$$

Question 4

Problem 2.3

Consider a system whose input u and output y are related by

$$y(t) = (P_\alpha u)(t) = \begin{cases} u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

where α is a fixed constant. The system is called a *truncation operation*, which chops off the input after time α. Is the system linear? Is it time-invariant? Is it causal?

Solution

a)

$$\begin{aligned} (P_\alpha (au_1 + bu_2))(t) =& au_1(t) + bu_2(t) \\ =& a(P_\alpha u_1)(t) + b(P_\alpha u_2)(t) \\ &\therefore \text{ linear} \end{aligned}$$

b)

The system is **causal** because $y(t)$ only depends on the current value of t, not future or past values.

c)

The system is **not time-invariant**. $(P_\alpha u)(t) \ne (P_\beta u)(t)$ unless $\alpha = \beta$.

Question 5

Problem 2.6

Consider a system whose input and output are related by

$$u(t) = \begin{cases} \frac{u^2(t)}{u(t-1)} & \text{if } u(t-1) \ne 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

for all t. Show that the system satisfies the homogeneity property, but not the additivity property.

Solution

Homogeneity:

$$\begin{aligned} f(at) =& \frac{(au)^2(t)}{au(t-1)} \\ =& \frac{au^2(t)}{u(t-1)} \\ =& af(t) \\ &\therefore \text{ satisfies homoegeniety} \end{aligned}$$

b)

Additivity

$$\begin{aligned}
g\left(u_1+u_2\right) =& \frac{\left(u_1+u_2\right)^2(t)}{u\left(t-1\right)} \\
=& \frac{\left(u_1+u_2\right)^2(t)}{\left(u_1+u_2\right)\left(t-1\right)} \\
\neq& g\left(u_1\right)+g\left(u_2\right) \\
\therefore& \text{ does } \textit{not} \text{ satisfy additivity}
\end{aligned}$$

Question 6

Problem 2.14

Consider a system described by

$$\ddot{y}+2\dot{y}-3y=\dot{u}-u \tag{5}$$

What are the transfer function and the impulse response of the system?

Solution

Transfer function:

To find the transfer function, we take the Laplace transform of Equation 5.

$$\begin{aligned}
s^2Y+2sY-3Y &= sU-U \\
\left(s^2+2s-3\right)Y &= \left(s-1\right)U \\
Y &= \frac{s-1}{\left(s^2+2s-3\right)}U \\
Y &= \frac{s-1}{\left(s+3\right)\left(s-1\right)}U \\
Y &= \frac{U}{s+3}
\end{aligned}$$

$$\boxed{G\left(s\right)=\frac{Y\left(s\right)}{U\left(s\right)}=\frac{1}{s+3}}$$

Note that since we're finding the transfer function all initial conditions are assumed to be 0.

Impulse Response:

For the impulse response, $u\left(t\right)=\delta\left(t\right)\xrightarrow{\mathcal{L}}U\left(s\right)=1$.

$$
\begin{aligned}
Y(s) &= \frac{U(s)}{s+3} \\
&= \frac{1}{s+3}
\end{aligned}
$$

and the inverse Laplace transform ($\mathcal{L}^{-1}$) is

$$
\boxed{y(t) = \begin{cases} e^{-3t} & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}}
$$

Question 7

Problem 2.15

Let $\bar{y}(t)$ be the unit-step response of a linear time-invariant system. Show that the impulse response of the system equals $\frac{d\bar{y}(t)}{dt}$.

Solution

$$
\begin{aligned}
\bar{y}(t) &= \int_0^t g(\tau)\, u(t-\tau)\, d\tau \\
&= \int_0^t g(\tau)\, d\tau
\end{aligned}
$$

$$
\boxed{\therefore \frac{d\bar{y}(t)}{dt} = g(t)}
$$

Question 8

Problem 2.20

The soft landing phase of a lunar module descending on the moon can be modeled as shown in Figure 2. The thrust generated is assumed to be proportional to $\dot{m}$, where m is the mass of the module. Then the system can be described by $m\ddot{y} = -k\dot{y} - mg$, where g is the gravity constant on the lunar surface. Define state variables of the system as $x_1 = y$, $x_2 = \dot{y}$, $x_3 = m$, and $u = \dot{m}$. Find a state-space equation to describe the system.

Solution

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m}u - g \\
\dot{x}_3 &= u
\end{aligned}
$$

Note that the g term in the x_2 state makes this system nonlinear and therefore it can't be written in matrix form.

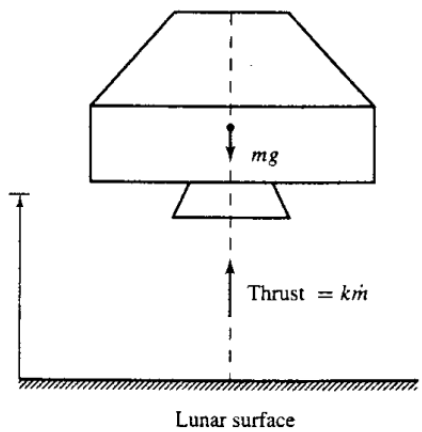

Figure 2: Soft landing phase of a lunar module descending on the moon