Linear Systems Analysis

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1 Linear Algebra Review

This is a review of linear algebra terms and concepts.

1.1 Terms

A review of linear algebra terms.

Consider an n-dimensional linear space $\mathbb{R}^n$. Every vector in $\mathbb{R}^n$ is an n-tuple of reals:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{1}$$

1.1.1 Linear Independence

Let V be a set of vectors

$$V = \{X_1, X_2, \cdots, X_n\} \in \mathbb{R}^n \tag{2}$$

whose element vectors are *linearly dependent* if

$$\exists\, \alpha_1, \alpha_2, \cdots, \alpha_n \neq 0 \tag{3}$$

such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0 \tag{4}$$

This means that a set of vectors V is linearly dependent if they can be composed of some combination of each other. We can take this definition and turn it around to find that a set of vectors V is *linearly independent* if Equation 4 is satisfied only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

1.1.2 Basis

A set of n linearly independent vectors in $\mathbb{R}^n$ is a *basis* if every vector in $\mathbb{R}^n$ can be expressed as a unique combination of this set (i.e., span the space). **Note:** in $\mathbb{R}^n$, any set of n linear independent vectors can be used as a basis.

1.1.3 Basis and Representation

Let $Q = \{q_1, q_2, \cdots, q_n\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. Now, any vector, $X \in \mathbb{R}^n$, can be expressed as

$$X = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n \tag{5}$$

such that $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$.

Assume that

$$\begin{aligned} Q &= \{q_1, q_2, \cdots, q_n\} \in \mathbb{R}^n \times \mathbb{R}^n, \quad \text{then} && (6) \\ X &= Q[\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]^T && (7) \\ &= Q\overline{X} && (8) \end{aligned}$$

where $\overline{X} = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]^T$ and is the *representation* of X with respect to the the basis Q.

Question: Consider svector $X, q_1, q_2 \in \mathbb{R}^2$ such that

$$X = [1 \quad 3]^T \tag{9}$$
$$q_1 = [3 \quad 1]^T \tag{10}$$
$$q_1 = [2 \quad 2]^T \tag{11}$$

a) Do q_1 and q_2 form a basis in $\mathbb{R}^2$?

b) If so, find the representatino of X with respect to the basis formed by q_1 and q_2.

1.1.4 Orthonormal Basis

An *orthonormal basis* is a basis in which the basis vectors are orthogonal to each other and has a unit length. For every $\mathbb{R}^n$ we can associate the following orthonormal basis

$$i_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad i_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

Question: If $X = [x_1 \; x_2 \; \cdots \; x_n]^T$, what is the representation of X with respect to the orthonormal basis?