Linear Systems Analysis

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1 Linear Algebra Review

This is a review of linear algebra terms and concepts.

1.1 Terms

A review of linear algebra terms.

Consider an n-dimensional linear space $\mathbb{R}^n$. Every vector in $\mathbb{R}^n$ is an n-tuple of reals:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{1}$$

1.1.1 Linear Independence

Let V be a set of vectors

$$V = \{X_1, X_2, \cdots, X_n\} \in \mathbb{R}^n \tag{2}$$

whose element vectors are *linearly dependent* if

$$\exists\, \alpha_1, \alpha_2, \cdots, \alpha_n \neq 0 \tag{3}$$

such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0 \tag{4}$$

This means that a set of vectors V is linearly dependent if they can be composed of some combination of each other. We can take this definition and turn it around to find that a set of vectors V is *linearly independent* if Equation 4 is satisfied only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

1.1.2 Basis

A set of n linearly independent vectors in $\mathbb{R}^n$ is a *basis* if every vector in $\mathbb{R}^n$ can be expressed as a unique combination of this set (i.e., span the space). **Note:** in $\mathbb{R}^n$, any set of n linear independent vectors can be used as a basis.

1.1.3 Basis and Representation

Let $Q = \{q_1, q_2, \cdots, q_n\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. Now, any vector, $X \in \mathbb{R}^n$, can be expressed as

$$X = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n \tag{5}$$

such that $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$.

Assume that

$$\begin{aligned} Q &= \{q_1, q_2, \cdots, q_n\} \in \mathbb{R}^n \times \mathbb{R}^n, \quad \text{then} && (6) \\ X &= Q[\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]^T && (7) \\ &= Q\overline{X} && (8) \end{aligned}$$

where $\overline{X} = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]^T$ and is the *representation* of X with respect to the the basis Q.

Question: Consider svector $X, q_1, q_2 \in \mathbb{R}^2$ such that

$$X = [1 \quad 3]^T \tag{9}$$
$$q_1 = [3 \quad 1]^T \tag{10}$$
$$q_1 = [2 \quad 2]^T \tag{11}$$

a) Do q_1 and q_2 form a basis in $\mathbb{R}^2$?

b) If so, find the representation of X with respect to the basis formed by q_1 and q_2.

1.1.4 Orthonormal Basis

An *orthonormal basis* is a basis in which the basis vectors are orthogonal to each other and has a unit length. For every $\mathbb{R}^n$ we can associate the following orthonormal basis

$$i_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad i_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

Note: two vectors x_1 and x_2 are orthogonal if $x_1^T \cdot x_2 = 0$ or $x_2^T \cdot x_1 = 0$ and a vector is normal if $x^T \cdot x = 1$

Question: If $X = [x_1 \; x_2 \; \cdots \; x_n]^T$, what is the representation of X with respect to the orthonormal basis?

1.1.5 Norm of a Vector

Any real-valued function of X can be defined as a *norm* if theem following properties are satisfied

1. $\|X\|^2 \geq 0 \quad \forall X$ and $\|X\| = 0$ iff $X = 0$
2. $\|\alpha X\| = |\alpha| \|X\| \quad \forall X, \quad X \in \mathbb{R}$
3. $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\| \quad \forall X_1, X_2$

Note that item number 3 is the triangle inequality.

Examples of Norms

$$\text{Let} \quad X = [x_1\, x_2\, \cdots\, x_n] \tag{13}$$

1. L_1 norm $\equiv \|X\|_1 = \sum_{i=1}^{n} |x_i|$
2. L_1 norm $\equiv \|X\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
3. L_1 norm $\equiv \|X\|_p = \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{p}}$
4. L_1 norm $\equiv \|X\|_\infty = max_i\, |x_i|$

Question

Let $X = \begin{bmatrix} 2 & 4 \end{bmatrix}^T$. Find

1. L_1 norm(X)
2. L_2 norm(X)
3. L_∞ norm(X)

1.2 Gram-Schmidt Process of Orthogonalization

Given a set of m linearly independent vectors $\{e_1, e_2, \cdots, e_m\}$, an orthonormal set can be obtained by using the following procedure

1. $u_1 = e_1, \quad q_1 = \dfrac{u_1}{\|u_1\|}$
2. $u_2 = e_2 - (q_1^T e_2), \quad q_2 = \dfrac{u_2}{\|u_2\|}$

   $\vdots$

3. $u_m = e_m - \sum_{k=1}^{m-1} (q_1^T\, e_2)\, q_k$

Note that the vectors $\{e_1, e_2, \cdots, e_m\}$ are not necessarily orthonormal.

1.3 Similarity Transformation

Let $X = Q\bar{X}$ where $Q = \{q_1, q_2, \cdots, q_n\}$ and is a set of basis vectors and $\bar{X}$ is the representation of X with respect to Q. Consider

$$AX = Y \tag{14}$$

If we write Equation 14 with respect to the basis Q, then we get

$$\begin{align}
\bar{A}\bar{X} &= \bar{Y} \tag{15}\\
X &= Q\bar{X} \tag{16}\\
Y &= Q\bar{Y} \tag{17}\\
A\,Q\,\bar{X} &= Q\,\bar{Y} \tag{18}\\
Q^{-1}\,A\,Q\,\bar{X} &= \bar{Y} \tag{19}
\end{align}$$

By examination of Equations 15 and 18, we can see that

$$\bar{A} = Q^{-1}\,A\,Q \tag{20}$$

where $\bar{A}$ is the similarity matrix of A with respect to Q.

1.3.1 Eigenvalues and Eigenvectors

Suppose

$$A\,X = \lambda\,X \tag{21}$$

Now, if we solve the equation

$$\begin{align}
\lambda\,X - A\,X &= 0 \tag{22}\\
(\lambda\,I - A)\,X &= 0 \tag{23}
\end{align}$$

which leads to a singular matrix

$$\lambda\,I - A = 0 \tag{24}$$

If we take the determinant of Equation 24

$$\Delta(\lambda) = |\lambda\, I - A| = 0 \tag{25}$$

which is the characteristic polynomial of degree n.

Question: Find the eigenvectors and values of A

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix} \tag{26}$$

1.4 Different Cases of Eigenvalues

Case 1: All eigenvalues are distinct

For an $n \times n$ matrix A with distince eigenvalues λ_1, $\lambda_2, \cdots, \lambda_n$ which are all real and distinct. Then $A\, q_i = \lambda_i\, q_i$, where q_i is the equivalent eigenvector associated with λ_i. Now, $Q = \{q_1, q_2, \cdots, q_n\}$ can be used as a basis and

$$\bar{A} \text{ (or } \hat{A}) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{27}$$

which is the representation of A with respect to Q (which is composed of the eigenvectors of A). Every matrix with distinct eigenvalues has a diagonal representation using its eigenvectors as a basis.

$$\hat{A} = Q^{-1}\, A\, Q \tag{28}$$

Case 2: All eigenvalues are *not* distinct, the representation is instead in *Jordan form*.

1.5 Range Space

The *range space* of a matrix A is all possible combinations of the columns of A.

1.6 Rank

The *rank* of a matrix A is the dimension of the range space of A, or the number of independent columns of A.

1.7 Null Vector

The *null vector* of a matrix A is a vector X such that $AX = 0$.

1.8 Null Space

The *null space* of a matrix A is the set off all the null vectors of A.

1.9 Nullity

The *nullity* of a matrix A is the dimension of the null space of A, or the number of columns of A minus the rank of A.

1.10 Determinant of a Matrix

The determinant of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{29}$$