EE 5600: Linear Systems Analysis - Assignment 1

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February 11, 2018

Question 1.

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

a)
First norm:

$$\begin{aligned} ||x_1||_1 &= \sum_1^3 x_{1i} && (1) \\ &= 2 - 3 - 1 && (2) \\ &= \textbf{-2} && (3) \end{aligned}$$

and

$$\begin{aligned} ||x_2||_1 &= \sum_1^3 x_{2i} && (4) \\ &= 1 + 1 - 1 && (5) \\ &= \mathbf{1} && (6) \end{aligned}$$

b)
Second norm:

$$\begin{aligned} ||x_1||_2 &= \sqrt{\sum_1^3 x_{1i}^2} && (7) \\ &= \sqrt{2^2 + (-3)^2 + (-1)^2} && (8) \\ &= \boldsymbol{\sqrt{14}} && (9) \end{aligned}$$

and

$$\begin{aligned} ||x_2||_2 &= \sqrt{\sum_1^3 x_{2i}^2} && (10)\\ &= \sqrt{1^2 + 1^2 + (-1)^2} && (11)\\ &= \mathbf{\sqrt{3}} && (12) \end{aligned}$$

c)
Infinite norm:

$$\begin{aligned} ||x_1||_\infty &= max(x_1) && (13)\\ &= \mathbf{2} && (14) \end{aligned}$$

and

$$\begin{aligned} ||x_2||_\infty &= max(x_2) && (15)\\ &= \mathbf{1} && (16) \end{aligned}$$

Question 2. Find two orthonormal vectors that span the same space as the two vectors, x_1 and x_2, in Problem 1.

Equation 17 shows is that the vectors x_1 and x_2 are orthogonal. Because x_1 and x_2 are orthogonal, they only need to be normalized, as shown below in Equations 18 to 19.

$$x_1^T \cdot x_2 = x_2^T \cdot x_1 = \mathbf{0} \tag{17}$$

The normalization process, where u_1 and u_2 are the normalized versions of x_1 and x_2, respectively:

$$\begin{aligned} u_1 &= \frac{x_1}{\|x_1\|} \\ &= \frac{x_1}{\sqrt{2^2 + (-3)^2 + (-1)^2}} \\ &= \frac{x_1}{\sqrt{4+9+1}} \\ &= \frac{x_1}{\sqrt{14}} \end{aligned} \tag{18}$$

$$\begin{aligned} \mathbf{u_1} &= \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix} \\ u_2 &= \frac{x_2}{\|x_2\|} \\ &= \frac{x_2}{\sqrt{1^2 + 1^2 + (-1)^2}} \\ &= \frac{x_2}{\sqrt{1+1+1}} \\ &= \frac{x_2}{\sqrt{3}} \\ \mathbf{u_2} &= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} \end{aligned} \tag{19}$$

Question 3.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{A_1}$:

By examination, it can be seen that the matrix A_1 has two linearly independent columns. Therefore, the **rank of $\mathbf{A_1}$ is 2**. There are four columns in A_1 and its rank is two, therefore **$\mathbf{A_1}$'s nullity is $\mathbf{4 - 2 = 2}$**.

$\mathbf{A_2}$:

Matrix A_2 can be transformed into an upper triangle using a sequence of elementary transformations as demonstrated by [1] and is given by Equation 20. According to [1], the rank of an upper triangular matrix is equal to the number of nonzero rows. The matrix A_{2ref} has three nonzero rows and therefore it and A_2 have a **rank of 3**. The **nullity of $\mathbf{A_2}$ is 0.**

$$A_2 \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_{2ref} \tag{20}$$

$\mathbf{A_3}$:

By examination, it can be seen that the matrix A_3 has three linearly independent columns. Therefore, the **rank of $\mathbf{A_3}$ is 3**. There are four columns in A_3 and its rank is three, therefore **$\mathbf{A_3}$'s nullity is $\mathbf{4 - 3 = 1}$**.

Question 4.

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The Cayley-Hamilton theorem can be used to compute powers of a matrix [1]. First, the eigenvalues of the matrix A_1 must be found.

$$|\mathbf{I}\lambda - A| \tag{21}$$

References

[1] C.-T. Chen, *Linear system theory and design*. Oxford University Press, Inc., 1998.