EE 5600: Linear Systems Analysis - Assignment 1

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February 11, 2018

Question 1.

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

a) First norm:

$$\begin{aligned} ||x_1||_1 &= \sum_1^3 x_{1i} && (1) \\ &= 2 - 3 - 1 && (2) \\ &= \textbf{-2} && (3) \end{aligned}$$

and

$$\begin{aligned} ||x_2||_1 &= \sum_1^3 x_{2i} && (4) \\ &= 1 + 1 - 1 && (5) \\ &= \mathbf{1} && (6) \end{aligned}$$

b) Second norm:

$$\begin{aligned} ||x_1||_2 &= \sqrt{\sum_1^3 x_{1i}^2} && (7) \\ &= \sqrt{2^2 + (-3)^2 + (-1)^2} && (8) \\ &= \sqrt{\mathbf{14}} && (9) \end{aligned}$$

and

$$||x_2||_2 = \sqrt{\sum_{1}^{3} x_{2i}^2} \tag{10}$$

$$= \sqrt{1^2 + 1^2 + (-1)^2} \tag{11}$$

$$= \boldsymbol{\sqrt{3}} \tag{12}$$

c) Infinite norm:

$$||x_1||_\infty = max(x_1) \tag{13}$$

$$= \mathbf{2} \tag{14}$$

and

$$||x_2||_\infty = max(x_2) \tag{15}$$

$$= \mathbf{1} \tag{16}$$

Question 2. Find two orthonormal vectors that span the same space as the two vectors, x_1 and x_2, in Problem 1.

Equation 17 shows is that the vectors x_1 and x_2 are orthogonal. Because x_1 and x_2 are orthogonal, they only need to be normalized, as shown below in Equations 18 to 19.

$$x_1^T \cdot x_2 = x_2^T \cdot x_1 = \mathbf{0} \tag{17}$$

The normalization process, where u_1 and u_2 are the normalized versions of x_1 and x_2, respectively:

$$
\begin{aligned}
u_1 &= \frac{x_1}{\|x_1\|} \\
&= \frac{x_1}{\sqrt{2^2 + (-3)^2 + (-1)^2}} \\
&= \frac{x_1}{\sqrt{4 + 9 + 1}} \\
&= \frac{x_1}{\sqrt{14}}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\mathbf{u_1} &= \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix} \\
u_2 &= \frac{x_2}{\|x_2\|} \\
&= \frac{x_2}{\sqrt{1^2 + 1^2 + (-1)^2}} \\
&= \frac{x_2}{\sqrt{1 + 1 + 1}} \\
&= \frac{x_2}{\sqrt{3}}
\end{aligned}
$$

$$
\mathbf{u_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} \tag{19}
$$

Question 3.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{A_1}$:

By examination, it can be seen that the matrix A_1 has two linearly independent columns. Therefore, the **rank of $\mathbf{A_1}$ is 2**. There are four columns in A_1 and its rank is two, therefore **$\mathbf{A_1}$'s nullity is $\mathbf{4 - 2 = 2}$**.

$\mathbf{A_2}$:

Matrix A_2 can be transformed into an upper triangle using a sequence of elementary transformations as demonstrated by [1] and is given by Equation 20. According to [1], the rank of an upper triangular matrix is equal to the number of nonzero rows. The matrix A_{2ref} has three nonzero rows and therefore it and A_2 have a **rank of 3**. The **nullity of $\mathbf{A_2}$ is 0.**

$$A_2 \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_{2ref} \tag{20}$$

$\mathbf{A_3}$:

By examination, it can be seen that the matrix A_3 has three linearly independent columns. Therefore, the **rank of $\mathbf{A_3}$ is 3**. There are four columns in A_3 and its rank is three, therefore **$\mathbf{A_3}$'s nullity is $\mathbf{4 - 3 = 1}$**.

Question 4.

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The Cayley-Hamilton theorem can be used to compute powers of a matrix [1]. First, the eigenvalues of the matrix A_1 must be found.

$$\begin{aligned} \left| \mathbf{I}\lambda - A_1 \right| &= \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & 1 - \lambda \end{vmatrix} \\ &= \lambda(\lambda - 1)^2 \\ \lambda &= 0, 1, 1 \end{aligned} \tag{21}$$

Now that the eigenvalues have been found, different powers of A_1 can be found by finding the β_is in Equation 22

$$h(\lambda_i) = \beta_0 + \beta_1\lambda_i + \beta_2\lambda_i^2 \tag{22}$$

using the previously found eigenvalues each corresponding to a λ_i . There is a slight modification that needs to be made in order to deal with the repeated eigenvalues. We can use a derivative of Equation 22 to allow us to solve for the coefficients

$$\frac{dh(\lambda_i)}{d\lambda_i} = \beta_1 + 2\beta_2\lambda_i \tag{23}$$

Once we know our coefficients (β_i) we can use the following formula to find any power of A_i

$$h(A_1) = \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \tag{24}$$

a) Find A_1^{10}.

Here, $h(\lambda_i) = \lambda_i^{10}$ and $\dfrac{dh(\lambda_i)}{d\lambda_i} = 10\lambda_i^9$. For $\lambda_i = 0$

$$\begin{aligned} (0)^{10} &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\ 0 &= \beta_0 \end{aligned} \tag{25}$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$\begin{aligned}(1)^{10} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\ 1 &= 0 + \beta_1 + \beta_2 \\ 1 &= \beta_1 + \beta_2\end{aligned} \tag{26}$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$\begin{aligned}10(1)^9 &= \beta_1 + 2\beta_2(1) \\ 10 &= 0 + \beta_1 + 2\beta_2 \\ 10 &= \beta_1 + 2\beta_2\end{aligned} \tag{27}$$

And solving the system of equations we get

$$\begin{aligned}\beta_0 &= 0 \\ \beta_1 &= -8 \\ \beta_2 &= 9\end{aligned} \tag{28}$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{10}

$$\begin{aligned}h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\ A_1^{10} &= -8A_1 + 9A_1^2 \\ &= -8\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix} + 9\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix}^2 \\ &= -8\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix} + 9\begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & 1 & 9\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix}\end{aligned} \tag{29}$$

b) Find A_1^{103}

Here $h(\lambda_i) = \lambda_i^{103}$, $\dfrac{dh(\lambda_i)}{d\lambda_i} = 103\lambda_i^{102}$, and $h(A_1) = A_1^{103}$.

For $\lambda_i = 0$

$$\begin{aligned} (0)^{103} &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\ 0 &= \beta_0 \end{aligned} \tag{30}$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$\begin{aligned} (1)^{103} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\ 1 &= 0 + \beta_1 + \beta_2 \\ 1 &= \beta_1 + \beta_2 \end{aligned} \tag{31}$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$\begin{aligned} 103(1)^{102} &= \beta_1 + 2\beta_2(1) \\ 103 &= 0 + \beta_1 + 2\beta_2 \\ 103 &= \beta_1 + 2\beta_2 \end{aligned} \tag{32}$$

And solving the system of equations we get

$$\begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -101 \\ \beta_2 &= 102 \end{aligned} \tag{33}$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{103}

$$
\begin{aligned}
h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\
A_1^{10} &= -101 A_1 + 102 A_1^2 \\
&= -101 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 102 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{34}
$$

c) Find $e^{A_1 t}$

Here $h(\lambda_i) = e^{\lambda_i t}$, $\dfrac{dh(\lambda_i)}{d\lambda_i} = te^{\lambda_i t}$, and $h(A_1) = e^{A_1 t}$.

For $\lambda_i = 0$

$$
\begin{aligned}
e^0 &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\
1 &= \beta_0
\end{aligned}
\tag{35}
$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$
\begin{aligned}
e^{(1)t} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\
e^t &= 1 + \beta_1 + \beta_2
\end{aligned}
\tag{36}
$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$
\begin{aligned}
te^{(1)t} &= \beta_1 + 2\beta_2(1) \\
te^t &= \beta_1 + 2\beta_2
\end{aligned}
\tag{37}
$$

And solving the system of equations we get

$$
\begin{aligned}
\beta_0 &= 1 \\
\beta_1 &= e^t(2 - t) \\
\beta_2 &= e^t(t - 1) + 1
\end{aligned}
\tag{38}
$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{103}

$$
\begin{aligned}
h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\
A_1^{10} &= 1\mathbb{I} + e^t(2-t)A_1 + [e^t(t-1)+1]A_1^2 \\
&= \mathbb{I} + \begin{bmatrix} e^t(2-t) & e^t(2-t) & 0 \\ 0 & 0 & e^t(2-t) \\ 0 & 0 & e^t(2-t) \end{bmatrix} + \begin{bmatrix} e^t(t-1)+1 & e^t(t-1)+1 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \end{bmatrix} \\
&= \begin{bmatrix} e^t+2 & e^t+1 & e^t+1 \\ 0 & 1 & e^t+1 \\ 0 & 0 & e^t+2 \end{bmatrix}
\end{aligned}
\tag{39}
$$

References

[1] C.-T. Chen, *Linear system theory and design.* Oxford University Press, Inc., 1998.