EE 5600: Linear Systems Analysis - Assignment 1

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Question 1.

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

a) First norm:

$$\begin{aligned} ||x_1||_1 &= \sum_{1}^{3} x_{1i} &(1)\\ &= 2 - 3 - 1 &(2)\\ &= \textbf{-2} &(3) \end{aligned}$$

and

$$\begin{aligned} ||x_2||_1 &= \sum_{1}^{3} x_{2i} &(4)\\ &= 1 + 1 - 1 &(5)\\ &= \mathbf{1} &(6) \end{aligned}$$

b) Second norm:

$$\begin{aligned} ||x_1||_2 &= \sqrt{\sum_{1}^{3} x_{1i}^2} &(7)\\ &= \sqrt{2^2 + (-3)^2 + (-1)^2} &(8)\\ &= \boldsymbol{\sqrt{14}} &(9) \end{aligned}$$

and

$$||x_2||_2 = \sqrt{\sum_{1}^{3} x_{2i}^2} \tag{10}$$
$$= \sqrt{1^2 + 1^2 + (-1)^2} \tag{11}$$
$$= \boldsymbol{\sqrt{3}} \tag{12}$$

c) Infinite norm:

$$||x_1||_\infty = max(x_1) \tag{13}$$
$$= \mathbf{2} \tag{14}$$

and

$$||x_2||_\infty = max(x_2) \tag{15}$$
$$= \mathbf{1} \tag{16}$$

Question 2. Find two orthonormal vectors that span the same space as the two vectors, x_1 and x_2, in Problem 1.

Equation 17 shows is that the vectors x_1 and x_2 are orthogonal. Because x_1 and x_2 are orthogonal, they only need to be normalized, as shown below in Equations 18 to 19.

$$x_1^T \cdot x_2 = x_2^T \cdot x_1 = \mathbf{0} \tag{17}$$

The normalization process, where u_1 and u_2 are the normalized versions of x_1 and x_2, respectively:

$$
\begin{aligned}
u_1 &= \frac{x_1}{\|x_1\|} \\
&= \frac{x_1}{\sqrt{2^2 + (-3)^2 + (-1)^2}} \\
&= \frac{x_1}{\sqrt{4 + 9 + 1}} \\
&= \frac{x_1}{\sqrt{14}}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\mathbf{u_1} &= \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix} \\
u_2 &= \frac{x_2}{\|x_2\|} \\
&= \frac{x_2}{\sqrt{1^2 + 1^2 + (-1)^2}} \\
&= \frac{x_2}{\sqrt{1 + 1 + 1}} \\
&= \frac{x_2}{\sqrt{3}}
\end{aligned}
$$

$$
\mathbf{u_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} \tag{19}
$$

Question 3.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{A_1}$:

By examination, it can be seen that the matrix A_1 has two linearly independent columns. Therefore, the **rank of $\mathbf{A_1}$ is 2**. There are four columns in A_1 and its rank is two, therefore **$\mathbf{A_1}$'s nullity is $\mathbf{4 - 2 = 2}$**.

$\mathbf{A_2}$:

Matrix A_2 can be transformed into an upper triangle using a sequence of elementary transformations as demonstrated by [1] and is given by Equation 20. According to [1], the rank of an upper triangular matrix is equal to the number of nonzero rows. The matrix A_{2ref} has three nonzero rows and therefore it and A_2 have a **rank of 3**. The **nullity of $\mathbf{A_2}$ is 0.**

$$A_2 \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_{2ref} \tag{20}$$

$\mathbf{A_3}$:

By examination, it can be seen that the matrix A_3 has three linearly independent columns. Therefore, the **rank of $\mathbf{A_3}$ is 3**. There are four columns in A_3 and its rank is three, therefore **$\mathbf{A_3}$'s nullity is $\mathbf{4 - 3 = 1}$**.

Question 4.

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The Cayley-Hamilton theorem can be used to compute powers of a matrix [1]. First, the eigenvalues of the matrix A_1 must be found.

$$\begin{aligned} \left|\mathbf{I}\lambda - A_1\right| &= \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & 1 - \lambda \end{vmatrix} \\ &= \lambda(\lambda - 1)^2 \\ \lambda &= 0, 1, 1 \end{aligned} \tag{21}$$

Now that the eigenvalues have been found, different powers of A_1 can be found by finding the β_is in Equation 22

$$h(\lambda_i) = \beta_0 + \beta_1\lambda_i + \beta_2\lambda_i^2 \tag{22}$$

using the previously found eigenvalues each corresponding to a λ_i . There is a slight modification that needs to be made in order to deal with the repeated eigenvalues. We can use a derivative of Equation 22 to allow us to solve for the coefficients

$$\frac{dh(\lambda_i)}{d\lambda_i} = \beta_1 + 2\beta_2\lambda_i \tag{23}$$

Once we know our coefficients (β_i) we can use the following formula to find any power of A_i

$$h(A_1) = \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \tag{24}$$

a) Find A_1^{10}.

Here, $h(\lambda_i) = \lambda_i^{10}$ and $\dfrac{dh(\lambda_i)}{d\lambda_i} = 10\lambda_i^9$. For $\lambda_i = 0$

$$\begin{aligned} (0)^{10} &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\ 0 &= \beta_0 \end{aligned} \tag{25}$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$\begin{aligned} (1)^{10} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\ 1 &= 0 + \beta_1 + \beta_2 \\ 1 &= \beta_1 + \beta_2 \end{aligned} \tag{26}$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$\begin{aligned} 10(1)^9 &= \beta_1 + 2\beta_2(1) \\ 10 &= 0 + \beta_1 + 2\beta_2 \\ 10 &= \beta_1 + 2\beta_2 \end{aligned} \tag{27}$$

And solving the system of equations we get

$$\begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -8 \\ \beta_2 &= 9 \end{aligned} \tag{28}$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{10}

$$\begin{aligned} h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\ A_1^{10} &= -8A_1 + 9A_1^2 \\ &= -8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 \\ &= -8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{29}$$

b) Find A_1^{103}

Here $h(\lambda_i) = \lambda_i^{103}$, $\dfrac{dh(\lambda_i)}{d\lambda_i} = 103\lambda_i^{102}$, and $h(A_1) = A_1^{103}$.

For $\lambda_i = 0$

$$\begin{aligned} (0)^{103} &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\ 0 &= \beta_0 \end{aligned} \tag{30}$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$\begin{aligned} (1)^{103} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\ 1 &= 0 + \beta_1 + \beta_2 \\ 1 &= \beta_1 + \beta_2 \end{aligned} \tag{31}$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$\begin{aligned} 103(1)^{102} &= \beta_1 + 2\beta_2(1) \\ 103 &= 0 + \beta_1 + 2\beta_2 \\ 103 &= \beta_1 + 2\beta_2 \end{aligned} \tag{32}$$

And solving the system of equations we get

$$\begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -101 \\ \beta_2 &= 102 \end{aligned} \tag{33}$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{103}

$$
\begin{aligned}
h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\
A_1^{10} &= -101 A_1 + 102 A_1^2 \\
&= -101 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 102 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{34}
$$

c) Find $e^{A_1 t}$

Here $h(\lambda_i) = e^{\lambda_i t}$, $\dfrac{dh(\lambda_i)}{d\lambda_i} = te^{\lambda_i t}$, and $h(A_1) = e^{A_1 t}$.

For $\lambda_i = 0$

$$
\begin{aligned}
e^0 &= \beta_0 + \beta_1(0) + \beta_2(0)^2 \\
1 &= \beta_0
\end{aligned} \tag{35}
$$

For $\lambda_i = 1$ using $h(\lambda_i)$

$$
\begin{aligned}
e^{(1)t} &= \beta_0 + \beta_1(1) + \beta_2(1)^2 \\
e^t &= 1 + \beta_1 + \beta_2
\end{aligned} \tag{36}
$$

For $\lambda_i = 1$ using $\dfrac{dh(\lambda_i)}{d\lambda_i}$

$$
\begin{aligned}
te^{(1)t} &= \beta_1 + 2\beta_2(1) \\
te^t &= \beta_1 + 2\beta_2
\end{aligned} \tag{37}
$$

And solving the system of equations we get

$$
\begin{aligned}
\beta_0 &= 1 \\
\beta_1 &= e^t(2 - t) \\
\beta_2 &= e^t(t - 1) + 1
\end{aligned} \tag{38}
$$

Now the matrix form of the Cayley-Hamilton theorm, Equation 24, can be prepared, then utilized to find A_1^{103}

$$\begin{aligned}
h(A_1) &= \beta_0 + \beta_1 A_1 + \beta_2 A_1^2 \\
A_1^{10} &= 1\mathbb{I} + e^t(2-t)A_1 + [e^t(t-1)+1]A_1^2 \\
&= \mathbb{I} + \begin{bmatrix} e^t(2-t) & e^t(2-t) & 0 \\ 0 & 0 & e^t(2-t) \\ 0 & 0 & e^t(2-t) \end{bmatrix} + \begin{bmatrix} e^t(t-1)+1 & e^t(t-1)+1 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \end{bmatrix} \\
&= \begin{bmatrix} e^t+2 & e^t+1 & e^t+1 \\ 0 & 1 & e^t+1 \\ 0 & 0 & e^t+2 \end{bmatrix}
\end{aligned} \tag{39}$$

Question 5. Find the unit-step response of the following system using two different methods.

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} X(t) \tag{40}$$

[Note: I'm going to be assuming that initial conditions are all zero.]

a) Using the Laplace Transform.

$$Y(s) = [C(s\mathbb{I} - A)^{-1}B + D]U(s) \tag{41}$$

Equation 41 gives the Laplace Transform of the output equation, Equation 40, where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad D = 0$$

If we simplify Equation 41, we arrive at

$$Y(s) = T(s)U(s)$$

where

$$\begin{aligned} T(s) &= C(s\mathbb{I} - A)^{-1}B + D \\ &= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \end{bmatrix} \left(\frac{1}{s^2 + 2s + 2} \right) \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{5s}{s^2 + 2s + 2} \end{aligned} \tag{42}$$

With $U(s) = \dfrac{1}{s}$ and Equations and we have

$$
\begin{aligned}
Y(s) &= \frac{5s}{s^2+2s+2} \cdot \frac{1}{s} \\
&= \frac{5}{s^2+2s+2} \\
&= \frac{5}{(s+1)^2+1}
\end{aligned} \tag{43}
$$

and by taking the inverse Laplace Transform of Equation 43, we finally arrive at

$$
y(t) = 5e^{-t}\sin(t) \tag{44}
$$

b) Using the Cayley-Hamilton theorem.

The general solution for a linear time-invariant (LTI) system is

$$
X(t) = e^{At}X(0) + \int_0^{\infty} e^{A(t-\tau)}BU(\tau)d\tau \tag{45}
$$

Using the Cayley-Hamilton theorem, e^{At} can be found. First, the eigenvalues must be determined.

$$
\begin{aligned}
\left|\lambda\mathbb{I} - A\right| &= \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{vmatrix} \\
&= \lambda(\lambda+2) - 2(-1) \\
&= \lambda^2 + 2\lambda + 2 = 0 \\
\therefore \lambda_{1,2} &= -1 \pm j
\end{aligned} \tag{46}
$$

$$
\text{let} \quad f(\lambda) = e^{\lambda t} = h(\lambda) = \beta_0 + \beta_1\lambda \tag{47}
$$

Plugging in the values for λ_1 and λ_2 we arrive at

$$
\begin{aligned}
e^{t(-1+j)} &= \beta_0 + \beta_1(-1+j) \\
e^{t(-1-j)} &= \beta_0 + \beta_1(-1-j)
\end{aligned}
$$

Solving the above system yields

$$\beta_0 = e^{-t}\sin t \tag{48}$$
$$\beta_1 = e^{-t}(\cos t + \sin t) \tag{49}$$

By the Cayley-Hamilton theorem, $h(\lambda) \mapsto h(A)$ and $f(\lambda) \mapsto f(A)$ which yields

$$\begin{aligned}
h(A) &= \beta_0 \mathbb{I} + \beta_1 A \\
&= \begin{bmatrix} e^{-t}[\cos t + \sin t] & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}[\cos t - \sin t] \end{bmatrix} \\
f(A) &= e^{At} \\
f(A) &= h(A) \\
e^{At} &= \begin{bmatrix} e^{-t}[\cos t + \sin t] & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}[\cos t - \sin t] \end{bmatrix} \qquad (50)
\end{aligned}$$

By solving Equation 51 the solution to the system will be found.

$$y(t) = \int_0^t Ce^{A(t-\tau)} BU(\tau) d\tau \tag{51}$$

Simplifying the integrand of Equation 51 gives

$$\begin{aligned}
y(t) &= \int_0^t 5e^{-(t-\tau)}(\cos(t-\tau) - \sin(t-\tau)) d\tau \\
y(t) &= 5e^{-t}\sin(t) \qquad (52)
\end{aligned}$$

By comparing Equations 44 and 52, we can see that they are the same.

Question 6. Are the two sets of state-space equations

$$\dot{X}(t) = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \tag{53}$$

$$\dot{X}(t) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \tag{54}$$

equivalent? Zero-state equivalent?

a) For equivalence, we must show that the A matrix of both systems have the same eigenvalues.

Starting with the system given by Equation 53, the eigenvalues are

$$\begin{aligned} [s\mathbb{I} - A]^{-1} &= \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix} \qquad (55) \\ &= (s-2)^2(s-1) \\ &= 0 \\ \therefore \lambda_{1,2,3} &= 2, 2, 1 \qquad (56) \end{aligned}$$

and the eigenvalues for the system given by Equation 54 are

$$\begin{aligned} [s\mathbb{I} - A]^{-1} &= \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix} \qquad (57) \\ &= (s-2)^2(s+1) \\ &= 0 \\ \therefore \lambda_{1,2,3} &= 2, 2, -1 \qquad (58) \end{aligned}$$

Equations 56 and 58 show that the eigenvalues of the two systems are different. Therefore, the two sets of state-space equations are not equivalent.

b) For zero-state equivalence, we must show that the two sets of state-space equations have the same transfer matrix. The transfer matrix is given by

$$\frac{Y(s)}{U(s)} = C[s\mathbb{I} - A]^{-1}B + D$$

The C, B, and D are the same for both systems which means in order for the two sets of state-space equations to be zero-state equivalent then the $[s\mathbb{I} - A]^{-1}$ must be equivalent. As is shown in Equations 55 and 57, these two matrices are *not* the same. Therefore, the two sets of state-space equations are not zero-state equivalent.

References

[1] C.-T. Chen, *Linear system theory and design.* Oxford University Press, Inc., 1998.