Assignment 3 - EE5600

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Question 1

The ability to balance actively is a key ingredient in the mobility of a device that hops and runs on one springy leg. The control of attitude of the device uses a gyroscope and a feedback such that $u(t) = Kx(t)$, where

$$K = \begin{bmatrix} -k & 0 \\ 0 & -2k \end{bmatrix} \text{ and}$$
$$\dot{X}(t) = AX(t) + BU(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; B = \mathbb{I}$$

Determine K so that response of the system is critically damped? (Use $\zeta = 1$)

Solution

The desired characteristic equation is given below in Equation 1

$$\begin{aligned} \lambda^2 + 2\zeta\omega_n\lambda + \omega_2^2 &= 0 \\ \lambda^2 + 2\omega_n\lambda + \omega_2^2 &= 0 \end{aligned} \tag{1}$$

Let $U(t) = KX(t)$, then we get

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \\ &= AX(t) + B(KX(t)) \\ &= AX(t) + BKX(t) \\ &= (A + BK)X(t) \end{aligned}$$

and

$$\begin{aligned} |\lambda\mathbb{I} - (A + BK)| &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -k & 0 \\ 0 & -2k \end{bmatrix} \right) \right| \\ &= \left| \begin{bmatrix} \lambda + k & 1 \\ -1 & \lambda + 2k \end{bmatrix} \right| \\ &= (\lambda + k)(\lambda + 2k) + 1 \\ &= \lambda^2 + 3k\lambda + \left(2k^2 + 1\right) \\ &= 0 \end{aligned}$$

We'll use matching with the desired characteristic equation, Equation 1, to determine k.

$$\begin{aligned} \lambda^2 + 2\omega_n\lambda + \omega_2^2 &= \lambda^2 + 3k\lambda + \left(2k^2 + 1\right) \\ 2\omega_n\lambda + \omega_2^2 &= 3k\lambda + \left(2k^2 + 1\right) \end{aligned}$$

Which gives

$$\begin{aligned} 2\omega_n &= 3k \\ \omega_n &= \frac{3}{2}k \end{aligned}$$

and

$$\begin{aligned} \omega_n^2 &= 2k^2 + 1 \\ \left(\frac{3}{2}k\right)^2 &= 2k^2 + 1 \\ \frac{1}{4}k^2 &= 1 \\ k &= \pm 2 \end{aligned}$$

k must be positive otherwise the system will be unstable. Therefore,

$$\boxed{k = 2}$$

Question 2

A system is described by the equations

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} x(t)$$

determine controllabilty and observability.

Solution

Observability

A system is *fully* observable iff $|P_o| \neq 0$ where $P_o = \begin{bmatrix} C & CA \end{bmatrix}^T$ for a system of order two (such as the system presented in this question). Therefore,

$$\begin{aligned} P_o &= \begin{bmatrix} C & CA \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 2 \\ 0 & -10 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} |P_o| &= \begin{vmatrix} 0 & 2 \\ 0 & -10 \end{vmatrix} \\ &= 0 \end{aligned}$$

Because $|P_o| = 0$, this system is **not *fully* observable**.

Controllability

A system is *fully* controllable iff $|P_c| \neq 0$ where $P_c = \begin{bmatrix} B & AB \end{bmatrix}$ for a system of order two (such as the system presented in this question). Therefore,

$$\begin{aligned} P_c &= \begin{bmatrix} B & AB \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & -10 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} |P_o| &= \begin{vmatrix} 0 & 2 \\ 2 & -10 \end{vmatrix} \\ &= -4 \end{aligned}$$

Because $|P_o| = -4 \neq 0$, this system is ***fully* controllable**.

Question 3

A system is described by the equations

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

determine controllabilty and observability

Solution

Observability

A system is *fully* observable iff $|P_o| \neq 0$ where $P_o = \begin{bmatrix} C & CA \end{bmatrix}^T$ for a system of order two (such as the system presented in this question). Therefore,

$$\begin{aligned} P_o &= \begin{bmatrix} C & CA \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} |P_o| &= \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \\ &= 1 \end{aligned}$$

Because $|P_o| = 1 \neq 0$, this system is ***fully* observable**.

Controllability

A system is *fully* controllable iff $|P_c| \neq 0$ where $P_c = \begin{bmatrix} B & AB \end{bmatrix}$ for a system of order two (such as the system presented in this question). Therefore,

$$\begin{aligned} P_c &= \begin{bmatrix} B & AB \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} |P_c| &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1 \end{aligned}$$

Because $|P_c| = 1 \neq 0$, this system is ***fully* controllable**.

Question 4

Hydraulic power actuators were used to drive the dinosaurs of the movie Jurassic park, the motions of the large monsters required high power actuators requiring 1200 watts. One specific limb motion has dynamics represented by

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + [0]\, u(t)\end{aligned}$$

We want to place the closed loop poles at $s = -1 \pm j3$. Determine the required state variable feedback using any method (Ackermann's formula or coefficient comparison). Assume that complete state vector is available for feedback.

Solution

Let's first check that the system is controllable. A system is *fully* controllable iff $|P_c| \neq 0$ where $P_c = \begin{bmatrix} B & AB \end{bmatrix}$.

$$\begin{aligned}|P_c| &= \left|\begin{bmatrix} B & AB \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\right| \\ &= 1 \\ &\neq 0 \;\therefore\; \text{fully controllable.}\end{aligned}$$

Now, let's use Ackermann's formula to determine the state variable feedback. Ackermann's formula (for a second-order system) is given by Equation 2.

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} P_c^{-1} p(A) \tag{2}$$

First, we need to find $p(\lambda)$ which will be used to construct $p(A)$.

$$\begin{aligned}(\lambda + 1 + j3)(\lambda + 1 - j3) &= \lambda^2 + 2\lambda + 10 \\ &= p(\lambda)\end{aligned}$$

Time to find $p(A)$

$$\begin{aligned}p(A) &= A^2 + 2A + 10\mathbb{I} \\ &= \begin{bmatrix} 16 & 0 \\ -5 & 1 \end{bmatrix} + 2\begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 18 & 0 \\ -3 & 9 \end{bmatrix}\end{aligned}$$

and P_c^{-1}

$$\begin{aligned}P_c^{-1} &= \frac{1}{\det(P_c)} \operatorname{adj}(P_c) \\ &= \frac{1}{1}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

We can now find K

$$\begin{aligned}K &= \begin{bmatrix} 0 & 1 \end{bmatrix} P_c^{-1} p(A) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 18 & 0 \\ -3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 30 & 36 \end{bmatrix}\end{aligned}$$

Therefore, the state variable feedback is

$$\boxed{K = \begin{bmatrix} 30 & 36 \end{bmatrix}}$$

Question 5

Consider

$$\begin{aligned}\dot{X}(t) &= AX(t) + BU(t)\\ Y(t) &= CX(t) + DU(t)\end{aligned}$$

Where

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ -5 & -10 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ C &= \begin{bmatrix} 1 & -4 \end{bmatrix}; D = [0]\end{aligned}$$

a) Comment on observability.

b) Design full state observer placing poles at $s_{1,2} = -1$.

c) Plot the response of the estimations error $e(t) = x(t) - \hat{x}(t)$ with initial estimation error $e(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ using Matlab and attach your result.

Solution

Observability

To test the observability of a system, we check if $|P_o| \neq 0$ where $P_o = \begin{bmatrix} C & CA \end{bmatrix}^T$ for a two-state system.

$$\begin{aligned}|P_o| &= \left|\begin{bmatrix} C & CA \end{bmatrix}^T\right|\\ &= \left|\begin{bmatrix} 1 & -4 \\ 21 & 44 \end{bmatrix}\right|\\ &= 128\\ &\neq 0 \;\therefore\; \textbf{fully observable}\end{aligned}$$

Full-State Observer

Equation 3 gives the Ackermann equation for observability.

$$L = q(A)\, P_o^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix}^T \tag{3}$$

First, we find P_o^{-1}

$$\begin{aligned}P_o^{-1} &= \frac{1}{\det(P_o)} \operatorname{adj}(P_o)\\ &= \frac{1}{128} \begin{bmatrix} 44 & 4 \\ -21 & 1 \end{bmatrix}\end{aligned}$$

Next, find $q(\lambda)$ which will then be used to construct $q(A)$.

$$(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1 = q(\lambda)$$

Therefore,

$$\begin{aligned}q(A) &= A^2 + 2A + \mathbb{I}\\ &= \begin{bmatrix} 0 & 1 \\ -5 & -10 \end{bmatrix}^2 + 2\begin{bmatrix} 0 & 1 \\ -5 & -10 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\ &= \begin{bmatrix} -16 & -28 \\ 35 & 61 \end{bmatrix}\end{aligned}$$

Putting it all together

$$\begin{aligned}L &= q(A)\, P_o^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix}^T\\ &= \begin{bmatrix} -16 & -28 \\ 35 & 61 \end{bmatrix} \frac{1}{128} \begin{bmatrix} 44 & 4 \\ -21 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ &= \begin{bmatrix} -0.71875 \\ 1.570131 \end{bmatrix}\end{aligned}$$

Therefore, the observer gains are

$$\boxed{L = \begin{bmatrix} -0.71875 \\ 1.570131 \end{bmatrix}}$$

MATLAB Plot

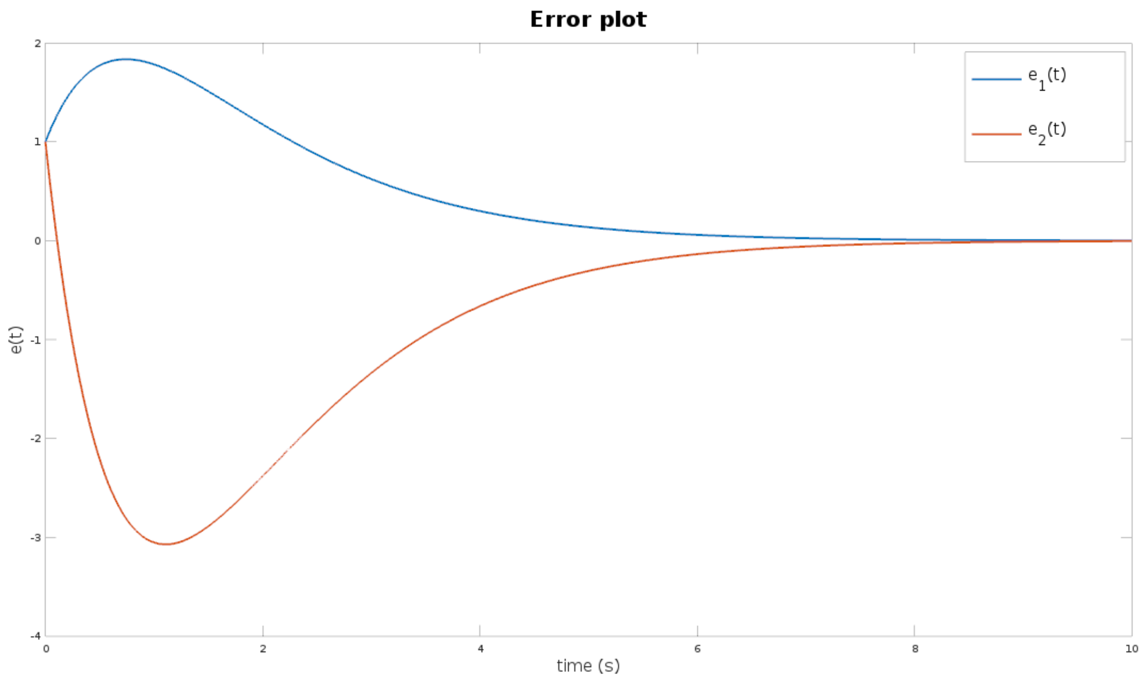

Fig. 1: Error plot

The code for the plot is located in the Appendix.

Question 6

A system is described by the equations

$$\dot{X}(t) = \begin{bmatrix} 1 & 0 \\ -5 & -20 \end{bmatrix} X(t) + \begin{bmatrix} 100 \\ 0 \end{bmatrix} U(t)$$
$$Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} X(t)$$

Determine observer gains to place the observer poles at $s_{1,2} = -5$.

Solution

Before constructing an observer, we need to make sure that our system is fully observable.

$$\begin{aligned} |P_o| &= \begin{vmatrix} 100 & 100 \\ 0 & -500 \end{vmatrix} \\ &= -50000 \\ &\neq 0 \;\therefore\; \text{fully observable} \end{aligned}$$

Now that we know that our system is observable, we can construct our observer using Ackermann's formula for observability for a two-state system, Equation 3.

$$\begin{aligned} P_o^{-1} &= \frac{1}{\det(P_o)} \operatorname{adj}(P_o) \\ &= \frac{1}{-50000} \begin{bmatrix} -500 & -100 \\ 0 & 100 \end{bmatrix} \\ &= \begin{bmatrix} 0.01 & 0.02 \\ 0 & -0.2 \end{bmatrix} \end{aligned}$$

Let's find $q(\lambda)$, which will be used to construct $q(A)$

$$(\lambda + 5)^2 = \lambda^2 + 10\lambda + 25 = q(\lambda)$$

Therefore,

$$\begin{aligned} q(A) &= A^2 + 10A + 25\mathbb{I} \\ &= \begin{bmatrix} 1 & 0 \\ 95 & 400 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 \\ -5 & -20 \end{bmatrix} + 25 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 \\ 45 & 225 \end{bmatrix} \end{aligned}$$

And finally

$$\begin{aligned} L &= q(A) P_o^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 36 & 0 \\ 45 & 225 \end{bmatrix} \begin{bmatrix} 0.01 & 0.02 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.072 \\ -0.36 \end{bmatrix} \end{aligned}$$

Therefore, the observer gains are

$$\boxed{L = \begin{bmatrix} 0.072 \\ -0.36 \end{bmatrix}}$$

Appendix

```
% Standard state-space matrices
A = [
 1,  4;
-5, -10];
B = [0; 1];
C = [1, -4];
D = [0];

poles = [-1, -1];

% Need to use acker() because the poles are at the same location
% We'll use the fact that the controllability of the dual of the system
% is the same as the observability of the original system
L = acker(A' ,C',poles)';

dt = 0.001;
total_time = 10;
t = 0:dt:total_time;
num_its = length(t);

U = 1e-3; % input as a step function

X = [1; 1];        % initial states
X_hat = [0; 0];    % initial estimate of states
e0 = X - X_hat;    % initial error

% Setup vectors
X_vec = X;
X_hat_vec = X_hat;
e_vec = e0;

for i = 1:num_its-1;
  Y = C*X + D*U;
  Y_hat = C*X_hat + D*U;

  dX = A*X + B*U;
  X = X + dX*dt;

  dX_hat = A*X_hat + B*U + L*(Y - Y_hat);
  X_hat = X_hat + dX_hat*dt;

  e = X - X_hat;

  X_vec = [X_vec, X];
  X_hat_vec = [X_hat_vec, X_hat];
  e_vec = [e_vec, e];
end

figure;

plot(t, e_vec(1,:), 'LineWidth', 2);
hold on;
plot(t, e_vec(2,:), 'LineWidth', 2);
title('Error plot', 'fontsize', 20)
xlabel('time (s)', 'fontsize', 16)
ylabel('e(t)', 'fontsize', 16)
leg = legend('e_{1}(t)', 'e_{2}(t)')
set (leg, "fontsize", 16);
```

```
% References:
% [1]: http://www.eecs.tufts.edu/~khan/Courses/Spring2013/EE194/Lecs/Lec6and7.
    pdf
% [2]: http://ctms.engin.umich.edu/CTMS/index.php?example=Introduction&section=
    ControlStateSpace#23
% [3]: http://cse.lab.imtlucca.it/~bemporad/teaching/ac/pdf/06b-estimator.pdf
```