Graphs

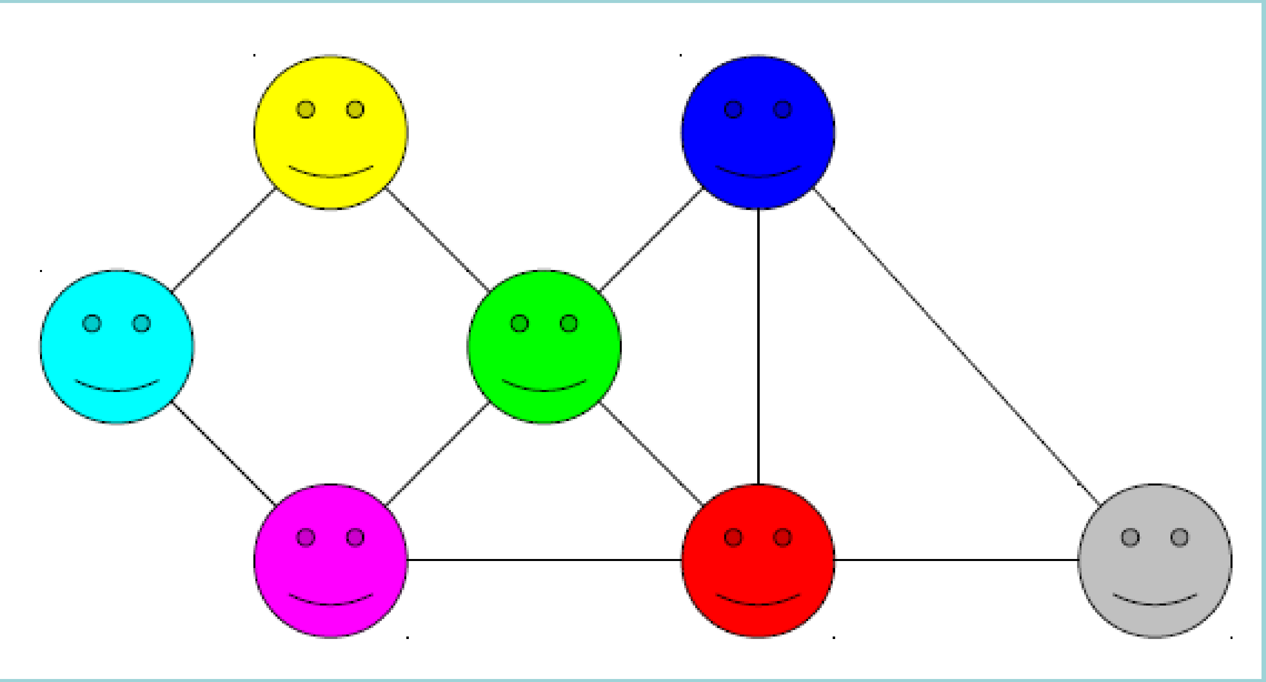
Objectives

- Learn about graphs
- Become familiar with the basic terminology of graph theory
- Discover how to represent graphs in computer memory
- Examine and implement various graph traversal algorithms
- Learn how to implement a shortest path algorithm
- Examine and implement the minimum spanning tree algorithm
- Explore topological sort
- Learn how to find Euler circuits in a graph

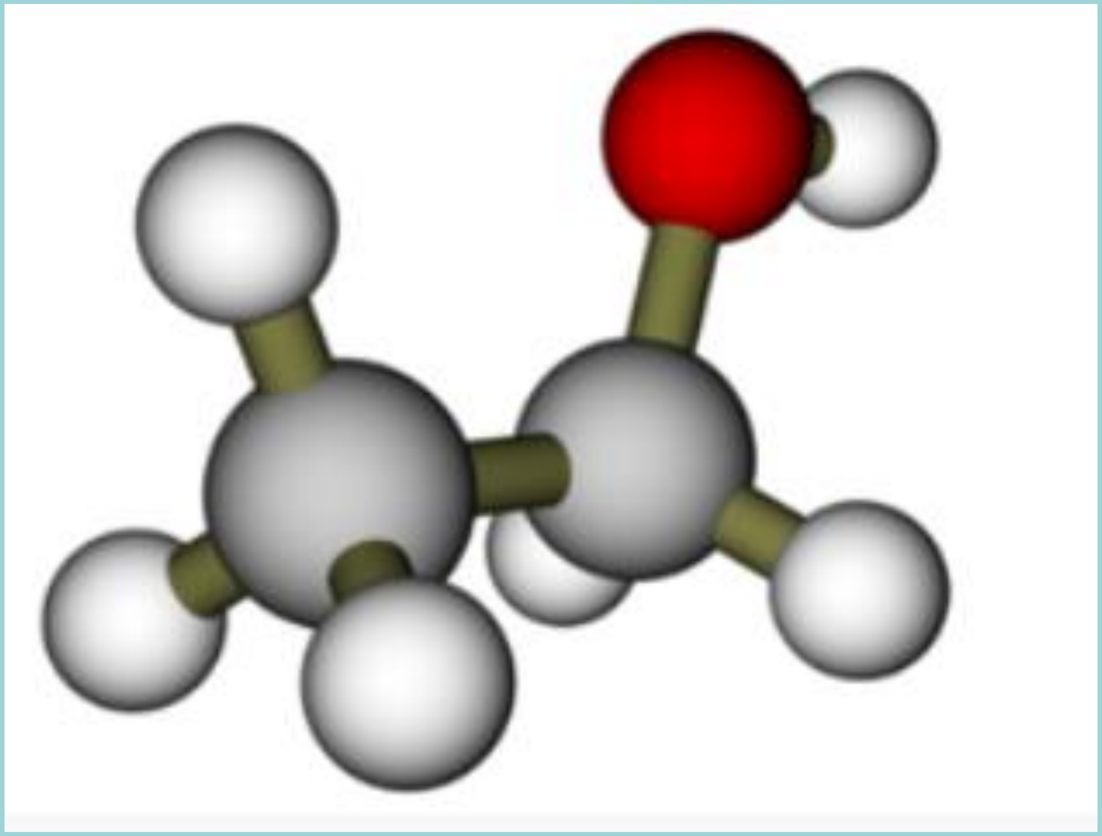
Introduction and Motivation

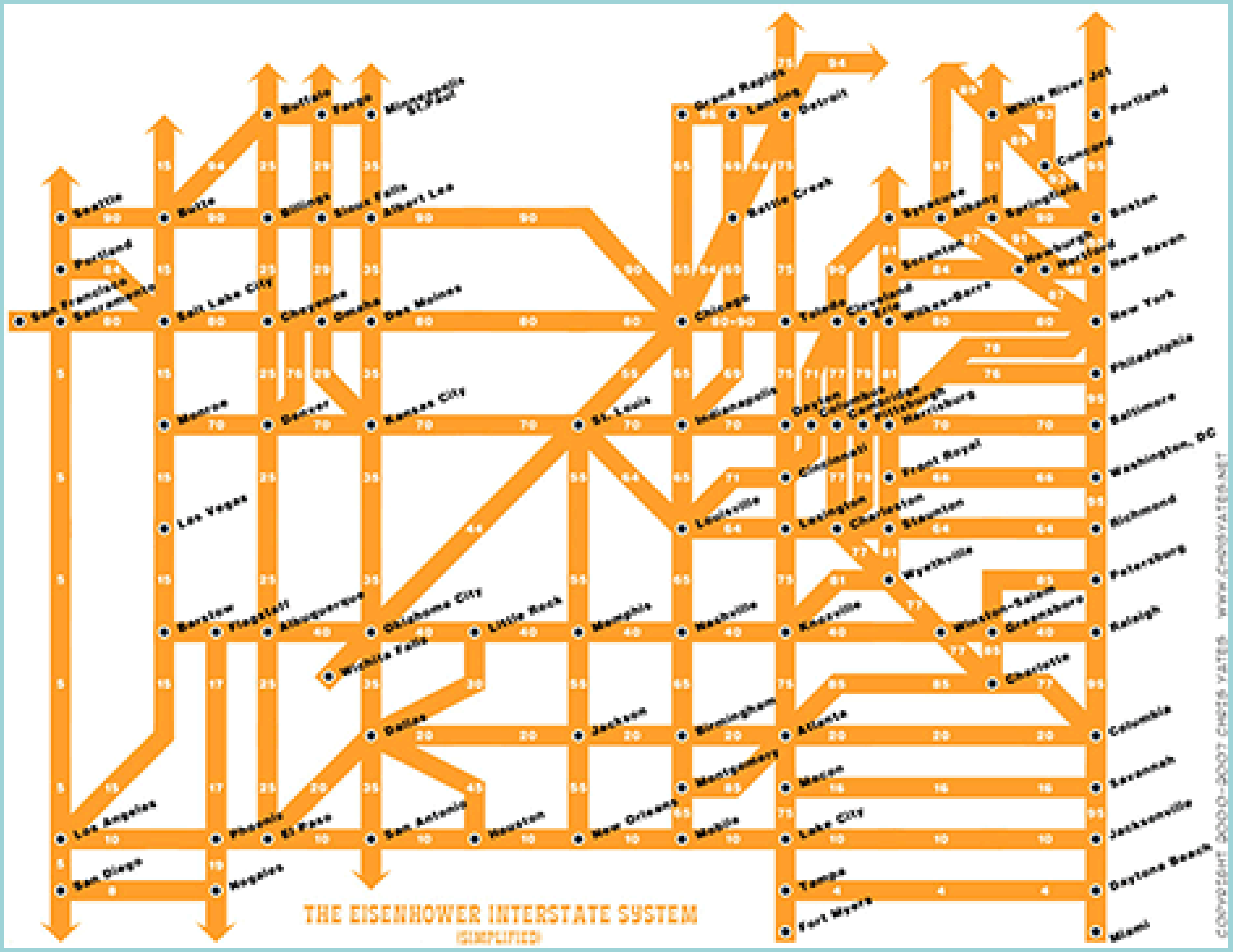
- Although trees are quite flexible, they have an inherent limitation in that they can only express hierarchical structures
- Fortunately, we can generalize a tree to form a ***graph***, in which this limitation is removed
- Informally, a graph is a collection of nodes and the connections between them
- Figure in next slide illustrates some examples of graphs; notice there is typically no limitation on the number of vertices or edges
- Consequently, graphs are extremely versatile and applicable to a wide variety of situations
- Graph theory has developed into a sophisticated field of study since its origins in the early 1700s

A Social Network



Chemical Bonds

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Graph Definitions and Notations

Graph Definitions and Notations

- Borrow definitions, terminology from set theory
- Subset
  - Set *Y* is a subset of *X*: $Y \subseteq X$
    - If every element of *Y* is also an element of *X*
- Intersection of sets *A* and *B*: $A \cap B$
  - Set of all elements that are in *A* and *B*
- Union of sets *A* and *B*: $A \cup B$
  - Set of all elements in *A* or in *B*
- Cartesian product: $A \times B$
  - Set of all ordered pairs of elements of *A* and *B*

Graph Definitions and Notations (cont'd.)

- Graph *G* pair $G = (V, E)$,
  - Where *V* is a finite nonempty set, called the set of vertices of *G*, and $E \subseteq V \times V$
    - Elements of *E* are pairs of elements of *V*.
  - *E* is called set of edges of *G*
    - *G* called trivial if it has only one vertex
- Graph *H* called subgraph of *G*
  - If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
  - Every vertex of *H*: vertex of *G*
  - Every edge in *H*: edge in *G*

Graph Definitions and Notations (cont'd.)

- A graph can be shown pictorially. The vertices are drawn as circles, and a label inside the circle represents the vertex.
- In an undirected graph, the edges are drawn using lines. Elements in set of edges of graph *G*: *un*ordered
- In a directed graph (digraph), the edges are drawn using arrows. Elements in set of edges of graph *G*: ordered

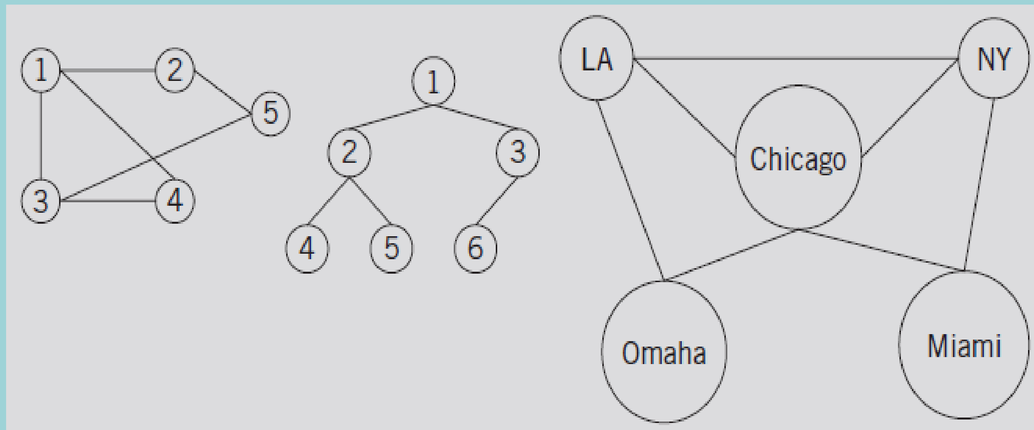


FIGURE 12-3 Various undirected graphs

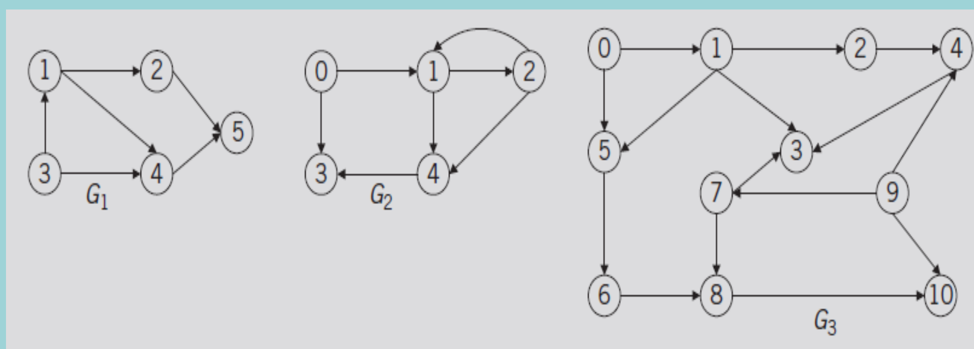


FIGURE 12-4 Various directed graphs

$V(G_1) = \{1, 2, 3, 4, 5\}$

$V(G_2) = \{0, 1, 2, 3, 4\}$

$V(G_3) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$E(G_1) = \{(1, 2), (1, 4), (2, 5), (3, 1), (3, 4), (4, 5)\}$

$E(G_2) = \{(0, 1), (0, 3), (1, 2), (1, 4), (2, 1), (2, 4), (4, 3)\}$

$E(G_3) = \{(0, 1), (0, 5), (1, 2), (1, 3), (1, 5), (2, 4), (4, 3), (5, 6), (6, 8), (7, 3), (7, 8), (8, 10), (9, 4), (9, 7), (9, 10)\}$

Graph Definitions and Notations (cont'd.)

- Let u and v be two vertices in G
  - u and v adjacent
    - If edge from one to the other exists: $(u, v) \in E$
- Loop
  - Edge incident on a single vertex
- e_1 and e_2 called parallel edges
  - If two edges e_1 and e_2 associate with same pair of vertices $\{u, v\}$
- Simple graph
  - No loops, no parallel edges

Graph Definitions and Notations (cont'd.)

- Let $e = (u, v)$ be an edge in G
  - Edge e is incident on the vertices u and v
  - Degree of u written deg(u) or d(u)
    - Number of edges incident with u
- Each loop on vertex u
  - Contributes two to the degree of u
- u is called an even (odd) degree vertex
  - If the degree of u is even (odd)

Graph Definitions and Notations (cont'd.)

- Path from u to v
  - If sequence of vertices $u_1, u_2, \ldots, u_n$ exists
    - Such that $u = u_1$, $u_\text{n} = v$ and $(u_i, u_i + 1)$ is an edge for all $i = 1, 2, \ldots, n - 1$
- Vertices u and v called connected
  - If path from u to v exists
- Simple path
  - All vertices distinct (except possibly first, last)
- Cycle in G
  - Simple path in which first and last vertices are the same

Graph Definitions and Notations (cont'd.)

- G is connected
  - If path from any vertex to any other vertex exists
- Component of G
  - Maximal subset of connected vertices
- Let G be a directed graph and let u and v be two vertices in G
  - If edge from u to v exists: $(u, v) \in E$
    - u is adjacent to v
    - v is adjacent from u

Graph Definitions and Notations (cont’d.)

- Definitions of paths and cycles in *G*
  - Similar to those for undirected graphs
- *G* is strongly connected
  - If any two vertices in *G* are connected

Graph Representation

Graph Representation

- Graphs represented in computer memory
  - Two common ways
    - Adjacency matrices
    - Adjacency lists

Adjacency Matrices

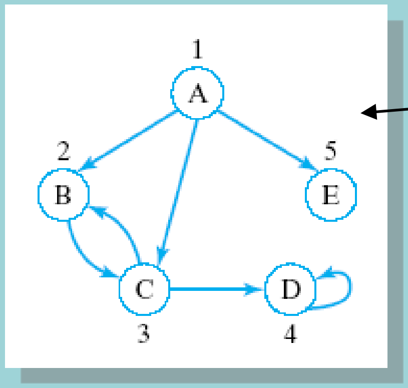
- Let G be a graph with n vertices where n > zero
- Let $V(G) = \{v_1, v_2, \ldots, v_n\}$
  - Adjacency matrix

$$A_G(i,j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

$$A_{G_1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{G_2} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Example: Adjacency matrix

Matrix [5][5] represents A, B, C, D, E vertices



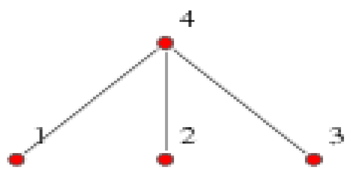
columns ***j***

rows ***i***

	1	*2*	*3*	*4*	*5*
1	0	1	1	0	1
2	0	0	1	0	0
3	0	0	0	1	0
4	0	0	1	0	0
5	0	0	0	0	0

- Entry [1, 5] set to true
- Edge from vertex 1 to vertex 5

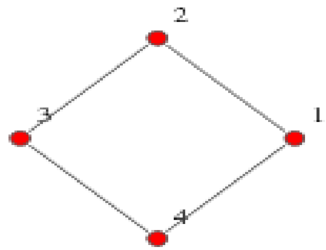
More Examples of Adjacency Matrix



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Claw graph

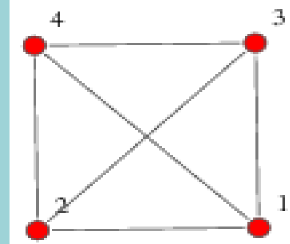
Matrix[4][4]



$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Cycle graph

Matrix[4][4]



$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

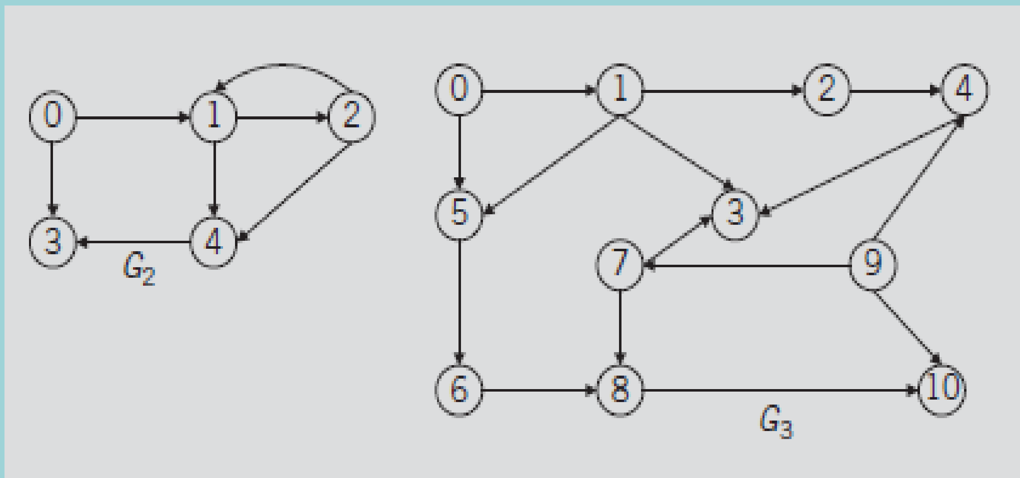
Complete graph

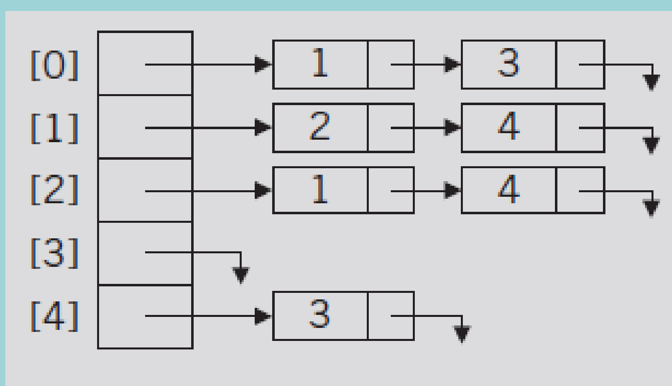
Matrix[4][4]

Adjacency Lists

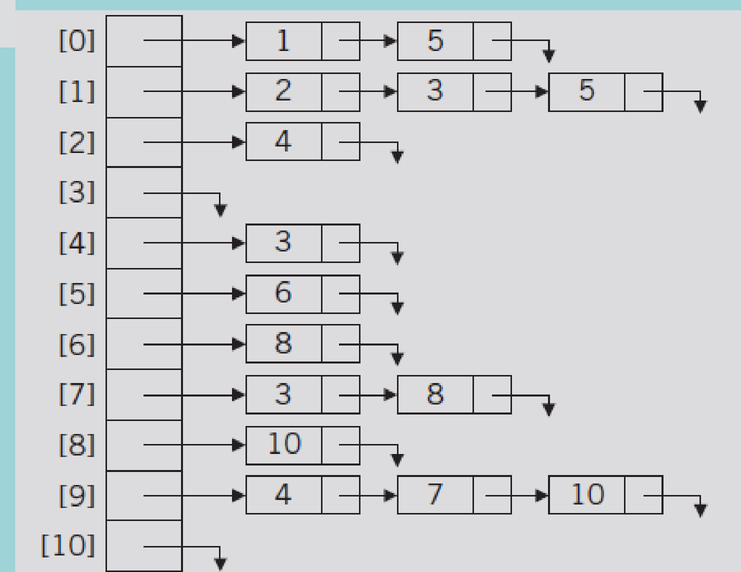
- Given:
  - Graph G with n vertices, where $n > zero$
  - $V(G) = \{v_1, v_2, ..., v_n\}$
- For each vertex v: linked list exists
  - Linked list node contains vertex u: $(v, u) \in E(G)$
- Use array A, of size n, such that $A[i]$
  - Reference variable pointing to first linked list node containing vertices to which v_i adjacent
- Each node has two components: vertex, link
  - Component vertex
    - Contains index of vertex adjacent to vertex i

Adjacency Lists- Examples





Adjacency list of graph *G2*



Adjacency list of graph *G3*

Operations on Graph

Operations on Graphs

- Commonly performed operations
  - Create graph
    - Store graph in computer memory using a particular graph representation
  - Clear graph
    - Makes graph empty
  - Determine if graph is empty
  - Traverse graph
  - Print graph

Operations on Graphs (cont'd.)

- Graph representation in computer memory
  - Depends on specific application
- Use linked list representation of graphs
  - For each vertex *v*
    - Vertices adjacent to *v* (directed graph: called immediate successors)
    - Stored in the linked list associated with *v*
- Managing data in a linked list
  - Use `class unorderedLinkedList`
- Labeling graph vertices
  - Depends on specific application

Graphs as ADTs

Defines a graph as an ADT

– Class specifying basic operations to implement a graph

- Definitions of the functions of the `class graphType`

```
bool graphType::isEmpty() const
{
    return (gSize == 0);
}
```

Graphs as ADTs (cont'd.)

- Function `createGraph`
  - Implementation
    - Depends on how data input into the program
- Function `clearGraph`
  - Empties the graph
    - Deallocates storage occupied by each linked list
    - Sets number of vertices to zero

Graph Traversals

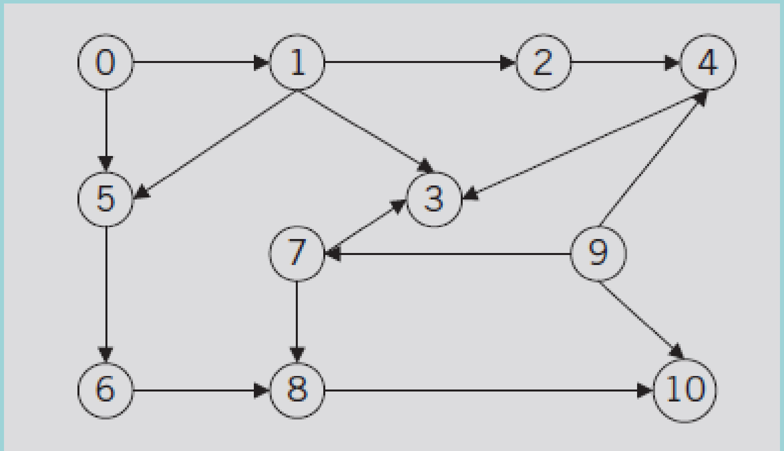
- Processing a graph
  - Requires ability to traverse the graph
- Traversing a graph
  - Similar to traversing a binary tree
    - A bit more complicated
- Two most common graph traversal algorithms
  - Depth first traversal
  - Breadth first traversal

Depth-First Traversal

Depth First Traversal

- Similar to binary tree preorder traversal – Goes as far as possible from a vertex before backing up.
- General algorithm

```
for each vertex, v, in the graph
        if v is not visited
                start the depth first traversal at v
```



Directed graph

Depth First Traversal (cont'd.)

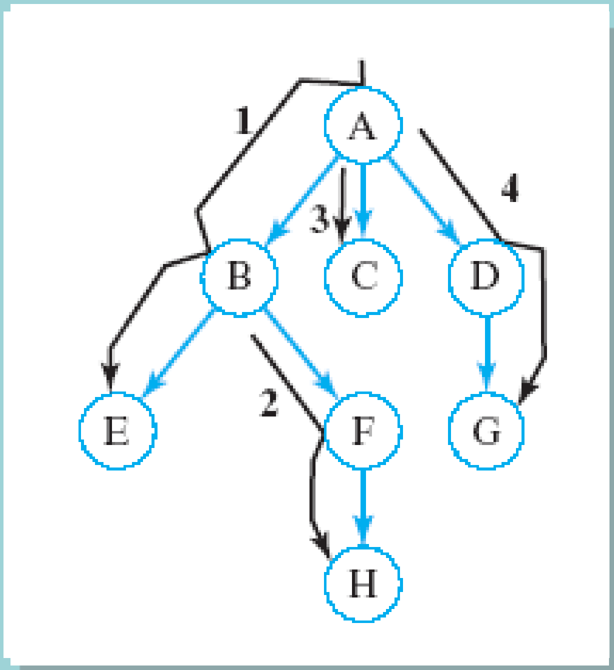
- General algorithm for depth first traversal at a given node *v*
  - Recursive algorithm

```
1. mark node v as visited
2. visit the node
3. for each vertex u adjacent to v
        if u is not visited
           start the depth first traversal at u
```

Depth-First Search

- Start from node 1
- What is a sequence of nodes which would be visited in DFS?

A, B, E, F, H, C, D, G



Depth First Traversal (cont'd.)

- Function `dft` implements algorithm

```
void graphType::dft(int v, bool visited[])
{
    visited[v] = true;
    cout << " " << v << " ";  //visit the vertex

    // declare graphIt to be an iterator to be used
    // to traverse a linked list to which graph[v] points
    linkedListIterator<int> graphIt;

        //for each vertex adjacent to v
    for (graphIt = graph[v].begin(); graphIt != graph[v].end();
                                     ++graphIt)
    {
        // *graphIT returns the label of the vertex
        int w = *graphIt;
        if (!visited[w])
            dft(w, visited);
    } //end while
} //end dft
```

Depth First Traversal (cont'd.)

- Function `depthFirstTraversal`
  - Implements depth first traversal of the graph

```
void graphType::depthFirstTraversal()
{
    bool *visited; //pointer to create the array to keep
                   //track of the visited vertices
    visited = new bool[gSize];

    for (int index = 0; index < gSize; index++)
        visited[index] = false;

        //For each vertex that is not visited, do a depth
        //first traverssal
    for (int index = 0; index < gSize; index++)
        if (!visited[index])
            dft(index,visited);
    delete [] visited;
} //end depthFirstTraversal
```

Depth First Traversal (cont'd.)

- Function `depthFirstTraversal`
  - Performs a depth first traversal of entire graph
- Function `dftAtVertex`
  - Performs a depth first traversal at a given vertex

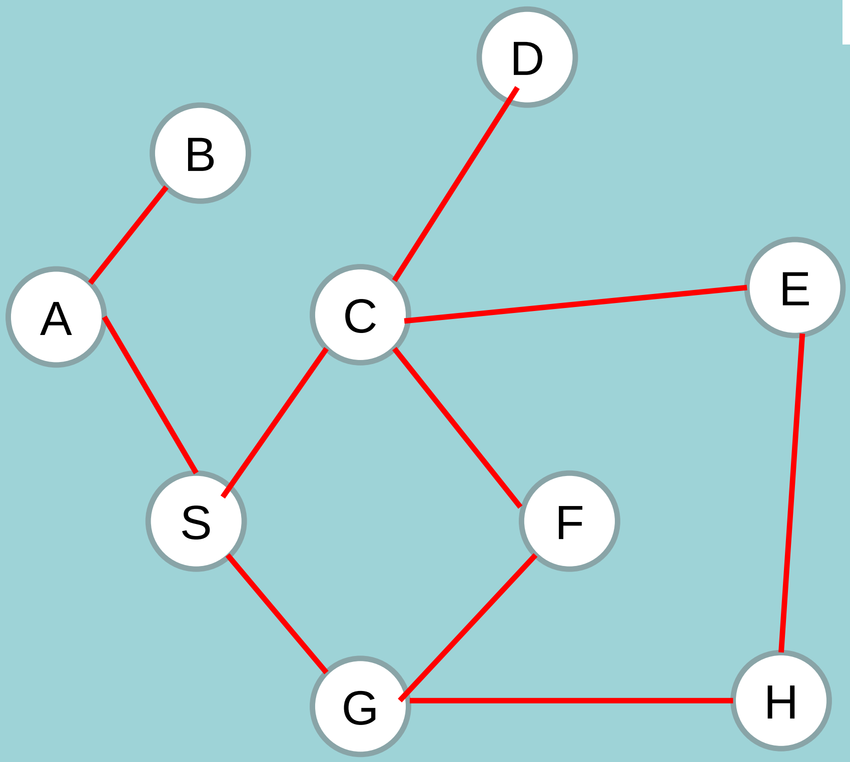
```
void graphType::dftAtVertex(int vertex)
{
    bool *visited;

    visited = new bool[gSize];
    for (int index = 0; index < gSize; index++)
        visited[index] = false;

    dft(vertex, visited);

    delete [] visited;
} // end dftAtVertex
```

Breadth-First Traversal

Breadth First Traversal

- Similar to traversing binary tree level-by-level
  - Visits all vertices adjacent to a vertex before going forward. BFT visits nodes by level.
    - Start from a given vertex of v; visit all neighbors of first neighbor of v
    - Then visit all neighbors of second neighbor x of v …

- Start from node A
- What is a sequence of nodes which would be visited in DFS?



A, B, S, C, D, E, F, G, H

Breadth First Traversal

- General search algorithm
  - Breadth first search algorithm with a queue

```
for each vertex v in the graph
        if v is not visited
                add v to the queue // start bf search at v
Mark v as visited
While the queue is not empty
        remove vertex u from the queue
        retrieve the vertices adjacent to u
        for each vertex w that is adjacent to u
                add w to the queue
                mark w as visited
```

- C++ function implements breadFirstTraversal this algorithm

```cpp
void graphType::breadthFirstTraversal()
{
    linkedQueueType<int> queue;

    bool *visited;
    visited = new bool[gSize];

    for (int ind = 0; ind < gSize; ind++)
        visited[ind] = false;   //initialize the array
                                //visited to false
    linkedListIterator<int> graphIt;
    for (int index = 0; index < gSize; index++)
        if (!visited[index])
        {
            queue.addQueue(index);
            visited[index] = true;
            cout << " " << index << " ";

            while (!queue.isEmptyQueue())
            {
                int u = queue.front();
                queue.deleteQueue();

                for (graphIt = graph[u].begin();
                     graphIt != graph[u].end(); ++graphIt)
                {
                    int w = *graphIt;
                    if (!visited[w])
                    {
                        queue.addQueue(w);
                        visited[w] = true;
                        cout << " " << w << " ";
                    }
                }
            } //end while
        }

    delete [] visited;
} //end breadthFirstTraversal
```

Shortest Path Algorithm

Shortest Path Algorithm

- Weight of the graph
  - Nonnegative real number assigned to the edges connecting to vertices.
- Weighted graphs
  - When a graph uses the weight to represent the distance between two places
- Weight of the path P
  - Given G as a weighted graph with vertices u and v in G and P as a path in G from u to v
    - Sum of the weights of all the edges on the path
- Shortest path: path with the smallest weight

Shortest Path Algorithm (cont'd.)

- Shortest path algorithm space (greedy algorithm)
- Using inheritance, we extend the definition of the `class graphType` by:
  - Extend definition of `class graphType`
  - Adds function `createWeightedGraph` to create graph and weight matrix associated with the graph
  - Call `class weightedGraphType`

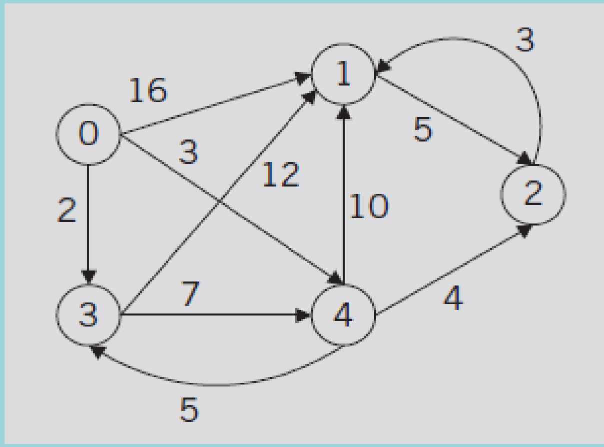
Sample of class weightedGraphType

```
class weightedGraphType: public graphType
{
public:
    void createWeightedGraph();
      //Function to create the graph and the weight matrix.
      //Postcondition: The graph using adjacency lists and
      //    its weight matrix is created.

    void shortestPath(int vertex);
      //Function to determine the weight of a shortest path
      //from vertex, that is, source, to every other vertex
      //in the graph.
      //Postcondition: The weight of the shortest path from vertex
      //    to every other vertex in the graph is determined.

    void printShortestDistance(int vertex);
      //Function to print the shortest weight from the vertex
      //specified by the parameter vertex to every other vertex in
      //the graph.
      //Postcondition: The weight of the shortest path from vertex
      //    to every other vertex in the graph is printed.

    weightedGraphType(int size = 0);
      //Constructor
      //Postcondition: gSize = 0; maxSize = size;
      // graph is an array of pointers to linked lists.
      // weights is a two-dimensional array to store the weights of the edges.
      //    smallestWeight is an array to store the smallest weight
      //    from source to vertices.

    ~weightedGraphType();
      //Destructor
      //The storage occupied by the vertices and the arrays
      //weights and smallestWeight is deallocated.

protected:
    double **weights;   //pointer to create weight matrix
    double *smallestWeight; //pointer to create the array to store
                     //the smallest weight from source to vertices
};
```

Shortest Path

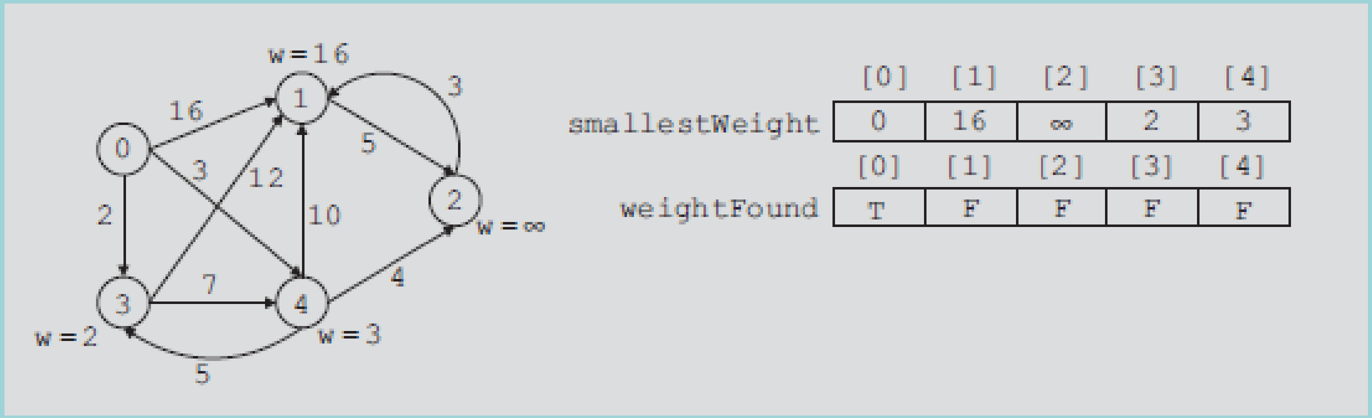
- General algorithm
  1. Initialize array *smallestWeight*
     - smallestWeight[*u*] = weights[*vertex*, *u*]
  2. Set smallestWeight[*vertex*] = zero
  3. Find vertex *v* closest to vertex where shortest path is not determined
  4. Mark *v* as the (next) vertex for which the smallest weight is found
  5. For each vertex *w* in *G*, such that the shortest path from vertex to w has not been determined and an edge (*v*, *w*) exists
     - If weight of the path to *w* via *v* smaller than its current weight
     - Update weight of *w* to the weight of *v* + weight of edge (*v*, *w*)

Shortest Path (cont'd.)

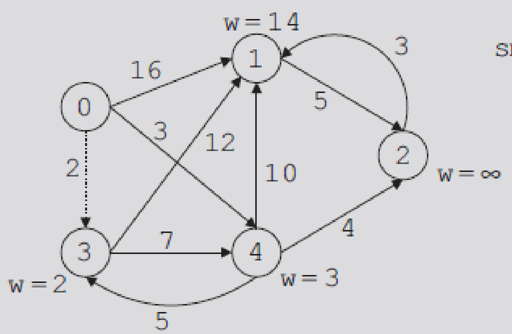


Weighted graph *G*

1. Initialize array *smallestWeight*
   - smallestWeight[*u*] = weights[*vertex*, *u*]
2. Set smallestWeight[*vertex*] = zero

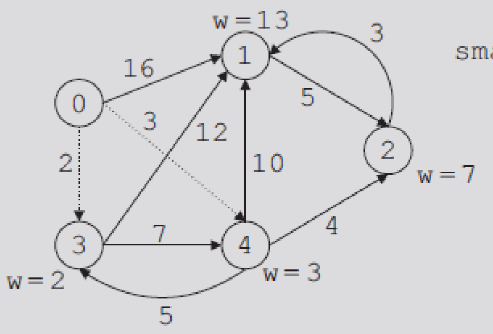


Shortest Path (cont'd.)



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	14	∞	2	3
weightFound	T	F	F	T	F

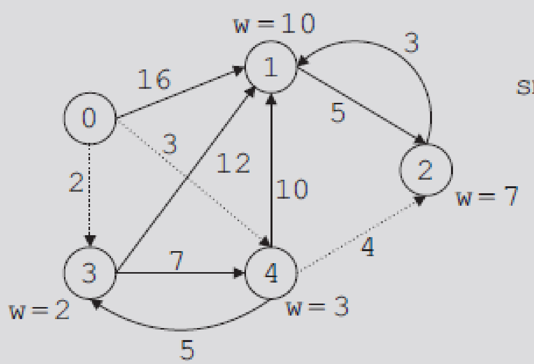
Graph after the first iteration of Steps 3 to 5



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	13	7	2	3
weightFound	T	F	F	T	T

Graph after the second iteration of Steps 3 to 5

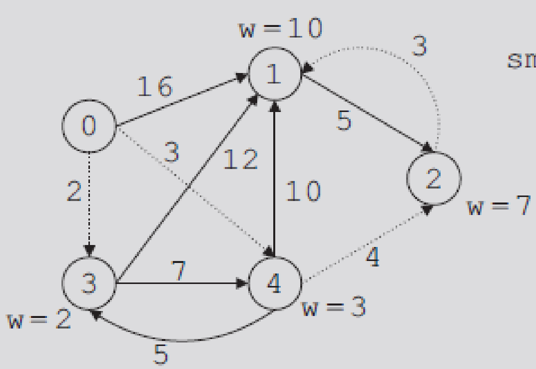
Shortest Path (cont'd.)



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	10	7	2	3

	[0]	[1]	[2]	[3]	[4]
weightFound	T	F	T	T	T

Graph after the third iteration of Steps 3 to 5



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	10	7	2	3

	[0]	[1]	[2]	[3]	[4]
weightFound	T	T	T	T	T

Graph after the fourth iteration of Steps 3 through 5

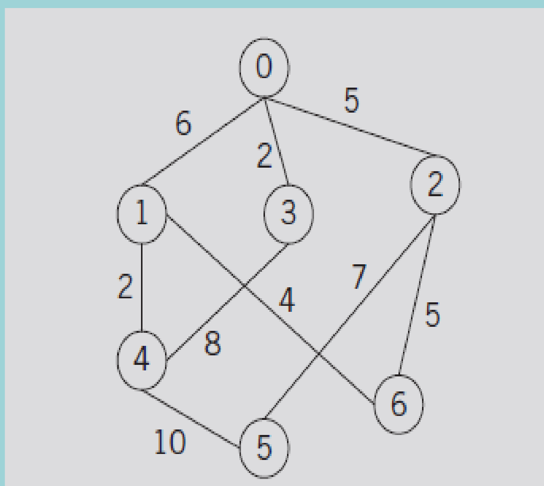
```cpp
void weightedGraphType::shortestPath(int vertex)
{
    for (int j = 0; j < gSize; j++)
        smallestWeight[j] = weights[vertex][j];

    bool *weightFound;
    weightFound = new bool[gSize];

    for (int j = 0; j < gSize; j++)
        weightFound[j] = false;

    weightFound[vertex] = true;
    smallestWeight[vertex] = 0;

    for (int i = 0; i < gSize - 1; i++)
    {
        double minWeight = DBL_MAX;
        int v;

        for (int j = 0; j < gSize; j++)
            if (!weightFound[j])
                if (smallestWeight[j] < minWeight)
                {
                    v = j;
                    minWeight = smallestWeight[v];
                }

        weightFound[v] = true;

        for (int j = 0; j < gSize; j++)
            if (!weightFound[j])
                if (minWeight + weights[v][j] < smallestWeight[j])
                    smallestWeight[j] = minWeight + weights[v][j];
    } //end for
} //end shortestPath
```

C++ function `shortestPath` implements previous algorithm

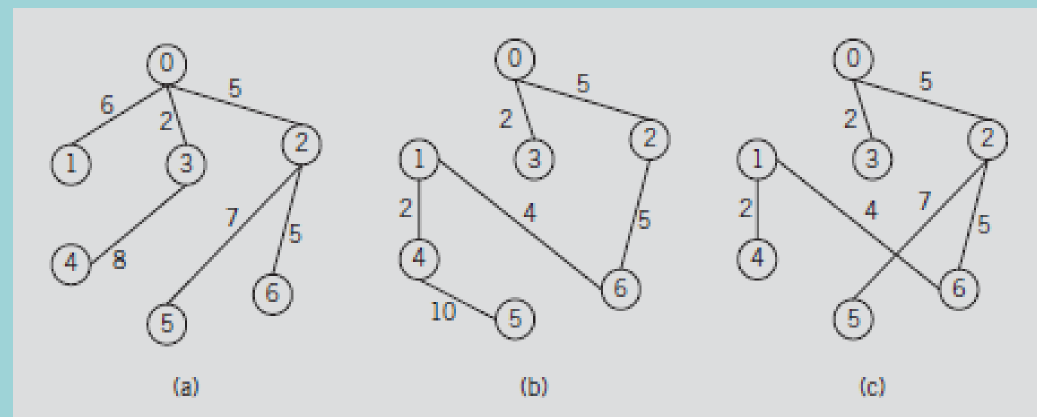
- Records only the weight of the shortest path from the source to a vertex

Minimum Spanning Tree

- Airline connections of a company
  - Between seven cities



Airline connections between cities and the cost factor of maintaining the connections



Possible solutions (a, b and c) if company must shutdown maximum number of connections.
Total cost of (a) = 33
Total cost of (b) = 28
Total cost of c) = 25
The desired solution would be (c) because it gives the lowest cost.
Graphs a, b and c are called spanning trees of the original graph.

Minimum Spanning Tree - Terminology

- Free tree *T*
  - Simple graph
  - If *u* and *v* are two vertices in *T*
    - Unique path from *u* to *v* exists
- Rooted tree
  - Tree with particular vertex designated as a root

Minimum Spanning Tree -Terminology

- Weighted tree T
  - Weight assigned to edges in T
  - Weight denoted by $W(T)$: sum of weights of all the edges in T
- Spanning tree T of graph G
  - T is a subgraph of G such that $V(T) = V(G)$

Minimum Spanning Tree -Terminology

- Spanning tree theorem
  - A graph *G* has a spanning tree if and only if *G* is connected
  - From this theorem, it follows that to determine a spanning tree of a graph
    - Graph must be connected
- Minimum (minimal) spanning tree of *G*
  - Spanning tree with the minimum weight

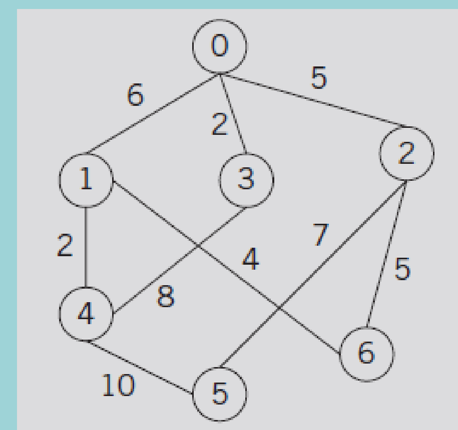
Minimum Spanning Tree (MST) - Algorithm

- Two well-known algorithms for finding a minimum spanning tree of a graph
  - Prim's algorithm
    - Builds the tree iteratively by adding edges until a minimum spanning tree obtained
  - Kruskal's algorithm – not cover in this lecture

MST – Prim's algorithm

- General form of Prim's algorithm

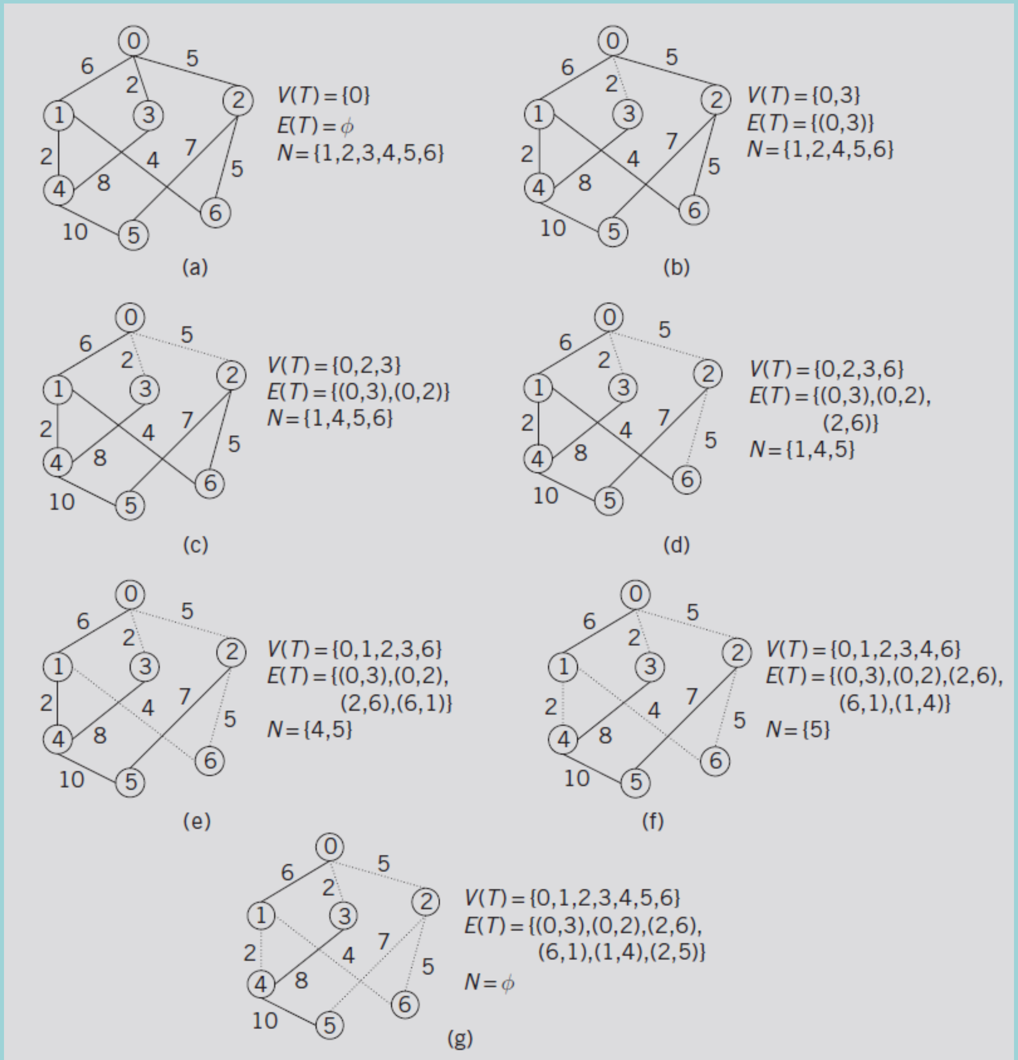
```
1.  Set V(T) = {source}
2.  Set E(T) = empty
3.  for i = 1 to n
    3.1.  minWeight = infinity;
    3.2.  for j = 1 to n
              if v_j is in V(T)
                for k = 1 to n
                 if v_k is not in T and weight[v_j, v_k] < minWeight
                 {
                     endVertex = v_k;
                     edge = (v_j, v_k);
                     minWeight = weight[v_j, v_k];
                 }
    3.3.  V(T) = V(T) ∪ {endVertex};
    3.4.  E(T) = E(T) ∪ {edge};
```



MST - Prim's algorithm Implementation

- `class msTreeType` defines spanning tree as an ADT
- C++ function `minimumSpanning` implementing Prim's algorithm
- Prim's algorithm given in this section: $O(n^3)$
  - Possible to design Prim's algorithm order $O(n^2)$

- See function `printTreeAndWeight` code
- See constructor and destructor code



Graph *G*, *V*(*T*), *E*(*T*), and *N* after Steps 1 and 2 execute

Topological Order

- Topological ordering of $V(G)$
  - Linear ordering $v_{i1}, v_{i2}, \ldots, v_{in}$ of the vertices such that
    - If v_{ij} is a predecessor of v_{ik}, $j \neq k$, $1 \leq j \leq n$, $1 \leq k \leq n$
    - Then v_{ij} precedes v_{ik}, that is, $j < k$ in this linear ordering
- Algorithm topological order
  - Outputs directed graph vertices in topological order
  - Assume graph has no cycles
    - There exists a vertex v in G such that v has no successor
    - There exists a vertex u in G such that u has no predecessor

Topological Order (cont'd.)

- Topological sort algorithm
  - Implemented with the depth first traversal or the breadth first traversal
- Extend `class graphType` definition (using inheritance)
  - Implement breadth first topological ordering algorithm
    - Called `class topologicalOrderType`
  - See code on pages 714-715
    - Illustrating class including functions to implement the topological ordering algorithm

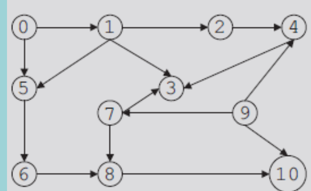
Breadth First Topological Ordering

- General algorithm

1. Create the array `predCount` and initialize it so that `predCount[i]` is the number of predecessors of the vertex v_i.
2. Initialize the queue, say `queue`, to all those vertices v_k so that `predCount[k]` is 0. (Clearly, `queue` is not empty because the graph has no cycles.)
3. `while` the queue is not empty
   - 3.1. Remove the front element, `u`, of the queue.
   - 3.2. Put `u` in the next available position, say `topologicalOrder[topIndex]`, and increment `topIndex`.
   - 3.3. For all the immediate successors `w` of `u`,
     - 3.3.1. Decrement the predecessor count of `w` by `1`.
     - 3.3.2. `if` the predecessor count of `w` is 0, add `w` to `queue`.

Breadth First Topological Ordering (cont'd.)

- `Breadth First Topological order`
  - 0 9 1 7 2 5 4 6 3 8 10



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	1	1	3	2	2	1	1	2	0	2

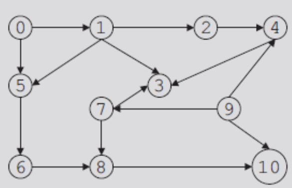
topologicalOrder

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

queue

0, 9

Arrays `predCount`, `topologicalOrder`, and `queue` after Steps 1 and 2 execute



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	1	3	2	1	1	1	2	0	2

topologicalOrder

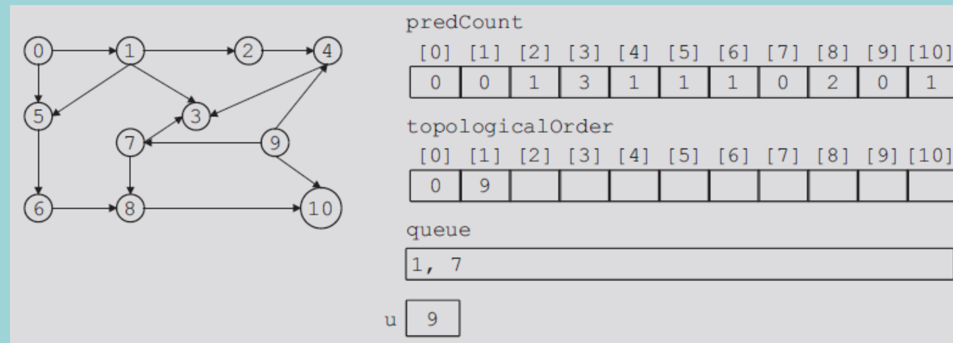
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0										

queue

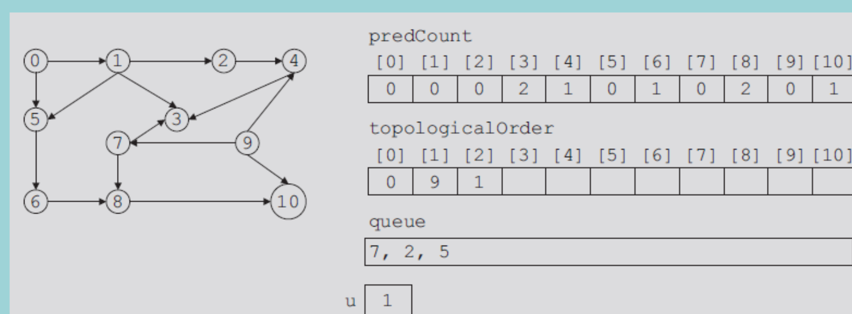
9, 1

u 0

Arrays `predCount`, `topologicalOrder`, and `queue` after the first iteration of Step 3



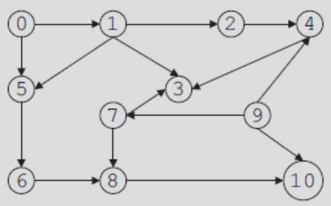
Arrays `predCount`, `topologicalOrder`, and `queue` after the second iteration of Step 3



Arrays `predCount`, `topologicalOrder`, and `queue` after the third iteration of Step 3

Breadth First Topological Ordering (cont'd.)

- See code on pages 718-719
  - Function implementing breadth first topological ordering algorithm



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	0	0	0	0	0	0	0	0	0

topologicalOrder

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	9	1	7	2	5	4	6	3	8	10

queue

Arrays `predCount`, `topologicalOrder`, and `queue` after Step 3 executes

Summary

- Many types of graphs
  - Directed, undirected, subgraph, weighted
- Graph theory borrows set theory notation
- Graph representation in memory
  - Adjacency matrices, adjacency lists
- Graph traversal
  - Depth first, breadth first
- Shortest path algorithm
- Prim's algorithm