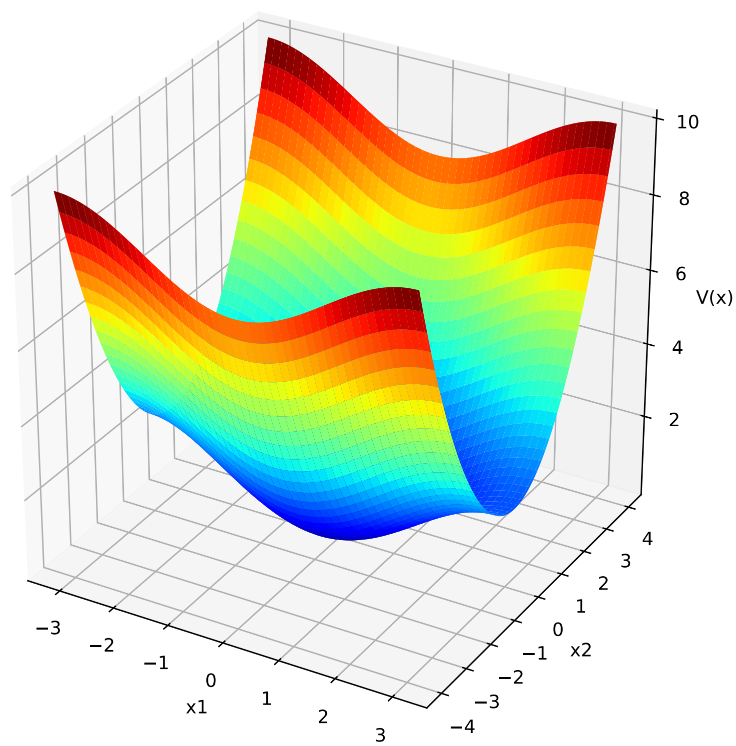
Notes on Applied Nonlinear Control



Joshua Saunders

August 14, 2018

Contents

[1] The material for these notes came from Slotine's text and online lectures [1, 2]

List of Figures

Chapter 1

Phase Plane Analysis

1.1 Introduction[1]

Used with second-order systems. Generate motion trajectories with different initial conditions. Good for

- visualization of nonlinear system
  - see what happens with various initial conditions with out solving differential equations
- small or smooth trajectories to strong nonlinearities and to "hard" trajectories
- control systems can be approximated as second-order systems

Disadvantage

- restricted to first- or second-order systems

1.2 Concepts of Phase Plane Analysis

1.2.1 Phase Portraits

Graphical study of system given by

$$\begin{aligned} \dot{x}_1 &= f_1\left(x_1, x_2\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2\right) \end{aligned} \tag{1.1}$$

where $\mathbf{x}\left(t\right)$ is a solution to Equation 1.1 with initial conditions $\mathbf{x}\left(0\right) = \mathbf{x}_0$ and is represented as a curve on the phase plane varying from $t \in [0, \infty)$ and is called a *phase plane trajectory*. A family of these curves (solutions with varying initial values) is a *phase portrait*.

[1] The material for these notes came from Slotine's text and online lectures [1, 2]

1.2.2 Singular Points

Singular point: equilibrium point in the phase plane (point where system states can stay forever, or $\dot{\mathbf{x}} = 0$). Linear systems usually only have one equilibrium point (or a *continuous* set of singular points). The slope of the singular point is given by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{1.2}$$

which is $\frac{0}{0}$ at a singular point and is therefore indeterminant (hence the name *singular*). Many trajectories may intersect at singular points and much information is gleaned from them as well (a linear system's stability is determined by its singular point).

1.2.3 Symmetry in Phase Plane Portraits

Phase portraits may have symmetry properties that are known *a priori*.

- Symmetry about x_1 axis: $f(x_1, x_2) = f(x_1, -x_2)$
- Symmetry about x_2 axis: $f(x_1, x_2) = -f(-x_1, x_2)$
- Symmetry about origin: $f(x_1, x_2) = -f(-x_1, -x_2)$

1.3 Constructing Phase Portraits

1.3.1 Analytical Method

Solve Equation 1.1 for x_1 and x_2 as functions of time

$$x_1 = g_1(t) \qquad x_2 = g_2(t)$$

Or, directly eliminate the time variable using Equation 1.2 and then obtaining a functional relation between x_1 and x_2.

1.3.2 Method of Isoclines

Isoclines are the set of all points (*locus*) with a given slope defined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha \tag{1.3}$$

Steps for generating the phase portrait

1. Obtain a field of directions of tangents to the trajectories
2. Form phase plane trajectories

1.4 Determining Time from Phase Portraits

1.4.1 Obtaining Time From $\Delta t \approx \Delta x / \dot{x}$

$\Delta x \approx \dot{x} \Delta t$ for a short time Δt

- use average value of $\dot{x}$ (*velocity*) during Δt for a finite magnitude Δx
- length of *time* corresponding to Δx is $\Delta t \approx \frac{\Delta x}{\dot{x}}$
- divide trajectory into small segments and add up the results
- each segment Δx may not be equally spaced

1.4.2 Obtaining Time from $t = \int (1/\dot{x})\, dx$

$\dot{x} = \frac{dx}{dt} \Longrightarrow dt = \frac{dx}{\dot{x}}$ therefore

$$t - t_0 = \int_{x_0}^{x} (1/\dot{x})\, dx$$

where x corresponds to time t and x_0 corresponds to time t_0.

1.5 Phase Plane Analysis of Linear Systems

The general for second-order linear system is

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2 \end{aligned} \tag{1.4}$$

which can be transformed into a scalar second-order system

$$b\dot{x}_2 = \; bcx_1 + d\,(\dot{x}_1 - ax_1)$$

and

$$\ddot{x}_1 = \; (a + d)\,\dot{x}_1 + (cb - ad)\,x_1$$

Consider the second-order linear system described by

$$\ddot{x} + a\dot{x} + bx = 0 \tag{1.5}$$

with the time history given by

$$x\,(t) = \begin{cases} k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} & \text{for } \lambda_1 \neq \lambda_2 \\ k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} & \text{for } \lambda_1 = \lambda_2 \end{cases} \tag{1.6}$$

and where λ_1 and λ_2 are solutions to the characteristic equation (poles)

$$s2 + as + b = (s - \lambda_1)\,(s - \lambda_2) = \; 0$$

And, finally

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4a}\right)$$
$$\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4a}\right)$$

The following cases for λ_1 and λ_2 are

1. λ_1 and λ_2 are both real and have the same sign
   (a) ***stable node*** if both negative
   (b) ***unstable node*** if both positive
2. λ_1 and λ_2 are both real and have opposing signs (***saddle point***)
3. λ_1 and λ_2 are complex conjugates with non-zero real parts
   (a) ***stable focus*** if real parts are negative
   (b) ***unstable node*** if real parts are positive
4. λ_1 and λ_2 are complex conjugates with real parts equal to zero (***center point***)

1.6 Phase Plane Analysis of Nonlinear Systems

Local behavior of nonlinear systems can be approximated by linear systems, but nonlinear systems exhibit much more complicated behavior (multiple singular points and *limit cycles*).

1.6.1 Local Behavior of Nonlinear Systems

Use a Taylor expansion of the system at singular point and at the origin (the system can always be shifted to the origin) the higher order terms can be neglected. The local behavior can now be analyzed using the cases given by linear systems.

1.6.2 Limit Cycles

Limit cycles are isolated closed curves.

- *Closed*: indicates periodicity of curve
- *Isolated*: indicates limiting nature of curve (nearby curves converge/diverge from it)

Classes of limit cycles

- ***Stable***: all nearby trajectories converge to it as $t \to \infty$
- ***Unstable***: all nearby trajectories diverge from it as $t \to \infty$
- ***Semi-stable***: some nearby trajectories converge to it and some diverge from it as $t \to \infty$

1.7 Existence of Limit Cycles

Note: the following theorems do not apply to systems of order greater than two.

1.7.1 Index Theorem (Poincare)

This describes a relation ship between the existence of a limit cycle and the number of singular points it encloses.

Theorem 1. *If a limit cycle exists in the second-order autonomous system Equation 1.1, then* $N = S + 1$.

N is the number of nodes, centers, and foci enclosed by a limit cycle and S is the number of enclosed saddle points. A consequence of this theorem is that a limit cycle must enclose at least one equilibrium point.

1.7.2 Poincare-Bendixson

This theorem is concerned about the asymptotic properties of trajectories of Equation 1.1.

Theorem 2. *If a trajectory of the second-order autonomous system remains in a finite region* $\mathbf{\Omega}$*, then one of the following is true:*

1. *the trajectory goes to an equilibrium point*
2. *the trajectory tends to an asymptotically stable limit cycle*
3. *the trajectory is itself a limit cycle*

1.7.3 Bendixson

This theorem provides a sufficient condition for the non-existence of limit cycles.

Theorem 3. *For the nonlinear system Equation 1.1, no limit cycle can exist in a region* $\mathbf{\Omega}$ *of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Chapter 2

Fundamentals of Lyapunov Theory

2.1 Nonlinear Systems and Equilibrium Points

A nonlinear system can be represented by a differential equation such as the on given by Equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right) \tag{2.1}$$

where $\mathbf{x}$ is the state vector and $\mathbf{f}$ is a nonlinear vector function, both of size $n \times 1$ (n is the number of states of $\mathbf{x}$). It should be noted, however, that Equation 2.1 does not explicitly depend on the control variable $\boldsymbol{u}$ although it still can represent the closed-loop dynamics of a control system if the input $\boldsymbol{u}$ is dependent on $\mathbf{x}$ and t. Equation 2.1 can be rewritten to explicitly depend on the control input as shown in Equation 2.2 and the closed-loop dynamics given in Equation 2.3 ,

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \boldsymbol{u}, t\right) \tag{2.2}$$

$$\dot{\mathbf{x}} = \mathbf{f}\left[\mathbf{x}, \boldsymbol{g}\left(\mathbf{x}, t\right), t\right] \tag{2.3}$$

where $\boldsymbol{u} = \boldsymbol{g}\left(\mathbf{x}, t\right)$. The dynamics of linear systems can be written in the form given by Equation 2.4

$$\dot{\mathbf{x}} = \mathbf{A}\left(t\right)\mathbf{x} \tag{2.4}$$

with $\mathbf{A}\left(t\right)$ being an $n \times n$ matrix.

2.1.1 Autonomous and Non-Autonomous Systems

Autonomous and non-autonomous systems are the generalized versions of time-invariant and time-varying systems found in linear systems. A system is autonomous if it does not explicitly depend on time (otherwise it is non-autonomous) as shown in Equation 2.5

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right). \tag{2.5}$$

All systems are non-autonomous, because no dynamic characteristics are time-invariant. It's also important to note that the above definition is made on the closed-loop dynamics for control systems. The non-autonomous nature of the system can either be due to the time-varying nature of the plant or the control law (the book gives an example of $u = -x^2 \sin\left(t\right)$). Fundamentally, the difference between autonomous and non-autonomous systems is the dependence on the initial time of the state trajectory for non-autonomous systems. This means that, for non-autonomous systems, the initial time must be considered for stability.

2.1.2 Equilibrium Points

Definition 1. $\boldsymbol{x}^*$ is an *equilibrium point* (or *state*) if once $\boldsymbol{x}(t) = \boldsymbol{x}^*$, $\boldsymbol{x}(t)$ stays at $\boldsymbol{x}^*$ for all remaining time.

To find $\boldsymbol{x}^*$, Equation 2.6 is used

$$\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}^*) . \tag{2.6}$$

Linear time-invariant (LTI) systems have one equilibrium point at the origin if the matrix $\boldsymbol{A}$ is nonsingular. LTI systems have infinite equilibrium points in the null-space of $\boldsymbol{A}$ if it is singular. Nonlinear systems can have several to infinitely many equilibrium points. Equilibrium points for linear and nonlinear systems can be transformed to the origin which can make analysis easier.

2.1.3 Nominal Motion

Sometimes we are concerned with the stability of a system to its original trajectory if subjected to slight perturbation (motion stability). This motion stability can be transformed to an equivalent stability problem about an equilibrium point, but the equivalent system will then be autonomous.

Let $\boldsymbol{x}^*(t)$ be a solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with initial value $\boldsymbol{x}^*(t) = \boldsymbol{x}_0$ and $\boldsymbol{x}(0) = \boldsymbol{x}_0 + \delta\boldsymbol{x}_0$ be a perturbation of the initial condition. The error associated with the variation of the motion error is given in Equation 2.7

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*(t) . \tag{2.7}$$

Both $\boldsymbol{x}^*(t)$ and $\boldsymbol{x}(t)$ are solutions to Equation 2.5 and therefore we have

$$\begin{aligned} \dot{\boldsymbol{x}}^* &= \boldsymbol{f}(\boldsymbol{x}^*) \qquad & \boldsymbol{x}(0) &= \boldsymbol{x}_0 \\ \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}^*) \qquad & \boldsymbol{x}(0) &= \boldsymbol{x}_0 + \delta\boldsymbol{x}_0 \end{aligned}$$

then $\boldsymbol{e}(t)$ satisfies the non-autonomous differential equation found in Equation

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{e}, t) - \boldsymbol{f}(\boldsymbol{x}^*, t) = \boldsymbol{g}(\boldsymbol{e}, t) \tag{2.8}$$

with initial conditions $\boldsymbol{e}(0) = \delta\boldsymbol{x}_0$. Because $\boldsymbol{g}(\mathbf{0}, t) = \mathbf{0}$, the equilibrium of the new system is at the origin and is non-autonomous due to the presence of $\boldsymbol{x}^*(t)$.

2.2 Concepts of Stability

Nonlinear systems may exhibit much more complex and exotic behavior than linear systems and, therefore, different stability concepts are needed to describe them. More concepts to describe this behavior such as asymptotic stability, exponential stability, and global asymptotic stability will be introduced for for autonomous systems.

Let $\boldsymbol{B}_R$ be a spherical region defined by $\|\boldsymbol{x}\| < R$ in state-space and $\boldsymbol{S}_R$ be the sphere itself defined by $\|\boldsymbol{x}\| = R$.

2.2.1 Stability and Instability

Definition 2. The equilibrium state $\boldsymbol{x} = \mathbf{0}$ is *stable* if, for any $R > 0$, $\exists r > 0$ such that if $\|\boldsymbol{x}(0)\| < r$, then $\|\boldsymbol{x}(t)\| < R \; \forall t \in [0, \infty)$. Otherwise, $\boldsymbol{x} = \mathbf{0}$ is *unstable.*

Stability (a.k.a. *Lyapunov stability*) means that the system trajectory can be kept arbitrarily close (R) to the origin by starting sufficiently close (r) to it. Definition 2 can be written as shown in Equation 2.9

$$\forall R > 0, \exists r > 0, \; \|\boldsymbol{x}(0)\| < r \quad \Rightarrow \quad \forall t \geq 0, \; \|\boldsymbol{x}(t)\| < R \tag{2.9}$$

or, as in Equation 2.10

$$\forall R > 0, \exists r > 0, \; \boldsymbol{x}(0) \in \boldsymbol{B}_r \quad \Rightarrow \quad \forall t \geq 0, \; \boldsymbol{x}(t) \in \boldsymbol{B}_R. \tag{2.10}$$

On the other hand, an *unstable* equilibrium is one if $\exists \boldsymbol{B}_R$ such that $\forall r > 0$, the trajectory can start in $\boldsymbol{B}_r$ and leave $\boldsymbol{B}_R$.

2.2.2 Asymptotic Stability and Exponential Stability

For many applications, Lyapunov stability may not be sufficient. For example, it is desired to not only know whether or not a satellite will maintain its attitude within a given range, but to also know if it will go back to its original attitude when disturbed. Figure 2.1 shows three types of stability that equilibrium points can be: trajectory 1 is unstable, trajectory 2 is asymptotically stable, and trajectory 3 is marginally stable where $\boldsymbol{S}_r$ and $\boldsymbol{S}_R$ are the spheres with radius r and R, respectively.

Definition 3. An equilibrium point $\mathbf{0}$ is *asymptotically stable* if it's stable and $\exists r > 0$ such that $\|\boldsymbol{x}(0)\| < r$ implies that $\boldsymbol{x}(t) \to \mathbf{0}$ as $t \to \infty$.

Basically, what Definition 3 is saying that the equilibrium is stable and states that start close to the equilibrium $\mathbf{0}$ will converge to $\mathbf{0}$ as time goes to infinity. In Figure 2.1, trajectories 2 starts within the sphere $\boldsymbol{S}_r$ and eventually converges to the equilibrium labeled $\mathbf{0}$. This region $\boldsymbol{S}_r$ is called the *domain of attraction* of the equilibrium point $\mathbf{0}$. The domain of attraction is the largest region such that all trajectories initiated in it will eventually converge to $\mathbf{0}$. A *marginally stable* equilibrium point is a Lyapunov stable equilibrium point that is not asymptotically stable (e.g., trajectory 3). If $r = \infty$ then it is called *global* asymptotic stability (see Subsection 2.2.3).

Definition 4. The equilibrium point $\mathbf{0}$ is *exponentially stable* if there exists two strictly positive numbers α and λ such that

$$\forall t > 0, \quad \|\boldsymbol{x}(t)\| \leq \alpha \|\boldsymbol{x}(0)\| \, \mathrm{e}^{-\lambda t} \tag{2.11}$$

in some ball $\boldsymbol{B}_r$ around the origin.

In other words, for an exponentially stable system the state vector converges to the origin faster than an exponential function (λ is the rate of exponential convergence). With exponential stability, not only do you know where the state goes (to the equilibrium point) but you also know how fast it goes.

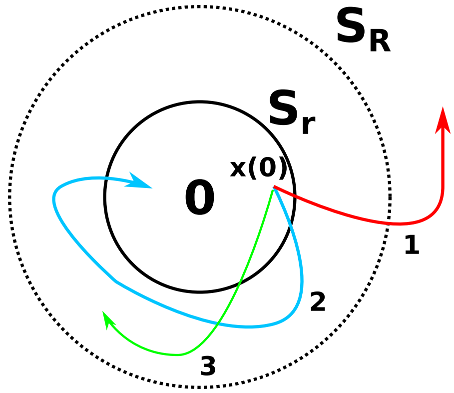


Figure 2.1: Examples of different types of stability

2.2.3 Local and Global Stability

So far we've been discussing the local behavior of systems which is concerned with how the state evolves after starting near the equilibrium point. But what about if the state doesn't start near the equilibrium point? Local stability does little to tell us how the system will behave in this condition.

Definition 5. If asymptotic or exponential stability holds for *any* initial states, the equilibrium point is asymptotically (or exponentially) stable *in the large* (or, *globally*).

LTI systems are either asymptotically stable, marginally stable, or unstable. This is because linear asymptotic stability is both asymptotic and exponential. If the system is unstable it blows up.

2.3 Linearization and Local Stability

Lyanpunov's linearization method is concerned with the local stability of a nonlinear system is the fundamental justification for using linear control techniques. It shows that stable the stability of the original physical system locally is guaranteed by linear control.

Let's assume that the autonomous system in Equation 2.5 is continuously differentiable. The system can then be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)_{\boldsymbol{x}=\boldsymbol{0}} \boldsymbol{x} + \boldsymbol{f}_{\text{h.o.t.}}(\boldsymbol{x}) \tag{2.12}$$

where $\boldsymbol{f}_{\text{h.o.t.}}$ is the higher-order-terms in $\boldsymbol{x}$ of the Taylor expansion. The Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x} = \boldsymbol{0}$ can be written as a constant matrix $\boldsymbol{A}$ (shown in Equation 2.13) giving the linearized system about the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$, found in Equation 2.14

$$\boldsymbol{A} = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)_{\boldsymbol{x}=\boldsymbol{0}} \tag{2.13}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{2.14}$$

which, of course, is ignoring the higher-order-terms $\boldsymbol{f}_{\text{h.o.t.}}$.

Theorem 4. *Lyapunov's linearization method*

- *If the linearized system is strictly stable (i.e., all the eigenvalues of $\boldsymbol{A}$ are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*

- *If the linearized system is unstable (i.e., if at least on eigenvalue of $\boldsymbol{A}$ is in the right-half complex plane), then the equilibrium point is unstable (for the nonlinear system).*

- *If the linearized system is marginally stable (i.e., all eigenvalues of $\boldsymbol{A}$ are in the left-half complex plane but at least one lies on the $j\omega$ axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system).*

2.4 Lyapunov's Direct Method

If the total energy of a system continuously dissipates then it must settle to an equilibrium point. This applies to both linear and nonlinear systems. Therefore, stability of a system may be concluded by examining a single scalar function, $V(\boldsymbol{x})$. There are relations between mechanical energy and stability:

- zero energy corresponds to an equilibrium point
- asymptotic stability corresponds to mechanical energy going to zero
- instability corresponds to the growth of mechanical energy

The basic method is to generate a scalar function, $V(\boldsymbol{x})$, that is "energy-like" for the dynamic system and examine the time-variation ($\dot{V}$) of that function.

2.4.1 Positive Definite Functions and Lyapunov Functions

The scalar function $V(\boldsymbol{x})$ has two properties

1. it is strictly positive unless $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are both zero (positive definite)
2. it is monotonically increasing

Definition 6. A scalar continuous function $V(\boldsymbol{x})$ is *locally positive definite* if $V(\mathbf{0}) = 0$ and, in a ball $\boldsymbol{B}_{R_0}$

$$\boldsymbol{x} \neq \mathbf{0} \qquad \Rightarrow \qquad V(\boldsymbol{x}) > 0 \tag{2.15}$$

If $V(\mathbf{0}) = 0$ and the above property holds over the whole state-space then $V(\boldsymbol{x})$ is *globally positive definite.*

Definition 6 implies that $V(\boldsymbol{x})$ has a unique minimum at the origin $\mathbf{0}$.

An example of a locally positive definite Lyapunov function, from the example given on page 59 of Slotine et al is given in Equation 2.16 and shown in Figure 2.2

$$V(\boldsymbol{x}) = \frac{1}{2} M R^2 x_2^2 + MRg\left(1 - \cos x_1\right). \tag{2.16}$$

Some related concepts are that of

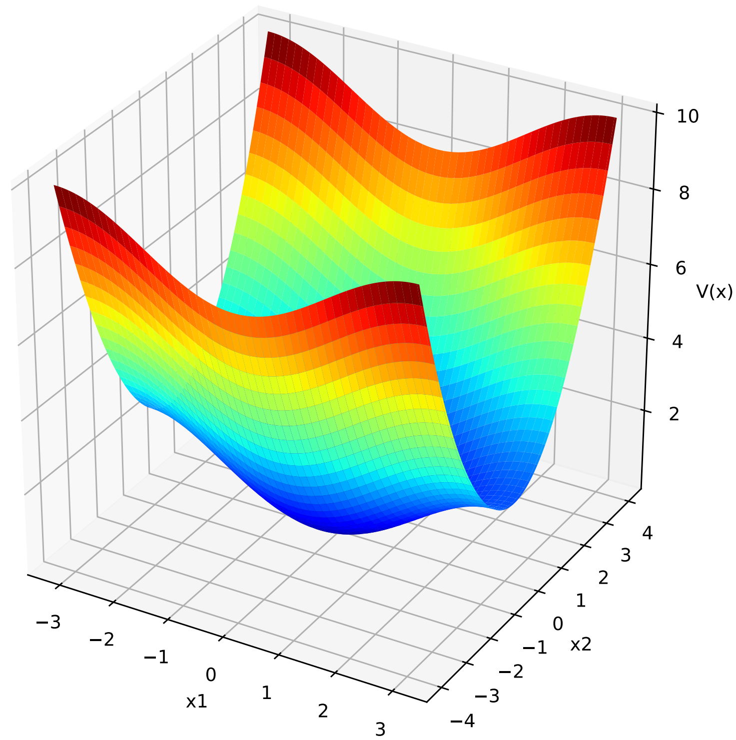


Figure 2.2: Locally positive definite Lyapunov function

- negative definite: $V(\boldsymbol{x})$ is negative definite if $-V(\boldsymbol{x})$ is positive definite
- positive semi-definite: if $V(0) = 0$ and $V(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \neq \mathbf{0}$
- negative semi-definite: if $V(\boldsymbol{x})$ is negative semi-definite if $-V(\boldsymbol{x})$ is positive semi-definite

$V(\boldsymbol{x})$ actually represents an implicit function of time t and its derivative with respect to time can be found (assuming that it is differentiable) by the chain rule and shown in Equation 2.17

$$\dot{V} = \frac{dV(\boldsymbol{x})}{dt} = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) . \tag{2.17}$$

Definition 7. If, in a ball $\boldsymbol{B}_{R_0}$, the function $V(\boldsymbol{x})$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of Equation 2.5 is negative semi-definite, or

$$\dot{V}(\boldsymbol{x}) \leq 0$$

then $V(\boldsymbol{x})$ is said to be a *Lyapunov function* for the system described by Equation 2.5.

2.4.2 Equilibrium Point Theorems

2.4.2.1 Lyapunov Theorem for Local Stability

Theorem 5. *If, in a ball $\boldsymbol{B}_{R_0}$, there exists a scalar function $V(\boldsymbol{x})$ with continuous first partial derivatives such that*

- *$V(\boldsymbol{x})$ is positive definite (locally in $\boldsymbol{B}_{R_0}$)*
- *$\dot{V}(\boldsymbol{x})$ is negative semi-definite (locally in $\boldsymbol{B}_{R_0}$)*

then the equilibrium point $\mathbf{0}$ *is stable. If, actually the derivative $\dot{V}(\boldsymbol{x})$ is locally negative definite in $\boldsymbol{B}_{R_0}$, then the stability is asymptotic.*

Let's look at a couple of examples. First, let's examine local stability.

Example 1. Local Stability

The dynamics of a damped pendulum is given in Equation 2.18

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0. \tag{2.18}$$

A possible Lyapunov function is one that is a combination of the kinetic and potential energies, given in Equation 2.19

$$V(\boldsymbol{x}) = (1 - \cos\theta) + \frac{\dot{\theta}^2}{2}. \tag{2.19}$$

Let's verify that $V(\boldsymbol{x})$ is locally positive definite. This can be done by examining the time-derivative of $V(\boldsymbol{x})$, noting that $\dot{V}(\boldsymbol{x})$ is the power dissipation and is shown in Equation 2.20

$$\dot{V}(\boldsymbol{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \leq 0. \tag{2.20}$$

Because Equation 2.20 represents the power dissipation of the system and it is always negative we can conclude that the system is stable because the energy will eventually go to zero. Because Equation 2.20 is only negative semi-definite, we cannot say anything about the systems asymptotic stability.

Next, let's examine asymptotic stability.

Example 2. Asymptotic Stability

Let a nonlinear system be defined by Equation 2.21

$$\begin{aligned} \dot{x}_1 &= x_1\left(x_1^2 + x_2^2 - 2\right) - 4x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2\left(x_1^2 + x_2^2 - 2\right) \end{aligned} \tag{2.21}$$

around its equilibrium point at the origin. Let the positive definite function in Equation 2.22 be the Lyapunov function

$$V(\boldsymbol{x}) = x_1^2 + x_2^2. \tag{2.22}$$

The time-derivative for Equation 2.22 is given in Equation 2.23

$$\dot{V}(\boldsymbol{x}) = 2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 2\right) \tag{2.23}$$

which is locally negative definite in the region $x_1^2 + x_2^2 < 2$. Therefore, the origin is asymptotically stable.

2.4.3 Lyapunov Theorem for Global Stability

For a system to have *global* stability, the radius of the ball $\boldsymbol{B}_{R_0}$ must be extended to infinity and $V(\boldsymbol{x})$ must be *radially bounded*. To be radially bounded, $V(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$ (as $\boldsymbol{x}$ tends to infinity in any direction).

Theorem 6. *Global Stability*

Assume that there exists a scalar function V of the state $\boldsymbol{x}$, with continuous first order derivatives such that

- $V(\boldsymbol{x})$ *is positive definite*
- $\dot{V}(\boldsymbol{x})$ *is negative definite*
- $V(\boldsymbol{x}) \to \infty$ *as* $\|\boldsymbol{x}\| \to \infty$ *($V(\boldsymbol{x})$ is radially bounded)*

then the equilibrium at the origin is globally asymptotically stable.

The reason that $V(\boldsymbol{x})$ needs to be radially bounded is so that the contours $V(\boldsymbol{x}) = V_\alpha$. Figure 2.3 shows the contour plot for the Lyapunov function, $V(\boldsymbol{x})$, found in Equation 2.16. The contour lines of this $V(\boldsymbol{x})$ are not all closed which means that $V(\boldsymbol{x})$ is not radially bounded system and the system is only locally stable, not globally. If all of the contour curves were closed , then it would not be possible for state trajectories to drift away from the equilibrium point.

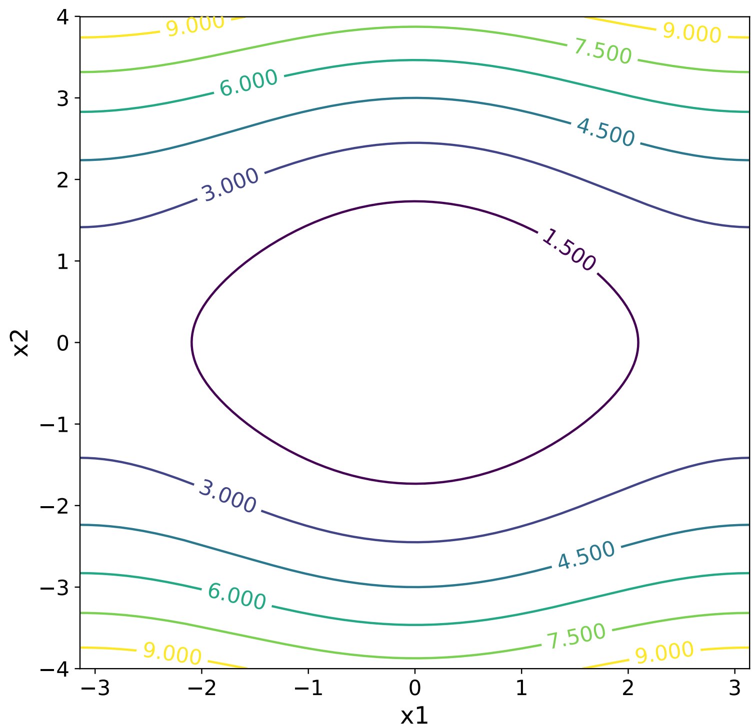


Figure 2.3: Contour plot for a non-radially bounded function

Many different Lyapunov functions may exist for a system (i.e., they are not unique) and some may reveal more about a system than others. For example, Equation 2.16 is a Lyapunov function for the damped pendulum system given in 2.18, but so is Equation 2.24

$$V(\boldsymbol{x}) = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\left(\dot{\theta} + \theta\right)^2 + 2\left(1 - \cos\theta\right). \tag{2.24}$$

In fact, the Lyapunov function given in Equation 2.24 shows that the system is asymptotically stable because it's time-derivative is negative definite, as shown in Equation 2.25

$$\dot{V}(\boldsymbol{x}) = -\left(\dot{\theta}^2 + \theta\sin\theta\right) \leq 0. \tag{2.25}$$

2.4.4 Invariant Set Theorems

The previous theorems can be difficult to apply to show asymptotic stability, often because $\dot{V}(x)$ of the candidate Lyapunov function is negative semi-definite. Despite this, it's still possible to show asymptotic stability using the invariant set theorems.

Definition 8. A set $\boldsymbol{G}$ is an *invariant set* for a dynamic system if every system trajectory which starts from a point in $\boldsymbol{G}$ remains in $\boldsymbol{G}$ for all future time.

This is motivated in part by the fact that requiring $\dot{V}(x)$ to be negative definite is too strong and to show more general behaviors such as convergence to limit cycles.

2.4.4.1 Local Invariant Set Theorem

Theorem 7. *Local Invariant Set Theorem*

Consider an autonomous system of the form found in Equation 2.5, with $\boldsymbol{f}$ continuous and let $V(\boldsymbol{x})$ be a scalar function with continuous first partial derivatives. Assume that

- *for some $l > 0$, the region $\boldsymbol{\Omega}_l$ defined by $V(\boldsymbol{x}) < l$ is bounded*
- *$\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$ in $\boldsymbol{\Omega}_l$*

Let $\boldsymbol{R}$ be the set of all points within $\boldsymbol{\Omega}_l$ where $\dot{V}(\boldsymbol{x}) = 0$, and $\boldsymbol{M}$ be the largest invariant set in $\boldsymbol{R}$. Then, every solution $\boldsymbol{x}(t)$ originating in $\boldsymbol{\Omega}_l$ tends to $\boldsymbol{M}$ as $t \to \infty$.

Here, $\boldsymbol{M}$ is the union of all invariant sets (e.g., equilibrium points or limit cycles) within $\boldsymbol{R}$. If $\boldsymbol{R}$ is the invariant set then $\boldsymbol{M} = \boldsymbol{R}$. Although V is still referred to as the Lyapunov function it is no longer required to be positive definite.

Example 3. Attractive Limit Cycle

Let's examine the system given by

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^7\left[x_1^4 + 2x_2^2 - 10\right]\\ \dot{x}_2 &= -x_1^3 - 3x_2^5\left[x_1^4 + 2x_2^2 - 10\right]\end{aligned}$$

Note that $\frac{d}{dt}\left[x_1^4 + 2x_2^2 - 10\right] = -\left(4x_1^{10} + 12x_2^6\right)\left[x_1^4 + 2x_2^2 - 10\right]$, therefore, $\left[x_1^4 + 2x_2^2 - 10\right]$ is an invariant set. The motion on this set can be described by either

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\,x_1^3$$

which, means that the invariant set is a limit cycle that rotates clockwise. The attractiveness of the limit cylce can be determined by choosing the candidate Lyapunov function to be

$$V = \left(x_1^4 + 2x_2^2 - 10\right)^2$$

which is the distance to the limit cycle. It can be seen that for any arbitrarily chosen positive number l the region surrounding the limit cycle, $\boldsymbol{R}_l$, is bounded.

$$\dot{V} = -8\left(x_1^{10} + 3x_2^6\right)\left(x_1^4 + 2x_2^2 - 10\right)$$

$\dot{V}$ is negative everywhere except if $\dot{V} = 0$, namely

$$x_1^4 + 2x_2^2 = 10 \qquad \text{or} \qquad x_1^{10} + 3x_2^6 = 0.$$

The first equation represents the equation of the motion of the limit cycle and the second is only true at the origin. This means that the system has two invariant sets: the limit cycle and the origin. Therefore, the invariant set of the system is the union of both sets. It should be noted that the equilibrium point at the origin is unstable and only trajectories starting there will converge to the origin, all others tend to the limit cycle.

Corollary 1. *Consider the autonomous system found in Equation 2.5, with $\boldsymbol{f}$ continuous and let $V(\boldsymbol{x})$ be a scalar function with continuous first partial derivatives. Assume that in a certain neighborhood $\boldsymbol{\Omega}$ of the origin*

- *$V(\boldsymbol{x})$ is locally positive definite*
- *$\dot{V}$ is negative semi-definite*
- *the set $\boldsymbol{R}$ defined by $\dot{V}(\boldsymbol{x}) = 0$ contains no trajectories of Equation 2.5 other than the trivial trajectory $\boldsymbol{x} = \mathbf{0}$*

Then, the equilibrium point $\mathbf{0}$ is asymptotically stable. Furthermore, the largest connected region of the form $\boldsymbol{\Omega}_l$ (defined by $V(\boldsymbol{x}) < l$) within $\boldsymbol{\Omega}$ is a domain of attraction of the equilibrium point.

In this case, the largest invariant set $\boldsymbol{M}$ in $\boldsymbol{R}$ only contains the equilibrium point $\mathbf{0}$. Note that

- Corollary 1 removes the condition that $\dot{V}$ must be negative definite and says instead that $\dot{V}$ only needs to be negative *semi*-definite
- the largest connected region of the form $\boldsymbol{\Omega}_l$ within $\boldsymbol{\Omega}$ is a domain of attraction of the equilibrium point, but not necessarily the entire domain of attraction due to the non-uniqueness of V
- $\boldsymbol{\Omega}$ is not necessarily a domain of attraction and is not guaranteed to be invariant. Some trajectories originating in $\boldsymbol{\Omega}$ but outside of $\boldsymbol{\Omega}_l$ may end up outside of $\boldsymbol{\Omega}$

2.4.4.2 Global Invariant Set Theorems

Local invariant set theorems can be extended to be global by requiring the radial unboundedness of the Lyapunov function V instead of the existence of a bounded $\boldsymbol{\Omega}_l$.

Theorem 8. *Global Invariant Set Theorem*

Consider an autonomous system of the form found in Equation 2.5, with $\boldsymbol{f}$ continuous and let $V(\boldsymbol{x})$ be a scalar function with continuous first partial derivatives. Assume that

- $V(\boldsymbol{x}) \to \infty$ *as* $\|\boldsymbol{x}\| \to \infty$
- $\dot{V}(\boldsymbol{x}) \leq 0$ *over the entire state-space*

Let $\boldsymbol{R}$ be the set of all points where $\dot{V}(\boldsymbol{x}) = 0$, and $\boldsymbol{M}$ be the largest invariant set in $\boldsymbol{R}$. Then, every solution globally asymptotically converges to $\boldsymbol{M}$ as $t \to \infty$.

Theorem 8 is important because it relaxes the constraint that $V(\boldsymbol{x})$ must be positive definite, allows for a single Lyapunov function to describe systems with multiple equilibria, and accounts more general behavior such as convergence to limit cycles.

Example 4. A class of second-order nonlinear systems

Consider a second-order system of the form

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

where b and c are linear continuous functions verifying the sign conditions

$$\begin{aligned} \dot{x}\, b(\dot{x}) > 0 \qquad &\text{for } \dot{x} \neq 0 \\ x\, c(x) > 0 \qquad &\text{for } x \neq 0. \end{aligned}$$

Let the Lyapunov function V be a combination of the kinetic and potential energies of the system and be described by

$$V(\boldsymbol{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x c(y)\, dy$$

with the time derivative

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= \dot{x}\ddot{x} + c(x)\,\dot{x} \\ &= -\,\dot{x}b(\dot{x}) - \dot{x}c(\dot{x}) + \dot{x}c(\dot{x}) \\ &= -\,\dot{x}b(\dot{x}) \leq 0 \end{aligned}$$

which represents the power dissipation of the system. Now we notice that $\dot{x}b(\dot{x}) = 0$ only if $\dot{x} = 0$ which implies that $\ddot{x} = -c(x)$ which is non-zero as long as $x \neq 0$. Therefore, system only converges to the equilibrium $x = 0$ and the set $\boldsymbol{R}$ is defined by $\dot{x} = 0$ and the largest invariant set, $\boldsymbol{M}$, in $\boldsymbol{R}$ contains only the point $[x = 0, \dot{x} = 0]$ and the system is locally asymptotically stable.

If the integral $\int_0^x c(y)\, dy$ is unbounded as $|x| \to \infty$ then V is radially unbounded and the origin $[x = 0, \dot{x} = 0]$ is globally asymptotically stable.

The previous example could represent a mass-spring damper system, an RLC circuit, or many other systems that have similar damping and spring effects.

Example 5. Multimodal Lyapunov Function

Consider the system

$$\ddot{x} + \left|x^2 - 1\right| \dot{x}^3 + x = \sin \frac{\pi x}{2}.$$

For this system, define the Lyapunov function to be

$$V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x \left(y - \sin \frac{\pi x}{2}\right) dy.$$

$V(x)$ has two minima at $[x = \pm 1; \dot{x} = 0]$, and a local maximum in x (which is a saddle point) at $[x = 0; \dot{x} = 0]$. The time-derivative of $V(x)$ is

$$\dot{V}(x) = -\left|x^2 - 1\right| \dot{x}^4$$

which is, again, is the power (or virtual power as this may not actually represent a physical system). Now,

$$\dot{V} = 0 \qquad \Rightarrow \qquad \dot{x} = 0 \quad \text{or} \quad x = \pm 1.$$

Consider the following cases

$$\begin{aligned} &\dot{x} = 0 \quad \Rightarrow \quad \ddot{x} = \sin \frac{\pi x}{2} - x \neq 0 \quad \text{except if} \quad x = 0 \text{ or } x = \pm 1 \\ &x = \pm 1 \quad \Rightarrow \quad \ddot{x} = 0 \end{aligned}$$

The invariant set theorem indicates that the system will converge to $[x = \pm 1; \dot{x} = 0]$ or $[x = 0; \dot{x} = 0]$ with the first set of equilibrium points being stable since they are local minima of V and the second set of equilibrium points being unstable as they lie in a saddle point.

As state earlier, many Lyapunov functions exist for the same system. As such, many invariant sets may be derived. Therefore, the system converges to the intersection the invariant sets.

2.5 System Analysis Based on Lyapunov's Direct Method

Before any system analysis is performed, a brief review of linear algebra will be given.

2.5.1 Brief Linear Algebra Review

A square matrix is *symmetric* if and only if $M^T = M$ and a matrix is *skew-symmetric* if and only if $M^T = -M$. Any square matrix M can be written as

$$\begin{aligned} M &= \frac{M + M^T}{2} + \frac{M - M^T}{2} \\ &= \frac{M_{\text{sym}}}{2} + \frac{M_{\text{skew}}}{2} \end{aligned}$$

where M_{sym} and M_{skew} are symmetric and skew-symmetric matrices, respectively.

Assume that M is skew-symmetric. Then,

$$\boldsymbol{x}^T M \boldsymbol{x} = \boldsymbol{x} M^T \boldsymbol{x}^T = -\boldsymbol{x}^T M \boldsymbol{x} \quad \forall \boldsymbol{x}$$

which implies that $\boldsymbol{x}^T M \boldsymbol{x} = 0,\ \forall \boldsymbol{x}$.

Quadratic functions of the form $\boldsymbol{x}^T M \boldsymbol{x}$ (where M is a square matrix) can be written as $\boldsymbol{x}^T M_{\text{sym}} \boldsymbol{x}$. A square matrix M is positive definite ($M > 0$) if

$$\boldsymbol{x} \neq \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{x}^T M \boldsymbol{x} > 0$$

which implies that to every positive definite matrix is associated a positive definite function, though the converse is not necessarily true. Sylvester's theorem shows that if M is symmetric, it is necessary for all of M's eigenvalues to be strictly positive for M to be positive definite.

2.5.2 Lyapunov Analysis of Linear Time-Invariant Systems

For a linear system of the form given in Equation 2.4, a candidate Lyapunov function of quadratic form is given in Equation 2.26

$$V = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \tag{2.26}$$

where $\boldsymbol{P}$ is a given symmetric positive definite matrix and $\boldsymbol{A}$ is strictly stable. The time-derivative is given in Equation 2.27

$$\begin{aligned}
\dot{V} &= \dot{\boldsymbol{x}}^T \boldsymbol{P} \boldsymbol{x} + \boldsymbol{x}^T \dot{\boldsymbol{P}} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P} \dot{\boldsymbol{x}} \\
&= \dot{\boldsymbol{x}}^T \boldsymbol{P} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P} \dot{\boldsymbol{x}} \\
&= \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{A} \boldsymbol{x} \\
&= -\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}.
\end{aligned} \tag{2.27}$$

The question that we want to answer is, given a symmetric positive definite matrix $\boldsymbol{P}$ and a strictly stable system (given by $\boldsymbol{A}$), can we find a positive definite matrix $\boldsymbol{Q}$ that is also symmetric positive definite? A counter example will show that the answer is "no."

Example 6. Consider a second-order linear system whose $\boldsymbol{A}$ matrix is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

If $\boldsymbol{P} = \mathbb{I}$, then

$$-\boldsymbol{Q} = \boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

which is *not* positive definite. Therefore, we cannot use any given symmetric positive definite $\boldsymbol{P}$ to find a positive definite $\boldsymbol{Q}$.

What can be done, however, is to go the other way around: start with a given symmetric positive definite $\boldsymbol{Q}$ and solve for $\boldsymbol{P}$ from using the equation $-\boldsymbol{Q} = \boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A}$ and to check that $\boldsymbol{P}$ is positive definite. This method will always lead to conclusive results for stable linear systems, as shown in Theorem 9.

Theorem 9. *A necessary and sufficient condition for an LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ to be strictly stable is that, for any symmetric positive definite $\boldsymbol{Q}$, the unique matrix $\boldsymbol{P}$ solution of $-\boldsymbol{Q} = \boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A}$ be symmetric positive definite.*

2.6 Control Design Based on Lyapunov's Direct Method

There are two ways to use Lyapunov's direct method for designing a controller: hypothesize a one form of control law and then find a Lyapunov function to justify it, and hypothesize a Lyapunov function candidate then find a control law to make the candidate function a real Lyapunov function.

Example 7. Regulator Design

Consider the problem of stabilizing the system

$$\ddot{x} + \dot{x}^3 + x^2 = u$$

where the desired behavior is to have the system go the equilibrium at $x \equiv 0$. It is sufficient to choose a continuous control law u of the form (see Example 4)

$$u = u_1(\dot{x}) + u_2(x)$$

where

$$\begin{aligned} \dot{x}\left(\dot{x}^3 + u_1(\dot{x})\right) &< 0 \qquad \text{for } \dot{x} \neq 0 \\ x\left(x^2 - u_2(x)\right) &> 0 \qquad \text{for } x \neq 0. \end{aligned}$$

A controller that can stabilize the system is

$$u = -2\dot{x}^3 - 5x\,|x|$$

Chapter 3

Advanced Stability Theory

The Lyapunov stability theorems will be extended to non-autonomous systems.

3.1 Concepts of Stability for Non-Autonomous Systems

The concepts of stability for non-autonomous systems is similar to those of autonomous systems, but for non-autonomous systems there is an explicit dependence on the initial time, t_0.

3.1.1 Equilibrium Points and Invariant Sets

For non-autonomous systems of the form given in Equation 3.1

$$\dot{x} = \boldsymbol{f}\left(\boldsymbol{x}, t\right) \tag{3.1}$$

equilibrium points, $\boldsymbol{x}^*$, are given by Equation 3.2

$$\dot{x} = \boldsymbol{f}\left(\boldsymbol{x}^*, t\right) \equiv \boldsymbol{0} \qquad \forall t \geq t_0. \tag{3.2}$$

The definition of an invariant set is the same for non-autonomous systems as it is for autonomous systems. (See Definition 8.)

3.1.2 Extensions of the Previous Stability Concepts

Definition 9. The equilibrium point $\boldsymbol{0}$ is *stable* at t_0 if for any $R > 0$, there exists a positive scalar $r\left(R, t_0\right)$ such that

$$\left\|\boldsymbol{x}\left(t_0\right)\right\| < r \quad \Rightarrow \quad \left\|\boldsymbol{x}\left(t\right)\right\| < R \qquad \forall t \geq t_0$$

otherwise, the equilibrium point $\boldsymbol{0}$ is *unstable*.

The difference between Definitions 1 and 9 is the explicit dependence on the initial time, t_0, in the latter.

Definition 10. The equilibrium point $\boldsymbol{0}$ is *asymptotically stable* at time t_0 if

- it is stable
- $\exists r(t_0) > 0$ such that $\|\boldsymbol{x}(t_0)\| < r(t_0) \quad \Rightarrow \quad \|\boldsymbol{x}(t)\| \to 0$ as $t \to \infty$

Definition 10 states that asymptotic stability for non-autonomous systems requires the existence of an attractive region for every initial time, t_0.

Definition 11. The equilibrium point $\mathbf{0}$ is *exponentially stable* if there exist two positive numbers, α and β, such that for sufficiently small $\boldsymbol{x}(t_0)$,

$$\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{x}_0\| \, \mathrm{e}^{-\lambda(t-t_0)} \qquad \forall t \geq t_0.$$

Definition 12. The equilibrium point $\mathbf{0}$ is *globally asymptotically stable* if $\forall \boldsymbol{x}(t_0)$

$$\boldsymbol{x}(t) \to 0 \qquad t \to \infty$$

3.1.3 Uniformity in Stability Concepts

Definition 13. The equilibrium point $\mathbf{0}$ is locally *uniformly stable* if the scalar r in Definition 9 can be chosen independently of t_0 , i.e., if $r = r(R)$

The reason behind the concept of uniformity is to rule out systems that are "less and less stable" for larger values of t_0. The definition of uniform asymptotic stability also tends to restrict the effect of the initial time t_0 on the state convergence pattern.

Definition 14. The equilibrium point at the origin is locally *uniformly asymptotically stable* if

- it is uniformly stable
- there exists a ball of attraction $\boldsymbol{B}_R$, whose radius is independent of t_0, such that any system trajectory with initial states in $\boldsymbol{B}_{R_0}$ converges to $\mathbf{0}$ uniformly in t_0

Uniform convergence in terms of t_0 means that for all R_1 and R_2 satisfying the condition $0 < R_2 < R_1 \leq R_0$, $\exists T(R_1, R_2) > 0$, such that $\forall t_0 \geq 0$,

$$\|\boldsymbol{x}(t_0)\| < R_1 \quad \Rightarrow \quad \|\boldsymbol{x}(t)\| < R_2 \qquad \forall t \geq t_0 + T(R_1, R_2)\,. \tag{3.3}$$

What Equation 3.3 means, is that a state trajectory that starts in the ball $\boldsymbol{B}_{R_1}$ will converge to the smaller ball $\boldsymbol{B}_{R_2}$. Uniform asymptotic stability always implies asymptotic stability, but not the other way around. Exponential stability always implies uniform asymptotic stability, and global asymptotic stability can be defined by replacing the ball of attraction $\boldsymbol{B}_{R_0}$ with the entire state-space.

3.2 Lyapunov Analysis of Non-Autonomous Systems

Although many of the ideas of Chapter 2 can be applied it is more care must be taken.

3.2.1 Lyapunov's Direct Method for Non-Autonomous Systems

The major difference between autonomous and non-autonomous systems for the direct method is that La Salle's theorems do not apply.

3.2.1.1 Time-Varying Positive Definite Functions and Decrescent Functions

Definition 15. To analyze non-autonomous systems using Lyapunov's direct method system a time-varying scalar function may have to be used.

A scalar time-varying function $V(\boldsymbol{x}, t)$ is *locally positive definite* if $V(\mathbf{0}, t) = 0$ and there exists a time-invariant positive definite function $V_0(\boldsymbol{x})$ such that

$$\forall t \geq t_0, \qquad V(\boldsymbol{x}, t) \geq V_0(\boldsymbol{x}) \tag{3.4}$$

Basically, a time-variant function is locally positive definite if it dominates a time-invariant locally positive definite one. Globally positive definite is defined similarly. Negative definite works similarly, it just needs to be shown that $-V(\boldsymbol{x}, t)$ is a positive definite function (similarly for negative semi-definite).

Definition 16. A scalar function $V(\boldsymbol{x}, t)$ is decerscent *if* $V(\mathbf{0}, t) = 0$ and if there exists a time-invariant positive definite function $V_1(\boldsymbol{x})$ such that

$$\forall t \geq 0, \qquad V(\boldsymbol{x}, t) \leq V_1(\boldsymbol{x})$$

Basically, a function is decrescent if it is dominated by a time-invariant positive definite function.

To find the time derivative of $V(\boldsymbol{x}, t)$, follow Equation 3.5

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}, t). \tag{3.5}$$

3.2.1.2 Lyapunov Theorem for Non-Autonomous System Stability

Theorem 10. *Lyapunov Theorem for Non-Autonomous Systems*

Stability*: If, in a ball $\boldsymbol{B}_{R_0}$ around the equilibrium point $\mathbf{0}$, there exists a scalar function $V(\boldsymbol{x}, t)$ with continuous partial derivatives such that*

1. *V is positive definite*
2. *$\dot{V}$ is negative semi-definite*

then the equilibrium point $\mathbf{0}$ is stable in the sense of Lyapunov.

Uniform stability and uniform asymptotic stability: *If, furthermore,*

3. *V is decrescent*

then the origin is uniformly stable. If condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then the equilibrium point is uniformly asymptotically stable.

Global uniform asymptotic stability*: If the ball $\boldsymbol{B}_{R_0}$ is replaced by the whole state-space, and condition 1, the strengthened condition 2, condition 3, and the condition*

4. $V(\boldsymbol{x}, t)$ is radially unbounded

are all satisfied, then the equilibrium point at **0** *is globally uniformly asymptotically stable.*

3.2.1.3 Lyapunov Analysis of Linear Time-Varying Systems

Consider the system given by Equation 3.6

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\,\boldsymbol{x}. \tag{3.6}$$

LTI systems are asymptotically stable if their eigenvalues all have negative real components. Therefore, it may be tempting to conclude that the same can be said about linear time-varying (LTV) systems as well, for all $t \geq 0$. This is simply not the case, as shown in Example 8.

Example 8. Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & \mathrm{e}^{2t} \\ 0 & -1 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalues of $\boldsymbol{A}(t)$ are -1 and -1. If one were to use the same LTI stability analysis using eigenvalues, then it would be concluded that this system is asymptotically stable. This is not the case, however, which can be shown by explicitly solving this simple system, as shown in Equation 3.7. Equation 3.7 shows that the x_1 is driven by an input that tends towards infinity as $t \to \infty$. This clearly shows that the tools used to determine the stability of an LTI system will not work with an LTV system.

$$\begin{aligned} x_1 + \dot{x}_1 &= x_2(0)\,\mathrm{e}^{t} \\ x_2 &= x_2(0)\,\mathrm{e}^{-t} \end{aligned} \tag{3.7}$$

The asymptotic stability of a time-varying system can, however, be determined by examining the eigenvalues of the symmetric matrix $\boldsymbol{A}(t) + \boldsymbol{A}^T(t)$. An time-varying system is asymptotically stable if the eigenvalues (all of which are real) remain strictly in the left-hand complex plane

$$\exists \lambda > 0,\ \forall i,\ \forall t \geq 0,\ \lambda_i\left(\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\right) \leq -\lambda. \tag{3.8}$$

Equation 3.8 shows that by assuming that all of the eigenvalues of $\boldsymbol{A}_{\text{sym}}(t) = \boldsymbol{A}(t) + \boldsymbol{A}^T(t)$ are less than some $-\lambda$ for all $t \geq 0$, then $\|\boldsymbol{x}(t)\| \to 0$ as $t \to \infty$. This can be proved by starting with a quadratic candidate Lyapunov function

$$V = \boldsymbol{x}^T\boldsymbol{x}$$

with time-derivative

$$\begin{aligned} \dot{V} &= \dot{\boldsymbol{x}}^T\boldsymbol{x} + \boldsymbol{x}^T\dot{\boldsymbol{x}} \\ &= \boldsymbol{x}^T\left(\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\right)\boldsymbol{x} \leq -2\lambda\boldsymbol{x}^T\boldsymbol{x} = -2\lambda V \\ &\Rightarrow \dot{V} + 2\lambda V \leq 0 \\ &\Rightarrow 0 \leq V \leq V(0)\,\mathrm{e}^{-2\lambda t}. \end{aligned}$$

The above result is *sufficient* to show asymptotic stability, but not necessary.

3.3 Lyapunov-Like Analysis Using Barbalat's Lemma

3.3.1 Asymptotic Properties of Functions and Their Derivatives

Some important facts to keep in mind

- $\dot{g}(t) \to 0 \;\; \not\Rightarrow \;\; g$ converges. $\dot{g}(t) \to 0$ means that the function $g(t)$ is getting flatter and flatter, but it does not imply that $g(t)$ converges to a limit

- g converges $\;\not\Rightarrow\; \dot{g}(t) \to 0$. For example, a function $g(t) = \mathrm{e}^{-t}\sin \mathrm{e}^{2t}$ converges to zero as $t \to \infty$, but its derivative $\dot{g}(t) = 2\mathrm{e}^{t}\cos \mathrm{e}^{2t} - \mathrm{e}^{-t}\sin \mathrm{e}^{2t}$ is unbounded

- If g is lower bounded and decreasing, then it converges to a limit.

3.3.1.1 Barbalat's Lemma

Lemma 1. *Barbalat's Lemma*

Suppose that $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ with scalar function $V(\boldsymbol{x}, t)$ that is lower bounded (i.e., has a finite limit as $t \to \infty$), and $\frac{d}{dt}V \leq 0$ always, and $\frac{d^2}{dt^2}V$ is bounded then $\frac{d}{dt}V \to 0$ as $t \to \infty$.

Typically, V can be chosen such that $\frac{d}{dt}V = -(\text{"error"})^2$ and the first two conditions of Lemma 1 imply the third condition.

Chapter 4

Sliding Control

This chapter will present a framework for controlling a model with both parametric (unknown quantities included in the model) and unmodeled dynamics (such as the underestimation of the order of the system).

4.1 Sliding Surfaces

Consider the system given in Equation 4.1

$$x^{(n)} = f\left(\boldsymbol{x}\right) + b\left(\boldsymbol{x}\right) u \tag{4.1}$$

with x and u as scalars. The tracking error is given by Equation 4.2

$$\tilde{\boldsymbol{x}}\left(t\right) = \boldsymbol{x}\left(t\right) - \boldsymbol{x}_d\left(t\right) \tag{4.2}$$

where $\boldsymbol{x}\left(t\right)$ is the actual state and $\boldsymbol{x}_d\left(t\right)$ is the desired state. The goal is to have $\tilde{\boldsymbol{x}}\left(t\right) \to 0$ as $t \to \infty$. If we call $v = f\left(\boldsymbol{x}\right) + b\left(\boldsymbol{x}\right)$ and v were known, then all that would need to be done is to solve for the control input u. This is not possible with real systems due to uncertainties with the parameters and/or simplifications of the model (i.e., unmodeled complexities).

4.1.1 Notational Simplification

A time-varying sliding surface $S\left(t\right)$ can be defined in the state-space $\boldsymbol{R}^{(n)}$ by the scalar equation $s\left(\boldsymbol{x};t\right) = 0$ where

$$s\left(\boldsymbol{x};t\right) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} \tag{4.3}$$

and $\lambda > 0$ and satisfies the following conditions

$$\begin{cases} \dot{s}\,\text{contains}\,u \\ s \to 0 \;\Rightarrow\; \tilde{x}\left(t\right) \to 0. \end{cases} \tag{4.4}$$

The introduction of the intermediate variable s reduces the problem from an n$^{\text{th}}$-order problem to a first-order one. Also, the bounds on s are directly related to the bounds on the tracking error $\tilde{\boldsymbol{x}}\left(t\right)$. By assuming that the initial conditions are zero

$$\forall t \geq 0 \; \left|s\left(t\right)\right| \leq \Phi \quad \Rightarrow \quad \forall t \geq 0, \; \left|\tilde{x}^{(i)}\left(t\right)\right| \leq \left(2\lambda\right)^{(i)} \epsilon, \quad \text{for}\, i = 0, \ldots, n-1$$

where $\epsilon = \frac{\Phi}{\lambda^{n-1}}$. Finally, for a first-order system s can be kept at zero by choosing u such that when outside of the sliding surface $S(t)$

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta\,|s| \tag{4.5}$$

where $\eta > 0$. Equation 4.5 is known as the "sliding condition."

4.1.2 Perfect Performance - At a Price

Consider the second-order system given by

$$\ddot{x} = f + u$$

where the dynamics f may or may not be time-varying and is not exactly known and is estimated by $\hat{f}$. The error is bounded by a known function $F = F(x, \dot{x})$ such that

$$\left|\hat{f} - f\right| \leq F.$$

An example system is

$$\ddot{x} + a(t)\,\dot{x}^2\cos 3x = u$$

where $a(t)$ is unknown but verifies

$$1 \leq a(t) \leq 2$$

then $\hat{f}$ and F can be

$$\hat{f} = -1.5\dot{x}^2\cos 3x \qquad F = 0.5\dot{x}^2\,|\cos 3x|\,.$$

Now, s and $\dot{s}$ can be defined as

$$\begin{aligned} s &= \dot{\tilde{x}} + \lambda\tilde{x} \\ \dot{s} &= \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = f + u - \ddot{x}_d\lambda\dot{\tilde{x}}. \end{aligned} \tag{4.6}$$

For $s > 0$, we have

$$\begin{aligned} s\,\dot{s} &\leq -\eta\,s \\ \dot{s} &\leq -\eta \end{aligned}$$

and $s = 0$ is reached in finite time $\left(\leq \frac{|s(t=0)|}{\eta}\right)$ starting from anywhere. $s = 0$ is an invariant set meaning that once the trajectory reaches the sliding surface, then it will not leave.

The control law u will be chosen such that it satisfies the sliding condition given in Equation 4.5. $\hat{u}$ is the approximate control law that makes $\dot{s} = 0$

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}. \tag{4.7}$$

Plugging Equation 4.7 into Equation 4.6 gives

$$\dot{s} = f - \hat{f}. \tag{4.8}$$

To satisfy the sliding condition, a discontinuous sgn() function will be added to Equation 4.8

$$\dot{s} = f - \hat{f} - k\mathrm{sgn}\,(s)$$

such that the sgn(s) function is defined as

$$\mathrm{sgn}(s) = \begin{cases} +1 & \text{if } s > 0 \\ -1 & \text{if } s < 0. \end{cases}$$

k can be chosen such that $k = F + \eta$, which leads to the controller

$$\begin{aligned} u &= \hat{u} - k\mathrm{sgn}\,(s) \\ &= 1.5\dot{x}^2 \cos 3x + \ddot{x}\lambda - \left(0.5\dot{x}^2 \left|\cos 3x\right| + \eta\right) \mathrm{sgn}\left(\dot{\tilde{x}} + \lambda\tilde{x}\right). \end{aligned} \tag{4.9}$$

The control law given in Equation 4.9 will ensure that when on the surface $s = 0$ that the trajectory remains there regardless of what is happening "outside" of it. However, there is a downside to this control law due to the addition of the discontinuous sgn$\,(s)$ function. The discontinuities added by sgn$\,(s)$ function, non-instantaneous switch of the system, and imperfect knowledge of the system leads to what is called chattering, and is show in Figure 4.1. In Figure 4.1, the state trajectory continually bounces back and forth around $s = 0$. This can be a problem in many systems because of the high control activity and this may cause the system to have some high-frequency dynamics become excited. A way around this is to have a trade off: a suitable smooth and continuous control law, but with loss of tracking precision.

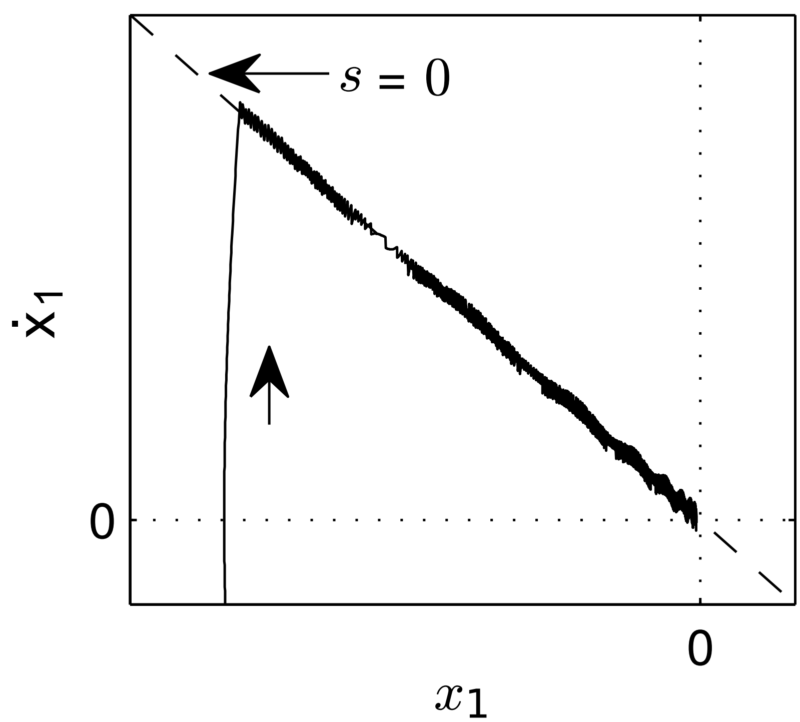


Figure 4.1: System with chattering[3]

The compromise will be to use a smoothing (saturation) function that provides a smooth transition between -1 and $+1$, as shown in Figure 4.2.

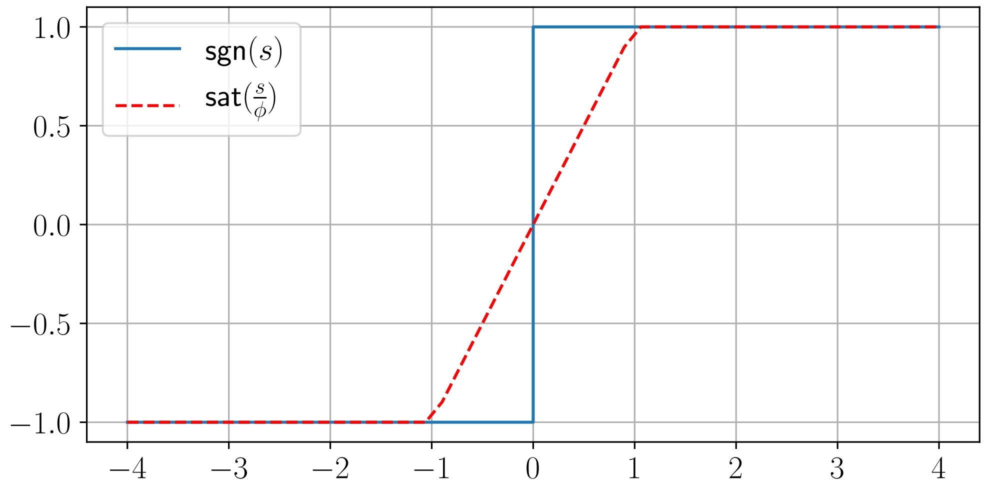


Figure 4.2: $\text{sgn}\,(s)$ and $\text{sat}\left(\frac{s}{\phi}\right)$ functions with $\phi = 1$

In this example, a control law u was developed such that inside the boundary layer

$$u = \hat{u} - k\text{sat}\left(\frac{s}{\phi}\right),$$

which means that instead of reaching the surface $s = 0$ the trajectory reaches a boundary layer $|s| \leq \phi$ in finite time and stays within it, presumably, with a small tracking error. If ϕ is time-varying because $k\,(\boldsymbol{x}, t)$ is time-varying and then $|s| \leq \phi \;\Rightarrow\; \frac{1}{2}\frac{d}{dt}s^2 \leq \left(\dot{\phi} - \eta\right)|s|$. This condition says that if the boundary layer ϕ is shrinking, then the trajectory needs to be faster than the boundary layer is shrinking. Once inside the boundary layer the control law becomes

$$u = \hat{u} - k\frac{s}{\phi}$$

and after the transients

$$\begin{aligned}\dot{s} &= \hat{f}_d - f_d - \left(k_d - \dot{\phi}\right)\frac{s}{\phi} + \mathcal{O}\left(\epsilon\right)\\ &= -\frac{k_d - \dot{\phi}}{\phi}s + \hat{f}_d - f_d + \mathcal{O}\left(\epsilon\right)\end{aligned}$$

where $\tilde{x} = \mathcal{O}\left(\epsilon\right)$. To find ϕ, it is necessary to solve

$$\dot{\phi} + \lambda\phi = k_d.$$

4.2 The Modeling/Performance Trade-Offs

The s-trajectories give insight into $\tilde{x}$ and ϕ can give insight into if the choices of uncertainty are too conservative. If the model is too conservative then the boundary layer will be too large and if the model is on the other end of the spectrum then the s-trajectories may not enter the boundary layer. These insights are given by

$$|\tilde{x}| \leq \frac{F_d}{\lambda^n}$$

where F_d is the uncertainty of the model parameters and λ is the uncertainty of the order of the system.

Some examples of limits on λ are

- structural unmodeled dynamics (λ_s) such as flexing and vibration
- motor time constant (λ_a)
- sampling rates (λ_{sample})

and it is desired to have λ be as small as possible therefore chose λ to be $\lambda = \min(\lambda_s, \lambda_a, \lambda_{sample}, \ldots)$. Each λ should also be as close to one another as possible (roughly the same order of magnitude) otherwise there may be some problems with the model.

Chapter 5

Adaptive Control

Adaptive control is how to exploit the fact that part of the uncertainty in $f(\boldsymbol{x}, t)$ can be described by simply as unknown constants. The question is, can the system be made to behave as if unknown constants are known? An example can demonstrate how this is possible.

Consider an underwater single-link robot arm such as the one shown in Figure 5.1 with dynamics given in Equation 5.1

$$J\ddot{x} + b\dot{x}\,|\dot{x}| + mgl\sin x = u \tag{5.1}$$

where J is the moment of inertia of the motor, b is the viscous fluid constant, mgl is the rotational torque. A vector of unknowns $\boldsymbol{a}$ can be defined such that

$$\begin{aligned} \boldsymbol{a}^T &= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T \\ &= \begin{bmatrix} J & b & mgl \end{bmatrix}^T. \end{aligned}$$

The goal of the control problem in this example is to get the robot arm to follow a given trajectory (i.e., this is a tracking control problem, similar to the one seen in Chapter 4) in the absence of knowing $\boldsymbol{a}$. Similar to the example given in Subsection 4.1.2, the first step will be to transform this second-order problem into a first-order one by defining s to be

$$\begin{aligned} s &= \dot{\tilde{x}} + \lambda\tilde{x} \\ &= \dot{x} - \dot{x}_r \end{aligned} \tag{5.2}$$

where $\dot{x}_r = \dot{x}_d - \lambda\dot{x}$. The s given in Equation 5.2 satisfies the conditions given in Equation 4.4 and the sliding condition given by Equation 4.5.

This is a non-autonomous system and therefore Barbalat's lemma will be used to construct a Lyapunov function V such that

- $V(\boldsymbol{x}, t)$ is lower bounded
- $\dot{V} \leq 0$
- $\ddot{V}$ is bounded

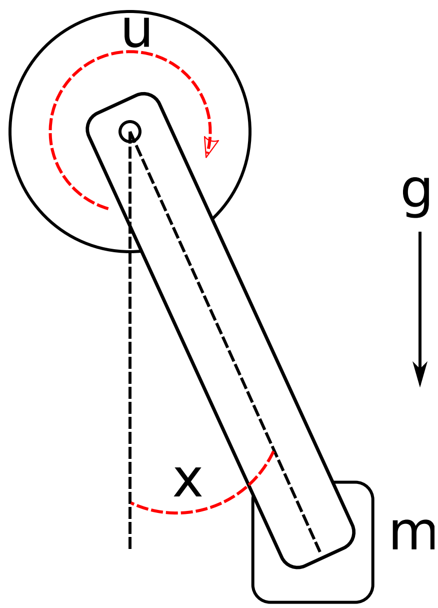


Figure 5.1: Single-link robot arm

which means that $V \to 0$. Let V be define as

$$\begin{aligned} V &= \frac{1}{2} J s^2 \\ \dot{V} &= s J \dot{s} \\ &= s J \left(\ddot{x} - \ddot{x}_r \right) \\ &= s \left(u - \boldsymbol{Y} \boldsymbol{a} \right) \end{aligned}$$

where $\boldsymbol{Y} = \left[\begin{array}{ccc} \ddot{x}_r & \dot{x}\left|\dot{x}\right| & \sin x \end{array} \right]$. For a moment, let's that we that the vector $\boldsymbol{a}$ is actually a known quantity. Then the control law u can be constructed as $u = \boldsymbol{Y}\boldsymbol{a} - ks$ where $k > 0$ such that now we get

$$\begin{aligned} \dot{V} &= s \left(u - \boldsymbol{Y} \boldsymbol{a} \right) \\ &= s \left(\boldsymbol{Y} \boldsymbol{a} - ks - \boldsymbol{Y} \boldsymbol{a} \right) \\ &= -ks^2 \end{aligned}$$

which, by Barbalat's lemma, means that $\dot{V} \to 0$, which implies that $s \to 0$, which implies that $\tilde{x} \to 0$ and $\dot{\tilde{x}} \to 0$ and our job would be done.

Because $\boldsymbol{a}$ is *not* known let the control law be $u = \boldsymbol{Y}\hat{\boldsymbol{a}} - ks$ where $\hat{\boldsymbol{a}}$ is the estimate for the unknown parameters and we will also get

$$\begin{aligned} \dot{V} &= s \left(u - \boldsymbol{Y} \boldsymbol{a} \right) \\ &= s \left(\boldsymbol{Y} \hat{\boldsymbol{a}} - ks - \boldsymbol{Y} \boldsymbol{a} \right) \\ &= -ks^2 + s \boldsymbol{Y} \tilde{\boldsymbol{a}} \end{aligned}$$

where $\tilde{\boldsymbol{a}}\left(t\right) = \hat{\boldsymbol{a}}\left(t\right) - \boldsymbol{a}\left(t\right)$ and is the parameter estimation error. Now the problem is the $\boldsymbol{Y}\tilde{\boldsymbol{a}}$ term in the definition of $\dot{V}$ which may prevent $\dot{V}$from being negative. The solution is to add another positive term to V such that after taking its derivative it cancels out the problem term in $\dot{V}$. Let's now redefine V and $\dot{V}$

$$
\begin{aligned}
V =& \frac{1}{2} J s^2 + \tilde{\boldsymbol{a}}^T \boldsymbol{P}^{-1} \tilde{\boldsymbol{a}} \\
\dot{V} =& - k s^2 + s \boldsymbol{Y} \tilde{\boldsymbol{a}} + \dot{\hat{\boldsymbol{a}}}^T \boldsymbol{P}^{-1} \tilde{\boldsymbol{a}} \\
=& - k s^2 + \left(s \boldsymbol{Y} + \dot{\hat{\boldsymbol{a}}}^T \boldsymbol{P}^{-1} \right) \tilde{\boldsymbol{a}}
\end{aligned}
$$

where $\boldsymbol{P}^{-1}$ is a symmetric positive definite matrix. The goal now is to have $\left(s\boldsymbol{Y} + \dot{\hat{\boldsymbol{a}}}^T \boldsymbol{P}^{-1} \right) \tilde{\boldsymbol{a}} = 0$ which leads to

$$
\dot{\hat{\boldsymbol{a}}} = -\boldsymbol{P} \boldsymbol{Y}^T s.
$$

We are now left with two equations: a control law $u = \boldsymbol{Y} \hat{\boldsymbol{a}} - ks$ and an adaptation law $\dot{\hat{\boldsymbol{a}}} = -\boldsymbol{P}\boldsymbol{Y}^T s$. The control does its best to cancel the dynamic parameters and replace them with something simple. The adaptation law which can take on any initial values is a differential equation that feeds the control law with the estimate of the unknown parameters $\hat{\boldsymbol{a}}$. The combination of the two laws guarantees that Barbalat's lemma works.

Now, starting anywhere an arbitrary desired trajectory and starting with arbitrary initial guesses for the parameters and after some transients the system will behave as if the parameters were known and the tracking errors will disappear.

Bibliography

[1]

[2] Jean-Jacques E Slotine, Weiping Li, et al. *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ, 1991.

[3] Wikimedia Commons. Reaching phase and sliding phase in first-order sliding mode control., 2009. File: Car.svg.