UNIT 11 SMALL SAMPLE TESTS

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11.1 INTRODUCTION

In previous unit, we have discussed the testing of hypothesis for large samples in details. Recall that throughout the unit, we were making an assumption that "if sample size is sufficiently large then test statistic follows approximately standard normal distribution". Also recall two points highlighted in this course, i.e.

- Cost of our study increases as sample size increases.
- Sometime nature of the units in the population under study is such that they destroyed under investigation.

If there are limited recourses in terms of money then first point listed above force us not to go for large sample size when items /units under study are very costly such as airplane, computer, etc. Second point listed above give an alarm for not to go for large sample if population units are destroyed under investigation.

So, we need an alternative technique which is used to test the hypothesis based on small sample(s). Small sample tests do this job for us. But in return they demand one basic assumption that population under study should be normal as you will see when you go through the unit. t, χ^2 and F -tests are some commonly used small sample tests.

In this unit, we will discuss t-test in details which is based on the t-distribution described in Unit 3 of this course. And χ^2 and F-tests will be discussed in next unit which are based on χ^2 and F-distributions described in Unit 3 and Unit 4 of this course respectively.

This unit is divided into eight sections. Section 11.1 is described the need of small sample tests. The general procedure of t-test for testing a hypothesis is described in Section 11.2. In Section 11.3, we discuss testing of hypothesis for population mean using t-test. Testing of hypothesis for difference of two population means when samples are independent is described in Section 11.4 whereas in Section 11.5, the paired t-test for difference of two population means when samples are dependent(paired) is discussed. In Section 11.6 testing of hypothesis for population correlation coefficient is explained. Unit ends by providing summary of what we have discussed in this unit in Section 11.7 and solution of exercises in Section 11.8.

Before moving further a humble suggestion to you that please revise what you have learned in previous two units. The concepts discussed there will help you a lot to better understand the concepts discussed in this unit.

Objectives

After studying this unit, you should be able to:

- realize the importance of small sample tests;
- know the procedure of t-test for testing a hypothesis;
- describe testing of hypothesis for population mean for using t-test;
- explain the testing of hypothesis for difference of two population means when samples are independent using t-test;
- describe the procedure for paired t-test for testing of hypothesis for difference of two population means when samples are dependent or paired; and
- explain the testing of hypothesis for population correlation coefficient using t-test.

11.2 GENERAL PROCEDURE OF t-TEST FOR TESTING A HYPOTHESIS

The general procedure of t-test for testing a hypothesis is similar as Z-test already explained in Unit 10. Let us give you similar details here.

For this purpose, let $X_1, X_2, \ldots, X_n$ be a random sample of **small size n (< 30)** selected from a **normal population** (recall the demand of small sample tests pointed out in previous Section 11.1) having parameter of interest, say, θ which is actually unknown but its hypothetical value, say, θ_0 estimated from some previous study or some other way is to be tested. t-test involves following steps for testing this hypothetical value:

Step I: First of all, we setup null and alternative hypotheses. Here, we want to test the hypothetical value θ_0 of parameter θ so we can take the null and alternative hypotheses as

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta \neq \theta_0 \qquad [\text{for two-tailed test}]$$

$$\text{or} \quad \left.\begin{array}{l} H_0 : \theta \leq \theta_0 \text{ and } H_1 : \theta > \theta_0 \\ H_0 : \theta \geq \theta_0 \text{ and } H_1 : \theta < \theta_0 \end{array}\right\} \qquad [\text{for one-tailed test}]$$

In case of comparing same parameter of two populations of interest, say, θ_1, and θ_2 then our null and alternative hypotheses would be

$$H_0 : \theta_1 = \theta_2 \text{ and } H_1 : \theta_1 \neq \theta_2 \qquad [\text{for two-tailed test}]$$

$$\text{or} \quad \left.\begin{array}{l} H_0 : \theta_1 \leq \theta_2 \text{ and } H_1 : \theta_1 > \theta_2 \\ H_0 : \theta_1 \geq \theta_2 \text{ and } H_1 : \theta_1 < \theta_2 \end{array}\right\} \qquad [\text{for one-tailed test}]$$

Step II: After setting the null and alternative hypotheses our next step is to decide a criteria for rejection or non-rejection of null hypothesis i.e.

decide the level of significance α, at which we want to test our null hypothesis. We generally take α = 5 % or 1%.

Step III: The third step is to determine an appropriate test statistic, say, t for testing the null hypothesis. Suppose T_n is the sample statistic (may be sample mean, sample correlation coefficient, etc. depending upon θ) for the parameter θ then test-statistic t is given by

$$t = \frac{T_n - E(T_n)}{SE(T_n)}$$

Step IV: As we know, t-test is based on t-distribution and t-distribution is described with the help of its degrees of freedom, therefore, test statistic t follows t-distribution with specified degrees of freedom as the case may be.

By putting the values of T_n, $E(T_n)$ and $SE(T_n)$ in above formula, we calculate the value of test statistic t. Let t_{cal} be the calculated value of test statistic t after putting these values.

Step V: After that, we obtain the critical (cut-off or tabulated) value(s) in the sampling distribution of the test statistic t corresponding to α assumed in Step II. The critical values for t-test are given in **Table-II (t-table)** of the Appendix at the end of Block 1 of this course corresponding to different level of significance (α). After that, we construct rejection (critical) region of size α in the probability curve of the sampling distribution of test statistic t.

Step VI: Take the decision about the null hypothesis based on calculated and critical value(s) of test statistic obtained in Step IV and Step V respectively. Since critical value depends upon the nature of the test that it is one-tailed test or two-tailed test so following cases arise:

In case of one-tailed test:

Case I: When $H_0 : \theta \le \theta_0$ and $H_1 : \theta > \theta_0$ (right-tailed test)

In this case, the rejection (critical) region falls under the right tail of the probability curve of the sampling distribution of test statistic t. Suppose $t_{(\nu),\alpha}$ is the critical value at α level of significance then entire region greater than or equal to $t_{(\nu),\alpha}$ is the rejection region and less than $t_{(\nu),\alpha}$ is the non-rejection region as shown in Fig. 11.1.

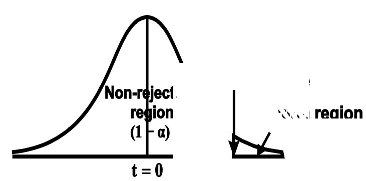

Fig. 11.1

If $t_{cal} \ge t_{(\nu,\alpha)}$, that means calculated value of test statistic t lies in the rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that sample data provides us sufficient evidence against the null hypothesis and there is a significant difference between hypothesized value and observed value of the parameter.

If $t_{cal} < t_{(\nu,\alpha)}$, that means calculated value of test statistic t lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that the sample data fails to provide us sufficient evidence against the null hypothesis and the difference between hypothesized value and observed value of the parameter due to fluctuation of sample.

Case II: When $H_0 : \theta \geq \theta_0$ and $H_1 : \theta < \theta_0$ (left-tailed test)

In this case, the rejection (critical) region falls under the left tail of the probability curve of the sampling distribution of test statistic t. Suppose $-t_{(\nu),\alpha}$ is the critical value at α level of significance then entire region less than or equal to $-t_{(\nu),\alpha}$ is the rejection region and greater than $-t_{(\nu),\alpha}$ is the non-rejection region as shown in Fig. 11.2.

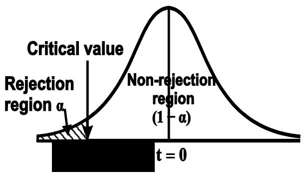

Fig. 11.2

If $t_{cal} \leq -t_{(\nu),\alpha}$, that means calculated value of test statistic t lies in the rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance.

If $t_{cal} > -t_{(\nu),\alpha}$, that means calculated value of test statistic t lies in the non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

In case of two-tailed test:

That is, when $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$

In this case, the rejection region falls under both tails of the probability curve of sampling distribution of the test statistic t. Half the area (α) i.e. $\alpha/2$ will lies under left tail and other half under the right tail. Suppose $-t_{(\nu),\alpha/2}$ and $t_{(\nu),\alpha/2}$ are the two critical values at the left- tailed and right-tailed respectively. Therefore, entire region less than or equal to $-t_{(\nu),\alpha/2}$ and greater than or equal to $t_{(\nu),\alpha/2}$ are the rejection regions and between $-t_{(\nu),\alpha/2}$ and $t_{(\nu),\alpha/2}$ is the non-rejection region as shown in Fig. 11.3.

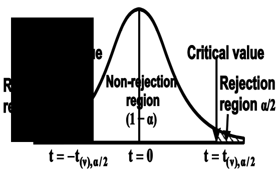

Fig. 11.3

If $t_{cal} \geq t_{(\nu),\alpha/2}$, or $t_{cal} \leq -t_{(\nu),\alpha/2}$, that means calculated value of test statistic t lies in the rejection(critical) region, then we reject the null hypothesis H_0 at α level of significance.

And if $-t_{(\nu),\alpha/2} < t_{cal} < t_{(\nu),\alpha/2}$, that means calculated value of test statistic t lies in the non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

Procedure of taking the decision about the null hypothesis on the basis of p-value:

To take the decision about the null hypothesis on the basis of p-value, the p-value is compared with given level of significance (α). And if p-value is less than or equal to α then we reject the null hypothesis and if p-value is greater than α then we do not reject the null hypothesis at α level of significance.

Since the distribution of test statistic t follows t-distribution with ν df and we also know that t-distribution is symmetrical about t = 0 line therefore, if t_{cal} represents calculated value of test statistic t then p-value can be defined as:

For one-tailed test:

For H_1: $\theta > \theta_0$ (right-tailed test)

$$\text{p-value} = P[t \geq t_{cal}]$$

For H_1: $\theta < \theta_0$ (left-tailed test)

$$\text{p-value} = P\left[t \leq t_{cal}\right]$$

For two-tailed test: For H_1: $\theta \neq \theta_0$

$$\text{p-value} = 2P\left[t \geq \left|t_{cal}\right|\right]$$

These p-values for t-test can be obtained with the help of **Table-II (t-table)** given in the Appendix at the end of Block 1 of this course. But this table gives the t-values corresponding to the standard values of α such as 0.10, 0.05, 0.025, 0.01 and 0.005 only, therefore, the exact p-values are not obtained with the help of this table and we can approximate the p-value for this test.

For example, if test is right-tailed and calculated (observed) value of test statistic t is 2.94 with 9 df then p-value is obtained as:

Since calculated value of test statistic t is based on the 9 df therefore, we use row for 9 df in the t-table and move across this row to find the values in which calculated t-value falls. Since calculated t-value falls between 2.821 and 3.250, which are corresponding to the values of one-tailed area α = 0.01 and 0.005 respectively, therefore, p-value will lie between 0.005 and 0.01, that is,

$$0.005 < \text{p-value} < 0.01$$

If in the above example, the test is two-tailed then the two values 0.01 and 0.005 would be doubled for p-value, that is,

$$0.005 \times 2 = 0.01 < \text{p-value} < 0.02 = 2 \times 0.01$$

Note 1: With the help of computer packages and softwares such as SPSS, SAS, MINITAB, EXCEL, etc. we can find the exact p-values for t-test.

Now, you can try the following exercise.

E1) If test is two-tailed and calculated value of test statistic t is 2.42 with 15 df then find the p-value for t-test.

11.3 TESTING OF HYPOTHESIS FOR POPULATION MEAN USING t-TEST

In Section 10.3 of the previous unit, we have discussed Z-test for testing the hypothesis about population mean when population variance σ^2 is known and unknown.

Recall form these, we have already pointed out that one basic difference between Z-test and t-test is that Z-test is used when population SD is known whether sample size is large or small and t-test is used when population SD is unknown whether sample size is small or large. But in case of large sample size Z-test is an appropriate of t-test as we did in previous unit. But in practice standard deviation of population is not known and sample size is small so in this situation, we use t-test provided population under study is normal.

Assumptions

Virtually every test has some assumptions which must be met prior to the application of the test. This t-test needs following assumptions to work:

(i) The characteristic under study follows normal distribution. In other words, populations from which random sample is drawn should be normal with respect to the characteristic of interest.

(ii) Sample observations are random and independent.

(iii) Population variance σ^2 is unknown.

For describing this test, let $X_1, X_2, \ldots, X_n$ be a random sample of **small size n (< 30)** selected from a **normal population** with mean μ and unknown variance σ^2.

Now, follow the same procedure as we have discussed in previous section, that is, first of all we setup the null and alternative hypotheses. Here, we want to test the claim about the specified value μ_0 of population mean μ so we can take the null and alternative hypotheses as

$$H_0 : \mu = \mu_0 \text{ and } H_1 : \mu \neq \mu_0 \; [\text{for two-tailed test}] \; \left[\begin{array}{l}\text{Here, } \theta = \mu \text{ and } \theta_0 = \mu_0 \\ \text{if we compare it with} \\ \text{general procedure.}\end{array}\right]$$

or
$$\left.\begin{array}{l} H_0 : \mu \leq \mu_0 \text{ and } H_1 : \mu > \mu_0 \\ H_0 : \mu \geq \mu_0 \text{ and } H_1 : \mu < \mu_0 \end{array}\right\} [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S / \sqrt{n}} \sim t_{(n-1)} \quad \text{under } H_0$$

where, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$ is the sample variance.

For computational simplicity, we may use the following formulae for $\bar{X}, S^2$:

$$\bar{X} = a + \frac{1}{n}\sum d \text{ and } S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right]$$

where, d = (X − a), 'a' being the assumed arbitrary value.

Here, the test statistic t follows t-distribution with $(n-1)$ degrees of freedom as we discussed in Unit 3 of this course.

After substituting values of $\bar{X}$, S and n , we get calculated value of test statistic t. Then we look for critical (or cut-off or tabulated) value(s) of test statistic t from the t-table. On comparing calculated value and critical value(s), we take the decision about the null hypothesis as discussed in previous section.

Let us do some examples of testing of hypothesis about population mean using t-test.

Example 1: A manufacturer claims that a special type of projector bulb has an average life 160 hours. To check this claim an investigator takes a sample of 20 such bulbs, puts on the test, and obtains an average life 167 hours with standard deviation 16 hours. Assuming that the life time of such bulbs follows normal distribution, does the investigator accept the manufacturer's claim at 5% level of significance?

Solution: Here, we are given that

$$\mu_0 = 160, \quad n = 20, \quad \bar{X} = 167 \quad \text{and} \quad S = 16$$

Here, we want to test the manufacturer claims that a special type of projector bulb has an average life (μ) 160 hours. So claim is $\mu = 160$ and its complement is $\mu \neq 160$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 160 \text{ and } H_1 : \mu \neq 160$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. Also sample size is small n = 20(n < 30) and population under study is normal, so we can go for t-test for testing the hypothesis about population mean.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

$$= \frac{167 - 160}{16/\sqrt{20}} = \frac{7}{3.58} = 1.96$$

The critical value of the test statistic t for various df and different level of significance α are given in **Table II** of the Appendix at the end of the Block 1of this course.

The critical (tabulated) values of test statistic for two-tailed test corresponding (n-1) = 19 df at 5% level of significance are $\pm t_{(n-1),\alpha/2} = \pm t_{(19),0.025} = \pm 2.093$.

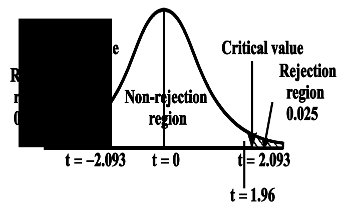

Fig. 11.4

Since calculated value of test statistic t (= 1.96) is greater than the critical value (= − 2.093) and is less than critical value (= 2.093), that means calculated value of test statistic lies in non-rejection region as shown in Fig. 11.4. So we do not reject the null hypothesis i.e. we support the manufacture's claim at 5% level of significance.

Decision according to p-value:

Since calculated value of test statistic t is based on 19 df therefore, we use row for 19 df in the t-table and move across this row to find the values in which calculated t-value falls. Since calculated t-value falls between 1.729 and 2.093 corresponding to one-tailed area $\alpha = 0.05$ and 0.025 respectively therefore p-value lies between 0.025 and 0.05, that is,

$$0.025 < \text{p-value} < 0.05$$

Since test is two-tailed so

$$2 \times 0.025 = 0.05 < \text{p-value} < 0.10 = 0.05 \times 2$$

Since p-value is greater than $\alpha(= 0.05)$ so we do not reject the null hypothesis at 5% level of significance.

Thus, we conclude that sample fails to provide us sufficient evidence against the null hypothesis so we may assume that the manufacture's claim is true so the investigator may accept the manufacturer's claim at 5% level of significance.

Example 2: The mean share price of companies of Pharma sector is Rs.70. The share prices of all companies were changed time to time. After a month, a sample of 10 Pharma companies was taken and their share prices were noted as below:

70, 76, 75, 69, 70, 72, 68, 65, 75, 72

Assuming that the distribution of share prices follows normal distribution, test whether mean share price is still the same at 1% level of significance?

Solution: Here, we wish to test that the mean share price (μ) of companies of Pharma sector is still Rs.70 besides all changes. So our claim is μ = 70 and its complement is $\mu \neq 70$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 70 \quad [\text{mean share price of companies is still Rs. } 70]$$

$$H_1 : \mu \neq \mu_0 = 70 \quad [\text{mean share price of companies is not still Rs. } 70]$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. Also sample size is small n = 10(n < 30) and population under study is normal, so we can go for t-test for testing the hypothesis about population mean.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)} \qquad \ldots (1)$$

Calculation for $\overline{X}$ and S:

S. No.	Sample value (X)	Deviation d = (X - a), a = 70	d^2
1	70	0	0
2	76	6	36
3	75	5	25
4	69	-1	1
5	70	0	0
6	72	2	4
7	68	-2	4
8	65	-5	25
9	75	5	25
10	72	2	4
Total		12	124

The assumed value of a is 70.

From the above calculation, we have

$$\overline{X} = a + \frac{1}{n}\sum d = 70 + \frac{1}{10} \times 12 = 71.2$$

$$S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right]$$

$$= \frac{1}{10-1}\left[124 - \frac{(12)^2}{10}\right] = \frac{1}{9}\left[124 - \frac{144}{10}\right] = 12.18$$

$$\Rightarrow S = \sqrt{12.18} = 3.49$$

Putting the values in equation (1), we have

$$t = \frac{71.2 - 70}{3.49/\sqrt{10}} = \frac{1.2}{1.10} = 1.09$$

The critical (tabulated) values of test statistic for two-tailed test corresponding (n-1) = 9 df at 1% level of significance are $\pm t_{(n-1),\alpha/2} = \pm t_{(9),0.005} = \pm 3.250$.

Since calculated value of test statistic t (= 1.09) is less than the critical value (= 3.250) and greater than the critical value (= –3.250), that means calculated value of t lies in non-rejection region as shown in Fig. 11.5. So we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

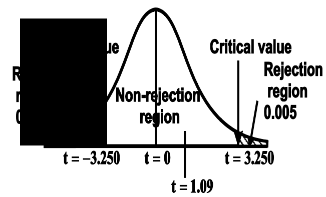

Fig. 11.5

Decision according to p-value:

Since calculated value of test statistic t is based on 9 df therefore, we use row for 9 df in the t-table and move across this row to find the values in which calculated t-value falls. Since all values in this row are greater than calculated t-value 1.09 and the smallest value is 1.383 corresponding to one-tailed area α = 0.10 therefore p-value is greater than 0.10, that is,

$$\text{p-value} > 0.10$$

Since test is two-tailed so

$$\text{p-value} > 2 \times 0.10 = 0.20$$

Since p-value (= 0.20) is greater than $\alpha(= 0.01)$ so we do not reject the null hypothesis at 1% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the claim so may assume that the mean share price is still Rs. 70.

Now, you can try the following exercises.

E2) A tyre manufacturer claims that the average life of a particular category of his tyre is 18000 km when used under normal driving conditions. A random sample of 16 tyres was tested. The mean and SD of life of the tyres in the sample were 20000 km and 6000 km respectively. Assuming that the life of the tyres is normally distributed, test the claim of the manufacturer at 1% level of significance using appropriate test.

E3) It is known that the average weight of cadets of a centre follows normal distribution. Weights of 10 randomly selected cadets from the same centre are as given below:

48, 50, 62, 75, 80, 60, 70, 56, 52, 77

Can we say that average weight of all cadets of the centre from which the above sample was taken is equal to 60 kg at 5% level of significance?

11.4 TESTING OF HYPOTHESIS FOR DIFFERENCE OF TWO POPULATION MEANS USING t-TEST

In Section 10.4 of the previous unit, we have discussed Z-test for testing the hypothesis about difference of two population means under different possibility of population variances σ_1^2 and σ_2^2. Recall from there, we have pointed out that one basic difference between Z-test and t-test is that, Z-test is used when standard deviations of both populations are known and t-test is used when standard deviations of both populations are unknown. But in practice standard deviations of both populations are not known, so in real life problems t-test is more suitable compared to Z-test.

Assumptions

This test works under following assumptions:

(i) The characteristic under study follows normal distribution in both the populations. In other words, both populations from which random samples are drawn should be normal with respect to the characteristic of interest.

(ii) Samples and their observations both are independent to each other.

(iii) Population variances σ_1^2 and σ_2^2 are both unknown but equal.

For describing this test, let there be two normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ under study. And we have to draw two independent random samples, say, $X_1, X_2, ..., X_{n_1}$ and $Y_1, Y_2, ..., Y_{n_2}$ of sizes n_1 and n_2 from these normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively. Let $\bar{X}$ and $\bar{Y}$ be the means of first and second sample respectively. Further, suppose the variances of both the populations are unknown but are equal, i.e., $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (say). In this case, σ^2 is estimated by value of pooled sample variance S_p^2 where,

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2\right]$$

and

$$S_1^2 = \frac{1}{(n_1 - 1)}\sum_{i=1}^{n_1}\left(X_i - \bar{X}\right)^2, \quad S_2^2 = \frac{1}{(n_2 - 1)}\sum_{i=1}^{n_2}\left(Y_i - \bar{Y}\right)^2$$

This can also be written as

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1}\left(X_i - \bar{X}\right)^2 + \sum_{i=1}^{n_2}\left(Y_i - \bar{Y}\right)^2\right]$$

For computational simplicity, use the following formulae for $\bar{X}, \bar{Y}$ and S_p^2:

$$\bar{X} = a + \frac{1}{n_1}\sum d_1, \quad \bar{Y} = b + \frac{1}{n_2}\sum d_2 \text{ and}$$

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\left\{\sum d_1^2 - \frac{\left(\sum d_1\right)^2}{n_1}\right\} + \left\{\sum d_2^2 - \frac{\left(\sum d_2\right)^2}{n_2}\right\}\right]$$

where, $d_1 = (X - a)$ and $d_2 = (Y - b)$, 'a' and 'b' are the assumed arbitrary values.

Now, follow the same procedure as we have discussed in Section 11.2, that is, first of all we have to setup null and alternative hypotheses. Here, we want to test the hypothesis about the difference of two population means so we can take the null hypothesis as

$$H_0 : \mu_1 = \mu_2 \text{ (no difference in means)} \begin{bmatrix} \text{Here, } \theta_1 = \mu_1 \text{ and } \theta_2 = \mu_2 \\ \text{if we compare it with} \\ \text{general procedure.} \end{bmatrix}$$

or $\quad H_0 : \mu_1 - \mu_2 = 0$ (difference in two means is 0) and the alternative hypothesis as

$$H_1: \mu_1 \neq \mu_2 \qquad [\text{for two-tailed test}]$$

or

$$\left.\begin{array}{l} H_0: \mu_1 \leq \mu_2 \text{ and } H_1: \mu_1 > \mu_2 \\ H_0: \mu_1 \geq \mu_2 \text{ and } H_1: \mu_1 < \mu_2 \end{array}\right\} [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} \quad \text{under } H_0$$

After substituting values of $\bar{X}$, $\bar{Y}$, S_p, n_1 and n_2, we get calculated value of test statistic t. Then we look for critical (or cut-off or tabulated) value(s) of test statistic t from the **t-table**. On comparing calculated value and critical value(s), we take the decision about the null hypothesis as discussed in Section 11.2.

Let us do some examples to become more user friendly with the test explained above.

Example 3: In a random sample of 10 pigs fed by diet A, the gain in weights (in pounds) in a certain period were

12, 8, 14, 16, 13, 12, 8, 14, 10, 9

In another random sample of 10 pigs fed by diet B, the gain in weights (in pounds) in the same period were

14, 13, 12, 15, 16, 14, 18, 17, 21, 15

Assuming that gain in the weights due to both foods follows normal distributions with equal variances, test whether diets A and B differ significantly regarding their effect on increase in weight at 5% level of significance.

Solution: Here, we can test that diets A and B differ significantly regarding their effect on increase in weight of pigs. If μ_1 and μ_2 denote the average gain in weights due to diet A and diet B respectively then our claim is $\mu_1 \neq \mu_2$ and its complement is $\mu_1 = \mu_2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0: \mu_1 = \mu_2 \text{ and } H_1: \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Since it is given that the increase in the weight due to both foods follows normal distributions and population variances are equal and unknown. And other assumptions of t-test for testing a hypothesis about difference of two population means also meet. So we can go for this test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} \quad \text{under } H_0 \qquad \ldots (2)$$

Calculation for $\bar{X}$, $\bar{Y}$ and S_p :

Diet A			Diet B		
X	**$d_1 = (X-a)$ a =12**	**d_1^2**	**Y**	**$d_2 = (Y-b)$ b = 16**	**d_2^2**
12	0	0	14	−2	4
8	−4	16	13	−3	9
14	2	4	12	−4	16
16	4	16	15	−1	1
13	1	1	16	0	0
12	0	0	14	−2	4
8	−4	16	18	2	4
14	2	4	17	1	1
10	−2	4	21	5	25
9	−3	9	15	−1	1
Total	−4	70		−5	65

Here, a =12, b = 16 are assumed values.

From above calculations, we have

$$\bar{X} = a + \frac{1}{n_1}\sum d_1 = 12 + \frac{(-4)}{10} = 11.6,$$

$$\bar{Y} = b + \frac{1}{n_2}\sum d_2 = 16 + \frac{(-5)}{10} = 15.5$$

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\left\{\sum d_1^2 - \frac{\left(\sum d_1\right)^2}{n_1}\right\} + \left\{\sum d_2^2 - \frac{\left(\sum d_2\right)^2}{n_2}\right\}\right]$$

$$= \frac{1}{10+10-2}\left[\left\{70 - \frac{(-4)^2}{10}\right\} + \left\{65 - \frac{(-5)^2}{10}\right\}\right]$$

$$= \frac{1}{18}(68.4 + 62.5) = 7.27$$

$$\Rightarrow S_p = \sqrt{7.27} = 2.70$$

Putting the values in equation (2), we have

$$t = \frac{11.6 - 15.5}{2.70\sqrt{\frac{1}{10} + \frac{1}{10}}}$$

$$= \frac{-3.90}{2.70 \times 0.45} = \frac{-3.90}{1.215} = -3.21$$

The critical values of test statistic t for two-tailed test corresponding $(n_1 + n_2 - 2) = 18$ df at 5% level of significance are $\pm t_{(n_1+n_2-2),\alpha/2} = \pm t_{(18),0.025} = \pm 2.101$.

Since calculated value of test statistic t (= −3.21) is less than critical values (± 2.101) that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. support our claim at 5% level of significance.

Thus, we conclude that samples do not provide us sufficient evidence against the claim so diets A and B differ significantly in terms of gain in weights of pigs.

Example 4: The means of two random samples of sizes 10 and 8 drawn from two normal populations are 210.40 and 208.92 respectively. The sum of squares of the deviations from their means is 26.94 and 24.50 respectively. Assuming that the populations are normal with equal variances, can samples be considered to have been drawn from normal populations having equal mean.

Solution: In usual notations, we are given that

$$n_1 = 10, \quad n_2 = 8, \quad \bar{X} = 210.40, \quad \bar{Y} = 208.92,$$

$$\sum\left(X - \bar{X}\right)^2 = 26.94, \qquad \sum\left(Y - \bar{Y}\right)^2 = 24.50$$

Therefore,

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum\left(X - \bar{X}\right)^2 + \sum\left(Y - \bar{Y}\right)^2\right]$$

$$= \frac{1}{10 + 8 - 2}\left[26.94 + 24.50\right] = \frac{1}{16} \times 51.44 = 3.215$$

$$\Rightarrow S_p = \sqrt{3.215} = 1.79$$

We wish to test that both the samples are drawn from normal populations having the same means. If μ_1 and μ_2 denote the means of both normal populations respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$H_0 : \mu_1 = \mu_2$ [mean of both populations is equal]

$H_1 : \mu_1 \neq \mu_2$ [mean of both populations is not equal]

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Since it is given that two populations are normal with equal and unknown variances and other assumptions of t-test for testing a hypothesis about difference of two population means also meet. So we can go for this test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t_{(n_1+n_2-2)} \quad \text{under } H_0$$

$$= \frac{210.40 - 208.92}{1.79\sqrt{\dfrac{1}{10} + \dfrac{1}{8}}} = \frac{1.48}{1.79 \times 0.47} = \frac{1.48}{0.84} = 1.76$$

The critical values of test statistic t for two-tailed test corresponding $(n_1 + n_2 - 2) = 16$ df at 5% level of significance are $\pm t_{(n_1+n_2-2),\alpha/2} = \pm t_{(16),0.025} = \pm 2.12$.

Since calculated value of test statistic t (= 1.76) is less than the critical value (= 2.12) and greater than the critical value (= –2.12), that means calculated value of test statistic t lies in non-rejection region so we do not reject the null hypothesis i.e. we support the claim.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that both samples are taken from normal populations having equal means.

Now, you can try the following exercises.

E4) Two different types of drugs A and B were tried on some patients for increasing their weights. Six persons were given drug A and other 7 persons were given drug B. The gain in weights (in ponds) is given below:

Drug A	5	8	7	10	9	6	−
Drug B	9	10	15	12	14	8	12

Assuming that increment in the weights due to both drugs follows normal distributions with equal variances, do the both drugs differ significantly with regard to their mean weights increment at 5% level of significance?

E5) To test the effect of fertilizer on wheat production, 26 plots of land with equal areas were chosen. Half of these plots were treated with fertilizer and the other half were untreated. Other conditions were the same. The mean yield of wheat on the untreated plots was 4.6 quintals with a standard deviation of 0.5 quintals, while the mean yield of the treated plots was 5.0 quintals with standard deviations of 0.3 quintals. Assuming that yields of wheat with and without fertilizer follow normal distributions with equal variances, can we conclude that there is significant improvement in wheat production due to effect of fertilizer at 1% level of significance?

11.5 PAIRED t-TEST

In the previous section, we have discussed t-test for equality of two population means in case of independent samples. However, there are so many situations where two samples are not independent and observations are recorded on the same individuals or items. Generally, such types of observations are recorded to assess the effectiveness of a particular training, diet, treatment, medicine, etc. In such situations, the observations are recorded **"before and after"** the insertion of training, treatment, etc. as the case may be. For example, if we wish to test a new diet on, say, 15 individuals then the weight of the individuals recorded before diet and after the diet will form two different samples in which observations will be paired as per each individual. Similarly, in the test of blood-sugar in human body, fasting sugar level before meal and sugar level after meal, both are recorded for a patient as paired observations, etc. The parametric test designed for this type of situation is known as paired t-test.

Now, come to the working principle of this test. This test first of all converts the two populations into a single population by taking the difference of paired observations.

Now, instead of two populations, we are left with one population, the population of differences. And the problem of testing equality of two population mean reduces to test the hypothesis that mean of the population of differences is equal to zero.

Assumptions

This test works under following assumptions:

(i) The population of differences follows normal distribution.

(ii) Samples are not independent.

(iii) Size of both the samples is equal.

(iv) Population variances are unknown but not necessarily equal.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a paired random sample of size n and the difference between paired observations X_i & Y_i be denoted by D_i, that is,

$$D_i = X_i - Y_i \quad \text{for all } i = 1, 2, \ldots, n$$

Hence, we can assume that $D_1, D_2, \ldots, D_n$ be a random sample from normal population of differences with mean μ_D and unknown variance σ_D^2. This is same as the case of testing of hypothesis for population mean when population variance is unknown which is described in Section 11.3 of this unit.

Here, we want to test that there is an effect of a diet, training, treatment, medicine, etc. So we can take the null hypothesis as

$$H_0: \mu_1 = \mu_2 \text{ or } H_0: \mu_D = \mu_1 - \mu_2 = 0$$

and the alternative hypothesis

$$H_1: \mu_1 \neq \mu_2 \text{ or } H_1: \mu_D \neq 0 \qquad [\text{for two-tailed test}]$$

$$\text{or} \quad \left.\begin{array}{l} H_0: \mu_1 \leq \mu_2 \text{ and } H_1: \mu_1 > \mu_2 \\ H_0: \mu_1 \geq \mu_2 \text{ and } H_1: \mu_1 < \mu_2 \end{array}\right\} \quad [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{D}}{S_D/\sqrt{n}} \sim t_{(n-1)} \quad \text{under } H_0$$

$$\text{where, } \bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i \text{ and } S_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(D_i - \bar{D}\right)^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} D^2 - \frac{\left(\sum_{i=1}^{n} D\right)^2}{n}\right]$$

After substituting values of $\bar{D}$, S_D and n we get calculated value of test statistic t. Then we look for critical (or cut-off or tabulated) value(s) of test statistic t from the **t-table**. On comparing calculated value and critical value(s), we take the decision about the null hypothesis as discussed in Section 11.2.

Let us do some examples to become more user friendly with paired t-test.

Example 5: A group of 12 children was tested to find out how many digits they would repeat from memory after hearing them once. They were given practice session for this test. Next week they were retested. The results obtained were as follows:

Child Number	1	2	3	4	5	6	7	8	9	10	11	12
Recall Before	6	4	5	7	6	4	3	7	8	4	6	5
Recall After	6	6	4	7	6	5	5	9	9	7	8	7

Assuming that the memories of the children before and after the practice session follow normal distributions, is the memory practice session improve the performance of children?

Solution: Here, we want to test that memory practice session improve the performance of children. If μ_1 and μ_2 denote the mean digit repetition before and after the practice so our claim is $\mu_1 < \mu_2$ and its complement is $\mu_1 \geq \mu_2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 \geq \mu_2 \text{ and } H_1 : \mu_1 < \mu_2$$

Since the alternative hypothesis is left-tailed so the test is left-tailed test.

It is a situation of before and after. Also, it is given that the memories of the children before and after the practice session follow normal distributions. So, population of differences will also be normal. Also all the assumptions of paired t-test meet so we can go for paired t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{D}}{S_D / \sqrt{n}} \sim t_{(n-1)} \text{ under } H_0 \qquad \ldots (3)$$

where, $\bar{D}$ and S_D are mean and standard deviation of the population of differences.

Calculation for $\bar{D}$ and S_D :

Child Number	Digit recall		D = (X−Y)	D^2
	Before (X)	After (Y)		
1	6	6	0	0
2	4	6	−2	4
3	5	4	1	1
4	7	7	0	0
5	6	6	0	0
6	4	5	−1	1
7	3	5	−2	4
8	7	9	−2	4
9	8	9	−1	1
10	4	7	−3	9
11	6	8	−2	4
12	5	7	−2	4
			$\sum D = -14$	$\sum D^2 = 32$

From above calculations, we have

$$\bar{D} = \frac{1}{n}\sum D = \frac{1}{12}(-14) = -1.17$$

$$S_D^2 = \frac{1}{n-1}\left\{\sum D^2 - \frac{\left(\sum D\right)^2}{n}\right\}$$

$$= \frac{1}{11}\left[32 - \frac{(-14)^2}{12}\right] = \frac{1}{11} \times 15.67 = 1.42$$

$$\Rightarrow \quad S_D = \sqrt{1.42} = 1.19$$

Substituting these values in equation (3), we have

$$t = \frac{-1.17}{1.19/\sqrt{12}} = \frac{-1.17}{0.34} = -3.44$$

The critical value of test statistic t for left-tailed test corresponding (n-1) = 11 df at 5% level of significance is $-t_{(n-1),\alpha} = -t_{(11),0.05} = -1.796$.

Since calculated value of test statistic t (= −3.44) is less than the critical value (=−1.796), that means calculated value of t lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. support the claim at 5% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that memory practice session improves the performance of children.

Example 6: Ten students were given a test in Statistics and after one month's coaching they were again given a test of the similar nature and the increase in their marks in the second test over the first are shown below:

Roll No.	1	2	3	4	5	6	7	8	9	10
Increase in Marks	6	−2	8	−4	10	2	5	−4	6	0

Assuming that increment in marks follows normal distribution. Do the data indicate that students have gained knowledge from the coaching at 1% level of significance?

Solution: Here, we want to test that students have gained knowledge from the coaching. If μ_D denotes the average increment in the marks due to one month's coaching then our claim is $\mu_D < 0$ but here we are given increment $\mathbf{D_i = (Y_i - X_i)}$ instead of $\mathbf{D_i = (X_i - Y_i)}$ so we take our claim is $\mu_D > 0$ and its complement is $\mu_D \leq 0$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_D \leq 0 \text{ and } H_1 : \mu_D > 0$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

It is given that increment in marks after one month coaching follows normal distribution and population variance is unknown. Also participants are same in both situations before and after the coaching. And all the assumption of paired t-test meet so we can go for paired t-test.

For testing H_0, the test statistic is given by

$$t = \frac{\bar{D}}{S_D/\sqrt{n}} \sim t_{(n-1)} \text{ under } H_0 \qquad \ldots (4)$$

Calculation for $\bar{D}$ and S_D:

Roll No.	1	2	3	4	5	6	7	8	9	10	Total
D	6	−2	8	−4	10	2	5	−4	6	0	$\sum D = 27$
D^2	36	4	64	16	100	4	25	16	36	0	$\sum D^2 = 301$

From above calculations, we have

$$\bar{D} = \frac{1}{n}\sum D = \frac{1}{10} \times 27 = 2.7$$

$$S_D^2 = \frac{1}{n-1}\left\{\sum D^2 - \frac{\left(\sum D\right)^2}{n}\right\}$$

$$= \frac{1}{10-1}\left[301 - \frac{(27)^2}{10}\right] = \frac{1}{9} \times 228.1 = 25.34$$

$$\Rightarrow \quad S_D = \sqrt{25.34} = 5.03$$

Substituting these values in equation (4), we have

$$t = \frac{2.7}{5.03/\sqrt{10}} = \frac{2.7}{1.59} = 1.70$$

The critical value of test statistic t for right-tailed test corresponding (n - 1) = 9 df at 1% level of significance is $t_{(n-1),\,\alpha} = t_{(9),\,0.01} = 2.821$.

Since calculated value of test statistic t (= 1.70) is less than the critical value (= 2.821), that means calculated value of test statistic t lies in non-rejection region, so we do not reject null hypothesis and reject the alternative hypothesis i.e. we reject our claim at 1% level of significance.

Thus, we conclude that sample provides us sufficient evidence against the claim so students are not gained knowledge from the coaching.

Now, you can try the following exercises.

E6) To verify whether the programme "Post Graduate Diploma in Applied Statistics (PGDAST)" improved performance of the graduate students in Statistics, a similar test was given to10 participants both before and after the programme. The original marks out of 100 (before course) recorded in an alphabetical order of the participants are 42, 46, 50, 36, 44, 60, 62, 43, 70 and 53. After the course the marks in the same order are 45, 46, 60, 42, 60, 72, 63, 43, 80 and 65. Assuming that marks of the students before and after the course follow normal distribution. Test whether the programme PGDAST has improved the performance of the graduate students in Statistics at 5% level of significance?

E7) A drug is given to 8 patients and the increments in their blood pressure are recorded to be 4, 0, 7, −2, 0, −3, 2, 0. Assume that increment in their blood pressure follows normal distribution. Is it reasonable to believe that the drug has no effect on the change of blood pressure at 5% level of significance?

11.6 TESTING OF HYPOTHESIS FOR POPULATION CORRELATION COEFFICIENT USING t-TEST

In Unit 6 of MST-002, we have discussed the concept of correlation. Where, we studied that if two variables are related in such a way that change in the value of one variable affects the value of another variable then the variables are said to be correlated or there is a correlation between these two variables. Correlation can be positive, which means the variables move together in the same direction, or negative, which means they move in opposite directions. And correlation coefficient is used to measure the intensity or degree of linear relationship between two variables. The value of correlation coefficient varies between −1 and +1, where −1 representing a perfect negative correlation, 0 representing no correlation, and +1 representing a perfect positive correlation.

Sometime, the sample data indicate for non-zero correlation but in population they are uncorrelated ($\rho = 0$).

For example, price of tomato in Delhi (X) and in London (Y) are not correlated in population ($\rho = 0$). But paired sample data of 20 days of prices of tomato at both places may show correlation coefficient (r) $\neq 0$. In general, in sample data $r \neq 0$ does not ensure in population $\rho \neq 0$ holds.

In this section, we will know how we test the hypothesis that population correlation coefficient is zero.

Assumptions

This test works under following assumptions:

(i) The characteristic under study follows normal distribution in both the populations. In other words, both populations from which random samples are drawn should be normal with respect to the characteristic of interest.

(ii) Samples observations are random.

Let us consider a random sample $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ of size n taken from a bivariate normal population. Let ρ and r be the correlation coefficients of population and sample data respectively.

Here, we wish to test the hypothesis about population correlation coefficient (ρ), that is, linear correlation between two variables X and Y in the population, so we can take the null hypothesis as

$$H_0 : \rho = 0 \text{ and } H_1 : \rho \neq 0 \left[\text{for two-tailed test}\right] \quad \left[\begin{array}{l}\text{Here, } \theta = \rho \text{ and } \theta_0 = 0 \text{ if}\\ \text{we compare it with general}\\ \text{procedure given in Section 11.2.}\end{array}\right]$$

or
$$\left.\begin{array}{l} H_0 : \rho \leq 0 \text{ and } H_1 : \rho > 0 \\ H_0 : \rho \geq 0 \text{ and } H_1 : \rho < 0 \end{array}\right\} \left[\text{for one-tailed test}\right]$$

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{(n-2)}$$

which follows t-distribution with n – 2 degrees of freedom.

After substituting values of r and n, we find out calculated value of test statistic t. Then we look for critical (or cut-off or tabulated) value(s) of test statistic t from the **t-table**. On comparing calculated value and critical value(s), we take the decision about the null hypothesis as discussed in Section 11.2.

Let us do some examples of testing of hypothesis that population correlation coefficient is zero.

Example 7: A random sample of 18 pairs of observations from a normal population gave a correlation coefficient of 0.7. Test whether the population correlation coefficient is zero at 5% level of significance.

Solution: Given that

$$n = 18, r = 0.7$$

Here, we wish to test that population correlation coefficient (ρ) is zero so our claim is $\rho = 0$ and its complement is $\rho \neq 0$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \rho = 0 \text{ and } H_1 : \rho \neq 0$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis regarding population correlation coefficient is zero and the populations under study follow normal distributions, so we can go for t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$= \frac{0.7\sqrt{18-2}}{\sqrt{1-(0.7)^2}} = \frac{0.7\times 4}{\sqrt{0.51}} = \frac{2.8}{0.71} = 3.94$$

The critical value of test statistic t for two-tailed test corresponding (n-2) = 16 df at 5% level of significance are $\pm t_{(n-2),\alpha/2} = \pm t_{(16),0.025} = \pm 2.120$.

Since calculated value of test statistic t (= 3.94) is greater than the critical values (= ± 2.120), that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis. i.e. we reject the claim at 5% level of significance.

Thus, we conclude that sample provides us sufficient evidence against the claim so there exists a relationship between two variables.

Example 8: A random sample of 15 married couples was taken from a population consisting of married couples between the ages of 30 and 40. The correlation coefficient between the IQs of husbands and wives was found to be 0.68. Assuming that the IQs of husbands and wives follow normal distributions then test that IQs of husbands and wives in the population are positively correlated at1% level of significance.

Solution: Given that

$$n = 15, r = 0.68$$

Here, we wish to test that IQs of husbands and wives in the population are positively correlated. If ρ denote the correlation coefficient between IQs of husbands and wives in the population then the claim is $\rho > 0$ and its complement is $\rho \leq 0$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \rho \leq 0 \text{ and } H_1 : \rho > 0$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Here, we want to test the hypothesis regarding population correlation coefficient is zero and the populations under study follow normal distributions, so we can go for t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$= \frac{0.68\sqrt{15-2}}{\sqrt{1-(0.68)^2}} = \frac{0.68 \times 3.61}{0.73} = 3.36$$

The critical value of test statistic t for right-tailed test corresponding (n-2) = 13 df at 1% level of significance is $t_{(n-2),\alpha} = t_{(13),0.01} = 2.650$.

Since calculated value of test statistic t (= 3.36) is greater than the critical value (= 2.650), that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. we support our claim at 1% level of significance.

Thus, we conclude that sample fail to provide us sufficient evidence against the claim so we may assume that the correlation between IQs of husbands and wives in the population is positive.

In the same way, you can try the following exercise.

E8) Twenty families were selected randomly from a colony to determine that correlation exists between family income and the amount of money spent per family member on food each month. The sample correlation coefficient was computed as r = 0.40. Assuming that the family income and the amount of money spent per family member on food each month follow normal distributions then test that there is a positive linear relationship between the family income and the amounts of money spent per family member on food each month in colony at 1% level of significance.

We now end this unit by giving a summary of what we have covered in it.

11.7 SUMMARY

In this unit, we have discussed the following points:

1. Need of small sample tests.
2. Procedure of testing a hypothesis for t-test.
3. Testing of hypothesis for population mean using t-test.
4. Testing of hypothesis for difference of two population means when samples are independent using t-test.
5. The procedure of paired t-test for testing of hypothesis for difference of two population means when samples are dependent or paired.
6. Testing of hypothesis for population correlation coefficient using t-test.

11.8 SOLUTIONS /ANSWERS

E1) Since calculated value of test statistic t is based on 15 df therefore, we use row for 15 df in the **t-table** and move across this row to find the values in which calculated t-value lies. Since calculated t-value falls between 2.131 and 2.602, which are corresponding to the values of one-tailed area α = 0.025 and 0.01 respectively, therefore p-value will lie between 0.01 and 0.025, that is, $0.01 < \text{p-value} < 0.025$

Since test is two-tailed, therefore, the values are doubled, so

$$0.02 = 2 \times 0.01 < \text{p-value} < 2 \times 0.025 = 0.05$$

E2) Here, we are given that

$$n = 16,\ \mu_0 = 18000,\ \bar{X} = 20000,\ S = 6000$$

Here, we want to test that manufacturer's claim is true that the average life (μ) of tyres is 18000 km. So claim is $\mu = 18000$ and its complement is $\mu \neq 18000$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$H_0: \mu = \mu_0 = 18000$ [average life of tyres is 18000 km]

$H_1: \mu \neq 18000$ [average life of tyres is not 18000 km]

Here, population SD is unknown and population under study is given to be normal. So we can go for t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{20000 - 18000}{6000/\sqrt{16}} = \frac{2000}{1500} = 1.33$$

The critical value of test statistic t for two-tailed test corresponding (n-1) = 15 df at 1% level of significance are $\pm\, t_{(15),0.005} = \pm\, 2.947$.

Since calculated value of test statistic t (= 1.33) is less than the critical (tabulated) value (= 2.947) and greater that critical value (= − 2.947), that means calculated value of test statistic lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the manufacture's claim at 1% level of significance.

Thus, we conclude that sample fails to provide sufficient evidence against the claim so we may assume that manufacturer's claim is true.

E3) Here, we want to test that the average weight (μ) of all cadets of the centre is 60 kg. So our claim is $\mu = 60$ and its complement is $\mu \neq 60$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$H_0: \mu = \mu_0 = 60$ [average weight of all cadets is 60 kg]

$H_1: \mu \neq 60$ [average weight of all cadets is not 60 kg]

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, population SD is unknown and population under study is given to be normal. So we can go for t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)} \text{ under } H_0 \qquad \ldots (5)$$

Before moving further, first we have to calculate value of $\bar{X}$ and S.

Calculation for $\bar{X}$ and S:

Sample value (X)	$(x - \bar{x})$	$(x - \bar{x})^2$
48	−15	225
50	−13	169
62	−1	1
75	12	144
80	17	289
60	−3	9
70	7	49

56	−7	49
52	−11	121
77	14	196
$\sum X = 630$		$\sum\left(X-\bar{X}\right)^2 = 1252$

From the above calculation, we have

$$\bar{X} = \frac{1}{n}\sum X = \frac{1}{10} \times 630 = 63$$

$$S^2 = \frac{1}{n-1}\sum\left(X-\bar{X}\right)^2$$

$$= \frac{1}{10-1} \times 1252 = 139.11$$

$$\Rightarrow S = \sqrt{139.11} = 11.79$$

Putting the values in equation (5), we have

$$t = \frac{63-60}{11.79/\sqrt{10}}$$

$$= \frac{3}{3.73} = 0.80$$

The critical values of test statistic t for two-tailed test corresponding $(n-1) = 9$ df at 5% level of significance are $\pm\, t_{(9),0.025} = \pm\, 2.262$.

Since calculated value of test statistic t (= 0.80) is less than the critical value (= 2.262) and greater than the critical value (= −2.262), that means calculated value of test statistic t lies in non-rejection region so we do not reject H_0 i.e. we support the claim at 5% level of significance.

Thus, we conclude that sample fails to provide sufficient evidence against the claim so we may assume that the average weight of all the cadets of given centre is 60 kg.

E4) Here, we want to test that there is no difference between drugs A and B with regard to their mean weight increment. If μ_1 and μ_2 denote the mean weight increment due to drug A and drug B respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \quad [\text{effect of both drugs is same}]$$

$$H_1 : \mu_1 \neq \mu_2 \quad [\text{effect of both drugs is not same}]$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Since it is given that increments in the weight due to both drugs follow normal distributions with equal and unknown variances and other assumptions of t-test for testing a hypothesis about difference of two population means also meet. So we can go for this test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} \text{ under } H_0 \qquad \dots (6)$$

Assume, a = 8, b = 12 and use short-cut method to find $\bar{X}$, $\bar{Y}$ and S_p.

Calculation for $\bar{X}$, $\bar{Y}$ and S_p :

Drug A			**Drug B**		
X	**d_1 = (X-a) a = 8**	**d_1^2**	**Y**	**d_2 = (Y-b) b = 12**	**d_2^2**
5	–3	9	9	–3	9
8	0	0	10	–2	4
7	–1	1	15	3	9
10	2	4	12	0	0
9	1	1	14	2	4
6	–2	4	8	–4	16
			12	0	0
	$\sum d_1 = -3$	$\sum d_1^2 = 19$		$\sum d_2 = -4$	$\sum d_2^2 = 42$

From above calculation, we have

$$\bar{X} = a + \frac{1}{n_1}\sum d_1 = 8 + \frac{1}{6}(-3) = 7.5,$$

$$\bar{Y} = b + \frac{1}{n_2}\sum d_2 = 12 + \frac{1}{7}(-4) = 11.43$$

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\left\{\sum d_1^2 - \frac{\left(\sum d_1\right)^2}{n_1}\right\} + \left\{\sum d_2^2 - \frac{\left(\sum d_2\right)^2}{n_2}\right\}\right]$$

$$= \frac{1}{6+7-2}\left[\left\{19 - \frac{(-3)^2}{6}\right\} + \left\{42 - \frac{(-4)^2}{7}\right\}\right]$$

$$= \frac{1}{11}(17.5 + 39.71) = 5.20$$

$$\Rightarrow S_p = \sqrt{5.20} = 2.28$$

Putting these values in equation (6), we have

$$t = \frac{7.5 - 11.43}{2.28\sqrt{\frac{1}{6} + \frac{1}{7}}} = \frac{-3.93}{2.28 \times 0.56} = \frac{-3.93}{1.28} = -3.07$$

The critical values of test statistic t for two-tailed test corresponding $(n_1 + n_2 - 2) = 11$ df at 5% level of significance are $\pm\, t_{(11),0.025} = \pm\, 2.201$.

Since calculated value of test statistic t (= −3.07) is less than the critical values (= ± 2.201) that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so drugs A and B differ significantly. Any one of them is better than other.

E5) Here, we are given that

$$n_1 = 13,\ \ \bar{X} = 4.6,\ \ S_1 = 0.5,$$

$$n_2 = 13,\ \ \bar{Y} = 5.0,\ \ S_2 = 0.3$$

Therefore, the pooled variance S_p^2 can be calculated as

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2\right]$$

$$= \frac{1}{13+13-2}\left[12\times(0.5)^2 + 12\times(0.3)^2\right]$$

$$= \frac{1}{24}(3.00 + 1.08) = 0.17$$

$$\Rightarrow S_p = \sqrt{0.17} = 0.41$$

We want to test that the there is significant improvement in wheat production due to fertilizer. If μ_1 and μ_2 denote the mean wheat productions without and with the fertilizer respectively then our claim is $\mu_1 < \mu_2$ and its complement is $\mu_1 \geq \mu_2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 \geq \mu_2 \text{ and } H_1 : \mu_1 < \mu_2$$

Since the alternative hypothesis is left-tailed so the test is left-tailed test.

Since it is given that yield of wheat with and without fertilizer follow normal distributions with equal and unknown variances and other assumptions of t-test for testing a hypothesis about difference of two population means also meet. So we can go for this test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} \quad \text{under } H_0$$

$$= \frac{4.6 - 5.0}{0.41\sqrt{\frac{1}{13} + \frac{1}{13}}} = \frac{-0.4}{0.41\times 0.39} = \frac{-0.4}{0.16} = -2.5$$

The critical values of test statistic t for left-tailed test corresponding $(n_1 + n_2 - 2) = 24$ df at 1% level of significance is $-t_{(24),\,0.01} = -2.492$.

Since calculated value of test statistic t (= −2.5) is less than the critical value (= −2.492), that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. we support our claim at 1% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that there is significant improvement in wheat production due to fertilizer.

E6) Here, we want to test whether the programme PGDAST has improved the performance of the graduate students in Statistics. If μ_1 and μ_2 denote the average marks before and after the programmed so our claim is $\mu_1 < \mu_2$ and its complement is $\mu_1 \geq \mu_2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 \geq \mu_2 \text{ and } H_1 : \mu_1 < \mu_2$$

Since the alternative hypothesis is left-tailed so the test is left-tailed test.

It is a situation of before and after. Also, the marks of the students before and after the programme PGDAST follow normal distributions. So, population of differences will also be normal. Also all the assumptions of paired t-test meet. So we can go for paired t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\overline{D}}{S_D / \sqrt{n}} \sim t_{(n-1)} \text{ under } H_0 \qquad \ldots (7)$$

Calculation for $\overline{D}$ and S_D:

Participant	Marks		D = (X−Y)	D^2
	Before (X)	After (Y)		
1	42	45	−3	9
2	46	46	0	0
3	50	60	−10	100
4	36	42	−6	36
5	44	60	−16	256
6	60	72	−12	144
7	62	63	−1	1
8	43	43	0	0
9	70	80	−10	100
10	53	65	−12	144
			$\sum D = -70$	$\sum D^2 = 790$

From above calculation, we have

$$\overline{D} = \frac{1}{n}\sum D = \frac{1}{10}(-70) = -7$$

$$S_D^2 = \frac{1}{n-1}\left[\sum D^2 - \frac{\left(\sum D\right)^2}{n}\right]$$

$$= \frac{1}{9}\left[790 - \frac{(-70)^2}{10}\right] = \frac{1}{9} \times 300 = 33.33$$

$$\Rightarrow S_D = \sqrt{33.33} = 5.77$$

Putting the values in equation (7), we have

$$t = \frac{-7.0}{5.77/\sqrt{10}} = -3.83$$

The critical value of test statistic t for left-tailed test corresponding $(n-1) = 9$ df at 5% level of significance is $-t_{(9),\,0.05} = -1.833$.

Since calculated value of test statistic t (= −3.83) is less than the critical (tabulated) value (= −1.833), that means calculated value of test statistic t lies in rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. we support our claim at 5% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that the participants have significant improvement after the programme "Post Graduate Diploma in Applied Statistics (PGDAST)".

E7) Here, we want to test that the drug has no effect on change in blood pressure. If μ_D denotes the average increment in the blood pressure before drug then our claim is $\mu_D = 0$ and its complement is $\mu_D \neq 0$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_D = \mu_1 - \mu_2 = 0 \; [\text{the drug has no effect}]$$

$$H_1 : \mu_D \neq 0 \qquad [\text{the drug has an effect}]$$

It is given that increment in the blood pressure follows normal distribution. Also patients are same in both situations before and after the drug. And all the assumption of paired t-test meet. So we can go for paired t-test.

For testing H_0, the test statistic is given by

$$t = \frac{\bar{D}}{S_D/\sqrt{n}} \sim t_{(n-1)} \text{ under } H_0 \qquad \ldots (8)$$

Calculation for $\bar{D}$ and S_D:

Patient Number	1	2	3	4	5	6	7	8	Total
D	4	0	7	−2	0	−3	2	0	$\sum D = 8$
$(\mathbf{D} - \bar{\mathbf{D}})$	3	−1	6	−3	−1	−4	1	−1	
$(\mathbf{D} - \bar{\mathbf{D}})^2$	9	1	36	9	1	16	1	1	$\sum (D-\bar{D})^2 = 74$

From above calculation, we have

$$\bar{D} = \frac{1}{n}\sum D = \frac{1}{8} \times 8 = 1$$

$$S_D^2 = \frac{1}{n-1}\sum (D-\bar{D})^2 = \frac{1}{7} \times 74 = 10.57$$

$$\Rightarrow S_D = \sqrt{10.57} = 3.25$$

Putting the values in test statistic, we have

$$t = \frac{1}{3.25/\sqrt{8}} = \frac{1}{1.15} = 0.87$$

The critical value of test statistic t for two-tailed test corresponding (n-1) = 7 df at 5% level of significance are $\pm t_{(7),\,0.025} = \pm 2.365$.

Since calculated value of test statistic t (= 0.87) is less than the critical value (= 2.365) and greater than the critical value (=−2.365) that means calculated value of test statistic t lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim at 5% level of significance.

Thus, we conclude that samples fail to provide us sufficient evidence against the claim so we may assume that the drug has no effect on the change of blood pressure of patients.

E8) We are given that

$$n = 20,\ r = 0.40$$

and we wish to test that there is a positive linear relationship between the family income and the amounts of money spent per family member on food each month in colony. If ρ denote the correlation coefficient between the family income and the amounts of money spent per family member then the claim is $\rho > 0$ and its complement is $\rho \le 0$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \rho \le 0 \text{ and } H_0 : \rho > 0$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$= \frac{0.40\sqrt{20-2}}{\sqrt{1-(0.40)^2}} = \frac{0.40 \times 4.24}{0.92} = 1.84$$

The critical value of test statistic t for right-tailed test corresponding (n-2) = 18 df at 1% level of significance is $t_{(n-2),\alpha} = t_{(18),0.01} = 2.552$.

Since calculated value of test statistic t (= 1.84) is less than the critical value (= 2.552), that means calculated value of test statistic t lies in non-rejection region, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject our claim at 1% level of significance.

Thus, we conclude that sample provide us sufficient evidence against the claim so there is no positive linear correlation between the family income and the amounts of money spent per family member on food each month in colony.