Date:

TASK 4: Commutation Relations and Euler Decomposition

Aim: To verify Pauli matrix commutation relations and decompose a gate using Euler angles.

1. Verify the fundamental commutation and anti-commutation relations of Pauli matrices (X, Y, Z)
2. Implement and validate Z-Y-Z Euler angle decomposition for arbitrary single-qubit gates
3. Demonstrate the decomposition on standard quantum gates (X, Y, Z, H, S, T) and Cirq operations

1. Mathematical Model

1.1 Pauli Matrix Commutation & Anti-Commutation Relations

- **Pauli Matrices**

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Commutator**

The commutator of two operators A,B is

$$[A, B] = AB - BA$$

For Pauli matrices

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

where ϵ_{ijk} is the Levi-Civita symbol.

Example:
$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$$

- **Anti-Commutator**

The anti-commutator is

$$\{A, B\} = AB + BA$$

For Pauli matrices

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\, I$$

where δ_{ij} is the Kronecker delta and I is the 2×2 identity.

This means

- If $i = j$: $\{\sigma_i, \sigma_j\} = 2I$
- If $i \neq j$: $\{\sigma_i, \sigma_j\} = 0$

1.2 Z–Y–Z Euler Decomposition for Single-Qubit Gates

Any single-qubit unitary $U \in SU(2)$ can be written (up to a global phase $e^{i\phi}$) as:

$$U = e^{i\phi} R_z(\alpha) R_y(\beta) R_z(\gamma)$$

where

- ϕ is a global phase.
- α, β, γ are rotation angles.
- Rotation operators

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

- **Decomposition steps**:

1. Extract global phase

$$\phi = \frac{\arg(\det(U))}{2}$$

2. Remove the global phase

$$U_0 = U\, e^{-i\phi}$$

3. From U0, solve for β using

$$\beta = 2\cos^{-1}(|U_{00}|)$$

4. Solve for α and γ from the argument (phase) of elements U_{00} and U_{01}.

2. Algorithm

2.1 Pauli Matrix Verification

a. Symbolically define Pauli matrices using SymPy
b. Compute commutators [A,B] = AB-BA and verify [σi,σj] = 2iεijkσk
c. Compute anti-commutators {A,B} = AB+BA and verify {σi,σj} = 2δijI

2.2 Z-Y-Z Decomposition

a. Check matrix unitarity: $U^T U = I$
b. Extract global phase from determinant
c. Solve for Euler angles (α, β, γ) in:
d. U = e^iφ Rz(α)Ry(β)Rz(γ)
e. Handle special cases when β ≈ 0 or π
f. Reconstruct matrix to validate decomposition

2.3 Testing

a. Standard gates: X, Y, Z, Hadamard (H), Phase (S), π/8 (T)
b. Random unitary matrices
c. Optional Cirq integration for hardware verification

3. Program

```
import numpy as np
import cmath

print("\n" + "="*50)
print("TASK 4: COMMUTATION RELATIONS AND EULER ANGLES")
print("="*50)

# --- Part 1: Verify Pauli commutation & anti-commutation with SymPy -
--
import sympy as sp

I = sp.eye(2)
sx = sp.Matrix([[0, 1], [1, 0]])
sy = sp.Matrix([[0, -sp.I], [sp.I, 0]])
```

```python
sz = sp.Matrix([[1, 0], [0, -1]])
paulis = {'X': sx, 'Y': sy, 'Z': sz}

def comm(A, B):
    return sp.simplify(A * B - B * A)

def anti(A, B):
    return sp.simplify(A * B + B * A)

print("\n=== Commutation relations ===")
for (a, b, k) in [('X', 'Y', 'Z'), ('Y', 'Z', 'X'), ('Z', 'X', 'Y')]:
    lhs = comm(paulis[a], paulis[b])
    rhs = 2 * sp.I * paulis[k]
    print(f"[{a},{b}] =\n{lhs}\nExpected:\n{rhs}\n")

print("\n=== Anti-commutation relations ===")
for i in ['X', 'Y', 'Z']:
    for j in ['X', 'Y', 'Z']:
        lhs = anti(paulis[i], paulis[j])
        rhs = 2 * (1 if i == j else 0) * I
        print(f"{{{i},{j}}} =\n{lhs}  Expected:\n{rhs}\n")

# --- Part 2: Z-Y-Z Euler decomposition ---
def is_unitary(U, tol=1e-8):
    return np.allclose(U.conj().T @ U, np.eye(2), atol=tol)

def decompose_zyz(U, tol=1e-8):
    """Return (phi, alpha, beta, gamma) such that
       U = e^{i phi} Rz(alpha) Ry(beta) Rz(gamma)
    """
    U = np.array(U, dtype=complex)
    if not is_unitary(U):
        raise ValueError("Matrix is not unitary.")
    detU = np.linalg.det(U)
    phi = cmath.phase(detU) / 2
    U0 = U * np.exp(-1j * phi)
    detU0 = np.linalg.det(U0)
    U0 = U0 / np.sqrt(detU0)
    a = U0[0, 0]
    b = U0[0, 1]
    beta = 2 * np.arccos(min(1.0, max(0.0, abs(a))))
    if np.isclose(np.sin(beta / 2), 0, atol=tol):
        alpha = 2 * (-cmath.phase(a))
        gamma = 0.0
    else:
        phi1 = -cmath.phase(a)
        phi2 = -cmath.phase(-b)
        alpha = phi1 + phi2
        gamma = phi1 - phi2
    return float(phi), float(alpha), float(beta), float(gamma)

def Rz(theta):
```

```
    return np.array([[np.exp(-1j * theta / 2), 0],
                     [0, np.exp(1j * theta / 2)]], dtype=complex)

def Ry(theta):
    return np.array([[np.cos(theta / 2), -np.sin(theta / 2)],
                     [np.sin(theta / 2), np.cos(theta / 2)]],
dtype=complex)

def reconstruct(phi, alpha, beta, gamma):
    return np.exp(1j * phi) @ (Rz(alpha) @ Ry(beta) @ Rz(gamma))

# --- Part 3: Test examples ---
def Rx(theta):
    return np.cos(theta / 2) * np.eye(2) - 1j * np.sin(theta / 2) * sx

examples = {
    "Rx(pi/3)": Rx(np.pi / 3),
    "Ry(pi/4)": Ry(np.pi / 4),
    "Rz(pi/2)": Rz(np.pi / 2),
    "H": (1 / np.sqrt(2)) * np.array([[1, 1], [1, -1]],
dtype=complex),
    "S": np.array([[1, 0], [0, 1j]], dtype=complex),
    "T": np.array([[1, 0], [0, np.exp(1j * np.pi / 4)]],
dtype=complex),
}

print("\n=== Z-Y-Z Euler Decomposition ===")
for name, U in examples.items():
    phi, alpha, beta, gamma = decompose_zyz(U)
    print(f"{name}:\n  φ={phi:.6f}, α={alpha:.6f}, β={beta:.6f},
γ={gamma:.6f}\n")

# Optional: Use Cirq if available
try:
    import cirq
    print("\nCirq example decomposition for H gate:")

    # Create a qubit and turn H into an operation
    q = cirq.LineQubit(0)
    H_op = cirq.H(q)

    # Extract the unitary matrix of H
    U = cirq.unitary(H_op)

    # Perform Z-Y-Z decomposition
    phi, alpha, beta, gamma = decompose_zyz(U)
    print(f"Cirq H: φ={phi:.6f}, α={alpha:.6f}, β={beta:.6f},
γ={gamma:.6f}")

except ImportError:
    print("\nCirq not installed. Skipping Cirq examples.")
```

4. Result

Commutation properties and Euler angle decomposition were successfully demonstrated.