E2-243
Programming Exercise - 2

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Instructions:

- Do not submit your code, output files, etc.
- There will one lab exam towards the end of the semester that will test your understanding of the concepts taught in class. The questions in the lab exam will be somewhat similar to these questions in both content and implementation complexity. If you do not program these exercises, handling the lab exam will not be easy! In a way, programming these assignments yourself will be your preparation for the lab exam.
- You may use any discrete plot like the 'stem' function in MATLAB.
- Whenever $x \in$ interval, you may have to take appropriate discrete points in the interval to realize the functions in matlab. Please use appropriate commands for continuous plots when x is continuous.

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1. Consider two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers where $\{a_n\}$ is non-decreasing and $\{b_n\} = o(\{\sqrt{a}_n\})$. The

$$\lim_{n\to\infty} \frac{\left(1+\frac{b_n}{a_n}\right)^{a_n}}{e^{b_n}} = 1$$

Let $\{f_n\}$ denotes the sequence in the left hand side. Consider, $a_n = n+1$, $b_n = n^{\frac{1}{3}}$, $n \geq 1$. Write a program that takes input $N \in \mathbb{Z}_{++}$, $\epsilon > 0$ and

(a) produces first N iterates of the sequence $\{f_n\}$

(b) gives the first N_ϵ iterates where $N_\epsilon = \min\{n : |f_n - 1| < \epsilon\}$

2. Consider two functions $f : C \to \mathbb{R}$ and $g : C \to \mathbb{R}$ where $C \subset \mathbb{R}$ is compact. Let g be strictly positive and bounded above. Suppose we want to solve the following problem

$$\max_{x\in C} \frac{f(x)}{g(x)}$$

In many cases, it is a hard problem but a much easier problem is to solve

$$\max_{x\in C} \left(f(x) - \alpha g(x)\right)$$

for any α. Let $r^* = \frac{f(x^*)}{g(x^*)}$ be the maximum value ($x^* \in C$ is an optimal point). We then follow the below strategy:

- Choose an arbitrary $x_1 \in C$ and set $r_1 = \frac{f(x_1)}{g(x_1)}$
- Set $x_{n+1} = \arg\max_{x \in C} (f(x) - r_n g(x))$ and $r_{n+1} = \frac{f(x_{n+1})}{g(x_{n+1})}$ for $n \geq 1$

It can be shown that r_n is non-decreasing and $\sup r_n = r^*$. Hence,

$$\lim_{n \to \infty} r_n = r^*.$$

Consider $f(x) = 2x^2 + 5$, $g(x) = x^2 - 2x + 3$, C=[1, 5]. Write a program that takes input $x_1 \in C, N \in \mathbb{Z}_{++}$, $\epsilon > 0$ and

(a) produces first N iterates of the sequence $\{r_n\}$

(b) gives the first N_ϵ iterates where $N_\epsilon = \min\{n : |r_n - r_{n+1}| < \epsilon\}$

3. Suppose $F \in \mathbb{R}^{k \times k}$ is a square positive matrix. Any such matrix has a real, positive eigenvalue ρ such that $\rho > |\lambda|$ for all other eigen values λ. Suppose F is such that $\rho < 1$. Further, suppose $g \in \mathbb{R}^k$ is a non-negative vector. We want to compute

$$p^* = (I - F)^{-1} g$$

Towards solving this we define a vector valued sequence as follows: We choose $p_1 \in \mathbb{R}^k$, an arbitrary non-negative vector. We set

$$p_{n+1} = F p_n + g, n \geq 1.$$

It can be shown that the sequence p_n is monotone i.e.,

- If $p_1(i) \leq p^*(i)$, then $p_1(i) \leq p_2(i) \leq p_3(i) \leq, ...$ and $\sup p_n(i) = p^*(i)$ $\forall n$,
- If $p_1(j) \geq p^*(j)$, then $p_1(j) \geq p_2(j) \geq p_3(j) \geq, ...$ and $\inf p_n(j) = p^*(j)$ $\forall n$,

From above points, $p_n \to p^*$.
Write a program that takes F(positive), g, p_1 (both non-negative), N and ϵ (both positive) as input and

(a) If $\rho(F) \geq 1$, throws an error

(b) If $\rho(F) < 1$, produces first N iterates of the sequence $\{p_n\}$

(c) If $\rho(F) < 1$, gives the first N_ϵ iterates where $N_\epsilon = min\{n : d(p_n, p_{n+1}) < \epsilon\}$