# E2-243

# Programming Exercise - 6

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**Instructions:**

- Do not submit your code, output files, etc.
- There will one lab exam towards the end of the semester that will test your understanding of the concepts taught in class. The questions in the lab exam will be somewhat similar to these questions in both content and implementation complexity. If you do not program these exercises, handling the lab exam will not be easy! In a way, programming these assignments yourself will be your preparation for the lab exam.
- You may use any discrete plot like the 'stem' function in MATLAB.
- Whenever $x \in$ interval, you may have to take appropriate discrete points in the interval to realize the functions in matlab. Please use appropriate commands for continuous plots when x is continuous.

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1. Consider a square matrix $A \in \mathbb{R}^{k \times k}$ with eigenvalues $\lambda_1, \ldots\ldots, \lambda_k$. We call an eigenvalue $\lambda_i = \lambda$ dominant if $|\lambda| > \max_{j \neq i} |\lambda_j|$. Let $v$ denote the eigenvector corresponding the dominant eigenvalue $\lambda$. Below we describe a way to obtain $\lambda$ and $v$. Let us arbitrarily choose a vector $x_1 \in \mathbb{R}^k$.
   Set
   $$x_{n+1} = \frac{Ax_n}{\|Ax_n\|}, \forall n \geq 1$$
   .
   If ${x_1}^T v \neq 0$ and $\{x_n\}$ converges then $x_n \to v$ and $\|Ax_n\| \to \lambda$.
   Write a program that takes input $A \in \mathbb{R}^{k \times k}, x_1 \in \mathbb{R}^k$, and $N > 0$, and
   (a) Generates first $N$ iterates of the sequence $\{x_n\}$.
   (b) Plots $z_n = \|x_{n+1} - x_n\|$ vs $n$ and infers from this plot whether $\{x_n\}$ converges.
   (c) If $\{x_n\}$ converges, outputs $\|Ax_N\|$ and $x_N$ as approximations to the dominant eigenvalue $\lambda$ and the corresponding eigenvector $v$. (Note that this conclusion could be wrong if ${x_1}^T v = 0$).

2. Consider $S = \{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n\}$ be a set of $\mathtt{n}$ vectors in $\mathbb{R}^m$, and $\mathtt{x}$ be another vector in $\mathbb{R}^m$.
   (a) Write a MATLAB function IsLinearCombination$(A, x)$ where $A$ is a $\mathtt{m} \times \mathtt{n}$ matrix that contains the vectors $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n$ as columns and outputs 1 or 0 depending on whether $\mathtt{x}$ is or is not (respectively) a linear combination of the vectors $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n$.

(b) Write a MATLAB function GetLinIndepVectors($A$) that takes a matrix $A$ whose columns are the vectors in $S$ as input, and returns a matrix $B$ whose columns are such that (i) they are a subset of the columns in matrix $A$, (ii) they are linearly independent, and (iii) the number of columns in matrix $B$ is the largest possible.

(c) Write a MATLAB function $[V, Q] = GramSchmidt(B)$ where $B$ is the matrix returned by the function `GetLinIndepVectors()`, and matrices `V` and `Q` respectively are the matrices containing orthogonal and orthonormal vectors obtained from the columns of matrix $B$ using the *GramSchmidt* procedure.

3. Plot a part of the plane in MATLAB (use the fill3() function) corresponding to the subspace $W = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix}, \in \mathbb{R}^3, \alpha, \beta \in \mathbb{R}$.
Let $u$ and $v$ be two vectors in this subspace. Plot the vectors $u$ and $v$ and $\alpha u + \beta v$ (use the quiver3() function) for a few values of $\alpha, \beta \in \mathbb{R}$. Use the MATLAB rotation toolbar button to rotate the 3D plot and visually verify that all these vectors are in the same plane corresponding to subspace $W$ .

4. Let $x, y, z, w \in \mathbb{R}^3$ be as defined below
$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, z = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

(a) Is the set $\{x, y\}$ linearly independent. Answer this first using hand calculation. Then form the matrix $A$ with the vectors $x; y$ as columns and calculate its rank: $A$=[x y], $r$=rank($A$).
What values for $r$ would show that the set $x, y$ is linearly dependent?

(b) Take an arbitrary linear combination $v$ of $x$ and $z$ and calculate the rank of the augmented matrix:
$v = rand(1) * x + rand(1) * z$, t=rank([$A$ $v$])
Can you conclude from the value obtained for $t$ that $v$ is not in the subspace spanned by $x, y$? Why?

(c) Form the matrix $B$ with the vectors $x; y; z; w$ as columns and calculate its rank:
$B$=[x y z w], $r$=rank($B$)
What can you conclude about the linear independence or dependence of the set of vectors $x; y; z; w$? Can you make this same conclusion for every set of four vectors in $\mathbb{R}^3$? Why?

5. Consider the matrix $H = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$
Find its eigenvalues and eigenvectors from the eigen equation $H\psi = \lambda\psi$, where

$\psi$ is an eigenvector of $H$ and $\lambda$ is the corresponding eigenvalue. Verify your results with the MATLAB function eig(). Examine whether the eigenvectors form a basis for $\mathbb{C}_3$.