CMO: Existence and Characterization of Minimum

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Let $S \subseteq \mathbb{R}^d$ and $f : S \mapsto \mathbb{R}$.
x^* is a local minimum $\iff \exists r > 0, \forall x \in N_r(x^*) \cap S, f(x^*) \le f(x)$.

We'll restrict our analysis in 2 ways:

- We'll only consider functions for which a global minimum exists. Here we'll discuss a sufficient condition for that.
- We'll only try to find a local minimum, since finding global minimum is difficult.

1 Necessary condition for local minimum of univariate function

Theorem 1. *If $f : \mathbb{R} \mapsto \mathbb{R}$ is differentiable, then x^* is the local minimum of $f \implies f'(x^*) = 0$.*

Proof. Let

$$h(t) = \frac{f(t) - f(x^*)}{t - x^*}$$

Then $f'(x^*) = \lim_{t \to x^*} h(t)$.

Suppose x^* is a local minimum in $(x - r, x + r)$. Then for $t \in (x - r, x)$, $h(t) \le 0$ and for $t \in (x, x + r)$, $h(t) \ge 0$. Therefore, left derivative of f at x^* is non-positive and right derivative of f at x^* is non-negative. Since f is differentiable, left and right derivatives are equal. Therefore, $f'(x^*) = 0$. $\square$

Theorem 2. *Let f be a C^2 function and x^* be a local minimum. Then $f''(x^*) \ge 0$.*

Proof. Using Taylor series near x^*, we get

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

$$\implies 0 \le f(x) - f(x^*) = \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

For this to hold true for all x near x^*, $f''(x^*) \ge 0$. $\square$

2 Characterization of functions which have a minimum

Consider a function from $\mathbb{R}^d$ to $\mathbb{R}$. Global minimum exists iff f is lower-bounded.

Definition 1.

$$\lim_{\|x\|\to\infty} f(x) = \infty \iff \forall F > 0, \exists M > 0, \forall x \in \mathbb{R}^d, (\|x\| > M \implies f(x) \geq F)$$

If $\lim_{\|x\|\to\infty} f(x) = \infty$*, then* f *is called a* ***coercive*** *function.*

Theorem 3 (Weierstrass' theorem)**.** *If a continuous function's domain is closed and bounded, the function has a global minimum and maximum.*

Theorem 4.

$$\lim_{\|x\|\to\infty} f(x) = \infty \wedge f \text{ is continuous} \implies f \text{ has global minimum}$$

Proof. Consider $F = f(0)$. Let $S_1 = \{x : \|x\| > M\}$ and $S_2 = \{x : \|x\| \leq M\}$.

Since f is coercive, $\forall x \in S_1, f(0) \leq f(x)$. By Weierstrass' theorem, a global minimum exists in S_2. Let it be x^*. Therefore, $f(x^*) \leq f(0)$. Therefore, x^* is a global minimum of $\mathbb{R}^d$. □

3 Sufficient condition for local minimum of univariate function

Theorem 5. $f'(x_0) = 0 \wedge f''(x_0) > 0 \implies x_0$ *is local minimum.*

Proof.

$$f(x) - f(x_0) = \frac{1}{2}(x - x^*)^2 f''(x^*) + o((x - x^*)^2)$$

In the neighborhood of x_0, the small-o term is negligible, so the $f''(x^*)$ makes $f(x) - f(x_0)$ positive. Therefore, x_0 is a local minimum in that neighborhood. □

4 Necessary condition for local minimum of multivariate function

Theorem 6. *Let* $f : \mathbb{R}^d \mapsto \mathbb{R}$ *be a differentiable function. Let* x^* *be a local minimum of* f*. Then* $\nabla_f(x^*) = 0$.

Proof. Let $u \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

$$x^* + tu \in N_r(x^*) \iff \|tu\| < r \iff |t| \leq \frac{r}{\|u\|} \iff t \in N_{\frac{r}{\|u\|}}(0)$$

Let $g(t) = f(x^* + tu)$.

$$
\begin{aligned}
& x^* \text{ is local minimum of } f \\
& \Rightarrow \forall x \in N_r(x^*), f(x^*) \le f(x) \\
& \Rightarrow \forall t \in N_{\frac{r}{\|u\|}}(0), f(x^*) \le f(x^* + tu) \\
& \Rightarrow \forall t \in N_{\frac{r}{\|u\|}}(0), g(0) \le g(t) \\
& \Rightarrow g \text{ has local minimum at } 0 \\
& \Rightarrow g'(0) = 0 \\
& \Rightarrow \nabla_f(x^*)^T u = 0
\end{aligned}
$$

Since this is true for all $u \in \mathbb{R}^d$, $\nabla_f(x^*) = 0$. ☐

Theorem 7. *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a differentiable function. Let x^* be a local minimum of f. Then $\mathrm{H}_f(x^*)$ is positive semi-definite.*

Proof. Similar to above proof. Use the fact that if g has a local minimum at 0, then $g''(0) \ge 0$. ☐

5 Sufficient condition for local minimum of multivariate function

Theorem 8. *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a differentiable function. Let $\nabla_f(x_0) = 0$ and $\mathrm{H}_f(x_0)$ be positive definite. Then x_0 is a local minimum of f.*

Proof. Proof follows directly from Taylor series. ☐