

# CMO: Constrained optimization for convex functions

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## 1 Convex function and convex constraints

Let's analyze the following problem:

$$\begin{array}{ll} \min_x & f(x) \\ \text{where} & c_i(x) \leq 0 \quad \forall i \in I \\ & h_j(x) = 0 \quad \forall j \in I \end{array}$$

Here  $f$  and  $c_i$  are convex and  $C^1$  and  $h_j$  is linear, i.e.  $h_j(x) = a_j^T x - b_j$ .

### 1.1 Feasible region is a convex set

**Lemma 1** (Homework). *The set  $S_i = \{x : c_i(x) \leq 0\}$  is convex.*

**Lemma 2** (Homework). *The set  $S_j = \{x : h_j(x) \leq 0\}$  is convex.*

**Lemma 3** (Homework). *The intersection of convex sets is convex.*

### 1.2 KKT point gives global minimum

Define the Lagrangian as

$$L(x, \lambda, \mu) = f(x) + \lambda^T c(x) + \mu^T h(x)$$

**Lemma 4.** *If  $\lambda_i \geq 0$  and  $x$  is a feasible point, then  $f(x) \geq L(x, \mu, \lambda)$ .*

*Proof.* Since  $x$  is a feasible point,

$$\begin{aligned} c_i(x) &\leq 0 \wedge h_j(x) = 0 \\ \implies \lambda^T c(x) &\leq 0 \wedge \mu^T h(x) = 0 \\ \implies f(x) + \lambda^T c(x) + \mu^T h(x) &\leq f(x) \\ \implies L(x, \lambda, \mu) &\leq f(x) \end{aligned}$$

□

**Lemma 5.** *Let  $(x^*, \lambda^*, \mu^*)$  be a KKT point. Then  $f(x^*) = L(x^*, \mu^*, \lambda^*)$ .*

*Proof.*

$$\begin{aligned}
& \lambda_i^* c_i(x^*) = 0 \wedge h_j(x^*) = 0 \quad (\text{complementary slackness and primal feasibility}) \\
& \implies \lambda^{*T} c(x^*) = 0 \wedge \mu^{*T} h(x^*) = 0 \\
& \implies f(x^*) + \lambda^{*T} c(x^*) + \mu^{*T} h(x^*) = f(x^*) \\
& \implies L(x^*, \lambda^*, \mu^*) = f(x^*)
\end{aligned}$$

□

**Theorem 6** (Proved previously). *Let  $f$  be  $C^1$  and convex. Then*

$$\forall u, v \in \mathbb{R}^d, f(v) \geq f(u) + \nabla f(u)^T (v - u)$$

**Theorem 7.** *Let  $(x^*, \lambda^*, \mu^*)$  be a KKT point. Then  $x^*$  is a constrained global minimum of  $f$ .*

*Proof.* Let  $x$  be a feasible point.

$$\begin{aligned}
f(x) & \geq L(x, \lambda^*, \mu^*) && (\text{by lemma 4}) \\
& = f(x) + \sum_i \lambda_i^* c_i(x) + \sum_j \mu_j^* (a_j^T x - b_j) \\
& \geq (f(x) + \nabla f(x^*)^T (x - x^*)) \\
& \quad + \sum_i \lambda_i^* (c_i(x^*) + \nabla_{c_i}(x^*)^T (x - x^*)) \\
& \quad + \sum_j \mu_j^* (a_j^T (x - x^*) + (a_j^T x^* - b_j)) && (\text{by theorem 6}) \\
& = \left( f(x^*) + \sum_i \lambda_i^* c_i(x^*) + \sum_j \mu_j^* (a_j^T x^* - b_j) \right) \\
& \quad + (x - x^*)^T \left( \nabla f(x^*) + \sum_i \lambda_i^* \nabla_{c_i}(x^*) + \sum_j \mu_j^* a_j \right) && (\text{rearrange terms}) \\
& = L(x^*, \lambda^*, \mu^*) + (x - x^*)^T (\nabla_x L)(x^*, \lambda^*, \mu^*) \\
& = f(x^*) && (\text{by lemma 5 and stationarity})
\end{aligned}$$

Since for all feasible points  $f(x) \geq f(x^*)$ ,  $x^*$  is a constrained global minimum of  $f$ . □

Note that unlike the necessary conditions for local minimum, here we do not require regularity.

### 1.3 Example: Projection over ball

Consider the optimization problem:

$$\min_x \frac{1}{2} \|x - z\|^2 \text{ where } \|x\|^2 \leq r^2$$

Here  $z$  lies outside the feasible region.

$\|x - z\|^2$  and  $\|x\|^2$  are convex functions (because their hessian is  $2I$ , which is positive definite), so this is a convex optimization problem.

$$L(x, \lambda) = \frac{1}{2} \|x - z\|^2 + \lambda(\|x\|^2 - r^2)$$

Applying the KKT conditions, we get

- Stationarity:  $z = (2\lambda + 1)x$ .
- Primal feasibility:  $\|x\|^2 \leq r^2$ .
- Dual feasibility:  $\lambda \geq 0$ .
- Complementary slackness:  $\lambda(\|x\|^2 - r^2) = 0$ .

If we take  $\lambda = 0$ , then stationarity gives us  $x = z$ . This violates feasibility, so this is not possible. Therefore, complementary slackness gives us  $\|x\|^2 = r^2$ . On simplifying, we get

$$z = \frac{r}{\|z\|} z \qquad \lambda = \frac{1}{2} \left( \frac{\|z\|}{r} - 1 \right) \qquad f(x) = \frac{1}{2} (r - \|z\|)^2$$

