Computational Methods of Optimization
Third Midterm(22nd Nov, 2019)

Instructions:

- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 90 mins.
- Answer the questions in the spaces provided. Answers outside the spaces provided will not be graded.
- Rough work can be done in the spaces provided at the end of the booklet

Name: ____

SRNO: Degree: Dept:

Question:	1	2	3	4	Total
Points:	10	10	15	10	45
Score:					

1. (10 points) Compute the projection of $\mathbf{z} \in \mathbb{R}^d$ onto the cube $[0,1]^d$.

Solution: Let $C = [0,1]^d$.

$$P_C(\mathbf{z}) = \operatorname{argmin}_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|^2 \text{ s.t. } 0 \leq x_i \leq 1 \quad i \in \{1,\ldots,d\}$$

The KKT condition of the problem suggests that there exists $\lambda_1 \geq 0, \lambda_2 \geq 0$

$$\mathbf{x} - \mathbf{z} - \lambda_1 + \lambda_2 = 0$$

which implies

$$\mathbf{x} = \mathbf{z} + \lambda_1 - \lambda_2$$

$$\lambda_{1i} x_i = 0,\ \lambda_{2i}(x_i - 1) = 0$$

We now construct a KKT point.
If $z_i > 1$ set $\lambda_{1i} = 0, \lambda_{2i} = z_i - 1$ and $x_i = 1$

If $z_i < 0$ set $\lambda_{2i} = 0, \lambda_{1i} = 0 - z_i$ and $x_i = 0$

If $0 < z_i < 1$ set $\lambda_{2i} = 0, \lambda_{1i} = 0$ and $x_i = z_i$ Hence the Projection is

$$(P_C(\mathbf{z}))_i = \begin{cases} 0 & z_i < 0 \\ z_i & 0 \leq z_i \leq 1 \\ 1 & z_i > 1 \end{cases}$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function. We are interested in

$$min_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x}) \text{ subject to } x_3^2 + 2x_2^2 + x_1 x_2 \geq 1$$

$$\mathbf{u}^\top \mathbf{x} + 4 = 0$$

It is given that at the point $\mathbf{x}^{(0)} = [0\ 1\ 0]^\top$ the gradient of f, denoted by, $g(\mathbf{x}^{(0)}) = [1 - 1\ 2]^\top$.The vector $\mathbf{u} \in \mathbb{R}^3$ is unknown.

(a) (3 points) State the KKT conditions for this problem?

Solution:

(b) (7 points) Can we find $\mathbf{u}$ such that $\mathbf{x}^{(0)}$ is a KKT point. Justify your answer

Solution: Note that the non-linear constraint is not active at $\mathbf{x}^{(0)}$. If $\mathbf{x}^{(0)}$ needs to be a KKT point then there needs to exist μ such that

$$g(\mathbf{x}^{(0)}) - \mu \mathbf{u} = 0$$

We note that the constraint is linear and hence the point is regular. As a result one can guarantee the existence of μ if a suitable $\mathbf{u}$ exist. Using the KKT condition we find that $\mathbf{u} = \frac{1}{\mu}[1 - 1\ 2]^\top$. Since the linear constraint is active one obtains that

$$\mathbf{u}^\top \mathbf{x}^{(0)} + 4 = 0$$

$$\frac{1}{\mu}[1 - 1\ 2]^\top \mathbf{x}^{(0)} + 4 = 0$$

which implies that $\mu = 4$. Thus $\mathbf{u} = \frac{1}{4}[1 - 1\ 2]^\top$ is the desired value.

3. Let the set of symmetric real valued $d \times d$ matrices be denoted by $\mathcal{S}^d$. Consider the following problem, which we will call *Primal problem*

$$min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \left(= \mathbf{x}^\top Q\mathbf{x}\right), \quad Q \in \mathcal{S}^d$$

$$\text{such that } \|\mathbf{x}\| = 1$$

(a) (7 points) Let μ be the dual variable. Find the dual function in terms of μ?

Solution: The constraint set is equivalent to $\{\mathbf{x} \in \mathbb{R}^d | \|\mathbf{x}\|^2 = 1\}$. (It is possible to work with the given constraint directly, but to simplify the algebra we work with the reformulated constraint). We define the Lagrangian as

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \mu(\|\mathbf{x}\|^2 - 1)$$

$$g(\mu) = min_{\mathbf{x}} L(\mathbf{x}, \mu) = \min_{\mathbf{x}} \mathbf{x}^\top (Q + \mu I)\mathbf{x} - \mu$$

The minimum exists only if $\hat{Q} = Q + \mu I$ is positive semidefinite(I is the identity matrix). In such a case the minimum of the quadratic form is attained at $\mathbf{x} = 0$ and

$$g(\mu) = \begin{cases} -\mu & \hat{Q} \succ 0 \\ -\infty & otheriwise \end{cases}$$

$\hat{Q}$ is positive semidefinite only if all eigenvalues of $\hat{Q}$ are non-negative, or in other words $\mu + \lambda_i \geq 0$ for all $i \in \{1, \ldots, d\}$, where λ_i are the eigenvalues of Q. Thus

$$\mu \geq -\lambda_{min} \text{ where } \lambda_{min} = min_i \lambda_i$$

As a consequence the dual function can be stated as

$$g(\mu) = \begin{cases} -\mu & \mu \geq -\lambda_{min} \\ -\infty & \text{otherwise} \end{cases}$$

(b) (3 points) Find the function value and the optimum point for the Dual optimization problem.

Solution: The dual problem is

$$max_{\mu} g(\lambda) \text{ such that } \mu \text{ is dual feasible}$$

Hence the dual problem can be stated as

$$max_{\mu} - \mu \text{ such that } \mu \geq -\lambda_{min}$$

The dual optimum value is λ_{min} and is attained at $\mu^* = -\lambda_{min}$ and .

(c) (3 points) Solve the *Primal problem* , report the function value and the point where it is attained.

Solution: Let $\mathbf{e}$ be the eigenvector corresponding to eigenvalue λ_{min}. Then $f(\mathbf{e}) = \lambda_{min} = g(\mu^*)$ and hence strong duality holds. As a consequence $\mathbf{x} = \mathbf{e}$ and $f(\mathbf{e}) = \lambda_{min}$.

(d) (2 points) If instead of minimization of f the original problem was posed as maximization then solve the problem. Only the final answer, i.e. the optimum value and the point where it is attained, is required with a brief justification

Solution: The maximization problem can be posed as minimization of $-f$ and the optimum is attained at $\mathbf{x} = \mathbf{e}$, where $\mathbf{e}$ is the eigenvector corresponding to the largest eigenvalue, λ_{max} of Q. The optimum value is λ_{max}.

4. Consider the following primal problem

$$min_{\mathbf{x}\in\mathbb{R}^d}\ c \ \text{ subject to } \mathbf{u}_i^\top \mathbf{x} \geq 1 \forall i \in \{1,\ldots,n\}$$

Note that c is a constant

(a) (5 points) Find the dual

Solution: The Dual function

$$g(\lambda) = min_{\mathbf{x}} L(\mathbf{x},\lambda) \left(= c - \sum_{i=1}^{n} \lambda_i(\mathbf{u}_i^\top \mathbf{x} - 1) \right)$$

Dual feasibility implies that

$$g(\lambda) = \begin{cases} c + \sum_{i=1}^{n} \lambda_i & \sum_{i=1}^{n} \lambda_i \mathbf{u}_i = 0 \\ -\infty & \text{otherwise} \end{cases}$$

(b) (5 points) Suppose it is given that the convex hull of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ contains 0 that is $0 \in \{\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{u}_i | \beta_i \geq 0, \sum_{i=1}^{n} \beta_i = 1\}$. Solve the dual problem and comment on the solution of the primal problem.

Solution: The dual problem is

$$max_{\lambda\geq 0} g(\lambda) = c + \sum_{i=1}^{n} \lambda_i$$

subject to $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i = 0$. Since 0 is a member of the convex hull then there exists $\beta \in \mathbb{R}^d$ with all non-negative entries such that $0 = \sum_{i=1}^{n} \beta_i \mathbf{u}_i$. Now for any $M > 0$, the choice of $\lambda = M\beta$ is dual feasible. Thus the Dual optimum is unbounded, i.e. achieved at $M = \infty$. This implies that the primal has no solution

Rough Sheet 1

Rough Sheet 2

Rough Sheet 3

Rough Sheet 4