CMO: Convex functions

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Definition 1 (Convex set). *Let $S \subseteq \mathbb{R}^d$, S is convex iff*

$$\forall u, v \in S, \forall \alpha \in (0,1), (1-\alpha)u + \alpha v \in S$$

Definition 2 (Convex function). *Let $f : S \mapsto \mathbb{R}$, where S is convex. f is convex iff*

$$\forall \alpha \in (0,1), \forall u, v \in S, f((1-\alpha)u + \alpha v) \leq (1-\alpha)f(u) + \alpha f(v)$$

Theorem 1. *f is convex $\iff$ $(\forall u, v \in S, \nabla_f(u)^T(v-u) \leq f(v) - f(u))$*

Proof.

$$\begin{aligned}
&f \text{ is convex} \\
&\Rightarrow f((1-\alpha)u + \alpha v) \leq (1-\alpha)f(u) + \alpha(v) \\
&\Rightarrow f(u + \alpha(v-u)) \leq f(u) + \alpha(f(v) - f(u)) \\
&\Rightarrow \frac{1}{\alpha}(f(u + \alpha(v-u)) - f(u)) \leq f(v) - f(u)
\end{aligned}$$

Let $g(\alpha) = f(u + \alpha(v-u))$.

$$\begin{aligned}
&f \text{ is convex} \\
&\Rightarrow \frac{g(\alpha) - g(0)}{\alpha} \leq g(1) - g(0) \\
&\Rightarrow \lim_{\alpha \to 0} \frac{g(\alpha) - g(0)}{\alpha} \leq \lim_{\alpha \to 0}(g(1) - g(0)) \\
&\Rightarrow g'(0) \leq g(1) - g(0) \\
&\Rightarrow \nabla_f(u)^T(v-u) \leq f(v) - f(u)
\end{aligned}$$

Suppose $\forall u, v \in S, \nabla_f(u)^T(v-u) \leq f(v) - f(u)$.

For any arbitrarily chosen x_1 and x_2 ($x_1 \neq x_2$), let $x = (1-\alpha)x_1 + \alpha x_2$. Then $x_1 - x = \alpha(x_1 - x_2)$ and $x_2 - x = (1-\alpha)(x_2 - x_1)$.

Setting $u = x$ and $v = x_1$, we get

$$\nabla_f(x)^T \alpha(x_1 - x_2) \leq f(x_1) - f(x)$$

Setting $u = x$ and $v = x_2$, we get

$$-\nabla_f(x)^T(1-\alpha)(x_1 - x_2) \leq f(x_2) - f(x)$$

Adding these equations with weights $1-\alpha$ and α, we get

$$\begin{aligned}
&(1-\alpha)\,\nabla_f(x)^T\alpha(x_1-x_2)-\alpha\,\nabla_f(x)^T(1-\alpha)(x_1-x_2)\\
&\quad\le(1-\alpha)(f(x_1)-f(x))+\alpha(f(x_2)-f(x))\\
&\Rightarrow 0\le(1-\alpha)f(x_1)+\alpha f(x_2)-f(x)\\
&\Rightarrow f((1-\alpha)x_1+\alpha x_2)\le(1-\alpha)f(x_1)+\alpha f(x_2)\\
&\Rightarrow f \text{ is convex}
\end{aligned}$$

□

Theorem 2. *If f is convex, and x^* is a local minimum, then x^* is also a global minimum.*

Proof. For all $x\in\mathbb{R}^d$,

$$0=f(x^*)^T(x-x^*)\le f(x)-f(x^*)$$

□

Theorem 3 (Proof omitted)**.** *Let $f:\mathbb{R}^d\mapsto\mathbb{R}$ and $f\in C^2$. Then f is convex iff H_f is positive semi-definite.*