Computational Methods of Optimization
First Midterm(6th Sep, 2017)

Answer all questions Time: 90 minutes

1. Consider minimization of the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows

$$f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

   All questions carry 1 mark

   (a) Compute the local minimum(minima)?

   (b) Compute the local maxima

   (c) Let M be a positive scalar. Is the set $\{x|f(x) \geq M\}$ empty?

   (d) Find Global minima of the problem

   (e) Is the function coercive?

2. Consider minimization of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows

$$f(x) = \frac{1}{2}\mathbf{x}^\top A\mathbf{x} - 2b^\top \mathbf{x}$$

   Where $A = \begin{bmatrix} 1 & 1 \\ -1 & a \end{bmatrix}$ and $b = [1, 0]^\top$. The value of a is not known.
   All questions carry 2 marks

   (a) For what values of a is the problem convex?

   (b) Compute all global minima if they exist for $a = 0$?

   (c) Given such a choice of a write one iteration of steepest descent direction algorithm with exact line search starting from $\mathbf{x} = [0, 0]^\top$.

   (d) Consider $a = -1$. Compute the critical points of f. Find $\mathbf{u}$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_c)$ where $\mathbf{x} = \mathbf{x}_c + \mathbf{u}$ and $\mathbf{x}_c$ is a critical point

   (e) In the previous question find $\mathbf{u}$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_c)$.

3. Let l, b, h be the length, breadth, and height of a cuboid.

   Recall that the volume of a cuboid is given by $V(l, b, h) = lbh$ and the surface area of a cuboid is given by $A(l, b, h) = 2(lb + bh + hl)$.

   Consider the problem of finding the volume of the cuboid with a fixed surface Area stated as follows

$$max_{l,b,h} V(l, b, h) \quad A(l, b, h) = 6a^2$$

(a) Restate the above problem as an equivalent maximization problem in 2 variables without the area constraint? 2 marks

(b) Derive a condition on the variables such that objective function is always negative whenever those conditions are satisified? 4 marks

(c) Find the global maxima of the function defined in first question? Compute the global maxima of the original problem,i.e. find l, b, h. Justify your answer. 4 marks

4. Consider the following problem

$$min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x})$$

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - b^\top \mathbf{x}$$

Lec $x^0 = b$

Let $Q = \begin{bmatrix} 3 & 1 \\ 1 & \frac{9}{2} \end{bmatrix}$, $b = [1, 2]^\top$

(a) What is the optimal $\mathbf{x}$? 2 marks

(b) Show that $f(\mathbf{x}) \in C_L^1$ and compute L. 2 marks

(c) Suppose the minimization is performed using the steepest descent direction with a stepsize of $\frac{1}{L}$. Find N, the number of iterations, such that $f(\mathbf{x}^{(k)}) \leq 0\ \forall k \geq N$. 3 marks

(d) How will your answer change if exact linesearch were used. 3 marks

5. It was found that while minimizing a C^1 function using descent directions the gradient of the iterates satisfied the following relationship,

$$\nabla f\left(\mathbf{x}^{(k+1)}\right)^\top \nabla f\left(\mathbf{x}^{(k)}\right) = 0$$

for all $k \geq 0$. What is the step-size selection procedure used? Explain your answer. 5 marks

Computational Methods of Optimization
Second Midterm(23rd Oct, 2017)

Answer all questions Time: 90 minutes

1. Consider applying Conjugate gradient method for solving the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - b^\top \mathbf{x}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $Q \succ 0$.

   (a) State the Conjugate gradient algorithm 2 marks

   (b) Suppose $b = \sum_{i=1}^{k} h_i \mathbf{e}_i$ where $\mathbf{e}_i$ is the eigenvector correspondig to the eigenvalue λ_i of Q and k is less than n. Assuming the starting point is at $\mathbf{x}^0 = 0$,can you estimate the number of iterations required to solve the problem. Justify your answer. 7 marks

2. Find the projection of point $\mathbf{z} \in \mathbb{R}^n$ on the sets

   (a) $C = \{\mathbf{x} \in \mathbb{R}^n | a_i \le x_i \le b_i, i \in \{1, \ldots, n\}\}$ 5 marks

   (b) $C = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mu + \kappa\mathbf{u},\ \|\mathbf{u}\| \le 1, \mathbf{u} \in \mathbb{R}^n\}$ where $\mu \in \mathbb{R}^n, \kappa \in \mathbb{R}$ are known. 5 marks

3. Derive the Dual problem for the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - b^\top \mathbf{x} \quad A\mathbf{x} = d$$

$A \in \mathbb{R}^{m\times n}, d \in \mathbb{R}^m, Q \succ 0, Q \in \mathbb{R}^{n\times n}$ symmetric matrix.

10 marks

4. Consider the convex problem

$$min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \text{ subject to } f_i(\mathbf{x}) \le 0 \; i = \{1, \ldots, m\}$$

where all functions are convex, differentiable and defined over $\mathbb{R}^n$.

   (a) State the Wolfe Dual of the problem?

   (b) State and prove any relationship between $\mathbf{x}^*$, the optimal solution, and the Wolfe dual optima

10 marks

5. What is the dual function of the following problem

$$min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\mathbf{x}^\top A\mathbf{x}, x_i^2 = 1, i \in \{1, \ldots, n\}$$

where A is any symmetric real valued matrix. Comment on the solvability of the problem. Compute a lower bound on the primal problem in terms of the eigenvalues of A. 10 marks

Computational Methods of Optimization
Final Exam

Answer all questions Time: 180 minutes

5th Dec, 2017

1. Consider unconstrained minimization of a convex Quadratic function using steepest descent procedure with exact line search. It is given that Eigen-values of the Hessian are all equal. In how many iterations will the algorithm converge starting from an arbitrary point? Justify your answer. 5 marks

2. Under what conditions a vector $\mathbf{u} \in \mathbb{R}^d$ is always a feasible direction for the set $C = \{\mathbf{x} | A\mathbf{x} \geq b, \mathbf{x} \geq 0\}$ where A is a $m \times d$ matrix and $b \in \mathbb{R}^m$. Is the set bounded? Justify 5 marks

3. For a $m \times n$ matrix A and $\mathbf{c} \in \mathbb{R}^n$, let there exist $\lambda \in \mathbb{R}^m$ such that

$$A^\top \lambda = \mathbf{c}, \; \lambda \geq 0$$

Show that the set

$$\{\mathbf{x} | A\mathbf{x} \geq 0, \; \mathbf{c}^\top \mathbf{x} < 0\}$$

is empty 5 marks

4. Let $f, g_1, g_2, g_3 : \mathbb{R}^3 \to \mathbb{R}$ be $\mathcal{C}^{(1)}$ functions. Consider the following problem.

$$min_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } g_i(\mathbf{x}) \geq 0, i \in \{1, 2, 3\}$$

However the inequalities maybe in the wrong direction. Given partial information about a point $\mathbf{x}^{(0)}$ we need to determine if the inequalities are in the proper directions using the following information

- $\mathbf{x}^{(0)}$ is a KKT point
- $g_1(\mathbf{x}^{(0)}) = g_2(\mathbf{x}^{(0)}) = 0$
- $g_3(\mathbf{x}^{(0)}) = -5$
- $\nabla f(\mathbf{x}^{(0)}) = [3, -7, 4]^\top, \nabla g_1(\mathbf{x}^{(0)}) = [1, -1, 0]^\top, \nabla g_2(\mathbf{x}^{(0)}) = [0, 1, -1]^\top$

From the above information can you decide if the inequalities are in the proper direction? If they are not state the correct inequality. Give reasons for your answer. 10 marks

5. (a) State one iteration of the Active set algorithm for minimizing a convex quadratic function with linear inequality constraints. 2 marks

(b) Consider the following problem

$$min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x})$$

$$f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$$

$$\mathbf{a}_1^\top \mathbf{x} \geq 3 \,, \mathbf{a}_2^\top \mathbf{x} \geq 3$$

$$\mathbf{a}_1 = [1,2]^\top, \mathbf{a}_2 = [2,1]^\top$$

Solve the problem using Active set method starting from $\mathbf{x} = [6,0]^\top$. You can compute at most three iterations. 8 marks

6. For any real $d \times m$ matrix A show that for every $\mathbf{x} \in \mathbb{R}^d$ there exists an unique $\mathbf{u} \in \mathbb{R}^m$ such that $\mathbf{x} = A\mathbf{u} + \mathbf{v}$ and $A^\top \mathbf{v} = 0$. 10 marks

7. Consider the following Linear programming problem

$$min_{\mathbf{x}\in\mathbb{R}^d} \mathbf{c}^\top \mathbf{x}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

where A is a $m \times d$ matrix, and $\mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^d$. Let $\mathbf{x}^0$ be a Basic Feasible point at the start of an iteration of an Simplex algorithm with Basis B and a_i be a column of A matrix not in B selected in that iteration.

(a) Express $\mathbf{x}^0$ in terms of matrix B and $\mathbf{b}$ 2 marks

(b) The Simplex algorithm can be viewed as choosing a Descent Direction, $\mathbf{u}$, such that $\mathbf{x}^1 = \mathbf{x}^0 + \alpha\mathbf{u}$ where $\alpha \geq 0$. State the Descent direction identified in the iteration. Give reasons 3 marks

(c) Suppose all elements of $\mathbf{u}$ are positive then what can you conclude about the optimality. Give reasons. 5 marks

(d) Let $(\mathbf{x}^*, \lambda^*)$ be a KKT point for the problem where λ^* are Lagrange multipliers. Suppose $\mathbf{x}^*$ is the optimal point obtained from the simplex algorithm, Determine λ^* from the simplex algorithm. 5 marks

8. Let $C \subset \mathbb{R}^d$ be a convex set. Consider the function $f : \mathbb{R}^d \to \mathbb{R}$ defined as follows

$$f(\mathbf{x}) = min_{\mathbf{z}\in C}\|\mathbf{x} - \mathbf{z}\|$$

(a) Show that the function is convex 5 marks

(b) Compute the subgradient 5 marks