CMO: Minimizing a function with bounded hessian

Eklavya Sharma

Objective: Minimize a C^2 function $f : \mathbb{R}^d \mapsto \mathbb{R}$ for which $AI - \mathrm{H}_f(x)$ and $\mathrm{H}_f(x) - aI$ are positive semi-definite for all $x \in \mathbb{R}^d$ $(0 < a \le A)$.

The trick we'll use is to lower-bound and upper-bound f.

Let $u = \nabla_f(x^{(i)})$. Let

$$f_l(x) = f(x^{(i)}) + u^T(x - x^{(i)}) + \frac{a}{2}\|x - x^{(i)}\|^2$$

$$f_h(x) = f(x^{(i)}) + u^T(x - x^{(i)}) + \frac{A}{2}\|x - x^{(i)}\|^2$$

By using Taylor series on f at $x^{(i)}$, we get that $\forall x \in \mathbb{R}^d, f_l(x) \le f(x) \le f_h(x)$.

Lemma 1.

$$f_l^* = \min_x f_l(x) = f(x^{(i)}) - \frac{\|u\|^2}{2a}$$

Proof sketch. Set $\nabla_{f_l}(x) = 0$ and solve for x. □

Lemma 2. *Let h_1 and h_2 be 2 functions such that $\forall x \in \mathbb{R}^d, h_1(x) \le h_2(x)$. Let $h_1^* = \min_x h_1(x)$ and $h_2^* = \min_x h_2(x)$. Then $h_1* \le h_2^*$.*

Proof. Let $x_2 = \operatorname{argmin}_x h_2(x)$. Then $h_1^* \le h_1(x_2) \le h_2(x_2) = h_2^*$. □

Let $x^* = \operatorname{argmin}_x f(x)$. Let $E(x) = f(x) - f(x^*)$.

Lemma 3.

$$E(x^{(i)}) \le \frac{\|u\|^2}{2a}$$

Proof sketch. By lemma 2, $f_l^* \le f(x^*)$. Now use lemma 1 to substitute f_l^*. □

Let $x^{(i+1)} = x^{(i)} - \frac{u}{A}$. (It can be proven that $x^{(i+1)}$ minimizes f_h, but we're not interested in that fact.)

Lemma 4.

$$E(x^{(i)}) - E(x^{(i+1)}) \ge \frac{\|u\|^2}{2A}$$

Proof.

$$\begin{aligned}
f(x^{(i+1)}) &\leq f_h(x^{(i+1)}) && (f_h \text{ upper-bounds } f) \\
&= f(x^{(i)}) + u^T(x^{(i+1)} - x^{(i)}) + \frac{A}{2}\|x^{(i+1)} - x^{(i)}\|^2 \\
&= f(x^{(i)}) - \frac{\|u\|^2}{A} + \frac{A}{2}\frac{\|u\|^2}{A^2} \\
&= f(x^{(i)}) - \frac{\|u\|^2}{2A}
\end{aligned}$$

$$\implies E(x^{(i)}) - E(x^{(i+1)}) = f(x^{(i)}) - f(x^{(i+1)}) \geq \frac{\|u\|^2}{2A}$$

$\square$

Therefore,

$$\frac{E(x^{(i+1)})}{E(x^{(i)})} = 1 - \frac{E(x^{(i)}) - E(x^{(i+1)})}{E(x^{(i)})} \leq 1 - \frac{a}{A}$$

This proves the convergence of our algorithm.