1.4 Coercive Functions and Global Min

Theorem 1.11 *(Theorem 1.4.1) A continuous function f on a closed bounded domain D has a global min and max.*

Definition 1.4 *(Def 1.4.2) f is coercive if $\lim_{\|x\|\to\infty} f(x) = \infty$.*

Example 1.8 *(Eg 1.4.3)*

1. $$f(x, y) = x^2 + y^2$$

   coercive

2. $$f(x, y) = x^4 + y^4 - 3xy = (x^4 + y^4)(1 - \frac{3xy}{x^4 + y^4})$$

   $x^4 + y^4$*: dominant. Hence coercive.*

3. $$f(x, y, z) = e^{x^2} + e^{y^2} + e^{z^2} - x^{100} - y^{100} - z^{100}$$

   coercive since e^x grows faster than x^n.

4. *$f(x, y) = ax + by + c(ab \neq 0)$ not coercive.*

   Proof: Let (x, y) satisfy $ax + by = 0$. Then

   $$\lim_{\|(x,y)\|\to\infty} f(x, y) = c\square$$

5. *$f(x, y, z) = x^4 + y^4 + z^4 - 3xyz - x^2 - y^2 - z^2$. $x^4 + y^4 + z^4$: dominant. Hence coercive.*

6. *$f(x, y) = x^2 - 2xy + y^2$ not coercive.*

   Proof: Let $x = y$ as $\| (x, y) \| \to \infty$. Then $f(x, y) = 0\square$.

   Note in this case,

   $$\lim_{|x|\to\infty} f(x, y_0) = \infty, \lim_{|y|\to\infty} f(x_0, y) = \infty$$

   Conclusion: $f(x) \to \infty$ as each coordinate tends to ∞ does not imply coercive.

Theorem 1.12 *(Them 1.4.4) Suppose f is continuous over R^n and coercive. f must has a global min.*

Proof: Since f is coercive, there exist $r > 0$ s.t.

$$f(x) > f(0), \forall \parallel x \parallel > r.$$

By Them (1.11), there is a global min $x^\star$ on $B(\bar{0}, r)$(closure of $B(0, r)$).

$$\Longrightarrow f(x) \geq f(x^\star), \forall \parallel x \parallel \leq r$$

In particular,

$$f(0) \geq f(x^\star)$$

$$\Longrightarrow f(x) > f(0) \geq f(x^\star), \forall \parallel x \parallel > r$$

Hence $x^\star$ is a global min on R^n. □

Example 1.9 *(Eg 1.4.5)*

$$F(x, y) = x^4 - 4xy + y^4$$

coercive.

$$\bigtriangledown f(x, y) = (4x^3 - 4y, -4x + 4y^3) = 0$$

$$\Longrightarrow y = x^3 \Longrightarrow -x + x^9 = 0 \Longrightarrow x = 0, \pm 1$$

f has a global min. Which critical point?

$$f(0, 0) = 0, f(-1, -1) = -2, f(1, 1) = -2$$

Global min: (-1,-1) and (1,1).

Note

$$Hf = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

$$Hf(0, 0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

indefinite. Hence (0,0) is a saddle point.

$$Hf(1, 1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

positive definite. Hence (1,1) local min.

$$Hf(-1, -1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

positive definite. Hence (-1,-1) local min.