MFDS ASSIGNMENT

Q1) i) prove that $I-B$ is non-Singular

Ans) Let us consider,

the possible Eigen values for a Skew-Symmetric matrix.

Lets 'λ' be the Eigen Value for B, then

$$Bx = \lambda x \quad -(1)$$

~~Consider L.H.~~

Multiply with Conjugate of x^T on both sides.

$$\Rightarrow \bar{x}^T B x = \bar{x}^T . \lambda . x$$

Consider R.H.S;

$$\Rightarrow \bar{x}^T . \lambda . x = \lambda \, \bar{x}^T . x$$

$$= \lambda ||x||^2$$

Consider L.H.S,

$$\bar{x}^T . B . x$$

This as good as dot product on $\bar{x}^T$ and Bx.

Apply Commutative on dot product

i.e., $A^T . B = B^T . A$

$$\Rightarrow (B.x)^T . \bar{x}^T$$

Q.P1

$\cancel{\Rightarrow x}$

Apply Transpose of a product i.e.,

$$(AB)^T = B^T \cdot A^T$$

$\Rightarrow \quad x^T \cdot B^T \cdot \bar{x} \quad \text{--- (2)}$

$\Rightarrow$ In skew symmetric matrix

$$A^T = -A \quad \text{--- (3)}$$

Substitute/apply (3) in (2)

$\Rightarrow \quad -x^T \cdot B \cdot \bar{x} \quad \text{--- (4)}$

Consider ~~RHS~~ Equation (1)

$\cancel{Ax = \lambda x}$

$$Bx = \lambda x$$

Apply conjugate on both sides

$$B\bar{x} = \bar{\lambda}\bar{x} \quad \text{--- (5)}$$

Here B was real

$\Rightarrow$ Substitute (5) in (4)

$\Rightarrow \quad -x^T \cdot \bar{\lambda} \cdot \bar{x}$

$\Rightarrow \quad -\bar{\lambda} \cdot ||x||^2$

Now try to Equate L.H.S & R.H.S

$\Rightarrow \quad -\bar{\lambda} \cdot ||x||^2 = -\lambda ||x||^2$

As x is Eigen Vector, it will be a non-zero Vector.

R. P.

$\Rightarrow -\bar{\lambda} = \lambda$

The matrix will be Equal to → -(Conjugate) only when it is Zero & λ is purely imaginary.

So the possible Eigen vectors for Skew Symmetric matrix will be either Zero or purely imaginary.

→ Let get back to proof

$I-B$ is non-singular.

The Eigen Values of $I-B$ will be of the for $1-\lambda$

Here λ will be either 0 or ib.

In both Cases, ~~the~~ The Eigen value won't be negitive.

So $I-B$ won't be Singular.

Q) 2) $A = (I+B)(I-B)^{-1}$ &

B is skew symmetric.

Apply Transpose on both sides

$A^T = \left((I+B).(I-B)^{-1}\right)^T$ — ①

Check R.H.S.

$\because (AB)^T = B^T . A^T$

Q_3 P_3

$\Rightarrow \quad = \left(\left(I-B\right)^{-1}\right)^{T} \cdot \left(I+B\right)^{T}$

$\Rightarrow \quad = \left(\left(I-B\right)^{T}\right)^{-1} \cdot \left(I+B\right)^{T}$

$\Rightarrow \quad = \left(I-B^{T}\right)^{-1} \cdot \left(I+B^{T}\right)$

$\left(\because \left(A+B\right)^{T} = \cancel{A^T} A^{T}+B^{T}\right)$

$\Rightarrow \cancel{= \left(I+B\right)^{-1} \cdot \left(I - B^{T}\right)}$

$\Rightarrow \quad = \left(I+B\right)^{-1} \cdot \left(I-B\right) \quad - (2)$

Multiply A & A^{T}

$\Rightarrow \cancel{A \cdot A^{T} = \left(I+B\right)\left(I-B\right)^{-1}}$

$\Rightarrow \quad A^{T} \cdot A = \left(I+B\right)^{-1} \cdot \left(I-B\right) \cdot \left(I+B\right)\left(I-B\right)^{-1}$

Apply Commutative on dot product of matrices

$\left(I-B\right) \cdot \left(I+B\right) = \left(I+B\right) \cdot \left(I-B\right)$

$\Rightarrow$

$\Rightarrow A \cdot A^{T} = \left(\left(I+B\right)^{-1}\left(I+B\right)\right)\left(\left(I-B\right)\left(I-B\right)^{-1}\right)$

$= \left(\because A \cdot A^{-1} = I\right)$

$A \cdot A^{T} = I \cdot I$

$\Rightarrow \quad A \cdot A^{T} = I$

Apply A^{-1} on both sides

$\Rightarrow \quad A^{-1} A \cdot A^{T} = A^{-1} \cdot I$

$\Rightarrow A' = -A^{-1}$.

Hence proved.

Q2) Given

$$M = \{m_1, m_2, m_3, \ldots m_r\}$$

$$N = \{m_1, m_2, m_3, m_4 \ldots m_{r+1}\}$$

be two vectors of same vector space.

Ans Let x be value formed by the variables $c_1, c_2 \ldots c_n$ on vector space M & $x \in \text{Span}\{M\}$

$$\Rightarrow x = c_1 m_1 + c_2 m_2 + \ldots + c_r m_r \quad (1)$$

Assume that $\text{Span}\{M\} = \text{Span}\{N\}$

Then there exists some constants for x $t_1, t_2 \ldots t_r, t_{r+1}$ such ~~that~~ that vectors it can form a linear combination over v

$$\Rightarrow x = t_1 m_1 + t_2 m_2 + \ldots + t_r m_r + t_{r+1} v \quad (2)$$

Equate (1) & (2)

$$\Rightarrow c_1 m_1 + c_2 m_2 + \ldots + c_r m_r = t_1 m_1 + \ldots + t_r m_r + t_{r+1} v$$

$$\Rightarrow t_{r+1} v = (c_1 - t_1) m_1 + (c_2 - t_2) m_2 + \ldots + (c_m - t_m) m_r$$

$$v = \frac{c_1 - t_1}{t_{r+1}} m_1 + \ldots + \frac{c_m - t_m}{t_{r+1}} m_r \quad (3)$$

Q2 P1

So $v = \sum k_i m_i$ where $k_i = \frac{t_i - t_{r+1}}{\dots}$

$\Rightarrow v = \text{Span}\{M\}$

(9)

try replacing (3) in (2)

$\Rightarrow \text{Span}\{N\} = t_1 m_1 + \dots + t_r m_r + \sum k_i m_i \cdot$

$= \sum t_i m_i + \sum k_i m_i \cdot t_{r+1}$

$= \text{Span}\{N\}$

(Q3) Ans) Let $\beta = \{v_1 - - - v_m\}$ be basis for Null (T)

→ Extend β to $\{v_1 - - v_{m+1} - - v_N\}$ for basis for of V.

→ Suppose $a_{m+1} T(v_{m+1}) + - - - + a_N T(v_N)$ Null vector

$\Rightarrow T(a_{m+1} v_{m+1} + - - + a_N v_N) = 0$ vector.

$\Rightarrow a_{m+1} v_{m+1} + - - - + a_N v_N \in Null(T)$ — ①

Earlier we already considered $\{v_1 - - v_n\}$ be the basis ⇒

⇒ $Span\{v_1 - - - v_m\} = Null(T)$ — ②

Compare ① & ②

⇒ $a_{m+1} v_{m+1} + - - - + a_N v_N = Span\{v_1 - - v_m\}$

⇒ $(\because Span\{v_1 - - v_n\} = c_1 v_1 + c_2 v_2 + - - + c_n v_n$ where $c_1, c_2 - -$ are constants)

⇒ $a_m v_{m+1} + - - + a_n v_N = + c_1 v_1 + c_2 v_2 + - - + c_m v_m$.

Q3)

$\Rightarrow -c_1v_1 - c_2v_2 - \cdots - c_mv_m + a_mv_{m+1} + \cdots + a_nv_n = 0$

Here $\{v_1, \cdots v_n\}$ will be basis for V and they are all in liner combination & linerly independent.

Now V ~~will b~~ can be written as the ~~Span~~ liner combination of $\{v_1 \cdots v_n\}$

$\Rightarrow \text{Span}\{v\} = \text{Span}\{v_1, v_2 \cdots v_n\} \cdot \{c_1 \cdots c_n\}$

$\Rightarrow v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

$\Rightarrow R(T(v)) = T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$

$\Rightarrow w = T(v) = a_1T(v_1) + \cdots + a_mT(v_m) + \cdots + a_nT(v_n)$

~~from~~ Equation (1)

~~$(a_1 T(v_1))$~~

We considered $\{v_1, v_2 \cdots v_n\}$ was basis over Null(T)

$\Rightarrow T(v_1) = 0, T(v_2) = 0 \cdots T(v_m) = 0$

$\Rightarrow w = T(v) = a_mT(v_m) + \cdots + a_nT(v_n)$

$\Rightarrow w = T(v) = \text{Span}\{T(v_{m+1}), \cdots T(v_n)\}$

Q.P

$\Rightarrow w = R(T(v)) \in \text{Span}\{T(v_{m+1}) - T(v_N)\}$

$\Rightarrow$ Span of T contains $N-m$ items.

So, dimention of w will be $N-m$.

$\Rightarrow$ The dimension of w of Rank of T will be $N-m$, Considering 'm' will be Nullity of T.

$\Rightarrow$ Adding Nullity of T & Rank of T dimensions

$\Rightarrow m + (N-m)$

$\Rightarrow N \Rightarrow$ which is the dimention of V

Q3 P3

Q4) Find the Eigen Values, eigen Vectors for the matrix $A_{n\times n}$ whose elements are given by $a_{ij} = \begin{cases} \alpha & \text{if } i=j \\ 1 & \text{if } |i-j|=1 \\ 0 & \text{otelse} \end{cases}$

Here α is constant.

Ans:- $\Rightarrow$ Let λ be the eigen value

$\Rightarrow a_n V = \lambda V \quad -(1)$

$\Rightarrow (a_n - \lambda I_n) V = 0$

As V is non-zero matrix.

$|(a_n - \lambda I_n)| = 0 \quad - (2)$

Let us consider $\det(a_n - I_n \lambda) = T_n$

So we can obtain recurrance relation from the above

$\Rightarrow T_{n+2} - (\alpha-\lambda) T_{n+1} + T_n = 0$ for all $n \geq 1$

where $T_1 = \alpha - \lambda$, $T_2 = (\alpha-\lambda)^2 - 1 \rightarrow (3)$

$T_n = \frac{1}{\sqrt{\Delta}} \left(p^{n+1} - q^{n+1} \right) \rightarrow (4)$

Here $\Delta = (\alpha-\lambda)^2 - 4$, when $\Delta \neq 0 \rightarrow (5)$

$p = \frac{\alpha - \lambda + \sqrt{(\alpha-\lambda)^2 - 4}}{2} \rightarrow (6)$

$q = \frac{\alpha - \lambda - \sqrt{(\alpha-\lambda)^2 - 4}}{2}$

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Condition (2) works only when

$$p^{n+1} - q^{n+1} = 0 \quad - (7)$$

$\Rightarrow \quad pq = 1$ (using p, q values of (6)) $- (8)$

from (7) & (8)

$$(p^2)^{n+1} = 1 \quad - (9)$$

The condition together with (8) leads to

$p+2$

$$p = \left(\cos \frac{s\pi}{n+1} + i \sin \frac{s\pi}{n+1} \right)$$

$$s = 1, 2, \dots (n+1) \quad - (10)$$

$$q = \left(\cos \frac{s\pi}{n+1} - i \sin \frac{s\pi}{n+1} \right) \quad - (11)$$

(8)

$$p = -\left(\cos \frac{s\pi}{n+1} + i \sin \frac{s\pi}{n+1} \right) \quad - (12)$$

$$q = -\left(\cos \frac{s\pi}{n+1} + \sin \frac{s\pi}{n+1} \right) \quad - (13)$$

By using (10), (11), (12), (13), we can get same eigen values.

$$p+q = 2 \cos \frac{s\pi}{n+1} \quad - (14)$$

where $s = 1, 2, \dots (n+1)$

from (6) we have

$$p+q = \alpha - \lambda$$

9 + 2

Replace $p+q$ ~~of~~ 14

$\Rightarrow \alpha - \lambda = 2\cos\frac{s\pi}{n+1}$

$\forall s = \{1, 2, \ldots n+1\}$

$\lambda = \alpha - 2\cos\frac{s\pi}{n+1}, \forall s = \{1, \ldots n\}$

Here at $(n+1)$ th we are getting $\Delta = 0$, so ~~for~~ the condition doesn't hold at $s = n+1$

$\Rightarrow \lambda = \alpha - 2\cos\frac{s\pi}{n+1} \quad \forall s = \{1, 2 \ldots n\}$.

So the components of eigen vectors will be of form

$V_1 = \sin\frac{nk\pi}{n+1}$ where $k = 1$ to n

p_4 3

Q5) Find the Singular Value decomposition of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{pmatrix}$ and determine the angle of rotation induced by U & V. Also, write the rank 1 decomposition of A in terms of the columns U & ~~V~~ rows of V. Can we do dimentionality reduction in this case?

A: Given $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

According to ~~SD~~ SVD, ~~A~~ can be written in the form of $U\Sigma V^T$.

$$\Rightarrow A = U\Sigma V^T \quad -(1) \quad \text{(Apply transpose)},$$

$$A^T = V\Sigma^T U^T \quad (2)$$

Multiply (1) & (2)

$$\Rightarrow A \cdot A^T = U\Sigma\Sigma^T U^T \quad -(3)$$

Calculate eigen vectors for $A \cdot A^T$

$$\Rightarrow A \cdot A^T x = \lambda \cdot x$$

$$\Rightarrow (A \cdot A^T - I\lambda)x = 0$$

As 'x' is Non-zero matrix

$$|A \cdot A^T - \lambda I| = 0 \quad -(4)$$

$\Rightarrow$ ~~A~~ Calculate $A \cdot A^T$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 17 & 16 \\ 16 & 17 \end{bmatrix} \quad - (5)$$

Replace Equation (5) in (4)

$$\Rightarrow \begin{vmatrix} 17-\lambda & 16 \\ 16 & 17-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (17-\lambda)^2 - 16^2 = 0$$

$$\Rightarrow \lambda = 1, 33$$

possible Eigen vectors will be

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Length of Eigen vectors $= \sqrt{\Sigma(\text{values in vector})}$

$\Rightarrow$ ~~$v_1 =$~~

length of $v_1 = \lambda_1 = \sqrt{1^2+1^2} = \sqrt{2}$

length of $v_2 = \lambda_2 = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\Rightarrow$ possible unit vectors will be

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Now Calculate the Eigen Values for

$$A^T \cdot A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 10 \\ 12 & 13 & 10 \\ 10 & 10 & 8 \end{bmatrix}$$

Calculate Eigen vectors for $A^T \cdot A$

$$\Rightarrow |A^T \cdot A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 13-\lambda & 12 & 10 \\ 12 & 13-\lambda & 10 \\ 10 & 10 & 8-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow = (13-\lambda)\left((13-\lambda)(8-\lambda) - 100\right) - 12\left(12(8-\lambda) - 100\right)$$

$$+ 10\left(120 - 10(13-\lambda)\right)$$

$$\Rightarrow = -\lambda^3 + 34\lambda^2 - 33\lambda = 0$$

$$\Rightarrow \lambda(\lambda-1)(\lambda-33) = 0$$

$$\Rightarrow \lambda = 0, 1, 33.$$

& replace λ values to find Eigen vectors.

$$\lambda_1 = 0 \Rightarrow v_1 = \begin{bmatrix} -0.4 \\ -0.4 \\ 1 \end{bmatrix}$$

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$\Rightarrow \lambda_2 = 1 \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \lambda_3 = 33 \Rightarrow v_3 = \begin{bmatrix} 1.2 \\ 1.2 \\ 1 \end{bmatrix}$

Length of Eigen vectors will be

$\sqrt{a^2 + b^2 + c^2}$

$\Rightarrow \lambda_1 = 0 \Rightarrow v_1 = \begin{bmatrix} -0.4 \\ -0.4 \\ 1 \end{bmatrix} \Rightarrow L_1 = \sqrt{(0.4)^2 + (0.4)^2 + 1}$

$L_1 = 1.1$

$\Rightarrow \lambda_2 = 1, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow L_2 = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}$

$\Rightarrow \lambda_3 = 33, v_3 = \begin{bmatrix} 1.2 \\ 1.2 \\ 1 \end{bmatrix} \Rightarrow L_3 = \sqrt{1.2^2 + 1.2^2 + 1^2}$

$= 1.9697$

$\Rightarrow$ Making into unit vectors

$\Rightarrow v_{12} = \begin{bmatrix} -0.4/1.1 \\ -0.4/1.1 \\ 1/1.1 \end{bmatrix} = \begin{bmatrix} -0.36 \\ -0.36 \\ 0.91 \end{bmatrix}$

$\Rightarrow v_{22} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -0.707 \\ 0.707 \\ 0 \end{bmatrix}$

$\Rightarrow v_{32} = \begin{bmatrix} 1.2/1.9697 \\ 1.2/1.9697 \\ 1/1.9697 \end{bmatrix} = \begin{bmatrix} 0.6092 \\ 0.6092 \\ 0.5076 \end{bmatrix}$

P.T.O.

$$\Sigma = \begin{bmatrix} \sqrt{33} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5.74 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$U = (u_1, u_2) = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \quad -(8)$$

$\Rightarrow$ V can be formed using

$$V_i = \frac{1}{\sigma} A^T V_i$$

Now, $$\Sigma = \begin{bmatrix} \sqrt{33} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5.74 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (9)$$

$$V = \begin{bmatrix} V_{12} & V_{22} & V_{32} \end{bmatrix} = \begin{bmatrix} 0.6092 & -0.707 & -0.36 \\ 0.6092 & 0.707 & -0.36 \\ 0.5076 & 0 & 0.91 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0.6092 & 0.6092 & 0.5076 \\ -0.707 & 0.707 & 0 \\ -0.36 & -0.36 & 0.91 \end{bmatrix} \quad -(10)$$

$A_{2\times3}$ Combining (8) (9) & (10)

$$A = U \Sigma V^T$$

$$\Rightarrow A_{2\times3} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \cdot \begin{bmatrix} 5.74 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.6092 & 0.6092 & 0.5076 \\ -0.707 & 0.707 & 0 \\ -0.36 & -0.36 & 0.91 \end{bmatrix}$$

$R_5 P_5$

(ii) For calculating the angle of rotation we need to calculate the trace of the diagonal.

$$\text{Trace of } U_{2\times2} = 0.707 + 0.0707$$
$$= 1.414$$

$$\text{Trace of } V_{3\times3} = 0.6092 + 0.707 + 0.91$$
$$= 1.3162.$$

$$\text{Angle of rotation} = \cos^{-1}\left(\frac{\text{Trace} - 1}{2}\right)$$

$$\text{Angle of rotation of } U = \cos^{-1}\left(\frac{1.414 - 1}{2}\right)$$
$$= \cos^{-1}\left(\frac{0.414}{2}\right)$$
$$= \cos^{-1}(0.207)$$
$$= 1.362$$

$$\text{Angle of rotation of } V = \cos^{-1}\left(\frac{0.3162}{2}\right)$$
$$= 0.1581$$
$$= 1.412$$

$q_5 \; p_6$

(viii) Rank 1 decomposition of A in terms of column matrix of U & row matrix of V

⇒ In SVD, the three rank one matrices in the form

$$A = U\Sigma V^T = 5.74 U_1 V_1^T + 1 \cdot u_2 V_2^T + 0 \cdot u_3 V_3^T$$

$$= 5.74 \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} 0.6092 & 0.6092 & 0.5026 \end{bmatrix}$$

$$+ 1 \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} -0.707 & 0.707 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.495 & 2.495 & 2 \\ 2.495 & 2.495 & 2 \end{bmatrix} +$$

$$\begin{bmatrix} 0.49 & -0.49 & 0 \\ -0.49 & 0.49 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$$

(ix) Can we do dimentionality reduction

Sol:- Yes, we can as one of singular values 0. hence the out come of

Since right singular matrices gives the new axis and the variance of data across new axis.

$P_5 P_7$

which means r not contributing at all in segregating the data, hence we can drop that axis.

we can validate by removing V_3 in Σ & V, still we will get same streach

$$\Rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}_{2\times3} = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}_{2\times2} \begin{bmatrix} 0.54 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} 0.609 & 0.609 & 0.492 \\ -0.707 & +0.707 & 0 \end{bmatrix}_{2\times3}$$

Rr Pr